A projection method for the computation of admissible measure valued solutions of the incompressible Euler equations

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A PROJECTION METHOD FOR THE COMPUTATION OF ADMISSIBLE MEASURE VALUED SOLUTIONS OF THE INCOMPRESSIBLE EULER EQUATIONS

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ABSTRACT. We formulate a fully discrete finite difference numerical method to approximate the incompressible Euler equations and prove that the sequence generated by the scheme converges to an admissible measure valued solution. The scheme combines an energy conservative flux with a velocity-projection temporal splitting in order to efficiently decouple the advection from the pressure gradient. With the use of robust Monte Carlo approximations, statistical quantities of the approximate solution can be computed. We present numerical results that agree with the theoretical findings obtained for the scheme.

1. INTRODUCTION

We consider the *incompressible Euler equations*, which model the motion of an inviscid, Newtonian fluid. These equations are of the form

{eq:euler}

$$(1a) \quad \{\{\texttt{eq:euler\_b}\}\}\text{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \text{ on } \mathcal{D} \times [0, T] =: \mathcal{D} \times \mathcal{I},$$

$$(1b) \quad \{\{\texttt{eq:euler\_a}\}\} \qquad \text{div}\,\mathbf{u} = 0, \text{ on } \mathcal{D} \times [0, T],$$

$$(1c) \qquad\qquad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \text{ on } \mathcal{D} \subseteq \mathbb{R}^d,$$

where $\mathbf{u}$ denotes the velocity field and p is the pressure, which can be viewed as a Lagrange multiplier for the divergence-free constraint (1b).

Incompressible Euler equations are a fundamental building block of fluid dynamics, and are considered a good model for flows with very low Mach number and high Reynolds number [22]. Yet, their mathematical and numerical understanding is still incomplete.

1.1. **Known results.** In the following, we summarize the main theoretical results about Euler equations.

Classical solutions, which require sufficient smoothness, fail to exist in general, for example on the 3D torus. For special cases (e.g. in 2D, with periodic boundary conditions) classical solutions exist globally and are unique [2, 22].

In the general case, one considers weak solutions to the Euler equations. Given a function $\mathbf{u}_0 \in L_x^2(\mathcal{D})$, a function $\mathbf{u}(t) \in L_x^2(\mathcal{D})$ is said to be a weak solution to the Euler equations, if it satisfies the equations in the sense of distributions, namely:

$$\int_{\mathcal{I} \times \mathcal{D}} \mathbf{u} \cdot \varphi_t + (\mathbf{u} \otimes \mathbf{u}) : \nabla\varphi \; dxdt - \int_{\mathcal{D}} \mathbf{u}_0 \cdot \varphi(x, 0)dx = 0,$$

for all test functions $\varphi \in C_0^\infty(\mathcal{I} \times \mathcal{D})^d$, with div$\varphi = 0$. The pressure can be recovered from the velocity up to a constant function depending only on the time.

In [26], Yudovich showed that, in $\mathbb{R}^2$, as long as the initial vorticity is bounded in L_x^∞, there is global existence and uniqueness of weak solutions to the Euler equations.

When the vorticity is unbounded, Delort [10] proved the existence of global weak solutions for 2D Euler equations, provided that the initial vorticity is a positive Radon measure in H^{-1}. This particular class of initial data is interesting for practical applications, since it encompasses the so called vortex-sheets.

In general, uniqueness has been proven to be false for Euler equations (see [23, 24], or the recent [20]). Numerically, there is also evidence for non-uniqueness of solutions of the vortex-sheet problem [21].

Our work will focus on the more general framework of (admissible) measure valued solutions (MVS). In this framework (introduced in [14]) the solution is not an integrable function, but rather a parametrized measure $\nu_{(x,t)}$ for $(x,t) \in \mathcal{D} \times \mathcal{I}$. Measure valued solutions are shown to exist globally, in 3D and to represent limits of interesting theoretical [13] and numerical approximation schemes [19].

1.2. **Numerical methods.** Many numerical approximation techniques have been applied successfully to the approximation of the Euler equations, even in cases where the underlying mathematical analysis in unavailable. However, numerical approximations of problems with rough initial data, may not converge (cf. eg. [6]). In such cases, it is necessary to question, whether the numerical approximation of weak solutions is an appropriate approach [15].

Existing numerical methods include spectral methods, which, for sufficiently regular solutions, has been shown to converge to the underlying solution with spectral accuracy (exponential accuracy in the number of modes) in a spatially semi-discrete framework [5]. The convergence of spectral methods to measure valued solutions was proved in [19]. However, spectral methods are only valid for periodic domains.

Finite element methods (cf. [16, 17]) are also a possible discretization technique, although they generally require some amount of regularity of the initial data.

Finite difference methods can be employed to approximate the equations. This, combined with operator splitting methods, yields efficient and robust methods, cf. [1, 6, 9, 18]. However, no convergence results for non-smooth initial data, even in the framework of measure valued solutions, are available for these type of schemes.

1.3. **Aim of the paper.** Our aim is to propose and analyse a variant of the finite-difference projection methods, and prove that it generates sequences that converge to a admissible measure valued solutions of the incompressible Euler equations.

The procedure used in this paper is based on a similar approach used for the computation of measure valued solutions in the context of hyperbolic conservation laws [15]. In the article, the authors devised a class of schemes that converge to measure value solutions. In [19], the authors present a semi-discrete spectral method that is able to approximate measure valued solutions for incompressible Euler equation in the 2-dimensional torus.

1.4. **Contents of the paper.** In section 2, we will recall basic definitions of measure valued solutions for incompressible Euler equations. In section 3, we will define the numerical scheme for the approximation of measure valued solutions. Finally, in section 4, we will provide numerical experiments of convergence of the scheme to measure valued solutions.

2. Admissible measure valued solutions

The concept of *measure valued solution* is based on that of (generalized) Young measure [25], namely, a probability measure parametrized in space and time.

This notion was initially introduced for conservation laws [11], subsequently extended to uniformly L^∞-bounded approximations to (incompressible) Euler equations [14] and then generalized to arbitrary L^2 sequences [12], after a small, technical modification taking into account the possibility of concentrations.

We define the family $\mathcal{F}$ of functions:

$$(2) \qquad \mathcal{F} := \{g(\mathbf{u})(1 + |\mathbf{u}|^2), g \in C_c(\mathbb{R}^d), \exists \lim_{r \to \infty} g(r\mathbf{u}) \in C(\mathcal{S}^{d-1})\}.$$

We define $g^\infty(\mathbf{u}) := \lim_{r \to \infty} g(r\mathbf{u})$ whenever the limit exists and is unique.

{thm:generalized_young}

Theorem 2.1 ((Generalized) Young measure theorem [14]). Given a sequence $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ of uniformly bounded functions in $L^2(\mathcal{D} \times [0, T))$, there exists:

- a measure $\lambda \in \mathcal{M}^+(\mathcal{D} \times [0, T))$ such that $|\mathbf{u}_n|^2 \rightharpoonup \lambda \in \mathcal{M}(\mathcal{D} \times [0, T))$;
- a weakly measurable map ν_x from $\mathcal{D} \times [0, T)$ to $\mathcal{P}(\mathbb{R}^d)$;
- a weakly measurable map, ν_x^∞ from $\mathcal{D} \times [0, T)$ to $\mathcal{P}(\mathcal{S}^{d-1})$,

such that, for a subsequence, relabelled as the original by abuse of notation, it holds:

$$(3) \quad g(\mathbf{u}_n)d\mathbf{x}dt \overset{*}{\rightharpoonup} \langle \nu, g \rangle (d\mathbf{x}dt + d\lambda) + \langle \nu^\infty, g^\infty \rangle d\lambda, \quad in \ \mathcal{M}(\mathcal{D} \times [0, T)), \ \forall g \in \mathcal{F}.$$

Definition 2.2 ((Generalized) Young measure). The triple $(\nu, \nu^\infty, \lambda)$ obtained from Theorem 2.1, is termed a *generalized Young measure* relative to the sequence $\mathbf{u}_n$.

Based on [15], we consider a general measure as initial data for the incompressible Euler equations. Having a measure as initial datum for the problem (rather than an integrable function), further generalizes the problem allowing to model uncertainty in the initial data. This automatically provides a framework for uncertainty quantification.

