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Weak convergence rates for numerical approximations of stochastic partial differential equations with nonlinear diffusion coefficients in UMD Banach spaces

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Abstract

Strong convergence rates for numerical approximations of semilinear stochastic partial differential equations (SPDEs) with smooth and regular nonlinearities are well understood in the literature. Weak convergence rates for numerical approximations of such SPDEs have been investigated for about two decades and are still not yet fully understood. In particular, no essentially sharp weak convergence rates are known for temporal or spatial numerical approximations of space-time white noise driven SPDEs with nonlinear multiplication operators in the diffusion coefficients. In this article we overcome this problem by establishing essentially sharp weak convergence rates for exponential Euler approximations of semilinear SPDEs with nonlinear multiplication operators in the diffusion coefficients. Key ingredients of our approach are applications of the mild Itô type formula in UMD Banach spaces with type 2.

1 Introduction

This article investigates weak convergence rates for time-discrete numerical approximations of semilinear stochastic partial differential equations (SPDEs). In the case of finite dimensional stochastic ordinary differential equations (SODEs) with smooth and regular nonlinearities both strong and numerically weak convergence rates of numerical approximations are well understood in the literature; see, e.g., the monographs Kloeden & Platen [30] and Milstein [41]. The situation is different in the case of SPDEs. While strong convergence rates for numerical approximations of semilinear SPDEs with smooth and regular nonlinearities are well understood in the literature, weak convergence rates for numerical approximations of such SPDEs have been investigated for about two decades and are still not yet fully understood. More specifically, to the best of our knowledge, there exist no result in the scientific literature which establishes essentially sharp weak convergence rates for temporal or spatial numerical approximations in the case of space-time white noise driven SPDEs with nonlinear multiplication operators in the diffusion coefficients. In this paper we overcome this problem in the case of time-discrete exponential Euler approximations for SPDEs (cf., e.g., Lord & Rougemont [39, Section 3], Cohen & Gauckler [13, Section 2.2], and Wang [48, Section 1]), which is illustrated in the following theorem.

Theorem 1.1. *Let $T \in (0, \infty)$, $p \in [2, \infty)$, let $f, b \colon \mathbb{R} \to \mathbb{R}$ and $\varphi \colon L^p((0,1);\mathbb{R}) \to \mathbb{R}$ be four times continuously differentiable functions with globally Lipschitz continuous and globally bounded derivatives, let $\xi \colon (0,1) \to \mathbb{R}$ be a $\mathcal{B}((0,1))/\mathcal{B}(\mathbb{R})$-measurable and globally bounded function, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an $\mathrm{Id}_{L^2((0,1);\mathbb{R})}$-cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $X \colon [0,T] \times \Omega \to L^p((0,1);\mathbb{R})$ be a continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted mild solution process of the SPDE*

$$dX_t(x) = \left[\tfrac{\partial^2}{\partial x^2} X_t(x) + f(X_t(x))\right] dt + b(X_t(x))\, dW_t(x) \tag{1}$$

with $X_t(0) = X_t(1) = 0$ *and* $X_0(x) = \xi(x)$ *for* $t \in [0,T]$, $x \in (0,1)$, *and for every* $N \in \mathbb{N}$ *let* $Y^N \colon \{0,1,\ldots,N\} \times \Omega \to L^p((0,1);\mathbb{R})$ *be a time-discrete exponential Euler approximation for the SPDE* (1) *with time step size* $^T\!/_N$ *(see, e.g., item* (iv) *of Theorem 9.3 in Section 9 below). Then for every* $\varepsilon \in (0,\infty)$ *there exists a real number* $C \in \mathbb{R}$ *such that for all* $N \in \mathbb{N}$ *it holds that*

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \leq C \cdot N^{(\varepsilon - 1/2)}. \tag{2}$$

Theorem 1.1 is an immediate consequence of Theorem 9.3 below. Theorem 1.1 establishes for every arbitrarily small $\varepsilon \in (0,\infty)$ the weak convergence rate $1/2 - \varepsilon$ for the exponential Euler approximations Y^N, $N \in \mathbb{N}$, (see, e.g., Lord & Rougemont [39, Section 3], Celledoni et al. [12, Section 2], Cohen & Gauckler [13, Section 2.2], Lord & Tambue [40, Section 2.3], Wang [48, Section 1], and item (iv) of Theorem 9.3 below) in the case of the SPDE (1). We would like to point out that the rate $1/2 - \varepsilon$ can, in general, not be essentially improved. More specifically, Corollary 9.8 in [29] and Theorem 1.1 above prove in the case $\forall\, x \in \mathbb{R} \colon f(x) = 0$, $\forall\, x \in \mathbb{R} \colon b(x) = 1$, and $\forall\, y \in (0,1) \colon \xi(y) = 0$ that there exist a four times continuously differentiable function $\varphi \colon L^p((0,1);\mathbb{R}) \to \mathbb{R}$ such that for all $\varepsilon \in (0,\infty)$ there exist real numbers $c, C \in (0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$c \cdot N^{-1/2} \leq \big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \leq C \cdot N^{(\varepsilon - 1/2)} \tag{3}$$

(cf., e.g., also the references mentioned in the overview article in Müller-Gronbach & Ritter [42] and in Section 9 in [29] for further lower bound results for numerical approximations of the SPDE (1)). The literature also contains a series of other results which establish essentially sharp weak convergence rates for temporal or spatial numerical approximations of the SPDE (1) in the case where the diffusion coefficient function $b \colon \mathbb{R} \to \mathbb{R}$ is affine linear, that is, in the case where it holds that

$$\exists\, \alpha, \beta \in \mathbb{R} \colon \forall\, x \in \mathbb{R} \colon b(x) = \alpha x + \beta \tag{4}$$

(cf., e.g., [45, 25, 18, 20, 24, 22, 26, 19, 32, 21, 7, 23, 50, 38, 33, 6, 9, 5, 35, 8, 34, 31, 49, 14, 48, 4, 29]). To the best of our knowledge, Theorem 1.1 is the first result in the scientific literature which establishes an essentially sharp weak convergence rate for the SPDE (1) in the case where (4) is not fulfilled. Note that Theorem 1.1 above and Theorem 9.3 below prove weak convergence rates only for exponential Euler approximations (see, e.g., item (v) of Theorem 9.3 below). However, their methods of proof extend to other kinds of numerical approximations for SPDEs such as linear-implicit Euler approximations (see, e.g., Da Prato et al. [17, Section 3.3.1]). Our proof of Theorem 1.1 above and Theorem 9.3 below, respectively, is based on the proof in [29] by extending the proof of Theorem 1.1 in [29] from Hilbert spaces to UMD Banach spaces (cf., e.g., Brzeźniak [10, (2.5)] and Van Neerven, Veraar, & Weis [47, (2.3)]) with type 2 (cf. Sections 2–4 and 6–8 in [29] with Sections 2–7 below).

1.1 Notation

Throughout this article the following notation is frequently used. For every set A we denote by $\mathcal{P}(A)$ the power set of A. For every set A we denote by $\#_A \in \{\infty, 0, 1, 2, \ldots\}$ the number of elements of A. For all sets A and B we denote by $\mathbb{M}(A,B)$ the set of all functions from A to B. For all measurable spaces $(A,\mathcal{A})$ and $(B,\mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A},\mathcal{B})$ the set of $\mathcal{A}/\mathcal{B}$-measurable functions. For every Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ we denote by $\lambda_A \colon \mathcal{B}(A) \to [0,\infty]$ the Lebesgue-Borel measure on A. We denote by $\lfloor \cdot \rfloor_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, the functions which satisfy for all $h \in (0,\infty)$, $t \in \mathbb{R}$ that $\lfloor t \rfloor_h = \max((-\infty,t] \cap \{0, h, -h, 2h, -2h, \ldots\})$. We denote by $\mathcal{E}_r \colon [0,\infty) \to [0,\infty)$, $r \in (0,\infty)$, the functions which satisfy for all $x \in [0,\infty)$, $r \in (0,\infty)$ that $\mathcal{E}_r(x) = \big[\sum_{n=0}^{\infty} \frac{x^{2n}\,\Gamma(r)^n}{\Gamma(nr+1)}\big]^{1/2}$ (cf. [27, Chapter 7] and [14, Section 1.2]). For all $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $\#_V > 1$ and every natural number $n \in \mathbb{N} = \{1, 2, \ldots\}$ we denote by

$|\cdot|_{C_b^n(V,W)} : C^n(V,W) \to [0,\infty]$ and $\|\cdot\|_{C_b^n(V,W)} : C^n(V,W) \to [0,\infty]$ the functions which satisfy for all $f \in C^n(V,W)$ that

$$|f|_{C_b^n(V,W)} = \sup_{x\in V} \big\| f^{(n)}(x) \big\|_{L^{(n)}(V,W)}, \qquad \|f\|_{C_b^n(V,W)} = \|f(0)\|_W + \sum_{k=1}^{n} |f|_{C_b^k(V,W)} \tag{5}$$

and we denote by $C_b^n(V,W)$ the set given by $C_b^n(V,W) = \{f \in C^n(V,W) \colon \|f\|_{C_b^n(V,W)} < \infty\}$. For all $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $\#_V > 1$ and every nonnegative integer $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,\ldots\}$ we denote by $|\cdot|_{\mathrm{Lip}^n(V,W)} : C^n(V,W) \to [0,\infty]$ and $\|\cdot\|_{\mathrm{Lip}^n(V,W)} : C^n(V,W) \to [0,\infty]$ the functions which satisfy for all $f \in C^n(V,W)$ that

$$\begin{aligned} |f|_{\mathrm{Lip}^n(V,W)} &= \begin{cases} \sup_{x,y\in V,\, x\neq y} \left(\frac{\|f(x)-f(y)\|_W}{\|x-y\|_V} \right) & : n = 0 \\ \sup_{x,y\in V,\, x\neq y} \left(\frac{\|f^{(n)}(x)-f^{(n)}(y)\|_{L^{(n)}(V,W)}}{\|x-y\|_V} \right) & : n \in \mathbb{N} \end{cases}, \\ \|f\|_{\mathrm{Lip}^n(V,W)} &= \|f(0)\|_W + \sum_{k=0}^{n} |f|_{\mathrm{Lip}^k(V,W)} \end{aligned} \tag{6}$$

and we denote by $\mathrm{Lip}^n(V,W)$ the set given by $\mathrm{Lip}^n(V,W) = \{f \in C^n(V,W) \colon \|f\|_{\mathrm{Lip}^n(V,W)} < \infty\}$. For every separable $\mathbb{R}$-Hilbert space $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ and every $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ we denote by $\gamma(U,V)$ the $\mathbb{R}$-Banach space of γ-radonifying operators from U to V (see, e.g., [47, Section 2]). For every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \mathcal{S})$, every set R, and every function $f \colon \Omega \to R$ we denote by $[f]_{\mu,\mathcal{S}}$ the set given by

$$[f]_{\mu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F},\mathcal{S}) \colon (\exists\, A \in \mathcal{F} \colon \mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)\}. \tag{7}$$

For every measure space $(\Omega, \mathcal{F}, \mu)$, every measurable space $(S, \mathcal{S})$, and every set R we do as usual often not distinguish between a function $f \colon \Omega \to R$ and its equivalence class $[f]_{\mu,\mathcal{S}}$.

1.2 General setting

Throughout this article the following setting is frequently used. Consider the notation in Section 1.1, let $(V, \|\cdot\|_V)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be separable UMD $\mathbb{R}$-Banach spaces with type 2, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $\angle = \{(t_1,t_2) \in [0,T]^2 \colon t_1 < t_2\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, and for every $p \in [2,\infty)$ let $\Upsilon_p \in [0,\infty)$ be the real number given by

$$\Upsilon_p = \sup\left(\left\{ \frac{\| \int_0^T X_t \, dW_t \|_{L^p(\mathbb{P};V)}}{\left(\int_0^T \|X_t\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))} \, dt\right)^{1/2}} : \begin{array}{c} (\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(\gamma(U,V))\text{-predictable} \\ X \colon [0,T]\times\Omega \to \gamma(U,V) \text{ with} \\ \int_0^T \|X_t\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))} \, dt \in (0,\infty) \end{array} \right\} \right) \tag{8}$$

(cf., e.g., [46, Corollary 3.10]).

1.3 An auxiliary lemma

Throughout this article we frequently use the following elementary lemma (see, e.g., [11, Lemma 2.3] and [15, Lemma 2.2]).

Lemma 1.2. *Consider the notation in Section 1.1, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $(V, \|\cdot\|_V)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be $\mathbb{R}$-Banach spaces, and let $\beta \in L^{(2)}(V, \mathcal{V})$. Then* *(i) it holds for all $A_1, A_2 \in \gamma(U,V)$ and all orthonormal sets $\mathbb{U} \subseteq U$ of U that there exists a unique $v \in \mathcal{V}$ such that*

$$\inf_{\substack{I\subseteq \mathbb{U},\\ \#_I<\infty}} \sup_{\substack{I\subseteq J\subseteq \mathbb{U},\\ \#_J<\infty}} \Big\| v - \sum_{u\in J} \beta(A_1u, A_2u) \Big\|_{\mathcal{V}} = 0, \tag{9}$$

(ii) it holds for all orthonormal bases $\mathbb{U}_1, \mathbb{U}_2 \subseteq U$ of U that

$$\sum_{u\in\mathbb{U}_1} \beta(A_1u, A_2u) = \sum_{u\in\mathbb{U}_2} \beta(A_1u, A_2u), \tag{10}$$

(iii) it holds for all $A_1, A_2 \in \gamma(U,V)$ and all orthonormal sets $\mathbb{U} \subseteq U$ of U that

$$\Big\| \sum_{u\in\mathbb{U}} \beta(A_1u, A_2u) \Big\|_{\mathcal{V}} \leq \|\beta\|_{L^{(2)}(V,\mathcal{V})} \|A_1\|_{\gamma(U,V)} \|A_2\|_{\gamma(U,V)}, \tag{11}$$

and

(iv) it holds for all orthonormal sets $\mathbb{U} \subseteq U$ of U that

$$\Big(\gamma(U,V) \times \gamma(U,V) \ni (A_1, A_2) \mapsto \sum_{u\in\mathbb{U}} \beta(A_1u, A_2u) \in \mathcal{V} \Big) \in L^{(2)}(\gamma(U,V), \mathcal{V}). \tag{12}$$

2 Strong a priori estimates for SPDEs

2.1 Setting

Assume the setting in Section 1.2, let $p \in [2,\infty)$, $\vartheta \in [0,1)$, $\mathbf{y}, \mathbf{z} \in [0,\infty)$, let $X\colon [0,T]\times\Omega \to V$ be a stochastic process with $\sup_{s\in[0,T]} \|X_s\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$, and for every $t \in (0,T]$ let $Y^t\colon [0,t]\times\Omega \to V$ and $Z^t\colon [0,t]\times\Omega \to \gamma(U,V)$ be $(\mathcal{F}_s)_{s\in[0,t]}$-predictable stochastic processes which satisfy for all $s \in (0,t)$ that

$$\|Y^t_s\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \tfrac{\mathbf{y} \sup_{u\in[0,s]} \|X_u\|_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^{\vartheta}} \qquad \text{and} \qquad \|Z^t_s\|_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))} \leq \tfrac{\mathbf{z} \sup_{u\in[0,s]} \|X_u\|_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^{\vartheta/2}}. \tag{13}$$

2.2 A strong a priori estimate

Proposition 2.1 (A strong a priori estimate)**.** *Assume the setting in Section 2.1. Then*

(i) it holds for all $t \in [0,T]$ that $\mathbb{P}\big(\int_0^t \|Y^t_s\|_V + \|Z^t_s\|^2_{\gamma(U,V)}\, ds < \infty\big) = 1$ and

(ii) it holds that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \sqrt{2}\, \mathcal{E}_{(1-\vartheta)}\Big[\tfrac{\mathbf{y}\sqrt{2}\, T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \sqrt{2T^{(1-\vartheta)}} \Big] \\
&\cdot \sup_{t\in[0,T]} \Big\| X_t - \Big[\int_0^t Y^t_s\, ds + \int_0^t Z^t_s\, dW_s \Big] \Big\|_{L^p(\mathbb{P};V)} \leq \Big[1 + \tfrac{\mathbf{y}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\mathbf{z}\Upsilon_p \sqrt{T^{(1-\vartheta)}}}{\sqrt{(1-\vartheta)}} \Big] \\
&\cdot \sqrt{2}\, \mathcal{E}_{(1-\vartheta)}\Big[\tfrac{\mathbf{y}\sqrt{2}\, T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \sqrt{2T^{(1-\vartheta)}} \Big] \sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty.
\end{aligned} \tag{14}$$

