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On arbitrarily slow convergence rates for strong numerical approximations of Cox-Ingersoll-Ross processes and squared Bessel processes

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Abstract

Cox-Ingersoll-Ross (CIR) processes are extensively used in state-of-the-art models for the approximative pricing of financial derivatives. In particular, CIR processes are day after day employed to model instantaneous variances (squared volatilities) of foreign exchange rates and stock prices in Heston-type models and they are also intensively used to model short-rate interest rates. The prices of the financial derivatives in the above mentioned models are very often approximately computed by means of explicit or implicit Euler- or Milstein-type discretization methods based on equidistant evaluations of the driving noise processes. In this article we study the strong convergence speeds of all such discretization methods. More specifically, the main result of this article reveals that each such discretization method achieves at most a strong convergence order of $\delta/2$, where $0 < \delta < 2$ is the dimension of the squared Bessel process associated to the considered CIR process. In particular, we thereby reveal that discretization methods currently employed in the financial industry may converge with arbitrarily slow strong convergence rates to the solution of the considered CIR process. This article thus discovers the alarming situation that discretization methods currently employed in the financial engineering industry are thus not capable to solve CIR processes in the strong sense in a reasonable computational time. We thereby lay open the need of the development of other more sophisticated approximation methods which are capable to solve CIR processes in the strong sense in a reasonable computational time and which thus can not belong to the class of algorithms which use equidistant evaluations of the driving noise processes.

1 Introduction

Stochastic differential equations (SDEs) are a key ingredient in a number of models from economics and the natural sciences. In particular, SDE based models are day after day used in the financial engineering industry to approximately compute prices of financial derivatives. The SDEs appearing in such models are typically highly nonlinear and contain non-Lipschitz nonlinearities in the drift or diffusion coefficient. Such SDEs can in almost all cases not be solved explicitly and it has been and still is a very active topic of research to approximate SDEs with non-Lipschitz nonlinearities; see, e.g., Hu [24], Gyöngy [14], Higham, Mao, & Stuart [21], Hutzenthaler, Jentzen, & Kloeden [27], Hutzenthaler & Jentzen [26], Sabanis [38, 39], and the references mentioned therein. In particular, in about the last five years several results have been obtained that demonstrate that approximation schemes may converge arbitrarily slow, see Hairer, Hutzenthaler, & Jentzen [16], Jentzen, Müller-Gronbach, & Yaroslavtseva [28], Müller-Gronbach & Yaroslavtseva [34], Yaroslavtseva [40], and Gerencsér, Jentzen, & Salimova [12]. For example, Theorem 1.2 in [28] demonstrates that there exists an SDE that has solutions with all moments bounded but for which all approximation schemes that use only evaluation points of the driving Brownian motion converge in the strong sense with an arbitrarily slow rate; see also [16, Theorem 1.3], [34, Theorem 3], [40, Theorem 1], and [12, Theorem 1.2] for related results. All the SDEs in the above examples are purely academic with no connection to applications. The key contribution of this work is to reveal that such slow convergence phenomena also arise in concrete models from applications. To be more specific, in this work we reveal that Cox-Ingersoll-Ross (CIR) processes and squared Bessel processes can in the strong sense in general not be solved approximately in a reasonable computational time by means of schemes using equidistant evaluations of the driving Brownian motion. The precise formulation of our result is the subject of the following theorem.

Theorem 1 (Cox-Ingersoll-Ross processes)**.** *Let $T, a, \sigma \in (0, \infty)$, $b, x \in [0, \infty)$ satisfy $2a < \sigma^2$, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0, T]\times\Omega \to \mathbb{R}$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X\colon [0, T]\times \Omega \to [0, \infty)$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [0, T]$ $\mathbb{P}$-a.s. that*

$$X_t = x + \int_0^t (a - bX_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\,\mathrm{d}W_s. \tag{1}$$

Then there exists a real number $c \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\inf_{\substack{\varphi\colon \mathbb{R}^N\to\mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T)\big|\Big] \geq c \cdot N^{-(2a)/\sigma^2}. \tag{2}$$

Theorem 1 is an immediate consequence of Theorem 34 in Section 5 below. Upper error bounds for strong approximation of CIR processes and squared Bessel processes, i.e., the opposite question of Theorem 1, have been intensively studied in the literature; see, e.g., Delstra & Delbaan [10], Alfonsi [1], Higham & Mao [22], Berkaoui, Bossy, & Diop [3], Gyöngy & Rásonyi [15], Dereich, Neuenkirch, & Szpruch [11], Alfonsi [2], Hutzenthaler, Jentzen, & Noll [25], Neuenkirch & Szpruch [36], Bossy & Olivero Quinteros [5], Hutzenthaler & Jentzen [26], Chassagneux, Jacquier, & Mihaylov [6], Hefter & Herzwurm [17], and Hefter & Herzwurm [18] (for further approximation results, see, e.g., Milstein & Schoenmakers [33], Cozma & Reisinger [9], and Kelly & Lord [31]). In the following we relate our result to these results.

Using the truncated Milstein scheme with the corresponding error bound from Hefter & Herzwurm [18] we get that the the lower bound obtained in (2) is essentially sharp. The precise formulation of this observation is the subject of the following corollary.

Corollary 2 (Cox-Ingersoll-Ross processes). *Let $T, a, \sigma \in (0,\infty)$, $b, x \in [0,\infty)$ satisfy $4a < \sigma^2$, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X\colon [0,T]\times \Omega \to [0,\infty)$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [0,T]$ $\mathbb{P}$-a.s. that*

$$X_t = x + \int_0^t (a - bX_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\,\mathrm{d}W_s. \tag{3}$$

Then there exist real numbers $c, C \in (0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$c \cdot N^{-2a/\sigma^2} \leq \inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R} \\ \textit{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T)\big|\Big] \leq C \cdot N^{-2a/\sigma^2}. \tag{4}$$

The lower bound in (4) is an immediate consequence of Theorem 1 and the upper bound in (4) is an immediate consequence of Hefter & Herzwurm [18, Theorem 2] using the truncated Milstein scheme. We conjecture that in the full parameter range $a, \sigma \in (0,\infty)$ the convergence order in (4) is equal to $\min\{2a/\sigma^2, 1\}$, since for scalar SDEs with coefficients satisfying standard assumptions a convergence order of one is optimal; see, e.g., Hofmann, Müller-Gronbach, & Ritter [23] and Müller-Gronbach [35]. Upper and lower error bounds for CIR processes are crucial due to the fact that CIR processes are a key ingredient in several models for the approximative pricing of financial derivatives on stocks (see, e.g., Heston [20]), interest rates (see, e.g., Cox, Ingersoll, & Ross [7]), and foreign exchange markets (see, e.g., Cozma & Reisinger [8]).

The remainder of this article is organized as follows. In Section 2 we review a few elementary properties of CIR processes and squared Bessel processes. In Section 3 we present some basic results for general SDEs. In Section 4 we prove the lower error bound for a specific parameter range, which is then generalized in Section 5.

2 Basics of Cox-Ingersoll-Ross (CIR) processes and squared Bessel processes

2.1 Setting

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a Brownian motion, and for every $\delta \in (0,\infty)$, $b, z \in [0,\infty)$ and every Brownian motion $V\colon [0,\infty)\times\Omega \to \mathbb{R}$ let $Z^{z,\delta,b,V}\colon [0,\infty)\times\Omega \to [0,\infty)$ be a $(\sigma_\Omega(\{\{V_s \leq a\}\colon a \in \mathbb{R}, s \in [0,t]\} \cup \{A \in \mathfrak{F}\colon \mathbb{P}(A) = 0\}))_{t\in[0,\infty)}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$Z_t^{z,\delta,b,V} = z + \int_0^t \big(\delta - b\, Z_s^{z,\delta,b,V}\big)\,\mathrm{d}s + \int_0^t 2\sqrt{Z_s^{z,\delta,b,V}}\,\mathrm{d}W_s. \tag{5}$$

2.2 A comparison principle

Lemma 3. *Assume the setting in Section 2.1 and let* $\delta \in (0,\infty)$, $b_1, b_2, z_1, z_2 \in [0,\infty)$ *satisfy* $z_1 \leq z_2$ *and* $b_1 \geq b_2$. *Then*

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon Z_t^{z_1,\delta,b_1,W} \leq Z_t^{z_2,\delta,b_2,W}\Big) = 1. \tag{6}$$

Proof of Lemma 3. Equation (6) is, e.g., an immediate consequence of Karatzas & Shreve [30, Proposition 5.2.18]. The proof of Lemma 3 is thus completed. ☐

2.3 A priori moment bounds

Lemma 4. *Assume the setting in Section 2.1 and let* $\delta, T \in (0,\infty)$, $b, z \in [0,\infty)$, $p \in [1,\infty)$. *Then*

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]} \big|Z_t^{z,\delta,b,W}\big|^p\Bigg] < \infty. \tag{7}$$

Proof of Lemma 4. Inequality (7) follows, e.g., from Mao [32, Corollary 2.4.2]. The proof of Lemma 4 is thus completed. ☐

2.4 Lipschitz continuity in the initial value

In the next result, Lemma 5, we recall a well-known explicit formula for the first moments of CIR processes and squared Bessel processes (cf., e.g., Cox, Ingersoll, & Ross [7, Equation (19)]).

Lemma 5 (An explicit formula for the first moment)**.** *Assume the setting in Section 2.1 and let* $\delta \in (0,\infty)$, $b, z, t \in [0,\infty)$. *Then*

$$\begin{aligned} &\mathbb{E}\big[Z_t^{z,\delta,b,W}\big] \\ &= z \cdot e^{-bt} + \delta \cdot \left(\int_0^t e^{-bs}\,\mathrm{d}s\right) = z \cdot e^{-bt} + \delta \cdot \begin{cases} (1-e^{-bt})/b & : b \neq 0 \\ t & : b = 0 \end{cases}. \end{aligned} \tag{8}$$

Proof of Lemma 5. Throughout this proof let $f\colon [0,\infty) \to \mathbb{R}$ be the function which satisfies for all $r \in [0,\infty)$ that

$$f(r) = \mathbb{E}\big[Z_r^{z,\delta,b,W}\big]. \tag{9}$$

Observe that Lemma 4, the fact that $Z_r^{z,\delta,b,W}$, $r \in [0,\infty)$, is a stochastic process with continuous sample paths, and Lebesgue's dominated convergence theorem ensure that f is a continuous function. This and (5) show that for all $r \in [0,\infty)$ it holds that

$$f(r) = z + \int_0^r \mathbb{E}\big[\delta - b\, Z_s^{z,\delta,b,V}\big]\,\mathrm{d}s = z + \int_0^r (\delta - b\, f(s))\,\mathrm{d}s. \tag{10}$$

This demonstrates that f is continuously differentiable and that for all $r \in [0,\infty)$ it holds that

$$f'(r) = \delta - b\, f(r). \tag{11}$$

Hence, we obtain that for all $r \in [0,\infty)$ it holds that

$$f(r) = e^{-br} z + \int_0^r e^{-b(r-s)}\,\delta\,\mathrm{d}s = e^{-br} z + \left(\int_0^r e^{-bs}\,\mathrm{d}s\right)\delta. \tag{12}$$

This and the fact that

$$\forall\,\beta \in (0,\infty)\colon \int_0^t e^{-\beta s}\,\mathrm{d}s = \frac{1}{-\beta}\int_0^t -\beta\, e^{-\beta s}\,\mathrm{d}s = \tfrac{1}{-\beta}\left[e^{-\beta s}\right]_{s=0}^{s=t} = \frac{e^{-\beta t}-1}{-\beta} = \frac{1-e^{-\beta t}}{\beta} \tag{13}$$

complete the proof of Lemma 5. □

Lemma 6 (L^1-Lipschitz continuity)**.** *Assume the setting in Section 2.1 and let $\delta \in (0,\infty)$, $b, z_1, z_2, t \in [0,\infty)$. Then*

$$\mathbb{E}\Big[\big|Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}\big|\Big] = e^{-bt}\cdot|z_1 - z_2|\,. \tag{14}$$

Proof of Lemma 6. Throughout this proof assume w.l.o.g. that $z_1 \geq z_2$. Next note that Lemma 3, Lemma 4, and Lemma 5 show that

$$\begin{aligned}&\mathbb{E}\Big[\big|Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}\big|\Big] = \mathbb{E}\Big[Z_t^{z_1,\delta,b,W} - Z_t^{z_2,\delta,b,W}\Big]\\&= \mathbb{E}\Big[Z_t^{z_1,\delta,b,W}\Big] - \mathbb{E}\Big[Z_t^{z_2,\delta,b,W}\Big] = z_1\cdot e^{-bt} - z_2\cdot e^{-bt} = e^{-bt}\cdot|z_1-z_2|\,.\end{aligned} \tag{15}$$

The proof of Lemma 6 is thus completed. □

2.5 The scaling property

Lemma 7. *Assume the setting in Section 2.1 and let $\delta, c \in (0,\infty)$, $b, z \in [0,\infty)$. Then*

$$\mathbb{P}\Big(\forall\, t\in[0,\infty)\colon c\cdot Z_{t/c}^{z/c,\delta,cb,(c^{-1/2}W_{cs})_{s\in[0,\infty)}} = Z_t^{z,\delta,b,W}\Big) = 1. \tag{16}$$

Proof of Lemma 7. Equation (16) follows directly from the corresponding scaling property of Brownian motion and the stochastic integral (cf., e.g., Revuz & Yor [37, Proposition XI.1.6]). The proof of Lemma 7 is thus completed. □

2.6 Hitting times

Lemma 8 (The Feller boundary condition)**.** *Assume the setting in Section 2.1 and let $\delta \in (0,\infty)$, $b, z \in [0,\infty)$. Then*

$$\begin{aligned}&\mathbb{P}\Big(\forall\, t\in(0,\infty)\colon Z_t^{z,\delta,b,W} > 0\Big)\\&= \mathbb{P}\Big(\forall\, t\in(0,\infty)\colon Z_t^{z,\delta,b,W} \neq 0\Big) = \begin{cases}1 & : \delta \geq 2\\ 0 & : \delta < 2\end{cases}.\end{aligned} \tag{17}$$

Proof of Lemma 8. First, observe that, e.g., Göing-Jaeschke & Yor [13, page 315, first paragraph] (cf. also Revuz & Yor [37, page 442]) shows that

$$\mathbb{P}\Big(\forall\, t\in(0,\infty)\colon Z_t^{z,\delta,b,W} > 0\Big) = \begin{cases}1 & : \delta \geq 2\\ 0 & : \delta < 2\end{cases}. \tag{18}$$

This and the fact that $\forall\, t \in [0,\infty)$, $\omega \in \Omega\colon Z_t^{z,\delta,b,W}(\omega) \in [0,\infty)$ complete the proof of Lemma 8. □

Lemma 9 (Bounds for hitting times)**.** *Assume the setting in Section 2.1 and let $\delta \in (0,2)$, $b \in [0,\infty)$, $T \in (0,\infty)$. Then there exists a real number $c \in (0,\infty)$ such that for every $\varepsilon \in (0,T]$ it holds that*

$$\mathbb{P}\bigg(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0 \bigg) \leq c \cdot \varepsilon^{1-\delta/2}. \tag{19}$$

Proof of Lemma 9. Throughout this proof let $\nu \in (0,1)$ be the real number given by $\nu = 1 - \delta/2$, let $\Gamma\colon (0,\infty) \to (0,\infty)$ be the function which satisfies for all $r \in (0,\infty)$ that

$$\Gamma(r) = \int_0^\infty x^{r-1}\, e^{-x}\, \mathrm{d}x \tag{20}$$

(Gamma function), and let $P\colon \mathbb{R} \times (0,\infty) \to [0,\infty)$ be the function which satisfies for all $z \in \mathbb{R}$, $r \in (0,\infty)$ that

$$P(z,r) = \begin{cases} \mathbb{P}\big(\inf_{t\in[0,r]} Z_t^{z,\delta,0,W} > 0 \big) & : z \geq 0 \\ 0 & : z < 0 \end{cases}. \tag{21}$$

There exists a real number $C \in (0,\infty)$ which satisfies for every $z, r \in (0,\infty)$ that

$$P(z,r) = C z^\nu \int_r^\infty t^{-\nu-1} \exp\!\big(\tfrac{-z}{2t}\big)\, \mathrm{d}t, \tag{22}$$

see, e.g., Borodin & Salminen [4, Part I, Chapter IV, Section 6, last equation in 46 on page 79] (with $\nu = \nu$, $y = \sqrt{z}$ in the notation of [4, Part I, Chapter IV, Section 6, last equation in 46 on page 79]). This and the fact that $\forall\, t,z \in (0,\infty)\colon \exp(\frac{-z}{2t}) \leq 1$ imply that for every $z, r \in (0,\infty)$ it holds that

$$P(z,r) \leq C z^\nu \int_r^\infty t^{-(1+\nu)}\, \mathrm{d}t. \tag{23}$$

In the next step we note that for every $\varepsilon \in (0,\infty)$ it holds that the random variable $\varepsilon^{-1} Z_\varepsilon^{0,\delta,0,W}$ is χ^2-distributed with δ degrees of freedom (see, e.g., Revuz & Yor [37, Corollary XI.1.4]). Hence, we obtain that for all $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned} \mathbb{E}\Big[\Big| \tfrac{Z_\varepsilon^{0,\delta,0,W}}{\varepsilon} \Big|^\nu \Big] &= \int_0^\infty x^\nu \left[\frac{\big[\frac{1}{2}\big]^{\delta/2}}{\Gamma(\delta/2)} x^{\delta/2-1} \exp\!\big(\tfrac{-x}{2}\big) \right] \mathrm{d}x \\ &= \frac{\big[\frac{1}{2}\big]^{\delta/2}}{\Gamma(\delta/2)} \int_0^\infty \exp\!\big(\tfrac{-x}{2}\big)\, \mathrm{d}x = \frac{2\,\big[\frac{1}{2}\big]^{\delta/2}}{\Gamma(\delta/2)} \int_0^\infty e^{-x}\, \mathrm{d}x = \frac{2^{1-\delta/2}}{\Gamma(\delta/2)}. \end{aligned} \tag{24}$$

This and (23) imply that for all $r \in (0,\infty)$, $\varepsilon \in (0,r)$ it holds that

$$\begin{aligned} &\mathbb{P}\bigg(\inf_{t\in[\varepsilon,r]} Z_t^{0,\delta,0,W} > 0 \bigg) \\ &= \mathbb{E}\big[P\big(Z_\varepsilon^{0,\delta,0,W}, r-\varepsilon\big) \big] \leq C \cdot \mathbb{E}\Big[\big| Z_\varepsilon^{0,\delta,0,W} \big|^\nu \Big] \cdot \int_{r-\varepsilon}^\infty t^{-(1+\nu)}\, \mathrm{d}t \\ &= C \cdot \int_{r-\varepsilon}^\infty t^{-(1+\nu)}\, \mathrm{d}t \cdot \mathbb{E}\Big[\Big| \tfrac{Z_\varepsilon^{0,\delta,0,W}}{\varepsilon} \Big|^\nu \Big] \cdot \varepsilon^\nu \\ &= C \cdot \int_{r-\varepsilon}^\infty t^{-(1+\nu)}\, \mathrm{d}t \cdot \left[\frac{2^{1-\delta/2}}{\Gamma(\delta/2)} \right] \cdot \varepsilon^\nu = C \cdot \left[\frac{(r-\varepsilon)^{-\nu}}{\nu} \right] \cdot \left[\frac{2^{1-\delta/2}}{\Gamma(\delta/2)} \right] \cdot \varepsilon^\nu. \end{aligned} \tag{25}$$