Definition 2.3 (Admissible measure valued solution). Let $\sigma_0 := \nu_{(x,0)}$ be an initial measure. A measure value solution for the Euler equations is a (generalized) Young measure ν such that

$$\int_{\mathcal{D} \times [0,T)} \phi_t \langle \nu_{(x,t)}, \mathrm{id} \rangle + \nabla \phi : \langle \nu_{(x,t)}, \xi \otimes \xi \rangle \, d\mathbf{x}dt \ + \int_{\mathcal{D}} \phi_0 \cdot \langle \sigma_0, Id \rangle \, d\mathbf{x} = 0,$$

for all divergence-free test functions $\phi \in C_c^\infty(D \times I)^d$. Moreover, if, for all $\varphi \in C_c(\mathcal{D} \times [0, T))$,

$$\int_{\mathcal{D} \times [0,T)} \varphi(x,t) \langle \nu_{(x,t)}, \xi^2 \rangle \, d\mathbf{x}dt + \int_{\mathcal{D} \times [0,T)} \varphi d\lambda \leq \|\varphi\|_{L^1(\mathcal{D} \times [0,T))} E(0)$$

then we call ν an *admissible measure valued solution*. Here, we denote $\lambda := \lambda_t \times dt$ and define $E(t) := \int_{\mathcal{D}} \langle \nu_{(x,t)}, \xi^2 \rangle \, d\mathbf{x} + \lambda_t(\mathcal{D})$ to be the mean *(Kinetic) energy* of the measure valued solution.

Admissible measure valued solution are not unique [20], thus a stronger notion of admissibility is necessary to recover uniqueness. However, at least in the case where a smooth solution exists, the notion of admissible measure valued solution coincides with it [7]. The vanishing viscosity approach can be exploited to *construct* approximations of measure valued solutions [8] from sequences of weak solutions to the Navier-Stokes problem. Another possible approach for the construction of measure valued solutions consists in using a numerical scheme that produces a

suitable sequence of approximations. The sequence can be then shown to converge, in the weak* sense, to admissible measure valued solutions.

3. Numerical scheme

In this section, we present a numerical scheme that is able to approximate (admissible) measure valued solutions. In order to compute measure valued solutions, we require two main ingredients:

- an "individual solver", able to approximate the Euler equations in their classical formulation, whenever such formulation is well-posed;
- a *statistical integrator*, capable of approximating measures exploiting stochastic sampling procedures.

In general, we cannot expect from the individual solver to converge (weakly or strongly) in the usual sense [6]. Instead, we require minimal stability properties allowing us to control key quantities of the approximated solutions, which, in turn, allow us to show consistency with measure valued solutions. Our individual solver is based upon the ideas of [9] and [6]. We utilize a projection method (also known as fractional step or pressure correction method) for an efficient enforcement of the divergence-free condition. The spatial discretization is enforced using a "mimetic" approach: we construct a scheme that possesses properties similar to the continuous equation.

We consider the domain $[0,1]^d \times [0,T]$. For simplicity, we assume a uniform grid in both space and time, with uniform grid spacing $\Delta x = \Delta y = 1/M > 0$ in space and $\Delta t > 0$ in time. Furthermore, we assume $O(\Delta x) = O(\Delta y) = O(\Delta t)$. The velocity will be approximated by a piecewise constant function in both space and time variables (in a cell $\mathfrak{C}_{i,j} := [i\Delta x, (i+1)\Delta x] \times [j\Delta y, (j+1)\Delta y]$). We will identify these velocities with a sequence $\{\mathbf{u}_{i,j}^n\}_{i,j,n}$ (denoted also $\mathbf{u}$ by abuse of notation) indexed by space variables $i \in I$ and $j \in J$ ($I = \{0, \ldots, M-1\}$, $J = \{0, \ldots, M-1\}$) and a time variable $n \in \{0, \ldots, N\}$, $T = N\Delta t$. We denote the space of grid functions for the choice $h = \Delta x = \Delta y$ as $\mathcal{G}_h$.

The scheme uses a *fractional step method*, in which the velocity is evolved, in a prediction step (4a), without imposing the divergence-free constraint. In turn, the divergence constraint is enforced in a second *projection step* (4b), which also provides an evolution equation for the pressure. The scheme reads:

$$(4a) \qquad \frac{\mathbf{u}^{*n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{C}(\mathbf{u}^n, \bar{\mathbf{u}}^{n+1/2}) = \mathbf{D}\bar{\mathbf{u}}^{n+1/2},$$

$$(4b) \qquad \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \mathbf{P}\left(\frac{\mathbf{u}^{*n+1} - \mathbf{u}^n}{\Delta t}\right),$$

$$(4c) \qquad \mathrm{grad}^h p^{n+1/2} = \mathbf{Q}\left(\frac{\mathbf{u}^{*n+1} - \mathbf{u}^n}{\Delta t}\right),$$

where $\mathbf{C}(\mathbf{u}^n, \bar{\mathbf{u}}^{n+1/2})_{i,j}$ is an approximation of the nonlinear convective term (which will be specified later). The operator $\mathbf{D}$ is an numerical dissipation term, which ensures the numerical stability of the scheme. The time-averaged velocity $\bar{\mathbf{u}}^{n+1/2} := \theta\mathbf{u}^n + (1-\theta)\mathbf{u}^{*,n+1}$ (with $\theta \in [0, 1/2]$ as free parameter) ensures that the scheme is sufficiently dissipative. The approximation of spatial differential operators exploits finite difference (FD) and finite volume approximations (FV).

3.1. **Definitions.** In this section, we introduce concrete discretizations of the spatial finite difference operators. We formulate the space discretization in 2 space dimensions, but remark that extension to 3D is straightforward.

3.1.1. *Discrete differential operators.* We denote by grad^h a standard cell-centred central, second order discrete gradient operator defined as

$$\text{grad}^h \varphi_{i,j} = \begin{pmatrix} \frac{\varphi_{i+1,j} - \varphi_{i-1,j}}{\Delta x} \\ \frac{\varphi_{i,j+1} - \varphi_{i,j-1}}{\Delta y}, \end{pmatrix}$$

and by div^h a cell-centred central divergence operator

$$\text{div}^h \mathbf{u}_{i,j} = \frac{(u_1)_{i+1,j} - (u_1)_{i-1,j}}{\Delta x} + \frac{(u_2)_{i,j+1} - (u_2)_{i,j-1}}{\Delta y}.$$

We introduce the notation Δ^h (resp. $\boldsymbol{\Delta}^h$) for a standard five point Laplace (resp. vector Laplace) operator:

$$\Delta^h \varphi_{i,j} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{\Delta y}.$$

3.1.2. *Notation.* In the following, for discrete (i.e. piecewise constant in cells $\mathfrak{C}_{i,j}$) quantities, we denote by

$$(5) \qquad \|w\|_p := \left(\sum_{\substack{i \in I \\ j \in J}} |w_{i,j}|^p \Delta x \Delta y \right)^{1/p}$$

the *discrete* L^p norm in space and with

$$(6) \qquad \|w\|_{p,q} := \left(\sum_{n=0}^{N-1} \left(\sum_{\substack{i \in I \\ j \in J}} |w_{i,j}^{(n)}|^p \Delta x \Delta y \right)^{q/p} \Delta t \right)^{1/q}$$

the $L^{p,q}$ norm in space-time (with standard extensions to ∞). We also introduce the L^2 inner product (in the spatial variables) for grid functions:

$$\langle \mathbf{a}, \mathbf{b} \rangle := \sum_{\substack{i \in I \\ j \in J}} \mathbf{a}_{i,j} \cdot \mathbf{b}_{i,j} \Delta x \Delta y.$$

We introduce the notion of jump $[\![a]\!]_{i+1/2,j}^x := a_{i+1,j} - a_{i,j}$ and average $\overline{a}_{i+1/2,j}^x = \frac{1}{2}(a_{i+1,j} - a_{i,j})$ (analogous definition for the y-direction).

