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# SYMMETRIC INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PROBLEMS IN POLYGONS

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Abstract. We analyze symmetric interior penalty discontinuous Galerkin finite element methods for linear, second-order elliptic boundary-value problems in polygonal domains $\Omega$ where solutions exhibit singular behavior near corners. To resolve corner singularities, we admit both, graded meshes and bisection refinement meshes. We prove that judiciously chosen refinement parameters in these mesh families imply optimal asymptotic rates of convergence with respect to the total number of degrees of freedom $N$, both for the DG energy norm error and the $L^2$-norm error. The sharpness of our asymptotic convergence rate estimates is confirmed in a series of numerical experiments.

## 1. Introduction

The error analysis of discontinuous Galerkin finite element methods (DGFEMs) for elliptic problems is by now well developed. For various $h$-version DG formulations, optimal energy norm and $L^2$-norm error estimates with respect to the mesh-width $h$ are provided in, e.g., [2, 17, 18], and the references therein. We remark that $L^2$-optimality typically requires *adjoint-consistent* DG discretizations, as introduced in [2].

Most of the DG error analyses available in the literature are based on sufficient smoothness of weak solutions and on quasi-uniformity assumptions for the mesh sequences. In addition, to derive $L^2$-norm error bounds, an $H^2$-regularity hypothesis for the solution of a suitable dual problem is usually imposed. While these smoothness properties hold true in smooth or convex domains, they are known to be false in general polygonal domains, due to the appearance of singular solution components near corners [10]. One way to characterize the singular behavior of solutions is by means of suitably weighted Sobolev spaces and corresponding elliptic regularity shifts of second, and higher order in these spaces. Here, we shall focus on the weighted spaces $H^{k,l}_{\boldsymbol{\delta}}(\Omega)$ and shifts as introduced and analyzed in [4, 3] in the context of conforming $hp$-version FEMs. For related, finite order elliptic regularity results in polygonal and polyhedral domains, we further refer to [10, 14, 8].

To resolve corner singularities in fixed order, conforming $h$-version discretizations, in recent years several types of local mesh refinement strategies have been proposed and investigated. In [5], conforming piecewise linear FEMs on so-called

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*graded* regular simplicial meshes were shown to capture singularities at the optimal $H^1$-norm convergence rate $N^{-1/2}$ with respect to $N$, the *number of degrees of freedom*. More recent variants of approximation rate bounds on families of graded meshes, also for FE spaces of polynomial degree $p \geq 1$ on graded, regular simplicial meshes in $\Omega$ can be found in [7, 1]. A public domain mesh generator for graded meshes in polygonal domains $\Omega$ is available in the LNG_FEM software package [13]. An alternative approach to build locally refined meshes is based on local refinement refinement via *newest vertex bisection*; see, e.g., [15, 16]. Indeed, in [9], a bisection refinement algorithm was proposed and it was proved that it creates mesh sequences which resolve singular solution components at optimal $H^1$-norm rates $N^{-p/2}$ with respect to $N$.

The work in [21] was the first to study (symmetric and non-symmetric) interior penalty (IP) discontinuous Galerkin methods for elliptic problems with solutions in the weighted spaces of [4, 3]. By introducing new technical tools to handle singular solution components, it established the boundedness and consistency of the interior penalty forms. It further showed and verified numerically that on graded meshes as in [5], algebraic convergence rates of the optimal order $N^{-p/2}$ are obtained for the DG norm errors. In [22], these consistency and stability properties were employed to show exponential convergence rates for *hp*-version interior penalty methods for problems with piecewise analytic solutions. For other work on the analysis of DG methods for elliptic problems with low-regularity solutions, we also refer to [17, Section 4.2.5], [23] and the refences therein.

In this paper, we build on, refine and extend the results of [21]. More specifically, we focus here on symmetric IP methods which are adjoint-consistent. Based on the weighted spaces of [4, 3], we consider IP approximations on graded and bisection refinement meshes which are locally refined towards corners of the domain. Using the techniques of [21], we establish continuity bounds for the IP forms with respect to suitable norms, show Galerkin orthogonality and derive optimal energy error estimates in terms of $N$. In addition, we derive an optimal $L^2$-norm error bound. While our approach proceeds roughly along the lines of standard arguments as in [2] and is based on duality, we now employ elliptic regularity with respect to the weighted spaces $H^{l,k}_{\boldsymbol{\delta}}(\Omega)$. Hence, unlike in previous works, we do not need to impose any extra (and unrealistic) regularity assumptions. Our continuity properties ensure the well-definedness of the various integral terms which appear in the derivation of the $L^2$-norm convergence rate bound.

To complete our error analysis, we present proofs that graded and bisection refinement meshes yield optimal approximation bounds, both for the primal and the dual solutions. To this end, we reexamine the error bound obtained in [21, Proposition 2.5.5] for graded meshes; we extend it to an estimate of the error in a slightly stronger norm and for a wider range of regularity orders. Moreover, for bisection refinement meshes, we establish a completely new variant of the approximation results in [9, Theorems 5.2 and 5.3] for weighted spaces and with respect to our consistency norm. Finally, we verify our theoretical statements in a series of numerical tests for bisection refinement meshes. Detailed numerical experiments for graded meshes are available in [21].

The paper is structured as follows: In Section 2, we introduce our elliptic model problem in polygonal domains and review regularity shifts in weighted spaces. In Section 3, we recall the symmetric IP method. Our main results are stated and discussed in Section 4 (Theorem 4.3). Section 5 contains the detailed proofs of our error estimates. In Section 6, numerical experiments are presented, verifying the sharpness of our theoretical estimates. Finally, in Section 7, conclusions are offered and potential extensions are outlined.

Throughout, we use standard notation. In particular, for a domain $G \subseteq \mathbb{R}^d$, $d = 1, 2$ and $q \in [1, \infty]$, the Lebesgue space of $q$-integrable functions is denoted by $L^q(G)$. For $k \in \mathbb{N}$, the classical Sobolev spaces of functions in $L^q(G)$ with $q$-integrable derivatives of order up to $k$ will be denoted by $W^{k,q}(G)$, and by $H^k(G)$ if $q = 2$.

## 2. Model Problem

We introduce polygonal domains, define our model problem and review regularity shifts in weighted Sobolev spaces.

2.1. **Polygonal domain.** An open and bounded domain $\Omega \subseteq \mathbb{R}^2$ is called *polygonal* if its boundary $\partial\Omega$ can be written as a finite union of $\mathsf{M} \in \mathbb{N}$ open and straight line segments $e_i$ of positive surface measure:

$$\partial\Omega = \bigcup_{i=1}^{\mathsf{M}} \overline{e}_i \quad \text{and} \quad \int_{e_i} dS > 0 \,, \quad 1 \le i \le \mathsf{M} \,. \tag{2.1}$$

The vertices of the polygon $\Omega$ are given by $\mathbf{c}_i := \overline{e}_i \cap \overline{e}_{i+1}$, $1 \le i \le \mathsf{M}$, with the understanding that $e_{\mathsf{M}+1} = e_1$. We assume the vertices to be numbered clockwise. We introduce the set of all vertices as $\mathcal{S} := \{\, \mathbf{c}_i \,:\, 1 \le i \le \mathsf{M} \,\}$. The interior opening angle of the domain at $\mathbf{c}_i$ is measured in positive orientation and denoted by $\omega_i \in (0, 2\pi]$. The case $\omega_i = 2\pi$ arises in models of fracture mechanics. Although in this case the domain $\Omega$ is not Lipschitz, it can be written as finite union of Lipschitz domains, so that all statements on variational formulations remain valid in this case. To discuss the solution regularity in the vicinity of corners, we associate with each corner local conical domains defined by

$$\Omega_i := \{\, \mathbf{x} \in \Omega \,:\, |\mathbf{x} - \mathbf{c}_i| < R_i \,\} \,, \qquad 1 \le i \le \mathsf{M}, \tag{2.2}$$

where $0 < R_i < \frac{1}{2} \min_{i \neq j} |\mathbf{c}_i - \mathbf{c}_j|$. This implies that the cones $\Omega_i$ are mutually disjoint and $\partial\Omega_i \cap \partial\Omega \subset \overline{e}_i \cup \overline{e}_{i+1}$. Hence, $\Omega_i$ is contained in a infinite cone with opening angle $\omega_i$ and vertex $\mathbf{c}_i$.

2.2. **Elliptic boundary-value problem.** Let now $\Omega$ be a polygonal domain. We denote by $\mathcal{D}$ and $\mathcal{N}$ the index sets of all boundary segments $e_i$, on which Dirichlet and Neumann boundary conditions will be applied, respectively. This leads to the partition $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where $\overline{\Gamma}_D = \cup_{i \in \mathcal{D}} \overline{e}_i$ and $\overline{\Gamma}_N = \cup_{i \in \mathcal{N}} \overline{e}_i$. We further denote by $\boldsymbol{\nu}$ the outward unit normal vector on the boundary $\partial\Omega$. Let $\mathsf{c} \in C^\infty(\overline{\Omega})$ be a smooth real-valued diffusion coefficient such that

$$c_* \le \mathsf{c}(\mathbf{x}) \le c^*, \qquad \mathbf{x} \in \overline{\Omega}, \tag{2.3}$$

for constants $0 < c_* < c^* < \infty$. Assume given in $\Omega$ a forcing term $f$, on $\Gamma_D$ a Dirichlet datum $g_D$ and on $\Gamma_N$ a Neumann boundary datum $g_N$. The smoothness assumptions on the data will be made precise subsequently in Proposition 2.3 and

Remark 2.4. We consider the diffusion problem:

$$-\nabla\cdot(\mathsf{c}\nabla u) = f \quad \text{in } \Omega\,, \tag{2.4}$$

$$u = g_D \quad \text{on } \Gamma_D\,, \tag{2.5}$$

$$\boldsymbol{\nu}\cdot(\mathsf{c}\nabla u) = g_N \quad \text{on } \Gamma_N\,. \tag{2.6}$$

The standard weak form of problem (2.4)–(2.6) reads: Find $u \in H^1(\Omega)$ such that $u|_{\Gamma_D} = g_D$ on $\Gamma_D$ and

$$a(u,v) := \int_\Omega \mathsf{c}\nabla u\cdot\nabla v\,d\mathbf{x} = \int_\Omega f\,v\,d\mathbf{x} + \int_{\Gamma_N} g_N\,v\,dS \tag{2.7}$$

for all $v \in H^1_{\Gamma_D}(\Omega) := \{\, v \in H^1(\Omega) \,:\, v|_{\Gamma_D} = 0\,\}$. If $\mathcal{D} \neq \emptyset$, problem (2.7) has a unique solution provided that $g_D$ can be stably lifted in $H^1(\Omega)$ and that the linear functionals on the right-hand side of (2.7) belong to $H^1_{\Gamma_D}(\Omega)^\star$, the dual space of $H^1_{\Gamma_D}(\Omega)$. In the pure Neumann case (i.e., $\mathcal{D} = \emptyset$), the trial and test functions $u$ and $v$ are taken in the factor space $H^1(\Omega)/\mathbb{R}$, i.e., weak solutions are equivalence classes which differ by constants. For existence, the data $f$ and $g_N$ must satisfy the compatibility condition $\int_\Omega f\,d\mathbf{x} + \int_{\partial\Omega} g_N\,dS = 0$, which will always be assumed to hold in this case.

2.3. **Weighted Sobolev spaces.** With each vertex $\mathbf{c}_i$ of $\Omega$, we assign a weight exponent $\delta_i \in \mathbb{R}$ and introduce the weight exponent vector $\boldsymbol{\delta} = \{\delta_i\}_{i=1}^{\mathsf{M}}$. For a scalar $\xi \in \mathbb{R}$, we define $\boldsymbol{\delta}+\xi$ by $\{\boldsymbol{\delta}+\xi\}_i := \delta_i + \xi$. Similarly, inequalities of the form $\boldsymbol{\delta} < \xi$ are understood componentwise. For $\boldsymbol{\delta} \in \mathbb{R}^{\mathsf{M}}$, we define the weighted distance function $\Phi_{\boldsymbol{\delta}}$ by

$$\Phi_{\boldsymbol{\delta}}(\mathbf{x}) := \prod_{i=1}^{\mathsf{M}} r_i(\mathbf{x})^{\delta_i}\,, \tag{2.8}$$

where $r_i(\mathbf{x}) = |\mathbf{x} - \mathbf{c}_i|$.

**Remark 2.1.** *To localize the weight function $\Phi_{\boldsymbol{\delta}}$ in* (2.8)*, we decompose $\Omega$ into*

$$\overline{\Omega} := \overline{\Omega}_0 \cup \big(\cup_{i=1}^{\mathsf{M}} \overline{\Omega}_i\big), \tag{2.9}$$

*where $\Omega_0 = \Omega \setminus \cup_{i=1}^{\mathsf{M}}\overline{\Omega}_i$ and where $\Omega_i$ is the cone in* (2.2)*. If $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$, then we have*

$$C_{dc}^{-1} \le r_i(\mathbf{x})^{\delta_i} \le C_{dc}, \qquad \mathbf{x}\in\Omega_0,\ 1\le i\le \mathsf{M}, \tag{2.10}$$

$$C_{dc}^{-1}\Phi_{\boldsymbol{\delta}}(\mathbf{x}) \le r_i^{\delta_i}(\mathbf{x}) \le C_{dc}\,\Phi_{\boldsymbol{\delta}}(\mathbf{x}), \qquad \mathbf{x}\in\Omega_i,\ 1\le i\le \mathsf{M}, \tag{2.11}$$

*for a constant $C_{dc} > 0$ depending on the radii $R_i$ in* (2.2)*.*

Given integers $k \ge l \ge 0$, we next define the weighted Sobolev spaces $H^{k,l}_{\boldsymbol{\delta}}(\Omega)$ as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm $\|v\|_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)}$ given by

$$\|v\|^2_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)} := \begin{cases} |v|^2_{H^{k,0}_{\boldsymbol{\delta}}(\Omega)}, & l = 0,\\ \|v\|^2_{H^{l-1}(\Omega)} + |v|^2_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)}, & l \ge 1. \end{cases} \tag{2.12}$$

The semi-norm $|v|_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)}$ is given by

$$|v|^2_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)} := \sum_{m=l}^{k} \|\Phi_{\boldsymbol{\delta}+m-l}\,|\mathsf{D}^m v|\|^2_{L^2(\Omega)}\,. \tag{2.13}$$

Here, we adopt the notation $|\mathsf{D}^m v|^2 := \sum_{|\boldsymbol{\alpha}|=m} |\mathsf{D}^{\boldsymbol{\alpha}} v|^2$, with $\mathsf{D}^{\boldsymbol{\alpha}} v$ denoting the partial derivative of $v$ with respect to the multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^2$. We shall also make use of the weighted spaces $H^{k,l}_{\delta_i}(\Omega)$, their associated norms $\|v\|_{H^{k,l}_{\delta_i}(\Omega)}$ and semi-norms $|v|_{H^{k,l}_{\delta_i}(\Omega)}$, which are defined completely analogously, but with respect to the weight $r_i(\mathbf{x})^{\delta_i}$. Furthermore, over subdomains $\Omega' \subseteq \Omega$ the weighted spaces and norms are defined by replacing the domains of integration by $\Omega'$.

Trace spaces of the weighted spaces $H^{k,l}_{\boldsymbol{\delta}}(\Omega)$ are defined as follows. Let $\gamma \subseteq \partial\Omega$ be the union of some line segments $e_i$. For $k \geq 1$, $k \geq l \geq 0$, we define $H^{k-1/2,l-1/2}_{\boldsymbol{\delta}}(\gamma)$ as the space of all functions $\phi : \gamma \to \mathbb{R}$ such that there is a function $\Phi \in H^{k,l}_{\boldsymbol{\delta}}(\Omega)$ with $\Phi|_\gamma = \phi$. The associated norm is defined by

$$\|\phi\|_{H^{k-1/2,l-1/2}_{\boldsymbol{\delta}}(\gamma)} := \inf\{\, \|\Phi\|_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)} \,:\, \Phi|_\gamma = \phi \,\}\,; \tag{2.14}$$

see also [4, Section 1.4]. The following properties hold.

