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Numerical methods for conservation laws with discontinuous coefficients

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Abstract

Conservation laws with discontinuous coefficients, such as fluxes and source terms, arise in a large number of problems in physics and engineering. We review some recent developments in the theory and numerical methods for these problems. The well-posedness theory for one-dimensional scalar conservation laws is briefly described, with a particular focus on the existence of infinitely many L^1 stable semi-groups of solutions. We also present both aligned and staggered versions of finite volume methods to approximate systems of conservation laws with discontinuous flux. We conclude with some illustrative numerical experiments and a set of open questions.

1 Introduction

Systems of balance laws are nonlinear partial differential equations of the generic form

$$\partial_t \mathbf{U} + \nabla_x \cdot \mathbf{F}(\mathbf{U}) = 0. \tag{1.1a}$$

$$\mathbf{U}(x,0) = \mathbf{U}_0(x). \tag{1.1b}$$

Here, the unknown $\mathbf{U} = \mathbf{U}(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$ is the vector of *conserved variables* and $\mathbf{F} = (\mathbf{F}^1, \ldots, \mathbf{F}^d) : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times d}$ is the *flux function*. We denote $\mathbb{R}_+ := [0,\infty)$. . Here, $\mathbf{U}_0$ denotes the prescribed initial data. Furthermore, the system needs to be supplemented with suitable boundary conditions.

The system (1.1) is termed *hyperbolic* if the flux Jacobian matrix has real eigenvalues [33]. Hyperbolic systems of conservation (balance) laws arise in a wide variety of models in physics and engineering. Prototypical examples include the compressible Euler equations of gas dynamics, the shallow water equations of oceanography, the magneto-hydrodynamics (MHD) equations of plasma physics and the equations of nonlinear elasticity [33].

It is well known that solutions of (1.1) develop discontinuities in finite time, even when the initial data is smooth [33]. Hence, solutions of (1.1) are sought (and computed) in the sense of distributions. These weak solutions are not necessarily unique. Additional admissibility criteria or *entropy conditions* need to be imposed in order to select physically relevant solutions. The well-posedness of entropy solutions has been established for multi-dimensional scalar conservation laws ($N = 1$) and for one-dimensional systems ($d = 1$).

In the absence of explicit solution formulas for these nonlinear problems, numerical methods have emerged as the main tools in the study of conservation laws and in their applications in science and engineering. A wide variety of numerical methods for approximating (1.1) are currently available. These include the finite volume, conservative finite difference, discontinuous Galerkin finite element, continuous Galerkin residual distribution schemes and spectral viscosity methods. Finite volume methods, based on approximate Riemann solvers, are very popular. Higher order spatial accuracy results from non-oscillatory piecewise polynomial reconstruction procedures such as TVD, ENO and WENO methods. High-order temporal accuracy can be achieved with strong stability preserving (SSP) Runge-Kutta methods. The reader is referred to other chapters of this handbook for a detailed survey of available numerical methods.

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1.1 Conservation laws with coefficients.

Although the form (1.1) suffices for many models of interest, there are a large number of problems in science and engineering, which need to modeled by the more general system of balance laws:

$$\begin{aligned} \partial_t \mathbf{U} + \nabla_x \cdot \mathbf{F}(\mathbf{k}(x,t), \mathbf{U}) &= \mathbf{S}(x,t,\mathbf{U}), \\ \mathbf{U}(x,0) &= \mathbf{U}_0(x). \end{aligned} \tag{1.2}$$

In addition to the vector of unknowns $\mathbf{U}$ and flux function $\mathbf{F}$, we also need $\mathbf{k} = (\mathbf{k}^1, \ldots, \mathbf{k}^d) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{N \times d}$, which is a spatio-temporal coefficient and $\mathbf{S} : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a source (sink) term. We term (1.2) as a *conservation law with a coefficient*.

It turns out that standard theoretical tools and numerical methods can be readily adapted to (1.2), as long as the spatio-temporal coefficient and the source term are smooth (atleast C^1) functions of their arguments. However, in many models of interest, the coefficient and the source term can be *rough* i.e, Hölder continuous or even *discontinuous*. Such *conservation laws with discontinuous coefficients* necessitate the use of novel theoretical tools and the design of new numerical methods. The development of theory and numerical methods for conservation laws with discontinuous coefficients has witnessed a large amount of research activity in recent years. The main aim of this chapter is to review these developments in a condensed and coherent manner.

The rest of this chapter is organized as follows: in section 2, we provide some motivating examples for the study of conservation laws with discontinuous coefficients. Available theory, particularly for the scalar case, is presented in section 3. We describe some of the widely used numerical methods for approximating conservation laws with discontinuous coefficients in section 4. Numerical experiments are presented in section 5.

2 Motivating examples

In this section, we will present many examples and models, that have motivated the study of conservation laws with discontinuous coefficients.

2.1 Multi-phase flows in porous media.

Water flooding into a oil reservoir is modeled in terms of the flow of two phases (oil and water) in a porous medium [29]. Using the Darcy's law and assuming that there is no capillary pressure and gravity, one can model the water saturation S and the pressure p in terms of the following system of PDEs:

$$\begin{aligned} S_t + \nabla_x \cdot (f(S)v) &= 0, \\ v &= -\lambda_T \nabla_x p, \\ \nabla_x \cdot v &= 0. \end{aligned} \tag{2.1}$$

Here, $f(S) = \frac{\lambda_w(S)}{\lambda_T(S)}$ is the fractional flow function, $\lambda_T = \lambda_w + \lambda_o$ is the total mobility and $\lambda_{w,o} = \lambda_{w,o}(x,S)$ are the phase mobilities, which are specified in terms of the absolute rock permeability and the relative permeabilities of the phases [29]. Note that (2.1) is a system coupling a hyperbolic equation for the saturation (with a non-local flux) and an elliptic equation for the pressure. The one-dimensional version of (2.1) (including gravity but still excluding capillary pressure) is given by [29],

$$S_t + \nabla_x f(k,S) = 0, \quad f(k,S) = \frac{k_w^r(S)(v + g(\rho_w - \rho_o)k k_o^r(S))}{k_w^r(S) + k_o^r(S)}. \tag{2.2}$$

Here, S still denotes the water saturation, v the total flow rate (determined by boundary conditions), $k = k(x)$, the rock permeability, $k^r_{w,o}$, the phase relative permeabilities and the phase densities are denoted by $\rho_{w,o}$. The acceleration due to gravity is denoted by g. Note that the spatial dependence of the coefficient in the fractional flow function in both (2.1) and (2.2) is on account of the variations in the rock permeability. It is well known [31] that rocks in most geological formations of interest are highly heterogeneous and characterized by a large variation in their properties, particularly in their permeability. This often results in rock permeabilities that are at best Hölder continuous. Moreover, for any layered medium, the absolute permeability of the rock is discontinuous across the layers [57, 43] and references therein. Thus, two-phase flows in a heterogeneous and layered porous medium was the original motivation for the study of conservation laws with discontinuous fluxes (coefficients) [43].

Moreover, many models in petroleum engineering consider the flow of multiple phases [29, 31]. As an example, consider tertiary oil recovery that involves the flow of oil, water and a gas (or polymer) in a porous medium. In this case, the resulting PDEs consist of a 2×2 hyperbolic system of conservation laws, coupled with suitable elliptic equations for the pressures [47, 51, 30] and references therein. Discontinuities in the rock permeability will result in discontinuous fluxes for the underlying system of conservation laws.