Proof. We first observe that (13), Hölder's inequality, and the assumption that $\sup_{s\in[0,T]} \|X_s\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ imply that for all $t \in [0,T]$ it holds that

$$\begin{aligned} \int_0^t \|Y_s^t\|_{\mathcal{L}^p(\mathbb{P};V)}\, ds &\leq \mathbf{y} \int_0^t \frac{\sup_{v\in[0,s]} \|X_v\|_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^\vartheta}\, ds \\ &\leq \mathbf{y} \left[\frac{t^{(1-\vartheta)}}{(1-\vartheta)} \int_0^t \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^\vartheta}\, ds \right]^{1/2} < \infty \end{aligned} \tag{15}$$

and

$$\left[\int_0^t \|Z_s^t\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))}\, ds \right]^{1/2} \leq \mathbf{z} \left[\int_0^t \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^\vartheta}\, ds \right]^{1/2} < \infty. \tag{16}$$

Combining (15)–(16) and the assumption that $p \geq 2$ proves that for all $t \in [0,T]$ it holds that $\int_0^t \|Y_s^t\|_{\mathcal{L}^1(\mathbb{P};V)} + \|Z_s^t\|^2_{\mathcal{L}^2(\mathbb{P};\gamma(U,V))}\, ds < \infty$. This, in turn, shows that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\int_0^t \|Y_s^t\|_V + \|Z_s^t\|^2_{\gamma(U,V)}\, ds < \infty. \tag{17}$$

It thus remains to prove (14) to complete the proof of Proposition 2.1. For this observe that (15)–(17) imply that for all $t \in [0,T]$ it holds that

$$\begin{aligned} &\left\| \int_0^t Y_s^t\, ds \right\|_{\mathcal{L}^p(\mathbb{P};V)} + \left\| \int_0^t Z_s^t\, dW_s \right\|_{L^p(\mathbb{P};V)} \\ &\leq \left[\frac{\mathbf{y}\, t^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \right] \left[\int_0^t \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^\vartheta}\, ds \right]^{1/2}. \end{aligned} \tag{18}$$

Next we observe that for all $t, u \in [0,T]$ with $t \leq u$ it holds that

$$\begin{aligned} \int_0^t \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^\vartheta}\, ds &= \int_{u-t}^u \frac{\sup_{v\in[0,s-u+t]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(u-s)^\vartheta}\, ds \\ \leq \int_{u-t}^u \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(u-s)^\vartheta}\, ds &\leq \int_0^u \frac{\sup_{v\in[0,s]} \|X_v\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(u-s)^\vartheta}\, ds. \end{aligned} \tag{19}$$

Moreover, we note that Minkowski's inequality ensures that for all $t \in [0,T]$ it holds that

$$\begin{aligned} &\|X_t\|_{\mathcal{L}^p(\mathbb{P};V)} \\ &\leq \left\| X_t - \left[\int_0^t Y_s^t\, ds + \int_0^t Z_s^t\, dW_s \right] \right\|_{L^p(\mathbb{P};V)} + \left\| \int_0^t Y_s^t\, ds \right\|_{\mathcal{L}^p(\mathbb{P};V)} + \left\| \int_0^t Z_s^t\, dW_s \right\|_{L^p(\mathbb{P};V)}. \end{aligned} \tag{20}$$

Combining (18)–(20) with the fact that $\forall\, a, b \in \mathbb{R}\colon\ (a+b)^2 \leq 2a^2 + 2b^2$ proves that for all $u \in [0,T]$ it holds that

$$\begin{aligned} \sup_{t\in[0,u]} \|X_t\|^2_{\mathcal{L}^p(\mathbb{P};V)} &\leq 2 \sup_{t\in[0,T]} \left\| X_t - \left[\int_0^t Y_s^t\, ds + \int_0^t Z_s^t\, dW_s \right] \right\|^2_{L^p(\mathbb{P};V)} \\ &\quad + 2 \left[\frac{\mathbf{y}\, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \right]^2 \int_0^u \frac{\sup_{t\in[0,s]} \|X_t\|^2_{\mathcal{L}^p(\mathbb{P};V)}}{(u-s)^\vartheta}\, ds. \end{aligned} \tag{21}$$

Combining this and the assumption that $\sup_{s\in[0,T]} \|X_s\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ with the generalized Gronwall lemma in Chapter 7 in Henry [27] (see, e.g., also Andersson et al. [2, Lemma 2.6]) proves the first inequality in (14). In the next step we note that (18) implies that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t - \left[\int_0^t Y_s^t \, ds + \int_0^t Z_s^t \, dW_s \right] \right\|_{L^p(\mathbb{P};V)} \\
&\leq \sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};V)} + \sup_{t\in[0,T]} \left[\left\| \int_0^t Y_s^t \, ds \right\|_{\mathcal{L}^p(\mathbb{P};V)} + \left\| \int_0^t Z_s^t \, dW_s \right\|_{L^p(\mathbb{P};V)} \right] \\
&\leq \left[1 + \frac{\mathbf{y}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \mathbf{z}\Upsilon_p \sqrt{\frac{T^{(1-\vartheta)}}{(1-\vartheta)}} \right] \sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};V)}.
\end{aligned} \tag{22}$$

This proves the second inequality in (14). The third inequality in (14) is an immediate consequence of the assumption that

$$\sup_{s\in[0,T]} \|X_s\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty. \tag{23}$$

The proof of Proposition 2.1 is thus completed. □

3 Strong perturbations for SPDEs

3.1 Setting

Assume the setting in Section 1.2, let $p \in [2,\infty)$, $\vartheta \in [0,1)$, $\mathbf{y},\mathbf{z} \in [0,\infty)$, let $X, \bar{X} \colon [0,T]\times\Omega \to V$ be stochastic processes with $\sup_{s\in[0,T]} \|X_s - \bar{X}_s\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$, and for every $t \in (0,T]$ let $Y^t, \bar{Y}^t \colon [0,t]\times\Omega \to V$, $Z^t, \bar{Z}^t \colon [0,t]\times\Omega \to \gamma(U,V)$ be $(\mathcal{F}_s)_{s\in[0,t]}$-predictable stochastic processes which satisfy for all $s \in (0,t)$ that $\mathbb{P}\big(\int_0^t \|Y_r^t\|_V + \|\bar{Y}_r^t\|_V + \|Z_r^t\|^2_{\gamma(U,V)} + \|\bar{Z}_r^t\|^2_{\gamma(U,V)} \, dr < \infty \big) = 1$ and

$$\|Y_s^t - \bar{Y}_s^t\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \frac{\mathbf{y} \sup_{u\in[0,s]} \|X_u - \bar{X}_u\|_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^{\vartheta}}, \quad \|Z_s^t - \bar{Z}_s^t\|_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))} \leq \frac{\mathbf{z} \sup_{u\in[0,s]} \|X_u - \bar{X}_u\|_{\mathcal{L}^p(\mathbb{P};V)}}{(t-s)^{\vartheta/2}}. \tag{24}$$

3.2 Strong perturbation estimates

The following result, Corollary 3.1, is an immediate consequence of Proposition 2.1 in Subsection 2.2 above.

Corollary 3.1 (A strong perturbation estimate)**.** *Assume the setting in Section 3.1. Then*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \sqrt{2}\, \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\mathbf{y}\sqrt{2}\, T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \sqrt{2T^{(1-\vartheta)}} \right] \\
&\cdot \sup_{t\in[0,T]} \left\| X_t - \left[\int_0^t Y_s^t \, ds + \int_0^t Z_s^t \, dW_s \right] + \left[\int_0^t \bar{Y}_s^t \, ds + \int_0^t \bar{Z}_s^t \, dW_s \right] - \bar{X}_t \right\|_{L^p(\mathbb{P};V)} \\
&\leq \sqrt{2}\, \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\mathbf{y}\sqrt{2}\, T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p \sqrt{2T^{(1-\vartheta)}} \right] \left[1 + \tfrac{\mathbf{y}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \mathbf{z}\Upsilon_p \sqrt{\tfrac{T^{(1-\vartheta)}}{(1-\vartheta)}} \right] \\
&\cdot \sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty.
\end{aligned} \tag{25}$$

The next result, Corollary 3.2, follows directly from Corollary 3.1 above.

Corollary 3.2. *Assume the setting in Section 3.1, let $S \in \mathbb{M}([0,T], L(V))$, and assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$$X_t = S_t\, X_0 + \int_0^t Y_s^t \, ds + \int_0^t Z_s^t \, dW_s, \qquad \bar{X}_t = S_t\, \bar{X}_0 + \int_0^t \bar{Y}_s^t \, ds + \int_0^t \bar{Z}_s^t \, dW_s. \tag{26}$$

Then

$$\sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{\mathcal{L}^p(\mathbb{P};V)}$$
$$\le \sqrt{2}\left[\sup_{t\in[0,T]} \|S_t\|_{L(V)}\right] \|X_0 - \bar{X}_0\|_{\mathcal{L}^p(\mathbb{P};V)}\, \mathcal{E}_{(1-\vartheta)}\Big[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\Upsilon_p\sqrt{2T^{(1-\vartheta)}}\Big]. \tag{27}$$

4 Strong convergence of mollified solutions for SPDEs

4.1 Setting

Assume the setting in Section 1.2, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with spectrum$(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$, let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$ (cf., e.g., [44, Section 3.7]), let $p \in [2,\infty)$, $\vartheta \in [0,1)$, $\varPi \in \mathcal{M}\big(\mathcal{B}([0,T]),\mathcal{B}([0,T])\big)$, $(C_r)_{r\in[0,1]} \subseteq [1,\infty)$, $F \in \mathrm{Lip}^0(V, V_{-\vartheta})$, $B \in \mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))$, $L \in \mathcal{M}\big(\mathcal{B}(\angle),\mathcal{B}(L(V_{-1}))\big)$ satisfy for all $t \in [0,T]$ that $\varPi(t) \le t$ and for all $(s,t) \in \big(\angle \cap (0,T]^2\big)$, $\rho \in [0,1)$ that $L_{0,t}(V) \subseteq V$, $L_{s,t}(V_{-\rho}) \subseteq V$, and $\|L_{s,t}\|_{L(V_{-\rho},V)} \le C_\rho\,(t-s)^{-\rho}$, let $\chi_r \in [1,\infty)$, $r \in [0,1]$, be the real numbers which satisfy for all $r \in [0,1]$ that $\chi_r = \max\{1, \sup_{t\in(0,T]} t^r\,\|(\eta-A)^r e^{tA}\|_{L(V)}, \sup_{t\in(0,T]} t^{-r}\,\|(\eta-A)^{-r}(e^{tA} - \mathrm{Id}_V)\|_{L(V)}\}$ (cf., e.g., [43, Lemma 11.36]), and let $Y^\kappa\colon [0,T]\times\Omega \to V$, $\kappa \in [0,T]$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for all $\kappa \in [0,T]$ that $\sup_{t\in[0,T]} \|Y^\kappa_{\Pi(t)}\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ and which satisfy that for all $\kappa \in [0,T]$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that $Y^\kappa_0 = Y^0_0$ and

$$Y^\kappa_t = L_{0,t}\,Y^\kappa_0 + \int_0^t L_{s,t}\,e^{\kappa A}F(Y^\kappa_{\varPi(s)})\,ds + \int_0^t L_{s,t}\,e^{\kappa A}B(Y^\kappa_{\varPi(s)})\,dW_s. \tag{28}$$

4.2 A priori bounds for the non-mollified process

Lemma 4.1. *Assume the setting in Section 4.1 and let $\kappa \in [0,T]$. Then* $\sup_{t\in[0,T]} \|Y^\kappa_t\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$.

Proof. We observe that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\|Y^\kappa_t\|_{\mathcal{L}^p(\mathbb{P};V)} \le \|L_{0,t}\,Y^\kappa_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \int_0^t \|L_{s,t}\,e^{\kappa A}F(Y^\kappa_{\varPi(s)})\|_{\mathcal{L}^p(\mathbb{P};V)}\,ds \\
&+ \Upsilon_p\left[\int_0^t \|L_{s,t}\,e^{\kappa A}B(Y^\kappa_{\varPi(s)})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))}\,ds\right]^{1/2} \\
&\le \chi_0\,\|Y^\kappa_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \int_0^t \tfrac{\chi_0\,C_\vartheta\,\|F(Y^\kappa_{\Pi(s)})\|_{\mathcal{L}^p(\mathbb{P};V_{-\vartheta})}}{(t-s)^\vartheta}\,ds \\
&\quad + \Upsilon_p\left[\int_0^t \tfrac{|\chi_0|^2\,|C_{\vartheta/2}|^2\,\|B(Y^\kappa_{\Pi(s)})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_{-\vartheta/2}))}}{(t-s)^\vartheta}\,ds\right]^{1/2} \\
&\le \left[\chi_0 + \tfrac{\chi_0\,C_\vartheta\,T^{(1-\vartheta)}\,\|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{(1-\vartheta)} + \tfrac{\chi_0\,C_{\vartheta/2}\Upsilon_p\sqrt{T^{(1-\vartheta)}}\,\|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}}\right] \\
&\quad \cdot \sup_{s\in[0,T]} \|\max\{1, \|Y^\kappa_{\Pi(s)}\|_V\}\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{29}$$

This and the fact that $\sup_{t\in[0,T]} \|\max\{1, \|Y^\kappa_{\Pi(t)}\|_V\}\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \le 1 + \sup_{t\in[0,T]} \|Y^\kappa_{\Pi(t)}\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ complete the proof of Lemma 4.1. □

Proposition 4.2 (An a priori bound for the non-mollified process)**.** *Assume the setting in Section 4.1. Then*

$$\sup_{t\in[0,T]} \|Y_t^0\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \sqrt{2}\left[\sup_{t\in(0,T]} \max\{1, \|L_{0,t}\|_{L(V)}\} \|Y_0^0\|_{\mathcal{L}^p(\mathbb{P};V)} + \tfrac{C_\vartheta\, T^{(1-\vartheta)}\, \|F(0)\|_{V_{-\vartheta}}}{(1-\vartheta)} + \tfrac{C_{\vartheta/2}\Upsilon_p \sqrt{T^{(1-\vartheta)}}\, \|B(0)\|_{\gamma(U,V_{-\vartheta/2})}}{\sqrt{1-\vartheta}}\right] \cdot \mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\, C_\vartheta\, T^{(1-\vartheta)}\, |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{\sqrt{1-\vartheta}} + \Upsilon_p\, C_{\vartheta/2}\, \sqrt{2T^{(1-\vartheta)}}\, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\right] < \infty. \tag{30}$$

Proof. Throughout this proof let $\tilde{L}\colon \{(t_1,t_2)\in[0,T]^2\colon t_1\leq t_2\} \to L(V_{-1})$ be the function which satisfies for all $t_0\in[0,T]$, $(t_1,t_2)\in\angle$, $v\in V_{-1}$ that

$$\tilde{L}_{t_1,t_2} v = L_{t_1,t_2} v \qquad \text{and} \qquad \tilde{L}_{t_0,t_0} = \mathrm{Id}_{V_{-1}}\,. \tag{31}$$

Combining Corollary 3.1 and Lemma 4.1 shows[1] that

$$\sup_{t\in[0,T]} \left\|Y_t^0\right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \sqrt{2} \sup_{t\in[0,T]} \left\| \tilde{L}_{0,t}\, Y_0^0 + \int_0^t \tilde{L}_{s,t}\, F(0)\, ds + \int_0^t \tilde{L}_{s,t}\, B(0)\, dW_s \right\|_{L^p(\mathbb{P};V)} \cdot \mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} C_\vartheta\, |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})} + \sqrt{2T^{(1-\vartheta)}}\, \Upsilon_p\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\right]. \tag{32}$$

Combining (32) with the triangle inequality completes the proof of Proposition 4.2. □

4.3 A strong convergence result

Proposition 4.3 (A bound on the difference between the mollified and the non-mollified processes)**.** *Assume the setting in Section 4.1 and let $\kappa\in[0,T]$, $\rho\in[0,\frac{1-\vartheta}{2})$. Then*

$$\sup_{t\in[0,T]} \left\|Y_t^0 - Y_t^\kappa\right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \tfrac{2\,\kappa^\rho}{T^\rho}\left[\sup\nolimits_{t\in(0,T]} \max\{1, \|L_{0,t}\|_{L(V)}\}\, \max\{1, \|Y_0^0\|_{\mathcal{L}^p(\mathbb{P};V)}\} + \tfrac{\chi_\rho\, C_\vartheta\, C_{\rho+\vartheta}\, T^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{(1-\vartheta-\rho)} + \tfrac{\Upsilon_p\, \chi_\rho\, C_{\vartheta/2}\, C_{\rho+\vartheta/2}\sqrt{T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta-2\rho}}\right]^2 \cdot \left|\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\,T^{(1-\vartheta)}\,\chi_0\, C_\vartheta}{\sqrt{1-\vartheta}} |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})} + \Upsilon_p \sqrt{2T^{(1-\vartheta)}}\, \chi_0\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\right]\right|^2. \tag{33}$$