**Lemma 2.2.** *Let $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$. There holds:*

(i) *We have the continuous embeddings*

$$H^{k,2}_{\boldsymbol{\delta}}(\Omega) \hookrightarrow H^{2,2}_{\boldsymbol{\delta}}(\Omega) \hookrightarrow C^0(\overline{\Omega})\,, \qquad k \geq 2. \tag{2.15}$$

(ii) *For $k \geq l \geq 1$, let $v \in H^{k,l}_{\boldsymbol{\delta}}(\Omega)$ and let $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ be such that $|\boldsymbol{\alpha}| \leq l$. Then we have $\mathsf{D}^{\boldsymbol{\alpha}} v \in H^{k-|\boldsymbol{\alpha}|,l-|\boldsymbol{\alpha}|}_{\boldsymbol{\delta}}(\Omega)$ and*

$$\|\mathsf{D}^{\boldsymbol{\alpha}} v\|_{H^{k-|\boldsymbol{\alpha}|,l-|\boldsymbol{\alpha}|}_{\boldsymbol{\delta}}(\Omega)} \leq \|v\|_{H^{k,l}_{\boldsymbol{\delta}}(\Omega)}\,. \tag{2.16}$$

(iii) *Let $f \in H^{0,0}_{\boldsymbol{\delta}}(\Omega)$. Then $\int_\Omega f v\, d\mathbf{x}$ is a linear continuous functional on $H^1(\Omega)$ and*

$$\Big|\int_\Omega f v\, d\mathbf{x}\Big| \leq C |f|_{H^{0,0}_{\boldsymbol{\delta}}(\Omega)} \|v\|_{H^1(\Omega)}, \qquad v \in H^1(\Omega), \tag{2.17}$$

*with $C > 0$ depending on $\boldsymbol{\delta}$.*

(iv) *Let $g_N \in H^{1/2,1/2}(\Gamma_N)$. Then $\int_\Omega g_N v\, dS$ is a linear continuous functional on $H^1(\Omega)$ and*

$$\Big|\int_{\Gamma_N} g_N v\, dS\Big| \leq C \|g_N\|_{H^{1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)} \|v\|_{H^1(\Omega)}, \qquad v \in H^1(\Omega), \tag{2.18}$$

*with $C > 0$ depending on $\boldsymbol{\delta}$.*

*Proof.* The proof of the second embedding in (2.15) can be found in [5, page 449]. The first inclusion in (2.15) is trivial. Property (2.16) is straightforward. Finally, the bounds (2.17) and (2.18) are proved in [4, Lemma 2.10 and Lemma 2.11]. □

2.4. **Regularity in weighted spaces.** We base our analysis on the following elliptic regularity shift in weighted Sobolev spaces, which is a consequence of Remark 3 and Lemma 3.2 in [4]. In the lowest-order case, we also refer to [5, Theorem 3.2]. Equivalent results (with differently defined weighted spaces) are established in [8, Sections 4 and 7.1].

**Proposition 2.3.** *There exist parameters $\delta_i^* > 0$ (depending on the domain $\Omega$, the sets $\mathcal{D}$, $\mathcal{N}$ and the coefficient $\mathsf{c}$) such that for weight exponents $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ with*

$$1 - \delta_i^* < \delta_i < 1, \qquad 1 \leq i \leq \mathsf{M}, \tag{2.19}$$

*the following elliptic regularity shifts hold: for $k \geq 1$, $f \in H^{k-1,0}_{\boldsymbol{\delta}}(\Omega)$, $g_D \in H^{k+1/2,3/2}_{\boldsymbol{\delta}}(\Gamma_D)$ and $g_N \in H^{k-1/2,1/2}_{\boldsymbol{\delta}}(\Omega)$, the weak solution $u \in H^1(\Omega)$ of* (2.7) *exists (see* (2.17) *and* (2.18)*) and belongs to $H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$. Moreover, we have the stability bound*

$$\|u\|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)} \leq C_{stab,k}\big(\|f\|_{H^{k-1,0}_{\boldsymbol{\delta}}(\Omega)} + \|g_D\|_{H^{k+1/2,3/2}_{\boldsymbol{\delta}}(\Gamma_D)} + \|g_N\|_{H^{k-1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)}\big),$$

*with a stability constant $C_{stab,k} > 0$ that is independent of the data (but depends on the order $k$, the domain $\Omega$, the sets $\mathcal{D}$, $\mathcal{N}$, and the coefficient $\mathsf{c}$).*

**Remark 2.4.** *Notice that $1-\delta_i^*$ in* (2.19) *can be negative. In this case, the condition $\delta_i > 1-\delta_i^*$ is considered void as it imposes no restriction on the range of $\delta_i \in [0,1)$. Moreover, in view of Proposition 2.3, we will always assume the minimum regularity*

$$f \in H^{0,0}_{\boldsymbol{\delta}}(\Omega), \qquad g_D \in H^{3/2,3/2}_{\boldsymbol{\delta}}(\Gamma_D), \qquad g_N \in H^{1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)\ . \tag{2.20}$$

*This ensures that the solution of* (2.7) *belongs to $H^{2,2}_{\boldsymbol{\delta}}(\Omega)$.*

**Remark 2.5.** *In the case of the Laplacian (where $\mathsf{c} = 1$), the parameter $\delta_i^*$ at corner $\mathbf{c}_i = \overline{e}_i \cap \overline{e}_{i+1}$ is well-known and given by*

$$\delta_i^* = \begin{cases} \frac{\pi}{\omega_i} & \text{if } \{i,i+1\} \in \mathcal{D} \text{ or } \{i,i+1\} \in \mathcal{N}, \\ \frac{\pi}{2\omega_i} & \text{otherwise}; \end{cases} \tag{2.21}$$

*see, e.g.,* [4, Remark 3] *or* [8, Example 7.2].

## 3. Discontinuous Galerkin Discretization

We introduce the symmetric interior penalty finite element method for the numerical approximation of problem (2.4)–(2.6), and review the discrete coercivity and continuity of the IP bilinear form.

### 3.1. Meshes, edges and trace operators.

Let $\mathcal{T}$ be a partition of $\Omega$ into straight-sided triangles $K$. For ease of presentation, we consider regular triangulations and comment on extensions to irregular meshes in Section 7. The triangulations are supposed to be sufficiently fine so that each element $K$ contains at most one vertex $\mathbf{c}_i$. For $K \in \mathcal{T}$, we denote by $\mathcal{P}^p(K)$ the polynomials on $K$ of total degree at most $p$, and by $\boldsymbol{\nu}_K$ the unit outward normal vector on $\partial K$.

Furthermore, we write $h_K$ and $\rho_K$ for the diameter and inradius of $K \in \mathcal{T}$, respectively. The mesh-width of $\mathcal{T}$ is given by $h = h(\mathcal{T}) := \max_{K\in\mathcal{T}} h_K$. We assume the triangulations to be shape-regular: There exists a constant $\kappa > 0$ such that there holds, for all $K \in \mathcal{T}$, and uniformly in the mesh sequence,

$$\kappa h_K \leq \rho_K \leq \kappa^{-1} h_K\ . \tag{3.1}$$

Edges are defined as follows. If $K$ and $K'$ are adjacent elements of the triangulation $\mathcal{T}$ with $\int_{\partial K\cap\partial K'} dS > 0$, we call the intersection $e = \partial K \cap \partial K'$ an interior edge. Elemental edges of $K$ are supposed to lie at most on one boundary segment $e_i$, and if $\int_{\partial K\cap e_i} dS > 0$, we call the intersection $e = \partial K \cap e_i$ a boundary edge; it belongs to either $\Gamma_D$ or $\Gamma_N$. Accordingly, we distinguish between Dirichlet and Neumann edges. The set of interior edges of a triangulation $\mathcal{T}$ is denoted by $\mathcal{E}_I(\mathcal{T})$, the set of Dirichlet boundary edges by $\mathcal{E}_D(\mathcal{T})$, and the set of Neumann boundary edges by $\mathcal{E}_N(\mathcal{T})$. Moreover, we define $\mathcal{E}_{ID}(\mathcal{T}) := \mathcal{E}_I(\mathcal{T}) \cup \mathcal{E}_D(\mathcal{T})$ and

$\mathcal{E}(\mathcal{T}) := \mathcal{E}_I(\mathcal{T}) \cup \mathcal{E}_D(\mathcal{T}) \cup \mathcal{E}_N(\mathcal{T})$. For $e \in \mathcal{E}(\mathcal{T})$, we denote by $\mathcal{P}^p(e)$ the polynomials of degree at most $p$ on $e$, and by $h_e = |e|$ the length of $e$. With the shape-regularity assumption (3.1), it can be readily verified that

$$\kappa h_K \le h_e \le h_K, \tag{3.2}$$

for all $e \subset \partial K$ with $e \in \mathcal{E}(\mathcal{T})$.

Following [2], we introduce the standard trace operators. Let $K^+, K^- \in \mathcal{T}$ be two adjacent elements which share the interior edge $e = \partial K^+ \cap \partial K^- \in \mathcal{E}_I(\mathcal{T})$. For a sufficiently smooth scalar function $v$ or vector field $\mathbf{q}$, we denote the traces of $v$ and $\mathbf{q}$ on $e$ taken from within $K^\pm$ by $v^\pm$ and $\mathbf{q}^\pm$, respectively. We then define the jumps and the averages of $v$ and $\mathbf{q}$ along $e$ by

$$[\![v]\!] := v^+ \boldsymbol{\nu}_{K^+} + v^- \boldsymbol{\nu}_{K^-}\ , \qquad \langle\!\langle v \rangle\!\rangle := \frac{1}{2}(v^+ + v^-)\ , \tag{3.3}$$

$$[\![\mathbf{q}]\!] := \mathbf{q}^+ \cdot \boldsymbol{\nu}_{K^+} + \mathbf{q}^- \cdot \boldsymbol{\nu}_{K^-}\ , \qquad \langle\!\langle \mathbf{q} \rangle\!\rangle := \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-)\ . \tag{3.4}$$

If $e \in \mathcal{E}_D(\mathcal{T})$ is a Dirichlet boundary edge, we similarly set $[\![v]\!] := v|_e \boldsymbol{\nu}$, $[\![\mathbf{q}]\!] = \mathbf{q}|_e \cdot \boldsymbol{\nu}$, as well as $\langle\!\langle v \rangle\!\rangle := v|_e$, $\langle\!\langle \mathbf{q} \rangle\!\rangle := \mathbf{q}|_e$.

3.2. **Discretization.** For an approximation order $p \ge 1$ and a given triangulation $\mathcal{T}$ of $\Omega$, we introduce the discontinuous finite element space

$$V_p(\mathcal{T}) := \{ v \in L^2(\Omega) \,:\, v|_K \in \mathcal{P}^p(K),\ K \in \mathcal{T} \}\ . \tag{3.5}$$

Then, the symmetric interior penalty discretization of (2.7) reads as follows: Find $u_N \in V_p(\mathcal{T})$ such that

$$a_{DG}(u_N, v_N) = l_{DG}(v_N) \tag{3.6}$$

for all $v_N \in V_p(\mathcal{T})$. Here, $a_{DG}(\cdot,\cdot)$ is the symmetric interior penalty form given by

$$\begin{aligned} a_{DG}(v,w) := &\sum_{K \in \mathcal{T}} \int_K \mathsf{c} \nabla v \cdot \nabla w \, d\mathbf{x} - r_{DG}(v,w) \\ &- r_{DG}(w,v) + \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \int_e \mathsf{j}_e [\![v]\!] \cdot [\![w]\!] \, dS\ , \end{aligned} \tag{3.7}$$

with the off-diagonal form

$$r_{DG}(v,w) := \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \int_e \langle\!\langle \mathsf{c} \nabla v \rangle\!\rangle \cdot [\![w]\!] \, dS\ . \tag{3.8}$$

The linear form $l_{DG}(\cdot)$ on the right-hand side in (3.6) is defined as

$$\begin{aligned} l_{DG}(w) := &\sum_{K \in \mathcal{T}} \int_K f w \, d\mathbf{x} - \sum_{e \in \mathcal{E}_D(\mathcal{T})} \int_e g_D (\mathsf{c} \nabla w) \cdot \boldsymbol{\nu} \, dS \\ &+ \sum_{e \in \mathcal{E}_D(\mathcal{T})} \int_e \mathsf{j}_e g_D w \, dS + \sum_{e \in \mathcal{E}_N(\mathcal{T})} \int_{\Gamma_N} g_N w \, dS\ . \end{aligned} \tag{3.9}$$

In (3.7), (3.9), we define the interior penalty function $\mathsf{j}$ edgewise as

$$\mathsf{j}_e := \mathsf{j}_0 \, \mathsf{c}|_e h_e^{-1}, \qquad e \in \mathcal{E}_{ID}(\mathcal{T})\ , \tag{3.10}$$

where $\mathsf{j}_0 > 0$ is a sufficiently large as specified below and where we recall that $h_e$ denotes the length of $e$.

**Remark 3.1.** *In Section 5.2, we show the boundedness of all the terms in* (3.7)–(3.9) *whenever $v$ or $w$ are weighted functions in $H^{2,2}_{\boldsymbol{\delta}}(\Omega)$. In particular, for elements $K$ abutting at corners and edges $e$ running into corners, the integrals in* (3.7)–(3.9) *are understood as bounded bilinear forms over $L^1(K) \times L^\infty(K)$ and $L^1(e) \times L^\infty(e)$, respectively.*

**Remark 3.2.** *The number of degrees of freedom of the discretization* (3.6) *is defined as*

$$N = N(p, \mathcal{T}) := \dim(V_p(\mathcal{T})). \tag{3.11}$$

*We are interested in achieving convergence $N \to \infty$ by reducing the mesh-width $h \to 0$ at a fixed (typically low) polynomial degree $p \geq 1$, which is known as the $h$-version of the finite element method. In the following, we derive optimal algebraic convergence rates with respect to $N$ for the DG method* (3.6) *on locally refined meshes.*

3.3. **DG norm and discrete stability.** For a partition $\mathcal{T}$ of $\Omega$, we introduce the broken $H^1$-space

$$H^1(\mathcal{T}) := \{\, v \in L^2(\Omega) \,:\, v|_K \in H^1(K),\ K \in \mathcal{T}\,\}, \tag{3.12}$$

which we endow with the DG energy norm

$$\|\!|v|\!\|_{DG}^2 := \sum_{K\in\mathcal{T}} \|\mathsf{c}^{1/2}\nabla v\|^2_{L^2(K)} + J(v), \qquad J(v) := \sum_{e\in\mathcal{E}_{ID}(\mathcal{T})} \|\mathsf{j}_e^{1/2}[\![v]\!]\|^2_{L^2(e)}\,. \tag{3.13}$$

Notice that for $\mathcal{D} \neq \emptyset$, the mapping $v \mapsto \|\!|v|\!\|_{DG}$ is a norm on $H^1(\mathcal{T})$. In the pure Neumann case, however, the expression $\|\!|v|\!\|_{DG}$ is zero if and only if $v$ is a constant, and hence it is a norm modulo constants.

The following discrete stability properties over the broken FE space $V_p(\mathcal{T})$ are well-known; we refer for example to [2] or [17].

**Lemma 3.3.** *There exists $\mathsf{j}_* > 0$ and constants $C_{coer} > 0$, $C_{cont} > 0$, which are independent of the mesh-widths, but depend on $\kappa$ in* (3.1)*, the bounds in* (2.3) *and the polynomial degree $p$, such that for $\mathsf{j}_0 > \mathsf{j}_*$ there holds*

$$a_{DG}(v_N, v_N) \geq C_{coer}\|\!|v_N|\!\|^2_{DG}, \qquad v_N \in V_p(\mathcal{T}), \tag{3.14}$$

$$|a_{DG}(v_N, w_N)| \leq C_{cont}\|\!|v_N|\!\|_{DG}\|\!|w_N|\!\|_{DG}, \qquad v_N, w_N \in V_p(\mathcal{T})\,. \tag{3.15}$$

## 4. Main Results

We introduce two types of mesh families with local refinement towards corners. Then, we state and discuss our main results: optimal energy norm and $L^2$-norm convergence rate estimates for IP discretizations of arbitrary order (see Theorem 4.3), on either type of mesh family with sufficiently strong refinement in the vicinity of corners.

4.1. **Graded mesh families.** We first recall the definition of graded mesh families as introduced in [5].

**Definition 4.1.** *A shape-regular family of regular triangulations $\mathcal{T}_{\boldsymbol{\beta}}$ is called graded towards the vertices in $\mathcal{S}$ with grading vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{\mathsf{M}})$, if there exists a uniform constant $C_G > 0$ such that for all elements $K \in \mathcal{T}_{\boldsymbol{\beta}}$ in each triangulation, one of the following conditions hold:*

(i) *If $K \in \mathcal{T}_{\boldsymbol{\beta}} \setminus \mathcal{K}(\mathcal{T}_{\boldsymbol{\beta}})$, then ${C_G}^{-1} h\Phi_{\boldsymbol{\beta}}(\mathbf{x}) \leq h_K \leq C_G h\Phi_{\boldsymbol{\beta}}(\mathbf{x})$ for all $\mathbf{x} \in K$.*

(ii) *If* $K \in \mathcal{K}(\mathcal{T}_{\boldsymbol{\beta}})$*, then* $C_G^{-1} \sup_{\mathbf{x}\in K} \Phi_{\boldsymbol{\beta}}(\mathbf{x}) \le h_K \le C_G h \sup_{\mathbf{x}\in K} \Phi_{\boldsymbol{\beta}}(\mathbf{x})$.