3.1.3. *The (numerical) diffusion operator.* The operator $\mathbf{D}$ is a second-order, energy dissipative diffusion operator evaluated at the time averaged quantity $\bar{\mathbf{u}}^{n+1/2} := \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)$ (this ensures energy dissipation in both space and time). We will use the 2nd order Lax-Wendroff diffusion, which has the form:

$$(7) \quad \{\{\mathbf{D}\bar{\mathbf{u}}\}\} \frac{c_{i+1/2,j} |[\![\mathbf{u}]\!]_{i+1/2,j}^x| \, [\![\mathbf{u}]\!]_{i+1/2,j}^x + c_{i-1/2,j} |[\![\mathbf{u}]\!]_{i-1/2,j}^x| \, [\![\mathbf{u}]\!]_{i-1/2,j}^x}{\Delta x}$$

$$(8) \qquad + \frac{c_{i,j+1/2} |[\![\mathbf{u}]\!]_{i,j+1/2}^y| \, [\![\mathbf{u}]\!]_{i,j+1/2}^y + c_{i,j-1/2} |[\![\mathbf{u}]\!]_{i,j-1/2}^y| \, [\![\mathbf{u}]\!]_{i,j-1/2}^y}{\Delta y},$$

with $0 < c_{i\pm\cdot,j\pm\cdot} < \infty$ as parameters of choice.

3.1.4. *The exact projection operator.* We denote by $\mathbf{P}$ the exact discrete projection operator of a vector field into a discretely divergence-free vector space (i.e. s.t. $\mathrm{div}^h \mathbf{P}(\mathbf{u}) = 0$) and $\mathbf{Q} := \mathbf{I} - \mathbf{P}$ (the gradient, or irrotational, part of the vector field, where $\mathbf{I}$ is the identity map). The discrete projection is implemented by solving: $\mathbf{Pu} := \mathbf{Iu} - \mathrm{grad}^h(\Delta_2^h)^{-1}\mathrm{div}^h\mathbf{u}$, where $\Delta_2^h = \mathrm{div}^h\mathrm{grad}^h$ is a cell-centred five point Laplacian acting on cell-centres in checker-board formation:

$$\Delta_2^h \varphi_{i,j} = \frac{\varphi_{i+2,j} - 2\varphi_{i,j} + \varphi_{i-2,j}}{\Delta x} + \frac{\varphi_{i,j+2} - 2\varphi_{i,j} + \varphi_{i,j-2}}{4\Delta y}.$$

3.1.5. *The non-linear (convective) term.* The non-linear term is evaluated at two different discrete quantities, namely the "lagged" velocity $\mathbf{u}^n$ and the time-averaged $\bar{\mathbf{u}}^{n+1/2}$. For those we use a second order, consistent flux term:

$$\mathbf{C}(\mathbf{u},\mathbf{v})_{i,j} := \frac{\mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}, \mathbf{u}_{i,j}, \mathbf{v}_{i+1,j}, \mathbf{v}_{i,j}) - \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i,j}, \mathbf{u}_{i-1,j}, \mathbf{v}_{i,j}, \mathbf{v}_{i-1,j})}{\Delta x}$$
$$+ \frac{\mathbf{F}^{\mathbf{y}}(\mathbf{u}_{i,j+1}, \mathbf{u}_{i,j}, \mathbf{v}_{i,j+1}, \mathbf{v}_{i,j}) - \mathbf{F}^{\mathbf{y}}(\mathbf{u}_{i,j}, \mathbf{u}_{i,j-1}, \mathbf{v}_{i,j}, \mathbf{v}_{i,j-1})}{\Delta y}$$

where

$$\mathbf{F}^{\mathbf{x}}(\mathbf{u}^+, \mathbf{u},^- , \mathbf{v}^+, \mathbf{v}^-) := \frac{1}{4}\begin{pmatrix} (u_1^+ + u_1^-)(v_1^+ + v_1^-) \\ (u_1^+ + u_1^-)(v_2^+ + v_2^-) \end{pmatrix},$$

$$\mathbf{F}^{\mathbf{y}}(\mathbf{u}^+, \mathbf{u},^- , \mathbf{v}^+, \mathbf{v}^-) := \frac{1}{4}\begin{pmatrix} (u_2^+ + u_2^-)(v_1^+ + v_1^-) \\ (u_2^+ + u_2^-)(v_2^+ + v_2^-). \end{pmatrix},$$

We define the cinvenience notation

$$\mathbf{F}^{\mathbf{x}\,n+1/2}_{i+1/2,j} := \mathbf{F}^{\mathbf{x}}(\mathbf{u}^n_{i+1,j}, \mathbf{u}^n_{i,j}, \mathbf{u}^{*\,n+1}_{i+1,j}, \mathbf{u}^{*\,n+1}_{i,j})$$
$$\mathbf{F}^{\mathbf{y}\,n+1/2}_{i,j+1/2} := \mathbf{F}^{\mathbf{y}}(\mathbf{u}^n_{i,j+1}, \mathbf{u}^n_{i,j}, \mathbf{u}^{*\,n+1}_{i,j+1}, \mathbf{u}^{*\,n+1}_{i,j}),$$
$$\mathbf{F}^{n+1/2}_{i+1/2,j+1/2} := \left(\mathbf{F}^{\mathbf{x}\,n+1/2}_{i+1/2,j}, \quad \mathbf{F}^{\mathbf{y}\,n+1/2}_{i,j+1/2} \right).$$

3.2. **Preliminaries.** In this subsection, we discuss some technical propositions which will be used in later sections. First of all, we remark that the fluxes are consistent with the continuous convective terms. Moreover, the fluxes are energy conservative, in the following sense.

Lemma 3.1. *The flux function is energy conservative, i.e. satisfies:*

$$\sum_{\substack{i \in I \\ j \in J}} \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}, \mathbf{u}_{i,j}, \mathbf{v}_{i+1,j}, \mathbf{v}_{i,j}) \cdot [\![\mathbf{v}_{\cdot,j}]\!]^x_{i+1/2} \Delta y$$
$$+ \sum_{\substack{i \in I \\ j \in J}} \mathbf{F}^{\mathbf{y}}(\mathbf{u}_{i,j+1}, \mathbf{u}_{i,j}, \mathbf{v}_{i,j+1}, \mathbf{v}_{i,j}) \cdot [\![\mathbf{v}_{i,\cdot}]\!]^y_{j+1/2} \Delta x = 0,$$

provided $\mathbf{u}$ is discretely divergence free (w.r.t. the operator div^h). Moreover,

$$\sum_{\substack{i \in I \\ j \in J}} \left(\frac{\mathbf{F}^{\mathbf{x}}(\mathbf{u},\mathbf{v})_{i+1/2,j} - \mathbf{F}^{\mathbf{x}}(\mathbf{u},\mathbf{v})_{i-1/2,j}}{\Delta x} \right.$$
$$\left. + \frac{\mathbf{F}^{\mathbf{y}}(\mathbf{u},\mathbf{v})_{i,j+1/2} - \mathbf{F}^{\mathbf{y}}(\mathbf{u},\mathbf{v})_{i,j-1/2}}{\Delta y} \right) \cdot \mathbf{v}_{i,j} = 0.$$

We omit the proof. Notice that the flux function $\mathbf{F^x}$ satisfies the following inequality:

(9) $|\mathbf{F^x}(\mathbf{A}, \mathbf{C}, \mathbf{X}, \mathbf{Y}) - \mathbf{F^x}(\mathbf{B}, \mathbf{C}, \mathbf{X}, \mathbf{Y})| \leq |\mathbf{A} - \mathbf{B}|(|\mathbf{X}| + |\mathbf{Y}|)$

and similarly for the other arguments of $\mathbf{F^x}$ and for the same arguments for $\mathbf{F^y}$.

3.2.1. *Projection.* We start by stating two useful lemmas on the exact projection operator.

Lemma 3.2. *There exists a grid function ψ such that:*

(10)   {{eq:lm:proj-has-grad-term}} $\mathbf{u}^* - \mathrm{grad}^h \psi.$

Proof. Observe that (4b), together with $\mathbf{P}(\mathbf{u}) = \mathbf{u}$, the linearity of the projection operator, and the fact $\mathbf{Q}(\mathbf{u}) = \mathrm{grad}^h \psi$ for some ψ, implies the claim. $\square$

Lemma 3.3. *The operators div^h and grad^h are self adjoint, namely:*

$$\sum_{\substack{i \in I \\ j \in J}} \mathbf{u} \cdot \mathrm{grad}^h \psi = -\sum_{\substack{i \in I \\ j \in J}} \mathrm{div}^h \mathbf{u} \; \psi, \quad \forall \mathbf{u} \in \mathcal{G}_h^2, \psi \in \mathcal{G}_h.$$

(For suitable boundaries: the lemma holds for a compactly supported velocity field, wall boundary conditions, periodic boundaries, or a compatible inflow/outflow boundary condition.)