Therefore, we obtain that for all $\varepsilon \in (0,T)$ it holds that

$$\begin{aligned}\mathbb{P}\Big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,0,W} > 0\Big) &\leq \left[\frac{(T-\varepsilon)^{\delta/2-1}}{(1-\delta/2)}\right]\cdot\left[\frac{C\cdot 2^{1-\delta/2}}{\Gamma(\delta/2)}\right]\cdot\varepsilon^{1-\delta/2} \\ &= \left[\frac{C\cdot 2^{1-\delta/2}\cdot (T-\varepsilon)^{\delta/2-1}}{\Gamma(\delta/2)\cdot(1-\delta/2)}\right]\cdot\varepsilon^{1-\delta/2}.\end{aligned}\tag{26}$$

This and Lemma 3 show that that for all $\varepsilon \in (0, T/2]$ it holds that

$$\begin{aligned}&\mathbb{P}\Big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0\Big) \leq \mathbb{P}\Big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,0,W} > 0\Big) \\ &\leq \left[\frac{C\cdot 2^{1-\delta/2}\cdot (T/2)^{\delta/2-1}}{\Gamma(\delta/2)\cdot(1-\delta/2)}\right]\cdot\varepsilon^{1-\delta/2} = \left[\frac{C\cdot 2^{(2-\delta)}\cdot T^{\delta/2-1}}{\Gamma(\delta/2)\cdot(1-\delta/2)}\right]\cdot\varepsilon^{1-\delta/2}.\end{aligned}\tag{27}$$

Hence, we obtain that

$$\sup_{\varepsilon\in(0,T/2]}\left[\frac{\mathbb{P}\big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0\big)}{\varepsilon^{1-\delta/2}}\right] < \infty. \tag{28}$$

This assures that

$$\begin{aligned}&\sup_{\varepsilon\in(0,T]}\left[\frac{\mathbb{P}\big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0\big)}{\varepsilon^{1-\delta/2}}\right] \\ &\leq \sup_{\varepsilon\in(0,T/2]}\left[\frac{\mathbb{P}\big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0\big)}{\varepsilon^{1-\delta/2}}\right] + \sup_{\varepsilon\in(T/2,T]}\left[\frac{1}{\varepsilon^{1-\delta/2}}\right] \\ &= \sup_{\varepsilon\in(0,T/2]}\left[\frac{\mathbb{P}\big(\inf_{t\in[\varepsilon,T]} Z_t^{0,\delta,b,W} > 0\big)}{\varepsilon^{1-\delta/2}}\right] + \left[\frac{1}{(T/2)^{1-\delta/2}}\right] < \infty.\end{aligned}\tag{29}$$

The proof of Lemma 9 is thus completed. □

3 Basics of general SDEs

3.1 Setting

Let $\mathcal{Z}^{(\cdot),(\cdot)} = (\mathcal{Z}^{z,v})_{z\in\mathbb{R},v\in C([0,\infty),\mathbb{R})}\colon \mathbb{R}\times C([0,\infty),\mathbb{R}) \to C([0,\infty),\mathbb{R})$ be a Borel-measurable and universally adapted function (see Kallenberg [29, page 423] for the notion of an universally adapted function), let $\alpha,\beta\colon\mathbb{R}\to\mathbb{R}$ be continuous functions, assume that for every complete probability space $(\Omega,\mathfrak{F},\mathbb{P})$, every normal filtration $\mathbb{F}=(\mathbb{F}_t)_{t\in[0,\infty)}$ on $(\Omega,\mathfrak{F},\mathbb{P})$, every $\mathbb{F}$-Brownian motion $W\colon[0,\infty)\times\Omega\to\mathbb{R}$, all sample paths continuous $\mathbb{F}$-adapted stochastic processes $Z^{(1)},Z^{(2)}\colon[0,\infty)\times\Omega\to\mathbb{R}$ with $\forall\, i\in\{1,2\}, t\in[0,\infty)\colon \mathbb{P}\big(Z_t^{(i)} = Z_0^{(1)} + \int_0^t \alpha(Z_s^{(i)})\,\mathrm{d}s + \int_0^t \beta(Z_s^{(i)})\,\mathrm{d}W_s\big) = 1$, and every $t\in[0,\infty)$ it holds that

$$\mathbb{P}\Big(Z_t^{(1)} = Z_t^{(2)}\Big) = 1, \tag{30}$$

assume that for every complete probability space $(\Omega,\mathfrak{F},\mathbb{P})$, every normal filtration $\mathbb{F}=(\mathbb{F}_t)_{t\in[0,\infty)}$ on $(\Omega,\mathfrak{F},\mathbb{P})$, every $\mathbb{F}$-Brownian motion $W\colon[0,\infty)\times\Omega\to\mathbb{R}$, every $\mathbb{F}_0/\mathcal{B}(\mathbb{R})$-measurable function $Z\colon \Omega \to \mathbb{R}$, and every $t \in [0,\infty)$ it holds that

$$\mathbb{P}\Big(\mathcal{Z}_t^{Z,W} = Z + \int_0^t \alpha(\mathcal{Z}_s^{Z,W})\,\mathrm{d}s + \int_0^t \beta(\mathcal{Z}_s^{Z,W})\,\mathrm{d}W_s\Big) = 1, \tag{31}$$

and let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space.

3.2 Brownian motion shifted by a stopping time

Lemma 10. *Assume the setting in Section 3.1, let $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,\infty)}$ be a normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}$ be a $\mathbb{F}$-Brownian motion, let $\tau\colon \Omega \to [0,\infty)$ be a $\mathbb{F}$-stopping time, let $Z\colon \Omega \to \mathbb{R}$ be a $\mathbb{F}_0/\mathcal{B}(\mathbb{R})$-measurable function, let $\tilde{W}\colon [0,\infty) \times \Omega \to \mathbb{R}$ be the stochastic process which satisfies for all $t \in [0,\infty)$ that $\tilde{W}_t = W_{t+\tau} - W_\tau$, and let $\tilde{Z}\colon \Omega \to \mathbb{R}$ be the random variable given by $\tilde{Z} = \mathcal{Z}_\tau^{Z,W}$. Then*

(i) it holds that $\tilde{W}$ is a Brownian motion,

(ii) it holds that $\tilde{W}$ and $\tilde{Z}$ are independent, and

(iii) it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon \mathcal{Z}_t^{\tilde{Z},\tilde{W}} = \mathcal{Z}_{t+\tau}^{Z,W}\Big) = 1. \tag{32}$$

Proof of Lemma 10. Throughout this proof let $\tilde{\mathbb{F}} = (\tilde{\mathbb{F}}_t)_{t\in[0,\infty)}$ be the normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$ which satisfies for all $t \in [0,\infty)$ that $\tilde{\mathbb{F}}_t = \mathbb{F}_{t+\tau}$. Observe that the fact that the function $\mathcal{Z}_\tau^{Z,W}$ is $\mathbb{F}_\tau/\mathcal{B}(\mathbb{R})$-measurable ensures that $\tilde{Z}$ is $\tilde{\mathbb{F}}_0/\mathcal{B}(\mathbb{R})$-measurable. In addition, note that, e.g., Kallenberg [29, Theorem 13.11] demonstrates that $\tilde{W}$ is a $\tilde{\mathbb{F}}$-Brownian motion. This and the fact that $\tilde{Z}$ is $\tilde{\mathbb{F}}_0/\mathcal{B}(\mathbb{R})$-measurable show that $\tilde{W}$ and $\tilde{Z}$ are independent. Next observe that the stochastic process $(\mathcal{Z}_{t+\tau}^{Z,W})_{t\in[0,\infty)}$ has continuous sample paths, is $\tilde{\mathbb{F}}$-adapted, and satisfies that for all $t \in [0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
\mathcal{Z}_{t+\tau}^{Z,W} &= Z + \int_0^{t+\tau} \alpha(\mathcal{Z}_s^{Z,W})\,\mathrm{d}s + \int_0^{t+\tau} \beta(\mathcal{Z}_s^{Z,W})\,\mathrm{d}W_s \\
&= Z + \int_0^{\tau} \alpha(\mathcal{Z}_s^{Z,W})\,\mathrm{d}s + \int_0^{\tau} \beta(\mathcal{Z}_s^{Z,W})\,\mathrm{d}W_s \\
&\quad + \int_\tau^{t+\tau} \alpha(\mathcal{Z}_s^{Z,W})\,\mathrm{d}s + \int_\tau^{t+\tau} \beta(\mathcal{Z}_s^{Z,W})\,\mathrm{d}W_s \\
&= \tilde{Z} + \int_0^{t} \alpha(\mathcal{Z}_{s+\tau}^{Z,W})\,\mathrm{d}s + \int_0^{t} \beta(\mathcal{Z}_{s+\tau}^{Z,W})\,\mathrm{d}\tilde{W}_s.
\end{aligned} \tag{33}$$

This establishes (32). The proof of Lemma 10 is thus completed. □

Lemma 11. *Assume the setting in Section 3.1, let $W, \tilde{W}\colon [0,\infty) \times \Omega \to \mathbb{R}$ be Brownian motions, let $\tau\colon \Omega \to [0,\infty]$ be a random variable, assume for all $t \in [0,\infty)$ that $\mathbb{P}(W_{t\wedge\tau} = \tilde{W}_{t\wedge\tau}) = 1$, let $Z\colon \Omega \to \mathbb{R}$ be a random variable, assume that W and Z are independent, and assume that $\tilde{W}$ and Z are independent. Then*

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon \big[\mathcal{Z}_t^{Z,W} - \mathcal{Z}_t^{Z,\tilde{W}}\big]\, \mathbb{1}_{\{t\leq\tau\}}^{\Omega} = 0\Big) = 1. \tag{34}$$

Proof of Lemma 11. Observe that it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty) \cap \mathbb{Q} \colon \big[t \leq \tau \implies W_t = \tilde{W}_t\big]\Big) = 1. \tag{35}$$

The fact that W and $\tilde{W}$ have continuous sample paths hence shows that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \big[t \leq \tau \implies W_t = \tilde{W}_t\big]\Big) = 1. \tag{36}$$

The assumption that $\mathcal{Z}^{(\cdot),(\cdot)}$ is universally adapted therefore proves that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty) \cap \mathbb{Q} \colon \big[\mathcal{Z}^{Z,W}_t - \mathcal{Z}^{Z,\tilde{W}}_t\big]\, \mathbb{1}^{\Omega}_{\{t \leq \tau\}} = 0\Big) = 1. \tag{37}$$

This and the fact that the stochastic process $\big[\mathcal{Z}^{Z,W}_t - \mathcal{Z}^{Z,\tilde{W}}_t\big]\, \mathbb{1}^{\Omega}_{\{t \leq \tau\}} \in \mathbb{R}$, $t \in [0,\infty)$, has left-continuous sample paths establishes (34). The proof of Lemma 11 is thus completed. ☐

Lemma 12. *Assume the setting in Section 3.1, for every $m \in \{0,1\}$ let $\mathbb{F}^{(m)} = (\mathbb{F}^{(m)}_t)_{t\in[0,\infty)}$ be a normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$, assume that $(\cup_{t\in[0,\infty)} \mathbb{F}^{(0)}_t) \subseteq \mathbb{F}^{(1)}_0$, for every $m \in \{0,1\}$ let $W^{(m)} \colon [0,\infty) \times \Omega \to \mathbb{R}$ be a $\mathbb{F}^{(m)}$-Brownian motion, for every $m \in \{0,1\}$ let $\tau^{(m)} \colon \Omega \to [0,\infty)$ be a $\mathbb{F}^{(m)}$-stopping time, let $\bar{\mathbb{F}} = (\bar{\mathbb{F}}_t)_{t\in[0,\infty)}$ be the normal filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ which satisfies for all $t \in [0,\infty)$ that*

$$\bar{\mathbb{F}}_t = \Big\{A \in \mathfrak{F} \colon A \cap \{t < \tau^{(0)}\} \in \mathbb{F}^{(0)}_t \text{ and } A \cap \{t \geq \tau^{(0)}\} \in \mathbb{F}^{(1)}_{\max\{t-\tau^{(0)},0\}}\Big\}, \tag{38}$$

let $\bar{\tau} \colon \Omega \to [0,\infty)$ be the random variable given by $\bar{\tau} = \tau^{(0)} + \tau^{(1)}$, let $\bar{W} \colon [0,\infty) \times \Omega \to \mathbb{R}$ be the stochastic process which satisfies for all $t \in [0,\infty)$ that

$$\bar{W}_t = W^{(0)}_t \mathbb{1}^{\Omega}_{\{t \leq \tau^{(0)}\}} + (W^{(1)}_{|t-\tau^{(0)}|} + W^{(0)}_{\tau^{(0)}}) \mathbb{1}^{\Omega}_{\{t > \tau^{(0)}\}}, \tag{39}$$

let $Z \colon \Omega \to \mathbb{R}$ be a random variable which is $\mathbb{F}^{(0)}_0/\mathcal{B}(\mathbb{R})$-measurable, and let $\tilde{Z} \colon \Omega \to \mathbb{R}$ be the random variable given by $\tilde{Z} = \mathcal{Z}^{Z,W^{(0)}}_{\tau^{(0)}}$. Then

(i) it holds that $\mathbb{F}^{(0)}_0 \subseteq \bar{\mathbb{F}}_0$,

(ii) it holds that $(\cup_{t\in[0,\infty)} \bar{\mathbb{F}}_t) \subseteq \sigma_\Omega\big(\cup_{t\in[0,\infty)} \mathbb{F}^{(1)}_t\big)$,

(iii) it holds that $\bar{\tau}$ is a $\bar{\mathbb{F}}$-stopping time,

(iv) it holds that $\tau^{(0)}$ is a $\bar{\mathbb{F}}$-stopping time,

(v) it holds that $\bar{W}$ is a $\bar{\mathbb{F}}$-Brownian motion,

(vi) it holds that $\bar{W}$ and Z are independent,

(vii) it holds that $W^{(1)}$ and $\tilde{Z}$ are independent, and

(viii) it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}^{Z,\bar{W}}_t = \mathcal{Z}^{Z,W^{(0)}}_t \mathbb{1}^{\Omega}_{\{t \leq \tau^{(0)}\}} + \mathcal{Z}^{\tilde{Z},W^{(1)}}_{|t-\tau^{(0)}|} \mathbb{1}^{\Omega}_{\{t > \tau^{(0)}\}}\Big) = 1. \tag{40}$$

Proof of Lemma 12. Throughout this proof let $F\colon \mathbb{R} \to (0,1)$ be the function which satisfies for all $x \in \mathbb{R}$ that $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$ (distribution function of the standard normal distribution) and for every $t \in [0,\infty)$ let $\rho_t\colon \Omega \to [0,\infty)$ be the random variable given by $\rho_t = \max\{t - \tau^{(0)}, 0\}$. Observe that for every $t \in [0,\infty)$ it holds that ρ_t is a $\mathbb{F}^{(1)}$-stopping time. The fact that $\{0 < \tau^{(0)}\} \in \mathbb{F}_0^{(0)}$ ensures that it holds for every $A \in \mathbb{F}_0^{(0)}$ that $A \cap \{0 < \tau^{(0)}\} \in \mathbb{F}_0^{(0)}$ and

$$A \cap \{0 \geq \tau^{(0)}\} \in \mathbb{F}_0^{(0)} \subseteq \mathbb{F}_0^{(1)} = \mathbb{F}^{(1)}_{\max\{0-\tau^{(0)},0\}}. \tag{41}$$

This proves item (i). Next observe that for every $t \in [0,\infty)$, $A \in \bar{\mathbb{F}}_t$ it holds that

$$A \cap \{t < \tau^{(0)}\} \in \mathbb{F}_t^{(0)} \subseteq \mathbb{F}_0^{(1)} \subseteq \sigma_\Omega(\cup_{s\in[0,\infty)} \mathbb{F}_s^{(1)}) \tag{42}$$

and

$$A \cap \{t \geq \tau^{(0)}\} \in \mathbb{F}^{(1)}_{\max\{t-\tau^{(0)},0\}} \subseteq \sigma_\Omega(\cup_{s\in[0,\infty)} \mathbb{F}_s^{(1)}). \tag{43}$$

Hence, we obtain for every $t \in [0,\infty)$, $A \in \bar{\mathbb{F}}_t$ that

$$A = (A \cap \{t < \tau^{(0)}\}) \cup (A \cap \{t \geq \tau^{(0)}\}) \in \sigma_\Omega(\cup_{s\in[0,\infty)} \mathbb{F}_s^{(1)}). \tag{44}$$

This proves item (ii). Observe that for every $t \in [0,\infty)$ it holds that

$$\{\bar{\tau} \leq t\} \cap \{t < \tau^{(0)}\} = \emptyset \in \mathbb{F}_t^{(0)} \tag{45}$$

and

$$\begin{aligned} &\{\bar{\tau} \leq t\} \cap \{t \geq \tau^{(0)}\} \\ &\quad = \{\bar{\tau} \leq t\} \\ &\quad = \{\tau^{(1)} \leq t - \tau^{(0)}\} \\ &\quad = \{\tau^{(1)} \leq \max\{t - \tau^{(0)}, 0\}\} \cap \{t - \tau^{(0)} \geq 0\} \in \mathbb{F}^{(1)}_{\max\{t-\tau^{(0)},0\}}. \end{aligned} \tag{46}$$

This proves item (iii). In the next step we note that for every $t \in [0,\infty)$ it holds that

$$\{\tau^{(0)} \leq t\} \cap \{t < \tau^{(0)}\} = \emptyset \in \mathbb{F}_t^{(0)} \tag{47}$$

and

$$\{\tau^{(0)} \leq t\} \cap \{t \geq \tau^{(0)}\} = \{\tau^{(0)} \leq t\} \in \mathbb{F}_t^{(0)} \subseteq \mathbb{F}_0^{(1)} \subseteq \mathbb{F}^{(1)}_{\max\{t-\tau^{(0)},0\}}. \tag{48}$$

This proves item (iv). The strong Markov property of Brownian motion (see, e.g., Kallenberg [29, Theorem 13.11]) implies that it holds for every $s \in [0,\infty)$ that $(W^{(1)}_{\rho_s+u} - W^{(1)}_{\rho_s})_{u\in[0,\infty)}$ is a Brownian motion independent of $\mathbb{F}^{(1)}_{\rho_s}$. This and the fact that for every $s \in [0,\infty)$, $A \in \bar{\mathbb{F}}_s$ it holds that $A \cap \{s \geq \tau^{(0)}\} \in \mathbb{F}^{(1)}_{\rho_s}$ demonstrate that for every $s \in [0,\infty)$, $t \in (s,\infty)$, $A \in \bar{\mathbb{F}}_s$, $a \in \mathbb{R}$ it holds that

$$\begin{aligned} &\mathbb{P}\Big(\{\bar{W}_t - \bar{W}_s \leq a\} \cap A \cap \{s \geq \tau^{(0)}\}\Big) \\ &= \mathbb{P}\Big(\{W^{(1)}_{\rho_s+t-s} - W^{(1)}_{\rho_s} \leq a\} \cap A \cap \{s \geq \tau^{(0)}\}\Big) \\ &= F(a/\sqrt{t-s}) \cdot \mathbb{P}\Big(A \cap \{s \geq \tau^{(0)}\}\Big)\,. \end{aligned} \tag{49}$$

Observe that for every $s \in [0,\infty)$, $t \in (s,\infty)$, $A \in \bar{\mathbb{F}}_s$ it holds

(a) that $W^{(0)}_{t\wedge\tau^{(0)}\vee s} - W^{(0)}_s$ is $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}/\mathcal{B}(\mathbb{R})$-measurable,

(b) that $A \cap \{s < \tau^{(0)}\} \in \mathbb{F}^{(0)}_s \subseteq \mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}$, and

(c) that $t - (t \wedge \tau^{(0)} \vee s)$ is $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}/\mathcal{B}(\mathbb{R})$-measurable.