2.2 Traffic flow

The popular Lighthill-Whitham-Richards (LWR) model for highway traffic results in the following one-dimensional scalar conservation law,

$$\rho_t + (\rho v(\rho))_x = 0. \tag{2.3}$$

Here ρ is the density of vehicles on the highway and $v(\rho)$ is the velocity flux function. Different models have proposed different forms of the flux v. The most commonly used form is the so-called *linear model*,

$$v(\rho) := v_{\max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

Here $v_{\max}$ is the maximum flow velocity (for instance the speed limit) and $\rho_{\max}$ is the maximum density that is related to the highway carrying capacity. As was first observed by Mochon [71], both these parameters are spatially dependent and one can experience abrupt variations (discontinuities) in them. For instance, the maximum velocity can be dramatically reduced by highway topography and the maximum density by construction work on a part of the highway. Furthermore, heavy rainfall or fog can affect the maximum velocity in a discontinuous and time-dependent manner. Thus, traffic flow provides a canonical example of conservation laws with discontinuous fluxes.

2.3 Other examples of scalar conservation laws with discontinuous flux.

A clarifier-thickener unit is an industrial unit widely used in waste water treatment plants and other chemical engineering scenarios. The aim of this machine is to separate a fluid-solid mixture by continuous sedimentation and provide a clear liquid at the top and a high-concentration of solids at the bottom, see [35, 34, 21] and references therein. Under idealized assumptions and using the batch sedimentation model of Kynch [21], one can describe continuous sedimentation inside the clarifier-thickener unit in terms of a scalar conservation law with a flux function that is discontinuous at three locations, namely the discharge, feed and overflow outlets, see [35, 21] and references therein for a detailed description of this model. The clarifier- thickener unit has also served as one of classical motivating examples for the study of scalar conservation laws with discontinuous flux.

A very interesting example of scalar conservation laws with discontinuous flux is provided by the industrial process of ion etching, that is heavily used in the fabrication of semi-conductor devices [75]. The unknown of interest is the slope of the free surface and the flux function is the so-called sputtering yield, which is modeled empirically. As the etching process involves two very different materials, the semi-conductor and the photo resist, the corresponding sputtering yields are discontinuous in space as well as in time, due to the motion of the free surface. Hence, we obtain a scalar conservation law with discontinuous time-dependent coefficient, coupled with an equation for a moving boundary. Another example of scalar conservation laws with a discontinuous flux function is provided by hydrodynamic limits of interacting particle systems [32] and references therein.

2.4 Wave propagation in heterogenous media

The propagation of acoustic waves in the sub-surface is the key element in seismic imaging . We can model this propagation in terms of the (linear) wave equation, which can also written in the first-order form as a system of (linear) conservation laws. The permeability of the sub-surface determines the wave speed and is dependent on the material properties of the medium. Most media of interest are highly heterogeneous, even uncertain, [70] and references therein. Hence, the resulting conservation law has rough, typically Hölder continuous fluxes. Moreover, many media of interest are layered. Consequently, the permeability is a discontinuous function in space and we obtain a variable coefficient linear system of conservation laws with discontinuous fluxes, [70] and references therein.

Similarly, the propagation of elastic waves in a heterogenous medium results in a (linear) system of conservation laws with either rough (merely Hölder continuous) or discontinuous fluxes [58] and references therein. In some cases, one needs to consider the propagation of nonlinear waves, such as shock waves, in a highly heterogenous medium. Examples include shock wave lithotripsy [37]. Such problems can result in multi-dimensional systems of conservation laws with discontinuous flux functions.

2.5 Systems of conservation laws with singular source terms.

A large set of examples for conservation laws with discontinuous coefficients is provided by balance laws with singular source terms. A prototypical example is provided by the single layer shallow water equations. In one space dimensions, they are of the form,

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right) &= -ghb_x. \end{aligned} \tag{2.4}$$

Here h is the height of the free surface, u is the flow velocity and $b = b(x)$ is a function that models bottom topography. The topography can be discontinuous, for instance if b models a steep underground slope (or a underground mountain chain). An added difficulty in this case is provided by the fact that the product hb_x is a product of distributions and needs to interpreted in a suitable manner. Other examples of conservation laws with singular source terms are provided by shallow-water equations modeling flow in a channel with a discontinuous channel geometry [17] and references therein, or the compressible Euler equations modeling flows in a nozzle with discontinuous nozzle geometry, [61, 60] and references therein.

2.6 Flows as perturbations of discontinuous steady states

Consider a flow that is modeled by the generic system of conservation laws (1.1). Assume that there exist a class of steady states $\hat{\mathbf{U}}$ that are of interest. It is natural to consider the perturbation around this steady state i.e $\mathbf{U} = \bar{\mathbf{U}} + \hat{\mathbf{U}}$. This perturbation is not necessarily small. The corresponding equations for the perturbed state $\bar{\mathbf{U}}$ are given by

$$\bar{\mathbf{U}}_t + \nabla_x \cdot (\mathbf{F}(\bar{\mathbf{U}} + \hat{\mathbf{U}})) = 0. \tag{2.5}$$

Clearly, (2.5) is an example of a conservation law with a coefficient, i.e, (1.2) with $\mathbf{k} = \hat{\mathbf{U}}$ and $\mathbf{F}(\mathbf{k}, \mathbf{U}) = \mathbf{F}(\mathbf{k}+\mathbf{U})$. In many problems of interest, the underlying steady state can be discontinuous. Thus, (2.5) will result in a conservation law with a discontinuous flux. A concrete example is provided by wave propagation in the outer solar atmosphere, [41] and references therein. In this article, waves in the outer solar atmosphere are modeled as perturbations of a steady state of the stratified Magneto-Hydro Dynamics (MHD) equations. The steady state of interest involves a discontinuity in the pressure and the temperature on account of the presence of the transition layer between the solar chromosphere and corona (across which the temperature jumps by two orders of magnitude).

In addition to the above examples, systems of conservation laws with discontinuous fluxes, also arise when St. Venant systems are used to model blood flow [38].

3 A brief review of available theoretical results

In this section, we will provide a very brief review of theoretical results for conservation laws with discontinuous coefficients. For simplicity of exposition, we focus on the simplest case of a scalar, one-dimensional conservation law with spatially dependent discontinuous flux,

$$\begin{aligned} u_t + f(k(x), u)_x &= 0. \\ u(x, 0) &= u_0(x). \end{aligned} \tag{3.1}$$

Here u, f denote the (scalar) unknown and flux function, respectively, and k is the spatially dependent coefficient. We assume that $k \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and has finitely many points of discontinuity. It is assumed to be C^1 outside this set of discontinuities. Even though (3.1) is the simplest example of conservation laws with discontinuous coefficients, it does model quite a few interesting applications such as two-phase flows in a heterogeneous porous medium, the action of the clarifier-thickener unit and traffic flows on highways with changing surface conditions, as outlined in the previous section. Moreover, the analysis of (3.1) is already quite involved on account of several key difficulties, some of which are outlined below,

- We note that the scalar conservation law (3.1) can be rewritten in terms of the following 2×2 system of conservation laws,
$$\begin{aligned} u_t + f(k,u)_x &= 0, \\ k_t &= 0, \\ (u,k)|_{t=0} &= (u_0(x), k(x))\,. \end{aligned} \tag{3.2}$$
It is straightforward to check that the Jacobian matrix of the above system (3.2) has two real eigenvalues f_u and 0. Hence, (3.2) is a 2×2 hyperbolic system of conservation laws. However, the system fails to be *strictly hyperbolic* around $f_u = 0$ (or critical points of the flux). It is well known that non-strictly hyperbolic or resonant systems are not covered under the standard theory for one-dimensional systems of conservation laws [77] and one needs to devise special tools for their study.