In [21, Proposition 2.5.5], it has been shown that IP methods for elliptic problems on graded mesh families converge optimally in the DG energy norm (but not in the $L^2$-norm). We will establish a refinement of this result for our convergence analysis. Other examples of graded mesh families and their constructions are well-known by now. We refer to [7, 1] and the software package LNG_FEM in [13]. The finite element spaces constructed on these mesh families, however, are not nested, i.e., an increase of accuracy in the numerical approximation requires construction of the entire finer mesh.

4.2. **Bisection refinement meshes.** An alternative are regular, simplicial mesh families which are produced by recursive bisection refinement; see, e.g., [15, 16] and the references therein. Our analysis will be based on the work [9], where a bisection refinement algorithm has been proposed and analyzed in the context of conforming finite element methods. Given an initial mesh $\mathcal{T}_0$, the algorithm there takes input parameters $h$, $p$, $L$ and a weight exponent $\gamma > 0$. In a first loop, it ensures that all elemental mesh-widths $h_K$ are smaller than $h$. In a second loop, the algorithm refines $2L+1$ times into the corners using newest vertex bisection, where $L$ is to be selected in dependence of $h$, $p$ and $\gamma$. This results in a regular mesh denoted by $\mathcal{T}_{h,2(L+1)}$. We emphasize that the bisection refinement meshes constructed from the regular, simplicial initial mesh $\mathcal{T}_0$ gives rise to a shape-regular mesh family, where the condition (3.1) is satisfied with a constant $\kappa$ depending on $\mathcal{T}_0$. For conforming finite element methods, it has been shown in [9] that the bisection refinement algorithm based on choosing suitable parameters captures solutions of elliptic problems with solutions which allow for decompositions into regular parts and corner singularities at optimal convergence orders in $N$. We will generalize this result to the discontinuous Galerkin framework and to functions in $H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$; see Proposition 5.18. Next, we introduce the notion of a locally adapted mesh.

**Definition 4.2.** *Let* $p \ge 1$ *and* $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ *be a weight exponent vector. We call a a family of triangulations* $\mathcal{T}$ locally adapted to $\mathcal{S}$ with respect to $\boldsymbol{\delta}$ and $p$ *if it is either*

(i) *a graded mesh family of meshes* $\mathcal{T}_{\boldsymbol{\beta}}$ *with grading parameters* $\beta_i \in (\beta_i^*, 1)$ *where*
$$\beta_i^* := 1 - \frac{1-\delta_i}{p} \;, \tag{4.1}$$

(ii) *or a family of bisection refinement meshes* $\mathcal{T}_{h,2(L+1)}$ *as in* [9]*, obtained by newest vertex bisection with parameters* $h$*,* $\gamma \in (0, \gamma^*]$ *and* $L$ *with*
$$\gamma^* := 1 - \max_{i=1}^{\mathsf{M}} \delta_i > 0 \quad \textit{and} \quad h \in [2^{-(L+1)\gamma/(p+1)}, 2^{-L\gamma/(p+1)}). \tag{4.2}$$

4.3. **Optimal error estimates.** Our main result establishes optimal energy norm and $L^2$-norm error bounds on locally adapted meshes with respect to the number $N$ of total degrees of freedom.

**Theorem 4.3.** *Let* $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ *be as in* (2.19)*. For* $p \ge 1$ *and* $1 \le k \le p$*, let* $f \in H^{k-1,0}_{\boldsymbol{\delta}}(\Omega)$*,* $g_D \in H^{k+1/2,3/2}_{\boldsymbol{\delta}}(\Gamma_D)$ *and* $g_N \in H^{k-1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)$*. Consider the solution* $u \in H^1(\Omega)$ *of* (2.4)–(2.6) *which is in* $H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$ *by Proposition 2.3. Let* $\mathcal{T}$ *be a mesh which is locally adapted to* $\mathcal{S}$ *with respect to* $\boldsymbol{\delta}$ *and* $p$ *either via mesh grading or via recursive bisection refinement as in Definition 4.2. Let* $u_N \in V_p(\mathcal{T})$

*be the symmetric IP approximation obtained in* (3.6) *with* $\mathrm{j}_0 > \mathrm{j}_*$*. Then, we have the error bound*

$$\|u - u_N\|_{DG} + N^{1/2}\|u - u_N\|_{L^2(\Omega)} \le CN^{-k/2}\|u\|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)} \,. \tag{4.3}$$

*The constant $C > 0$ is independent of $N$, but depends on $\kappa$ in* (3.1)*, on the parameter $\mathrm{j}_0$ in* (3.10)*, the bounds in* (2.3)*, the regularity parameter $k$, the polynomial degree $p$, the parameter $C_{dc}$ in* (2.10)*,* (2.11) *the stability constant $C_{stab,2}$ in Proposition 2.3, the vector $\boldsymbol{\delta}$, and on the parameters for graded and bisection refinement meshes in Definition 4.2.*

**Remark 4.4.** *For $k = p$ (i.e., $u \in H^{p+1,2}_{\boldsymbol{\delta}}(\Omega)$), the estimate* (4.3) *gives optimal convergence rates of order $N^{-p/2}$ and of order $N^{-(p+1)/2}$ for the $L^2$-norm error, respectively. We remark further that Theorem 4.3 is based on the regularity bound* (3.1)*. Hence, no additional regularity assumptions on the domains are necessary for the $L^2$-norm estimate.*

*We further note that in the proof of the $L^2$-norm bound, we implicitly use the adjoint-consistency of the symmetric IP method in the sense of* [2]*. This property does not hold for non-symmetric IP discretizations of* (2.4)–(2.6)*. As a consequence, $L^2$-norm optimality as in Theorem 4.3 and the arguments for its proof cannot in general be expected to apply for non-symmetric IP methods or for other DG formulations that do not afford adjoint-consistency. We refer to* [18, Section 2.8.2] *for a detailed discussion on $L^2$-norm error estimation for non-symmetric and so-called incomplete interior penalty methods.*

**Remark 4.5.** *The results in Theorem 4.3 remain valid in the pure Neumann case due to the fact that the nodal interpolants used in our analysis reproduce constants.*

## 5. Proofs

In this section, we detail the proof of Theorem 4.3. We shall frequently use the short-hand notation $a \lesssim b$ for inequalities of the form $a \le Cb$, where $C > 0$ solely may depend on $\kappa$ in (3.1), the bounds in (2.3), the parameter $\mathrm{j}_0$ in (3.10), the polynomial degree $p$, and the particular exponent vector $\boldsymbol{\delta}$ under consideration.

5.1. **Preliminaries.** In this section, we introduce discrete neighborhoods and establish some essential embedding properties and trace results for the weighted space $H^{k,l}_{\boldsymbol{\delta}}(\Omega)$. We further introduce the broken consistency norm which is appropriate for our analysis; it is a generalization to weighted spaces of the norm used in [2, Section 4.1].

5.1.1. *Auxiliary results.* We first recall a number of technical estimates which are relevant in our analysis. The first result is the trace inequality:

$$\|v\|^q_{L^q(\partial K)} \lesssim h_K^{-1}\big(\|v\|^q_{L^q(K)} + h_K^q\|\nabla v\|^q_{L^q(K)}\big), \qquad v \in W^{1,q}(K),\ 1 \le q < \infty \,, \tag{5.1}$$

where the implied constant is independent of $h_K$, but also depends on $\kappa$ in (3.1) and on $q$. This bound follows readily by the trace inequality on a reference triangle $\widehat{K}$ combined with scaling arguments employing affine equivalence. We will use the polynomial trace inequality

$$\|q\|_{L^2(\partial K)} \lesssim h_K^{-1/2}\|q\|_{L^2(K)}, \qquad q \in \mathcal{P}^p(K) \,, \tag{5.2}$$

as well as the inverse inequality

$$\|q\|_{L^\infty(e)} \lesssim h_e^{-1/2}\|q\|_{L^2(e)}, \qquad q \in \mathcal{P}^p(e)\ ; \tag{5.3}$$

see, e.g., [17, Lemmas 1.46 and 1.50].

5.1.2. *Discrete neighborhoods.* For a partition $\mathcal{T}$ of $\Omega$, we introduce the discrete neighborhoods

$$\begin{aligned} \mathcal{N}_i(\mathcal{T}) &:= \{\, K \in \mathcal{T} \,:\, K \cap \Omega_i \neq \emptyset \,\}, \quad 1 \le i \le \mathsf{M}, \\ \mathcal{N}_0(\mathcal{T}) &:= \mathcal{T} \setminus \big( \cup_{i=1}^{M} \mathcal{N}_i(\mathcal{T})\big). \end{aligned} \tag{5.4}$$

We always assume the meshes to be sufficiently refined, so that $\mathcal{N}_i(\mathcal{T}) \cap \mathcal{N}_j(\mathcal{T}) = \emptyset$ for $i \neq j$.

**Remark 5.1.** *For $1 \le i \le \mathsf{M}$, an element $K \in \mathcal{N}_i(\mathcal{T})$ can be written as $\overline{K} = (\overline{K \cap \Omega_0}) \cup (\overline{K \cap \Omega_i})$ with $K \cap \Omega_i \neq \emptyset$. Then, due to* (2.10), (2.11), *we find that*

$$\begin{aligned} |v|^2_{H^{k+1,2}_{\delta_i}(K)} &= |v|^2_{H^{k+1,2}_{\delta_i}(K\cap\Omega_0)} + |v|^2_{H^{k+1,2}_{\delta_i}(K\cap\Omega_i)} \\ &\lesssim |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K\cap\Omega_0)} + |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K\cap\Omega_i)} \lesssim |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)}, \end{aligned} \tag{5.5}$$

*for any regularity index $k \ge 1$.*

The following properties are generalizations of the results in [21, Lemmas 1.3.2 and 1.3.4].

**Lemma 5.2.** *For $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$, there holds:*

(i) *Let $v \in H^{0,0}_{\boldsymbol{\delta}}(K)$ for $K \in \mathcal{T}$. Then we have $v|_K \in L^1(K)$ and*

$$\|v\|_{L^1(K)} \lesssim \begin{cases} h_K \|v\|_{H^{0,0}_{\boldsymbol{\delta}}(K)}, & K \in \mathcal{N}_0(\mathcal{T})\ , \\ h_K^{1-\delta_i}\|v\|_{H^{0,0}_{\boldsymbol{\delta}}(K)}, & K \in \mathcal{N}_i(\mathcal{T}),\ 1 \le i \le \mathsf{M}\ . \end{cases} \tag{5.6}$$

(ii) *Let $v \in H^{1,1}_{\boldsymbol{\delta}}(K)$ for $K \in \mathcal{T}$. Then we have $v|_{\partial K} \in L^1(\partial K)$ and*

$$\|v\|_{L^1(\partial K)} \lesssim \begin{cases} \|v\|_{L^2(K)} + h_K |v|_{H^{1,1}_{\boldsymbol{\delta}}(K)}, & K \in \mathcal{N}_0(\mathcal{T}), \\ \|v\|_{L^2(K)} + h_K^{1-\delta_i}|v|_{H^{1,1}_{\boldsymbol{\delta}}(K)}, & K \in \mathcal{N}_i(\mathcal{T}),\ 1 \le i \le \mathsf{M}. \end{cases} \tag{5.7}$$

(iii) *Let $v \in H^{1,1}_{\boldsymbol{\delta}}(\Omega)$. Then for any edge $e \in \mathcal{E}_I(\mathcal{T})$, we have*

$$[\![v]\!]|_e = 0 \quad \text{in } L^1(e). \tag{5.8}$$

**Remark 5.3.** *The properties in Lemma 5.2 are valid for elements abutting at corners, for which they have been proved in* [21]. *We further remark that the first case inequalities in* (5.6) *and* (5.7) *hold true for all elements $K \in \mathcal{T}$ away from corners, but without uniform control in weighted norms.*

*Proof.* We prove each item separately.

*Proof of* (5.6)*:* Let $K \in \mathcal{N}_i(\mathcal{T})$ be such that $\overline{K} = (\overline{K \cap \Omega_i}) \cup (\overline{K \cap \Omega_0})$, where either one or both of the intersections are non-empty. Clearly,

$$\|v\|_{L^1(K)} = \|v\|_{L^1(K\cap\Omega_0)} + \|v\|_{L^1(K\cap\Omega_i)}.$$

For the first term, we use the Cauchy-Schwarz inequality and property (2.10) to obtain

$$\|v\|_{L^1(K\cap\Omega_0)} \lesssim \mathrm{area}(K \cap \Omega_0)^{1/2}\|v\|_{H^{0,0}_{\boldsymbol{\delta}}(K\cap\Omega_0)} \lesssim h_K \|v\|_{H^{0,0}_{\boldsymbol{\delta}}(K)}\ .$$

To estimate the second term, we introduce polar coordinates $(r_i, \vartheta_i)$ centered at the vertex $\mathbf{c}_i$ and apply the Cauchy-Schwarz in combination with (2.11). This yields

$$\|v\|_{L^1(K\cap\Omega_i)} = \int_{K\cap\Omega_i} r_i^{-\delta_i} r_i^{\delta_i} |v|\, d\mathbf{x} \lesssim \|r_i^{-\delta_i}\|_{L^2(K)} \|v\|_{H^{0,0}_{\boldsymbol{\delta}}(K)}.$$

To bound $\|r_i^{-\delta_i}\|_{L^2(K)}$, we set $r_K = \operatorname{dist}(K, \mathbf{c}_i) = \inf_{\mathbf{y}\in K} |\mathbf{y} - \mathbf{c}_i| \geq 0$. With the shape-regularity assumption (3.1) and by integrating out over the angular variable $\vartheta_i$, we conclude that

$$\|r_i^{-\delta_i}\|^2_{L^2(K)} \lesssim \int_{r_i=r_K}^{r_i=r_K+h_K} r_i^{-2\delta_i+1}\, dr_i \lesssim r_i^{2-2\delta_i}\Big|_{r_i=r_K}^{r_i=r_K+h_K} \lesssim h_K^{2-2\delta_i},$$

where the last inequality follows since $|x^\alpha - y^\alpha| \leq |x-y|^\alpha$ for any $x, y \in [0,\infty)$ and $\alpha \in [0,1)$. These bounds imply (5.6) in all cases.

*Proof of* (5.7)*:* The inequality (5.7) is a consequence of the standard trace inequality (5.1) with $q = 1$ and the embedding (5.6) applied to $\nabla v \in H^{0,0}_{\boldsymbol{\delta}}(K)^2$, taking into account that $\|v\|_{L^1(K)} \lesssim h_K \|v\|_{L^2(K)}$ for $v \in H^{1,1}_{\boldsymbol{\delta}}(K)$.