Next note that the fact that for every $s \in [0,\infty)$, $t \in (s,\infty)$ it holds that $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s} \subseteq \sigma_\Omega(\cup_{u\in[0,\infty)}\mathbb{F}^{(0)}_u) \subseteq \mathbb{F}^{(1)}_0$ ensures that for every $s \in [0,\infty)$, $t \in (s,\infty)$ it holds that the Brownian motion $W^{(1)}$ is independent of $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}$. Moreover, observe that the strong Markov property of Brownian motion (see, e.g., Kallenberg [29, Theorem 13.11]) demonstrates that for every $s \in [0,\infty)$, $t \in (s,\infty)$ it holds that $(W^{(0)}_{u+(t\wedge\tau^{(0)}\vee s)} - W^{(0)}_{t\wedge\tau^{(0)}\vee s})_{u\in[0,\infty)}$ is a Brownian motion which is independent of $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}$. Combining items (a)-(c) with the fact that for every $s \in [0,\infty)$, $t \in (s,\infty)$ it holds that the Brownian motion $W^{(1)}$ is independent of $\mathbb{F}^{(0)}_{t\wedge\tau^{(0)}\vee s}$ therefore demonstrates that for every $s \in [0,\infty)$, $t \in (s,\infty)$, $A \in \bar{\mathbb{F}}_s$, $a \in \mathbb{R}$ it holds that

$$\begin{aligned}
&\mathbb{P}\Big(\{\bar{W}_t - \bar{W}_s \le a\} \cap A \cap \{s < \tau^{(0)}\}\Big) \\
&= \mathbb{P}\Big(\{W^{(1)}_{t-(t\wedge\tau^{(0)}\vee s)} + W^{(0)}_{t\wedge\tau^{(0)}\vee s} - W^{(0)}_s \le a\} \cap A \cap \{s < \tau^{(0)}\}\Big) \\
&= \mathbb{P}\Big(\{W^{(0)}_t - W^{(0)}_s \le a\} \cap A \cap \{s < \tau^{(0)}\}\Big).
\end{aligned} \tag{50}$$

The fact that for every $s \in [0,\infty)$, $A \in \bar{\mathbb{F}}_s$ it holds that $A \cap \{s < \tau^{(0)}\} \in \mathbb{F}^{(0)}_s$ hence shows that for all $s \in [0,\infty)$, $t \in (s,\infty)$, $A \in \bar{\mathbb{F}}_s$, $a \in \mathbb{R}$ it holds that

$$\begin{aligned}
&\mathbb{P}\Big(\{\bar{W}_t - \bar{W}_s \le a\} \cap A \cap \{s < \tau^{(0)}\}\Big) \\
&= F(a/\sqrt{t-s}) \cdot \mathbb{P}\Big(A \cap \{s < \tau^{(0)}\}\Big).
\end{aligned} \tag{51}$$

Combining this and (49) imply that it holds for every $s \in [0,\infty)$, $t \in (s,\infty)$, $A \in \bar{\mathbb{F}}_s$, $a \in \mathbb{R}$ that

$$\begin{aligned}
&\mathbb{P}\big(\{\bar{W}_t - \bar{W}_s \le a\} \cap A\big) \\
&= \mathbb{P}\Big(\{\bar{W}_t - \bar{W}_s \le a\} \cap A \cap \{s \ge \tau^{(0)}\}\Big) \\
&\quad + \mathbb{P}\Big(\{\bar{W}_t - \bar{W}_s \le a\} \cap A \cap \{s < \tau^{(0)}\}\Big) \\
&= F(a/\sqrt{t-s}) \cdot \mathbb{P}(A).
\end{aligned} \tag{52}$$

This proves item (v). Item (i) implies that it holds that Z is $\bar{\mathbb{F}}_0/\mathcal{B}(\mathbb{R})$-measurable. This proves item (vi). Observe that $\tilde{Z}$ is $\sigma_\Omega(\cup_{u\in[0,\infty)}\mathbb{F}^{(0)}_u)/\mathcal{B}(\mathbb{R})$-measurable. The fact that $\sigma_\Omega(\cup_{u\in[0,\infty)}\mathbb{F}^{(0)}_u) \subseteq \mathbb{F}^{(1)}_0$ hence shows that $W^{(1)}$ and $\tilde{Z}$ are independent. This proves item (vii). Lemma 11 implies that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon t \le \tau^{(0)} \implies \mathcal{Z}^{Z,\bar{W}}_t = \mathcal{Z}^{Z,W^{(0)}}_t\Big) = 1. \tag{53}$$

Therefore, we obtain that

$$\mathbb{P}(\tilde{Z} = \mathcal{Z}^{Z,\bar{W}}_{\tau^{(0)}}) = 1. \tag{54}$$

Next observe that it holds for all $t \in [0,\infty)$ that $\bar{W}_{t+\tau^{(0)}} - \bar{W}_{\tau^{(0)}} = W^{(1)}_t$. Lemma 10 and (54) hence imply that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon \mathcal{Z}^{Z,\bar{W}}_{t+\tau^{(0)}} = \mathcal{Z}^{\tilde{Z},W^{(1)}}_t\Big) = 1. \tag{55}$$

Combining (53) and (55) establishes item (viii). The proof of Lemma 12 is thus completed. ☐

3.3 A piecewise construction of a Brownian motion

Lemma 13. *Assume the setting in Section 3.1, for every $m \in \mathbb{N}_0$ let $\mathbb{F}^{(m)} = (\mathbb{F}^{(m)}_t)_{t\in[0,\infty)}$ be a normal filtration on $(\Omega,\mathfrak{F},\mathbb{P})$, assume for all $m \in \mathbb{N}_0$ that $(\cup_{u\in[0,\infty)}\mathbb{F}^{(m)}_u) \subseteq \mathbb{F}^{(m+1)}_0$, for every $m \in \mathbb{N}_0$ let $W^{(m)}\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a $\mathbb{F}^{(m)}$-Brownian motion, for every $m \in \mathbb{N}_0$ let $\tau^{(m)}\colon \Omega \to [0,\infty)$ be a $\mathbb{F}^{(m)}$-stopping time, assume that $\sum_{m=0}^{\infty}\tau^{(m)} = \infty$, for every $m \in \mathbb{N}_0$ let $T^{(m)}\colon \Omega \to [0,\infty)$ be the random variable given by $T^{(m)} = \sum_{i=0}^{m-1}\tau^{(i)}$, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be the stochastic process which satisfies for all $m \in \mathbb{N}_0$, $t \in [0,\infty)$ that $W_0 = 0$ and*

$$\big[W_t - W_{T^{(m)}} - W^{(m)}_{|t-T^{(m)}|}\big]\,\mathbb{1}^{\Omega}_{\{T^{(m)}\le t\le T^{(m+1)}\}} = 0, \tag{56}$$

let $\bar{Z}\colon \Omega \to \mathbb{R}$ be a $\mathbb{F}^{(0)}_0/\mathcal{B}(\mathbb{R})$-measurable function, let $Z^{(m)}\colon \Omega \to \mathbb{R}$, $m \in \mathbb{N}_0$, be the random variables which satisfy for every $m \in \mathbb{N}_0$ that $Z^{(m)}$ is $\mathbb{F}^{(m)}_0/\mathcal{B}(\mathbb{R})$-measurable, $Z^{(0)} = \bar{Z}$, and

$$Z^{(m+1)} = \mathcal{Z}^{Z^{(m)},W^{(m)}}_{\tau^{(m)}}, \tag{57}$$

and let $\tilde{Z}\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a stochastic process with continuous sample paths which satisfies that

$$\mathbb{P}\Big(\forall\, m \in \mathbb{N}_0, t \in [0,\infty)\colon \big[\tilde{Z}_t - \mathcal{Z}^{Z^{(m)},W^{(m)}}_{|t-T^{(m)}|}\big]\,\mathbb{1}^{\Omega}_{\{T^{(m)}\le t\le T^{(m+1)}\}} = 0\Big) = 1. \tag{58}$$

Then

(i) it holds that W is a a Brownian motion,

(ii) it holds that W and $\bar{Z}$ are independent, and

(iii) it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon \mathcal{Z}^{\bar{Z},W}_t = \tilde{Z}_t\Big) = 1. \tag{59}$$

Proof of Lemma 13. Throughout this proof let $\mathcal{D}$ be the set given by

$$\begin{aligned}\mathcal{D} = \Big\{ \big(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}\big)\colon \\ &\forall\, i \in \{1,2\}\colon \mathfrak{G}^{(i)} = (\mathfrak{G}^{(i)}_t)_{t\in[0,\infty)} \text{ is a normal filtration on } (\Omega,\mathfrak{F},\mathbb{P}),\\ &\forall\, i \in \{1,2\}\colon V^{(i)}\colon [0,\infty)\times\Omega \to \mathbb{R} \text{ is a } \mathfrak{G}^{(i)}\text{-Brownian motion},\\ &\forall\, i \in \{1,2\}\colon \upsilon^{(i)}\colon \Omega \to [0,\infty) \text{ is a } \mathfrak{G}^{(i)}\text{-stopping time},\\ &\text{and } (\cup_{u\in[0,\infty)}\mathfrak{G}^{(0)}_u) \subseteq \mathfrak{G}^{(1)}_0 \Big\},\end{aligned} \tag{60}$$

let

$$\bar{\mathbb{F}}\colon \mathcal{D} \to \{\mathfrak{G} = (\mathfrak{G}_t)_{t\in[0,\infty)} \text{ is a normal filtration on } (\Omega, \mathfrak{F}, \mathbb{P})\} \tag{61}$$

be the function which satisfies for all $(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}) \in \mathcal{D}$, $t \in [0, \infty)$ that

$$\begin{split}(\bar{\mathbb{F}}(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}))_t = \Big\{ A \in \mathfrak{F} \colon \\ \big(A \cap \{t < \upsilon^{(0)}\} \in \mathfrak{G}^{(0)}_t\big) \text{ and } \big(A \cap \{t \geq \upsilon^{(0)}\} \in \mathfrak{G}^{(1)}_{\max\{t-\upsilon^{(0)},0\}}\big)\Big\},\end{split} \tag{62}$$

let

$$\bar{\tau}\colon \mathcal{D} \to \{\upsilon\colon \Omega \to [0, \infty) \text{ is a random variable}\} \tag{63}$$

be the function which satisfies for all $(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}) \in \mathcal{D}$ that

$$\bar{\tau}(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}) = \upsilon^{(0)} + \upsilon^{(1)}, \tag{64}$$

let

$$\bar{W}\colon \mathcal{D} \to \{V\colon [0, \infty) \times \Omega \to \mathbb{R} \text{ is a stochastic process}\} \tag{65}$$

be the function which satisfies for all $(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}) \in \mathcal{D}$, $t \in [0, \infty)$ that

$$(\bar{W}(\mathfrak{G}^{(0)}, \mathfrak{G}^{(1)}, V^{(0)}, V^{(1)}, \upsilon^{(0)}, \upsilon^{(1)}))_t = \begin{cases} V^{(0)}_t & : t \leq \upsilon^{(0)} \\ V^{(1)}_{t-\upsilon^{(0)}} + V^{(0)}_{\upsilon^{(0)}} & : t \geq \upsilon^{(0)} \end{cases}, \tag{66}$$

and for every $m \in \mathbb{N}_0$ let $\bar{\mathbb{F}}^{(m)} = (\bar{\mathbb{F}}^{(m)})_{t\in[0,\infty)}$ be the normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$, $\bar{W}^{(m)}\colon [0, \infty) \times \Omega \to \mathbb{R}$ be the $\bar{\mathbb{F}}^{(m)}$-Brownian motion, and $\bar{\tau}^{(m)}\colon \Omega \to [0, \infty)$ be the $\bar{\mathbb{F}}^{(m)}$-stopping time which satisfy for all $m \in \mathbb{N}$ that

$$(\bar{\mathbb{F}}^{(0)}, \bar{W}^{(0)}, \bar{\tau}^{(0)}) = (\mathbb{F}^{(0)}, W^{(0)}, \tau^{(0)}) \tag{67}$$

and

$$\begin{aligned} \bar{\mathbb{F}}^{(m)} &= \bar{\mathbb{F}}\Big(\bar{\mathbb{F}}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)}\Big), \\ \bar{W}^{(m)} &= \bar{W}\Big(\bar{\mathbb{F}}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)}\Big), \\ \bar{\tau}^{(m)} &= \bar{\tau}\Big(\bar{\mathbb{F}}^{(m-1)}, \mathbb{F}^{(m)}, \bar{W}^{(m-1)}, W^{(m)}, \bar{\tau}^{(m-1)}, \tau^{(m)}\Big) \end{aligned} \tag{68}$$

(the unique existence of $(\bar{\mathbb{F}}^{(m)}, \bar{W}^{(m)}, \bar{\tau}^{(m)})$, $m \in \mathbb{N}_0$, follows from Lemma 12). Observe that for every $m \in \mathbb{N}_0$, $t \in [0, \infty)$ it holds that $\bar{\tau}^{(m)} = T^{(m+1)}$ and

$$\big[W_t - \bar{W}^{(m)}_t\big] \mathbb{1}^{\Omega}_{\{t \leq \bar{\tau}^{(m)}\}} = 0. \tag{69}$$

Hence, we obtain for every $t \in [0, \infty)$ that

$$\lim_{m\to\infty} \bar{W}^{(m)}_t = W_t. \tag{70}$$

This shows that W is a Brownian motion. Lemma 12 implies that for every $m \in \mathbb{N}_0$ it holds that $\bar{Z}$ is $\bar{\mathbb{F}}_0^{(m)}/\mathcal{B}(\mathbb{R})$-measurable. Therefore, we obtain for every $m \in \mathbb{N}_0$ that $\bar{Z}$ and $\bar{W}^{(m)}$ are independent. Combining this with (70) implies that $\bar{Z}$ and W are independent. In the following we show by induction that for all $m \in \mathbb{N}_0$ it holds

$$\mathbb{P}\big(\forall\, t \in [0,\infty)\colon \big[\mathcal{Z}_t^{\bar{Z},\bar{W}^{(m)}} - \tilde{Z}_t\big]\mathbb{1}^{\Omega}_{\{t \le \bar{\tau}^{(m)}\}} = 0\big) = 1. \tag{71}$$

The induction base case $m = 0$ is clear. For the induction step $\mathbb{N}_0 \ni m \to m+1 \in \mathbb{N}$, assume that (71) holds for some $m \in \mathbb{N}_0$. Note that the induction hypothesis implies that

$$\mathbb{P}\big(\mathcal{Z}_{\bar{\tau}^{(m)}}^{\bar{Z},\bar{W}^{(m)}} = \tilde{Z}_{\bar{\tau}^{(m)}} = \mathcal{Z}_{\tau^{(m)}}^{Z^{(m)},W^{(m)}} = Z^{(m+1)}\big) = 1. \tag{72}$$

Lemma 12 hence implies that it holds $\mathbb{P}$-a.s. for all $t \in [0,\infty)$ it holds that

$$\mathcal{Z}_t^{\bar{Z},\bar{W}^{(m+1)}} = \begin{cases} \mathcal{Z}_t^{\bar{Z},\bar{W}^{(m)}} & : t \le \bar{\tau}^{(m)} \\ \mathcal{Z}_{t-\bar{\tau}^{(m)}}^{Z^{(m+1)},W^{(m+1)}} & : t \ge \bar{\tau}^{(m)} \end{cases}. \tag{73}$$

This proves (71) in the case $m+1$. Induction thus establishes (71). Combining (69) and (71) with Lemma 11 demonstrates (59). The proof of Lemma 13 is thus completed. □

4 Lower error bounds for CIR processes and squared Bessel processes in the case of a special choice of the parameters

4.1 Setting

For every $\delta \in (0,2)$, $b \in [0,\infty)$ let $\mathcal{Z}^{(\cdot),\delta,b,(\cdot)} = (\mathcal{Z}^{z,\delta,b,v})_{z\in\mathbb{R},v\in C([0,\infty),\mathbb{R})}\colon \mathbb{R} \times C([0,\infty),\mathbb{R}) \to C([0,\infty),\mathbb{R})$ be a Borel-measurable and universally adapted function (see Kallenberg [29, page 423] for the notion of an universally adapted function) which satisfies that for every complete probability space $(\Omega,\mathfrak{F},\mathbb{P})$, every normal filtration $(\mathbb{F}_t)_{t\in[0,\infty)}$ on $(\Omega,\mathfrak{F},\mathbb{P})$, every $\mathbb{F}_0/\mathcal{B}(\mathbb{R})$-measurable function $Z\colon \Omega \to \mathbb{R}$, every $(\mathbb{F}_t)_{t\in[0,\infty)}$-Brownian motion $W\colon [0,\infty)\times\Omega \to \mathbb{R}$, and every $t \in [0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$\mathcal{Z}_t^{Z,\delta,b,W} = X + \int_0^t \big(\delta - b\cdot \mathcal{Z}_s^{Z,\delta,b,W}\big)\,\mathrm{d}s + \int_0^t 2\sqrt{|\mathcal{Z}_s^{Z,\delta,b,W}|}\,\mathrm{d}W_s, \tag{74}$$

let $\delta \in (0,2)$, $b \in [0,\infty)$, let $\mathcal{C}_0$ and $\mathcal{C}_{00}$ be the sets given by $\mathcal{C}_0 = \{f \in C([0,\infty),\mathbb{R})\colon f(0) = 0\}$ and $\mathcal{C}_{00} = \{f \in C([0,1],\mathbb{R})\colon f(0) = f(1) = 0\}$, let $\mathfrak{v}\colon \{\triangle,\square\} \to \mathbb{N}$ be the function which satisfies $\mathfrak{v}(\triangle) = 3$ and $\mathfrak{v}(\square) = 4$, for every $n \in \mathbb{N}$, $* \in \{\triangle,\square\}$ let $G_n^*\colon \mathcal{C}_0 \times \mathcal{C}_{00} \to \mathcal{C}_0$ be the function which satisfies for all $w \in \mathcal{C}_0$, $f \in \mathcal{C}_{00}$, $t \in [0,\infty)$ that

$$\begin{aligned}&(G_n^*(w,f))_t \\ &= \begin{cases} \big(n\cdot w_{1/n}\cdot t + \frac{1}{\sqrt{n}}\cdot f_{nt}\big)\cdot(\mathfrak{v}(*)-3) + w_t\cdot(4-\mathfrak{v}(*)) & : 0 \le t \le \frac{1}{n} \\ w_t & : \frac{1}{n} \le t < \infty \end{cases},\end{aligned} \tag{75}$$

for every $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ let $F_n^*\colon [0,\infty) \times [\mathcal{C}_0]^3 \times \mathcal{C}_{00} \to \mathcal{C}_0$ be the function which satisfies for all $w^{(1)}, w^{\triangle}, w^{(2)} \in \mathcal{C}_0$, $f \in \mathcal{C}_{00}$, $r, t \in [0,\infty)$ that

$$\begin{aligned} &(F_n^*(r, w^{(1)}, w^{\triangle}, w^{(2)}, f))_t \\ &= \begin{cases} w_t^{(1)} & : t \le r \\ (G_n^*(w^{\triangle}, f))_{t-r} + w_r^{(1)} & : r \le t \le r + \frac{1}{n} \\ w_{t-(r+1/n)}^{(2)} + w_{1/n}^{\triangle} + w_r^{(1)} & : r + \frac{1}{n} \le t \end{cases}, \end{aligned} \tag{76}$$

for every $n \in \mathbb{N}$, $k \in \{1,2\}$ let $\mathfrak{T}_n^k\colon [0,\infty) \to [{}^{(k-1)}/n, {}^{k}/n)$ be the function which satisfies for all $t \in [0,\infty)$ that

$$\mathfrak{T}_n^k(t) = \min\bigl(\{0, \tfrac{1}{n}, \tfrac{2}{n}, \tfrac{3}{n}, \dots\} \cap [t,\infty)\bigr) - t + \tfrac{(k-1)}{n}, \tag{77}$$