- *Generation of Oscillations;* Consider the following example. Let the flux in (3.1) be $f(k,u) = (1-H(x))u(1-u) + 2H(x)u(1-u)$, with initial data $u_0 \equiv 0.5$ and H denoting the standard Heaviside function,
$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$$
In this case, the only admissible weak solution is given by,
$$u(x,t) = \begin{cases} \frac{1}{2}, & x < 0, \\ \theta, & 0 < x < \sigma t \\ \frac{1}{2}, & x > \sigma t \end{cases} \tag{3.3}$$
Here, $\theta \in [0, 0.5]$ is a root of the quadratic equation $\theta(1-\theta) = \frac{1}{2}$ and the shock speed σ is determined by the classical Rankine-Hugoniot relation applied to the flux $f_R(u) = 2u(1-u)$. Note that the above solution satisfies mass conservation as the Rankine-Hugoniot condition at the discontinuous interface $x = 0$ is satisfied. It also satisfies the usual Lax-Oleinik entropy condition, away from the interface. Clearly, the total variation of the solution is greater than the initial total variation. Thus, a scalar conservation law with discontinuous flux can lead to an increase in variation, in complete contrast to the standard scalar conservation law, where the solution operator is total variation diminishing (TVD) in time.

- *Non-Uniqueness.* However (3.3) is not the only admissible solution for the preceding example of scalar conservation law with discontinuous flux. Following [3, 67], we let $A, B \in [0,1]$ such that $A \in [1/2, 1]$ and $B \in [0, \theta]$ such that $A(1-A) = 2B(1-B)$. Then the following,
$$u(x,t) = \begin{cases} \frac{1}{2}, & x < \sigma_L t, \\ A, & \sigma_L t < x < 0, \\ B, & 0 < x < \sigma_R t \\ \frac{1}{2}, & x > \sigma_T t \end{cases} \tag{3.4}$$
is also a weak solution of (3.1) that satisfies the Lax-Oleinik entropy condition, away from the interface $x = 0$. The shock speeds σ_L, σ_R are calculated using the standard Rankine-Hugoniot condition on the fluxes $f_L(u) = u(1-u)$ and $f_R(u)$, respectively. Such an admissible weak solution can be constructed for any pair (A, B), providing a considerable source of non-unique admissible solutions.

Given the above difficulties, many different notion of solutions for scalar conservation laws with discontinuous flux have been proposed in recent years. Given the limited space, we will focus on the most popular solution framework [3, 12, 13]. All notions agree on the following standard definition of weak solution of (3.1).

Definition 3.1. Weak Solution. A function $u \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$, such that $f(k, u) \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution of the scalar conservation law with discontinuous flux (3.1) if the following integral identity,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} u\varphi_t + f(k(x), u)\varphi_x dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x, 0)dx = 0, \tag{3.5}$$

holds for all test functions $\varphi \in C^1_c(\mathbb{R} \times \mathbb{R}_+)$. ■

If in addition, we assume that the set of discontinuities D of the coefficient k is finite and of the form $D = \{x_k\}_{k=1}^K$, and assume that the function u has traces at each of the points (x_k, t) for almost every $t \in \mathbb{R}_+$, then, it is straightforward to see that these trace values $u_k(^{\pm}(t)$ satisfy the interface Rankine-Hugoniot relation,

$$f(k(x_k^-), u^-(t)) = f(k(x_k^+), u^+(t)), \quad 1 \leqslant k \leqslant K. \tag{3.6}$$

Furthermore, all the proposed solution frameworks agree that one should impose the standard Kruzkhov entropy condition, away from the set of the discontinuities of the coefficient k. This leads to the following definition,

Definition 3.2. Interior entropy condition. A weak solution u of (3.1) satisfies the interior entropy condition if for all constants $c \in \mathbb{R}$, the following integral identity,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - c|\varphi_t + \text{sign}(u - c)\,(f(k(x), u) - f(k(x), c))\,\varphi_x dxdt + \int_{\mathbb{R}} |u_0(x) - c|\varphi(x, 0)dx \geqslant 0, \tag{3.7}$$

for all test functions $0 \leqslant \varphi \in C^1_c((\mathbb{R} \setminus D) \times \mathbb{R}_+)$. Note that the admissible test functions are supported outside the set of discontinuities D of the coefficient k. ■

This interior entropy condition amounts to requiring the standard Lax-Oleinik entropy conditions at jump discontinuities of the solution, away from the set of discontinuities.

However, the interior entropy condition does not suffice in guaranteeing uniqueness of solutions, as shown by example (3.4), presented above. Thus, one needs to also specify some admissibility criteria or entropy conditions at the *interfaces* i.e, points of discontinuity of the coefficient k.

For the sake of notational simplicity, we focus on a very concrete case, where $f(k, u) = (1 - H(x))g(u) + H(x)f(u)$, where H is the Heaviside function and the constituent fluxes (f, g) are such that there exist $s, S \in \mathbb{R}$ with $f(s) = g(s), f(S) = g(S)$. This assumption guarantees an L^∞ bound on the solution. Furthermore, we assume that both g and f have a single minimum and no maxima in the interval $[s, S]$. The minima of g and f are attained at the points θ_g and θ_f, respectively, see figure 1 for an illustration. These hypothesis on the fluxes are motivated by examples from two-phase flows in heterogeneous porous media [2, 3, 57, 68] and references therein. The case where g, f have exactly one maximum and no minima can be treated analogously.

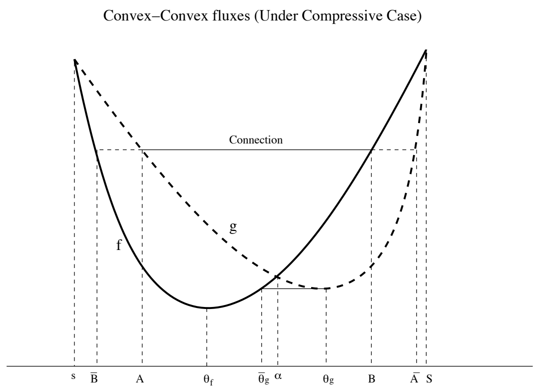


Figure 1: Left flux g and right flux f for a scalar conservation law (3.1) with flux $(1 - H(x))g(u) + H(x)f(u)$ (H being the Heaviside function). Three possible interface connections (A, B), (α, α) and $\theta_g, \bar{\theta}_g$ are also identified.