Proof. First of all, we observe that Lemma 4.1 allows us to apply Corollary 3.1 to obtain[2] that

$$\sup_{t\in[0,T]} \left\|Y_t^0 - Y_t^\kappa\right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \mathcal{E}_{(1-\vartheta)}\left[C_\vartheta\, |e^{\kappa A}F|_{\mathrm{Lip}^0(V,V_{-\vartheta})} \tfrac{\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + C_{\vartheta/2}\, \Upsilon_p |e^{\kappa A}B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \sqrt{2T^{(1-\vartheta)}}\right] \cdot \sqrt{2} \sup_{t\in[0,T]} \left\|\int_0^t L_{s,t}\left(\mathrm{Id}_V - e^{\kappa A}\right) F(Y_{\Pi(s)}^0)\, ds + \int_0^t L_{s,t}\left(\mathrm{Id}_V - e^{\kappa A}\right) B(Y_{\Pi(s)}^0)\, dW_s\right\|_{L^p(\mathbb{P};V)}. \tag{34}$$

Moreover, we observe that for all $t\in(0,T]$ it holds that

$$\left\|\int_0^t L_{s,t}\left(\mathrm{Id}_V - e^{\kappa A}\right) F(Y_{\Pi(s)}^0)\, ds\right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \int_0^t \frac{\chi_\rho\, C_{\rho+\vartheta}\, \kappa^\rho}{(t-s)^{(\rho+\vartheta)}} \|F(Y_{\Pi(s)}^0)\|_{\mathcal{L}^p(\mathbb{P};V_{-\vartheta})}\, ds \leq \tfrac{\chi_\rho\, C_{\rho+\vartheta}\, t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})} \sup_{s\in[0,T]} \max\{1, \|Y_s^0\|_{\mathcal{L}^p(\mathbb{P};V)}\}\, \kappa^\rho. \tag{35}$$

[1]with $\bar{X}_t = 0$, $\bar{Y}_s^t = \tilde{L}_{s,t}F(0)$, $\bar{Z}_s^t = \tilde{L}_{s,t}B(0)$ for $s\in(0,t)$, $t\in(0,T]$ in the notation of Corollary 3.1

[2]with $\bar{X}_t = Y_t^\kappa$, $\bar{Y}_s^t = L_{s,t}\, e^{\kappa A}F(Y_{\Pi(s)}^\kappa)$, $\bar{Z}_s^t = L_{s,t}\, e^{\kappa A}B(Y_{\Pi(s)}^\kappa)$ for $s\in(0,t)$, $t\in(0,T]$ in the notation of Corollary 3.1

In addition, Lemma 4.1 ensures that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\left\| \int_0^t L_{s,t} \left(\mathrm{Id}_V - e^{\kappa A} \right) B(Y^0_{\varPi(s)}) \, dW_s \right\|_{L^p(\mathbb{P};V)} \\
&\leq \Upsilon_p \left[\int_0^t \frac{|\chi_\rho \, C_{\rho+\vartheta/2} \, \kappa^\rho|^2}{(t-s)^{(2\rho+\vartheta)}} \, \|B(Y^0_{\varPi(s)})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_{-\vartheta/2}))} \, ds \right]^{1/2} \\
&\leq \tfrac{\Upsilon_p \, \chi_\rho \, C_{\rho+\vartheta/2} \sqrt{t^{(1-\vartheta-2\rho)}}}{\sqrt{1-\vartheta-2\rho}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \sup_{s\in[0,T]} \max\!\left\{1, \|Y^0_s\|_{\mathcal{L}^p(\mathbb{P};V)}\right\} \kappa^\rho .
\end{aligned} \tag{36}$$

Putting (35) and (36) into (34) yields that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\| Y^0_t - Y^\kappa_t \right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \sqrt{2} \, \kappa^\rho \sup_{t\in[0,T]} \max\!\left\{1, \|Y^0_t\|_{\mathcal{L}^p(\mathbb{P};V)}\right\} \\
&\cdot \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\sqrt{2}\, T^{(1-\vartheta)} \chi_0 \, C_\vartheta}{\sqrt{1-\vartheta}} |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})} + \Upsilon_p \sqrt{2T^{(1-\vartheta)}} \, \chi_0 \, C_{\vartheta/2} \, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \right] \\
&\cdot \left[\tfrac{\chi_\rho \, C_{\rho+\vartheta} \, T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})} + \tfrac{\Upsilon_p \, \chi_\rho \, C_{\rho+\vartheta/2} \sqrt{T^{(1-\vartheta-2\rho)}}}{\sqrt{1-\vartheta-2\rho}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \right] .
\end{aligned} \tag{37}$$

Combining Proposition 4.2 and (37) proves that

$$\begin{aligned}
&\left\| Y^0_T - Y^\kappa_T \right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq 2 \, \kappa^\rho \left[\sup_{t\in(0,T]} \max\!\left\{1, \|L_{0,t}\|_{L(V)}\right\} \max\{1, \|Y^0_0\|_{\mathcal{L}^p(\mathbb{P};V)}\} \right. \\
&\left. + \tfrac{C_\vartheta \, T^{(1-\vartheta)} \, \|F(0)\|_{V_{-\vartheta}}}{(1-\vartheta)} + \tfrac{\Upsilon_p \, C_{\vartheta/2} \sqrt{T^{(1-\vartheta)}} \, \|B(0)\|_{\gamma(U,V_{-\vartheta/2})}}{\sqrt{1-\vartheta}} \right] \\
&\cdot \left[\tfrac{\chi_\rho \, C_{\rho+\vartheta} \, T^{(1-\vartheta-\rho)} \, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{(1-\vartheta-\rho)} + \tfrac{\Upsilon_p \, \chi_\rho \, C_{\rho+\vartheta/2} \sqrt{T^{(1-\vartheta-2\rho)}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta-2\rho}} \right] \\
&\cdot \left| \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\sqrt{2}\, T^{(1-\vartheta)} \, \chi_0 \, C_\vartheta \, |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{\sqrt{1-\vartheta}} + \Upsilon_p \sqrt{2T^{(1-\vartheta)}} \, \chi_0 \, C_{\vartheta/2} \, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \right] \right|^2 .
\end{aligned} \tag{38}$$

Hence, we obtain that

$$\begin{aligned}
&\left\| Y^0_T - Y^\kappa_T \right\|_{\mathcal{L}^p(\mathbb{P};V)} \leq \tfrac{2\,\kappa^\rho}{T^\rho} \left[\sup_{t\in(0,T]} \max\!\left\{1, \|L_{0,t}\|_{L(V)}\right\} \max\{1, \|Y^0_0\|_{\mathcal{L}^p(\mathbb{P};V)}\} \right. \\
&\left. + \tfrac{C_\vartheta \, T^{(1-\vartheta)} \, \|F(0)\|_{V_{-\vartheta}}}{(1-\vartheta)} + \tfrac{\Upsilon_p \, C_{\vartheta/2} \sqrt{T^{(1-\vartheta)}} \, \|B(0)\|_{\gamma(U,V_{-\vartheta/2})}}{\sqrt{1-\vartheta}} \right] \\
&\cdot \left[\tfrac{\chi_\rho \, C_{\rho+\vartheta} \, T^{(1-\vartheta)} \, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{(1-\vartheta-\rho)} + \tfrac{\Upsilon_p \, \chi_\rho \, C_{\rho+\vartheta/2} \sqrt{T^{(1-\vartheta)}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta-2\rho}} \right] \\
&\cdot \left| \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\sqrt{2}\, T^{(1-\vartheta)} \, \chi_0 \, C_\vartheta \, |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{\sqrt{1-\vartheta}} + \Upsilon_p \sqrt{2T^{(1-\vartheta)}} \, \chi_0 \, C_{\vartheta/2} \, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \right] \right|^2 .
\end{aligned} \tag{39}$$

This implies (33). The proof of Proposition 4.3 is thus completed. □

5 Weak temporal regularity and analysis of the weak distance between exponential Euler approximations of SPDEs and their semilinear integrated counterparts

5.1 Setting

Assume the setting in Section 1.2, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $A \colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with spectrum$(A) \subseteq \{z \in \mathbb{C} \colon \mathrm{Re}(z) <$

$\eta\}$, let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $h \in (0,\infty)$, $p \in [2,\infty)$, $\vartheta \in [0,1)$, $F \in \mathrm{Lip}^0(V, V_{-\vartheta})$, $B \in \mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))$, let $(B^b)_{b\in\mathbb{U}} \subseteq C(V, V_{-\vartheta/2})$ be the functions which satisfy for all $b \in \mathbb{U}$, $v \in V$ that $B^b(v) = B(v)\,b$, let $\varsigma_{F,B} \in \mathbb{R}$ be the real number given by $\varsigma_{F,B} = \max\{1, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\}$, let $\chi_r \in [1,\infty)$, $r \in [-1,1]$, be the real numbers which satisfy for all $r \in [-1,1]$ that $\chi_r = \max\{1, \sup_{t\in(0,T]} t^{\max\{r,0\}} \|(\eta - A)^r e^{tA}\|_{L(V)}, \sup_{t\in(0,T]} t^{-\max\{r,0\}} \|(\eta - A)^{-\max\{r,0\}}(e^{tA} - \mathrm{Id}_V)\|_{L(V)}\}$, let $Y, \bar{Y} \colon [0,T] \times \Omega \to V$ be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy $\|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ and $\bar{Y}_0 = Y_0$ and which satisfy that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$Y_t = e^{tA}\,Y_0 + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,F(Y_{\lfloor s\rfloor_h})\,ds + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,B(Y_{\lfloor s\rfloor_h})\,dW_s, \tag{40}$$

$$\bar{Y}_t = e^{tA}\,\bar{Y}_0 + \int_0^t e^{(t-s)A}\,F(Y_{\lfloor s\rfloor_h})\,ds + \int_0^t e^{(t-s)A}\,B(Y_{\lfloor s\rfloor_h})\,dW_s, \tag{41}$$

and let $(K_r)_{r\in[0,\infty)} \subseteq [0,\infty]$ be the extended real numbers which satisfy for all $r \in [0,\infty)$ that $K_r = \sup_{s,t\in[0,T]} \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_V^r, \|Y_t\|_V^r\}\big]$.

5.2 Weak temporal regularity of semilinear integrated exponential Euler approximations

In Proposition 5.2 below we establish a weak temporal regularity result for the process $\bar{Y}$ in Subsection 5.1. The proof of Proposition 5.2 uses the following elementary result.

Lemma 5.1. *Assume the setting in Section 5.1. Then*

$$\begin{aligned}
&\sup_{r\in[0,p]} K_r = K_p \\
&\leq \left[\chi_0 \max\{1, \|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)}\} + \tfrac{\chi_\vartheta \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\Upsilon_p\,\chi_{\vartheta/2}\sqrt{T^{(1-\vartheta)}}\,\|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}}\right]^{2p} \\
&\cdot 2^{(\frac{p}{2}+1)} \left|\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\,\chi_\vartheta\,T^{(1-\vartheta)}\,|F|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{\sqrt{1-\vartheta}} + \Upsilon_p\,\chi_{\vartheta/2}\sqrt{2T^{(1-\vartheta)}}\,|B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\right]\right|^p < \infty.
\end{aligned} \tag{42}$$

Proof. We first observe that the equality in (42) follows from the fact that for all $x \in V$, $r, s \in [0,\infty)$ with $r \leq s$ it holds that $\max\{1, \|x\|_V^r\} \leq \max\{1, \|x\|_V^s\}$. Moreover, we note that the second inequality in (42) is an immediate consequence of the assumption that $\|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$. It thus remains to prove the first inequality in (42). For this we claim that for all $k \in \{0,1,\ldots,\lfloor T\rfloor_h/h\}$ it holds that

$$\|Y_{kh}\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty. \tag{43}$$

We now prove (43) by induction on $k \in \{0,1,\ldots,\lfloor T\rfloor_h/h\}$. The assumption that $\|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty$ establishes (43) in the base case $k = 0$. For the induction step $\mathbb{N}_0 \cap (-\infty, \lfloor T\rfloor_h/h) \ni k \to k+1 \in \mathbb{N} \cap [0, \lfloor T\rfloor_h/h]$ assume that there exists a nonnegative integer $k \in \mathbb{N}_0 \cap (-\infty, \lfloor T\rfloor_h/h)$ such that (43) holds for $k = 0$, $k = 1, \dots, k = k$. This ensures that

$$
\begin{aligned}
&\|Y_{(k+1)h}\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\le \|e^{(k+1)hA}\, Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \left\| \int_0^{(k+1)h} e^{((k+1)h - \lfloor s \rfloor_h)}\, F(Y_{\lfloor s \rfloor_h})\, ds \right\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&+ \left\| \int_0^{(k+1)h} e^{((k+1)h - \lfloor s \rfloor_h)}\, B(Y_{\lfloor s \rfloor_h})\, dW_s \right\|_{L^p(\mathbb{P};V)} \\
&\le \chi_0\, \|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \int_0^{(k+1)h} \|e^{((k+1)h - \lfloor s \rfloor_h)}\, F(Y_{\lfloor s \rfloor_h})\|_{\mathcal{L}^p(\mathbb{P};V)}\, ds \\
&+ \Upsilon_p \left[\int_0^{(k+1)h} \|e^{((k+1)h - \lfloor s \rfloor_h)}\, B(Y_{\lfloor s \rfloor_h})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V))}\, ds \right]^{1/2} \qquad (44)\\
&\le \chi_0\, \|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})} \left[\max_{j \in \{0,1,\dots,k\}} \max\{1, \|Y_{jh}\|_{\mathcal{L}^p(\mathbb{P};V)}\} \right] \int_0^{(k+1)h} \tfrac{1}{((k+1)h - s)^\vartheta}\, ds \\
&+ \Upsilon_p\, \chi_{\vartheta/2}\, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \left[\max_{j \in \{0,1,\dots,k\}} \max\{1, \|Y_{jh}\|_{\mathcal{L}^p(\mathbb{P};V)}\} \right] \left[\int_0^{(k+1)h} \tfrac{1}{((k+1)h - s)^\vartheta}\, ds \right]^{1/2} \\
&\le \left[\chi_0 + \tfrac{\chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, |(k+1)h|^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\Upsilon_p\, \chi_{\vartheta/2}\, |(k+1)h|^{(1-\vartheta)/2}\, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}} \right] \\
&\cdot \max_{j \in \{0,1,\dots,k\}} \max\{1, \|Y_{jh}\|_{\mathcal{L}^p(\mathbb{P};V)}\} < \infty.
\end{aligned}
$$

This proves (43) in the case $k+1$. Induction hence proves (43).