The following proposition provides a discrete energy bound (i.e. an L^2 bound for the velocity) and a bound on the gradient pressure for the scheme (4).

{lm:energy-proj}

Proposition 3.1. *The projection $\mathbf{P}$ is energy dissipative, namely:*

(11)   {{energy-part1}} $\|\mathbf{u}\|_2^2 = \|\mathbf{u}^*\|_2^2 - \left\|\mathrm{grad}^h \psi\right\|_2^2.$

Proof. Take the L^2 discrete inner product of (10) with $\mathbf{u}$ and use the fact that $\mathbf{u}$ is divergence-free:

$$\langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}^*, \mathbf{u} \rangle = 0.$$

The usual reformulation implies

$$\|\mathbf{u}\|_2^2 - \|\mathbf{u}^*\|_2^2 + \|\mathbf{u} - \mathbf{u}^*\|_2^2 = 0.$$

$\square$

3.2.2. *Energy conservation.*

Proposition 3.2. *The approximate solution for the projection scheme satisfies the energy balance:*

(12) $\left\|\mathbf{u}^{n+1}\right\|_2^2 = \|\mathbf{u}^n\|_2^2 + 2\Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \right\rangle$

(13) $+ \frac{1}{2}(1 - 2\theta) \left\|\mathbf{u}^{*,n+1} - \mathbf{u}^n\right\|_2^2 - \left\|\mathrm{grad}^h \psi^{n+1/2}\right\|_2^2.$

Moreover, we have

(14) $\left\|\mathbf{u}^{n+1}\right\|_2^2 \leq \|\mathbf{u}^n\|_2^2,$

by the sign property of the diffusion term and positivity of the other terms.

Proof. Taking the L^2 discrete inner product of (4a) with $\Delta t \bar{\mathbf{u}}^{n+1/2}$, we obtain

$$\left\langle \mathbf{u}^{*\,n+1}_{i,j} - \mathbf{u}^n_{i,j}, \bar{\mathbf{u}}^{n+1/2} \right\rangle + \Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{C}(\mathbf{u}^n, \bar{\mathbf{u}}^{n+1/2})_{i,j} \right\rangle$$
$$= \Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \right\rangle,$$

which, by the conservation property of the nonlinear term, reads

$$\left\| \mathbf{u}^{*n+1} \right\|_2^2 - \left\| \mathbf{u}^n \right\|_2^2 = 2\Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}_1 \bar{\mathbf{u}}^{n+1/2} \right\rangle + \frac{1}{2}(1-2\theta) \left\| \mathbf{u}^{*,n+1} - \mathbf{u}^n \right\|_2^2 .$$

Using Theorem 3.1, we obtain

$$\left\| \mathbf{u}^{n+1} \right\|_2^2 = \left\| \mathbf{u}^n \right\|_2^2 + 2\Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}_1 \bar{\mathbf{u}}^{n+1/2} \right\rangle$$
$$+ \frac{1}{2}(1-2\theta) \left\| \mathbf{u}^{*,n+1} - \mathbf{u}^n \right\|_2^2 - \left\| \mathrm{grad}^h \psi^{n+1/2} \right\|_2^2 .$$

$\square$

3.2.3. *Weak space-time total variation bounds.* The following proposition provides discrete total variation limits that will be used in the following sections.

{fully:tv-bound}

Proposition 3.3. The following limit hold with a second order Lax-Wendroff diffusion operator $\mathbf{D}$:

$$(15) \qquad \sum_{n=0}^{N-1} \sum_{\substack{i \in I \\ j \in J}} \left| \bar{\mathbf{u}}^{n+1}_{j+1,i} - \bar{\mathbf{u}}^{n+1}_{j,i} \right|^3 \Delta t \Delta x \Delta y \overset{h \to 0}{\to} 0,$$

$$(16) \qquad \sum_{n=0}^{N-1} \sum_{\substack{i \in I \\ j \in J}} \left| \bar{\mathbf{u}}^{n+1}_{j,i+1} - \bar{\mathbf{u}}^{n+1}_{j,i} \right|^3 \Delta t \Delta x \Delta y \overset{h \to 0}{\to} 0,$$

$$(17) \qquad \sum_{n=0}^{N-1} \sum_{\substack{i \in I \\ j \in J}} \left| \mathbf{u}^{*\,n+1}_{j,i+1} - \mathbf{u}^n_{j,i} \right|^2 \Delta t \Delta x \Delta y \overset{h \to 0}{\to} 0.$$

We denote this limits as space (resp. time) weak total variation bounds.

Proof. The discrete diffusion operator (of Lax-Wendroff type) provides the spatial total variation limits. Hence, using the energy estimate, we can write

$$(18)$$
$$-2\Delta t \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \right\rangle - \Delta t (\frac{1}{2} - \theta) \left\| \mathbf{u}^{*,n+1} - \mathbf{u}^n \right\|_2^2 \le - \left\| \mathbf{u}^{n+1} \right\|_2^2 + \left\| \mathbf{u}^n \right\|_2^2 .$$

Summing over each time-step and exploiting the telescoping summation, we obtain

$$(19)$$
$$-\sum_{n=0}^{N-1} 2 \left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \right\rangle + (\frac{1}{2} - \theta) \left\| \mathbf{u}^{*,n+1} - \mathbf{u}^n \right\|_2^2 \Delta t \le - \left\| \mathbf{u}^N \right\|_2^2 + \left\| \mathbf{u}^0 \right\|_2^2 .$$

Using the Lax-Wendroff diffusion (7), we obtain

$$\left\langle \bar{\mathbf{u}}^{n+1/2}, \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \right\rangle$$

$$= \sum_{\substack{i \in I \\ j \in J}} \frac{\bar{\mathbf{u}}^{n+1/2} \cdot c_{i+1/2,j} | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket}{\Delta x}$$

$$+ \quad \frac{\bar{\mathbf{u}}^{n+1/2} \cdot c_{i-1/2,j} | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket}{\Delta y} \Delta x \Delta y$$

$$+ \sum_{\substack{i \in I \\ j \in J}} \frac{\bar{\mathbf{u}}^{n+1/2} \cdot c_{i,j+1/2} | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket}{\Delta x}$$

$$+ \quad \frac{\bar{\mathbf{u}}^{n+1/2} \cdot c_{i,j+-1/2} | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket | \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket}{\Delta y} \Delta x \Delta y$$

$$= \quad - \sum_{\substack{i \in I \\ j \in J}} c_{i,j} \frac{| \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket |^3}{2\Delta x} + c_{i,j} \frac{| \llbracket \bar{\mathbf{u}}^{n+1/2} \rrbracket |^3}{2\Delta y} \Delta x \Delta y.$$

With $c_{i,j} > 0$ we obtain

$$(20) \quad \sum_{n=0}^{N-1} \left(\left\| \bar{\mathbf{u}}_{i+1/2,j}^{n+1/2} - \bar{\mathbf{u}}_{i-1/2,j}^{n+1/2} \right\|_3^3 + \left\| \bar{\mathbf{u}}_{i,j+1/2}^{n+1/2} - \bar{\mathbf{u}}_{i,j-1/2}^{n+1/2} \right\|_3^3 \right) \Delta t \leq \left\| \mathbf{u}^0 \right\|_2^2 2h \to 0.$$

This proves our claim. $\qquad \square$

3.3. **Convergence to admissible measure valued solutions.** In this section we prove that the limit of the sequence, as the mesh is refined, of piecewise constant functions generated by the scheme is consistent with admissible measure valued solutions.