Following [3, 67], we define the following,

Definition 3.3. Interface connection. The pair (A, B) is termed an interface connection for the scalar conservation law with discontinuous flux if $A \in [s, \theta_g]$, $B \in [\theta_f, S]$ and $g(A) = f(B)$. ■

For every (A, B)-interface connection, we follow [3] and define the following interface entropy condition,

Definition 3.4. Interface entropy condition. Let u be a weak solution of the conservation law with discontinuous flux (3.1) such that the traces $u^{\pm}(t)$ at the flux interface $x = 0$, exist for almost every $t \in \mathbb{R}_+$. Furthermore, for every connection (A, B), the trace values $u^{\pm}$ satisfy,

$$\mathcal{I}_{AB}(t) := \mathrm{sign}(u^-(t) - A)(g(u^-(t)) - g(A)) - \mathrm{sign}(u^+(t) - B)(f(u^+(t)) - f(B)) \geqslant 0, \quad a.e\ t > 0. \tag{3.8}$$

■

An AB-entropy solution of the scalar conservation law with discontinuous flux (3.1) is a weak solution that satisfies both the interior as well as interface entropy conditions.

In [3], the authors proved that for every connection (A, B), the corresponding AB-entropy solution exists, is unique and forms a contractive semi-group in $L^1(\mathbb{R})$. Thus, there can be *infinitely many L^1-stable entropy solutions* for scalar conservation laws with discontinuous fluxes. This should be contrasted with the fact that the Kruzkhov entropy solution for scalar conservation laws is unique. Although, we only presented the concept of AB-entropy solutions for a specail case of scalar conservation law with discontinuous flux, namely with one spatial discontinuity in the flux and a convex flux geometry, this theory has been extended to the more general case of finitely many discontinuities and arbitrary flux geometries in [4, 5, 6, 7] and references therein.

Given the fact that any AB-entropy solution is stable, which of these solutions correspond to a physically admissible solution ? Many proposals have been made to single out the physically relevant solution. A very incomplete list includes the minimal jump condition of [43], the Γ-condition of [35, 34, 36], crossing conditions of [55] and references therein, the optimal entropy conditions of [3], the vanishing capillarity conditions of [57, 26, 27], the conditions of [76, 15, 42] and references therein. However, the emerging consensus in the community accepts the possibility that there is no unique physically relevant solution for scalar conservation laws with discontinuous flux. Instead, the relevant AB-connection depends on the physics of the underlying problem. In particular, the same PDEs model two-phase flows in heterogeneous porous medium and the action of the clarifier-thickener unit. However, the underlying physics might lead to different connections being chosen as physically relevant, as shown in [23, 24] and references therein. This situation is reminiscent of models that involve small-scale dependent shock waves such as non-strictly hyperbolic systems [63], non-conservative hyperbolic systems [28] and references therein, and boundary-value problems for systems of conservation laws [44, 69] and references therein.

The above entropy framework relies on the existence of traces and special admissibility conditions at each interface. An alternative was proposed in [16, 14, 12] and references therein, where a global entropy condition is proposed. The basis of this entropy condition is the presence of a rich family of steady states (stationary solutions) $\bar{u}$ of (3.1) that satisfy

$$f(k(x), \bar{u}_\alpha(x)) = \alpha, \quad \forall \alpha \in \mathbb{R}. \tag{3.9}$$

Note that each interface connection (A, B) gives rise to the following stationary solution,

$$\bar{u}_{AB}(x) = \begin{cases} A & \text{if } x < 0, \\ B & \text{if } x > 0. \end{cases} \tag{3.10}$$

Once a suitable family of stationary solutions is available, one can define the global entropy condition in terms of the following integral inequality,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - \bar{u}_\alpha(x)| \varphi_t + \mathrm{sign}(u - \bar{u}_\alpha(x)) \left(f(k(x), u) - \alpha \right) \varphi_x \, dx dt + \int_{\mathbb{R}} |u_0(x) - \bar{u}_\alpha(x)| \varphi(x, 0) dx \geqslant 0, \tag{3.11}$$

for all test functions $0 \leqslant \varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$.

One can follow the arguments of [14, 12] and prove that these entropy solutions are unique and stable in L^1. The equivalence of these approaches, the global with adapted entropies and the local, with interface entropy conditions, was shown in [12] for the special case of a single discontinuity of the coefficient.

The global modified Kruzkhov entropy condition is amenable for extensions to several space dimensions, such as in [73, 74, 72] and references therein.

Most of the theory for conservation laws with discontinuous coefficients is only available for the scalar case. Similar theoretical development for nonlinear systems with discontinuous flux, even in one-space dimension, awaits further research. A notable exception is provided by the results of [48, 49] that study compressible Euler flows in a pipe.

4 Numerical Schemes.

In this section, we will present some of the commonly used numerical methods for approximating conservation laws with discontinuous coefficients (1.2). For simplicity of notation and of exposition, we will focus on the one-dimensional version of conservation laws with spatially-dependent and discontinuous flux:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{k}(x), \mathbf{U})_x = 0. \tag{4.1}$$

We consider a computational domain $\Omega = (X_L, X_R)$ and assume that the coefficient $\mathbf{k} \in L^\infty(\Omega) \cap BV(\Omega)$. We discretize the computational domain Ω into $N+1$ points $x_j = X_L + j\Delta x$ with $\Delta x = 1/N$. Denoting, $x_{j+1/2} = (x_j + x_{j+1})/2$, we have discretized the computational domain into N equally sized intervals or *cells*, $C_j := [x_{j-1/2}, x_{j+1/2}]$. Note that we choose an uniform mesh size Δx only for the sake of simplicity. The following discussion is straightforward to extend to non-uniform grids. In fact, most realistic computations involve coefficients $\mathbf{k}$ with very strong non-uniformity and necessitate the use of non-uniform grids.

We start with the following simple, semi-discrete, first-order in space, finite volume (difference) scheme for approximating (4.1),

$$\frac{d}{dt}\mathbf{U}_j(t) + \frac{1}{\Delta x}\left(\hat{\mathbf{F}}_{j+1/2}(t) - \hat{\mathbf{F}}_{j-1/2}(t)\right) = 0, \quad 1 \leqslant j \leqslant N-1. \tag{4.2}$$

The ODE system (4.2) is initialized in terms of cell averages,

$$\mathbf{U}_j(0) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}_0(x)dx,$$

for a finite volume scheme or point values $\mathbf{U}_j(0) = \mathbf{U}_0(x_j)$ for a (conservative) finite difference scheme. We will only discuss finite volume schemes here.

We need to specify the numerical flux function $\hat{\mathbf{F}}_{j+1/2}$, for all j, in order to define the finite volume scheme (4.2). Two principal types of fluxes leading to two different sets of schemes have been proposed in the literature. We outline them below.

4.1 Aligned Schemes.

In aligned schemes, for instance those proposed in [3, 5, 2, 1] and references therein, the idea is to align the discretization of the unknown $\mathbf{U}$ and the flux coefficient $\mathbf{k}$. The aligned schemes are based on the following numerical flux (time-dependence is suppressed for notational convenience),

$$\begin{aligned} \hat{\mathbf{F}}_{j+1/2} &:= \hat{\mathbf{F}}\left(\mathbf{k}_j, \mathbf{U}_j, \mathbf{k}_{j+1}, \mathbf{U}_{j+1}\right), \\ \hat{\mathbf{F}}(\mathbf{k}, \mathbf{U}, \mathbf{k}, \mathbf{U}) &= \mathbf{F}\left(\mathbf{k}, \mathbf{U}\right). \end{aligned} \tag{4.3}$$

Here, the coefficient $\mathbf{k}$ is discretized in terms of its cell average,

$$\mathbf{k}_j = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{k}(x)dx.$$

Note that the second condition of (4.3) amounts to requiring consistency with the underlying flux function in (4.1).