*Proof of* (5.8)*:* The identity (5.8) is clear for edges $e \in \mathcal{E}_I(\mathcal{T})$ away from corners. For an edge $e \in \mathcal{E}_I(\mathcal{T})$ with $\overline{e} \cap \mathbf{c}_i \neq \emptyset$, let $\mathbf{x} : [0,1] \to e$, $t \mapsto \mathbf{x}(t)$ be an affine parametrization of $e$ with $\mathbf{x}(0) = \mathbf{c}_i$. It can be readily seen that $\int_\varepsilon^1 |[\![v]\!]|_e|\, dt = 0$ for all $\varepsilon > 0$. By Lebesgue's dominated convergence theorem (using (5.7)), it follows that $\int_e |[\![v]\!]|\, dS = |e| \int_0^1 |[\![v]\!]|\, dt = 0$, which is (5.8) for this case. $\square$

5.1.3. *Corner elements and consistency norm.* In the sequel, a particular role will be played by the subset $\mathcal{K}_i(\mathcal{T})$ of elements of $\mathcal{N}_i(\mathcal{T})$ abutting at $\mathbf{c}_i$, defined by

$$\mathcal{K}_i(\mathcal{T}) := \{K \in \mathcal{N}_i(\mathcal{T}) \,:\, \overline{K} \cap \mathbf{c}_i \neq \emptyset\}, \qquad 1 \leq i \leq \mathsf{M}. \tag{5.9}$$

We also have $\mathcal{K}_i(\mathcal{T}) \cap \mathcal{K}_j(\mathcal{T}) = \emptyset$ for $i \neq j$. We further may assume that $K \in \mathcal{K}_i(\mathcal{T})$ is located in the cone $\Omega_i$ (i.e., $\overline{K} \subset \Omega_i$). We then set

$$\mathcal{K}(\mathcal{T}) := \bigcup_{i=1}^{\mathsf{M}} \mathcal{K}_i(\mathcal{T}). \tag{5.10}$$

For elements away from corners, we next introduce the weighted elemental norm

$$M_K[v]^2 := h_K^{-2}\|v\|^2_{L^2(K)} + \|\nabla v\|^2_{L^2(K)} + h_K^2 \|\mathsf{D}^2 v\|^2_{L^2(K)}, \qquad K \in \mathcal{T} \setminus \mathcal{K}(\mathcal{T}). \tag{5.11}$$

For a corner element $K \in \mathcal{K}_i(\mathcal{T})$ and $\delta_i \in [0,1)$, we define

$$N_{K,\delta_i}[v]^2 := h_K^{-2}\|v\|^2_{L^2(K)} + \|\nabla v\|^2_{L^2(K)} + h_K^{2-2\delta_i} |v|^2_{H^{2,2}_{\delta_i}}(K), \qquad K \in \mathcal{K}_i(\mathcal{T}). \tag{5.12}$$

**Lemma 5.4.** *Let $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$, and $v = v_0 + v_N$ with $v_0 \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ and $v_N \in V_p(\mathcal{T})$. Then:*

(i) *We have $v \in C^0(\overline{K})$ for $K \in \mathcal{T}$ and*

$$\|v\|^2_{C^0(\overline{K})} \lesssim \begin{cases} M_K[v]^2, & K \in \mathcal{T} \setminus \mathcal{K}(\mathcal{T}), \\ N_{K,\delta_i}[v]^2, & K \in \mathcal{K}_i(\mathcal{T}),\ 1 \leq i \leq \mathsf{M}. \end{cases} \tag{5.13}$$

(ii) *We have $\nabla v \in L^1(\partial K)^2$ for $K \in \mathcal{T}$ and*

$$\|\nabla v\|^2_{L^1(\partial K)} \lesssim \begin{cases} M_K[v]^2, & K \in \mathcal{T} \setminus \mathcal{K}(\mathcal{T}), \\ N_{K,\delta_i}[v]^2, & K \in \mathcal{K}_i(\mathcal{T}),\ 1 \leq i \leq \mathsf{M}. \end{cases} \tag{5.14}$$

*Proof.* The bound (5.13) follows from the continuous embeddings $H^2(\widehat{K}) \hookrightarrow C^0(\overline{\widehat{K}})$ and $H^{2,2}_{\delta_i}(\widehat{K}) \hookrightarrow C^0(\overline{\widehat{K}})$, respectively, formulated for a reference triangle $\widehat{K}$ and combined with an affine scaling argument. To show (5.14), we note that for $K \in \mathcal{T} \setminus \mathcal{K}(\mathcal{T})$, the trace inequality (5.1) in $L^1(\partial K)$ combined with $h_K^{-2}\|\nabla v\|^2_{L^1(K)} \lesssim \|\nabla v\|^2_{L^2(K)}$, $\|\mathsf{D}^2 v\|^2_{L^1(K)} \lesssim h_K^2 \|\mathsf{D}^2 v\|^2_{L^2(K)}$ readily yields $\|\nabla v\|^2_{L^1(\partial K)} \lesssim M_K[v]^2$. For $K \in \mathcal{K}_i(\mathcal{T})$, the inequality (5.14) follows from (5.7). $\square$

For $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ and a subset $\mathcal{T}' \subseteq \mathcal{T}$ of elements, we now introduce the broken consistency norm

$$\|v\|^2_{\mathcal{T}',\boldsymbol{\delta}} := \sum_{K \in (\mathcal{T}' \cap \mathcal{K}(\mathcal{T})) \setminus \mathcal{K}(\mathcal{T})} M_K[v]^2 + \sum_{i=1}^{\mathsf{M}} \sum_{K \in \mathcal{T}' \cap \mathcal{K}_i(\mathcal{T})} N_{K,\delta_i}[v]^2 \,. \tag{5.15}$$

We further show the following bound, which implies that DG energy norm (3.13) is bounded by the norm (5.15) for sufficiently smooth functions $v$.

**Lemma 5.5.** *There holds*

$$\|v\|^2_{DG} \lesssim \sum_{K \in \mathcal{T}} \left( h_K^{-2} \|v\|^2_{L^2(K)} + \|\nabla v\|^2_{L^2(K)} \right) , \tag{5.16}$$

*for all* $v \in H^1(\mathcal{T})$.

*Proof.* With (2.3), it suffices to bound the jump terms appearing in the energy norm $\|v\|_{DG}$. To this end, let $e = \partial K \cap \partial K' \in \mathcal{E}_I(\mathcal{T})$ be an interior edge. By applying the trace inequality (5.1) (with $q = 2$), the property (3.2) and the bounds in (2.3), we obtain

$$\|\mathsf{j}|_e^{1/2} [\![v]\!]\|^2_{L^2(e)} \lesssim h_K^{-2}\|v\|^2_{L^2(K)} + \|\nabla v\|^2_{L^2(K)} + h_{K'}^{-2}\|v\|^2_{L^2(K')} + \|\nabla v\|^2_{L^2(K')} \,.$$

A similar argument holds for Dirichlet edges, which yields (5.16). $\square$

5.2. **Boundedness.** We next establish several continuity results for the IP bilinear form $a_{DG}(\cdot,\cdot)$ and for the right-hand side $l_{DG}(\cdot)$.

5.2.1. *Continuity bounds for* $a_{DG}(\cdot,\cdot)$. To establish continuity properties of $a_{DG}$ in (3.7), we first note that, by the Cauchy-Schwarz inequality and by (2.3),

$$\sum_{K \in \mathcal{T}} \int_K |\mathsf{c}\nabla v \cdot \nabla w| \, d\mathbf{x} \le \|v\|_{DG} \|w\|_{DG}, \tag{5.17}$$

$$\sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \int_e \mathsf{j}_e |[\![v]\!] \cdot [\![w]\!]| \, dS \le \|v\|_{DG} \|w\|_{DG}, \tag{5.18}$$

for all $v, w \in H^1(\mathcal{T})$.

The boundedness of the off-diagonal form $r_{DG}(v,w)$ in (3.8) is more involved and will be discussed next. The first bound (5.19) below is a standard result, see [2]. A proof is included to render the error analysis self-contained. On the other hand, the second estimate (5.20) is a generalization to weighted spaces of standard consistency bounds as, e.g., in [2, Section 4.1] or [17, Section 4.2]. To prove it, we use the approach in [21, Proposition 2.4.1], which is based on using Hölder's inequality in $L^1(e) \times L^\infty(e)$; see also Lemma 5.2. The third property (5.21) is new and will be crucial to derive $L^2$-norm error estimates.

**Proposition 5.6.** *For* $r_{DG}(\cdot,\cdot)$ *defined in* (3.8)*, there holds:*

(i) *For* $v_N \in V_p(\mathcal{T})$, *we have*

$$|r_{DG}(v_N, w)| \leq C_{r_{DG},1} J(w) \|v_N\|_{DG}, \qquad w \in H^1(\mathcal{T}) \ . \tag{5.19}$$

(ii) *Let* $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$. *For every* $v = v_0 + v_N$ *with* $v_0 \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ *and* $v_N \in V_p(\mathcal{T})$, *there holds for every* $w_N \in V_p(\mathcal{T})$

$$|r_{DG}(v, w_N)| \leq C_{r_{DG},2} J(w_N) \|v\|_{\mathcal{T},\boldsymbol{\delta}}. \tag{5.20}$$

(iii) *Let* $\boldsymbol{\delta}, \boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$. *For every* $v = v_0 + v_N$ *and* $w = w_0 + w_N$ *with* $v_0 \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$, $w_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ *and* $v_N, w_N \in V_p(\mathcal{T})$, *we have*

$$|r_{DG}(v, w)| \leq C_{r_{DG},3} \|w\|_{\mathcal{T},\boldsymbol{\delta}'} \|v\|_{\mathcal{T},\boldsymbol{\delta}} \ . \tag{5.21}$$

*In* (5.19)–(5.21), *the constants* $C_{r_{DG},1}, C_{r_{DG},2}, C_{r_{DG},3} > 0$ *depend on* $\kappa$ *in* (3.1), *the bounds in* (2.3), *on the parameter* $\mathfrak{j}_0$ *in* (3.10) *and on the polynomial degree* $p$. *The constants* $C_{r_{DG},2}, C_{r_{DG},3}$ *further depend on the exponents* $\boldsymbol{\delta}, \boldsymbol{\delta}'$.

*Proof.* We proceed in two steps:

*Proof of* (5.19)*:* For $v_N \in V_p(\mathcal{T})$ and for $w \in H^1(\mathcal{T})$, the Cauchy-Schwarz inequality, the bounds in (2.3), the definition of $\mathfrak{j}$ in (3.10), the equivalence (3.2) and the polynomial trace inequality (5.2) yield

$$\begin{aligned}
|r_{DG}(v_N, w)| &\lesssim \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} h_e^{-1/2} \|[\![w]\!]\|_{L^2(e)} h_e^{1/2} \|\langle\!\langle \nabla v_N \rangle\!\rangle\|_{L^2(e)} \\
&\lesssim J(w) \Big( \sum_{K \in \mathcal{T}} h_K \|\nabla v_N\|^2_{L^2(\partial K)} \Big)^{1/2} \lesssim J(w) \Big( \sum_{K \in \mathcal{T}} \|\nabla v_N\|^2_{L^2(K)} \Big)^{1/2},
\end{aligned}$$

which, together with (2.3), implies (5.19).

*Proof of* (5.20) *and* (5.21)*:* Let

$$w = w_0 + w_N \quad \text{with} \quad w_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega) \text{ and } w_N \in V_p(\mathcal{T}). \tag{5.22}$$

We invoke Hölder's inequality as in [21, Proposition 2.4.1], the bound (2.3) and the discrete Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|r_{DG}(v, w)| &\lesssim \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \int_e |[\![w]\!]| |\langle\!\langle \nabla v \rangle\!\rangle| \, dS \\
&\lesssim \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \|[\![w]\!]\|_{C^0(\overline{e})} \|\langle\!\langle \nabla v \rangle\!\rangle\|_{L^1(e)} \\
&\lesssim \Big( \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \|[\![w]\!]\|^2_{L^\infty(e)} \Big)^{1/2} \Big( \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \|\langle\!\langle \nabla v \rangle\!\rangle\|^2_{L^1(e)} \Big)^{1/2} \\
&\lesssim \Big( \sum_{e \in \mathcal{E}_{ID}(\mathcal{T})} \|[\![w]\!]\|^2_{L^\infty(e)} \Big)^{1/2} \Big( \sum_{K \in \mathcal{T}} \|\nabla v\|^2_{L^1(\partial K)} \Big)^{1/2}
\end{aligned}. \tag{5.23}$$

In the last term in (5.23) above, we apply the bounds in (5.14) and find that

$$\Big( \sum_{K \in \mathcal{T}} \|\nabla v\|^2_{L^1(\partial K)} \Big)^{1/2} \lesssim \|v\|_{\mathcal{T},\boldsymbol{\delta}}. \tag{5.24}$$

Let us now establish (5.20) and take $w_0 = 0$ in (5.22), (5.23). Then, for each edge $e \in \mathcal{E}_{ID}(\mathcal{T})$, the jump $[\![w]\!]|_e = [\![w_N]\!]|_e \in \mathcal{P}^p(e)$ is a univariate polynomial.

Therefore, the inverse estimate (5.3), the bounds in (2.3) and the definition of $\mathfrak{j}$ in (3.10) yield

$$\Big(\sum_{e\in\mathcal{E}_{ID}(\mathcal{T})}\|[\![w_N]\!]\|^2_{L^\infty(e)}\Big)^{1/2}\lesssim\Big(\sum_{e\in\mathcal{E}_{ID}(\mathcal{T})}h_e^{-1/2}\|[\![w_N]\!]\|^2_{L^2(e)}\Big)^{1/2}\lesssim J(w_N)\ . \tag{5.25}$$

The combination of (5.23), (5.24) and (5.25) implies (5.20).

Finally, to prove (5.21), we now take in (5.22), (5.23) a generic $w = w_0 + w_N$, where $0 \neq w_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$. Then, the embeddings (5.13) and the definition (5.15) show that

$$\Big(\sum_{e\in\mathcal{E}_{ID}(\mathcal{T})}\|[\![w]\!]\|^2_{L^\infty(e)}\Big)^{1/2}\lesssim\Big(\sum_{K\in\mathcal{T}}\|w\|^2_{L^\infty(\partial K)}\Big)^{1/2}\lesssim|\!|\!|w|\!|\!|_{\mathcal{T},\boldsymbol{\delta}'}\ . \tag{5.26}$$

The equations (5.23), (5.24) and (5.26) yield the bound (5.21). □

Proposition 5.6 and the bounds in (5.17), (5.18) imply the following result.

**Proposition 5.7.** *Let $\boldsymbol{\delta},\boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$. Then:*

(i) *For $v = v_0 + v_N$ with $v_0 \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ and $v_N \in V_p(\mathcal{T})$, we have*

$$|a_{DG}(v,w_N)|\le C_{a_{DG},1}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}|\!|\!|w_N|\!|\!|_{DG}\ ,\qquad w_N\in V_p(\mathcal{T})\ . \tag{5.27}$$

(ii) *For $v = v_0 + v_N$ and $w = w_0 + w_N$ with $v_0 \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$, $w_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ and $v_N, w_N \in V_p(\mathcal{T})$, we have*

$$|a_{DG}(v,w)|\le C_{a_{DG},2}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}|\!|\!|w|\!|\!|_{\mathcal{T},\boldsymbol{\delta}'}\ , \tag{5.28}$$

*The constants $C_{a_{DG},1}, C_{a_{DG},2} > 0$ depend on $\kappa$ in (3.1), the bounds in (2.3), the parameter $\mathfrak{j}_0$ in (3.10), the polynomial degree $p$ and the exponents $\boldsymbol{\delta},\boldsymbol{\delta}'$.*

*Proof.* To show (5.27), we employ the continuity bounds (5.19) and (5.20), respectively. We find that

$$|r_{DG}(w_N,v)|\le C_{r_{DG},1}J(v)|\!|\!|w_N|\!|\!|_{DG}\le C_{r_{DG},1}|\!|\!|v|\!|\!|_{DG}|\!|\!|w_N|\!|\!|_{DG}\ ,$$
$$|r_{DG}(v,w_N)|\le C_{r_{DG},2}J(w_N)|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}\le C_{r_{DG},2}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}|\!|\!|w_N|\!|\!|_{DG}\ .$$

These bounds in conjunction with (5.17), (5.18) and (5.16) imply (5.27).

For the proof of (5.28), we apply (5.21) twice and obtain

$$|r_{DG}(v,w)|\le C_{r_{DG},3}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}|\!|\!|w|\!|\!|_{\mathcal{T},\boldsymbol{\delta}'}\ ,$$
$$|r_{DG}(w,v)|\le C_{r_{DG},3}|\!|\!|w|\!|\!|_{\mathcal{T},\boldsymbol{\delta}'}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}}\ .$$

The estimate (5.28) follows from (5.17), (5.18) and (5.16). □

5.2.2. *Continuity bounds for $l_{DG}(\cdot)$.* With the same arguments used to prove Proposition 5.6, we further derive bounds for the right-hand side $l_{DG}(\cdot)$ in (3.9).

For the functionals associated with the source term $f$ and the Neumann datum $g_N$, we establish bounds, which are expressed in terms of weighted norms of the data.

**Proposition 5.8.** *Let $\boldsymbol{\delta},\boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$, $f \in H^{0,0}_{\boldsymbol{\delta}}(\Omega)$ and $g_N \in H^{1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)$. Let $v = v_0 + v_N$ with $v_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ and $v_N \in V_p(\mathcal{T})$. Then:*

(i) *We have*

$$\sum_{K\in\mathcal{T}}\int_K|fv|\,d\mathbf{x}\le C_f\|f\|_{H^{0,0}_{\boldsymbol{\delta}}(\Omega)}|\!|\!|v|\!|\!|_{\mathcal{T},\boldsymbol{\delta}'}\ . \tag{5.29}$$

(ii) *For* $\mathcal{T}_N := \{ K \in \mathcal{T} : \partial K \cap e \in \mathcal{E}_N(\mathcal{T}) \text{ for } e \in \mathcal{E}(\mathcal{T}) \}$, *we have*

$$\sum_{e \in \mathcal{E}_N(\mathcal{T})} \int_e |g_N v| \, dS \le C_{g_N} \|g_N\|_{H^{1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)} \|v\|_{\mathcal{T}_N, \boldsymbol{\delta}'} \,. \tag{5.30}$$

*The constants* $C_f > 0$ *and* $C_{g_N} > 0$ *depend on* $\kappa$ *in* (3.1), *the polynomial degree* $p$ *and the exponents* $\boldsymbol{\delta}, \boldsymbol{\delta}'$.

*Proof.* We prove each item separately.