Lemma 3.4. *The vector valued piecewise constant function* $\mathbf{R}$*, defined as*

$$\mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \bar{\mathbf{u}}_{i+1,j}^{n+1}, \bar{\mathbf{u}}_{i,j}^{n+1}) - \mathbf{f}(\mathbf{u}_{i,j}^n) =: \mathbf{R}$$

satisfies $\left\| \mathbf{R} : \mathrm{grad}^h \varphi^h \right\|_1 \to 0$*, for a suitable piecewise approximation (on the grid with size h) of any smooth test function* φ^h*.*

Proof. Consider the term:

$$\mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \bar{\mathbf{u}}_{i+1,j}^{n+1/2}, \bar{\mathbf{u}}_{i,j}^{n+1/2})$$

$$= \quad \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \bar{\mathbf{u}}_{i+1,j}^{n+1/2}, \bar{\mathbf{u}}_{i,j}^{n+1/2}) - \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \bar{\mathbf{u}}_{i,j}^{n+1/2}, \bar{\mathbf{u}}_{i,j}^{n+1/2})$$

$$+ \quad \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \bar{\mathbf{u}}_{i,j}^{n+1/2}, \bar{\mathbf{u}}_{i,j}^{n+1/2}) - \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}^n_{i,j})$$

$$+ \quad \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i+1,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n) - \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n)$$

$$+ \quad \mathbf{F}^{\mathbf{x}}(\mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n, \mathbf{u}_{i,j}^n).$$

Using the consistency of the fluxes, the last term is exactly $\mathbf{f}(\mathbf{u}_{i,j}^n)$. Then (using Equation 9):

$$|\mathbf{R}| \leq \quad |(|\mathbf{u}_{i+1,j}^n| + |\mathbf{u}_{i,j}^n|)|\bar{\mathbf{u}}_{i+1,j}^{n+1/2} - \bar{\mathbf{u}}_{i,j}^{n+1/2})| + 2|\mathbf{u}_{i+1,j}^n + \mathbf{u}_{i,j}^n||\bar{\mathbf{u}}_{i,j}^{n+1/2} - \mathbf{u}_{i,j}^n|$$

$$+ \quad 2|\mathbf{u}_{i+1,j}^n - \mathbf{u}_{i,j}^n||\mathbf{u}_{i,j}^n|.$$

Using the discrete weak total variation bounds, all terms converge to zero after summation over the space-time indices and integration over the probability space Ω. In fact, we have (we consider the first term only, the rest is completely analogous):

$$\int_\Omega \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \left| \mathbf{u}_{i,j}^n \right| \left| \left[\!\left[\bar{\mathbf{u}}_{\cdot,j}^{n+1/2} \right]\!\right]_{i+1/2} \right| \left| \mathrm{grad}_+^h \varphi_{i,j}^n \right| \Delta x \Delta y \Delta t \, dP(\omega)$$

$$\leq \sup_{\omega \in \Omega} \left\| \mathbf{u}_0^h \right\|_p \left\| \left[\!\left[\bar{\mathbf{u}}_{\cdot,j}^{n+1/2} \right]\!\right]_{i+1/2} \right\|_q \left\| \mathrm{grad}_+^h \varphi^h \right\|_{r,s} \overset{h \to 0}{\to} 0,$$

using the fact that φ is sufficiently smooth (i.e. of class C_0^∞). Here, $p, q, r, s > 0$ are the exponents obtained from the application of the discrete Hölder inequalities. $\quad \square$

In the following, we will establish that the weak*-limit of the approximate solution is a measure valued solution.

For the approximation of measure valued solutions with initial data σ_x, we proceed as following (cf. [15]): from the law $\mathbf{u}_0(\omega; x, t)$ of σ_x, we draw a sufficient number of i.i.d. samples $\mathbf{u}_0(\omega_i; x)$, for $\omega_i \in \Omega, i \in I$ for some index set I. We evolve each sample with scheme (4), and find a discretization $\mathbf{u}^h(\omega_i; x, t)$, where $h > 0$ represent a discretization parameter. From the individual samples, we reconstruct the approximate measure valued solution as:

$$\nu_{(x,t)} \approx \sum_{i \in I} \delta_{\mathbf{u}^h(\omega_i; x, t)}.$$

From this approximation, we can compute statistical quantities of interest, such as mean, variance and mean mean kinetic energy.

The following theorem holds:

Theorem 3.4. Let σ_x be an initial Young measure in $L_w^\infty(\mathcal{D}; \mathcal{P}(\mathbb{R}^d))$. Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a random variable $\mathbf{u}_0(\omega; x)$ with law σ_x. For fixed $\omega \in \Omega$, let $\mathbf{u}^h(\omega; x, t)$ be the approximate solution with initial data $\mathbf{u}_0(\omega; x)$ obtained with scheme (4). Let $\nu_{(x,t)}^h$ be its law. Then there exists a subsequence (which we denote by abuse of notation by $\nu_{(x,t)}^h$) and a limit Young measure $\nu_{(x,t)}$, such that, for every divergence-free test vector field $\varphi \in C_0^\infty(\mathcal{D} \times [0,T))^d$ and $q \in C_0^\infty(\mathcal{D} \times [0,T))$, the following holds:

(21a)
$$\int_{[0,T)} \left\langle \left\langle \nu_{(x,t)}, \mathrm{id} \right\rangle, \varphi_t \right\rangle dt - \left\langle \left\langle \sigma_x, \mathrm{id} \right\rangle, \varphi(x,0) \right\rangle + \int_{[0,T)} \left\langle \left\langle \nu_{(x,t)}, \xi \otimes \xi \right\rangle, \nabla \varphi \right\rangle dt = 0$$

(21b)
$$\int_{[0,T)} \left\langle \left\langle \nu_{(x,t)}, \mathrm{id} \right\rangle, \nabla q \right\rangle dt = 0,$$

moreover the admissibility criterion is satisfied. The limit is therefore an admissible measure valued solution.

Proof. Denote by $\mathbf{u}^h(\omega; x, t)$ and $\mathrm{grad}^h p^h(\omega; x, t)$ the approximated step vector fields generated by the numerical scheme (depending on the outcome ω and on the step size h). The fundamental theorem of Young measures provides a measure $\nu_{(x,t)}$, which is the narrow limit of a subsequence $\nu_{(x,t)}^h$, the law of the approximation $\mathbf{u}^h$.

The evolution equations for the projection scheme are condensed into (removing dependency on ω for ease of notation):

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = - \mathbf{C}(\mathbf{u}^n, \bar{\mathbf{u}}^{n+1/2}) - \text{grad}^h p^{n+1/2} + \mathbf{D}\bar{\mathbf{u}}^{n+1/2},$$
$$\text{div}^h \mathbf{u}^n = 0.$$

We denote by $\mathfrak{C}_{i,j}$ the cell $[x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ and by $\mathfrak{I}^n$ the temporal interval $[t^n, t^{n+1}]$. Let $\varphi := (\varphi_1, \varphi_2) \in C_0^\infty(\mathcal{D} \times [0, T))^2$ be a solenoidal test vector field and construct a sequence sequence φ^h with limit φ, such that $\text{div}^h_+ \varphi^h = 0$. Multiplying the evolution equation by the corresponding approximate test function and by the cell-indicator function $\mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n}(x, t)$, summing over each time step $n = 0, ..., N - 1$ and each spatial index, and integrating in space, time, and on the probability space Ω, we obtain

$$0 = \int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \varphi^h \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt \, dP(\omega)$$

$$+ \int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \text{grad}^h p^{n+1/2} \cdot \varphi^h \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt \, dP(\omega)$$

$$+ \int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \mathbf{C}(\mathbf{u}^n, \bar{\mathbf{u}}^{n+1/2}) \cdot \varphi^h \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt \, dP(\omega)$$

$$+ \int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \mathbf{D}\bar{\mathbf{u}}^{n+1/2} \cdot \varphi^h \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt \, dP(\omega).$$

Let us consider each term of the sum individually.

- For the first term, we define:

$$\overline{\varphi}_{i,j}^n := \int_{\mathcal{D} \times [0,T)} \varphi \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt.$$

Consider:

$$\int_{\mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \varphi \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt$$

$$= \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \int_{\mathcal{D} \times [0,T)} \varphi \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} dx dt$$

$$= \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \overline{\varphi}_{i,j}^n \Delta x \Delta y \Delta t$$

$$= \sum_{\substack{i \in I \\ j \in J}} \left(\overline{\varphi}_{i,j}^N \cdot \mathbf{u}_{i,j}^N - \overline{\varphi}_{i,j}^0 \cdot \mathbf{u}_{i,j}^0 \right) \Delta x \Delta y - \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\overline{\varphi}_{i,j}^{n+1} - \overline{\varphi}_{i,j}^n}{\Delta t} \cdot \mathbf{u}_{i,j}^{n+1} \Delta t \Delta x \Delta y.$$