The flux function $\hat{\mathbf{F}}\left(\mathbf{k}_L, \mathbf{U}_L, \mathbf{k}_R, \mathbf{U}_R\right)$is determined by using exact or approximate (local) solutions of the following Riemann problems,

$$\begin{aligned} \mathbf{U}_t + \mathbf{F}(\mathbf{k}, \mathbf{U})_x &= 0, \\ \mathbf{k}_t &= 0, \\ (\mathbf{U}(x,0), \mathbf{k}(x,0)) &= \begin{cases} (\mathbf{U}_L, \mathbf{k}_L), & \text{if } x < 0, \\ (\mathbf{U}_R, \mathbf{k}_R), & \text{if } x > 0. \end{cases} \end{aligned} \tag{4.4}$$

For scalar conservation laws, the above Riemann problem (4.4) can be solved analytically in a very large number of cases, see [3, 4, 5, 67] for a detailed description. In fact,the AB-entropy solution for the above Riemann problem can be constructed for any given interface connection (A, B). The resulting finite volume scheme is a Godunov scheme [45]. We can write the corresponding numerical flux in terms of a particularly simple formula in the case where the flux function in the scalar version of (4.4) is of the form, $\mathbf{F}(\mathbf{k}, \mathbf{U}) = f(k, u) = (1-H(x))g(u)+H(x)f(u)$, with H being the Heaviside function and g, f satisfying the hypothesis that they have exactly one minimum and no maxima. In this case, the numerical flux is given by,

$$\hat{\mathbf{F}}(\mathbf{k}_L, \mathbf{U}_L, \mathbf{k}_R, \mathbf{U}_R) = \hat{F}(k_L, u_L, k_R, u_R) = \max\left(G(u_L, A), F(B, u_R)\right). \tag{4.5}$$

Here, G, F are the standard scalar Godunov fluxes, associated to the underlying flux functions g and f respectively. The Godunov flux H corresponding to a flux h is given by the formula,

$$H(a, b) = \begin{cases} \min\limits_{\theta\in[a,b]} h(\theta), & \text{if } a \leqslant b, \\ \max\limits_{\theta\in[b,a]} h(\theta), & \text{if } a \geqslant b, \end{cases} \tag{4.6}$$

More complicated but explicit numerical flux formulas were proposed for very general flux geometries in [6] and references therein.

It is more difficult to define approximate Riemann solvers that are consistent with AB-entropy solutions for scalar conservation laws with discontinuous flux. Enquist-Osher schemes with this property were proposed in [24] and Lax-Friedrichs type schemes in [11].

Rigorous convergence results for aligned schemes have been obtained. For instance, in [5], the authors were able to prove that the approximate solutions converge to an AB-entropy solution for a fully discrete version of (4.2) (with forward Euler time stepping). It needs to be emphasized it is not straightforward to prove convergence even in the scalar case as the solution operator is no longer TVD (see section 3). Instead, one uses the singular mapping technique with a Temple functional to prove estimates on the total variation of flux of the approximate solutions. This technique and its variants have been employed in [2, 3, 5, 66, 24] and many other papers. It was shown in [8] that the total variation of the approximate solution can indeed blow up, even for a single discontinuity in the scalar flux coefficient.

Although the numerical flux function for scalar problems can be readily obtained in terms of Godunov type Riemann solvers, it is much more difficult to do so in the case of systems. The following class of schemes are more useful in the context of systems.

4.2 Staggered schemes.

An alternative strategy is based on using the following numerical flux in (4.2),

$$\begin{aligned} \hat{\mathbf{F}}_{j+1/2} &:= \hat{\mathbf{F}}\left(\mathbf{k}_{j+1/2}, \mathbf{U}_j, \mathbf{U}_{j+1}\right), \\ \hat{\mathbf{F}}(\mathbf{k}, \mathbf{U}, \mathbf{U}) &= \mathbf{F}(\mathbf{k}, \mathbf{U}). \end{aligned} \tag{4.7}$$

Here the coefficient $\mathbf{k}_{j+1/2}$ is defined as the cell average of the coefficient on a staggered cell i.e,

$$\mathbf{k}_{j+1/2} = \frac{1}{\Delta x} \int\limits_{x_j}^{x_{j+1}} \mathbf{k}(x) dx.$$

Thus, the discretizations of the unknowns and the coefficients are *staggered*. The great advantage of staggered schemes is their simplicity. This is on account of the fact that any standard numerical flux can be used in (4.7). In particular, it is rather straightforward to modify the well-known Godunov, Roe and HLL type fluxes to accommodate a constant coefficient. These fluxes $\hat{F}(\mathbf{k}, \mathbf{U}_L, \mathbf{U}_R)$ will be based on the solution of the following Riemann problem,

$$\begin{aligned} \mathbf{U}_t + \mathbf{F}(\mathbf{k}, \mathbf{U})_x &= 0, \\ \mathbf{U}(x, 0) &= \begin{cases} \mathbf{U}_L, & \text{if } x < 0, \\ \mathbf{U}_R, & \text{if } x > 0. \end{cases} \end{aligned} \tag{4.8}$$

Note that $\mathbf{k}$ in (4.8) is only a constant and any standard (approximate) Riemann solver can be modified slightly in-order to accommodate it. We present the example of a very simple Rusanov (Local Lax-Friedrichs) flux,

$$\hat{\mathbf{F}}(\mathbf{k}, \mathbf{U}_L, \mathbf{U}_r) = \frac{1}{2}\left(\mathbf{F}(\mathbf{k}, \mathbf{U}_L) + \mathbf{F}(\mathbf{k}, \mathbf{U}_R)\right) - \frac{|c|}{2}\left(\mathbf{U}_R - \mathbf{U}_L\right). \tag{4.9}$$

Here c stands for the maximum eigenvalue of the flux Jacobain $\partial_{\mathbf{U}}\mathbf{F}\left(\mathbf{k}, \frac{\mathbf{U}_L+\mathbf{U}_R}{2}\right)$.

Staggered schemes were first proposed for scalar conservation laws in [78, 79] and have been used in a very large body of literature, see [55, 21] for two proto-typical examples. They are also widely used for systems of conservation laws with coefficients, see [30, 51, 41] and references therein.

Although simple to design and implement, staggered schemes possess one major deficiency. It is unclear if one can modify them in-order to approximate arbitrary AB-entropy solutions, even for scalar conservation laws with discontinuous flux. The schemes have an inbuilt numerical viscosity and may not take into account the correct small scale effects. As small scale dependent shock waves are fundamental to conservation laws with discontinuous flux, it is unclear if staggered schemes can approximate the physically relevant solution for these problems.