for every $n \in \mathbb{N}$ let $\mathcal{S}_n\colon [0,\infty) \times [\mathcal{C}_0]^3 \times \mathcal{C}_{00} \to [{}^1/n, \infty]$, $\mathcal{T}_n\colon [0,\infty) \times [\mathcal{C}_0]^3 \times \mathcal{C}_{00} \to [{}^1/n, \infty)$, and $\Phi_n = (\Phi_{n,1}, \dots, \Phi_{n,6})\colon [0,\infty) \times [\mathcal{C}_0]^3 \times \mathcal{C}_{00} \to [0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2$ be the functions which satisfy for all $t \in [0,\infty)$, $y \in [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$ that

$$\mathcal{S}_n(t,y) = \max_{* \in \{\triangle,\square\}} \inf\Bigl(\Bigl\{s \in [\mathfrak{T}_n^2(t), \infty)\colon \mathcal{Z}_s^{0,\delta,b,F_n^*(\mathfrak{T}_n^1(t),y)} = 0\Bigr\} \cup \{\infty\}\Bigr), \tag{78}$$

$$\mathcal{T}_n(t,y) = \begin{cases} \mathcal{S}_n(t,y) & : \mathcal{S}_n(t,y) \neq \infty \\ \mathfrak{T}_n^2(t) & : \mathcal{S}_n(t,y) = \infty \end{cases}, \tag{79}$$

and

$$\begin{aligned} \Phi_n(t,y) &= (\Phi_{n,1}(t,y), \dots, \Phi_{n,6}(t,y)) = \bigl(t, t + \mathcal{T}_n(t,y), \\ &\quad F_n^{\triangle}(\mathfrak{T}_n^1(t), y), F_n^{\square}(\mathfrak{T}_n^1(t), y), \mathcal{Z}^{0,\delta,b,F_n^{\triangle}(\mathfrak{T}_n^1(t),y)}, \mathcal{Z}^{0,\delta,b,F_n^{\square}(\mathfrak{T}_n^1(t),y)}\bigr), \end{aligned} \tag{80}$$

let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, let $\tilde{W}, \tilde{W}^{(1)}, \tilde{W}^{\triangle}, \tilde{W}^{(2)}\colon \Omega \to \mathcal{C}_0$ be Brownian motions, let $B\colon \Omega \to \mathcal{C}_{00}$ be a Brownian bridge, let $Z\colon \Omega \to [0,\infty)$ be a random variable, let $Y^{[n]}\colon \Omega \to [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$, $n \in \mathbb{N}_0$, be i.i.d. random variables with $Y^{[0]} = (\tilde{W}^{(1)}, \tilde{W}^{\triangle}, \tilde{W}^{(2)}, B)$, let $X^{(n),[m]} = (X_1^{(n),[m]}, \dots, X_6^{(n),[m]})\colon \Omega \to [0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, be the random variables which satisfy for all $n, m \in \mathbb{N}$ that $X^{(n),[m]} = \Phi_n(X_2^{(n),[m-1]}, Y^{[m]})$ and

$$X^{(n),[0]} = \begin{cases} (0, 0, \tilde{W}, \tilde{W}, \mathcal{Z}^{Z,\delta,b,\tilde{W}}, \mathcal{Z}^{Z,\delta,b,\tilde{W}}) \\ \quad : (\forall\, t \in [0,\infty)\colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} \neq 0) \\ (0, \inf\{t \in [0,\infty)\colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0\}, \tilde{W}, \tilde{W}, \mathcal{Z}^{Z,\delta,b,\tilde{W}}, \mathcal{Z}^{Z,\delta,b,\tilde{W}}) \\ \quad : (\exists\, t \in [0,\infty)\colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0) \end{cases}, \tag{81}$$

for every $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ let $W^{(n),*}\colon \Omega \to \mathcal{C}_0$ be a stochastic process which satisfies for all $m \in \mathbb{N}_0$, $t \in [0,\infty)$ that $W_0^{(n),*} = 0$ and

$$\Bigl[W_t^{(n),*} - W_{X_1^{(n),[m]}}^{(n),*} - (X_{\mathfrak{v}(*)}^{(n),[m]})_{|t - X_1^{(n),[m]}|}\Bigr] \mathbb{1}_{\{X_1^{(n),[m]} \le t \le X_2^{(n),[m]}\}}^{\Omega} = 0, \tag{82}$$

for every $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ let $Z^{(n),*}\colon [0,\infty) \times \Omega \to \mathbb{R}$ be a stochastic process with continuous sample paths which satisfies for all $m \in \mathbb{N}_0$ that

$$\mathbb{P}\Bigl(\forall\, t \in [0,\infty)\colon \bigl[Z_t^{(n),*} - (X_{\mathfrak{v}(*)+2}^{(n),[m]})_{|t - X_1^{(n),[m]}|}\bigr] \mathbb{1}_{\{X_1^{(n),[m]} \le t \le X_2^{(n),[m]}\}}^{\Omega} = 0\Bigr) = 1, \tag{83}$$

for every $n \in \mathbb{N}$ let $\mathcal{M}_n \colon \Omega \to \mathbb{N}_0$ and $\gamma_n \colon \Omega \to [0,1] \cup \{\infty\}$ be the random variables given by $\mathcal{M}_n = \sup(\{0\} \cup \{m \in \{0,1,\ldots,n+1\} \colon X_1^{(n),[m]} \leq 1\})$ and

$$\gamma_n = \begin{cases} X_1^{(n),[\mathcal{M}_n]} & : \mathcal{M}_n \neq 0 \\ \infty & : \mathcal{M}_n = 0 \end{cases}, \tag{84}$$

and assume that $\tilde{W}$, $\tilde{W}^{(1)}$, $\tilde{W}^{\triangle}$, $\tilde{W}^{(2)}$, B, Z, $Y^{[1]}$, $Y^{[2]}$, $\ldots$ are independent.

4.2 Properties of the constructed random objects

4.2.1 The Feller boundary condition revisited

Lemma 14 (Hit of the zero boundary)**.** *Assume the setting in Section 4.1. Then*

$$\mathbb{P}\Big(\exists\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0\Big) = 1. \tag{85}$$

Proof of Lemma 14. Note that the assumption that $\delta \in (0,2)$ and Lemma 8 ensure that for all $z \in [0,\infty)$ it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{z,\delta,b,\tilde{W}} \neq 0\Big) = 0. \tag{86}$$

Next observe that the integral transformation theorem, the fact that Z and $\tilde{W}$ are independent, and Fubini's theorem ensure that

$$\begin{aligned}
&\mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} \neq 0\Big) = \mathbb{E}\Big[\mathbb{1}^{\Omega}_{\{\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} \neq 0\}}\Big] \\
&= \mathbb{E}\Big[\mathbb{1}^{C([0,\infty),\mathbb{R})}_{\{v \in C([0,\infty),\mathbb{R}) \colon (\forall\, t \in [0,\infty) \colon v(t) \neq 0)\}}(\mathcal{Z}^{Z,\delta,b,\tilde{W}})\Big] \\
&= \int_{[0,\infty) \times C([0,\infty),\mathbb{R})} \mathbb{1}^{C([0,\infty),\mathbb{R})}_{\{v \in C([0,\infty),\mathbb{R}) \colon (\forall\, t \in [0,\infty) \colon v(t) \neq 0)\}}(\mathcal{Z}^{z,\delta,b,w}) \\
&\quad \big((Z,\tilde{W})(\mathbb{P})_{\mathcal{B}([0,\infty)) \otimes \mathcal{B}(C([0,\infty),\mathbb{R}))}\big)(dz,dw) \\
&= \int_{[0,\infty)} \int_{C([0,\infty),\mathbb{R})} \mathbb{1}^{C([0,\infty),\mathbb{R})}_{\{v \in C([0,\infty),\mathbb{R}) \colon (\forall\, t \in [0,\infty) \colon v(t) \neq 0)\}}(\mathcal{Z}^{z,\delta,b,w}) \\
&\quad \tilde{W}(\mathbb{P})_{\mathcal{B}(C([0,\infty),\mathbb{R}))}(dw)\, Z(\mathbb{P})_{\mathcal{B}([0,\infty))}(dz).
\end{aligned} \tag{87}$$

Combining this and (86) assures that

$$\begin{aligned}
&\mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} \neq 0\Big) \\
&= \int_0^\infty \mathbb{E}\Big[\mathbb{1}^{C([0,\infty),\mathbb{R})}_{\{v \in C([0,\infty),\mathbb{R}) \colon (\forall\, t \in [0,\infty) \colon v(t) \neq 0)\}}(\mathcal{Z}^{z,\delta,b,\tilde{W}})\Big]\, Z(\mathbb{P})_{\mathcal{B}([0,\infty))}(dz) \\
&= \int_0^\infty \mathbb{E}\Big[\mathbb{1}^{\Omega}_{\big\{\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{z,\delta,b,\tilde{W}} \neq 0\big\}}\Big]\, Z(\mathbb{P})_{\mathcal{B}([0,\infty))}(dz) \\
&= \int_0^\infty \mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{z,\delta,b,\tilde{W}} \neq 0\Big)\, Z(\mathbb{P})_{\mathcal{B}([0,\infty))}(dz) = 0.
\end{aligned} \tag{88}$$

Hence, we obtain that

$$\mathbb{P}\Big(\exists\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0\Big) = 1 - \mathbb{P}\Big(\forall\, t \in [0,\infty) \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} \neq 0\Big) = 1. \tag{89}$$

The proof of Lemma 14 is thus completed. ∎

4.2.2 One step in the construction of the Brownian motions

In the next well-known lemma we briefly recall the covariance matrix associated to a Brownian bridge.

Lemma 15 (Covariance associated to a Brownian bridge)**.** *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space, let* $T \in (0, \infty)$*, let* $W : [0, T] \times \Omega \to \mathbb{R}$ *be a Brownian motion, and let* $B : [0, T] \times \Omega \to \mathbb{R}$ *be the function which satisfies for all* $t \in [0, T]$ *that*

$$B_t = W_t - \tfrac{t}{T} W_T. \tag{90}$$

Then it holds for all $s, t \in [0, T]$ *that*

$$\mathbb{E}[B_s B_t] = \min\{s, t\} - \tfrac{st}{T} \tag{91}$$

Proof of Lemma 15. Observe that the fact that

$$\forall\, s, t \in [0, T] \colon \quad \mathbb{E}[W_s W_t] = \min\{s, t\} \tag{92}$$

ensures that for all $s, t \in [0, T]$ it holds that

$$\begin{aligned} \mathbb{E}[B_s B_t] &= \mathbb{E}\big[(W_s - \tfrac{s}{T} W_T)(W_t - \tfrac{t}{T} W_T)\big] \\ &= \mathbb{E}[W_s W_t] - \tfrac{s}{T}\mathbb{E}[W_T W_t] - \tfrac{t}{T}\mathbb{E}[W_s W_T] + \tfrac{st}{T^2}\mathbb{E}\big[(W_T)^2\big] \\ &= \min\{s, t\} - \tfrac{st}{T} - \tfrac{st}{T} + \tfrac{st}{T} = \min\{s, t\} - \tfrac{st}{T}. \end{aligned} \tag{93}$$

The proof of Lemma 15 is thus completed. □

Lemma 16 (Construction of a Brownian motion)**.** *Let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space, let* $T \in (0, 1]$*, let* $W : [0, 1] \times \Omega \to \mathbb{R}$ *be a Brownian motion, let* $B : [0, 1] \times \Omega \to \mathbb{R}$ *be a Brownian bridge, assume that* W *and* B *are independent, and let* $\mathcal{W} : [0, T] \times \Omega \to \mathbb{R}$ *be the function which satisfies for all* $t \in [0, T]$ *that*

$$\mathcal{W}_t = \tfrac{t}{T} \cdot W_T + \sqrt{T} \cdot B_{\frac{t}{T}}. \tag{94}$$

Then it holds that $\mathcal{W}$ *is a Brownian motion.*

Proof of Lemma 16. Note that Lemma 15 and the assumption that W and B are independent ensure that for all $s, t \in [0, T]$ it holds that

$$\begin{aligned} &\mathbb{E}[\mathcal{W}_s \mathcal{W}_t] = \mathbb{E}\Big[(\tfrac{s}{T} \cdot W_T + \sqrt{T} \cdot B_{\frac{s}{T}})(\tfrac{t}{T} \cdot W_T + \sqrt{T} \cdot B_{\frac{t}{T}})\Big] \\ &= \tfrac{st}{T^2} \cdot \mathbb{E}\big[(W_T)^2\big] + \tfrac{t}{\sqrt{T}} \cdot \mathbb{E}\big[B_{\frac{s}{T}} W_T\big] + \tfrac{s}{\sqrt{T}} \cdot \mathbb{E}\big[W_T B_{\frac{t}{T}}\big] + T \cdot \mathbb{E}\big[B_{\frac{s}{T}} B_{\frac{t}{T}}\big] \\ &= \tfrac{st}{T} + \tfrac{t}{\sqrt{T}} \cdot \mathbb{E}\big[B_{\frac{s}{T}}\big] \cdot \mathbb{E}\big[W_T\big] + \tfrac{s}{\sqrt{T}} \cdot \mathbb{E}\big[W_T\big] \cdot \mathbb{E}\big[B_{\frac{t}{T}}\big] + T\left(\min\{\tfrac{s}{T}, \tfrac{t}{T}\} - \tfrac{st}{T^2}\right) \\ &= \tfrac{st}{T} + \min\{s, t\} - \tfrac{st}{T} = \min\{s, t\}. \end{aligned} \tag{95}$$

The proof of Lemma 16 is thus completed. □

Lemma 17 (Construction of Brownian motions)**.** *Assume the setting in Section 4.1, let* $n \in \mathbb{N}$*, let* $\tau : \Omega \to [0, \infty)$ *be a random variable, assume that* $Y^{[0]}$ *and* τ *are independent, and let* $\tilde{W}^{\square}, W^{\triangle}, W^{\square} : \Omega \to \mathcal{C}_0$ *be the random variables given by*

$$\tilde{W}^{\square} = G_n^{\square}(\tilde{W}^{\triangle}, B), \quad W^{\triangle} = F_n^{\triangle}(\tau, Y^{[0]}), \quad \textit{and} \quad W^{\square} = F_n^{\square}(\tau, Y^{[0]}). \tag{96}$$

Then it holds that the stochastic processes $\tilde{W}^{\square}$*,* $W^{\triangle}$*, and* $W^{\square}$ *are Brownian motions.*

Proof of Lemma 17. In the case of a constant random variable τ the claim follows from Lemma 16. The case of a general τ follows from the corresponding claim with a constant τ by using the independence of $Y^{[0]}$ and τ. The proof of Lemma 17 is thus completed. $\square$

Lemma 18 (One step in the construction of the Brownian motions)**.** *Assume the setting in Section 4.1, let $n \in \mathbb{N}$, let $\tau\colon \Omega \to [0,\infty)$ be a random variable, assume that $Y^{[0]}$ and τ are independent, let $\rho\colon \Omega \to [1/n, 2/n)$ be the random variable given by $\rho = \mathfrak{T}_n^2(\tau)$, and for every $* \in \{\triangle, \square\}$ let $W^*\colon \Omega \to \mathcal{C}_0$ and $Z^*\colon \Omega \to C([0,\infty),\mathbb{R})$ be the random variables given by*

$$W^* = F_n^*(\mathfrak{T}_n^1(\tau), Y^{[0]}) \qquad \textit{and} \qquad Z^* = \mathcal{Z}^{0,\delta,b,W^*}. \tag{97}$$

Then

(i) it holds that $\tilde{W}^{(2)}$ and $(Z_\rho^\triangle, Z_\rho^\square)$ are independent,

(ii) it holds for every $* \in \{\triangle, \square\}$ that

$$\mathbb{P}\Big(\forall\, t \in [0,\infty)\colon Z_{t+\rho}^* = \mathcal{Z}_t^{Z_\rho^*,\delta,b,\tilde{W}^{(2)}}\Big) = 1, \tag{98}$$

(iii) it holds that

$$\mathbb{P}\Big(\big[Z_\rho^\triangle \ge Z_\rho^\square\big] \iff \big[\forall\, t \in [0,\infty)\colon Z_{t+\rho}^\triangle \ge Z_{t+\rho}^\square\big]\Big) = 1, \tag{99}$$

(iv) it holds that

$$\mathbb{P}\Big(\big[Z_\rho^\square \ge Z_\rho^\triangle\big] \iff \big[\forall\, t \in [0,\infty)\colon Z_{t+\rho}^\square \ge Z_{t+\rho}^\triangle\big]\Big) = 1, \tag{100}$$

and

(v) it holds that

$$\begin{aligned}&\mathbb{P}\Big(\mathcal{S}_n(\tau, Y^{[0]}) = \mathcal{T}_n(\tau, Y^{[0]}) = \\ &\inf\big(\{\infty\} \cup \big\{t \in [0,\infty)\colon t \ge \rho \textit{ and } \max\nolimits_{*\in\{\triangle,\square\}} Z_t^* = 0\big\}\big)\Big) = 1.\end{aligned} \tag{101}$$

Proof of Lemma 18. We prove Lemma 18 in two steps. In the first step we assume that there exists a real number $\mathfrak{t} \in [0,\infty)$ such that for all $\omega \in \Omega$ it holds that $\tau(\omega) = \mathfrak{t}$. Observe that

$$\tilde{W}^{(2)} \qquad \text{and} \qquad \big(W^\triangle|_{[0,\mathfrak{T}_n^2(\mathfrak{t})]\times\Omega}, W^\square|_{[0,\mathfrak{T}_n^2(\mathfrak{t})]\times\Omega}\big) \tag{102}$$

are independent. Moreover, note that for every $* \in \{\triangle, \square\}$, $t \in [0,\infty)$ it holds that

$$W_{t+\mathfrak{T}_n^2(\mathfrak{t})}^* - W_{\mathfrak{T}_n^2(\mathfrak{t})}^* = \tilde{W}_t^{(2)}. \tag{103}$$

Combining this and (102) proves items (i)–(ii). Next note that Lemma 3, item (i), and item (ii) establish item (iii) and item (iv). Moreover, observe that Lemma 8 implies that

$$\mathbb{P}(\mathcal{S}_n(\tau, Y^{[0]}) = \mathcal{T}_n(\tau, Y^{[0]})) = 1. \tag{104}$$

This, item (iii), and item (iv) establish item (v). The case of a general τ follows immediately from the case of a constant τ by using the fact that $Y^{[0]}$ and τ are independent. The proof of Lemma 18 is thus completed. $\square$

4.2.3 Properties of the constructed random times

Lemma 19. *Assume the setting in Section 4.1 and let $n \in \mathbb{N}$. Then*

(i) it holds for all $m \in \mathbb{N}_0$ that

$$0 \leq X_1^{(n),[m]} \leq X_2^{(n),[m]} = X_1^{(n),[m+1]} \leq X_2^{(n),[m+1]}, \tag{105}$$

(ii) it holds that

$$\sup_{m \in \mathbb{N}_0} X_1^{(n),[m]} = \sup_{m \in \mathbb{N}_0} X_2^{(n),[m]} = \infty, \tag{106}$$

(iii) it holds for all $m \in \mathbb{N}$, $i \in \{5, 6\}$ that

$$\mathbb{P}\Big(\big(X_i^{(n),[m]}\big)_0 = 0\Big) = 1, \tag{107}$$

and

(iv) it holds for all $m \in \mathbb{N}_0$, $i \in \{5, 6\}$ that

$$\mathbb{P}\Big(\big(X_i^{(n),[m]}\big)_{X_2^{(n),[m]} - X_1^{(n),[m]}} = 0\Big) = 1. \tag{108}$$