In the next step we observe that (43) shows that

$$
\sup_{t \in [0,T]} \|Y_{\lfloor t \rfloor_h}\|_{\mathcal{L}^p(\mathbb{P};V)} = \max_{k \in \{0,1,\dots,\lfloor T \rfloor_h / h\}} \|Y_{kh}\|_{\mathcal{L}^p(\mathbb{P};V)} < \infty. \qquad (45)
$$

Proposition 4.2 hence yields[3] that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|Y_t\|_{\mathcal{L}^p(\mathbb{P};V)} \le \sqrt{2} \\
&\cdot \left[\chi_0\, \|Y_0\|_{\mathcal{L}^p(\mathbb{P};V)} + \tfrac{\chi_\vartheta\, T^{(1-\vartheta)} \|F(0)\|_{V_{-\vartheta}}}{(1-\vartheta)} + \Upsilon_p\, \chi_{\vartheta/2} \sqrt{\tfrac{T^{(1-\vartheta)}}{(1-\vartheta)}}\, \|B(0)\|_{\gamma(U,V_{-\vartheta/2})} \right] \qquad (46)\\
&\cdot \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\sqrt{2}\, \chi_\vartheta\, T^{(1-\vartheta)}\, |F|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{\sqrt{1-\vartheta}} + \Upsilon_p\, \chi_{\vartheta/2} \sqrt{2T^{(1-\vartheta)}}\, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \right].
\end{aligned}
$$

Furthermore, note that (45) ensures that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|\bar{Y}_t\|_{\mathcal{L}^p(\mathbb{P};V)} \le \sup_{t \in [0,T]} \big\| \max\{1, \|Y_t\|_V\} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\cdot \left[\chi_0 + \tfrac{\chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\Upsilon_p\, \chi_{\vartheta/2}\, \sqrt{T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}} \right]. \qquad (47)
\end{aligned}
$$

Moreover, we observe that for all $s, t \in [0,T]$ it holds that

$$
\begin{aligned}
\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_V^p, \|Y_t\|_V^p\}\big] &\le \mathbb{E}\big[\|\bar{Y}_s\|_V^p\big] + \mathbb{E}\big[\max\{1, \|Y_t\|_V^p\}\big] \\
&\le \sup_{u \in [0,T]} \|\bar{Y}_u\|^p_{\mathcal{L}^p(\mathbb{P};V)} + \sup_{u \in [0,T]} \big\|\max\{1, \|Y_u\|_V\}\big\|^p_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}. \qquad (48)
\end{aligned}
$$

This together with (46) and (47) proves the first inequality in (42). The proof of Lemma 5.1 is thus completed. □

[3] with $\kappa = 0$, $L_{0,t} = e^{tA}$, $L_{s,t} = e^{(t - \lfloor s \rfloor_h)A}$, $\Pi(s) = \lfloor s \rfloor_h$ for $(s,t) \in (\angle \cap (0,T]^2)$ in the notation of Proposition 4.2

Proposition 5.2. *Assume the setting in Section 5.1 and let $\Xi \in [0,\infty)$, $q \in [0,\infty) \cap (-\infty, p-3]$, $\rho \in [0, 1-\vartheta)$, $\psi = (\psi(x,y))_{x,y\in V} \in C^2(V \times V, \mathcal{V})$ satisfy for all $x_1, x_2, y \in V$, $i, j \in \{0,1,2\}$ with $i + j \leq 2$ that*

$$\left\| \left(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j} \psi\right)(x_1, y) - \left(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j} \psi\right)(x_2, y) \right\|_{L^{(i+j)}(V,\mathcal{V})} \leq \Xi \max\{1, \|x_1\|_V^q, \|x_2\|_V^q, \|y\|_V^q\} \, \|x_1 - x_2\|_V . \quad (49)$$

Then it holds for all $(s,t) \in \angle$ that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\|_{\mathcal{V}}\big] < \infty$ and

$$\begin{aligned} &\left\| \mathbb{E}\big[\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\big] \right\|_{\mathcal{V}} \leq \Xi \, |\chi_0|^{(q+1)} \, |\chi_\rho|^2 \, \varsigma_{F,B} \, K_{q+3} \, (t-s)^\rho \\ &\cdot \left[\tfrac{2^\rho}{t^\rho} + \tfrac{\left(2\,\chi_\vartheta + \chi_{\rho+\vartheta} + 2\,|\chi_{\vartheta/2}|^2 + 2\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)} + \left(\chi_\vartheta + \frac{1}{2}|\chi_{\vartheta/2}|^2\right) |t-s|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right]. \end{aligned} \quad (50)$$

Proof. Throughout this proof let $(g_r)_{r\in[0,\infty)} \subseteq C(V, \mathbb{R})$ be the functions which satisfy for all $r \in [0,\infty)$, $x \in V$ that $g_r(x) = \max\{1, \|x\|_V^r\}$ and let $\psi_{1,0} \colon V \times V \to L(V,\mathcal{V})$, $\psi_{0,1} \colon V \times V \to L(V,\mathcal{V})$, $\psi_{2,0} \colon V \times V \to L^{(2)}(V,\mathcal{V})$, $\psi_{0,2} \colon V \times V \to L^{(2)}(V,\mathcal{V})$, $\psi_{1,1} \colon V \times V \to L^{(2)}(V,\mathcal{V})$ be the functions which satisfy for all $x, y, v_1, v_2 \in V$ that

$$\begin{aligned} \psi_{1,0}(x,y)\, v_1 &= \left(\tfrac{\partial}{\partial x}\psi\right)(x,y)\, v_1, \qquad & \psi_{0,1}(x,y)\, v_1 &= \left(\tfrac{\partial}{\partial y}\psi\right)(x,y)\, v_1, \\ \psi_{2,0}(x,y)(v_1,v_2) &= \left(\tfrac{\partial^2}{\partial x^2}\psi\right)(x,y)(v_1,v_2), \qquad & \psi_{0,2}(x,y)(v_1,v_2) &= \left(\tfrac{\partial^2}{\partial y^2}\psi\right)(x,y)(v_1,v_2), \\ \psi_{1,1}(x,y)(v_1,v_2) &= \left(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\psi\right)(x,y)(v_1,v_2). \end{aligned} \quad (51)$$

Next we observe that Lemma 5.1 and the assumption that $q \leq p-3$ ensure that $K_{q+1} \leq K_{q+3} < \infty$. Combining this with the fact that

$$\forall\, x_1, x_2, y \in V \colon \|\psi(x_1,y) - \psi(x_2,y)\|_{\mathcal{V}} \leq 2\,\Xi \max\big\{1, \|x_1\|_V^{q+1}, \|x_2\|_V^{q+1}, \|y\|_V^{q+1}\big\} \quad (52)$$

shows that for all $(s,t) \in \angle$ it holds that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\|_{\mathcal{V}}\big] < \infty$. It thus remains to prove (50). To do so, we apply the mild Itô formula in [15]. More formally, an application of Proposition 3.11 in [15] shows that for all $(s,t) \in \angle$ it holds that $\mathbb{E}\big[\|\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\|_{\mathcal{V}}\big] < \infty$ and

$$\begin{aligned} &\left\|\mathbb{E}\big[\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\right\|_{\mathcal{V}} \leq \left\|\mathbb{E}\big[\psi\big(e^{(t-s)A}\,\bar{Y}_s, Y_s\big) - \psi(\bar{Y}_s, Y_s)\big]\right\|_{\mathcal{V}} \\ &+ \int_s^t \mathbb{E}\big[\big\|\psi_{1,0}(e^{(t-r)A}\,\bar{Y}_r, Y_s)\, e^{(t-r)A} F(Y_{\lfloor r \rfloor_h})\big\|_{\mathcal{V}}\big]\, dr \\ &+ \int_s^t \mathbb{E}\left[\left\| \tfrac{1}{2} \sum_{b\in\mathbb{U}} \psi_{2,0}(e^{(t-r)A}\,\bar{Y}_r, Y_s)\big(e^{(t-r)A}\, B^b(Y_{\lfloor r \rfloor_h}), e^{(t-r)A}\, B^b(Y_{\lfloor r \rfloor_h})\big) \right\|_{\mathcal{V}}\right] dr. \end{aligned} \quad (53)$$

In the following we establish suitable estimates for the three summands appearing on the right hand side of (53). Combining these estimates with (53) will then allow us to establish (50). We begin with the second and the third summands on the right hand side of (53). We note that the assumption that

$$\forall\, x_1, x_2, y \in V \colon \ \|\psi(x_1,y) - \psi(x_2,y)\|_{\mathcal{V}} \leq \Xi \max\{1, \|x_1\|_V^q, \|x_2\|_V^q, \|y\|_V^q\}\, \|x_1 - x_2\|_V \quad (54)$$

implies that $\forall\, x, y \in V \colon \|\psi_{1,0}(x,y)\|_{L(V,\mathcal{V})} \leq \Xi \max\{1, \|x\|_V^q, \|y\|_V^q\}$. This, in turn, proves that for all $(r,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\big\|\psi_{1,0}(e^{(t-r)A}\, u, v)\, e^{(t-r)A}\, F(w)\big\|_{\mathcal{V}} \\ &\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \|e^{(t-r)A}\|_{L(V,V_\vartheta)}\, \|F(w)\|_{V_{-\vartheta}} \\ &\leq \frac{\Xi\, |\chi_0|^q\, \chi_\vartheta \max\{1, \|u\|_V^q, \|v\|_V^q\} \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, g_1(w)}{(t-r)^\vartheta}. \end{aligned} \quad (55)$$

Next we observe that the assumption that

$$\forall\, x_1, x_2, y \in V\colon\ \|\psi_{1,0}(x_1,y) - \psi_{1,0}(x_2,y)\|_{L(V,\mathcal{V})} \le \Xi \max\{1, \|x_1\|_V^q, \|x_2\|_V^q, \|y\|_V^q\} \|x_1 - x_2\|_V \quad (56)$$

shows that $\forall\, x, y \in V\colon \|\psi_{2,0}(x,y)\|_{L^{(2)}(V,\mathcal{V})} \le \Xi \max\{1, \|x\|_V^q, \|y\|_V^q\}$. This and Lemma 1.2 prove that for all $(r,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\tfrac{1}{2}\Big\| \sum_{b\in\mathbb{U}} \psi_{2,0}\big(e^{(t-r)A}\, u, v\big)\big(e^{(t-r)A}\, B^b(w), e^{(t-r)A}\, B^b(w)\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{1}{2}\, \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \|e^{(t-r)A} B(w)\|^2_{\gamma(U,V)} \\ &\le \frac{\Xi\, |\chi_0|^q\, |\chi_{\vartheta/2}|^2 \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, g_2(w)}{2(t-r)^{\vartheta}}. \end{aligned} \quad (57)$$

Furthermore, we note that Hölder's inequality implies that for all $r, l \in (0,\infty)$, $s,t \in [0,T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_V^r, \|Y_t\|_V^r\big\}\, g_l(Y_{\lfloor s\rfloor_h})\big] \\ &\le \bigg(\sup_{u,v\in[0,T]} \big\|\max\big\{1, \|\bar{Y}_u\|_V^r, \|Y_v\|_V^r\big\}\big\|_{\mathcal{L}^{1+l/r}(\mathbb{P};\mathbb{R})}\bigg) \bigg(\sup_{u\in[0,T]} \big\|\max\big\{1, \|Y_u\|_V^l\big\}\big\|_{\mathcal{L}^{1+r/l}(\mathbb{P};\mathbb{R})}\bigg) \\ &\le |K_{r+l}|^{\frac{1}{1+l/r}}\, |K_{r+l}|^{\frac{1}{1+r/l}} = K_{r+l}. \end{aligned} \quad (58)$$

This and the fact that for all $l \in [0,\infty)$ it holds that $\sup_{s\in[0,T]} \mathbb{E}\big[g_l(Y_{\lfloor s\rfloor_h})\big] \le K_l$ prove that for all $r, l \in [0,\infty)$, $s,t \in [0,T]$ it holds that

$$\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_V^r, \|Y_t\|_V^r\big\}\, g_l(Y_{\lfloor s\rfloor_h})\big] \le K_{r+l}. \quad (59)$$

Combining (55), (57), and (59) implies that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned} &\int_s^t \mathbb{E}\big[\big\|\psi_{1,0}(e^{(t-r)A}\, \bar{Y}_r, Y_s)\, e^{(t-r)A} F(Y_{\lfloor r\rfloor_h})\big\|_{\mathcal{V}}\big]\, dr \\ &+ \int_s^t \mathbb{E}\bigg[\Big\|\tfrac{1}{2}\sum_{b\in\mathbb{U}} \psi_{2,0}(e^{(t-r)A}\, \bar{Y}_r, Y_s)\big(e^{(t-r)A}\, B^b(Y_{\lfloor r\rfloor_h}), e^{(t-r)A}\, B^b(Y_{\lfloor r\rfloor_h})\big)\Big\|_{\mathcal{V}}\bigg]\, dr \\ &\le \Xi\, |\chi_0|^q \Big(\chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})} + \tfrac{1}{2}\, |\chi_{\vartheta/2}|^2\, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\Big)\, K_{q+2} \int_s^t \frac{1}{(t-r)^{\vartheta}}\, dr \\ &\le \frac{\Xi\, |\chi_0|^q\, \big(\chi_\vartheta + \frac{1}{2}|\chi_{\vartheta/2}|^2\big)\, \varsigma_{F,B}\, K_{q+2}\, (t-s)^{(1-\vartheta)}}{(1-\vartheta)}. \end{aligned} \quad (60)$$

Inequality (60) provides us an appropriate estimate for the second and the third summand on the right hand side of (53). It thus remains to provide a suitable estimate for the first summand on the right hand side of (53). For this we will employ the mild Itô formula in [15] again and this will allow us to obtain an appropriate upper bound for $\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_{\mathcal{V}}$ for $(s,t) \in \angle$. More formally, let $\tilde{F}_{r,s,t}\colon V \times V \times V \to \mathcal{V}$, $r \in [0,s)$, $s \in (0,t)$, $t \in (0,T]$, be the functions which satisfy for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u, v, w \in V$ that

$$\begin{aligned} \tilde{F}_{r,s,t}(u,v,w) &= \psi_{1,0}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big)\, e^{(t-r)A}\, F(w) - \psi_{1,0}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\, e^{(s-r)A}\, F(w) \\ &\quad + \big[\psi_{0,1}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big) - \psi_{0,1}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big]\, e^{(s-\lfloor r\rfloor_h)A}\, F(w) \end{aligned} \quad (61)$$

and let $\tilde{B}_{r,s,t}\colon V \times V \times V \to \mathcal{V}$, $r \in [0,s)$, $s \in (0,t)$, $t \in (0,T]$, be the functions which satisfy for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in V$ that

$$
\begin{aligned}
&\tilde{B}_{r,s,t}(u,v,w) = \tfrac{1}{2} \sum_{b \in \mathbb{U}} \psi_{2,0}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big)\big(e^{(t-r)A}\, B^b(w), e^{(t-r)A}\, B^b(w)\big)\\
&- \tfrac{1}{2} \sum_{b \in \mathbb{U}} \psi_{2,0}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big(e^{(s-r)A}\, B^b(w), e^{(s-r)A}\, B^b(w)\big)\\
&+ \tfrac{1}{2} \sum_{b \in \mathbb{U}} \big[\psi_{0,2}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big) - \psi_{0,2}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big]\big(e^{(s-\lfloor r \rfloor_h)A}\, B^b(w), e^{(s-\lfloor r \rfloor_h)A}\, B^b(w)\big)\\
&+ \sum_{b \in \mathbb{U}} \psi_{1,1}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big)\big(e^{(t-r)A}\, B^b(w), e^{(s-\lfloor r \rfloor_h)A}\, B^b(w)\big)\\
&- \sum_{b \in \mathbb{U}} \psi_{1,1}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big(e^{(s-r)A}\, B^b(w), e^{(s-\lfloor r \rfloor_h)A}\, B^b(w)\big).
\end{aligned}
\tag{62}
$$

An application of Proposition 3.11 in [15] shows that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_{\mathcal{V}} \leq \mathbb{E}\big[\big\|\psi\big(e^{tA}\, Y_0, e^{sA}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, e^{sA}\, Y_0\big)\big\|_{\mathcal{V}}\big]\\
&+ \int_0^s \mathbb{E}\big[\|\tilde{F}_{r,s,t}\big(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h}\big)\|_{\mathcal{V}}\big]\, dr + \int_0^s \mathbb{E}\big[\|\tilde{B}_{r,s,t}\big(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h}\big)\|_{\mathcal{V}}\big]\, dr.
\end{aligned}
\tag{63}
$$

In the next step we estimate the summands on the right hand side of (63). We observe that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$
\begin{aligned}
&\big\|\psi\big(e^{tA}\, Y_0, e^{sA}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, e^{sA}\, Y_0\big)\big\|_{\mathcal{V}}\\
&\leq \Xi \max\big\{1, \|e^{tA}\, Y_0\|_V^q, \|e^{sA}\, Y_0\|_V^q\big\}\, \|e^{tA}\, Y_0 - e^{sA}\, Y_0\|_V\\
&\leq \Xi\, |\chi_0|^q\, g_q(Y_0)\, \|e^{tA} - e^{sA}\|_{L(V)}\, \|Y_0\|_V \leq \Xi\, |\chi_0|^q\, g_{q+1}(Y_0)\, \tfrac{|\chi_\rho|^2\, (t-s)^\rho}{s^\rho}.
\end{aligned}
\tag{64}
$$

This and the fact that $\mathbb{E}\big[g_{q+1}(Y_0)\big] \leq K_{q+1}$ imply that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$
\mathbb{E}\big[\|\psi\big(e^{tA}\, Y_0, e^{sA}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, e^{sA}\, Y_0\big)\|_{\mathcal{V}}\big] \leq \Xi\, |\chi_0|^q\, K_{q+1}\, \tfrac{|\chi_\rho|^2}{s^\rho}\, (t-s)^\rho\,.
\tag{65}
$$