4.3 Higher order schemes.

We can readily extend the aligned schemes to second-order of accuracy. To do this, we modify the numerical flux (4.3) to be

$$\hat{\mathbf{F}}_{j+1/2} := \hat{\mathbf{F}}\left(\mathbf{k}_j^+, \mathbf{U}_j^+, \mathbf{k}_{j+1}^-, \mathbf{U}_{j+1}^-\right), \tag{4.10}$$

Here, $\hat{\mathbf{F}}$ is still the numerical flux defined as an (approximate) solution of the Riemann problem (4.4) whereas the edge values are determined by a piecewise linear reconstruction, i.e,

$$\mathbf{U}_j^{\pm} = \mathbf{p}_j(x \pm 1/2), \quad \mathbf{k}_j^{\pm} = \mathbf{q}_j(x \pm 1/2).$$

Here, $\mathbf{p}_j, \mathbf{q}_j$ are linear functions with cell averages $\mathbf{U}_j, \mathbf{k}_j$ and slopes given by some suitable TVD limiter such as the usual minmod, MC and superbee limiters [64]. Time integration can be performed using a standard strong stability preserving (SSP) Runge-Kutta second-order method.

It is difficult to prove rigorously that this second-order scheme converges to a weak solution, even for scalar conservation laws as the underlying solution operator is not TVD. A modification, in terms of a *flux TVD* limiter was proposed fairly recently in [25]. This limiter is global and the underlying second order scheme is shown to converge to a weak solution. However, it is unclear if the resulting second-order scheme will converge to an AB-entropy solution. An alternative second-order scheme was also proposed in [10].

Even higher-order of spatial accuracy can be obtained by employing ENO and WENO reconstruction procedures. However, proving convergence even for scalar conservation laws with discontinuous flux for these approximations is still open.

On the other hand, extending staggered schemes to second or higher (formal) order of accuracy is far from straightforward as the numerical flux is defined in terms of the Riemann problem (4.8) that is based on a constant coefficient. Use of linear coefficients is possible for some simple fluxes such as the Rusanov flux (4.9).

4.4 Extensions and other approaches

4.4.1 Time dependent coefficients.

The above discussion was based on a flux coefficient that is only spatially dependent. One can readily extend both the aligned and staggered schemes of this section to cover time-dependent coefficients. However, rigorous convergence results for such schemes are hard to obtain. Some results were obtained in [56] for the Lax-Friedrichs scheme and in [51, 52] for Godunov type schemes for triangular systems of conservation laws and for balance laws with singular source terms. Both sets of convergence results rely on the method of compensated compactness. We also remark that the well-posedness theory for problems with time-dependent coefficients is not complete, even in the scalar case.

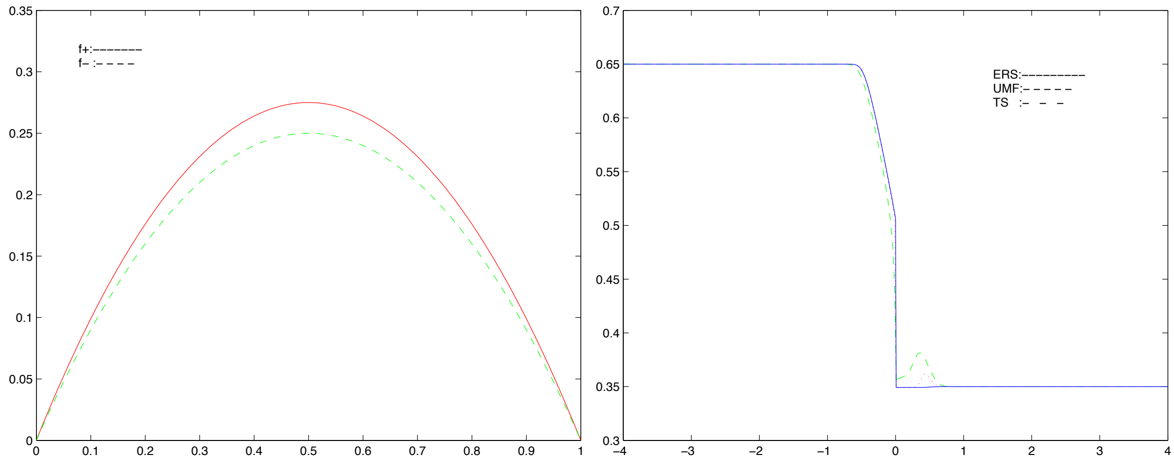


Figure 2: Left: Flux functions used in numerical experiment 1, Solid line (Right flux) and Dashed line (Left flux). Right: Approximate solutions computed with the aligned Godunov scheme of [3] (solid line), the upstream mobility flux scheme of [2] (dotted line) and the Engquist-Osher type scheme of [78, 55] (dashed line), at time $T = 2$ on a uniform grid with $\Delta x = 0.01$. This figure is reproduced from [68]

4.4.2 Multi-dimensional problems.

The finite volume scheme (4.2) can be extended to several space dimensions in a standard fashion. If the underlying geometry is Cartesian, one can extend (4.2) using the standard dimension by dimension finite volume approach or use an dimension splitting method [50] and references therein. Moreover, complex domain geometries can be discretized by unstructured grids and the finite volume method is readily extended to such grids. Both the aligned (4.3) and staggered (4.5) can be readily used in this finite volume scheme. Rigorous convergence results are lacking at the moment. A notable exception are the results of [30] where the authors prove that an Engquist-Osher type scheme for a two-dimensional triangular system of conservation laws, converges to a weak solution, by adapting the method of H-measures. It is to be noted that a triangular system can be viewed as a scalar conservation law with time-dependent and possibly discontinuous coefficient.

4.4.3 Other approaches

An alternative approach to the design of finite volume schemes is provided in [65] and references therein. In these papers, the authors propose a flux splitting approach based on f-waves. In this approach, the finite volume scheme is written in fluctuation form and the instead of considering the fluctuations of the unknown, fluctuations of the flux are taken into account. A different approach of designing schemes for some systems with coefficients, such as the polymer flooding model [80], based on splitting into coupled scalar problems was proposed in [9].

A completely different method is provided by the front tracking scheme [46]. Applications of front tracking to conservation laws with discontinuous flux are provided in [43, 59, 48] and references therein.

5 Numerical experiments.

In this section, we will present a few numerical experiments that illustrates the schemes described in the last section.

5.1 Numerical experiment 1.

We consider the one-dimensional scalar conservation law (3.1) with a flux function of the form $f(k,u) = (1 - H(x))g(u) + H(x)f(u)$. Thus, the flux has a single discontinuity at the point $x = 0$ and the component fluxes are given by,