*Proof of* (5.29)*:* For $K \in \mathcal{T}$, property (5.6) and Hölder's inequality yield

$$\int_K |fv| \, d\mathbf{x} \lesssim \|f\|_{L^1(K)} \|v\|_{C^0(\overline{K})} \lesssim \|f\|_{H^{0,0}_{\boldsymbol{\delta}}(K)} \|v\|_{C^0(\overline{K})}.$$

The summation of these bounds over all elements $K \in \mathcal{T}$, the discrete Cauchy-Schwarz inequality and the bounds in (5.13) now imply (5.29).

*Proof of* (5.30)*:* Let $G_N \in H^{1,1}_{\boldsymbol{\delta}}(\Omega)$ be such that $G_N|_{\Gamma_N} = g_N$. For a Neumann edge $\partial K \cap e \in \mathcal{E}_N(\mathcal{T})$, we apply (5.7) and obtain

$$\int_e |g_N v| \, dS \lesssim \|G_N\|_{L^1(\partial K)} \|v\|_{C^0(\overline{K})} \lesssim \|G_N\|_{H^{1,1}_{\boldsymbol{\delta}}(K)} \|v\|_{C^0(\overline{K})}.$$

Hence, by summing this bound over all $K \in \mathcal{T}_N$, using the discrete Cauchy-Schwarz inequality, applying (5.13) and taking the infimimum over all possible $G_N$ as above, the bound (5.30) follows. ☐

**Remark 5.9.** *We note that in the proof of* (5.29) *some powers of* $h$ *(i.e.,* $h^{2-2\delta_i}$ *for vertex* $\mathbf{c}_i$*) were dropped after application of* (5.6)*. The resulting bound* (5.29) *will be sufficient for our purposes.*

For the Dirichlet datum $g_D$, we introduce the natural boundary jump term

$$J_D(v)^2 := \sum_{e \in \mathcal{E}_D(\mathcal{T})} \|\mathsf{j}_e^{1/2} v\|^2_{L^2(e)} \,. \tag{5.31}$$

As in (5.18), we then have

$$\sum_{e \in \mathcal{E}_D(\mathcal{T})} \int_e \mathsf{j}_e |g_D v| \, dS \le J_D(g_D) J_D(v) \tag{5.32}$$

for any $v \in H^1(\mathcal{T})$. The remaining functional associated with $g_D$ in (3.9) can be bounded completely analogously to $r_{DG}(\cdot, \cdot)$ in Proposition 5.6.

**Proposition 5.10.** *Let* $\boldsymbol{\delta}, \boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$ *and define* $\mathcal{T}_D := \{ K \in \mathcal{T} : \partial K \cap e \in \mathcal{E}_D(\mathcal{T}) \text{ for } e \in \mathcal{E}(\mathcal{T}) \}$. *Then:*

(i) *We have for all* $v_N \in V_p(\mathcal{T})$

$$\sum_{e \in \mathcal{E}_D(\mathcal{T})} \int_e |g_D| |\mathsf{c} \nabla v_N| \, dS \le C_{g_D,1} J_D(g_D) \Big( \sum_{K \in \mathcal{T}_D} \|\mathsf{c}^{1/2} \nabla v_N\|^2_{L^2(K)} \Big)^{1/2}. \tag{5.33}$$

(ii) *Let* $g_D \in \mathcal{P}^p(e)$ *for* $e \in \mathcal{E}_D(\mathcal{T})$*. For any* $v = v_0 + v_N$ *with* $v_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ *and* $v_N \in V_p(\mathcal{T})$, *we have*

$$\sum_{e \in \mathcal{E}_D(\mathcal{T})} \int_e |g_D| |\mathsf{c} \nabla v| \, dS \le C_{g_D,2} J_D(g_D) \|v\|_{\mathcal{T}_D, \boldsymbol{\delta}'}. \tag{5.34}$$

(iii) *Let* $G_D \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ *be such that* $G_D|_{\Gamma_D} = g_D \in H^{3/2,3/2}_{\boldsymbol{\delta}}(\Gamma_D)$. *Let* $v = v_0 + v_N$ *with* $v_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ *and* $v_N \in V_p(\mathcal{T})$. *Then we have*

$$\sum_{e\in\mathcal{E}_D(\mathcal{T})} \int_e |g_D||\mathsf{c}\nabla v|\, dS \leq C_{g_D,3} \| G_D \|_{\mathcal{T}_D,\boldsymbol{\delta}} \| v \|_{\mathcal{T}_D,\boldsymbol{\delta}'}, \tag{5.35}$$

*The constants* $C_{g_D,1}, C_{g_D,2}, C_{g_D,3} > 0$ *depend on* $\kappa$ *in* (3.1)*, the bounds in* (2.3)*, the parameter* $\mathsf{j}_0$ *in* (3.10)*, the polynomial degree* $p$ *and the exponents* $\boldsymbol{\delta}, \boldsymbol{\delta}'$.

*Proof.* We prove each item separately.

*Proof of* (5.33)*:* This bound is obtained completely analogously to (5.19). That is, with the help of (2.3), the definition of $\mathsf{j}$ in (3.10), the discrete Cauchy-Schwarz inequality, the equivalence (3.2) and the polynomial trace inequality (5.2), we obtain for $v_N \in V_p(\mathcal{T})$

$$\begin{aligned}\sum_{e\in\mathcal{E}_D(\mathcal{T})} \int_e |g_D||\mathsf{c}\nabla v_N|\, dS &\lesssim \sum_{e\in\mathcal{E}_D(\mathcal{T})} \mathsf{j}_e^{1/2} \|g_D\|_{L^2(e)} h_e \|\langle\!\langle \nabla v_N \rangle\!\rangle\|_{L^2(e)} \\ &\lesssim J_D(g_D) \Big( \sum_{K\in\mathcal{T}_D} h_K \|\nabla v_N\|^2_{L^2(\partial K)} \Big)^{1/2},\end{aligned}$$

which yields (5.33).

*Proof of* (5.34)*:* Following (5.20), we use Hölder's inequality in $L^\infty(e)\times L^1(e)$, the discrete Cauchy-Schwarz inequality, the bounds in (2.3), the inverse inequality (5.3) (since $g_D$ is piecewise polynomial) and the definition of $\mathsf{j}_0$ in (3.10). This yields

$$\sum_{e\in\mathcal{E}_D(\mathcal{T})} \int_e |g_D||\mathsf{c}\nabla v|\, dS \lesssim \Big( \sum_{e\in\mathcal{E}_D(\mathcal{T})} \|\mathsf{j}_e^{1/2} g_D\|^2_{L^2(e)} \Big)^{1/2} \Big( \sum_{K\in\mathcal{T}} \|\nabla v\|^2_{L^1(\partial K)} \Big)^{1/2}.$$

Invoking (5.14) shows (5.34).

*Proof of* (5.35)*:* This bound is similar to (5.21). By employing Hölder's inequality and the discrete Cauchy-Schwarz inequality, we conclude that

$$\sum_{e\in\mathcal{E}_D(\mathcal{T})} \int_e |g_D||\mathsf{c}\nabla v|\, dS \lesssim \Big( \sum_{K\in\mathcal{T}_D} \|G_D\|^2_{L^\infty(\partial K)} \Big)^{1/2} \Big( \sum_{K\in\mathcal{T}_D} \|\nabla v\|^2_{L^1(\partial K)} \Big)^{1/2}.$$

The inequalities (5.13) and (5.14) imply (5.35). □

**Remark 5.11.** *Let* $v = v_0 + v_N$*, with* $v_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ *and* $v_N \in V_p(\mathcal{T})$*. Under assumption* (2.20) *and from the bound* (5.32)*, Proposition 5.8 and Proposition 5.10, we find the boundedness of* $l_{DG}(\cdot)$ *in the sense that*

$$\big| l_{DG}(v) \big| \leq C_{l_{DG}} \big( \|f\|_{H^{0,0}_{\boldsymbol{\delta}}(\Omega)} + \|g_N\|_{H^{1/2,1/2}_{\boldsymbol{\delta}}(\Gamma_N)} + J_D(g_D) + \| G_D \|_{\mathcal{T}_D,\boldsymbol{\delta}} \big) \| v \|_{\mathcal{T},\boldsymbol{\delta}'}, \tag{5.36}$$

*for a constant* $C_{l_{DG}} > 0$ *depending on* $\kappa$ *in* (3.1)*, the bounds in* (2.3)*, the parameter* $\mathsf{j}_0$ *in* (3.10)*, the polynomial degree* $p$ *and the exponents* $\boldsymbol{\delta}, \boldsymbol{\delta}'$.

5.3. **Consistency.** We verify the Galerkin orthogonality property of the DG discretization (3.6).

5.3.1. *Integration by parts.* We will use the following variant of Green's formula in [21, Lemma A.2.1]: For $\boldsymbol{q} \in H^{1,1}_{\boldsymbol{\delta}}(\Omega)^2$, we have

$$\int_K \boldsymbol{q} \cdot \nabla v \, d\mathbf{x} = -\int_K (\nabla \cdot \boldsymbol{q}) v \, d\mathbf{x} + \int_{\partial K} \boldsymbol{q} \cdot \boldsymbol{\nu}_K v \, dS, \tag{5.37}$$

for all $K \in \mathcal{T}$ and for $v \in C^0(\overline{K})$ with $\nabla v \in L^2(K)^2$. In particular, if $K \in \mathcal{K}(\mathcal{T})$ is a corner element, all integrals in (5.37) are well-defined in $L^1(K) \times L^\infty(K)$ respectively $L^1(\partial K) \times L^\infty(\partial K)$, due to Lemma 5.2 and Hölder's inequality. The identity (5.37) is readily proved using the density of $C^\infty(\overline{\Omega})^2$ in $H^{1,1}_{\boldsymbol{\delta}}(\Omega)^2$; cf. [21, Lemma A.2.1]. A consequence of (5.37) is the following integration-by-parts formula.

**Lemma 5.12.** *Let $\boldsymbol{\delta}, \boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$. For $v \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$, we have $\mathsf{c}\nabla v \in H^{1,1}_{\boldsymbol{\delta}}(\Omega)^2$, $\nabla \cdot (\mathsf{c}\nabla v) \in H^{0,0}_{\boldsymbol{\delta}}(\Omega)$ and*

$$\sum_{K\in\mathcal{T}} \int_K \mathsf{c}\nabla v \cdot \nabla w \, d\mathbf{x} = -\sum_{K\in\mathcal{T}} \int_K \nabla \cdot (\mathsf{c}\nabla v) w \, d\mathbf{x} + \sum_{e\in\mathcal{E}(\mathcal{T})} \int_e \langle\!\langle \mathsf{c}\nabla v \rangle\!\rangle \cdot [\![ w ]\!] \, dS \tag{5.38}$$

*for any $w = w_0 + w_N$ with $w_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ and $w_N \in V_p(\mathcal{T})$. Here, all the terms are well-defined, which follows from Lemma 5.2, Lemma 5.4, and Proposition 5.8. In particular, as in Remark 3.1, for corner elements $K \in \mathcal{K}(\mathcal{T})$, the volume integrals on the right-hand sides are understood in the sense of $L^1(K) \times L^\infty(K)$, and the integrals over edges $e \in \mathcal{E}(\mathcal{T})$ running into corners are well-defined as bilinear forms over $L^1(e) \times L^\infty(e)$.*

*Proof.* Since $v \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ and $\mathsf{c}$ is smooth, we clearly have $\boldsymbol{q} = \mathsf{c}\nabla v \in H^{1,1}_{\boldsymbol{\delta}}(\Omega)^2$, and $\nabla \cdot \boldsymbol{q} \in H^{0,0}_{\boldsymbol{\delta}}(\Omega)$, see Lemma 2.2. Moreover, the test function $w$ in (5.38) belongs to $L^\infty(\Omega)$ and satisfies for all $K \in \mathcal{T}$: $w|_K \in C^0(\overline{K})$, cf. (2.15), and $\nabla w|_K \in L^2(K)^2$. Hence, by applying (5.37) and summing over all elements, we obtain

$$\sum_{K\in\mathcal{T}} \int_K \mathsf{c}\nabla v \cdot \nabla w \, d\mathbf{x} = -\sum_{K\in\mathcal{T}} \int_K \nabla \cdot (\mathsf{c}\nabla v) w \, d\mathbf{x} + \sum_{K\in\mathcal{T}} \int_{\partial K} (\mathsf{c}\nabla v) \cdot \boldsymbol{\nu}_K w \, dS.$$

All the terms above are well-defined thanks to Lemma 5.2, Lemma 5.4 and Proposition 5.8. In the last term, we express integrals over elemental boundaries by integrals over edges using the identity in [2, Equation (3.3)]. We conclude that

$$\sum_{K\in\mathcal{T}} \int_{\partial K} (\mathsf{c}\nabla v) \cdot \boldsymbol{\nu}_K w \, dS = \sum_{e\in\mathcal{E}(\mathcal{T})} \int_e \langle\!\langle \mathsf{c}\nabla v \rangle\!\rangle \cdot [\![ w ]\!] \, dS + \sum_{e\in\mathcal{E}_I(\mathcal{T})} \int_e [\![ \mathsf{c}\nabla v ]\!] \langle\!\langle w \rangle\!\rangle \, dS,$$

where again all terms are well-defined over $L^1(e) \times L^\infty(e)$. The equality (5.38) now follows from (5.8). □

5.3.2. *Galerkin orthogonality.* The following properties hold.

**Lemma 5.13.** *For $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$, let $u \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ be the solution of (2.4)–(2.6). There holds:*

(i) *Let $\boldsymbol{\delta}' \in [0,1)^{\mathsf{M}}$ and $v = v_0 + v_N$ with $v_0 \in H^{2,2}_{\boldsymbol{\delta}'}(\Omega)$ and $v_N \in V_p(\mathcal{T})$. Then we have*

$$a_{DG}(u,v) = \sum_{K\in\mathcal{T}} \int_K \mathsf{c}\nabla u \cdot \nabla v \, d\mathbf{x} - r_{DG}(u,v) = l_{DG}(v) \,, \tag{5.39}$$

*where* $r_{DG}(u,v)$ *and* $l_{DG}(v)$ *are bounded as in* (5.21) *and* (5.36) *respectively.*

(ii) *Let* $u_N \in V_p(\mathcal{T})$ *be the IP approximation in* (3.6). *Then the error* $u - u_N$ *satisfies*

$$a_{DG}(u - u_N, v_N) = 0, \qquad v_N \in V_p(\mathcal{T}). \tag{5.40}$$

*Proof.* The identity (5.39) is an immediate consequence of the formula (5.38) and the PDE (2.4), by taking into account the boundary conditions in (2.5) and (2.6), respectively, and by noting that $[\![u]\!]|_e = \mathbf{0}$ for $e \in \mathcal{E}_I(\mathcal{T})$ and $[\![u]\!]|_e = g_D \cdot \boldsymbol{\nu}$ for $e \in \mathcal{E}_D(\mathcal{T})$.

The Galerkin orthogonality (5.40) follows immediately from (5.39) and the definition of the IP method (3.6). $\square$

**Remark 5.14.** *We emphasize that identity* (5.39) *in conjunction with the symmetry of the IP form* $a_{DG}(\cdot,\cdot)$ *yields adjoint-consistency of the symmetric interior penalty method* (3.6) *in the sense of* [2].

5.4. **Error estimates.** We now derive generic and quasi-optimal error bounds.

5.4.1. *Energy norm error.* We begin by proving the following energy norm bound.

**Lemma 5.15.** *For* $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$, *let* $u \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ *be the solution of* (2.4)–(2.6). *Let* $u_N \in V_p(\mathcal{T})$ *be the IP approximation in* (3.6). *Then we have the energy norm error bound*

$$|\!|\!| u - u_N |\!|\!|_{DG} \le C \inf_{v_N \in V_p(\mathcal{T})} |\!|\!| u - v_N |\!|\!|_{\mathcal{T},\boldsymbol{\delta}}, \tag{5.41}$$

*with a constant* $C > 0$ *depending on* $\kappa$ *in* (3.1), *the bounds in* (2.3), *the parameter* $\mathsf{j}_0$ *in* (3.10), *the polynomial degree* $p$, *and the exponent* $\boldsymbol{\delta}$.