Proof of Lemma 19. First, observe that (105) is a direct consequence from (81). Next note that for all $t \in [0, \infty)$, $y \in [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$ it holds that

$$\mathcal{T}_n(t, y) \geq \tfrac{1}{n}. \tag{109}$$

Combining this with (80) establishes (106). In the next step we observe that for every $m \in \mathbb{N}$ and every $i \in \{3, 4\}$ it holds that the stochastic process $X_i^{(n),[m]}$ is a Brownian motion. This establishes (107). It thus remains to prove (108). For this we note that Lemma 14 assures that for all $i \in \{5, 6\}$ it holds that

$$\mathbb{P}\Big(\big(X_i^{(n),[0]}\big)_{X_2^{(n),[0]} - X_1^{(n),[0]}} = 0\Big) = 1. \tag{110}$$

In addition, observe that item (v) of Lemma 18 ensures that for all $m \in \mathbb{N}$, $i \in \{5, 6\}$ it holds that

$$\mathbb{P}\Big(\big(X_i^{(n),[m]}\big)_{X_2^{(n),[m]} - X_1^{(n),[m]}} = 0\Big) = 1. \tag{111}$$

Combining (110) and (111) establishes (108). The proof of Lemma 19 is thus completed. ☐

4.2.4 Properties of the constructed Brownian motions

Lemma 20. *Assume the setting in Section 4.1 and let $n \in \mathbb{N}$. Then it holds for every $t \in \{0, 1/n, 2/n, \dots\}$ that*

$$W_t^{(n),\triangle} = W_t^{(n),\square}. \tag{112}$$

Proof of Lemma 20. First, observe that it holds for all $t \in [0, \infty)$ that

$$(X_3^{(n),[0]})_t = (X_4^{(n),[0]})_t. \tag{113}$$

Hence, we obtain for all $k \in \mathbb{N}_0$ that

$$(X_3^{(n),[0]})_{k/n} = (X_4^{(n),[0]})_{k/n} \tag{114}$$

and

$$(X_3^{(n),[0]})_{X_2^{(n),[0]}} = (X_4^{(n),[0]})_{X_2^{(n),[0]}}. \tag{115}$$

Next note that it holds for all $r \in [0, \infty)$, $y \in [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$, $t \in [0, r] \cup [r + 1/n, \infty)$ that

$$\big(F_n^{\triangle}(r, y)\big)_t = \big(F_n^{\square}(r, y)\big)_t. \tag{116}$$

Moreover, observe that it holds for all $m \in \mathbb{N}$ that

$$\begin{aligned} X_2^{(n),[m]} - X_1^{(n),[m]} &= \mathcal{T}_n\big(X_2^{(n),[m-1]}, Y^{[m]}\big) \\ &\geq \mathfrak{T}_n^2\big(X_2^{(n),[m-1]}\big) = \mathfrak{T}_n^1(X_1^{(n),[m]}) + 1/n. \end{aligned} \tag{117}$$

This and (116) yield that for all $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ it holds that

$$\begin{aligned} (X_3^{(n),[m]})_{\mathfrak{T}_n^1(X_1^{(n),[m]})+k/n} &= \Big(F_n^{\triangle}(\mathfrak{T}_n^1(X_1^{(n),[m]}), Y^{[m]})\Big)_{\mathfrak{T}_n^1(X_1^{(n),[m]})+k/n} \\ &= \Big(F_n^{\square}(\mathfrak{T}_n^1(X_1^{(n),[m]}), Y^{[m]})\Big)_{\mathfrak{T}_n^1(X_1^{(n),[m]})+k/n} \\ &= (X_4^{(n),[m]})_{\mathfrak{T}_n^1(X_1^{(n),[m]})+k/n} \end{aligned} \tag{118}$$

and

$$\begin{aligned} (X_3^{(n),[m]})_{X_2^{(n),[m]}-X_1^{(n),[m]}} &= \Big(F_n^{\triangle}(\mathfrak{T}_n^1(X_1^{(n),[m]}), Y^{[m]})\Big)_{X_2^{(n),[m]}-X_1^{(n),[m]}} \\ &= \Big(F_n^{\square}(\mathfrak{T}_n^1(X_1^{(n),[m]}), Y^{[m]})\Big)_{X_2^{(n),[m]}-X_1^{(n),[m]}} \\ &= (X_4^{(n),[m]})_{X_2^{(n),[m]}-X_1^{(n),[m]}}. \end{aligned} \tag{119}$$

Combining (114), (115), (118), and (119) proves (112). The proof of Lemma 20 is thus completed. □

4.2.5 Properties of the constructed squared Bessel processes

Lemma 21. *Assume the setting in Section 4.1 and let $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$. Then*

(i) it holds that $W^{(n),*}$ is a Brownian motion,

(ii) it holds that $W^{(n),*}$ and Z are independent, and

(iii) it holds that

$$\mathbb{P}\Big(\forall\, t \in [0, \infty)\colon Z_t^{(n),*} = \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}}\Big) = 1. \tag{120}$$

Proof of Lemma 21. We present the proof of Lemma 21 in the case $* = \triangle$. The case $* = \square$ is handled similarly. Throughout this proof for every $m \in \mathbb{N}_0$ let $W^{(m)} \colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $W^{(m)} = X_3^{(n),[m]}$, for every $m \in \mathbb{N}_0$ let $\tau^{(m)} \colon \Omega \to [0,\infty)$ be the random variable given by $\tau^{(m)} = X_2^{(n),[m]} - X_1^{(n),[m]}$, for every $m \in \mathbb{N}_0$ let $\mathbb{F}^{(m)} = (\mathbb{F}_t^{(m)})_{t\in[0,\infty)}$ be the normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$ which satisfies for all $t \in [0,\infty)$, $m \in \mathbb{N}$ that

$$\mathbb{F}_t^{(0)} = \sigma_\Omega\Big(\sigma_\Omega\big(W_s^{(0)} \colon s \in [0,t]\big) \cup \sigma_\Omega(Z) \cup \big\{A \in \mathfrak{F} \colon \mathbb{P}(A) = 0\big\}\Big) \tag{121}$$

and

$$\begin{aligned}\mathbb{F}_t^{(m)} = \sigma_\Omega\Big(&\sigma_\Omega\big(W_s^{(m)} \colon s \in [0,t]\big) \cup \sigma_\Omega\big(Z, \tilde{W}, Y_4^{[m]}\big) \\ &\cup \sigma_\Omega\big(Y^{[k]} \colon k \in \mathbb{N} \cap [1, m-1]\big) \cup \big\{A \in \mathfrak{F} \colon \mathbb{P}(A) = 0\big\}\Big),\end{aligned} \tag{122}$$

and for every $m \in \mathbb{N}$ let $\tilde{t}_1^{(m)} \colon \Omega \to [0, 1/n)$ be the random variable given by $\tilde{t}_1^{(m)} = \mathfrak{T}_n^1(X_2^{(n),[m-1]})$ and let $\tilde{t}_2^{(m)} \colon \Omega \to [1/n, 2/n)$ be the random variable given by $\tilde{t}_2^{(m)} = \mathfrak{T}_n^2(X_2^{(n),[m-1]})$. Note that for every $m \in \mathbb{N}_0$ it holds that $W^{(m)}$ is a $\mathbb{F}^{(m)}$-Brownian motion. Next note that for every $m \in \mathbb{N}_0$ it holds that

$$(\cup_{u\in[0,\infty)} \mathbb{F}_u^{(m)}) \subseteq \mathbb{F}_0^{(m+1)}. \tag{123}$$

Lemma 14 implies that $\tau^{(0)}$ is a $\mathbb{F}^{(0)}$-stopping time. Observe that for every $m \in \mathbb{N}$ it holds that $\tilde{t}_1^{(m)}$ is $\mathbb{F}_0^{(m)}/\mathcal{B}([0, 1/n))$-measurable. Moreover, note that for every $m \in \mathbb{N}$ it holds that $\tilde{t}_2^{(m)}$ is $\mathbb{F}_0^{(m)}/\mathcal{B}([1/n, 2/n))$-measurable. Item (v) of Lemma 18 implies that for all $m \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\tau^{(m)} &= X_2^{(n),[m]} - X_1^{(n),[m]} = \mathcal{T}_n(X_2^{(n),[m-1]}, Y^{[m]}) = \mathcal{S}_n(X_2^{(n),[m-1]}, Y^{[m]}) \\ &= \max_{*\in\{\triangle,\square\}} \Bigg[\inf\bigg(\Big\{t \in [0,\infty) \colon t \geq \mathfrak{T}_n^2(X_2^{(n),[m-1]}) \text{ and} \\ &\qquad \mathcal{Z}_t^{0,\delta,b,F_n^*(\mathfrak{T}_n^1(X_2^{(n),[m-1]}),Y^{[m]})} = 0\Big\} \cup \{\infty\}\bigg)\Bigg] \\ &= \max_{i\in\{5,6\}} \Big[\inf\big(\{t \in [0,\infty) \colon t \geq \tilde{t}_2^{(m)} \text{ and } X_i^{(n),[m]} = 0\} \cup \{\infty\}\big)\Big].\end{aligned} \tag{124}$$

Observe that for every $m \in \mathbb{N}$, $t \in [0,\infty)$ it holds that

$$\begin{aligned}&(X_4^{(n),[m]})_t \\ &= \begin{cases} (X_3^{(n),[m]})_t & : 0 \leq t \leq \tilde{t}_1^{(m)} \\ n\Big[(X_3^{(n),[m]})_{\tilde{t}_2^{(m)}} - (X_3^{(n),[m]})_{\tilde{t}_1^{(m)}}\Big] \cdot [t - \tilde{t}_1^{(m)}] & \\ + \frac{1}{\sqrt{n}}(Y_4^{[m]})_{n(t-\tilde{t}_1^{(m)})} + (X_3^{(n),[m]})_{\tilde{t}_1^{(m)}} & : \tilde{t}_1^{(m)} \leq t \leq \tilde{t}_2^{(m)} \\ (X_3^{(n),[m]})_t & : \tilde{t}_2^{(m)} \leq t < \infty \end{cases}.\end{aligned} \tag{125}$$

Hence, we obtain for every $m \in \mathbb{N}$, $t \in [0,\infty)$, $s \in [0,t]$ that

$$\begin{aligned}
&(X_4^{(n),[m]})_s \cdot \mathbb{1}^{\Omega}_{\{t \geq \tilde{t}_2^{(m)}\}} \\
&= W_s^{(m)} \cdot \mathbb{1}^{\Omega}_{\{s \leq \tilde{t}_1^{(m)} \text{ or } s \geq \tilde{t}_2^{(m)}\}} \cdot \mathbb{1}^{\Omega}_{\{t \geq \tilde{t}_2^{(m)}\}} \\
&\quad + \left(n(W^{(m)}_{t \wedge \tilde{t}_2^{(m)}} - W^{(m)}_{t \wedge \tilde{t}_1^{(m)}})(s - \tilde{t}_1^{(m)}) + \frac{1}{\sqrt{n}} (Y_4^{[m]})_{1 \wedge (n(s - \tilde{t}_1^{(m)})) \vee 0} + W^{(m)}_{t \wedge \tilde{t}_1^{(m)}} \right) \\
&\qquad \cdot \mathbb{1}^{\Omega}_{\{\tilde{t}_1^{(m)} < s < \tilde{t}_2^{(m)}\}} \cdot \mathbb{1}^{\Omega}_{\{t \geq \tilde{t}_2^{(m)}\}}.
\end{aligned} \tag{126}$$

This demonstrates for every $m \in \mathbb{N}$, $t \in [0,\infty)$, $s \in [0,t]$ that the function $\Omega \ni \omega \mapsto (X_4^{(n),[m]})_s(\omega) \cdot \mathbb{1}^{\Omega}_{\{t \geq \tilde{t}_2^{(m)}\}}(\omega) \in \mathbb{R}$ is $\mathbb{F}_t^{(m)}/\mathcal{B}(\mathbb{R})$-measurable. Therefore, we obtain that for every $i \in \{5,6\}$, $m \in \mathbb{N}$ it holds that $((X_i^{(n),[m]})_t \cdot \mathbb{1}_{\{t \geq \tilde{t}_2^{(m)}\}})_{t \in [0,\infty)}$ is $\mathbb{F}^{(m)}$-adapted. This implies that for every $i \in \{5,6\}$, $m \in \mathbb{N}$, $t \in [0,\infty)$ it holds that

$$\begin{aligned}
&\sup\{(X_i^{(n),[m]})_s \colon s \in [0,t] \text{ and } s \geq \tilde{t}_2^{(m)}\} \\
&\qquad = \sup\{(X_i^{(n),[m]})_s \cdot \mathbb{1}^{\Omega}_{\{s \geq \tilde{t}_2^{(m)}\}} \colon s \in [0,t] \text{ and } s \geq \tilde{t}_2^{(m)}\}
\end{aligned} \tag{127}$$

and

$$\begin{aligned}
&\inf\{(X_i^{(n),[m]})_s \colon s \in [0,t] \text{ and } s \geq \tilde{t}_2^{(m)}\} \\
&\qquad = \inf\{(X_i^{(n),[m]})_s \cdot \mathbb{1}^{\Omega}_{\{s \geq \tilde{t}_2^{(m)}\}} \colon s \in [0,t] \text{ and } s \geq \tilde{t}_2^{(m)}\}
\end{aligned} \tag{128}$$

are $\mathbb{F}_t^{(m)}/\mathcal{B}([-\infty,\infty])$-measurable. Hence, we get that for all $i \in \{5,6\}$, $m \in \mathbb{N}$ it holds that $\inf\{t \in [0,\infty) \colon t \geq \tilde{t}_2^{(m)} \text{ and } X_i^{(n),[m]} = 0\}$ is a $\mathbb{F}^{(m)}$-stopping time. This implies that for all $m \in \mathbb{N}$ it holds that

$$\max_{i \in \{5,6\}} \big(\inf\{t \in [0,\infty) \colon t \geq \tilde{t}_2^{(m)} \text{ and } X_i^{(n),[m]} = 0\} \big) \tag{129}$$

is a $\mathbb{F}^{(m)}$-stopping time. Combining this and (124) assures that for all $m \in \mathbb{N}$ it holds that $\tau^{(m)}$ is a $\mathbb{F}^{(m)}$-stopping time. Next observe that it holds that

$$\sum_{m \in \mathbb{N}_0} \tau^{(m)} = \infty \tag{130}$$

and Z is $\mathbb{F}_0^{(0)}/\mathcal{B}([0,\infty))$-measurable. Item (iv) of Lemma 19 implies that for all $m \in \mathbb{N}_0$ it holds that

$$\mathbb{P}\Big((X_5^{(n),[m]})_{\tau^{(m)}} = 0 \Big) = 1. \tag{131}$$

Combining (123), the fact that for every $m \in \mathbb{N}_0$ it holds that $W^{(m)}$ is a $\mathbb{F}^{(m)}$-Brownian motion, the fact that for every $m \in \mathbb{N}_0$ it holds that $\tau^{(m)}$ is a $\mathbb{F}^{(m)}$-stopping time, (130), the fact that Z is $\mathbb{F}_0^{(0)}/\mathcal{B}([0,\infty))$-measurable, (131), and Lemma 13 completes the proof of Lemma 21. □

4.2.6 On conditional distributions of the considered random objects

Lemma 22. *Assume the setting in Section 4.1, let $n \in \mathbb{N}$, and for every $r \in [0,\infty)$ let $\mathbb{P}_r \colon \mathcal{B}([0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2) \to [0,1]$ be the probability measure which satisfies for all $B \in \mathcal{B}([0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2)$ that*

$$\mathbb{P}_r(B) = \mathbb{P}\Big(\big\{\Phi_n\big(r, Y^{[0]}\big) \in B\big\} \,|\, \big\{\Phi_{n,2}\big(r, Y^{[0]}\big) > 1\big\}\Big). \tag{132}$$

Then it holds for all $B \in \mathcal{B}([0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2)$ that

$$\mathbb{P}\Big(\mathbb{1}^{\Omega}_{\{0 \le \gamma_n \le 1\}} \mathbb{P}\big(X^{(n),[\mathcal{M}_n]} \in B \,|\, \sigma_\Omega(\gamma_n)\big) = \mathbb{1}^{\Omega}_{\{0 \le \gamma_n \le 1\}} \mathbb{P}_{\min\{\gamma_n,1\}}(B)\Big) = 1. \tag{133}$$

Proof of Lemma 22. Throughout this proof let

$$\mathbb{B} = [0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2, \tag{134}$$

let $\mathbb{P}_\infty \colon \mathcal{B}(\mathbb{B}) \to [0,1]$ be the function which satisfies for all $B \in \mathcal{B}(\mathbb{B})$ that $\mathbb{P}_\infty(B) = 0$, let $\mathbb{Q}_r \colon \mathcal{B}(\mathbb{B}) \to [0,1]$, $r \in [0,\infty]$, be the functions which satisfy for all $r \in [0,\infty)$, $B \in \mathcal{B}(\mathbb{B})$ that

$$\mathbb{Q}_r(B) = \mathbb{P}\Big(\Big\{\Phi_n\big(r, Y^{[0]}\big) \in B\Big\} \cap \Big\{\Phi_{n,2}\big(r, Y^{[0]}\big) > 1\Big\}\Big) \tag{135}$$

and $\mathbb{Q}_\infty(B) = 0$, and let $A \in \mathcal{B}([0,1])$, $B \in \mathcal{B}(\mathbb{B})$. To establish Lemma 22, we need to prove that

$$\mathbb{P}\Big(\big\{X^{(n),[\mathcal{M}_n]} \in B\big\} \cap \big\{\gamma_n \in A\big\}\Big) = \mathbb{E}\Big[\mathbb{P}_{\gamma_n}(B) \cdot \mathbb{1}^{[0,\infty]}_A(\gamma_n)\Big]. \tag{136}$$

For this we observe that

$$\begin{aligned}
&\mathbb{P}\Big(\big\{X^{(n),[\mathcal{M}_n]} \in B\big\} \cap \big\{\gamma_n \in A\big\}\Big) = \mathbb{E}\Big[\mathbb{1}^{\mathbb{B}}_B(X^{(n),[\mathcal{M}_n]}) \cdot \mathbb{1}^{[0,\infty]}_A(\gamma_n)\Big] \\
&= \sum_{m=0}^{\infty} \mathbb{E}\Big[\mathbb{1}^{\mathbb{B}}_B(X^{(n),[\mathcal{M}_n]}) \cdot \mathbb{1}^{[0,\infty]}_A(\gamma_n) \cdot \mathbb{1}^{\mathbb{R}}_{\{m\}}(\mathcal{M}_n)\Big] \\
&= \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbb{1}^{\mathbb{B}}_B(X^{(n),[\mathcal{M}_n]}) \cdot \mathbb{1}^{[0,\infty]}_A(\gamma_n) \cdot \mathbb{1}^{\mathbb{R}}_{\{m\}}(\mathcal{M}_n)\Big] \\
&= \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbb{1}^{\mathbb{B}}_B(X^{(n),[m]}) \cdot \mathbb{1}^{\mathbb{R}}_A(X^{(n),[m]}_1) \cdot \mathbb{1}^{\mathbb{R}}_{[0,1]}(X^{(n),[m]}_1) \cdot \mathbb{1}^{\mathbb{R}}_{(1,\infty)}(X^{(n),[m+1]}_1)\Big].
\end{aligned} \tag{137}$$

Next we recall that for all $m \in \mathbb{N}$ it holds

(a) that
$$X^{(n),[m]} = \Phi_n\big(X^{(n),[m-1]}_2, Y^{[m]}\big), \tag{138}$$

(b) that
$$Y^{[m]}(\mathbb{P})_{\mathcal{B}([\mathcal{C}_0]^3 \times \mathcal{C}_{00})} = Y^{[0]}(\mathbb{P})_{\mathcal{B}([\mathcal{C}_0]^3 \times \mathcal{C}_{00})}, \tag{139}$$

and

(c) that $X^{(n),[m-1]}_2$ and $Y^{[m]}$ are independent.