$$g(u) = u(1-u), \quad f(u) = 1.1u(1-u). \tag{5.1}$$

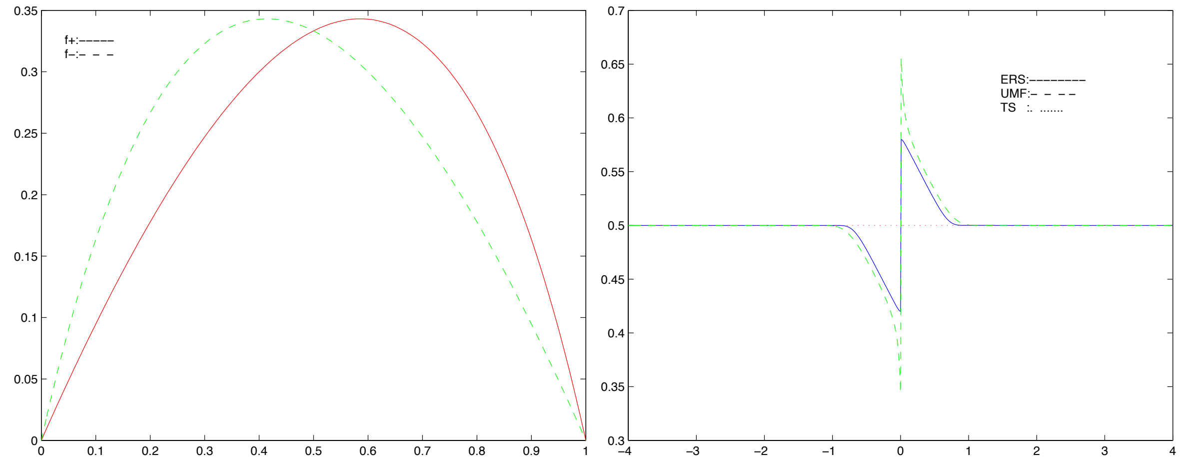


Figure 3: Left: Flux functions used in numerical experiment 2, Solid line (Right flux) and Dashed line (Left flux). Right: Approximate solutions computed with the aligned Godunov scheme of [3] (solid line), the upstream mobility flux scheme of [2] (dashed line) and the Engquist-Osher type scheme of [78, 55] (dotted line), at time $T = 3$ on a uniform grid with $\Delta x = 0.01$. This figure is reproduced from [68]

The flux functions are shown in figure 2 (left). We consider (3.1) with the above fluxes and the initial Riemann data,

$$u_0(x) = \begin{cases} 0.65, & \text{if } x < 0, \\ 0.35, & \text{if } x > 0. \end{cases} \tag{5.2}$$

We compute the approximate solutions with an aligned Godunov type scheme with numerical flux (4.5), proposed in [3], a staggered Engquist-Osher type scheme proposed in [79, 55] and the so-called upstream mobility flux scheme of [20, 2, 68]. The results, at time 2, on a grid with $\Delta x = 0.01$ are shown in figure 2 (right). The Godunov type scheme is based on an optimal entropy connection given by the pair $(0.5, 0.35)$. All three schemes approximate the solution quite well and converge to the same optimal entropy solution in this case. The staggered scheme generates a spurious traveling wave, that moves to the right.

5.2 Numerical experiment 2.

As in the previous numerical experiment, we consider the same one-dimensional scalar conservation law with discontinuous flux (3.1), with a single discontinuity at $x = 0$. The left and right fluxes are given by,

$$g(u) = \frac{2u(1-u)}{1+u}, \quad f(u) = \frac{2u(1-u)}{2-u} \tag{5.3}$$

The flux functions are shown in figure 3 (left). We consider (3.1) with the above fluxes and the initial Riemann data,

$$u_0(x) \equiv 0.5 \tag{5.4}$$

We compute the approximate solutions with an aligned Godunov type scheme with numerical flux (4.5), proposed in [3], a staggered Engquist-Osher type scheme proposed in [79, 55] and the heuristically derived upstream mobility flux scheme of [20, 68, 2]. The results, at time 3, on a grid with $\Delta x = 0.01$, are shown in figure 3 (right). The Godunov type scheme is based on an optimal entropy connection given by the pair $(0.42, 0.58)$. In this case, there is a big difference in the approximations computed with the three schemes. The staggered scheme always returns the steady state solution $u \equiv 0.5$. However, this is an undercompressive solution and corresponds to the interface connection $(0.5, 0.5)$. On the other hand, the aligned Godunov type scheme of [3] does converge to the optimal entropy solution. Moreover, the upstream mobility flux schemes converges to yet another AB-entropy solution. This example illustrates that numerical methods can converge to different entropy solutions and care must exercised in designing schemes that converge to the physically relevant solution of the underlying model.

5.3 Numerical experiment 3.

For the final numerical experiment, we will consider a multi-dimensional system of conservation laws with a source term as well as a spatially dependent coefficient, that can be discontinuous. Our aim is to simulate the propagation of nonlinear waves in the upper solar atmosphere. Following [19, 41], the dynamical core of this wave propagation model is the stratified magneto-hydrodynamics (MHD) equation given by,

$$
\begin{aligned}
\rho_t + \nabla_x \cdot (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \nabla_x \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + \left(p + \frac{1}{2} |\mathbf{B}|^2 \right) I - \mathbf{B} \otimes \mathbf{B} \right) &= -\rho g \mathbf{e}_3, \\
\mathbf{B}_t + \nabla_x \cdot (\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) &= 0, \\
E_t + \nabla_x \cdot \left(\left(E + p + \frac{1}{2} |\mathbf{B}|^2 \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right) &= -\rho g \left(\mathbf{u} \cdot \mathbf{e}_3 \right), \\
\nabla_x \cdot \mathbf{B} &= 0.
\end{aligned}
\tag{5.5}
$$

Here ρ is the density, $\mathbf{u} = \{u_1, u_2, u_3\}$ and $\mathbf{B} = \{B_1, B_2, B_3\}$ are the velocity and magnetic fields respectively, p is the thermal pressure, g is the constant acceleration due to gravity , $\mathbf{e}_3$ represents the unit vector in the vertical (z-) direction. E is the total energy, for simplicity determined by the ideal gas equation of state:

$$
E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{2} |\mathbf{B}|^2, \tag{5.6}
$$

where $\gamma > 1$ is the adiabatic gas constant.

As is fairly standard in solar wave propagation models, one considers the waves as (possibly large) perturbations of a steady state of interest. We consider the following steady state,

$$
\begin{aligned}
&\mathbf{u} \equiv \mathbf{0}, \quad \nabla_x \cdot \mathbf{B} \equiv \mathbf{0}, \quad \nabla_x \times \mathbf{B} \equiv 0, \\
&\rho(z) = \frac{\rho_0 T_0}{T(z)} e^{-\frac{\alpha(z)}{H}}, \quad p(z) = p_0 e^{-\frac{\alpha(z)}{H}}.
\end{aligned}
\tag{5.7}
$$

Here, H denotes the constant scale height, $T = T(z)$ the temperature that only varies along the vertical and α is a potential given by,

$$
\alpha(x, y, z) = \alpha(z) = \int_0^z \frac{1}{T(s)} ds. \tag{5.8}
$$

Hence, the perturbation of the steady state (5.7), evolves according to a conservation (balance) law with a coefficient (2.5) and an additional source term. The steady state (5.7) plays the role of the coefficient in the flux. It is well known that in the sun, there is a massive (two orders of magnitude) jump in the mean temperature between the outer chromosphere and the corona, across a thin transition region. Consequently we can model the steady state temperature (in a simplified manner) as,