Inequality (65) provides us an appropriate estimate for the first summand on the right hand side of (63). In the next step we establish a suitable bound for the second summand on the right hand side of (63). Note that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in V$ it holds that

$$
\begin{aligned}
&\big\|\psi_{1,0}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big)\, e^{(t-r)A}\, F(w) - \psi_{1,0}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\, e^{(s-r)A}\, F(w)\big\|_{\mathcal{V}}\\
&\leq \big\|\big[\psi_{1,0}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big) - \psi_{1,0}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big]\, e^{(t-r)A}\, F(w)\big\|_{\mathcal{V}}\\
&+ \big\|\psi_{1,0}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\, e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_V\big)\, F(w)\big\|_{\mathcal{V}}\\
&\leq \Xi\, \max\big\{1, \|e^{(t-r)A}u\|_V^q, \|e^{(s-r)A}u\|_V^q, \|e^{(s-r)A}v\|_V^q\big\}\, \big\|e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_V\big)\, u\big\|_V\\
&\cdot \|e^{(t-r)A}F(w)\|_V + \Xi\, \max\big\{1, \|e^{(s-r)A}u\|_V^q, \|e^{(s-r)A}v\|_V^q\big\}\, \big\|e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_V\big)\, F(w)\big\|_V\\
&\leq \frac{\Xi\, |\chi_0|^q \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, |\chi_\rho|^2\, (t-s)^\rho\, \|u\|_V\, \chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, g_1(w)}{(s-r)^\rho\, (t-r)^\vartheta}\\
&+ \frac{\Xi\, |\chi_0|^q \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, \chi_{\rho+\vartheta}\, \chi_\rho\, (t-s)^\rho\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, g_1(w)}{(s-r)^{(\rho+\vartheta)}},
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
&\big\|\big[\psi_{0,1}\big(e^{(t-r)A}\, u, e^{(s-r)A}\, v\big) - \psi_{0,1}\big(e^{(s-r)A}\, u, e^{(s-r)A}\, v\big)\big]\, e^{(s-\lfloor r \rfloor_h)A}\, F(w)\big\|_{\mathcal{V}}\\
&\leq \Xi\, |\chi_0|^q \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, \big\|e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_V\big)\, u\big\|_V\, \|e^{(s-\lfloor r \rfloor_h)A}\|_{L(V_{-\vartheta},V)}\, \|F(w)\|_{V_{-\vartheta}}\\
&\leq \frac{\Xi\, |\chi_0|^q \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, |\chi_\rho|^2\, (t-s)^\rho\, \|u\|_V\, \chi_\vartheta\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, g_1(w)}{(s-r)^{(\rho+\vartheta)}}\\
&\leq \frac{\Xi\, |\chi_0|^q\, |\chi_\rho|^2\, \chi_\vartheta \max\big\{1, \|u\|_V^{q+1}, \|v\|_V^{q+1}\big\}\, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}\, g_1(w)\, (t-s)^\rho}{(s-r)^{(\rho+\vartheta)}}.
\end{aligned}
\tag{67}
$$

Inequalities (66) and (67) prove that for all $t \in (0, T]$, $s \in (0, t)$, $r \in [0, s)$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\|\tilde{F}_{r,s,t}(u, v, w)\|_{\mathcal{V}} \\ &\leq \Xi\, |\chi_0|^q \left[\frac{|\chi_\rho|^2\, \chi_\vartheta}{(s-r)^\rho\, (t-r)^\vartheta} + \frac{\chi_{\rho+\vartheta}\, \chi_\rho}{(s-r)^{(\rho+\vartheta)}} + \frac{|\chi_\rho|^2\, \chi_\vartheta}{(s-r)^{(\rho+\vartheta)}} \right] \\ &\quad \cdot \max\bigl\{1, \|u\|_V^{q+1}, \|v\|_V^{q+1}\bigr\}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, g_1(w)\, (t-s)^\rho \\ &\leq \Xi\, |\chi_0|^q \left[\frac{\chi_\rho\, (2\, \chi_\rho\, \chi_\vartheta + \chi_{\rho+\vartheta})}{(s-r)^{(\rho+\vartheta)}} \right] \\ &\quad \cdot \max\bigl\{1, \|u\|_V^{q+1}, \|v\|_V^{q+1}\bigr\}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, g_1(w)\, (t-s)^\rho. \end{aligned} \tag{68}$$

This and (59) prove that for all $t \in (0, T]$, $s \in (0, t)$ it holds that

$$\begin{aligned} &\int_0^s \mathbb{E}\bigl[\|\tilde{F}_{r,s,t}(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h})\|_{\mathcal{V}}\bigr]\, dr \\ &\leq \frac{\Xi\, |\chi_0|^q\, \chi_\rho\, (2\, \chi_\rho\, \chi_\vartheta + \chi_{\rho+\vartheta})}{(1-\vartheta-\rho)}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, K_{q+2}\, (t-s)^\rho\, s^{(1-\vartheta-\rho)}. \end{aligned} \tag{69}$$

Next we provide an appropriate bound for the third summand on the right hand side of (63). Observe that Lemma 1.2 shows that for all $t \in (0, T]$, $s \in (0, t)$, $r \in [0, s)$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\Bigl\| \sum_{b \in \mathbb{U}} \psi_{2,0}\bigl(e^{(t-r)A} u, e^{(s-r)A} v\bigr) \bigl(e^{(t-r)A} B^b(w), e^{(t-r)A} B^b(w)\bigr) \\ &- \sum_{b \in \mathbb{U}} \psi_{2,0}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr) \bigl(e^{(s-r)A} B^b(w), e^{(s-r)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\leq \Bigl\| \sum_{b \in \mathbb{U}} \bigl[\psi_{2,0}\bigl(e^{(t-r)A} u, e^{(s-r)A} v\bigr) - \psi_{2,0}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr)\bigr] \bigl(e^{(t-r)A} B^b(w), e^{(t-r)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &+ \Bigl\| \sum_{b \in \mathbb{U}} \psi_{2,0}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr) \bigl((e^{(t-r)A} + e^{(s-r)A})\, B^b(w), e^{(s-r)A}\, (e^{(t-s)A} - \mathrm{Id}_V)\, B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\leq \frac{\Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, |\chi_\rho|^2\, (t-s)^\rho\, \|u\|_V\, |\chi_{\vartheta/2}|^2\, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\, g_2(w)}{(s-r)^\rho\, (t-r)^\vartheta} \\ &+ \frac{\Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, 2\, \chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2}\, \chi_\rho\, (t-s)^\rho\, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\, g_2(w)}{(s-r)^{(\rho+\vartheta)}}, \end{aligned} \tag{70}$$

$$\begin{aligned} &\Bigl\| \sum_{b \in \mathbb{U}} \bigl[\psi_{0,2}\bigl(e^{(t-r)A} u, e^{(s-r)A} v\bigr) - \psi_{0,2}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr)\bigr] \bigl(e^{(s-\lfloor r \rfloor_h)A} B^b(w), e^{(s-\lfloor r \rfloor_h)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\leq \frac{\Xi\, |\chi_0|^q \max\bigl\{1, \|u\|_V^q, \|v\|_V^q\bigr\}\, |\chi_\rho|^2\, (t-s)^\rho\, \|u\|_V\, |\chi_{\vartheta/2}|^2\, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\, g_2(w)}{(s-r)^{(\rho+\vartheta)}}, \end{aligned} \tag{71}$$

$$\begin{aligned} &\Bigl\| \sum_{b \in \mathbb{U}} \psi_{1,1}\bigl(e^{(t-r)A} u, e^{(s-r)A} v\bigr) \bigl(e^{(t-r)A} B^b(w), e^{(s-\lfloor r \rfloor_h)A} B^b(w)\bigr) \\ &- \sum_{b \in \mathbb{U}} \psi_{1,1}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr) \bigl(e^{(s-r)A} B^b(w), e^{(s-\lfloor r \rfloor_h)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\leq \Bigl\| \sum_{b \in \mathbb{U}} \bigl[\psi_{1,1}\bigl(e^{(t-r)A} u, e^{(s-r)A} v\bigr) - \psi_{1,1}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr)\bigr] \bigl(e^{(t-r)A} B^b(w), e^{(s-\lfloor r \rfloor_h)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\quad + \Bigl\| \sum_{b \in \mathbb{U}} \psi_{1,1}\bigl(e^{(s-r)A} u, e^{(s-r)A} v\bigr) \bigl(e^{(s-r)A}\, (e^{(t-s)A} - \mathrm{Id}_V)\, B^b(w), e^{(s-\lfloor r \rfloor_h)A} B^b(w)\bigr) \Bigr\|_{\mathcal{V}} \\ &\leq \frac{\Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, |\chi_\rho|^2\, (t-s)^\rho\, \|u\|_V\, |\chi_{\vartheta/2}|^2\, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\, g_2(w)}{(s-r)^{(\rho+\vartheta/2)}\, (t-r)^{\vartheta/2}} \end{aligned} \tag{72}$$

$$+ \frac{\Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \chi_{\rho+\vartheta/2}\, \chi_\rho\, (t-s)^\rho\, \chi_{\vartheta/2}\, \|B\|^2_{\operatorname{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, g_2(w)}{(s-r)^{(\rho+\vartheta)}}.$$

Inequalities (70)–(72) imply that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in V$ it holds that

$$\begin{aligned} &\|\tilde{B}_{r,s,t}(u,v,w)\|_{\mathcal{V}} \le \Xi\, |\chi_0|^q \max\{1, \|u\|_V^{q+1}, \|v\|_V^{q+1}\}\, (t-s)^\rho\, \|B\|^2_{\operatorname{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, g_2(w) \\ &\cdot \left[\tfrac{\frac{1}{2}\,|\chi_\rho|^2\, |\chi_{\vartheta/2}|^2}{(s-r)^\rho\, (t-r)^\vartheta} + \tfrac{\chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2}\, \chi_\rho}{(s-r)^{(\rho+\vartheta)}} + \tfrac{\frac{1}{2}\,|\chi_\rho|^2\, |\chi_{\vartheta/2}|^2}{(s-r)^{(\rho+\vartheta)}} + \tfrac{|\chi_\rho|^2\, |\chi_{\vartheta/2}|^2}{(s-r)^{(\rho+\vartheta/2)}\, (t-r)^{\vartheta/2}} + \tfrac{\chi_{\rho+\vartheta/2}\, \chi_\rho\, \chi_{\vartheta/2}}{(s-r)^{(\rho+\vartheta)}} \right] \\ &\le \Xi\, |\chi_0|^q \max\{1, \|u\|_V^{q+1}, \|v\|_V^{q+1}\}\, (t-s)^\rho\, \|B\|^2_{\operatorname{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, g_2(w) \\ &\cdot \frac{2\, \chi_\rho\, \chi_{\vartheta/2}\, \big(\chi_\rho\, \chi_{\vartheta/2} + \chi_{\rho+\vartheta/2}\big)}{(s-r)^{(\rho+\vartheta)}}. \end{aligned} \tag{73}$$

This and (59) prove that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned} &\int_0^s \big\|\mathbb{E}\big[\tilde{B}_{r,s,t}(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h})\big]\big\|_{\mathcal{V}}\, dr \\ &\le \Xi\, |\chi_0|^q\, K_{q+3}\, (t-s)^\rho\, \|B\|^2_{\operatorname{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, \frac{2\, \chi_\rho\, \chi_{\vartheta/2}\, \big(\chi_\rho\, \chi_{\vartheta/2} + \chi_{\rho+\vartheta/2}\big)\, s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}. \end{aligned} \tag{74}$$

Combining (63) with the estimates (65), (69), and (74) yields that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_{\mathcal{V}} \le \Xi\, |\chi_0|^q\, K_{q+1}\, \tfrac{|\chi_\rho|^2}{s^\rho}\, (t-s)^\rho \\ &+ \frac{\Xi\, |\chi_0|^q\, \chi_\rho\, (2\, \chi_\rho \chi_\vartheta + \chi_{\rho+\vartheta})}{(1-\vartheta-\rho)}\, \|F\|_{\operatorname{Lip}^0(V,V_{-\vartheta})}\, K_{q+2}\, (t-s)^\rho\, s^{(1-\vartheta-\rho)} \\ &+ \frac{2\, \Xi\, |\chi_0|^q\, \chi_\rho\, \chi_{\vartheta/2}\, \big(\chi_\rho\, \chi_{\vartheta/2} + \chi_{\rho+\vartheta/2}\big)}{(1-\vartheta-\rho)}\, \|B\|^2_{\operatorname{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\, K_{q+3}\, (t-s)^\rho\, s^{(1-\vartheta-\rho)} \\ &\le \Xi\, |\chi_0|^q\, \varsigma_{F,B}\, K_{q+3}\, (t-s)^\rho \left[\tfrac{|\chi_\rho|^2}{s^\rho} + \tfrac{\chi_\rho \left(2\, \chi_\rho \chi_\vartheta + \chi_{\rho+\vartheta} + 2\, \chi_\rho\, |\chi_{\vartheta/2}|^2 + 2\, \chi_{\rho+\vartheta/2}\, \chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right] \\ &\le \Xi\, |\chi_0|^q\, |\chi_\rho|^2\, \varsigma_{F,B}\, K_{q+3}\, (t-s)^\rho \left[\tfrac{1}{s^\rho} + \tfrac{\left(2\, \chi_\vartheta + \chi_{\rho+\vartheta} + 2\, |\chi_{\vartheta/2}|^2 + 2\, \chi_{\rho+\vartheta/2}\, \chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right]. \end{aligned} \tag{75}$$

In addition, we note that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_{\mathcal{V}} \\ &\le \Xi\, \mathbb{E}\big[\max\{1, \|e^{(t-s)A} \bar{Y}_s\|_V^q, \|\bar{Y}_s\|_V^q, \|Y_s\|_V^q\}\, \|e^{(t-s)A} - \operatorname{Id}_V\|_{L(V)}\, \|\bar{Y}_s\|_V\big] \\ &\le \Xi\, |\chi_0|^{(q+1)}\, \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_V^q, \|Y_s\|_V^q\}\, \|\bar{Y}_s\|_V\big] \\ &\le \Xi\, |\chi_0|^{(q+1)}\, \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_V^{q+1}, \|Y_s\|_V^{q+1}\}\big] \le \Xi\, |\chi_0|^{(q+1)} K_{q+1}. \end{aligned} \tag{76}$$

Combining this with (75) proves that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_{\mathcal{V}} \\ &\le \Xi\, |\chi_0|^{(q+1)}\, |\chi_\rho|^2\, \varsigma_{F,B}\, K_{q+3} \left[\min\left\{1, \tfrac{(t-s)^\rho}{s^\rho}\right\} + \tfrac{(t-s)^\rho \left(2\, \chi_\vartheta + \chi_{\rho+\vartheta} + 2\, |\chi_{\vartheta/2}|^2 + 2\, \chi_{\rho+\vartheta/2}\, \chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right] \\ &= \Xi\, |\chi_0|^{(q+1)}\, |\chi_\rho|^2\, \varsigma_{F,B}\, K_{q+3} \\ &\cdot \left[\mathbb{1}_{[\frac{t}{2},T]}(s) \cdot \tfrac{(t-s)^\rho}{s^\rho} + \mathbb{1}_{[0,\frac{t}{2})}(s) \cdot \tfrac{(t-s)^\rho}{(t-s)^\rho} + \tfrac{(t-s)^\rho \left(2\, \chi_\vartheta + \chi_{\rho+\vartheta} + 2\, |\chi_{\vartheta/2}|^2 + 2\, \chi_{\rho+\vartheta/2}\, \chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right] \\ &\le \Xi\, |\chi_0|^{(q+1)}\, |\chi_\rho|^2\, \varsigma_{F,B}\, K_{q+3} \left[\tfrac{(t-s)^\rho}{(t/2)^\rho} + \tfrac{(t-s)^\rho \left(2\, \chi_\vartheta + \chi_{\rho+\vartheta} + 2\, |\chi_{\vartheta/2}|^2 + 2\, \chi_{\rho+\vartheta/2}\, \chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right]. \end{aligned} \tag{77}$$

Combining this, (60), and (53) establishes that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_s\big) - \psi\big(\bar{Y}_s, Y_s\big)\big]\big\|_{\mathcal{V}} \leq \Xi \, |\chi_0|^{(q+1)} \, |\chi_\rho|^2 \, \varsigma_{F,B} \, K_{q+3} \, (t-s)^\rho \\ &\cdot \left[\left|\tfrac{2}{t}\right|^\rho + \tfrac{\left(2\,\chi_\vartheta + \chi_{\rho+\vartheta} + 2\,|\chi_{\vartheta/2}|^2 + 2\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2}\right) s^{(1-\vartheta-\rho)} + \left(\chi_\vartheta + \frac{1}{2}|\chi_{\vartheta/2}|^2\right) |t-s|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \right]. \end{aligned} \tag{78}$$