Item (a) ensures that for all $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_B^{\mathbb{B}}(X^{(n),[m]}) \cdot \mathbb{1}_A^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}(X_1^{(n),[m+1]})\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_B^{\mathbb{B}}(X^{(n),[m]}) \cdot \mathbb{1}_A^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}(X_2^{(n),[m]})\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}\big(X_2^{(n),[m-1]}\big) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_2^{(n),[m-1]}) \\
&\quad \cdot \mathbb{1}_B^{\mathbb{B}}\big(\Phi_n(X_2^{(n),[m-1]}, Y^{[m]})\big) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}\big(\Phi_{n,2}(X_2^{(n),[m-1]}, Y^{[m]})\big)\Big].
\end{aligned} \tag{140}$$

Items (b) and (c) hence show that for all $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_B^{\mathbb{B}}(X^{(n),[m]}) \cdot \mathbb{1}_A^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}(X_1^{(n),[m+1]})\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{Q}_{X_2^{(n),[m-1]}}(B)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{P}_{X_2^{(n),[m-1]}}(B) \\
&\quad \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}\big(\Phi_{n,2}(X_2^{(n),[m-1]}, Y^{[m]})\big)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{P}_{X_2^{(n),[m-1]}}(B) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}(X_2^{(n),[m]})\Big].
\end{aligned} \tag{141}$$

Item (i) in Lemma 19 therefore proves that for all $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_B^{\mathbb{B}}(X^{(n),[m]}) \cdot \mathbb{1}_A^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{[0,1]}^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{1}_{(1,\infty)}^{\mathbb{R}}(X_1^{(n),[m+1]})\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}(X_2^{(n),[m-1]}) \cdot \mathbb{P}_{X_2^{(n),[m-1]}}(B) \cdot \mathbb{1}_{\{m\}}^{\mathbb{R}}(\mathcal{M}_n)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{\mathbb{R}}(X_1^{(n),[m]}) \cdot \mathbb{P}_{X_1^{(n),[m]}}(B) \cdot \mathbb{1}_{\{m\}}^{\mathbb{R}}(\mathcal{M}_n)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{[0,\infty]}(\gamma_n) \cdot \mathbb{P}_{\gamma_n}(B) \cdot \mathbb{1}_{\{m\}}^{\mathbb{R}}(\mathcal{M}_n)\Big].
\end{aligned} \tag{142}$$

Combining (137) with (142) yields that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_B^{\mathbb{B}}(X^{(n),[\mathcal{M}_n]}) \cdot \mathbb{1}_A^{[0,\infty]}(\gamma_n)\Big] = \sum_{m=1}^{\infty} \mathbb{E}\Big[\mathbb{1}_A^{[0,\infty]}(\gamma_n) \cdot \mathbb{P}_{\gamma_n}(B) \cdot \mathbb{1}_{\{m\}}^{\mathbb{R}}(\mathcal{M}_n)\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_A^{[0,\infty]}(\gamma_n) \cdot \mathbb{P}_{\gamma_n}(B)\Big] = \mathbb{E}\Big[\mathbb{P}_{\gamma_n}(B) \cdot \mathbb{1}_A^{[0,\infty]}(\gamma_n)\Big].
\end{aligned} \tag{143}$$

This establishes (136). The proof of Lemma 22 is thus completed. □

4.3 Lower bounds for strong L^1-distances between the constructed squared Bessel processes

4.3.1 A first very rough lower bound for strong L^1-distances between the constructed squared Bessel processes

Lemma 23. *Assume the setting in Section 4.1, let $z \in [0, \infty)$, and let $\tilde{W}^{\square} \colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $\tilde{W}^{\square} = G_1^{\square}(\tilde{W}^{\triangle}, B)$. Then the following two statements are equivalent:*

(i) It holds that

$$\mathbb{E}\Big[\big|\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} - \mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box}\big|\Big] = 0. \tag{144}$$

(ii) There exists a $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies

$$\mathbb{P}\Big(\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} = f(\tilde{W}_1^\triangle)\Big) = 1. \tag{145}$$

Proof of Lemma 23. First, we prove that $\big(\text{(ii)} \implies \text{(i)}\big)$. Item (ii) ensures

$$\mathbb{P}\Big(\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box} = f(\tilde{W}_1^\Box)\Big) = 1. \tag{146}$$

Combining item (ii) and the fact that $\tilde{W}_1^\triangle = \tilde{W}_1^\Box$ hence ensures that

$$\begin{aligned}\mathbb{E}\Big[\big|\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} - \mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box}\big|\Big] &= \mathbb{E}\Big[\big|f(\tilde{W}_1^\triangle) - f(\tilde{W}_1^\Box)\big|\Big] \\ &= \mathbb{E}\Big[\big|f(\tilde{W}_1^\triangle) - f(\tilde{W}_1^\triangle)\big|\Big] = 0.\end{aligned} \tag{147}$$

This establishes that $\big(\text{(ii)} \implies \text{(i)}\big)$. Next we prove that $\big(\text{(i)} \implies \text{(ii)}\big)$. Combining the fact that $\tilde{W}^\triangle$ and B are independent and item (i) assures that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\mathbb{E}\Big[\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} \;\big|\; \sigma_\Omega(\tilde{W}_1^\triangle)\Big] &= \mathbb{E}\Big[\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} \;\big|\; \sigma_\Omega(\tilde{W}_1^\triangle, B)\Big] \\ &= \mathbb{E}\Big[\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box} \;\big|\; \sigma_\Omega(\tilde{W}_1^\triangle, B)\Big] \\ &= \mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box} \\ &= \mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle}.\end{aligned} \tag{148}$$

This together with the factorization lemma for conditional expectations establishes item (ii). This demonstrates that $\big(\text{(i)} \implies \text{(ii)}\big)$. The proof of Lemma 23 is thus completed. ☐

Lemma 24. *Assume the setting in Section 4.1, let $z \in [0,\infty)$, and let $\tilde{W}^\Box\colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $\tilde{W}^\Box = G_1^\Box(\tilde{W}^\triangle, B)$. Then*

$$\mathbb{E}\Big[\big|\mathcal{Z}_1^{z,\delta,b,\tilde{W}^\triangle} - \mathcal{Z}_1^{z,\delta,b,\tilde{W}^\Box}\big|\Big] > 0. \tag{149}$$

Proof of Lemma 24. In the case $(\delta, b) = (1,0)$ inequality (149) follows from Lemma 23 and, e.g., Hefter & Herzwurm [17, Equation (13)] and in the case $(\delta, b) \in \big((0,2) \times [0,\infty)\big) \setminus \{(1,0)\}$ inequality (149) follows from Lemma 23 and, e.g., Hefter, Herzwurm, & Müller-Gronbach [19]. The proof of Lemma 24 is thus completed. ☐

Lemma 25. *Assume the setting in Section 4.1 and for every $r \in [0,1]$, $* \in \{\triangle, \Box\}$ let $\mathcal{W}^{r,*}\colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $\mathcal{W}^{r,*} = F_1^*(r, Y^{[0]})$. Then*

$$\inf_{r\in[0,1]} \inf_{\beta\in[0,b]} \mathbb{E}\Big[\big|\mathcal{Z}_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,\triangle}} - \mathcal{Z}_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,\Box}}\big|\Big] > 0. \tag{150}$$

Proof of Lemma 25. Throughout this proof let $U^\triangle, U^\square, V\colon [0,1]\times[0,\infty)\times\Omega\to\mathbb{R}$ be the random fields which satisfy for all $*\in\{\triangle,\square\}$, $r\in[0,1]$, $\beta\in[0,\infty)$ that

$$U^*(r,\beta)=\mathcal{Z}_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,*}} \tag{151}$$

and

$$V(r,\beta)=\mathcal{Z}_{r}^{0,\delta,\beta,\tilde{W}^{(1)}}, \tag{152}$$

let $g\colon[0,1]\times[0,\infty)\to\mathbb{R}$ be the function which satisfies for all $r\in[0,1]$, $\beta\in[0,\infty)$ that

$$g(r,\beta)=\mathbb{E}\Big[\big|U^\triangle(r,\beta)-U^\square(r,\beta)\big|\Big], \tag{153}$$

and let $\tilde{W}^\square\colon\Omega\to\mathcal{C}_0$ be the Brownian motion given by $\tilde{W}^\square=G_1^\square(\tilde{W}^\triangle,B)$. Observe that for every $*\in\{\triangle,\square\}$, $r,t\in[0,1]$ it holds that

$$\left(\mathcal{W}^{r,*}\right)_{t+r}-\left(\mathcal{W}^{r,*}\right)_{r}=\tilde{W}_t^*. \tag{154}$$

Moreover, note that for every $*\in\{\triangle,\square\}$, $r\in[0,1]$, $t\in[0,r]$ it holds that

$$\left(\mathcal{W}^{r,*}\right)_{t}=\tilde{W}_t^{(1)}. \tag{155}$$

Hence, we obtain that for every $*\in\{\triangle,\square\}$, $r\in[0,1]$, $\beta\in[0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$U^*(r,\beta)=\mathcal{Z}_{r+1}^{0,\delta,\beta,\mathcal{W}^{r,*}}=\mathcal{Z}_1^{\mathcal{Z}_r^{0,\delta,\beta,\mathcal{W}^{r,*}},\delta,\beta,\tilde{W}^*}=\mathcal{Z}_1^{V(r,\beta),\delta,\beta,\tilde{W}^*}. \tag{156}$$

Combining Lemma 24, (156), and the fact that for every $r\in[0,1]$, $\beta\in[0,\infty)$ it holds that $V(r,\beta)$ and $(\tilde{W}^\triangle,\tilde{W}^\square)$ are independent yields that for every $r\in[0,1]$, $\beta\in[0,\infty)$ it holds that

$$\begin{aligned}g(r,\beta)&=\mathbb{E}\Big[\big|U^\triangle(r,\beta)-U^\square(r,\beta)\big|\Big]\\&=\mathbb{E}\Big[\big|\mathcal{Z}_1^{V(r,\beta),\delta,\beta,\tilde{W}^\triangle}-\mathcal{Z}_1^{V(r,\beta),\delta,\beta,\tilde{W}^\square}\big|\Big]>0.\end{aligned} \tag{157}$$

In the next step we combine Lemma 6, (156), and the fact that for every $*\in\{\triangle,\square\}$, $r,t\in[0,1]$, $\beta\in[0,\infty)$ it holds that $\tilde{W}^*$ and $(V(r,\beta),V(t,\beta))$ are independent to obtain that for every $*\in\{\triangle,\square\}$, $r,t\in[0,1]$, $\beta\in[0,\infty)$ it holds that

$$\begin{aligned}\mathbb{E}\big[\big|U^*(r,\beta)-U^*(t,\beta)\big|\big]&=\mathbb{E}\Big[\big|\mathcal{Z}_1^{V(r,\beta),\delta,\beta,\tilde{W}^*}-\mathcal{Z}_1^{V(t,\beta),\delta,\beta,\tilde{W}^*}\big|\Big]\\&=e^{-\beta}\cdot\mathbb{E}[|V(r,\beta)-V(t,\beta)|]\\&=e^{-\beta}\cdot\mathbb{E}\Big[\big|\mathcal{Z}_r^{0,\delta,\beta,\tilde{W}^{(1)}}-\mathcal{Z}_t^{0,\delta,\beta,\tilde{W}^{(1)}}\big|\Big].\end{aligned} \tag{158}$$

In addition, we note that Lemma 3 ensures that for all $*\in\{\triangle,\square\}$, $t\in[0,1]$, $\beta_1,\beta_2\in[0,\infty)$ it holds that

$$\mathbb{E}\big[|U^*(t,\beta_1)-U^*(t,\beta_2)|\big]=\big|\mathbb{E}\big[U^*(t,\beta_1)\big]-\mathbb{E}\big[U^*(t,\beta_2)\big]\big|. \tag{159}$$

The triangle inequality and (158) thereby imply that for all $* \in \{\triangle, \square\}$, $r, t \in [0,1]$, $\beta_1, \beta_2 \in [0,\infty)$ it holds that

$$\begin{aligned} &\mathbb{E}\big[|U^*(r,\beta_1) - U^*(t,\beta_2)|\big] \\ &\leq \mathbb{E}\big[|U^*(r,\beta_1) - U^*(t,\beta_1)|\big] + \mathbb{E}\big[|U^*(t,\beta_1) - U^*(t,\beta_2)|\big] \\ &= e^{-\beta_1} \cdot \mathbb{E}\Big[\big|\mathcal{Z}_r^{0,\delta,\beta_1,\tilde{W}^{(1)}} - \mathcal{Z}_t^{0,\delta,\beta_1,\tilde{W}^{(1)}}\big|\Big] + \big|\mathbb{E}\big[U^*(t,\beta_1)\big] - \mathbb{E}\big[U^*(t,\beta_2)\big]\big| \\ &\leq \mathbb{E}\Big[\big|\mathcal{Z}_r^{0,\delta,\beta_1,\tilde{W}^{(1)}} - \mathcal{Z}_t^{0,\delta,\beta_1,\tilde{W}^{(1)}}\big|\Big] + \big|\mathbb{E}\big[U^*(t,\beta_1)\big] - \mathbb{E}\big[U^*(t,\beta_2)\big]\big|. \end{aligned} \tag{160}$$

Next observe that for all $r \in [0,1]$, $\beta \in [0,\infty)$ it holds that

$$\limsup_{\substack{t \to r, \\ t \in [0,1]}} \mathbb{E}\Big[\big|\mathcal{Z}_t^{0,\delta,\beta,\tilde{W}^{(1)}} - \mathcal{Z}_r^{0,\delta,\beta,\tilde{W}^{(1)}}\big|\Big] = 0 \tag{161}$$

(cf., e.g., Mao [32, Theorem 2.4.3]). Moreover, we note that Lemma 5 ensures that for all $* \in \{\triangle, \square\}$, $r \in [0,1]$, $\beta_1 \in [0,\infty)$ it holds that

$$\limsup_{\substack{(t,\beta_2)\to(r,\beta_1), \\ (t,\beta_2)\in[0,1]\times[0,\infty)}} \big|\mathbb{E}[U^*(t,\beta_2)] - \mathbb{E}[U^*(r,\beta_1)]\big| = 0. \tag{162}$$

Combining (160), (161), and (162) yields that for all $* \in \{\triangle, \square\}$, $r \in [0,1]$, $\beta_1 \in [0,\infty)$ it holds that

$$\limsup_{\substack{(t,\beta_2)\to(r,\beta_1), \\ (t,\beta_2)\in[0,1]\times[0,\infty)}} \mathbb{E}[|U^*(t,\beta_2) - U^*(r,\beta_1)|] = 0. \tag{163}$$

This proves that g is continuous. Combining this and (157) establishes (150). The proof of Lemma 25 is thus completed. □

4.3.2 On conditional L^1-distances between the constructed squared Bessel processes

Lemma 26. *Assume the setting in Section 4.1, let $n \in \mathbb{N} \cap [5,\infty)$, $t_0 \in [0, 1/2]$, $t_1 = \mathfrak{T}_n^1(t_0) \in [0, 1/n)$, $t_2 = \mathfrak{T}_n^2(t_0) \in [1/n, 2/n)$, $t_3 = 1 - t_0 \in [1/2, 1]$, and for every $* \in \{\triangle, \square\}$ let $W^*\colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $W^* = F_n^*(t_1, Y^{[0]})$. Then*

(i) it holds that $t_1 < t_2 < t_3$,

(ii) it holds that $\mathbb{P}\big(\inf_{s\in[t_2,t_3]} \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0\big) > 0$, and

(iii) it holds that

$$\begin{aligned} &\mathbb{E}\Big[\big|\mathcal{Z}_{t_3}^{0,\delta,b,W^\triangle} - \mathcal{Z}_{t_3}^{0,\delta,b,W^\square}\big| \Big| \Big\{\inf_{s\in[t_2,t_3]} \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0\Big\}\Big] \\ &\qquad \geq \frac{\mathbb{E}\Big[\big|\mathcal{Z}_{t_1 n+1}^{0,\delta,b/n,F_1^\triangle(t_1 n, Y^{[0]})} - \mathcal{Z}_{t_1 n+1}^{0,\delta,b/n,F_1^\square(t_1 n, Y^{[0]})}\big|\Big]}{2\, n\, e^{b(t_3-t_2)}\, \mathbb{P}\Big(\forall\, s \in [2/n, 1/2]\colon \mathcal{Z}_s^{0,\delta,b,W^\triangle} > 0\Big)}. \end{aligned} \tag{164}$$

Proof of Lemma 26. Throughout this proof for every $* \in \{\triangle, \square\}$ let $U^* \colon \Omega \to \mathbb{R}$ be the random variable given by

$$U^* = \mathcal{Z}_{t_2}^{0,\delta,b,W^*} \tag{165}$$

and let $\tilde{Y} \colon \Omega \to [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$ be the random variable given by

$$\tilde{Y} = \Big(\big(\sqrt{n}\, \tilde{W}^{(1)}_{t/n}\big)_{t\in[0,\infty)}, \big(\sqrt{n}\, \tilde{W}^{\triangle}_{t/n}\big)_{t\in[0,\infty)}, \big(\sqrt{n}\, \tilde{W}^{(2)}_{t/n}\big)_{t\in[0,\infty)}, B \Big). \tag{166}$$

Observe that the fact that $n \geq 5$ ensures that

$$t_2 \leq 2/n \leq 2/5 < 1/2 \leq t_3. \tag{167}$$

This and the fact that $0 \leq t_1 < 1/n \leq t_2$ prove that

$$0 \leq t_1 < t_2 < t_3. \tag{168}$$

Moreover, note that

$$\begin{split} &\mathbb{P}\big(\inf\nolimits_{s\in[t_2,t_3]} \max\nolimits_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0 \big) \\ &\geq \mathbb{P}\big(\inf\nolimits_{s\in[t_2,t_3]} \mathcal{Z}_s^{0,\delta,b,W^\triangle} > 0 \big) = \mathbb{P}\big(\forall\, s \in [t_2,t_3] \colon \mathcal{Z}_s^{0,\delta,b,W^\triangle} > 0 \big) > 0. \end{split} \tag{169}$$

Next observe that items (i)–(ii) of Lemma 18 imply that

$$\begin{split} &\mathbb{E}\Big[\big| \mathcal{Z}_{t_3}^{0,\delta,b,W^\triangle} - \mathcal{Z}_{t_3}^{0,\delta,b,W^\square} \big| \, \mathbb{1}^{\Omega}_{\{\exists\, s\in[t_2,t_3] \colon \max\{\mathcal{Z}_s^{0,\delta,b,W^\triangle}, \mathcal{Z}_s^{0,\delta,b,W^\square}\} = 0\}} \Big] \\ &= \mathbb{E}\Big[\big| \mathcal{Z}_{t_3-t_2}^{U^\triangle,\delta,b,\tilde{W}^{(2)}} - \mathcal{Z}_{t_3-t_2}^{U^\square,\delta,b,\tilde{W}^{(2)}} \big| \, \mathbb{1}^{\Omega}_{\{\exists\, s\in[0,t_3-t_2] \colon \mathcal{Z}_s^{U^\triangle,\delta,b,\tilde{W}^{(2)}} = \mathcal{Z}_s^{U^\square,\delta,b,\tilde{W}^{(2)}} = 0\}} \Big] \\ &= 0. \end{split} \tag{170}$$