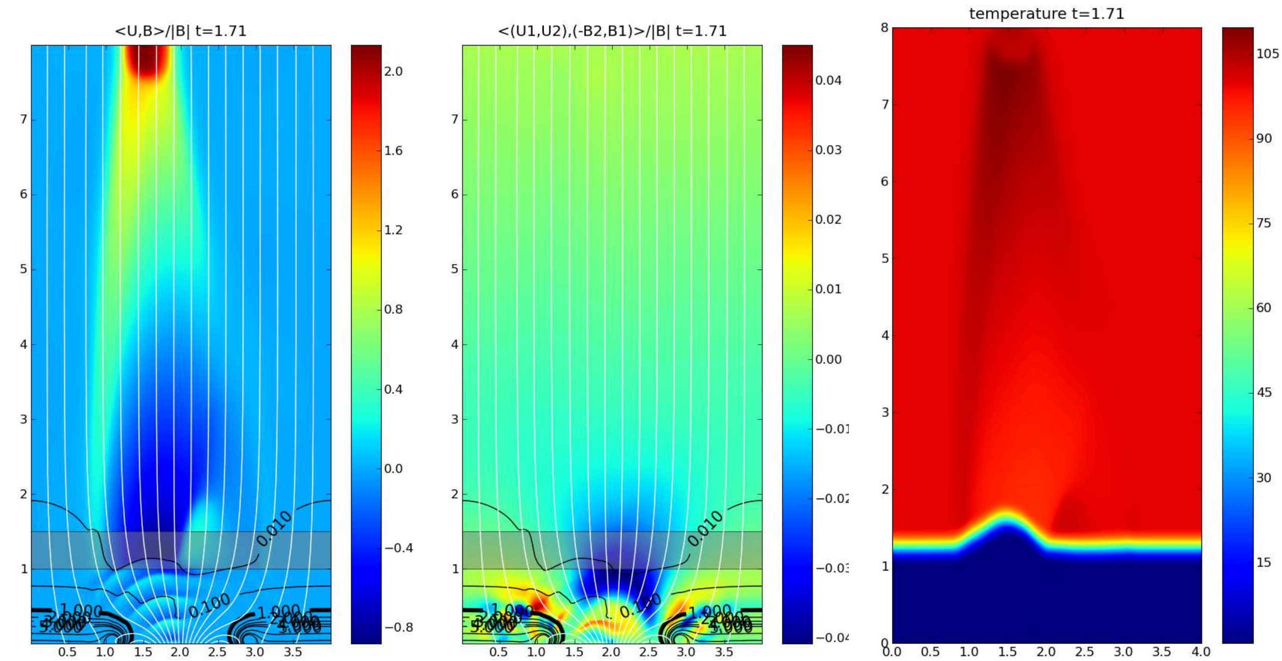
$$
T(z) = \begin{cases} 1, & \text{if } z \leqslant 1 \\ 100, & \text{if } 1 < z \end{cases} \tag{5.9}
$$

Thus, the resulting model does involve a multi-dimensional system of balance laws with a discontinuous flux.

We follow [41] and approximate this system of PDEs with a numerical framework that includes,

- A cleverly designed HLL three-wave solver for the ideal MHD equations [39].
- An upwind discretization of the Godunov-Powell source term for divergence cleaning [39].
- A positivity preserving second-order WENO reconstruction procedure [39].
- A well-balanced evaluation of the flux and source terms [40].
- Non-reflecting numerical boundary conditions at the (top) vertical boundary [41].
- A staggered discretization of the coefficient, analogous to (4.7), (4.8).

Given these ingredients, we consider a two-dimensional model outer solar atmosphere, with steady state temperature distribution given by (5.9) and potential divergence-free and curl-free magentic field, expressed in terms of a truncated Fourier expansion [41]. Finally, waves at introduced at the bottom vertical boundary by sinusoidal in time perturbations of the magnetic field as described in [41]. The incident waves are aligned with magnetic field. The resulting solution is shown in figure 4, where different components of the velocity and the temperature are plotted at time $T = 1.71$ on a Cartesian 400×800 mesh. We remark that the numerical results show robust approximation with our proposed method. In particular, the strong magnetic field is seen to focus the incident waves in the chromosphere and corona. Furthermore, the deviation of the transition layer temperature by energy transfer through the incoming wave is also nicely captured. Three dimensional versions of the same numerical framework were used in [41] to simulate solar wave propagation, with initial magnetic field, temperature and velocity and magnetic field boundary conditions, determined from observed SOHO (satellite) data sets.



(a) $\mathbf{u}^{\parallel}$ (b) $\mathbf{u}^{\perp}$ (c) T

Figure 4: Wave propagation on a model two-dimensional solar upper atmosphere with a 400×800 mesh at time $t = 1.71$. Left: Velocity, parallel to magnetic field Center: Velocity perpendicular to magnetic field and Right: Temperature.

6 Summary and open problems.

Conservation laws with discontinuous coefficients, such as fluxes and sources, arise in a large number of problems including multi-phase flows in heterogenous porous media, traffic flows on highways with changing surface conditions, wave propagation in heterogeneous media, flows in geometries such as channels, pipes and nozzles and flows realized as perturbations of discontinuous steady states. The mathematical and numerical analysis of these problems is quite challenging as many difficulties already arise while dealing with the simplest case of one-dimensional scalar conservation laws with discontinuous flux. We highlight some of these key issues, particularly the lack of uniqueness of admissible solutions. We review the available solution frameworks in the scalar case with a focus on the concept of AB-entropy solutions of [3]. We overview numerical methods for these problems and classify popular numerical methods into the aligned and staggered finite volume types. Aligned methods, in which the unknown and coefficient are collocated, can be designed to be consistent with any prescribed entropy connection in the scalar case. However, these methods are rather complicated and not easy to extend to systems of conservation laws. Instead, methods in which the unknown and coefficient are discretized on staggered grids are more amenable to deal with systems. However, these method maynot converge to a physically relevant solution. We present some numerical experiments in order to illustrate available schemes. One set of experiments focus on the model case of a one-dimensional scalar conservation law with discontinuous flux while another experiment deals with a multi-dimensional and complicated system of balance laws.

It needs to be reiterated that research on conservation laws with discontinuous coefficients is fairly recent. A large number of open questions remain and we summarize some of them below,

- The well-posedness theory for one-dimensional scalar conservation laws is fairly complete. However, there are very few, if any, uniqueness results for scalar conservation laws in several space dimensions. It would be of interest to characterize the counterparts of entropy connections or adapted entropies in this case.
- There are very few rigorous results for one-dimensional systems of conservation laws with discontinuous flux. In particular, the possible presence of multiple entropy solutions and small-scale dependent shock waves needs to be investigated.
- Although numerical methods, approximating multi-dimensional scalar conservation laws and one-dimensional systems have been devised, there are very few rigorous convergence results for these approximations.
- Moreover, there are very few rigorous convergence results for high-order schemes approximating even one-dimensional scalar conservation laws with discontinuous flux.
- If there are multiple entropy solutions and small-scale dependent shock waves for systems of conservation laws with discontinuous flux, it would be interesting to design numerical methods that converge to the correct (physically relevant) solution, under mesh refinement. For scalar conservation laws, use of aligned Godunov type schemes sufficed. However, it might be difficult to design exact Riemann solvers for systems of conservation laws. A possible alternative approach could be to modify schemes with well-controlled dissipation that have been devised for non-strictly hyperbolic systems, see [62] for a review of WCD schemes.
- The coefficient $\mathbf{k}$ and the source $\mathbf{S}$ in (1.1) need to be measured. Measurements are inherently uncertain. This input uncertainty propagates into the solution. The modeling, analysis and computation of uncertainty falls under the rubric of uncertainty quantification (UQ). UQ for hyperbolic PDEs is a rapidly emerging topic, see [18] and a companion chapter in this handbook for a review. However, there have been very few attempts to carry out UQ for conservation laws with discontinuous coefficients. First attempts for wave propagation in heterogenous uncertain media were reported in a recent paper [70]. The area of UQ for uncertain and heterogeneous coefficients needs to be pursued.

Acknowledgements

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