As φ is compactly supported in $[0, T)$ we have $\overline{\varphi}_{i,j}^N = 0, \forall i \in I, j \in J$ for a large enough N (i.e. for small enough Δt). Hence (using Fubini's theorem):

$$\int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \cdot \varphi \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} \, dxdt \, dP(\omega)$$

$$= \quad -\sum_{\substack{i \in I \\ j \in J}} \overline{\varphi}_{i,j}^0 \cdot \int_\Omega \mathbf{u}_{i,j}^0 \, dP(\omega) \Delta x \Delta y$$

$$- \quad \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\overline{\varphi}_{i,j}^{n+1} - \overline{\varphi}_{i,j}^n}{\Delta t} \cdot \int_\Omega \mathbf{u}_{i,j}^{n+1} \, dP(\omega) \Delta t \Delta x \Delta y$$

$$= \quad -\int_{\mathcal{D}} \varphi^h(x,0) \cdot \left\langle \nu_{(x,0)}^h, \mathrm{id} \right\rangle dx$$

$$- \quad \int_{\mathcal{D} \times [0,T)} \Delta_t \varphi^h(x,t) \cdot \left\langle \nu_{(x,t)}^h, \mathrm{id} \right\rangle dxdt.$$

Thus, using the weak*-convergence and the consistency with the initial data, we obtain

$$\int_{\Omega \times \mathcal{D} \times [0,T)} \sum_{\substack{i \in I \\ j \in J}} \sum_{n=0}^{N-1} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} \varphi \mathbf{1}_{\mathfrak{C}_{i,j} \times \mathfrak{I}^n} \, dxdt \, dP(\omega)$$

$$\overset{h \to 0}{\to} -\int_{[0,T)} \left\langle \left\langle \nu(x,t), \mathrm{id} \right\rangle, \varphi_t \right\rangle dt - \left\langle \left\langle \sigma(x), \mathrm{id} \right\rangle, \varphi(x,0) \right\rangle.$$

- The pressure term is tested against discretely divergence-free vector field, hence the term drops upon integration.
- Let us now consider the convection term. We notice that the divergence for the flux form is considered as an edge-centred sequence. Thus, integration by parts with a cell-centred quantity leads to a forward gradient. The

convective term becomes:

$$-\int_{\Omega\times\mathcal{D}\times[0,T)}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{C}_{i,j}^{n+1/2}\cdot\varphi\mathbf{1}_{\mathcal{D}\times[t^n,t^{n+1}]}dxdt\,dP(\omega)$$

$$=-\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{C}_{i,j}^{n+1/2}\cdot\overline{\varphi}_{i,j}^n\Delta x\Delta y\Delta t\,dP(\omega)$$

$$=\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{F}_{i+1/2,j+1/2}^{n+1/2}:\operatorname{grad}_+^h\overline{\varphi}_{i,j}^n\Delta x\Delta y\Delta t\,dP(\omega)$$

$$=\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{u}^n\otimes\mathbf{u}^n:\operatorname{grad}_+^h\overline{\varphi}_{i,j}^n\Delta x\Delta y\Delta t\,dP(\omega)+\left\|R:\operatorname{grad}_+^h\varphi^h\right\|_1$$

$$=\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{f}(\mathbf{u}^h(x_{i,j},t^n)):\operatorname{grad}_+^h\overline{\varphi}^h(x_{i,j},t^n)\Delta x\Delta y\Delta t\,dP(\omega)$$

$$=\int_{\mathcal{D}\times[0,T)}\left\langle\nu_{(x,t)}^h,\mathbf{f}\right\rangle:\operatorname{grad}_+^h\overline{\varphi}_{i,j}^n dxdt$$

$$\overset{h\to0}{\to}\int_{\mathcal{D}\times[0,T)}\left\langle\nu_{(x,t)},\mathbf{f}\right\rangle:\nabla\varphi dxdt,$$

using consistency, the weak*-convergence of the Young measure (by Young's theorem), Fubini's theorem, and the fact that $\varphi\in C_0^\infty$.

• For the dissipation term, consider:

$$\int_{\Omega\times\mathcal{D}\times[0,T)}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\mathbf{D}\overline{\mathbf{u}}^{n+1/2}\cdot\varphi\mathbf{1}_{\mathfrak{C}_{i,j}\times\mathfrak{I}^n}dxdt\,dP(\omega).$$

The first term can be bounded in the following way (assuming boundedness of the coefficients $c_{i,j}$ of the diffusion):

$$\left|\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\left[\!\left[(\mathbf{u}^{*n+1}+\mathbf{u}^n)\right]\!\right]^2\cdot\operatorname{grad}^h\varphi_{i,j}^n\Delta x\Delta y\Delta t\,dP(\omega)\right|$$

$$\leq\int_{\Omega}\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\left|\left[\!\left[(\mathbf{u}^{*n+1}+\mathbf{u}^n)\right]\!\right]\right|^2\left|\operatorname{grad}^h\varphi_{i,j}^n\right|\Delta x\Delta y\Delta t\,dP(\omega)$$

$$\leq\sup_{\Omega}\left(\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\left(\left|\left[\!\left[(\mathbf{u}^{*n+1}+\mathbf{u}^n)\right]\!\right]\right|^3\right)\cdot\sum_{\substack{i\in I\\j\in J}}\sum_{n=0}^{N-1}\left(|\operatorname{grad}^h\varphi_{i,j}^n|^3\right)\right)^{\frac{1}{3}}(\Delta x\Delta y\Delta t)^{2/3}$$

$$\overset{h\to0}{\to}0,$$

by the time total variation limit and smoothness of φ. Having space bounds on the averaged state it follows immediately that the term goes to zero.

- The divergence-free constraint is treated analogously.

As $h \to 0$, the approximate measure generated by the scheme converges to a measure valued solution. $\hfill\square$

Given the previous theorem, we have a tool for the consistent approximation of measure valued solutions from a measure as initial data. For an atomic initial data (i.e. $\sigma_0 = \delta_{\mathbf{u}_0}$), we can define an analogous algorithm.

The algorithm is the following (cf. also [15, 19]) and uses a Monte-Carlo sampling procedure:

(1) set a small discretization parameter $\delta > 0$, and create a new random initial data $\mathbf{u}_{\delta,0}(\omega; x)$, adding a random perturbation of size δ to the initial data $\mathbf{u}_0(x)$;

(2) draw $M > 0$ i.i.d. samples from the random field arising from the perturbed $\mathbf{u}_{\delta,0}(\omega; x)$;

(3) evolve each sample, individually, using (4), after choosing suitable space-time discretization parameters;

(4) compute mean, variance and other quantities of interest form the samples.

4. Numerical experiments

The following computations are performed on the Cray XC40 Piz Dora of the Swiss National Supercomputing Center and on the ETH Zürich cluster Euler. The code was developed in C++ and exploits the parallel framework PETSc [3, 4] to achieve efficient load balancing on distributed domains.

4.1. **Vortex patch.** As first set of experiments for computation of measure valued solutions, we consider a vortex patch. We call vortex patch an initial data whose vorticity takes the form of an indicator function of a set. The vorticity is bounded and weak solutions with vortex patches as initial data are unique and well defined in 2D and with periodic boundary conditions [26].

We perform experiments in the domain $[0,1)^2$ augmented with periodic boundary conditions. We consider a random initial data given by a random field for the vorticity $\eta := \nabla \times \mathbf{u}$. For a radius $r > 0$ and a perturbation parameter $\delta > 0$, the random "vortex-patch" will be $\eta_{0,\delta}(\omega; x, y) := \mathbb{1}_{S_\delta(\omega)}$, where

$$S_\delta(\omega) := \{(r\cos\theta, r\sin\theta) \in [0,1)^2 \mid r < r_0 + f_\delta(\omega; \theta), \theta \in [0, 2\pi)\},$$

(a perturbed ball of radius r_0). We define a random perturbation function f_δ as

$$f_\delta : [-1,1]^{2M} \times [0, 2\pi) \to \mathbb{R},$$

$$f_\delta(\omega, \theta) := \sum_{m=0}^{M} \frac{\delta}{S} \omega_{2m} \sin(2\pi(\omega_{2m+1} + \theta)), \text{ with } \sum_{m=0}^{M} \omega_{2m}^2 =: S^2.$$

We choose the final time $T = 1$ and $M = 20$. The initial velocity $\mathbf{u}_0$ is reconstructed from the vorticity η_0.

In order to test our simulation, we perform $M = 400$ i.i.d. draws for $\omega \sim \mathcal{U}(-1, 1)$. We use the Monte-Carlo estimator to approximate the mean and variance (cf. Figure 1) of the underlying random field for the velocity and the vorticity.