The proof of Proposition 5.2 is thus completed. □

5.3 Analysis of the weak distance between exponential Euler approximations and their semilinear integrated counterparts

Lemma 5.3 (Analysis of the analytically weak but probabilistically strong distance between exponential Euler approximations and their semilinear integrated counterparts)**.** *Assume the setting in Section 5.1 and let $\rho \in [0,1)$, $\varrho \in \big[0, 1 - \max\{\frac{1+\vartheta}{2} - \rho, 0\}\big)$, $t \in (0,T]$. Then*

$$\begin{aligned} &\|Y_t - \bar{Y}_t\|_{\mathcal{L}^p(\mathbb{P};V_{-\rho})} \leq |K_p|^{\frac{1}{p}} \; \chi_\varrho \, h^\varrho \\ &\cdot \left[\frac{\chi_{\varrho+\vartheta-\rho} \, t^{1-\max\{\vartheta+\varrho-\rho,0\}} \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})}}{(1-\max\{\vartheta+\varrho-\rho,0\})} + \frac{\Upsilon_p \, \chi_{\varrho+\vartheta/2-\rho} \, \sqrt{t^{1-\max\{\vartheta+2\varrho-2\rho,0\}}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}}{\sqrt{1-\max\{\vartheta+2\varrho-2\rho,0\}}} \right]. \end{aligned} \tag{79}$$

Proof. First of all, we observe that

$$\begin{aligned} \big\|Y_t - \bar{Y}_t\big\|_{\mathcal{L}^p(\mathbb{P};V_{-\rho})} \leq & \left\| \int_0^t \left(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\right) F(Y_{\lfloor s \rfloor_h}) \, ds \right\|_{\mathcal{L}^p(\mathbb{P};V_{-\rho})} \\ &+ \left\| \int_0^t \left(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\right) B(Y_{\lfloor s \rfloor_h}) \, dW_s \right\|_{L^p(\mathbb{P};V_{-\rho})}. \end{aligned} \tag{80}$$

Next we note that

$$\begin{aligned} &\left\| \int_0^t \left(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\right) F(Y_{\lfloor s \rfloor_h}) \, ds \right\|_{\mathcal{L}^p(\mathbb{P};V_{-\rho})} \\ &\leq \int_0^t \big\|(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}) F(Y_{\lfloor s \rfloor_h})\big\|_{\mathcal{L}^p(\mathbb{P};V_{-\rho})} \, ds \\ &\leq \int_0^t \frac{\chi_{\varrho+\vartheta-\rho} \, \chi_\varrho \, h^\varrho \, \|F(Y_{\lfloor s \rfloor_h})\|_{\mathcal{L}^p(\mathbb{P};V_{-\vartheta})}}{(t-s)^{\max\{\vartheta+\varrho-\rho,0\}}} \, ds \\ &\leq \frac{\chi_{\varrho+\vartheta-\rho} \, \chi_\varrho \, t^{1-\max\{\vartheta+\varrho-\rho,0\}}}{(1-\max\{\vartheta+\varrho-\rho,0\})} \, \|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})} \, |K_p|^{\frac{1}{p}} \, h^\varrho. \end{aligned} \tag{81}$$

Moreover, Lemma 5.1 ensures that $K_p < \infty$. This assures that

$$\begin{aligned} &\left\| \int_0^t (e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}) B\big(Y_{\lfloor s \rfloor_h}\big) \, dW_s \right\|_{L^p(\mathbb{P};V_{-\rho})} \\ &\leq \Upsilon_p \left[\int_0^t \big\|\big(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\big) B(Y_{\lfloor s \rfloor_h})\big\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_{-\rho}))} \, ds \right]^{1/2} \\ &\leq \Upsilon_p \left[\int_0^t \frac{|\chi_{\varrho+\vartheta/2-\rho} \, \chi_\varrho|^2 \, h^{2\varrho} \, \|B(Y_{\lfloor s \rfloor_h})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_{-\vartheta/2}))}}{(t-s)^{\max\{\vartheta+2\varrho-2\rho,0\}}} \, ds \right]^{1/2} \\ &\leq \frac{\Upsilon_p \, \chi_{\varrho+\vartheta/2-\rho} \, \chi_\varrho \, \sqrt{t^{1-\max\{\vartheta+2\varrho-2\rho,0\}}}}{\sqrt{1-\max\{\vartheta+2\varrho-2\rho,0\}}} \, \|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \, |K_p|^{\frac{1}{p}} \, h^\varrho. \end{aligned} \tag{82}$$

Combining (80)–(82) completes the proof of Lemma 5.3. □

Proposition 5.4 (Weak distance between exponential Euler approximations and their semilinear integrated counterparts)**.** *Assume the setting in Section 5.1 and let $\Xi \in [0,\infty)$, $q \in [0,\infty) \cap (-\infty, p-3]$, $\rho \in [0, 1-\vartheta)$, $\psi = (\psi(x,y))_{x,y\in V} \in C^2(V \times V, \mathcal{V})$ satisfy for all $x, y_1, y_2 \in V$, $i, j \in \mathbb{N}_0$ with $i + j \leq 2$ that*

$$\left\| \left(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\right)(x, y_1) - \left(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\right)(x, y_2) \right\|_{L^{(i+j)}(V,\mathcal{V})} \leq \Xi \max\{1, \|x\|_V^q, \|y_1\|_V^q, \|y_2\|_V^q\} \, \|y_1 - y_2\|_V. \quad (83)$$

Then it holds for all $t \in (0, T]$ that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_t) - \psi(\bar{Y}_t, \bar{Y}_t)\|_{\mathcal{V}}\big] < \infty$ and

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_t\big) - \psi\big(\bar{Y}_t, \bar{Y}_t\big)\big]\big\|_{\mathcal{V}} \leq \Xi \, |\chi_0|^q \, \chi_\rho \, \varsigma_{F,B} \, K_{q+3} \, h^\rho \\ &\cdot \frac{t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \Bigg[\chi_{\rho+\vartheta} + 2\, \chi_{\vartheta/2} \, \chi_{\rho+\vartheta/2} + 2\, \chi_\rho \left(|\chi_{\vartheta/2}|^2 + \chi_\vartheta \right) \\ &\cdot \left(\frac{\chi_\vartheta \, t^{(1-\vartheta)} \, \|F\|_{\operatorname{Lip}^0(V, V_{-\vartheta})}}{(1-\vartheta)} + \frac{\Upsilon_{q+3} \, \chi_{\vartheta/2} \sqrt{t^{(1-\vartheta)}} \, \|B\|_{\operatorname{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}} \right) \Bigg]. \end{aligned} \quad (84)$$

Proof. Throughout this proof let $(g_r)_{r\in[0,\infty)} \subseteq C(V, \mathbb{R})$ be the functions which satisfy for all $r \in [0,\infty)$, $x \in V$ that $g_r(x) = \max\{1, \|x\|_V^r\}$ and let $\psi_{1,0} \colon V \times V \to L(V, \mathcal{V})$, $\psi_{0,1} \colon V \times V \to L(V, \mathcal{V})$, $\psi_{2,0} \colon V \times V \to L^{(2)}(V, \mathcal{V})$, $\psi_{0,2} \colon V \times V \to L^{(2)}(V, \mathcal{V})$, $\psi_{1,1} \colon V \times V \to L^{(2)}(V, \mathcal{V})$ be the functions which satisfy for all $x, y, v_1, v_2 \in V$ that

$$\begin{aligned} \psi_{1,0}(x,y)\, v_1 &= \left(\tfrac{\partial}{\partial x}\psi\right)(x,y)\, v_1, \qquad & \psi_{0,1}(x,y)\, v_1 &= \left(\tfrac{\partial}{\partial y}\psi\right)(x,y)\, v_1, \\ \psi_{2,0}(x,y)(v_1,v_2) &= \left(\tfrac{\partial^2}{\partial x^2}\psi\right)(x,y)(v_1,v_2), \qquad & \psi_{0,2}(x,y)(v_1,v_2) &= \left(\tfrac{\partial^2}{\partial y^2}\psi\right)(x,y)(v_1,v_2), \\ \psi_{1,1}(x,y)(v_1,v_2) &= \left(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\psi\right)(x,y)(v_1,v_2). \end{aligned} \quad (85)$$

Next we observe that Lemma 5.1 and the assumption that $q \leq p-3$ ensure that $K_{q+1} \leq K_{q+3} < \infty$. Combining this with the fact that

$$\forall\, x, y_1, y_2 \in V \colon \|\psi(x,y_1) - \psi(x,y_2)\|_{\mathcal{V}} \leq 2\,\Xi \max\big\{1, \|x\|_V^{q+1}, \|y_1\|_V^{q+1}, \|y_2\|_V^{q+1}\big\} \quad (86)$$

shows that for all $t \in (0,T]$ it holds that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_t) - \psi(\bar{Y}_t, \bar{Y}_t)\|_{\mathcal{V}}\big] < \infty$. It thus remains to prove (84). To do so, we make use of the mild Itô formula in [15]. For this let $\tilde{F}_{s,t} \colon V \times V \times V \to \mathcal{V}$, $(s,t) \in \angle$, be the functions which satisfy for all $(s,t) \in \angle$, $u, v, w \in V$ that

$$\begin{aligned} \tilde{F}_{s,t}(u,v,w) &= \big[\psi_{1,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{1,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\, e^{(t-s)A}\, F(w) \\ &\quad + \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\, e^{(t-\lfloor s \rfloor_h)A}\, F(w) - \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\, e^{(t-s)A}\, F(w) \end{aligned} \quad (87)$$

and let $\tilde{B}_{s,t} \colon V \times V \times V \to \mathcal{V}$, $(s,t) \in \angle$, be the functions which satisfy for all $(s,t) \in \angle$, $u, v, w \in V$ that

$$\begin{aligned} &\tilde{B}_{s,t}(u,v,w) \\ &= \tfrac{1}{2} \sum_{b\in\mathbb{U}} \big[\psi_{2,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{2,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \\ &+ \tfrac{1}{2} \sum_{b\in\mathbb{U}} \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\big(e^{(t-\lfloor s \rfloor_h)A}\, B^b(w), e^{(t-\lfloor s \rfloor_h)A}\, B^b(w)\big) \\ &- \tfrac{1}{2} \sum_{b\in\mathbb{U}} \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \\ &+ \sum_{b\in\mathbb{U}} \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-\lfloor s \rfloor_h)A}\, B^b(w)\big) \\ &- \sum_{b\in\mathbb{U}} \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big). \end{aligned} \quad (88)$$

An application of Proposition 3.11 in [15] shows that for all $t \in (0,T]$ it holds that

$$\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_t\big) - \psi\big(\bar{Y}_t, \bar{Y}_t\big)\big]\big\|_{\mathcal{V}} \leq \int_0^t \mathbb{E}\Big[\big\|\tilde{F}_{s,t}\big(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h}\big)\big\|_{\mathcal{V}}\Big] + \mathbb{E}\Big[\big\|\tilde{B}_{s,t}\big(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h}\big)\big\|_{\mathcal{V}}\Big]\, ds. \quad (89)$$

In the following we establish suitable estimates for the two integrands on the right hand side of (89). We begin with the first integrand on the right hand side of (89). Observe that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\big\| \big[\psi_{1,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{1,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\, e^{(t-s)A}\, F(w) \big\|_{\mathcal{V}} \\ &\leq \Xi \max\big\{1, \|e^{(t-s)A}\, v\|_V^q, \|e^{(t-s)A}\, u\|_V^q\big\} \|e^{(t-s)A}(v-u)\|_V\, \|e^{(t-s)A}\, F(w)\|_V \\ &\leq \frac{\Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\} \|e^{(t-s)A}(v-u)\|_V\, \chi_\vartheta\, \|F(w)\|_{V_{-\vartheta}}}{(t-s)^\vartheta} \\ &\leq \frac{\Xi\, |\chi_0|^q\, \chi_\vartheta\, \chi_\rho\, \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, \|v-u\|_{V_{-\rho}} \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, g_1(w)}{(t-s)^{(\rho+\vartheta)}}. \end{aligned} \tag{90}$$

Moreover, we note that the assumption that

$$\forall\, x, y_1, y_2 \in V \colon \|\psi(x,y_1) - \psi(x,y_2)\|_{\mathcal{V}} \leq \Xi \max\big\{1, \|x\|_V^q, \|y_1\|_V^q, \|y_2\|_V^q\big\}\, \|y_1 - y_2\|_V \tag{91}$$

implies that for all $x, y \in V$ it holds that $\|\psi_{0,1}(x,y)\|_{L(V,\mathcal{V})} \leq \Xi \max\{1, \|x\|_V^q, \|y\|_V^q\}$. This, in turn, proves that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\big\| \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\, e^{(t-\lfloor s \rfloor_h)A}\, F(w) - \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\, e^{(t-s)A}\, F(w) \big\|_{\mathcal{V}} \\ &\leq \big\| \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) \big[e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\big] F(w) \big\|_{\mathcal{V}} \\ &+ \big\| \big[\psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\, e^{(t-s)A}\, F(w) \big\|_{\mathcal{V}} \\ &\leq \Xi \max\big\{1, \|e^{(t-s)A}\, v\|_V^q, \|e^{(t-s)A}\, u\|_V^q\big\}\, \big\| \big[e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}\big] F(w) \big\|_V \\ &+ \Xi \max\big\{1, \|e^{(t-s)A}\, v\|_V^q, \|e^{(t-s)A}\, u\|_V^q\big\}\, \|e^{(t-s)A}(v-u)\|_V\, \|e^{(t-s)A}\, F(w)\|_V \\ &\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, g_1(w) \\ &\cdot \left[\frac{\chi_{\rho+\vartheta}\, \chi_\rho}{(t-s)^{(\rho+\vartheta)}}\, h^\rho + \frac{\chi_\vartheta\, \chi_\rho\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right]. \end{aligned} \tag{92}$$

Inequalities (90) and (92) imply that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} \|\tilde{F}_{s,t}(u,v,w)\|_{\mathcal{V}} &\leq \Xi\, |\chi_0|^q\, \chi_\rho \max\big\{1, \|u\|_V^q, \|v\|_V^q\big\}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}\, g_1(w) \\ &\cdot \left[\frac{\chi_{\rho+\vartheta}}{(t-s)^{(\rho+\vartheta)}}\, h^\rho + \frac{2\, \chi_\vartheta\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right]. \end{aligned} \tag{93}$$

Next we estimate the second integrand on the right hand side of (89). Next we observe that Lemma 1.2 shows that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$\begin{aligned} &\Big\| \sum_{b \in \mathbb{U}} \big[\psi_{2,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{2,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big] \big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \Big\|_{\mathcal{V}} \\ &\leq \frac{\Xi\, |\chi_0|^q\, |\chi_{\vartheta/2}|^2\, \chi_\rho \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \|v-u\|_{V_{-\rho}}\, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\, g_2(w)}{(t-s)^{(\rho+\vartheta)}}, \end{aligned} \tag{94}$$

$$\begin{aligned} &\Big\| \sum_{b \in \mathbb{U}} \big[\psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) \big(e^{(t-\lfloor s \rfloor_h)A}\, B^b(w), e^{(t-\lfloor s \rfloor_h)A}\, B^b(w)\big) \\ &- \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big) \big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big)\big] \Big\|_{\mathcal{V}} \\ &\leq \Big\| \sum_{b \in \mathbb{U}} \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) \big([e^{(t-\lfloor s \rfloor_h)A} + e^{(t-s)A}]\, B^b(w), [e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}]\, B^b(w)\big) \Big\|_{\mathcal{V}} \\ &+ \Big\| \sum_{b \in \mathbb{U}} \big[\psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big] \big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \Big\|_{\mathcal{V}} \end{aligned} \tag{95}$$

$$
\begin{aligned}
&\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\} \Big[\|e^{(t-s)A}(v-u)\|_V \, \|e^{(t-s)A}\, B(w)\|^2_{\gamma(U,V)} \\
&+ \|(e^{(t-\lfloor s \rfloor_h)A} + e^{(t-s)A})\, B(w)\|_{\gamma(U,V)} \, \|(e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A})\, B(w)\|_{\gamma(U,V)} \Big] \\
&\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\} \, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \, g_2(w) \\
&\cdot \left[\frac{2\, \chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2}\, \chi_\rho}{(t-s)^{(\rho+\vartheta)}}\, h^\rho + \frac{|\chi_{\vartheta/2}|^2\, \chi_\rho\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\| \sum_{b\in\mathbb{U}} \big[\psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-\lfloor s \rfloor_h)A}\, B^b(w)\big) \\
&- \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big)\big] \Big\|_{\mathcal{V}} \\
&\leq \Big\| \sum_{b\in\mathbb{U}} \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big)\big(e^{(t-s)A}\, B^b(w), [e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}]\, B^b(w)\big) \Big\|_{\mathcal{V}} \\
&+ \Big\| \sum_{b\in\mathbb{U}} \big[\psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, v\big) - \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big] \big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \Big\|_{\mathcal{V}} \\
&\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\} \Big[\|e^{(t-s)A}(v-u)\|_V \, \|e^{(t-s)A}\, B(w)\|^2_{\gamma(U,V)} \\
&+ \|e^{(t-s)A}\, B(w)\|_{\gamma(U,V)} \, \|[e^{(t-\lfloor s \rfloor_h)A} - e^{(t-s)A}]\, B(w)\|_{\gamma(U,V)} \Big] \\
&\leq \Xi\, |\chi_0|^q \max\{1, \|u\|_V^q, \|v\|_V^q\} \, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \, g_2(w) \\
&\cdot \left[\frac{\chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2}\, \chi_\rho}{(t-s)^{(\rho+\vartheta)}}\, h^\rho + \frac{|\chi_{\vartheta/2}|^2\, \chi_\rho\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right].
\end{aligned} \tag{96}
$$