Hence, we obtain that

$$\begin{split} &\mathbb{E}\Big[\big| \mathcal{Z}_{t_3}^{0,\delta,b,W^\triangle} - \mathcal{Z}_{t_3}^{0,\delta,b,W^\square} \big| \, \Big| \Big\{ \inf\nolimits_{s\in[t_2,t_3]} \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0 \Big\} \Big] \\ &= \frac{\mathbb{E}\Big[\big| \mathcal{Z}_{t_3}^{0,\delta,b,W^\triangle} - \mathcal{Z}_{t_3}^{0,\delta,b,W^\square} \big| \, \mathbb{1}^{\Omega}_{\{\inf_{s\in[t_2,t_3]} \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0\}} \Big]}{\mathbb{P}\Big(\forall\, s \in [t_2,t_3] \colon \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0 \Big)} \\ &= \frac{\mathbb{E}\Big[\big| \mathcal{Z}_{t_3}^{0,\delta,b,W^\triangle} - \mathcal{Z}_{t_3}^{0,\delta,b,W^\square} \big| \Big]}{\mathbb{P}\Big(\forall\, s \in [t_2,t_3] \colon \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0 \Big)}. \end{split} \tag{171}$$

In the next step we note that items (iii) and (iv) of Lemma 18 and (167) imply that

$$\begin{split} &\mathbb{P}\Big(\forall\, s \in [t_2,t_3] \colon \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,W^*} > 0 \Big) \\ &\leq 2 \cdot \mathbb{P}\Big(\forall\, s \in [t_2,t_3] \colon \mathcal{Z}_s^{0,\delta,b,W^\triangle} > 0 \Big) \\ &\leq 2 \cdot \mathbb{P}\Big(\forall\, s \in [\tfrac{2}{n}, \tfrac{1}{2}] \colon \mathcal{Z}_s^{0,\delta,b,W^\triangle} > 0 \Big). \end{split} \tag{172}$$

In addition, we observe that item (ii) of Lemma 18 ensures that

$$\mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,W^{\triangle}}_{t_3} - \mathcal{Z}^{0,\delta,b,W^{\square}}_{t_3}\big|\Big] = \mathbb{E}\Big[\big|\mathcal{Z}^{U^{\triangle},\delta,b,\tilde{W}^{(2)}}_{t_3-t_2} - \mathcal{Z}^{U^{\square},\delta,b,\tilde{W}^{(2)}}_{t_3-t_2}\big|\Big]. \tag{173}$$

Furthermore, we note that item (i) of Lemma 18 implies that $(U^{\triangle}, U^{\square})$ and $\tilde{W}^{(2)}$ are independent. Combining this with (173) and Lemma 6 assures that

$$\begin{aligned}\mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,W^{\triangle}}_{t_3} - \mathcal{Z}^{0,\delta,b,W^{\square}}_{t_3}\big|\Big] &= e^{-b(t_3-t_2)}\cdot\mathbb{E}\Big[\big|U^{\triangle} - U^{\square}\big|\Big]\\ &= e^{-b(t_3-t_2)}\cdot\mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,W^{\triangle}}_{t_2} - \mathcal{Z}^{0,\delta,b,W^{\square}}_{t_2}\big|\Big].\end{aligned} \tag{174}$$

Combing this with Lemma 7 assures that

$$\begin{aligned}&\mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,W^{\triangle}}_{t_3} - \mathcal{Z}^{0,\delta,b,W^{\square}}_{t_3}\big|\Big]\\ &= e^{-b(t_3-t_2)}\cdot\tfrac{1}{n}\cdot\mathbb{E}\Bigg[\Big|\mathcal{Z}^{0,\delta,b/n,(\sqrt{n}W^{\triangle}_{t/n})_{t\in[0,\infty)}}_{n\cdot t_2} - \mathcal{Z}^{0,\delta,b/n,(\sqrt{n}W^{\square}_{t/n})_{t\in[0,\infty)}}_{n\cdot t_2}\Big|\Bigg].\end{aligned} \tag{175}$$

The fact that

$$t_2 n = t_1 n + 1 \tag{176}$$

and the fact that for every $* \in \{\triangle, \square\}$, $t \in [0,\infty)$ it holds that

$$\sqrt{n}W^{*}_{t/n} = \sqrt{n}(F^{*}_{n}(t_1, Y^{[0]}))_{t/n} = (F^{*}_{1}(t_1 n, \tilde{Y}))_t \tag{177}$$

therefore demonstrate that

$$\begin{aligned}&\mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,W^{\triangle}}_{t_3} - \mathcal{Z}^{0,\delta,b,W^{\square}}_{t_3}\big|\Big]\\ &= e^{-b(t_3-t_2)}\cdot\tfrac{1}{n}\cdot\mathbb{E}\Big[\Big|\mathcal{Z}^{0,\delta,b/n,F^{\triangle}_1(t_1 n,\tilde{Y})}_{t_1 n+1} - \mathcal{Z}^{0,\delta,b/n,F^{\square}_1(t_1 n,\tilde{Y})}_{t_1 n+1}\Big|\Big]\\ &= e^{-b(t_3-t_2)}\cdot\tfrac{1}{n}\cdot\mathbb{E}\Big[\Big|\mathcal{Z}^{0,\delta,b/n,F^{\triangle}_1(t_1 n,Y^{[0]})}_{t_1 n+1} - \mathcal{Z}^{0,\delta,b/n,F^{\square}_1(t_1 n,Y^{[0]})}_{t_1 n+1}\Big|\Big].\end{aligned} \tag{178}$$

Combining (171), (172), and (178) yields (164). The proof of Lemma 26 is thus completed. □

Lemma 27. *Assume the setting in Section 4.1. Then*

$$\inf_{n\in\mathbb{N}\cap[5,\infty)}\Bigg(n^{\delta/2}\cdot\inf_{r\in[0,1/2]}\mathbb{E}\Bigg[\big|\mathcal{Z}^{0,\delta,b,F^{\triangle}_n(\mathfrak{T}^1_n(r),Y^{[0]})}_{1-r} - \mathcal{Z}^{0,\delta,b,F^{\square}_n(\mathfrak{T}^1_n(r),Y^{[0]})}_{1-r}\big|\,\Big|\, \Big\{\forall\, s\in[\mathfrak{T}^2_n(r),1-r]\colon \max\nolimits_{*\in\{\triangle,\square\}}\mathcal{Z}^{0,\delta,b,F^{*}_n(\mathfrak{T}^1_n(r),Y^{[0]})}_{s} > 0\Big\}\Bigg]\Bigg) > 0. \tag{179}$$

Proof of Lemma 27. Inequality (179) is an immediate consequence of Lemma 9, Lemma 25, and Lemma 26. The proof of Lemma 27 is thus completed. □

Lemma 28. *Assume the setting in Section 4.1, let $n\in\mathbb{N}$, for every $*\in\{\triangle,\square\}$, $r\in[0,1]$ let $\mathcal{W}^{*,r}\colon\Omega\to\mathcal{C}_0$ be the Brownian motion given by $\mathcal{W}^{*,r} = F^{*}_n(\mathfrak{T}^1_n(r), Y^{[0]})$, and for every $r\in[0,1]$ let $E_r\in\mathbb{R}$ be the real number given by*

$$E_r = \mathbb{E}\Big[\big|\mathcal{Z}^{0,\delta,b,\mathcal{W}^{\triangle,r}}_{1-r} - \mathcal{Z}^{0,\delta,b,\mathcal{W}^{\square,r}}_{1-r}\big|\,\Big|\,\Big\{\forall\, s\in[\mathfrak{T}^n_2(r),1-r]\colon \max_{*\in\{\triangle,\square\}}\mathcal{Z}^{0,\delta,b,\mathcal{W}^{*,r}}_{s} > 0\Big\}\Big]. \tag{180}$$

Then it holds $\mathbb{P}$-a.s. that

$$\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big] \mathbb{1}^{\Omega}_{\{0\leq\gamma_n\leq 1\}} = E_{\min\{\gamma_n,1\}}\, \mathbb{1}^{\Omega}_{\{0\leq\gamma_n\leq 1\}}. \tag{181}$$

Proof of Lemma 28. Throughout this proof let E_∞ be the real number given by $E_\infty = 0$, let $\mathbb{B} = [0,\infty)^2 \times [\mathcal{C}_0]^2 \times [C([0,\infty),\mathbb{R})]^2$, for every $r \in [0,\infty]$ let $\mathbb{P}_r\colon \mathcal{B}(\mathbb{B}) \to [0,1]$ be the probability measures which satisfy for all $B \in \mathcal{B}(\mathbb{B})$ that

$$\mathbb{P}_r(B) = \begin{cases} \mathbb{P}\big(\Phi_n(r,Y^{[0]}) \in B \,\big|\, \{\Phi_{n,2}(r,Y^{[0]}) > 1\}\big) & : r < \infty \\ \mathbb{P}\big(\Phi_n(0,Y^{[0]}) \in B \,\big|\, \{\Phi_{n,2}(0,Y^{[0]}) > 1\}\big) & : r = \infty \end{cases}, \tag{182}$$

and let $G\colon \mathbb{B} \to [0,\infty)$ be the function which satisfies for all $x = (x_1,\ldots,x_6) \in \mathbb{B}$ that

$$G(x) = |x_5(|1-x_1|) - x_6(|1-x_1|)|\,. \tag{183}$$

Observe that for all $* \in \{\triangle,\square\}$ it holds that

$$\mathbb{P}\Big(Z_1^{(n),*} = (X^{(n),[\mathcal{M}_n]}_{\mathfrak{v}(*)+2})_{1-X_1^{(n),[\mathcal{M}_n]}}\Big) = 1. \tag{184}$$

Lemma 21 hence implies that for all $* \in \{\triangle,\square\}$ it holds that

$$\mathbb{P}\Big(\mathcal{Z}_1^{Z,\delta,b,W^{(n),*}} = (X^{(n),[\mathcal{M}_n]}_{\mathfrak{v}(*)+2})_{1-X_1^{(n),[\mathcal{M}_n]}}\Big) = 1. \tag{185}$$

Next observe for every $r \in [0,1]$ that it holds that $\Phi_{n,2}(r,Y^{[0]}) > 1$ if and only if it holds that $\mathcal{T}_n(r,Y^{[0]}) > 1-r$. Item (v) of Lemma 18 therefore assures that for all $r \in [0,1]$ it holds that

$$\mathbb{P}\bigg(\Big[\Phi_{n,2}(r,Y^{[0]}) > 1\Big] \Leftrightarrow \Big[\forall\, s \in [\mathfrak{T}_2^n(r), 1-r]\colon \max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,\mathcal{W}^{r,*}} > 0\Big]\bigg) = 1. \tag{186}$$

Hence, we obtain that for all $r \in [0,1]$ it holds that

$$\begin{aligned} &\int G(x)\,\mathbb{P}_r(\mathrm{d}x) \\ &= \mathbb{E}\Big[\,G\big(\Phi_n(r,Y^{[0]})\big) \,\big|\, \big\{\Phi_{n,2}(r,Y^{[0]}) > 1\big\}\Big] \\ &= \mathbb{E}\bigg[\big|\mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{\triangle,r}} - \mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{\square,r}}\big| \,\Big|\, \cap_{s\in[\mathfrak{T}_2^n(r),1-r]} \Big\{\max_{*\in\{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,\mathcal{W}^{*,r}} > 0\Big\}\bigg] \\ &= E_r. \end{aligned} \tag{187}$$

Next observe that (185) yields that

$$\mathbb{P}\Big(\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big| = G(X^{(n),[\mathcal{M}_n]})\Big) = 1. \tag{188}$$

Hence, we obtain that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\mathbb{1}^{[0,\infty]}_{[0,1]}(\gamma_n)\cdot \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big] \\ &= \mathbb{1}^{[0,\infty]}_{[0,1]}(\gamma_n)\cdot \mathbb{E}\Big[G(X^{(n),[\mathcal{M}_n]}) \,\big|\, \sigma_\Omega(\gamma_n)\Big] \\ &= \mathbb{1}^{[0,\infty]}_{[0,1]}(\gamma_n)\cdot \int G(x)\,\mathbb{P}\big(X^{(n),[\mathcal{M}_n]} \in \mathrm{d}x \,\big|\, \sigma_\Omega(\gamma_n)\big). \end{aligned} \tag{189}$$

Lemma 22 therefore assures that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\mathbb{1}_{[0,1]}^{[0,\infty]}(\gamma_n) \cdot \mathbb{E}\Big[\Big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\Big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big] \\ &= \Big[\int G \,\mathrm{d}\mathbb{P}_{\min\{\gamma_n,1\}}\Big] \,\mathbb{1}_{[0,1]}^{[0,\infty]}(\gamma_n). \end{aligned} \tag{190}$$

Equation (187) hence demonstrates that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\mathbb{1}_{[0,1]}^{[0,\infty]}(\gamma_n) \cdot \mathbb{E}\Big[\Big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\Big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big] \\ &\qquad = E_{\min\{\gamma_n,1\}} \cdot \mathbb{1}_{[0,1]}^{[0,\infty]}(\gamma_n). \end{aligned} \tag{191}$$

This establishes (181). The proof of Lemma 28 is thus completed. □

4.3.3 A lower bound for hitting time probabilities

Lemma 29. *Assume the setting in Section 4.1. Then*

$$\inf_{n\in\mathbb{N}} \Big[\mathbb{P}\big(0 \le \gamma_n \le 1/2\big)\Big] > 0. \tag{192}$$

Proof of Lemma 29. First, observe that for all $n \in \mathbb{N}$ it holds that

$$\big(\gamma_n \in [0,1]\big) \iff \big(\mathcal{M}_n \neq 0\big) \iff \big(X_1^{(n),[1]} \le 1\big) \iff \big(X_2^{(n),[0]} \le 1\big). \tag{193}$$

Hence, we obtain that for all $n \in \mathbb{N}$ it holds that

$$\mathbb{P}\Big(\big(\gamma_n \in [0,1]\big) \iff \big(\exists\, t \in [0,1] \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0\big)\Big) = 1. \tag{194}$$

This ensures that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$\mathbb{P}\Big(\big(\gamma_n \in [0,1]\big) \iff \big(\exists\, t \in [0,1] \colon Z_t^{(n),*} = 0\big)\Big) = 1. \tag{195}$$

This and Lemma 21 demonstrate that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$\mathbb{P}\Big(\big(\gamma_n \in [0,1]\big) \iff \big(\exists\, t \in [0,1] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} = 0\big)\Big) = 1. \tag{196}$$

Next note that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$\mathbb{P}\bigg(\forall\, m \in \mathbb{N} \colon Z^{(n),*}_{X_1^{(n),[m]}} = 0\bigg) = 1. \tag{197}$$

This shows that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$\mathbb{P}\Big(\big(\gamma_n \in [0,1]\big) \implies \big(Z^{(n),*}_{\min\{\gamma_n,1\}} = 0\big)\Big) = 1. \tag{198}$$

Lemma 21 hence proves that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$\mathbb{P}\Big(\big(\gamma_n \in [0,1]\big) \implies \big(\mathcal{Z}^{Z,\delta,b,W^{(n),*}}_{\min\{\gamma_n,1\}} = 0\big)\Big) = 1. \tag{199}$$

Combining (196) and (199) demonstrates that for all $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$ it holds that

$$
\begin{aligned}
&\mathbb{P}\big(0 \leq \gamma_n \leq 1/2\big) \\
&= \mathbb{P}\Big(\big\{0 \leq \gamma_n \leq 1/2\big\} \\
&\quad \cap \Big\{\big(\gamma_n \in [0,1]\big) \iff \big(\exists\, t \in [0,1] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} = 0\big)\Big\} \\
&\quad \cap \Big\{\big(\gamma_n \in [0,1]\big) \implies \big(\mathcal{Z}_{\min\{\gamma_n,1\}}^{Z,\delta,b,W^{(n),*}} = 0\big)\Big\}\Big) \\
&\geq \mathbb{P}\Big(\Big\{\exists\, t \in [0,1/2] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} = 0\Big\} \cap \Big\{\forall\, t \in [1/2,1] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} > 0\Big\} \\
&\quad \cap \Big\{\big(\gamma_n \in [0,1]\big) \iff \big(\exists\, t \in [0,1] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} = 0\big)\Big\} \\
&\quad \cap \Big\{\big(\gamma_n \in [0,1]\big) \implies \big(\mathcal{Z}_{\min\{\gamma_n,1\}}^{Z,\delta,b,W^{(n),*}} = 0\big)\Big\}\Big) \\
&= \mathbb{P}\Big(\Big\{\exists\, t \in [0,1/2] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} = 0\Big\} \cap \Big\{\forall\, t \in [1/2,1] \colon \mathcal{Z}_t^{Z,\delta,b,W^{(n),*}} > 0\Big\}\Big).
\end{aligned} \tag{200}
$$

This and Lemma 21 show that for all $n \in \mathbb{N}$ it holds that

$$
\begin{aligned}
&\mathbb{P}\big(0 \leq \gamma_n \leq 1/2\big) \\
&\geq \mathbb{P}\Big(\Big\{\exists\, t \in [0,1/2] \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} = 0\Big\} \cap \Big\{\forall\, t \in [1/2,1] \colon \mathcal{Z}_t^{Z,\delta,b,\tilde{W}} > 0\Big\}\Big) > 0.
\end{aligned} \tag{201}
$$

The proof of Lemma 29 is thus completed. □

4.3.4 A refined lower bound for strong L^1-distances between the constructed squared Bessel processes

Lemma 30. *Assume the setting in Section 4.1. Then*

$$
\inf_{n \in \mathbb{N} \cap [5,\infty)} \Big(n^{\delta/2} \cdot \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big]\Big) > 0. \tag{202}
$$

Proof of Lemma 30. Throughout this proof for every $n \in \mathbb{N} \cap [5,\infty)$, $* \in \{\triangle, \square\}$, $r \in [0,1]$ let $\mathcal{W}^{n,*,r} \colon \Omega \to \mathcal{C}_0$ be the Brownian motion given by $W^{n,*,r} = F_n^*(\mathfrak{T}_n^1(r), Y^{[0]})$ and for every $n \in \mathbb{N} \cap [5,\infty)$, $r \in [0,1]$ let $E_{n,r} \in \mathbb{R}$ be the real number given by

$$
\begin{aligned}
&E_{n,r} = \\
&\mathbb{E}\Bigg[\big|\mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{n,\triangle,r}} - \mathcal{Z}_{1-r}^{0,\delta,b,\mathcal{W}^{n,\square,r}}\big| \,\Big|\, \cap_{s \in [\mathfrak{T}_2^n(r), 1-r]} \Big\{\max_{* \in \{\triangle,\square\}} \mathcal{Z}_s^{0,\delta,b,\mathcal{W}^{n,*,r}} > 0\Big\}\Bigg].
\end{aligned} \tag{203}
$$

Next observe that the tower property for conditional expectations ensures that for all $n \in \mathbb{N} \cap [5,\infty)$ it holds that

$$
\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big]\Bigg].
\end{aligned} \tag{204}
$$

Combining this with Lemma 28 implies that for all $n \in \mathbb{N} \cap [5, \infty)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big] \\
&\geq \mathbb{E}\Big[\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big| \,\Big|\, \sigma_\Omega(\gamma_n)\Big] \mathbb{1}^\Omega_{\{0\leq\gamma_n\leq 1\}}\Big] \\
&= \mathbb{E}\Big[E_{n,\gamma_n} \mathbb{1}^\Omega_{\{0\leq\gamma_n\leq 1\}}\Big] \geq \mathbb{E}\Big[E_{n,\gamma_n} \mathbb{1}^\Omega_{\{0\leq\gamma_n\leq 1/2\}}\Big].
\end{aligned} \tag{205}$$

Hence, we obtain that for all $n \in \mathbb{N} \cap [5, \infty)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big] \\
&\geq \mathbb{E}\Big[\mathbb{1}^\Omega_{\{0\leq\gamma_n\leq 1/2\}}\Big]\Big[\inf_{r\in[0,1/2]} E_{n,r}\Big] = \mathbb{P}\Big(\gamma_n \in [0, 1/2]\Big)\Big[\inf_{r\in[0,1/2]} E_{n,r}\Big].
\end{aligned} \tag{206}$$