Numerical evidence indicates that, as $\delta \to 0$, the approximate measure valued solution converges towards a limit measure (cf. Figure 2). Moreover, the perturbed measure valued solution converges towards an atomic solution (1): i.e. towards the
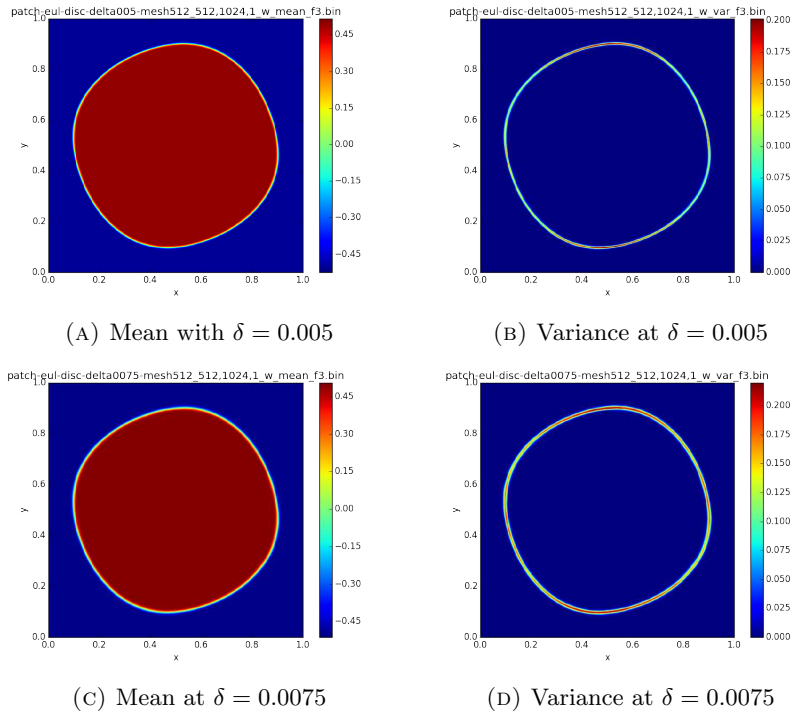
(A) Mean with $\delta = 0.005$

(B) Variance at $\delta = 0.005$

(C) Mean at $\delta = 0.0075$

(D) Variance at $\delta = 0.0075$

FIGURE 1. Color map of mean and variance of the vorticity at time $T = 1$ for two different perturbation sizes δ.

{fig:patch-map}

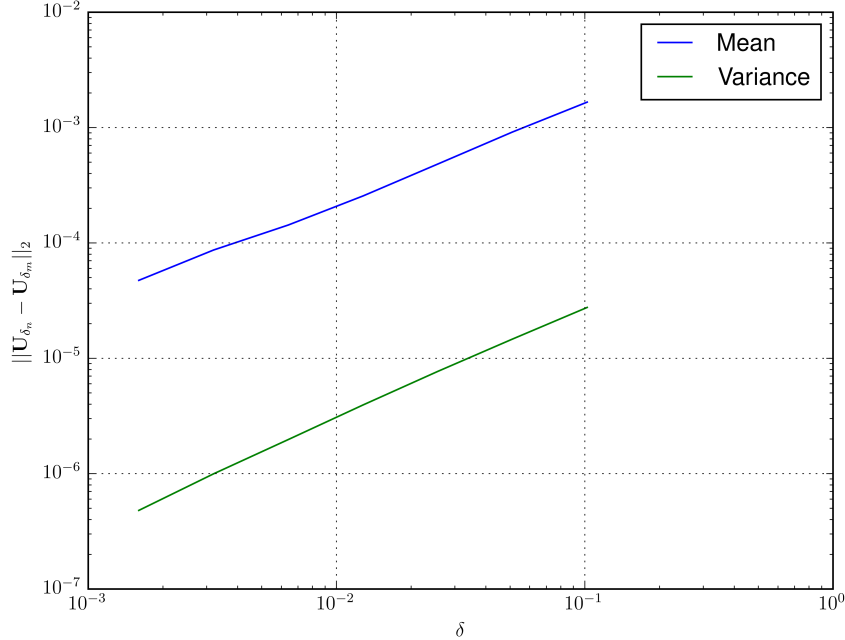Dirac measure of the indicator function of the ball of radius r_0 (the initial data with $\delta = 0$).

FIGURE 2. L^2-norm of the error of the mean and variance of the
velocity at time $T = 1$ with different values of δ, for the vortex-
patch initial data w.r.t. to a reference solution with $\delta = 10e - 5$.

{fig:patch-conf}

(A) $\delta = 0.01$ (B) $\delta = 0.0075$ (C) $\delta = 0.005$ (D) $\delta = 0.0025$ (E) $\delta = 0.001$
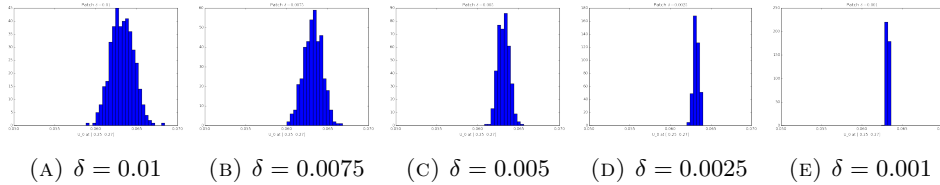
FIGURE 3. Histograms of the vorticity at the point $(0.8, 0.8)$ and
final time $T = 1$ for the perturbed vortex patch.

{fig:patch-histo}

4.2. **Perturbed vortex sheet.** We now consider a so-called vortex-sheet. The
vortex-sheet is a standard benchmark commonly used in engineering. Contrary to
the vortex-patch, the vortex-sheet is ill-defined and present large instabilities for
small times. The initial data for the velocity is

$$\mathbf{u}_0(x, y) := (H(y), 0),$$

where $H(y) = 1$ for $y > 0$ and $H(y) = -1$ for $y \leq 0$. This initial data is a steady
state for the Euler equations.

We consider a small random perturbation of this initial state. To this end, we
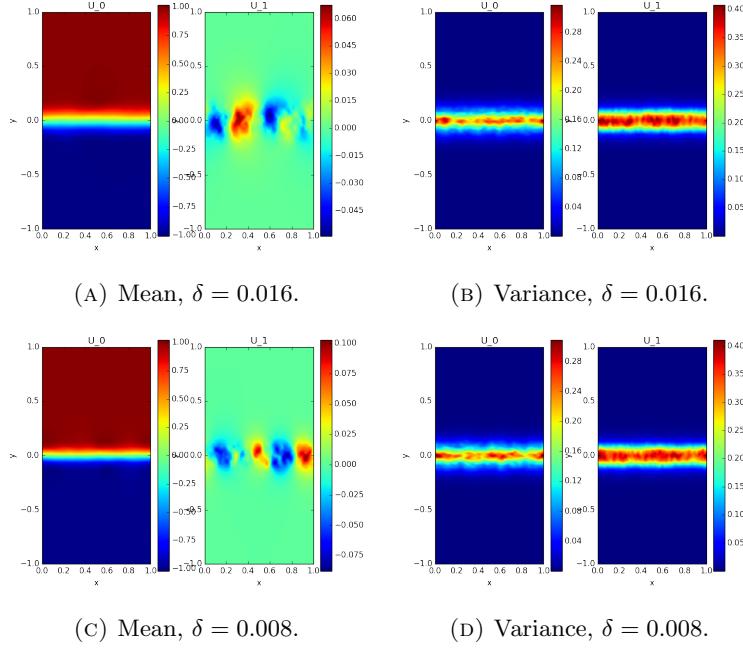define a perturbation by setting the initial data for the velocity to be the random

(A) Mean, $\delta = 0.016$.   (B) Variance, $\delta = 0.016$.

(C) Mean, $\delta = 0.008$.   (D) Variance, $\delta = 0.008$.

FIGURE 4. Color map of the mean and the variance of both components of the velocity $\mathbf{u}$, for different values of δ at time $T = 1$, for a fixed mesh resolution of 512×1024.

{fig:sl-mav}

field:

$$\mathbf{u}_0^\delta(\omega; x, y) := \mathcal{P}(\mathbf{u}_0(y + f_\delta(\omega; 2\pi x)), 0), \tag{22}$$

where f_δ is the same perturbation function used in the case of the vortex patch. Here $\mathcal{P}$ denotes the Leray projection. As boundary conditions, we impose periodic condition in the x-direction and wall boundaries in the y-direction (where $\mathbf{u} = 0$). We choose, as final time, $T = 1$ and as domain $[0,1] \times [-1,1]$ to ensure the turbulence zone does not reach the wall boundaries. We perform experiments with a random space of dimension $M = 20$. We compute the sample mean and the sample variance of the ensemble (cf. Figure 4).