*Proof.* We proceed in a standard manner. For $v_N \in V_p(\mathcal{T})$, the triangle inequality gives

$$|\!|\!| u - u_N |\!|\!|_{DG} \le |\!|\!| u - v_N |\!|\!|_{DG} + |\!|\!| v_N - u_N |\!|\!|_{DG} \, .$$

Hence, from the coercivity in Lemma 3.3 and the Galerkin orthogonality (5.40), we find that

$$C_{coer} |\!|\!| v_N - u_N |\!|\!|^2_{DG} \le a_{DG}(v_N - u_N, v_N - u_N) = -a_{DG}(u - v_N, v_N - u_N) \, .$$

By the continuity of the form $a_{DG}(\cdot,\cdot)$ in (5.27), we have

$$|a_{DG}(u - v_N, v_N - u_N)| \le C_{a_{DG},1} |\!|\!| u - v_N |\!|\!|_{\mathcal{T},\boldsymbol{\delta}} |\!|\!| v_N - u_N |\!|\!|_{DG}.$$

The bound (5.41) follows. $\square$

5.4.2. *$L^2$-norm error.* To prove an $L^2$-norm error estimate for $u - u_N$, we use regularity in weighted spaces for the dual problem:

$$-\nabla \cdot (\mathsf{c}\nabla z) = u - u_N \qquad \text{in } \Omega \,, \tag{5.42}$$

$$z = 0 \qquad \text{on } \Gamma_D \,, \tag{5.43}$$

$$(\mathsf{c}\nabla z) \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \Gamma_N \, . \tag{5.44}$$

For $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ as in (2.19), the stability bound in Proposition 2.3 and the continuous embedding $L^2(\Omega) \hookrightarrow H^{0,0}_{\boldsymbol{\delta}}(\Omega)$ ensure that $z \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ and that

$$\|z\|_{H^{2,2}_{\boldsymbol{\delta}}(\Omega)} \le C_{stab,2} \|u - u_N\|_{H^{0,0}_{\boldsymbol{\delta}}(\Omega)} \le C_{dc} C_{stab,2} \|u - u_N\|_{L^2(\Omega)} \, . \tag{5.45}$$

**Lemma 5.16.** *For $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ as in* (2.19)*, let $u \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ be the solution of* (2.4)–(2.6)*. Let $z \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$ be the dual solution of* (5.42)–(5.44)*. Assume the approximation property*

$$\inf_{z_N \in V_p(\mathcal{T})} \|\!|z - z_N\|\!|_{\mathcal{T},\boldsymbol{\delta}} \leq C_{approx} d(p,\mathcal{T},\boldsymbol{\delta}) \|z\|_{H^{2,2}_{\boldsymbol{\delta}}(\Omega)}, \tag{5.46}$$

*Let further $u_N \in V_p(\mathcal{T})$ be the IP approximation in* (3.6)*. Then we have the $L^2$-norm error bound*

$$\|u - u_N\|_{L^2(\Omega)} \leq C d(p,\mathcal{T},\boldsymbol{\delta}) \inf_{v_N \in V_p(\mathcal{T})} \|\!|u - v_N\|\!|_{\mathcal{T},\boldsymbol{\delta}}, \tag{5.47}$$

*with a constant $C > 0$ depending on $\kappa$ in* (3.1)*, the bounds in* (2.3)*, the constants $C_{stab,2}$ and $C_{approx}$ in* (5.45) *and* (5.46)*, respectively, the polynomial degree $p$, and the weight exponent vector $\boldsymbol{\delta}$.*

*Proof.* We invoke identity (5.39) for $v = u - u_N$ with respect to the dual problem (5.42)–(5.44) and employ the symmetry of the IP form. This results in

$$\|u - u_N\|^2_{L^2(\Omega)} = a_{DG}(u - u_N, z) .$$

From here on, we proceed in a usual manner and apply the Galerkin orthogonality (5.40) to conclude that

$$\|u - u_N\|^2_{L^2(\Omega)} = a_{DG}(u - u_N, z - z_N)$$

for all $z_N \in V_p(\mathcal{T})$. Hence, by employing (5.28), assumption (5.46) and the stability bound (5.45), we conclude that

$$\begin{aligned}
|a_{DG}(u - u_N, z - z_N)| &\leq C_{a_{DG},2} \|\!|u - u_N\|\!|_{\mathcal{T},\boldsymbol{\delta}} \|\!|z - z_N\|\!|_{\mathcal{T},\boldsymbol{\delta}} \\
&\leq C_{a_{DG},2} C_{approx} d(p,\mathcal{T},\boldsymbol{\delta}) \|z\|_{H^{2,2}_{\boldsymbol{\delta}}(\Omega)} \|\!|u - u_N\|\!|_{\mathcal{T},\boldsymbol{\delta}} \\
&\leq C_{a_{DG},2} C_{approx} C_{dc} C_{stab,2} d(p,\mathcal{T},\boldsymbol{\delta}) \|u - u_N\|_{L^2(\Omega)} \|\!|u - u_N\|\!|_{\mathcal{T},\boldsymbol{\delta}}.
\end{aligned}$$

These bounds imply (5.47). ☐

5.5. **Nodal interpolation.** In view of the error bounds (5.41) and (5.47), it remains to establish optimal interpolation estimates on graded and bisection refinement meshes with respect to the norm $\|\!|\cdot\|\!|_{\mathcal{T},\boldsymbol{\delta}}$.

For an element $K \in \mathcal{T}$, let $\mathcal{I}^p_K : C^0(\overline{K}) \to \mathcal{P}^p(K)$ denote the elemental nodal interpolant into polynomials of degree at most $p$. By standard interpolation in elements away from $\mathcal{S}$ and for $1 \leq k \leq p$, there holds

$$M_K[v - \mathcal{I}^p_K v]^2 \lesssim h_K^{2k} \|\mathsf{D}^{k+1} v\|^2_{L^2(K)}, \qquad K \in \mathcal{T} \setminus \mathcal{K}(\mathcal{T}). \tag{5.48}$$

In corner elements, the interpolation bounds in [19, Lemma 4.16] for the linear interpolant in the weighted spaces $H^{2,2}_{\delta_i}(K)$ for $\delta_i \in [0,1)$ give

$$N_{K,\boldsymbol{\delta}}[v - \mathcal{I}^1_K v]^2 \lesssim h_K^{2-2\delta_i} |v|^2_{H^{2,2}_{\delta_i}(K)}, \qquad K \in \mathcal{K}_i(\mathcal{T}). \tag{5.49}$$

Here, we note that $\mathcal{I}^1_K v$ is well-defined for $v \in H^{2,2}_{\delta_i}(K)$ due to (2.15). Due to (5.48), (5.49), we then define the global interpolant $\mathcal{I}_p : C^0(\overline{\Omega}) \to V_p(\mathcal{T})$ by

$$\mathcal{I}_p v|_K := \begin{cases} \mathcal{I}^1_K(v|_K) & \text{if } K \in \mathcal{K}(\mathcal{T}) , \\ \mathcal{I}^p_K(v|_K) & \text{otherwise} . \end{cases} \tag{5.50}$$

We note again that $\mathcal{I}_p v$ is well-defined for $v \in H^{2,2}_{\boldsymbol{\delta}}(\Omega) \subset C^0(\overline{\Omega})$. Next, we derive interpolation estimates for $\mathcal{I}_p v$ on locally adapted meshes.

5.6. **Interpolation estimates on graded mesh families.** We first consider graded mesh families and prove approximation bounds which are slightly generalized versions of those in [21, Proposition 2.5.5]. Analogous results (on different mesh families and for conforming FEMs) are obtained in [7].

**Proposition 5.17.** *Let $p \geq 1$, $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ and $v \in H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$ for $1 \leq k \leq p$. Let $\mathcal{T}_{\boldsymbol{\beta}}$ be a graded mesh with grading vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\mathsf{M}})$ chosen as in* (4.1)*, i.e., with $\beta_i \in (\beta_i^*, 1)$ where*

$$\beta_i^* := 1 - \frac{1-\delta_i}{p}. \tag{5.51}$$

*Let $\mathcal{I}_p v$ be the interpolant defined in* (5.50)*. Then we have the approximation bound*

$$\| v - \mathcal{I}_p v \|_{\mathcal{T}_{\boldsymbol{\beta}}, \boldsymbol{\delta}} \leq C_{grad} N^{-k/2} |v|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)} \,. \tag{5.52}$$

*The constant $C_{grad} > 0$ is independent of $N$, but depends on $\kappa$ in* (3.1)*, on the constant $C_{dc}$ in* (2.10), (2.11)*, on the vector $\boldsymbol{\delta}$, the parameters $C_G$, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^*$ in Definition 4.2, the regularity parameter $k$, and the polynomial degree $p$.*

*Proof.* For an interior element $K \in \mathcal{N}_0(\mathcal{T}_{\boldsymbol{\beta}})$, due to (5.48) and (2.10), there holds

$$M_K[v - \mathcal{I}_p v]^2 \lesssim h_K^{2k} \|\mathsf{D}^{k+1} v\|^2_{L^2(K)} \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)} \,. \tag{5.53}$$

Next, for $1 \leq i \leq \mathsf{M}$, let $K \in \mathcal{N}_i(\mathcal{T}_{\boldsymbol{\beta}})$, the discrete corner neighborhood defined in (5.4). Let first $K \in \mathcal{N}_i(\mathcal{T}_{\boldsymbol{\beta}}) \setminus \mathcal{K}_i(\mathcal{T}_{\boldsymbol{\beta}})$. Then, the bounds (5.48), the first property in Definition 4.1, and the fact that $\beta_i \in (\beta_i^*, 1)$ imply

$$\begin{aligned} M_K[v - \mathcal{I}_p v]^2 &\lesssim h_K^{2k} \|\mathsf{D}^{k+1} v\|^2_{L^2(K)} \lesssim h^{2k} \| r_i^{\beta_i k} |\mathsf{D}^{k+1} v| \|^2_{L^2(K)} \\ &\lesssim h^{2k} \| r_i^{\beta_i^* k} |\mathsf{D}^{k+1} v| \|^2_{L^2(K)} \lesssim h^{2k} \| r_i^{k - (1-\delta_i)\frac{k}{p}} |\mathsf{D}^{k+1} v| \|^2_{L^2(K)}. \end{aligned}$$

Since $k - (1-\delta_i)\frac{k}{p} \geq k - (1-\delta_i)$ and due to (5.5), we obtain

$$\begin{aligned} M_K[v - \mathcal{I}_p v]^2 &\lesssim h^{2k} \| r_i^{\delta_i + k - 1} |\mathsf{D}^{k+1} v| \|^2_{L^2(K)} \\ &\lesssim h^{2k} |v|^2_{H^{k+1,2}_{\delta_i}(K)} \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)}. \end{aligned} \tag{5.54}$$

Second, let $K \in \mathcal{K}_i(\mathcal{T}_{\boldsymbol{\beta}})$. Since $r_i(\mathbf{x}) \leq h_K$ for all $\mathbf{x} \in K$, the second assumption in Definition 4.1 yields

$$h_K \lesssim h \sup_{\mathbf{x} \in K} r_i(\mathbf{x})^{\beta_i^*} \lesssim h h_K^{\beta_i^*},$$

and then $h_K \lesssim h^{\frac{1}{1-\beta_i^*}}$, as well as

$$\sup_{\mathbf{x} \in K} r_i(\mathbf{x})^{\beta_i^*} \lesssim h^{\frac{\beta_i^*}{1-\beta_i^*}} = h^{\frac{p}{1-\delta_i} - 1}. \tag{5.55}$$

The estimate (5.49), the second property in Definition 4.1, the bound (5.55) and property (2.11) now yield

$$\begin{aligned} N_{K,\boldsymbol{\delta}}[v - \mathcal{I}_p v]^2 &\lesssim h_K^{2-2\delta_i} |v|^2_{H^{2,2}_{\delta_i}(K)} \\ &\lesssim h^{2-2\delta_i} \sup_{\mathbf{x} \in K} r_i(\mathbf{x})^{\beta_i(2-2\delta_i)} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)} \\ &\lesssim h^{2-2\delta_i} \sup_{\mathbf{x} \in K} r_i(\mathbf{x})^{\beta_i^*(2-2\delta_i)} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)} \\ &\lesssim h^{2p} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)} \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)}. \end{aligned} \tag{5.56}$$

Summing the bounds in (5.53), (5.54) and (5.56) over all elements yields

$$\|v - \mathcal{I}_p v\|^2_{\mathcal{T}_\beta,\boldsymbol{\delta}} \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}. \tag{5.57}$$

Finally, an elementary counting argument (see, e.g., [21, Lemma 2.5.6]) reveals that $N = \dim(V_p(\mathcal{T})) \lesssim p^2 h^{-2}$, which implies (5.52). □

5.7. **Interpolation estimates on bisection refinement meshes.** We shall establish the following variant of [9, Theorem 5.3], which is based solely on the properties of the algorithm in [9]. We also recall from Section 4.2 that bisection refinement mesh family thus constructed is shape-regular with $\kappa$ in (3.1) depending on the initial mesh $\mathcal{T}_0$.

**Proposition 5.18.** *Let $p \geq 1$, $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ and $v \in H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$ for $1 \leq k \leq p$. For parameters $h$, $\gamma \in (0, \gamma^*]$ and $L$ as in* (4.2)*, i.e., with*

$$\gamma := 1 - \max_{i=1}^{\mathsf{M}} \delta_i > 0 \quad \textit{and} \quad h \in [2^{-(L+1)\gamma/(p+1)}, 2^{-L\gamma/(p+1)}). \tag{5.58}$$

*consider the triangulations $\mathcal{T}_{h,2(L+1)}$ constructed by the bisection refinement algorithm of* [9] *starting from an initial triangulation $\mathcal{T}_0$ with $\#\mathcal{T}_0 \lesssim h^{-2}$. Let $\mathcal{I}_p v$ be the interpolant in* (5.50)*. Then we have the approximation bound*

$$\|v - \mathcal{I}_p v\|_{\mathcal{T}_{2(L+1)},\boldsymbol{\delta}} \leq C_{bis} N^{-k/2} |v|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}. \tag{5.59}$$

*The constant $C_{bis} > 0$ is independent of $N$, but depends on the shape regularity parameter $\kappa$ in* (3.1)*, the constant $C_{dc}$ in* (2.10)*,* (2.11)*, the vector $\boldsymbol{\delta}$, the initial mesh $\mathcal{T}_0$, the parameter $\gamma$, the sufficiently large refinement parameter $L$, the regularity parameter $k$, and on the polynomial degree $p$.*

*Proof.* Following [9, Theorems 5.2 and 5.3], we proceed in several steps.

*Interior elements:* We first bound the errors over elements in the interior neighborhood $\mathcal{N}_0(\mathcal{T}_{h,2(L+1)})$. To this end, we recall from [9, Lemma 4.4] that the first loop of the bisection refinement algorithm ensures $|K| \simeq h_K^2 \lesssim h^2$. Hence, with (5.48) and (2.10), we find that

$$\begin{aligned}
\sum_{K \in \mathcal{N}_0(\mathcal{T}_{h,2(L+1)})} M_K[v - \mathcal{I}_p v]^2 &\lesssim \sum_{K \in \mathcal{N}_0(\mathcal{T}_{h,2(L+1)})} h_K^{2k} \|\mathsf{D}^{k+1} v\|^2_{L^2(K)} \\
&\lesssim h^{2k} \sum_{K \in \mathcal{N}_0(\mathcal{T}_{h,2(L+1)})} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(K)} \\
&\lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}.
\end{aligned} \tag{5.60}$$

*Corner neighborhoods:* Let $1 \leq i \leq \mathsf{M}$ be a fixed corner index. For $K \in \mathcal{N}_i(\mathcal{T}_{h,2(L+1)})$, we set

$$r_K := \operatorname{dist}(K, \mathbf{c}_i) = \inf_{\mathbf{y} \in K} r_i(\mathbf{y}). \tag{5.61}$$

As in [9, page 933], we then consider the following concentric neighborhoods at $\mathbf{c}_i$:

$$D_\ell := \cup\{K \in \mathcal{N}_i(\mathcal{T}_{h,2(L+1)}) \,:\, 2^{-\frac{\ell+1}{2}} < r_K \leq 2^{-\frac{\ell}{2}}\}, \;\; \ell = 0, \ldots, 2L+1,$$

and

$$D_{2L+2} := \cup\{K \in \mathcal{N}_i(\mathcal{T}^i_{h,2(L+1)}) \,:\, r_K \leq 2^{-(L+1)}\}.$$

Here and as in [9], we assume without loss of generality that $\mathcal{N}_i(\mathcal{T}_{h,2(L+1)}) \subseteq \cup_{\ell=0}^{2L+2} D_\ell$.