This assures that

$$\begin{aligned}
&\inf_{n\in\mathbb{N}\cap[5,\infty)} \Big(n^{\delta/2} \cdot \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big]\Big) \\
&\geq \inf_{n\in\mathbb{N}\cap[5,\infty)} \Big(\mathbb{P}\Big(\gamma_n \in [0, 1/2]\Big) \cdot \Big(n^{\delta/2} \cdot \inf_{r\in[0,1/2]} E_{n,r}\Big)\Big) \\
&\geq \Big[\inf_{n\in\mathbb{N}\cap[5,\infty)} \mathbb{P}\Big(\gamma_n \in [0, 1/2]\Big)\Big] \cdot \Big[\inf_{n\in\mathbb{N}\cap[5,\infty)} \Big(n^{\delta/2} \cdot \inf_{r\in[0,1/2]} E_{n,r}\Big)\Big].
\end{aligned} \tag{207}$$

Combining this with Lemma 29 and Lemma 27 establishes (202). The proof of Lemma 30 is thus completed. □

4.4 Proofs for the lower error bounds

Lemma 31. *Assume the setting in Section 4.1. Then there exists a real number $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$ it holds that*

$$\inf_{\substack{\varphi\colon \mathbb{R}^n\to\mathbb{R} \\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{2/n}, \ldots, \tilde{W}_1)\big|\Big] \geq c \cdot n^{-\delta/2}. \tag{208}$$

Proof of Lemma 31. Throughout this proof let $e = (e_n)_{n\in\mathbb{N}} \colon \mathbb{N} \to [0, \infty]$ be the function which satisfies for all $n \in \mathbb{N}$ that

$$e_n = \inf_{\substack{\varphi\colon \mathbb{R}^n\to\mathbb{R} \\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{2/n}, \ldots, \tilde{W}_1)\big|\Big] \tag{209}$$

and let $c, C \in [0, \infty]$ be the real numbers given by

$$C = \inf_{n\in\mathbb{N}\cap[5,\infty)} \Big(n^{\delta/2} \cdot \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big]\Big) \tag{210}$$

and $c = \frac{C}{24}$. Note that items (i) and (ii) of Lemma 21 ensure that for all $n \in \mathbb{N}$ and all Borel-measurable functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathcal{Z}^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)\big|\Big] \\
&= \tfrac{1}{2}\Big(2 \cdot \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)\big|\Big]\Big) \\
&= \tfrac{1}{2}\Big(\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \varphi\big(W^{(n),\triangle}_{1/n}, W^{(n),\triangle}_{2/n}, \ldots, W^{(n),\triangle}_1\big)\big|\Big] \\
&\quad + \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}} - \varphi\big(W^{(n),\square}_{1/n}, W^{(n),\square}_{2/n}, \ldots, W^{(n),\square}_1\big)\big|\Big]\Big).
\end{aligned} \tag{211}$$

Next observe that Lemma 20 ensures that for all $n \in \mathbb{N}$ and all Borel-measurable functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ it holds that

$$\varphi\big(W_{1/n}^{(n),\triangle}, W_{2/n}^{(n),\triangle}, \ldots, W_1^{(n),\triangle}\big) = \varphi\big(W_{1/n}^{(n),\square}, W_{2/n}^{(n),\square}, \ldots, W_1^{(n),\square}\big). \tag{212}$$

Combining this with (211) proves that for all $n \in \mathbb{N}$ and all Borel-measurable functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ it holds that

$$\begin{aligned} &\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)\big|\Big] \\ &= \tfrac{1}{2}\left(\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \varphi\big(W_{1/n}^{(n),\triangle}, W_{2/n}^{(n),\triangle}, \ldots, W_1^{(n),\triangle}\big)\big|\Big]\right. \\ &\quad \left.+ \mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}} - \varphi\big(W_{1/n}^{(n),\triangle}, W_{2/n}^{(n),\triangle}, \ldots, W_1^{(n),\triangle}\big)\big|\Big]\right). \end{aligned} \tag{213}$$

The triangle inequality hence implies that for all $n \in \mathbb{N}$ and all Borel-measurable functions $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ it holds that

$$\begin{aligned} &\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,\tilde{W}} - \varphi(\tilde{W}_{1/n}, \tilde{W}_{1/n}, \ldots, \tilde{W}_1)\big|\Big] \\ &\geq \tfrac{1}{2}\,\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big]. \end{aligned} \tag{214}$$

This establishes that for all $n \in \mathbb{N} \cap [5, \infty)$ it holds that

$$e_n \geq \tfrac{1}{2}\,\mathbb{E}\Big[\big|\mathcal{Z}_1^{Z,\delta,b,W^{(n),\triangle}} - \mathcal{Z}_1^{Z,\delta,b,W^{(n),\square}}\big|\Big] \geq \big[\tfrac{C}{2}\big] \cdot n^{-\delta/2}. \tag{215}$$

The fact that $\forall\, n \in \{1, 2, 3, 4\}\colon e_n \geq e_{12}$ hence assures that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} e_n &\geq \min\Big\{e_1, e_2, e_3, e_4, \big[\tfrac{C}{2}\big] \cdot n^{-\delta/2}\Big\} \geq \min\Big\{e_{12}, \big[\tfrac{C}{2}\big] \cdot n^{-\delta/2}\Big\} \\ &\geq \min\Big\{\big[\tfrac{C}{2}\big] \cdot 12^{-\delta/2}, \big[\tfrac{C}{2}\big] \cdot n^{-\delta/2}\Big\} \geq \big[\tfrac{C}{2}\big] \cdot 12^{-\delta/2} \cdot n^{-\delta/2} \\ &\geq \big[\tfrac{C}{24}\big] \cdot n^{-\delta/2} = c \cdot n^{-\delta/2}. \end{aligned} \tag{216}$$

In the next step we observe that Lemma 30 proves that $C > 0$. Hence, we obtain that $c \in (0, \infty)$. This and (216) complete the proof of Lemma 31. ☐

5 Lower error bounds for CIR processes and squared Bessel processes in the general case

Lemma 32. *let $\delta \in (0, 2)$, $b, x \in [0, \infty)$, let $\mathcal{C}_0$ and $\mathcal{C}_{00}$ be the sets given by $\mathcal{C}_0 = \{f \in C([0, \infty), \mathbb{R})\colon f(0) = 0\}$ and $\mathcal{C}_{00} = \{f \in C([0, 1], \mathbb{R})\colon f(0) = f(1) = 0\}$, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, let $\tilde{W}, \tilde{W}^{(1)}, \tilde{W}^{\triangle}, \tilde{W}^{(2)}\colon \Omega \to \mathcal{C}_0$ be Brownian motions, let $B\colon \Omega \to \mathcal{C}_{00}$ be a Brownian bridge, let $Y^{[n]}\colon \Omega \to [\mathcal{C}_0]^3 \times \mathcal{C}_{00}$, $n \in \mathbb{N}_0$, be i.i.d. random variables with $Y^{[0]} = (\tilde{W}^{(1)}, \tilde{W}^{\triangle}, \tilde{W}^{(2)}, B)$, assume that $\tilde{W}$, $\tilde{W}^{(1)}$, $\tilde{W}^{\triangle}$, $\tilde{W}^{(2)}$, B, $Y^{[1]}$, $Y^{[2]}$, $\ldots$ are independent, let $X\colon [0, \infty) \times \Omega \to [0, \infty)$ be a $(\sigma_\Omega(\{\{\tilde{W}_s \leq a\}\colon a \in \mathbb{R}, s \in [0, t]\} \cup \{A \in \mathfrak{F}\colon \mathbb{P}(A) = 0\}))_{t \in [0,\infty)}$-adapted stochastic process with continuous sample paths which satisfies that for all $t \in [0, \infty)$ it holds $\mathbb{P}$-a.s. that*

$$X_t = x + \int_0^t (\delta - bX_s)\,\mathrm{d}s + \int_0^t 2\sqrt{X_s}\,\mathrm{d}\tilde{W}_s. \tag{217}$$

Then there exists a real number $c \in (0,\infty)$ *such that for all* $N \in \mathbb{N}$ *it holds that*

$$\inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_1 - \varphi(\tilde{W}_{\frac{1}{N}}, \tilde{W}_{\frac{2}{N}}, \ldots, \tilde{W}_1)\big|\Big] \geq c \cdot N^{-\delta/2}. \tag{218}$$

Proof of Lemma 32. Inequality (218) is a consequence of Kallenberg [29, Theorem 21.14] and Lemma 31 (observe that all objects in Section 4.1 exist, cf. Lemma 19 for the existence of $W^{(n),*}\colon \Omega \to \mathcal{C}_0$ and $Z^{(n),*}\colon [0,\infty) \times \Omega \to \mathbb{R}$ for $n \in \mathbb{N}$, $* \in \{\triangle, \square\}$). The proof of Lemma 32 is thus completed. □

Corollary 33. *Let* $\delta \in (0,2)$*,* $b, x \in [0,\infty)$*, let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,1]}$*, let* $W\colon [0,1] \times \Omega \to \mathbb{R}$ *be a* $(\mathbb{F}_t)_{t\in[0,1]}$*-Brownian motion, let* $X\colon [0,1] \times \Omega \to [0,\infty)$ *be a* $(\mathbb{F}_t)_{t\in[0,1]}$*-adapted stochastic process with continuous sample paths which satisfies that for all* $t \in [0,1]$ *it holds* $\mathbb{P}$*-a.s. that*

$$X_t = x + \int_0^t (\delta - bX_s)\,\mathrm{d}s + \int_0^t 2\sqrt{X_s}\,\mathrm{d}W_s. \tag{219}$$

Then there exists a real number $c \in (0,\infty)$ *such that for all* $N \in \mathbb{N}$ *it holds that*

$$\inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_1 - \varphi(W_{\frac{1}{N}}, W_{\frac{2}{N}}, \ldots, W_1)\big|\Big] \geq c \cdot N^{-\delta/2}. \tag{220}$$

Proof of Corollary 33. The claim follows directly from Lemma 32. □

Theorem 34 (Cox-Ingersoll-Ross processes)**.** *Let* $T, a, \sigma \in (0,\infty)$*,* $b, x \in [0,\infty)$ *satisfy* $2a < \sigma^2$*, let* $(\Omega, \mathfrak{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathbb{F}_t)_{t\in[0,T]}$*, let* $W\colon [0,T]\times\Omega \to \mathbb{R}$ *be a* $(\mathbb{F}_t)_{t\in[0,T]}$*-Brownian motion, let* $X\colon [0,T]\times\Omega \to [0,\infty)$ *be a* $(\mathbb{F}_t)_{t\in[0,T]}$*-adapted stochastic process with continuous sample paths which satisfies that for all* $t \in [0,T]$ *it holds* $\mathbb{P}$*-a.s. that*

$$X_t = x + \int_0^t (a - bX_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\,\mathrm{d}W_s. \tag{221}$$

Then there exists a real number $c \in (0,\infty)$ *such that for all* $N \in \mathbb{N}$ *it holds that*

$$\inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T)\big|\Big] \geq c \cdot N^{-(2a)/\sigma^2}. \tag{222}$$

Proof of Theorem 34. Throughout this proof let $(\mathbf{F}_t)_{t\in[0,1]}$ be the normal filtration on $(\Omega, \mathfrak{F}, \mathbb{P})$ which satisfies for all $t \in [0,1]$ that $\mathbf{F}_t = \mathbb{F}_{tT}$, let $\mathbf{W}\colon [0,1]\times\Omega \to \mathbb{R}$ be the $(\mathbf{F}_t)_{t\in[0,1]}$-Brownian motion which satisfies for all $t \in [0,1]$ that $\mathbf{W}_t = \frac{1}{\sqrt{T}} W_{tT}$, let $\delta = {}^{4a}/_{\sigma^2}$, $\mathbf{b} = Tb$, $\rho = {}^{4}/_{(T\sigma^2)} \in (0,\infty)$, $\mathbf{x} = \rho x$, let $\mathbf{X}\colon [0,1] \times \Omega \to [0,\infty)$ be the $(\mathbf{F}_t)_{t\in[0,1]}$-adapted stochastic process with continuous sample paths which satisfies for all $t \in [0,1]$ that $\mathbf{X}_t = \rho X_{tT}$. Observe that it holds that

$$\delta \in (0,2), \qquad \mathbf{b} \in [0,\infty), \text{ and } \qquad \mathbf{x} \in [0,\infty). \tag{223}$$

Moreover, note that for all $t \in [0,1]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
\mathbf{X}_t &= \rho X_{tT} \\
&= \rho x + \rho \int_0^{tT} (a - bX_s)\,\mathrm{d}s + \rho \int_0^{tT} \sigma \sqrt{X_s}\,\mathrm{d}W_s \\
&= \rho x + \rho T \int_0^{t} (a - bX_{sT})\,\mathrm{d}s + \rho\sqrt{T} \int_0^{t} \sigma \sqrt{X_{sT}}\,\mathrm{d}\mathbf{W}_s \\
&= \rho x + \rho T \int_0^{t} (a - b\mathbf{X}_s/\rho)\,\mathrm{d}s + \rho\sqrt{T} \int_0^{t} \sigma \sqrt{\mathbf{X}_s/\rho}\,\mathrm{d}\mathbf{W}_s \\
&= \mathbf{x} + \int_0^{t} (\delta - \mathbf{b}\mathbf{X}_s)\,\mathrm{d}s + 2 \int_0^{t} \sqrt{\mathbf{X}_s}\,\mathrm{d}\mathbf{W}_s.
\end{aligned} \tag{224}$$

Next observe that for all $N \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R} \\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi\big(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T\big)\big|\Big] \\
&= \frac{1}{\rho} \cdot \inf_{\substack{\varphi\colon \mathbb{R}^N \to \mathbb{R} \\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|\mathbf{X}_1 - \varphi(\mathbf{W}_{\frac{1}{N}}, \mathbf{W}_{\frac{2}{N}}, \ldots, \mathbf{W}_1)\big|\Big].
\end{aligned} \tag{225}$$

Combining (223), (224), and (225) with Corollary 33 establishes (222). The proof of Theorem 34 is thus completed. □

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References

[1] Aurélien Alfonsi. On the discretization schemes for the CIR (and Bessel squared) processes. *Monte Carlo Methods Appl.*, 11(4):355–384, 2005.

[2] Aurélien Alfonsi. Strong order one convergence of a drift implicit Euler scheme: application to the CIR process. *Statist. Probab. Lett.*, 83(2):602–607, 2013.

[3] Abdel Berkaoui, Mireille Bossy, and Awa Diop. Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence. *ESAIM Probab. Stat.*, 12:1–11 (electronic), 2008.

[4] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, second edition, 2015.

[5] M. Bossy and H. Olivero Quinteros. Strong convergence of the symmetrized Milstein scheme for some CEV-like SDEs. *ArXiv e-prints*, August 2015.

[6] Jean-François Chassagneux, Antoine Jacquier, and Ivo Mihaylov. An Explicit Euler Scheme with Strong Rate of Convergence for Financial SDEs with Non-Lipschitz Coefficients. *SIAM J. Financial Math.*, 7(1):993–1021, 2016.

[7] John C. Cox, Jonathan E. Ingersoll, Jr., and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.

[8] A. Cozma and C. Reisinger. A mixed Monte Carlo and PDE variance reduction method for foreign exchange options under the Heston-CIR model. *ArXiv e-prints*, September 2015.

[9] A. Cozma and C. Reisinger. Exponential integrability properties of Euler discretization schemes for the Cox-Ingersoll-Ross process. *ArXiv e-prints*, December 2016.

[10] G. Deelstra and F. Delbaen. Convergence of discretized stochastic (interest rate) processes with stochastic drift term. *Appl. Stochastic Models Data Anal.*, 14(1):77–84, 1998.

[11] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 468(2140):1105–1115, 2012.

[12] M. Gerencsér, A. Jentzen, and D. Salimova. On stochastic differential equations with arbitrarily slow convergence rates for strong approximation in two space dimensions. *ArXiv e-prints*, February 2017.

[13] Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[14] István Gyöngy. A note on Euler's approximations. *Potential Anal.*, 8(3):205–216, 1998.

[15] István Gyöngy and Miklós Rásonyi. A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. *Stochastic Process. Appl.*, 121(10):2189–2200, 2011.

[16] Martin Hairer, Martin Hutzenthaler, and Arnulf Jentzen. Loss of regularity for Kolmogorov equations. *Ann. Probab.*, 43(2):468–527, 2015.

[17] M. Hefter and A. Herzwurm. Optimal Strong Approximation of the One-dimensional Squared Bessel Process. *ArXiv e-prints*, January 2016.

[18] M. Hefter and A. Herzwurm. Strong Convergence Rates for Cox-Ingersoll-Ross Processes - Full Parameter Range. *ArXiv e-prints*, August 2016.

[19] M. Hefter, A. Herzwurm, and T. Müller-Gronbach. Lower error bounds for strong approximation of scalar sdes with non-lipschitzian coefficients. *In preparation*, 2017.

[20] Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.

[21] Desmond J. Higham, Xuerong Mao, and Andrew M. Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.*, 40(3):1041–1063 (electronic), 2002.

[22] D.J. Higham and X. Mao. Convergence of monte carlo simulations involving the mean-reverting square root process. *Journal of Computational Finance*, 8(3):35–61, 2005.

[23] Norbert Hofmann, Thomas Müller-Gronbach, and Klaus Ritter. The optimal discretization of stochastic differential equations. *J. Complexity*, 17(1):117–153, 2001.

[24] Yaozhong Hu. Semi-implicit Euler-Maruyama scheme for stiff stochastic equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, volume 38 of *Progr. Probab.*, pages 183–202. Birkhäuser Boston, Boston, MA, 1996.

[25] M. Hutzenthaler, A. Jentzen, and M. Noll. Strong convergence rates and temporal regularity for Cox-Ingersoll-Ross processes and Bessel processes with accessible boundaries. *ArXiv e-prints*, March 2014.

[26] Martin Hutzenthaler and Arnulf Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc.*, 236(1112):v+99, 2015.

[27] Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.*, 22(4):1611–1641, 2012.

[28] Arnulf Jentzen, Thomas Müller-Gronbach, and Larisa Yaroslavtseva. On stochastic differential equations with arbitrary slow convergence rates for strong approximation. *Commun. Math. Sci.*, 14(6):1477–1500, 2016.

[29] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[30] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[31] C. Kelly and G. J. Lord. Adaptive timestepping strategies for nonlinear stochastic systems. *ArXiv e-prints*, October 2016.

[32] Xuerong Mao. *Stochastic differential equations and their applications*. Horwood Publishing Series in Mathematics & Applications. Horwood Publishing Limited, Chichester, 1997.

[33] Grigori N. Milstein and John Schoenmakers. Uniform approximation of the Cox-Ingersoll-Ross process. *Adv. in Appl. Probab.*, 47(4):1132–1156, 2015.

[34] T. Müller-Gronbach and L. Yaroslavtseva. On hard quadrature problems for marginal distributions of SDEs with bounded smooth coefficients. *ArXiv e-prints*, March 2016.

[35] Thomas Müller-Gronbach. Optimal pointwise approximation of SDEs based on Brownian motion at discrete points. *Ann. Appl. Probab.*, 14(4):1605–1642, 2004.

[36] Andreas Neuenkirch and Lukasz Szpruch. First order strong approximations of scalar SDEs defined in a domain. *Numer. Math.*, 128(1):103–136, 2014.

[37] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[38] Sotirios Sabanis. A note on tamed Euler approximations. *Electron. Commun. Probab.*, 18:no. 47, 10, 2013.

[39] Sotirios Sabanis. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab.*, 26(4):2083–2105, 2016.

[40] L. Yaroslavtseva. On non-polynomial lower error bounds for adaptive strong approximation of SDEs. *ArXiv e-prints*, September 2016.