The error of the mean and of the variance of the approximate solution converges to zero for any time (cf. Figure 5), even if the individual samples do not. Moreover, looking at the distributions of the components of the velocity at fixed points in time and space, the variance does not converge to zero and, in fact, the limit measure appear to be non-atomic (cf. Figure 6).
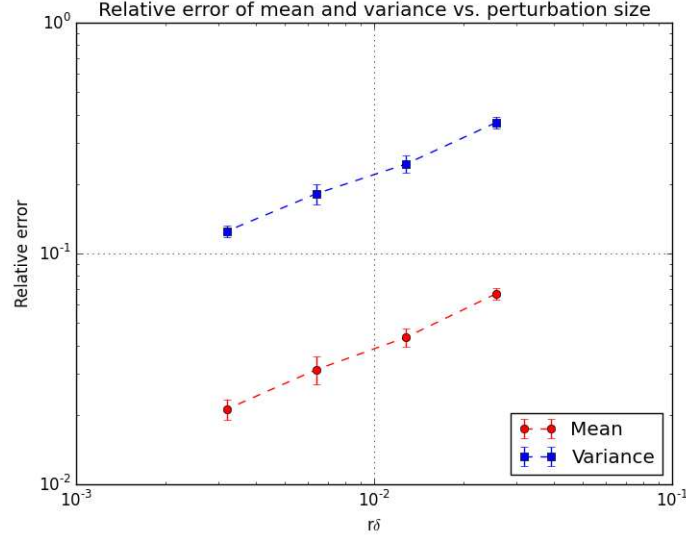
FIGURE 5. L^2-error of the mean and variance for the perturbed
vortex-sheet (with root mean square error) as the perturbation size
δ goes to zero. The error is computed against a reference pertur-
bation of size $\delta = 10e - 4$.

{fig:mavconv_sl_eul}

(A) $\delta = 0.0032$ (B) $\delta = 0.0016$ (C) $\delta = 0.0008$ (D) $\delta = 0.0004$
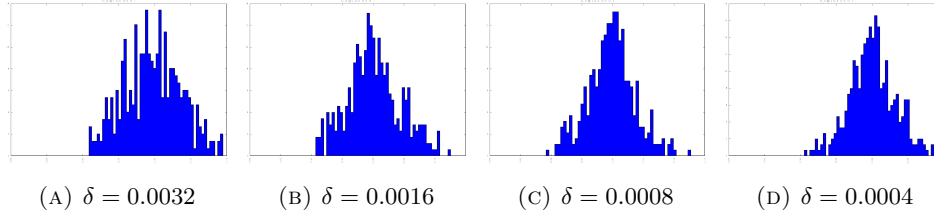
FIGURE 6. Histograms for the shear-layer at the point $p = (0.5, 0.5)$ for the first component of the velocity and for different
perturbation sizes (from $\delta = 0.0016$ to 0.0128) at the time $T = 1$.

{sl:histo}

5. CONCLUSION

In this paper, we presented a fully discrete Monte Carlo finite difference projec-
tion method for the approximation of measure valued solutions of the incompressible
Euler equations. We showed the consistency and weak* convergence of our scheme.
Those properties are derived from the interesting stability features of our scheme.

Numerical experiments demonstrate the difference between vortex-patches (well
defined in the standard weak sense) and vortex-sheets: the first appear to converge
to atomic measure valued solutions (i.e. to standard weak solutions), whilst the sec-
ond appear to converge to non-atomic weak solutions. Out findings strongly suggest
that some initial data, even if atomic, may generate measure valued solutions that
are non-atomic. Our results are consistent with the ones illustrated in [19].

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References

[1] A. S. Almgren, J. B. Bell, and W. G. Szymczak, *A Numerical Method for the Incompressible Navier–Stokes Equations Based on an Approximate Projection*, SIAM J. Sci. Comput., 17 (1996), pp. 358–369.

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, vol. 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg, 2011.

[3] S. Balay, J. Brown, K. Buschelman, V. Eijkhout, W. D. Gropp, D. Kaushik, M. G. Knepley, L. C. McInnes, B. F. Smith, and H. Zhang, *PETSc Users Manual*, Tech. Rep. ANL-95/11 - Revision 3.4, Argonne National Laboratory, 2013.

[4] S. Balay, W. D. Gropp, L. C. McInnes, and B. F. Smith, *Efficient Management of Parallelism in Object Oriented Numerical Software Libraries*, in Modern Software Tools in Scientific Computing, E. Arge, A. M. Bruaset, and H. P. Langtangen, eds., Birkhäuser Press, 1997, pp. 163–202.

[5] C. Bardos and E. Tadmor, *Stability and spectral convergence of Fourier method for nonlinear problems: on the shortcomings of the 2/3 de-aliasing method*, Numerische Mathematik, 129 (2015), p. 749.

[6] J. B. Bell, P. Colella, and H. M. Glaz, *A second-order projection method for the incompressible navier-stokes equations*, Journal of Computational Physics, 85 (1989), pp. 257–283.

[7] Y. Brenier, C. D. Lellis, and L. Székelyhidi, Jr, *Weak-strong uniqueness for measure-valued solutions*, Dec. 2009.

[8] D. Chae, *The vanishing viscosity limit of statistical solutions of the Navier-Stokes equations. II. The general case*, Journal of Mathematical Analysis and Applications, 155 (1991), pp. 460–484.

[9] A. J. Chorin, *Numerical solution of the Navier-Stokes Equations*, Math. Comp., 22 (1968), pp. 745–762.

[10] J.-M. Delort, *Existence de mappes de tourbillon en dimension deux*, Journal of the American Mathematical Society, 4 (1991), pp. 553–586.

[11] R. J. DiPerna, *Measure valued solutions to conservation laws*, Arch. Rational Mech. Anal., 88 (1985), pp. 223–270.

[12] R. J. Diperna and A. Majda, *Reduced Hausdorff Dimension and Concentration-Cancellation for Two Dimensional Incompressible Flow*, Journal of the American Mathematical Society, 1 (1988), pp. 59–95.

[13] R. J. DiPerna and A. J. Majda, *Concentrations in regularizations for 2-D incompressible flow*, Communications on Pure and Applied Mathematics, 40 (1987), pp. 301–345.

[14] R. J. DiPerna and A. J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Communications in Mathematical Physics, 108 (1987), pp. 667–689.

[15] U. S. Fjordholm, R. Käppeli, S. Mishra, and E. Tadmor, *Construction of approximate entropy measure valued solutions for hyperbolic systems of conservation laws*, arXiv preprint arXiv:1402.0909, (2014).

[16] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations: theory and algorithms*, NASA STI/Recon Technical Report A, 87 (1986), p. 52227.

[17] R. Glowinski, *Finite element methods for incompressible viscous flow*, Handbook of numerical analysis, 9 (2003), pp. 3–1176.

[18] A. Krzhivitski and O. A. Ladyzhenskaya, *A grid method for the Navier-Stokes equations*, Soviet Physics Dokl., 11 (1966), pp. 212–213.

[19] S. Lanthaler and S. Mishra, *Computation of measure-valued solutions for the incompressible Euler equations*, Nov. 2014.

[20] C. Lellis and L. Székelyhidi, *On Admissibility Criteria for Weak Solutions of the Euler Equations*, Archive for Rational Mechanics and Analysis, 195 (2009), p. 225.

[21] M. C. Lopes Filho, J. Lowengrub, H. J. Nussenzveig Lopes, and Y. Zheng, *Numerical evidence of nonuniqueness in the evolution of vortex sheets*, ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique, 40 (2006), pp. 225–237.

[22] A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.

[23] V. Scheffer, *An inviscid flow with compact support in space-time*, The Journal of Geometric Analysis, 3 (1993), pp. 343–401.

[24] A. Shnirelman, *On the nonuniqueness of weak solution of the Euler equation*, Communications on Pure and Applied Mathematics, 50 (1997), pp. 1261–1286.

[25] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, vol. 39 of Res. Notes in Math., Pitman, Boston, Mass.-London, 1979, pp. 136–212.

[26] V. Yudovich, *Non-stationary flow of an ideal incompressible liquid*, USSR Computational Mathematics and Mathematical Physics, 3 (1963), pp. 1407–1456.

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