Combining (94)–(96) implies that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$
\begin{aligned}
&\|\tilde{B}_{s,t}(u,v,w)\|_{\mathcal{V}} \\
&\leq 2\, \Xi\, |\chi_0|^q\, \chi_{\vartheta/2}\, \chi_\rho \, \max\{1, \|u\|_V^q, \|v\|_V^q\} \, \|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))} \, g_2(w) \\
&\cdot \left[\frac{\chi_{\rho+\vartheta/2}}{(t-s)^{(\rho+\vartheta)}}\, h^\rho + \frac{\chi_{\vartheta/2}\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right].
\end{aligned} \tag{97}
$$

Next observe that (93) and (97) show that for all $(s,t) \in \angle$, $u, v, w \in V$ it holds that

$$
\begin{aligned}
&\|\tilde{F}_{s,t}(u,v,w)\|_{\mathcal{V}} + \|\tilde{B}_{s,t}(u,v,w)\|_{\mathcal{V}} \leq \Xi\, |\chi_0|^q\, \chi_\rho \, \max\{1, \|u\|_V^q, \|v\|_V^q\}\, \varsigma_{F,B}\, g_2(w) \\
&\cdot \left[\left[\frac{\chi_{\rho+\vartheta} + 2\, \chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2}}{(t-s)^{(\rho+\vartheta)}} \right] h^\rho + \frac{2\, (|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\, \|v-u\|_{V_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right].
\end{aligned} \tag{98}
$$

In addition, note that Hölder's inequality ensures that for all $r \in (0,\infty)$, $s \in [0,T]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_V^r, \|Y_s\|_V^r\big\}\, g_2\big(Y_{\lfloor s \rfloor_h}\big)\big] \\
&\leq \left(\sup_{u,v\in[0,T]} \big\|\max\big\{1, \|\bar{Y}_u\|_V^r, \|Y_v\|_V^r\big\}\big\|_{\mathcal{L}^{1+2/r}(\mathbb{P};\mathbb{R})} \right) \left(\sup_{u\in[0,T]} \big\|\max\big\{1, \|Y_u\|_V^2\big\}\big\|_{\mathcal{L}^{1+r/2}(\mathbb{P};\mathbb{R})} \right) \\
&\leq |K_{r+2}|^{\frac{1}{1+2/r}}\, |K_{r+2}|^{\frac{1}{1+r/2}} = K_{r+2},
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
\mathbb{E}\big[g_2(Y_{\lfloor s \rfloor_h})\, \|Y_s - \bar{Y}_s\|_{V_{-\rho}}\big] &\leq \|g_2(Y_{\lfloor s \rfloor_h})\|_{\mathcal{L}^{3/2}(\mathbb{P};\mathbb{R})} \, \|Y_s - \bar{Y}_s\|_{\mathcal{L}^3(\mathbb{P};V_{-\rho})} \\
&\leq |K_3|^{2/3} \left(\sup_{u\in[0,T]} \big\|Y_u - \bar{Y}_u\big\|_{\mathcal{L}^3(\mathbb{P};V_{-\rho})} \right),
\end{aligned} \tag{100}
$$

and

$$
\begin{aligned}
&\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_V^r, \|Y_s\|_V^r\big\}\, g_2(Y_{\lfloor s \rfloor_h})\, \|Y_s - \bar{Y}_s\|_{V_{-\rho}}\big] \\
&\leq \|\max\{1, \|\bar{Y}_s\|_V^r, \|Y_s\|_V^r\}\|_{\mathcal{L}^{1+3/r}(\mathbb{P};\mathbb{R})}\, \|g_2(Y_{\lfloor s \rfloor_h})\|_{\mathcal{L}^{(r+3)/2}(\mathbb{P};\mathbb{R})}\, \|Y_s - \bar{Y}_s\|_{\mathcal{L}^{r+3}(\mathbb{P};V_{-\rho})} \\
&\leq |K_{r+3}|^{\frac{r+2}{r+3}} \left(\sup_{u\in[0,T]} \big\|Y_u - \bar{Y}_u\big\|_{\mathcal{L}^{r+3}(\mathbb{P};V_{-\rho})} \right).
\end{aligned}
\tag{101}
$$

Combining (98)–(101) with Lemma 5.3 and the fact that $1 - \max\{\frac{1+\vartheta}{2} - \rho, 0\} > \rho$ yields that for all $t \in (0, T]$ it holds that

$$
\begin{aligned}
&\int_0^t \mathbb{E}\Big[\big\|\tilde{F}_{s,t}(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h})\big\|_{\mathcal{V}}\Big] + \mathbb{E}\Big[\big\|\tilde{B}_{s,t}(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h})\big\|_{\mathcal{V}}\Big]\, ds \\
&\leq \tfrac{\Xi\, |\chi_0|^q\, \chi_\rho\, \varsigma_{F,B}\, K_{q+3}\, h^\rho\, t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \Bigg[\chi_{\rho+\vartheta} + 2\, \chi_{\vartheta/2}\, \chi_{\rho+\vartheta/2} + 2\, \chi_\rho \left(|\chi_{\vartheta/2}|^2 + \chi_\vartheta \right) \\
&\cdot \left(\tfrac{\chi_\vartheta\, t^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}}{(1-\vartheta)} + \tfrac{\Upsilon_{q+3}\, \chi_{\vartheta/2}\, \sqrt{t^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}}{\sqrt{1-\vartheta}} \right) \Bigg].
\end{aligned}
\tag{102}
$$

Putting (102) into (89) proves (84). This finishes the proof of Proposition 5.4. □

6 Weak error estimates for exponential Euler approximations of SPDEs with mollified nonlinearities

6.1 Regularity properties for solutions of infinite dimensional Kolmogorov equations in Banach spaces

Lemma 6.1. *Assume the setting in Section 1.2, let $\varphi \in \mathrm{Lip}^0(V, \mathcal{V})$, $F \in \mathrm{Lip}^0(V, V)$, $B \in \mathrm{Lip}^0(V, \gamma(U, V))$, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with* spectrum$(A) \subseteq \{z \in \mathbb{C}\colon Re(z) < \eta\}$*, let $X^x\colon [0, T] \times \Omega \to V$, $x \in V$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for all $x \in V$ that $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t^x\|_V^2\big] < \infty$ and which satisfy that for all $x \in V$, $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that*

$$
X_t^x = e^{tA} x + \int_0^t e^{(t-s)A} F(X_s^x)\, ds + \int_0^t e^{(t-s)A} B(X_s^x)\, dW_s, \tag{103}
$$

let $Y\colon [0, T] \times \Omega \to V$ be a continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies for all $t \in [0, T]$ that $\mathbb{E}\big[\|Y_t\|_V\big] < \infty$ and which satisfies that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$
Y_t = e^{tA} Y_0 + \int_0^t e^{(t-s)A} F(Y_s)\, ds + \int_0^t e^{(t-s)A} B(Y_s)\, dW_s, \tag{104}
$$

and let $u\colon [0, T] \times V \to \mathcal{V}$ be the function which satisfies for all $x \in V$, $t \in [0, T]$ that $u(t, x) = \mathbb{E}[\varphi(X_{T-t}^x)]$. Then

(i) it holds for all $s, t \in [0, T]$ that $\mathbb{E}\big[\|\varphi(Y_t)\|_{\mathcal{V}} + \|u(t, Y_s)\|_{\mathcal{V}}\big] < \infty$ and

(ii) it holds for all $t, h \in [0, T]$ with $t + h \leq T$ that

$$
\mathbb{E}[\varphi(Y_{T-t})] = \mathbb{E}[u(t + h, Y_h)]. \tag{105}
$$

Proof. Throughout this proof let $\psi_n\colon \mathcal{V} \to \mathcal{V}$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$, $v \in \mathcal{V}$ that

$$\psi_n(v) = \begin{cases} v & : \|v\|_{\mathcal{V}} \leq n \\ \frac{nv}{\|v\|_{\mathcal{V}}} & : \|v\|_{\mathcal{V}} > n \end{cases}, \tag{106}$$

let $\varphi_n\colon V \to \mathcal{V}$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$, $v \in \mathcal{V}$ that

$$\varphi_n(v) = \psi_n(\varphi(v)), \tag{107}$$

and let $u_n\colon [0,T] \times V \to \mathcal{V}$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$, $x \in V$, $t \in [0,T]$ that

$$u_n(t,x) = \mathbb{E}[\varphi_n(X^x_{T-t})]. \tag{108}$$

Observe that for all $n \in \mathbb{N}$ it holds that $\varphi_n \in \mathcal{M}(\mathcal{B}(V), \mathcal{B}(\mathcal{V}))$ and

$$\sup_{v\in V} \|\varphi_n(v)\|_{\mathcal{V}} \leq n. \tag{109}$$

We note that the Burkholder-Davis-Gundy type inequality in, e.g., [46, Corollary 3.10], Gronwall's lemma, Fatou's lemma, and the fact that $F \in \operatorname{Lip}^0(V,V)$ and $B \in \operatorname{Lip}^0(V, \gamma(U,V))$ ensure that for every probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with a normal filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$, every Id_U-cylindrical $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]})$-Wiener process $(\tilde{W}_t)_{t\in[0,T]}$, and all continuous $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$-adapted stochastic processes $\tilde{X}^{(i)}\colon [0,T] \times \tilde{\Omega} \to V$, $i \in \{1,2\}$, which satisfy $\tilde{\mathbb{P}}(\tilde{X}^{(1)}_0 = \tilde{X}^{(2)}_0) = 1$ and which satisfy that for all $i \in \{1,2\}$, $t \in [0,T]$ it holds $\tilde{\mathbb{P}}$-a.s. that

$$\tilde{X}^{(i)}_t = e^{tA}\tilde{X}^{(i)}_0 + \int_0^t e^{(t-s)A} F(\tilde{X}^{(i)}_s)\, ds + \int_0^t e^{(t-s)A} B(\tilde{X}^{(i)}_s)\, d\tilde{W}_s \tag{110}$$

it holds that $\tilde{\mathbb{P}}(\sup_{t\in[0,T]} \|\tilde{X}^{(1)}_t - \tilde{X}^{(2)}_t\|_V = 0) = 1$ (cf., e.g., Kunze [36, Theorem 5.6]). This and, e.g., Kunze [37, Theorem 3.6, Theorem 5.3, & Proposition 6.9] guarantee the uniqueness in law for solutions of the local martingale problem associated to (A, F, B) (see, e.g., [37, (3.2)]). Moreover, note that, e.g., Theorem 6.2 in Van Neerven et al. [47] ensure that for every probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with a normal filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ and every Id_U-cylindrical $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]})$-Wiener process $(\tilde{W}_t)_{t\in[0,T]}$ there exist continuous $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$-adapted stochastic processes $\tilde{X}^x\colon [0,T] \times \tilde{\Omega} \to V$, $x \in V$, which satisfy that for all $x \in V$, $t \in [0,T]$ it holds $\tilde{\mathbb{P}}$-a.s. that

$$\tilde{X}^x_t = e^{tA}x + \int_0^t e^{(t-s)A} F(\tilde{X}^x_s)\, ds + \int_0^t e^{(t-s)A} B(\tilde{X}^x_s)\, d\tilde{W}_s. \tag{111}$$

This and, e.g., [37, Theorem 3.6 & Proposition 6.9] assure that the local martingale problem associated to (A, F, B) is well-posed (see, e.g., [37, Definition 2.3]). Combining this with (109) and, e.g., [37, Theorem 4.2 and item (4) of Theorem 2.2] implies that for all $n \in \mathbb{N}$, $t, h \in [0,T]$ with $t + h \leq T$ it holds that

$$\mathbb{E}[\varphi_n(Y_{T-t})] = \mathbb{E}\big[\mathbb{E}[\varphi(Y_{(T-t-h)+h})|\mathcal{F}_h]\big] = \mathbb{E}[u_n(t+h, Y_h)]. \tag{112}$$

Next note that for all $n \in \mathbb{N}$, $v \in V$ it holds that

$$\|\varphi_n(v) - \varphi(v)\|_{\mathcal{V}} \leq 2\mathbb{1}_{\{y\in\mathcal{V}\colon \|y\|_{\mathcal{V}}>n\}}(\varphi(v))\, \|\varphi(v)\|_{\mathcal{V}} \leq 2\|\varphi\|_{\operatorname{Lip}^0(V,\mathcal{V})}\, (1 + \|v\|_V). \tag{113}$$

This implies that for all $n \in \mathbb{N}$, $x \in V$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|\varphi_n(Y_t) - \varphi(Y_t)\|_{\mathcal{V}}\big] \leq 2\|\varphi\|_{\operatorname{Lip}^0(V,\mathcal{V})}\, \big(1 + \mathbb{E}\big[\|Y_t\|_V\big]\big) < \infty \tag{114}$$

and

$$\mathbb{E}\big[\|\varphi_n(X^x_t) - \varphi(X^x_t)\|_{\mathcal{V}}\big] \leq 2\|\varphi\|_{\operatorname{Lip}^0(V,\mathcal{V})}\, \big(1 + \mathbb{E}\big[\|X^x_t\|_V\big]\big) < \infty. \tag{115}$$

Note that (109) and (114) show that for all $x \in V$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|\varphi(Y_t)\|_{\mathcal{V}}\big] < \infty. \tag{116}$$

Moreover, combining (114)–(115) with Lebesgue's theorem of dominated convergence and the fact that

$$\forall\, v \in V\colon\ \limsup_{n\to\infty} \|\varphi_n(v) - \varphi(v)\|_{\mathcal{V}} = 0 \tag{117}$$

yields that for all $x \in V$, $t \in [0,T]$ it holds that

$$\limsup_{n\to\infty} \big\|\mathbb{E}[\varphi_n(Y_t)] - \mathbb{E}[\varphi(Y_t)]\big\|_{\mathcal{V}} + \limsup_{n\to\infty} \|u_n(t,x) - u(t,x)\|_{\mathcal{V}} = 0. \tag{118}$$

Next observe that for all $t \in [0,T]$ it holds that

$$\sup_{x\in V}\left[\frac{\mathbb{E}\big[\|X_t^x\|_V\big]}{(1+\|x\|_V)}\right] \leq \sup_{x\in V}\left[\frac{\|X_t^x\|_{\mathcal{L}^2(\mathbb{P};V)}}{\max\{1,\|x\|_V\}}\right] < \infty \tag{119}$$

(cf., e.g., Cox & Van Neerven [16, (2.1) and Theorem 2.7]). Next observe that (115) imply that for all $n \in \mathbb{N}$, $x \in V$, $t \in [0,T]$ it holds that

$$\begin{aligned}\|u_n(t,x) - u(t,x)\|_{\mathcal{V}} &\leq \mathbb{E}\big[\|\varphi_n(X_{T-t}^x) - \varphi(X_{T-t}^x)\|_{\mathcal{V}}\big]\\ &\leq 2\|\varphi\|_{\mathrm{Lip}^0(V,\mathcal{V})}\big(1 + \mathbb{E}\big[\|X_{T-t}^x\|_V\big]\big)\\ &\leq 2\|\varphi\|_{\mathrm{Lip}^0(V,\mathcal{V})}\left(1 + \sup_{v\in V}\left[\frac{\mathbb{E}\big[\|X_{T-t}^v\|_V\big]}{(1+\|v\|_V)}\right](1+\|x\|_V)\right).\end{aligned} \tag{120}$$

This and (119) yield that for all $n \in \mathbb{N}$, $s,t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|u_n(t,Y_s) - u(t,Y_s)\|_{\mathcal{V}}\big] \leq 2\|\varphi\|_{\mathrm{Lip}^0(V,\mathcal{V})}\left(1 + \sup_{v\in V}\left[\frac{\mathbb{E}\big[\|X_t^v\|_V\big]}{(1+\|v\|_V)}\right](1+\mathbb{E}\big[\|Y_s\|_V\big])\right) < \infty. \tag{121}$$