*Elements in $D_{2L+2}$:* We first bound the consistency errors in the elements in the innermost neighborhood $D_{2L+2}$. To do so, we need to estimate the terms $T_1$ and $T_2$ given by

$$T_1 = \sum_{\substack{K \in D_{2L+2} \\ r_K > 0}} M_K[v - \mathcal{I}_p v]^2, \tag{5.62}$$

$$T_2 = \sum_{\substack{K \in D_{2L+2} \\ r_K = 0}} N_{K,\boldsymbol{\delta}}[v - \mathcal{I}_p v]^2 . \tag{5.63}$$

Let $K \in D_{2L+2}$ and first consider the case $r_K > 0$. By [9, Lemma 4.6], there holds $h_K \lesssim r_K$. Moreover, $r_K \le r_i(\mathbf{x})$ for all $\mathbf{x} \in K$. With (5.48), we thus obtain

$$\begin{aligned} M_K[v - \mathcal{I}_p v]^2 &\lesssim h_K^{2k} \|\mathsf{D}^{k+1} v\|_{L^2(K)}^2 \\ &\lesssim r_K^{2k} r_K^{2-2\delta_i-2k} \|r_i^{\delta_i+k-1} |\mathsf{D}^{k+1} v|\|_{L^2(K)}^2 \lesssim r_K^{2-2\delta_i} |v|^2_{H^{k+1,2}_{\delta_i}(K)}. \end{aligned}$$

Using the definition of $D_{2L+2}$ and $\gamma$, as well as the condition on $h$ from (5.58), we find that

$$\begin{aligned} M_K[v - \mathcal{I}_p v]^2 &\lesssim 2^{-(2-2\delta_i)(L+1)} |v|^2_{H^{k+1,2}_{\delta_i}(K)} \\ &\lesssim 2^{-2\gamma(L+1)} |v|^2_{H^{k+1,2}_{\delta_i}(K)} \lesssim h^{2p+2} |v|^2_{H^{k+1,2}_{\delta_i}(K)}. \end{aligned} \tag{5.64}$$

Second, let $r_K = 0$. We now use (5.49) and proceed as before. This results in

$$\begin{aligned} N_{K,\boldsymbol{\delta}}[v - \mathcal{I}_p v]^2 &\lesssim h_K^{2-2\delta_i} |v|^2_{H^{2,2}_{\delta_i}(K)} \\ &\lesssim 2^{-2\gamma(L+1)} |v|^2_{H^{2,2}_{\delta_i}(K)} \lesssim h^{2p+2} |v|^2_{H^{k+1,2}_{\delta_i}(K)}. \end{aligned} \tag{5.65}$$

Summing the estimates in (5.64) and (5.65) over all relevant elements and taking into account (5.5) yield

$$T_1 + T_2 \lesssim h^{2p+2} \sum_{K \in D_{2L+2}} |v|^2_{H^{k+1,2}_{\delta_i}(K)} \lesssim h^{2p+2} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}. \tag{5.66}$$

Notice that the bound (5.66) corresponds to [9, Equation (5.5)].

*Elements in $D_\ell$:* Next, consider an element $K \in D_\ell$, for $0 \le \ell \le 2L+1$. Since $r_K \le 2^{-\frac{\ell}{2}}$, the result in [9, Lemma 4.7] and the definition of $\gamma \in (0, \gamma^*]$ imply

$$|K| \lesssim h_K^2 \lesssim h^2 2^{-\ell \frac{p+1-\gamma}{p+1}} \lesssim h^2 2^{-\ell \frac{p+1-\gamma^*}{p+1}} \lesssim h^2 2^{-\ell \frac{p+\delta_i}{p+1}}.$$

Then, inserting the appropriate power of $r_i$, employing the bound above and noticing that $r_K \ge 2^{-\frac{\ell+1}{2}}$, we conclude that

$$\begin{aligned} M_K[v - \mathcal{I}_p v]^2 &\lesssim h_K^{2k} \|\mathsf{D}^{k+1} v\|_{L^2(K)}^2 \\ &\lesssim h_K^{2k} r_K^{2-2\delta_i-2k} \|r_i^{\delta_i+k-1} \mathsf{D}^{k+1} v\|_{L^2(K)}^2 \\ &\lesssim h^{2k} 2^{-\ell k \frac{p+\delta_i}{p+1}} 2^{(\ell+1)(k-1+\delta_i)} |v|^2_{H^{k+1,2}_{\delta_i}(K)} \\ &\lesssim h^{2k} 2^{-\ell k \frac{p+\delta_i}{p+1}} 2^{\ell(k-1+\delta_i)} |v|^2_{H^{k+1,2}_{\delta_i}(K)} . \end{aligned}$$

Since

$$2^{-\ell\left(k \frac{p+\delta_i}{p+1} - (k-1+\delta_i)\right)} = 2^{-\ell \frac{(p-k)(1-\delta_i)+(1-\delta_i)}{p+1}} \le 1 ,$$

for $1 \leq k \leq p$ and $\delta_i \in [0,1)$, it follows that

$$M_K[v - \mathcal{I}_p v]^2 \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\delta_i}(K)}, \qquad K \in D_\ell. \tag{5.67}$$

From (5.67) and (5.5), we obtain

$$T_3 = \sum_{\ell=0}^{2L+1} \sum_{K \in D_\ell} M_K[v - \mathcal{I}_p v]^2 \lesssim h^{2k} |v|^2_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}. \tag{5.68}$$

*Conclusion:* The bounds (5.66) and (5.68) yield

$$|\!|\!| v - \mathcal{I}_p v |\!|\!|_{\mathcal{T}_{h,2(L+1)},\boldsymbol{\delta}} \lesssim h^k |v|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}. \tag{5.69}$$

Moreover, in [9, Lemma 4.9], it has been proved that

$$\#\mathcal{T}_{h,2(L+1)} - \#\mathcal{T}_0 \lesssim h^{-2}, \tag{5.70}$$

with an implied constant also depending on $\mathcal{T}_0$. With (5.69), it follows that

$$|\!|\!| v - \mathcal{I}_p v |\!|\!|_{\mathcal{T}_{h,2(L+1)},\boldsymbol{\delta}} \lesssim (\#\mathcal{T}_{h,2(L+1)} - \#\mathcal{T}_0)^{-k/2} |v|_{H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)}\ . \tag{5.71}$$

Noting that $N \simeq \#\mathcal{T}_{h,2(L+1)}$ and that there is a constant $C > 0$ depending on $\#\mathcal{T}_0$ and $p$ such that

$$N \leq C\big(\#\mathcal{T}_{h,2(L+1)} - \#\mathcal{T}_0\big),$$

for $\#\mathcal{T}_{h,2(L+1)}$ sufficiently large, the bound (5.59) follows. □

5.8. **Proof of Theorem 4.3.** The energy norm estimates contained in (4.3) follow immediately from the error bound (5.41) in Lemma (5.15) and the consistency statements in (5.52) and (5.59) for $1 \leq k \leq p$.

To establish the $L^2$-norm bound in (4.3), we start from Lemma 5.16. Let $\boldsymbol{\delta} \in [0,1)^{\mathsf{M}}$ be as in (2.19). Then, due to Proposition 2.3, the primal solution $u$ of (2.4)–(2.6) and the dual solution $z$ of (5.42)–(5.44) belong to weighted spaces with the same exponent $\boldsymbol{\delta}$, i.e., $u \in H^{k+1,2}_{\boldsymbol{\delta}}(\Omega)$ and $z \in H^{2,2}_{\boldsymbol{\delta}}(\Omega)$. From (5.52) and (5.59) for $k = 1$, we have $d(p, \mathcal{T}, \boldsymbol{\delta}) \lesssim N^{-1/2}$ and (5.47) implies the $L^2$-norm estimate in (4.3).

## 6. Numerical Experiments

In this section, we present a series of numerical experiments to confirm the results in Theorem 4.3 empirically. We first note that for graded mesh families, an exhaustive series of numerical experiments can be found in [21, Section 2.5.3], for the lowest-order case (i.e., $p = 1$). The tests there numerically confirm the sharpness of the convergence rates in Theorem 4.3, both for the DG energy norm errors and for the $L^2$-norm errors (although corresponding $L^2$-norm error bounds were not proved in [21]). We therefore restrict our experiments to bisection refinement meshes.

6.1. **Setting.** Throughout, we consider the case $\mathsf{c} \equiv 1$ in problem (2.4)–(2.6), i.e., we numerically solve the Poisson equation. Our goal is to verify the theoretically predicted convergence rates in Theorem 4.3 for a range of corner angles. To this end, given an angle $\omega^* \in (0, 2\pi)$, we choose the polygonal domain $\Omega$ to be the square $(-1,1)^2$ minus a cone centered at $\mathbf{c}^* = (0,0)$ with interior opening angle of $2\pi - \omega^*$. This results in a polygon $\Omega$ with the given interior opening angle $\omega^*$ at the corner $\mathbf{c}^*$. For $\omega^* \in [\frac{3\pi}{2}, \frac{7\pi}{4}]$, the domain $\Omega$ is illustrated in Figure 6.1; the only non-convex corner is given by $\mathbf{c}^* = (0,0)$. For $\omega^* = \frac{3\pi}{2}$, we obtain a standard L-shaped domain. If $\omega^* > \frac{7\pi}{4}$, the domain $\Omega$ is constructed analogously upon introducing two additional convex corners.

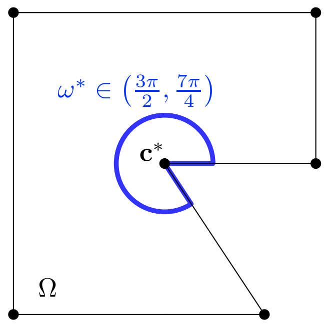



FIGURE 1. Illustration of the domain $\Omega$ for opening angles $\omega^*$ in $[\frac{3\pi}{2}, \frac{7\pi}{4}]$ and reentrant corner $\mathbf{c}^* = (0,0)$. The highlighted points are convex corners of $\Omega$; they contain the corner points $(-1, \pm 1)$ and $(1,1)$ of the square $(-1,1)^2$.

On a polygon $\Omega$ as constructed above, we numerically solve the Dirichlet problem

$$-\Delta u = 1 \quad \text{in } \Omega\,, \qquad u|_{\partial\Omega} = 0\,, \tag{6.1}$$

by employing the piecewise linear and quadratic ($p \in \{1,2\}$) symmetric interior penalty DG method (3.6). For the penalty parameter $\mathfrak{j}_0$ in (3.10), we consider the standard choice $\mathfrak{j}_0 = 10p^2$. This will be done on five regular triangulations of $\Omega$ generated by local bisection tree refinement towards the re-entrant corner $\mathbf{c}^*$, using the same initial triangulation $\mathcal{T}_0$ in the mesh generation.

For each mesh, we proceed to compute the error to a reference solution in the DG energy norm $\|\!|\cdot|\!\|_{DG}$ as well as in the $L^2(\Omega)$-norm. The reference solution is computed on a mesh which is obtained by refining the finest mesh in the family two more times. We will visualize the obtained *relative* errors on a bi-logarithmic scale with the dimension of the FE space on the abscissa and the quantity of the error on the ordinate axis. We term the slope of a line fitted through the data points in the least-squares sense the *empirical convergence rate*.

6.2. **Specifications of the code.** The code used for our experiments is written in Python 2.7 and depends on the libraries NumPy 1.10.1 and SciPy 0.16.1, see [12]. The resulting linear systems of equations are solved using the direct solver `spsolve` included in the SciPy submodule `scipy.sparse.linalg`.

We first generate a regular and quasi-uniform triangulation $\mathcal{T}_0$ of $\Omega$ with mesh-width $h = 0.1$, by using the Delaunay-based mesh generator contained in the Python library `triangle`, see [20]. This serves as the initial triangulation in the local bisection refinement algorithm described in Section 4.2 and introduced in [9]. Elements are bisected according to the rule of newest vertex bisection (NVB). Note that the elementary bisection of elements needs to include a recursive call in order to ensure the output of regular triangulations without hanging nodes. We refer to [16] for a detailed overview on NVB.

The bisection refinement algorithm takes as input parameters the uniform mesh-width bound $h$, the parameter $L$ specifying the number of dyadic refinements towards corner $\mathbf{c}^*$, and the problem-dependent exponent $\gamma$ in (4.2). In order to fulfill the second condition in (4.2), we choose $L$ in dependence of $h$, $p$ and $\gamma$ as

$$L = \lceil -\frac{\log_2(h)}{\gamma(p+1)} - 1 \rceil. \tag{6.2}$$

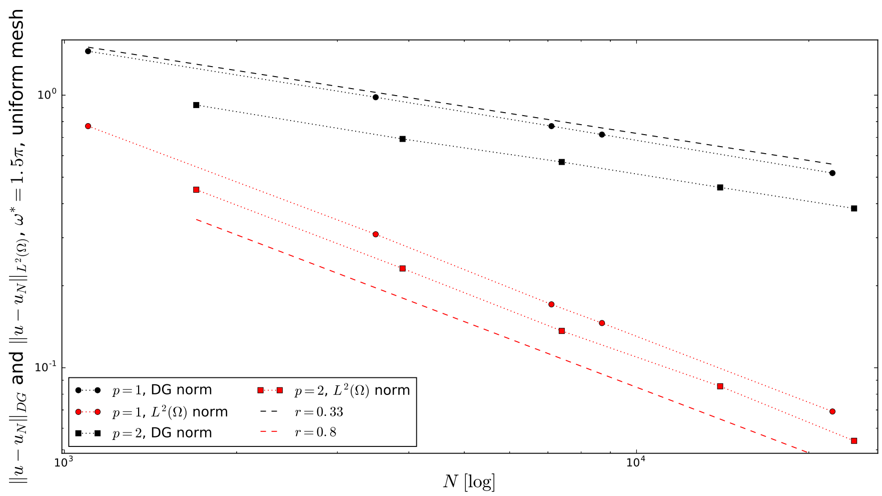
Note that, by construction, a smaller value of $h$ results in a triangulation which is a refinement of the previous triangulation where a larger value for $h$ has been used. Therefore, the reference solution is straightforwardly projected on coarser grids for error computation.

6.3. **Results.** We compute empirical convergence rates on the two domains corresponding to the non-convex angles $\omega^* \in \{1.5, 1.9\}\pi$ at the re-entrant corner $\mathbf{c}^*$.

6.3.1. *Quasi-uniform refinement.* We first discuss and show results in the absence of local bisection refinement towards $\mathbf{c}^*$. In this case, we expect the empirical convergence rates to coincide with the convergence rates on quasi-uniform mesh families as discussed, e.g., in [18, Theorems 2.13 and 2.14]. According to these results, the symmetric IP method with $\mathfrak{j}_0 > \mathfrak{j}_*$ converges asymptotically as $N^{-(\min(p+1,s)-1)/2}$ in the DG norm and as $N^{-\min(p+1,s)/2}$ in the $L^2(\Omega)$-norm, where $s > 0$ is such that $u \in H^s(\Omega)$. As our domain $\Omega$ has the non-convex corner $\mathbf{c}^*$ with interior opening angle $\omega^* > \pi$, the solution of the Dirichlet problem for the Poisson equation (6.1) in general only belongs to the Sobolev space $H^s(\Omega)$ with $s < 1 + \frac{\pi}{\omega^*}$, see e.g. [10, Chapter 4]. For $\omega^* = 1.5\pi$, this bound reads $s < 1+\frac{2}{3}$, thus $\min(p+1, s) = s < 1+\frac{2}{3}$ for both $p \in \{1, 2\}$. Therefore, the expected asymptotic error bounds are $N^{-1/3}$ in the DG norm, and $N^{-5/6}$, where $\frac{5}{6} \simeq 0.83$, in the $L^2(\Omega)$-norm. Similarly, for $\omega^* = 1.9\pi$, $\min(p+1, s) = s < 1 + \frac{10}{19} \simeq 1.526$, and we expect the error to behave asymptotically as $N^{-0.26}$ in the DG norm, and as $N^{-0.76}$ in the $L^2(\Omega)$ norm. The experimental results are depicted in Figure 2 and confirm these expected (suboptimal) convergence rates for our example.

6.3.2. *Bisection refinement.* Next, we show results for the full bisection refinement algorithm, with local refinement towards $\mathbf{c}^*$, with dyadic refinement level $L$ chosen in dependence of $h, \gamma, p$ as in (6.2). Due to the imposition of Dirichlet boundary conditions in (6.1), with Remark 2.5 the coefficient $\delta^*_{\mathbf{c}^*}$ associated with corner $\mathbf{c}^*$ as in Proposition 2.3 is given by $\delta^*_{\mathbf{c}^*} = \frac{\pi}{\omega^*} \in \{2/3, 10/19\} \simeq \{0.67, 0.53\}$. Hence, by taking as weight exponent $\delta_{\mathbf{c}^*}$ associated with $\mathbf{c}^*$ the lower bound $1 - \delta^*_{\mathbf{c}^*} \in \{1/3, 9/19\}$ in (2.19), the definition of $\gamma$ in (4.2) leads to the grading parameter $\gamma^* \in \{2/3, 10/19\}$ in the bisection refinement algorithm.

We present convergence plots for bisection refinement meshes with the grading parameters $\gamma \in \{1/2, 2/3, 4/5\}$ for $\omega^* = 1.5\pi$, and the values $\gamma \in \{2/5, 10/19, 2/3\}$ for $\omega^* = 1.9\pi$. The results obtained for $p = 1$ are depicted in Figure 3 as bi-logarithmic plots of the errors in the DG norm and the $L^2$-norm versus $N$. In agreement with our error analysis, for $\gamma^*$, we observe the optimal convergence rate $N^{-1/2}$ for the DG norm and $N^{-1}$ for the $L^2$-norm. Notice that Theorem 4.3 ensures optimal convergence rates for $0 < \gamma < \gamma^*$. This is corroborated in Figure 3 for $\gamma = 0.5$ when $\gamma^* = 2/3$, and $\gamma = 0.4$ when $\gamma^* = 10/19$, respectively, although the resulting constants are slightly worse. Finally, we see that using the value $\gamma > \gamma^*$ (i.e., $\gamma \in \{4/5, 2/3\}$) leads to suboptimal empirical convergence rates with respect to $N$, while still larger than the empirical convergence rates observed on quasi-uniform meshes. This is indicative of the sharpness of our convergence analysis.



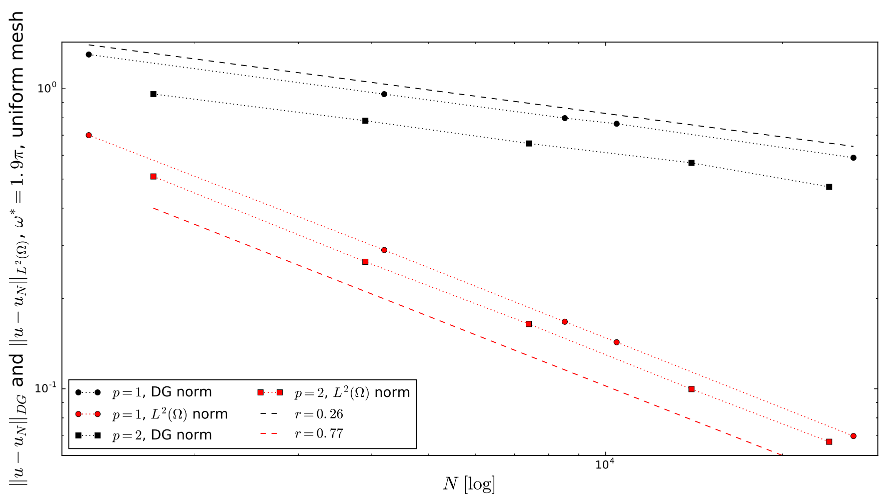




FIGURE 2. Empirical relative errors on a quasi-uniform mesh family in DG norm and $L^2(\Omega)$ norm, for piecewise linear ($p = 1$) and piecewise quadratic ($p = 2$) approximations, and on a domain with $\omega^* = 1.5\pi$ (top) and $\omega^* = 1.9\pi$ (bottom). We choose $\mathrm{j}_0 = 10$ for $p = 1$ and $\mathrm{j}_0 = 40$ for $p = 2$. The dashed lines have fixed slope $-r$ and serve as reference for the asymptotic behavior $N^{-r}$.

Analogous results are obtained for $p = 2$ as depicted in Figure 4. Here, the quasi-optimal convergence rate is $N^{-1}$ for the DG norm and $N^{-1.5}$ for the $L^2$-norm. We observe that the experimental values lie on a line parallel to the reference line with the quasi-optimal convergence rate as slope. We find again that the value $\gamma < \gamma^*$ yields optimal rates (with worse constants) and that the meshes constructed using the value $\gamma > \gamma^*$ do not yield quasi-optimal empirical convergence rate.

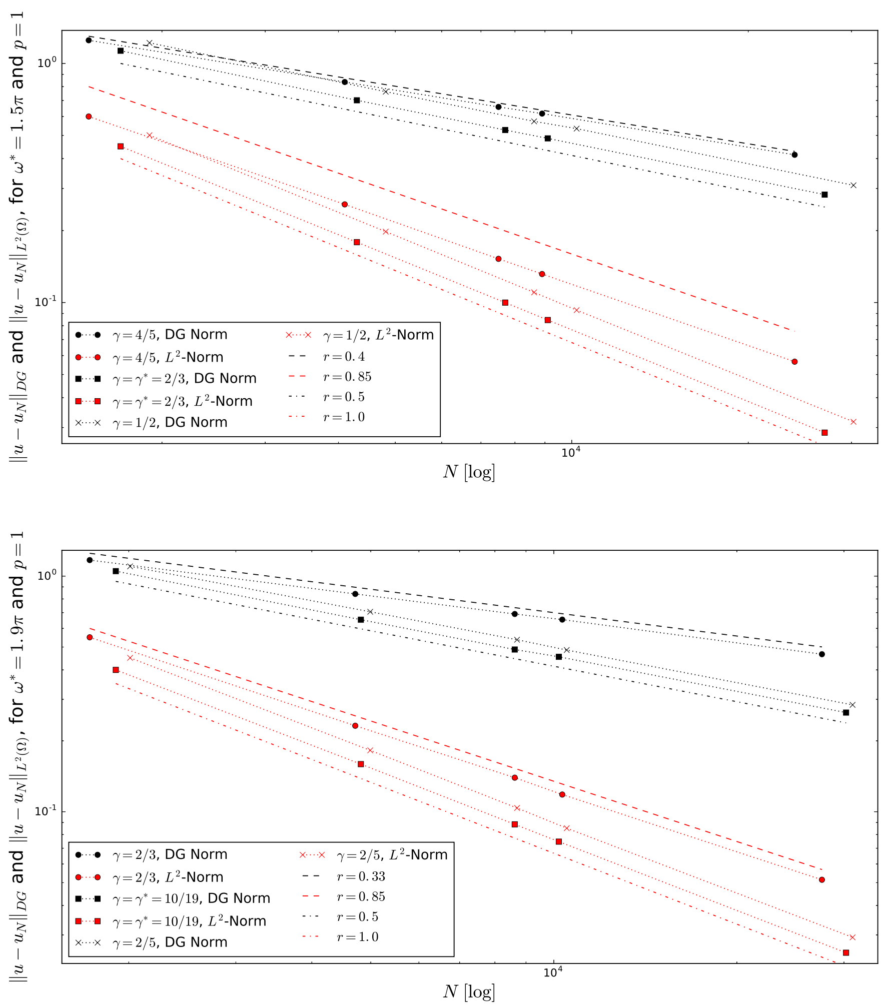
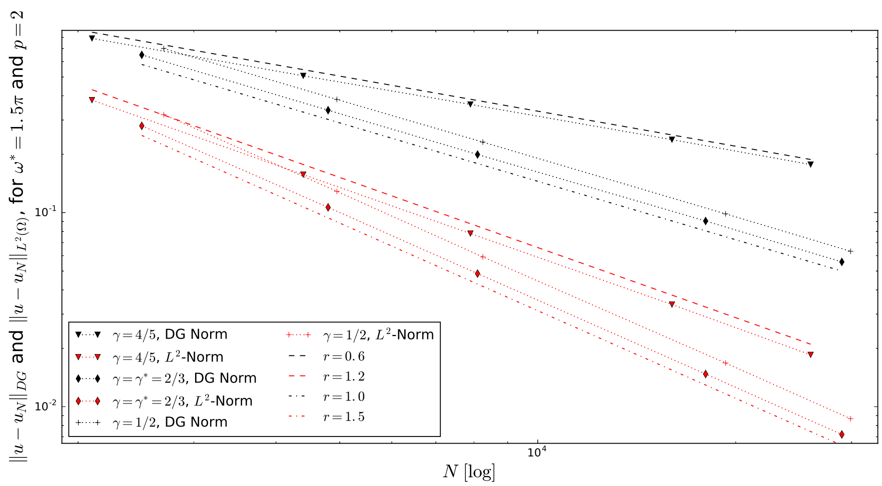



FIGURE 3. Results for piecewise linear approximations (p=1) and $j_0 = 10$. Empirical relative errors are shown in the DG energy norm and the $L^2$-norm for $\omega^* = 1.5\pi$ (top) and $\omega^* = 1.9\pi$ (bottom). The dashed lines have fixed slope $-r$ and serve as reference for the asymptotic behavior $N^{-r}$.

## 7. Conclusions and Extensions

We developed an a-priori error analysis for symmetric interior penalty DGFEM discretizations of the linear and second-order elliptic boundary-value problem (2.4) with mixed boundary conditions (2.5), (2.6) in a plane polygon. We showed that IP methods based on either graded families or bisection refinement families of regular, simplicial triangulations with local refinement near the corners of the domain $\Omega$ result in optimal asymptotic convergence rates, for any order $p \geq 1$ of the approximations. We also showed that the same families of locally adapted meshes imply



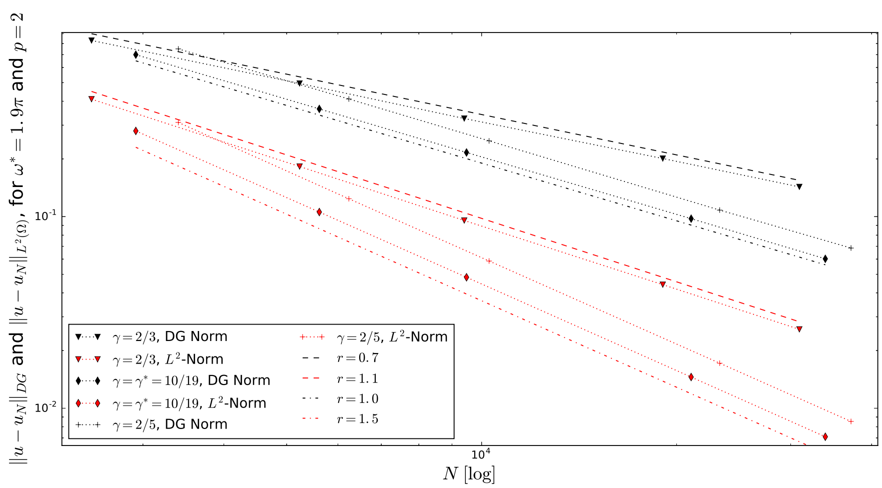




FIGURE 4. Results for piecewise quadratic approximations (p=2) and $j_0 = 40$. Empirical relative errors are shown in the DG energy norm and the $L^2$-norm for $\omega^* = 1.5\pi$ (top) and $\omega^* = 1.9\pi$ (bottom). The dashed lines have fixed slope $-r$ and serve as reference for the asymptotic behavior $N^{-r}$.

optimal convergence rates in $L^2(\Omega)$, by generalizing an Aubin-Nitsche duality estimate and by using the adjoint-consistency of the symmetric interior penalty formulation (cf. Remark 4.4) in combination with adjoint-regularity shifts in weighted Sobolev spaces. This result is crucial in the convergence rate bounds for linear parabolic and second order hyperbolic evolution problems in polygonal domains.

We provided a set of numerical experiments with dG elements of polynomial degree $p = 1$ and $p = 2$, for a family of model elliptic boundary value problems in polygonal domains. The numerical tests confirmed the theoretical results. They indicated strongly that the sufficient conditions on mesh grading to ensure optimal asymptotic convergence rate bounds that were established in the present paper are also necessary.

We considered only the second-order, elliptic PDE (2.4), but emphasize that the present consistency and duality error analysis can be extended to symmetric interior penalty dG discretizations of problem (2.4) with first order and absolute terms. Following the non-symmetric approach [11], symmetric IP methods can be readily designed for such problems.

Also, for linear, second order elliptic systems in polygonal domains $\Omega$ which afford elliptic regularity shifts in the presently considered scales of weighted Sobolev spaces, the present consistency and duality error analysis of corresponding symmetric IP methods (as can be readily devised along the lines of [24] for, e.g., plane linear elasticity) on graded and bisection refined regular simplicial meshes generalizes with analogous results.

The assumption $\mathsf{c} \in C^\infty(\overline{\Omega})$ used in this paper as well as the assumptions in [11] on the smoothness of the coefficients can be substantially relaxed; for the regularity, we refer to [6]; however, details on corresponding DG formulations and convergence rate estimates are beyond the scope of the present paper and will be provided elsewhere.

Although we focussed on the symmetric interior penalty DG method in this work, our techniques are equally well and *mutatis mutandis* applicable to a wider range of DG methods, for example, by employing the unifying framework in [2]. We emphasize again that in this setting, the methods are required to be adjoint-consistent in order to achieve $L^2$-norm error optimality on locally adapted meshes.

For simplicity, our analysis was carried out for regular meshes. However, with only minor modifications, it can be extended to simplicial mesh families with $k$-irregular nodes, which are a particular case of the shape-regular and contact-regular mesh families introduced in [17, Section 1.4].

## References

1. J. Adler and V. Nistor, *Graded mesh approximation in weighted Sobolev spaces and elliptic equations in 2D*, Math. Comp. **84** (2015), no. 295, 2191–2220. MR 3356024
2. D. Arnold, F. Brezzi, B. Cockburn, and D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2002), no. 5, 1749–1779. MR 1885715 (2002k:65183)
3. I. Babuška and B. Q. Guo, *The h-p version of the finite element method for domains with curved boundaries*, SIAM J. Numer. Anal. **25** (1988), no. 4, 837–861. MR 954788
4. ———, *Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order*, SIAM J. Math. Anal. **19** (1988), no. 1, 172–203. MR 924554
5. I. Babuška, R.B. Kellogg, and J. Pitkäranta, *Direct and inverse error estimates for finite elements with mesh refinements*, Numer. Math. **33** (1979), no. 4, 447–471. MR 553353 (81c:65054)
6. C. Bacuta, H. Li, and V. Nistor, *Differential operators on domains with conical points: precise uniform regularity estimates*, arXiv preprint (2016), no. 1605.07907.
7. C. Băcuţă, V. Nistor, and L.T. Zikatanov, *Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps*, Numer. Math. **100** (2005), no. 2, 165–184. MR 2135780 (2006d:65130)
8. M. Costabel, M. Dauge, and S. Nicaise, *Analytic regularity for linear elliptic systems in polygons and polyhedra*, Math. Models Methods Appl. Sci. **22** (2012), no. 8.
9. F.D. Gaspoz and P. Morin, *Convergence rates for adaptive finite elements*, IMA J. Numer. Anal. **29** (2009), no. 4, 917–936. MR 2557050 (2010k:65257)

10. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, Reprint of the 1985 original [ MR0775683], With a foreword by Susanne C. Brenner. MR 3396210
11. P. Houston, Ch. Schwab, and E. Süli, *Discontinuous hp-finite element methods for advection–diffusion–reaction problems*, SIAM J. Numer. Anal. **39** (2002), 2133–2163.
12. E. Jones, T. Oliphant, P. Peterson, et al., *SciPy: Open source scientific tools for Python*, 2001–, [Online; accessed 2017-02-20].
13. H. Li and V. Nistor, *LNG_FEM: graded meshes on domains of polygonal structures*, Recent Advances in Scientific Computing and Applications, Contemp. Math., vol. 586, Amer. Math. Soc., Providence, RI, 2013, pp. 239–246. MR 3075874
14. V. Maz'ya and J. Rossmann, *Elliptic equations in polyhedral domains*, Mathematical Surveys and Monographs, vol. 162, American Mathematical Society, Providence, RI, 2010. MR 2641539 (2011h:35002)
15. W.F. Mitchell, *Unified Multilevel Adaptive Finite Element Methods for Elliptic Problems*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 1988, p. 119. MR 2637128
16. R.H. Nochetto, K.G. Siebert, and A. Veeser, *Theory of adaptive finite element methods: an introduction*, Multiscale, Nonlinear and Adaptive Approximation, Springer, Berlin, 2009, pp. 409–542. MR 2648380 (2011k:65164)
17. D.A. Di Pietro and A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 69, Springer, Heidelberg, 2012. MR 2882148
18. B. Rivière, *Discontinuous galerkin methods for solving elliptic and parabolic problems: Theory and implementation*, Frontiers in Applied Mathematics, SIAM, 2008.
19. Ch. Schwab, *p- and hp-FEM – Theory and application to solid and fluid mechanics*, Oxford University Press, Oxford, 1998.
20. J. R. Shewchuk, *Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator*, Applied Computational Geometry: Towards Geometric Engineering (M. C. Lin and D. Manocha, eds.), Lecture Notes in Computer Science, vol. 1148, Springer-Verlag, May 1996, First ACM Workshop on Applied Computational Geometry, pp. 203–222.
21. T. P. Wihler, *Discontinuous Galerkin FEM for Elliptic Problems in Polygonal Domains*, Ph.D. thesis, Swiss Federal Institute of Technology Zurich, 2002, Diss. ETH No. 14973.
22. T. P. Wihler, P. Frauenfelder, and C. Schwab, *Exponential convergence of the hp-DGFEM for diffusion problems*, Comput. Math. Appl. **46** (2003), 183–205.
23. T. P. Wihler and B. Rivière, *Discontinuous Galerkin methods for second-order elliptic PDE with low-regularity solutions*, J. Sci. Comput. **46** (2011), no. 2, 151–165. MR 2753240
24. Thomas P. Wihler, *Locking-free DGFEM for elasticity problems in polygons*, IMA J. Numer. Anal. **24** (2004), no. 1, 45–75. MR 2027288

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