This and (109) show that for all $s,t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|u(t,Y_s)\|_{\mathcal{V}}\big] < \infty. \tag{122}$$

This and (116) prove item (i). Next we combine (118) and (121) with Lebesgue's theorem of dominated convergence to obtain that for all $s,t \in [0,T]$ it holds that

$$\limsup_{n\to\infty} \big\|\mathbb{E}\big[u_n(t,Y_s) - u(t,Y_s)\big]\big\|_{\mathcal{V}} = 0. \tag{123}$$

This, (112), and (118) yield that for all $t,h \in [0,T]$ with $t+h \leq T$ it holds that

$$\mathbb{E}[\varphi(Y_{T-t})] = \mathbb{E}[u(t+h,Y_h)]. \tag{124}$$

This proves item (ii). The proof of Lemma 6.1 is thus completed. □

Lemma 6.2. *Assume the setting in Section 1.2, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with* spectrum$(A) \subseteq \{z \in \mathbb{C}\colon Re(z) < \eta\}$, *let $(V_r, \|\cdot\|_{V_r})$, $r \in [0,\infty)$, be the $\mathbb{R}$-Banach spaces which satisfy for all $r \in [0,\infty)$ that $(V_r, \|\cdot\|_{V_r}) = (D((\eta - A)^r), \|(\eta - A)^r(\cdot)\|_V)$, let $\varphi \in \mathrm{Lip}^4(V,\mathcal{V})$, $F \in \mathrm{Lip}^4(V,V_1)$, $B \in \mathrm{Lip}^4(V,\gamma(U,V_1))$, let $\Pi_k \in \mathcal{P}\big(\mathcal{P}\big(\mathcal{P}(\mathbb{N})\big)\big)$, $k \in \mathbb{N}_0$, be the sets which satisfy for all $k \in \mathbb{N}$ that $\Pi_0 = \emptyset$ and*

$$\Pi_k = \big\{C \subseteq \mathcal{P}(\mathbb{N})\colon [\emptyset \notin C] \wedge [\cup_{a\in C} a = \{1,\ldots,k\}] \wedge [\forall\, a,b \in C\colon (a \neq b \Rightarrow a \cap b = \emptyset)]\big\}, \tag{125}$$

and for every $k \in \mathbb{N}$, $\varpi \in \Pi_k$ let $I_i^\varpi \in \varpi$, $i \in \{1, \ldots, \#_\varpi\}$, be the sets which satisfy that $\min(I_1^\varpi) < \cdots < \min(I_{\#_\varpi}^\varpi)$, let $I_{i,j}^\varpi \in I_i^\varpi$, $j \in \{1, \ldots, \#_{I_i^\varpi}\}$, $i \in \{1, \ldots, \#_\varpi\}$, be the natural numbers which satisfy for all $i \in \{1, \ldots, \#_\varpi\}$ that $I_{i,1}^\varpi < I_{i,2}^\varpi < \cdots < I_{i,\#_{I_i^\varpi}}^\varpi$, and let $[\cdot]_i^\varpi \colon V^{k+1} \to V^{\#_{I_i^\varpi}+1}$, $i \in \{1, \ldots, \#_\varpi\}$, be the functions which satisfy for all $i \in \{1, \ldots, \#_\varpi\}$, $\mathbf{v} = (v_0, v_1, \ldots, v_k) \in V^{k+1}$ that $[\mathbf{v}]_i^\varpi = (v_0, v_{I_{i,1}^\varpi}, \ldots, v_{I_{i,\#_{I_i^\varpi}}^\varpi})$. Then

(i) there exist up-to-modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic processes $X^{k,\mathbf{v}} \colon [0,T] \times \Omega \to V$, $\mathbf{v} \in V^{k+1}$, $k \in \{0,1,2\}$, which satisfy for all $k \in \{0,1,2\}$, $\mathbf{v} \in V^{k+1}$, $p \in (0,\infty)$ that $\sup_{t \in [0,T]} \mathbb{E}\big[\|X_t^{k,\mathbf{v}}\|_V^p\big] < \infty$ and which satisfy that for all $k \in \{0,1,2\}$, $\mathbf{v} = (v_0, v_1, \ldots, v_k) \in V^{k+1}$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
X_t^{k,\mathbf{v}} &= \mathbb{1}_{\{0,1\}}(k)\, e^{tA} v_k \\
&+ \int_0^t e^{(t-s)A}\Big[\mathbb{1}_{\{0\}}(k)\, F(X_s^{0,v_0}) + \sum_{\varpi \in \Pi_k} F^{(\#_\varpi)}(X_s^{0,v_0})\big(X_s^{\#_{I_1^\varpi}, [\mathbf{v}]_1^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{v}]_{\#_\varpi}^\varpi}\big)\Big]\, ds \\
&+ \int_0^t e^{(t-s)A}\Big[\mathbb{1}_{\{0\}}(k)\, B(X_s^{0,v_0}) + \sum_{\varpi \in \Pi_k} B^{(\#_\varpi)}(X_s^{0,v_0})\big(X_s^{\#_{I_1^\varpi}, [\mathbf{v}]_1^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{v}]_{\#_\varpi}^\varpi}\big)\Big]\, dW_s,
\end{aligned}
\tag{126}
$$

(ii) there exists a unique function $\phi \colon [0,T] \times V \to \mathcal{V}$ which satisfies for all $x \in V$, $t \in [0,T]$ that $\phi(t,x) = \mathbb{E}[\varphi(X_t^{0,x})]$,

(iii) it holds for all $t \in [0,T]$ that $(V \ni x \mapsto \phi(t,x) \in \mathcal{V}) \in C_b^4(V, \mathcal{V})$,

(iv) it holds for all $k \in \{1,2\}$, $\mathbf{v} = (v_0, v_1, \ldots, v_k) \in V^{k+1}$, $t \in [0,T]$ that

$$
\sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\big\|\varphi^{(\#_\varpi)}(X_t^{0,v_0})\big(X_t^{\#_{I_1^\varpi}, [\mathbf{v}]_1^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{v}]_{\#_\varpi}^\varpi}\big)\big\|_{\mathcal{V}}\Big] < \infty,
\tag{127}
$$

(v) it holds for all $k \in \{1,2\}$, $\mathbf{v} \in V^k$, $x \in V$, $t \in [0,T]$ that

$$
\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)\mathbf{v} = \sum_{\varpi \in \Pi_k} \mathbb{E}\Big[\varphi^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi}, [(x,\mathbf{v})]_1^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [(x,\mathbf{v})]_{\#_\varpi}^\varpi}\big)\Big],
\tag{128}
$$

(vi) it holds for all $k \in \{1,2,3,4\}$, $\delta_1, \ldots, \delta_k \in (-1/2, 0]$ with $\sum_{i=1}^k \delta_i > -1/2$ that

$$
\sup_{t \in (0,T]} \sup_{x \in V} \sup_{v_1, \ldots, v_k \in V \setminus \{0\}} \left[\frac{\big\|\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x)(v_1, \ldots, v_k)\big\|_{\mathcal{V}}}{t^{(\delta_1 + \ldots + \delta_k)}\, \|(\eta - A)^{\delta_1} v_1\|_V \cdot \ldots \cdot \|(\eta - A)^{\delta_k} v_k\|_V}\right] < \infty,
\tag{129}
$$

(vii) it holds for all $k \in \{1,2,3,4\}$, $\delta_1, \ldots, \delta_k \in (-1/2, 0]$ with $\sum_{i=1}^k \delta_i > -1/2$ that

$$
\sup_{t \in (0,T]} \sup_{\substack{x,y \in V,\\ x \neq y}} \sup_{v_1, \ldots, v_k \in V \setminus \{0\}} \left[\frac{\big\|\big[\big(\tfrac{\partial^4}{\partial x^4}\phi\big)(t,x) - \big(\tfrac{\partial^4}{\partial x^4}\phi\big)(t,y)\big](v_1, \ldots, v_k)\big\|_{\mathcal{V}}}{t^{(\delta_1 + \ldots + \delta_k)}\, \|x - y\|_V \cdot \|(\eta - A)^{\delta_1} v_1\|_V \cdot \ldots \cdot \|(\eta - A)^{\delta_k} v_k\|_V}\right] < \infty,
\tag{130}
$$

(viii) it holds for all $p \in (0,\infty)$ that

$$
\sup_{\substack{x,y \in V,\\ x \neq y}} \sup_{t \in [0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|x - y\|_V}\right] < \infty,
\tag{131}
$$

(ix) it holds for all $k \in \{1, 2\}$, $p \in (0, \infty)$ that

$$\sup_{t\in[0,T]} \sup_{x\in V} \sup_{v_1,\dots,v_k\in V\setminus\{0\}} \left[\frac{\|X_t^{k,(x,v_1,\dots,v_k)}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \right] < \infty, \tag{132}$$

(x) it holds for all $k \in \{1, 2\}$, $p \in (0, \infty)$ that

$$\sup_{\substack{x,y\in V,\\ x\neq y}} \sup_{t\in[0,T]} \sup_{v_1,\dots,v_k\in V\setminus\{0\}} \left[\frac{\|X_t^{k,(x,v_1,\dots,v_k)} - X_t^{k,(y,v_1,\dots,v_k)}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|x-y\|_V \cdot \|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \right] < \infty, \tag{133}$$

(xi) it holds for all $x \in V_1$, $t \in [0, T]$ that $\mathbb{P}(X_t^{0,x} \in V_1) = 1$,

(xii) it holds for all $p \in (0, \infty)$, $x \in V_1$, $t \in [0, T]$ that $\mathbb{E}\big[\|X_t^{0,x} \mathbb{1}_{\{X_t^{0,x}\in V_1\}}\|_{V_1}^p\big] < \infty$,

(xiii) it holds for all $l \in \{0, 1\}$, $p \in [1, \infty)$, $x \in V_l$ that

$$([0, T] \ni t \mapsto [X_t^{0,x}]_{\mathbb{P},\mathcal{B}(V_l)} \in L^p(\mathbb{P}; V_l)) \in C([0, T], L^p(\mathbb{P}; V_l)), \tag{134}$$

(xiv) it holds for all $k \in \{1, 2\}$, $p, r \in (0, \infty)$, $x \in V$, $t \in [0, T]$ that

$$\limsup_{[0,T]\ni s\to t} \sup_{v_1,\dots,v_k\in V_{r\mathbb{1}_{\{1\}}(k)}\setminus\{0\}} \left[\frac{\|X_s^{k,(x,v_1,\dots,v_k)} - X_t^{k,(x,v_1,\dots,v_k)}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|v_1\|_{V_{r\mathbb{1}_{\{1\}}(k)}} \cdot \ldots \cdot \|v_k\|_{V_{r\mathbb{1}_{\{1\}}(k)}}} \right] = 0, \tag{135}$$

(xv) it holds for all $x \in V_1$ that $([0, T] \ni t \mapsto \phi(t, x) \in \mathcal{V}) \in C^1([0, T], \mathcal{V})$,

(xvi) it holds that $\big([0, T] \times V_1 \ni (t, x) \mapsto \big(\tfrac{\partial}{\partial t}\phi\big)(t, x) \in \mathcal{V}\big) \in C([0, T] \times V_1, \mathcal{V})$,

(xvii) it holds for all $k \in \{1, 2\}$, $r \in (0, \infty)$ that

$$\begin{aligned}\big([0, T] \times V_r \ni (t, x) \mapsto \big((V_r)^k \ni \mathbf{u} \mapsto \big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t, x)\mathbf{u} \in \mathcal{V}\big) \in L^{(k)}(V_r, \mathcal{V})\big)& \\ \in C([0, T] \times V_r, L^{(k)}(V_r, \mathcal{V})),&\end{aligned} \tag{136}$$

(xviii) it holds for all $k \in \{1, 2\}$ that $\sup_{t\in[0,T]} \sup_{x\in V} \big\|\big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t, x)\big\|_{L^{(k)}(V,\mathcal{V})} < \infty$, and

(xix) it holds for all $x \in V_1$, $t \in (0, T]$ that

$$\big(\tfrac{\partial}{\partial t}\phi\big)(t, x) = \big(\tfrac{\partial}{\partial x}\phi\big)(t, x)(Ax + F(x)) + \frac{1}{2}\sum_{b\in\mathbb{U}} \big(\tfrac{\partial^2}{\partial x^2}\phi\big)(t, x)(B(x)b, B(x)b). \tag{137}$$

Proof. Throughout this proof let $\chi \in \mathbb{R}$ be the real number given by $\chi = \sup_{t\in[0,T]} \|e^{tA}\|_{L(V)}$. The proof of items (i)–(vii) is entirely analogous to the proof of items (i)–(v), (vii), & (x) of Theorem 3.3 in Andersson et al. [1]. Item (ii) ensures that there exists a unique function $\psi\colon [0, T] \times V \to \mathcal{V}$ which satisfies for all $x \in V$, $t \in [0, T]$ that

$$\psi(t, x) = \phi(T - t, x) = \mathbb{E}[\varphi(X_{T-t}^{0,x})]. \tag{138}$$

The proof of item (viii) is entirely analogous to the proof of item (iii) of Corollary 2.10 in Andersson et al. [2]. The proof of items (ix)–(x) is entirely analogous to the proof of items (ii) & (iv) of Theorem 2.1 in Andersson et al. [3]. The fact that $F \in \mathrm{Lip}^0(V, V_1)$, the fact that $B \in \mathrm{Lip}^0(V, \gamma(U, V_1))$, and the fact that $\forall\, x \in V,\, p \in (0,\infty)\colon \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t^{0,x}\|_V^p\big] < \infty$ show that for all $x \in V$, $t \in [0,T]$ it holds that

$$\int_0^t \|e^{(t-s)A} F(X_s^{0,x})\|_{\mathcal{L}^p(\mathbb{P};V_1)} + \|e^{(t-s)A} B(X_s^{0,x})\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_1))}\, ds < \infty. \tag{139}$$

This, (126), Jensen's inequality, and the Burkholder-Davis-Gundy type inequality in, e.g., [46, Corollary 3.10] prove items (xi)–(xii). Next note that (126) implies that for all $x \in V$, $s,t \in [0,T]$ with $s \le t$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &X_t^{0,x} - X_s^{0,x} \\ &= e^{sA}(e^{(t-s)A} - \mathrm{Id}_V)x + \int_s^t e^{(t-u)A} F(X_u^{0,x})\, du + \int_s^t e^{(t-u)A} B(X_u^{0,x})\, dW_u \\ &+ \int_0^s e^{(s-u)A}(e^{(t-s)A} - \mathrm{Id}_V) F(X_u^{0,x})\, du + \int_0^s e^{(s-u)A}(e^{(t-s)A} - \mathrm{Id}_V) B(X_u^{0,x})\, dW_u. \end{aligned} \tag{140}$$

Combining the Burkholder-Davis-Gundy type inequality in, e.g., [46, Corollary 3.10] with the fact that $F \in \mathrm{Lip}^0(V,V_1)$, the fact that $B \in \mathrm{Lip}^0(V,\gamma(U,V_1))$, the fact that $\forall\, l \in \{0,1\},\, x \in V_l\colon \limsup_{t\searrow 0} \|(e^{tA} - \mathrm{Id}_V)(\eta - A)^l x\|_V = 0$, and the fact that $\forall\, x \in V,\, p \in (0,\infty)\colon \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t^{0,x}\|_V^p\big] < \infty$ hence proves item (xiii). Next note that (126), the Burkholder-Davis-Gundy type inequality in, e.g., [46, Corollary 3.10], the fact that $F \in \mathrm{Lip}^0(V,V_1)$, and the fact that $B \in \mathrm{Lip}^0(V,\gamma(U,V_1))$ show that for all $p \in [2,\infty)$, $x,y \in V_1$, $t \in [0,T]$ it holds that

$$\begin{aligned} \|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V_1)} &\le \|e^{tA}(x-y)\|_{V_1} + \left\| \int_0^t e^{(t-s)A}(F(X_s^{0,x}) - F(X_s^{0,y}))\, ds \right\|_{\mathcal{L}^p(\mathbb{P};V_1)} \\ &\quad + \left\| \int_0^t e^{(t-s)A}(B(X_s^{0,x}) - B(X_s^{0,y}))\, dW_s \right\|_{L^p(\mathbb{P};V_1)} \\ &\le \chi\, \|x-y\|_{V_1} + \int_0^t \|e^{(t-s)A}(F(X_s^{0,x}) - F(X_s^{0,y}))\|_{\mathcal{L}^p(\mathbb{P};V_1)}\, ds \\ &\quad + \Upsilon_p \left[\int_0^t \|e^{(t-s)A}(B(X_s^{0,x}) - B(X_s^{0,y}))\|^2_{\mathcal{L}^p(\mathbb{P};\gamma(U,V_1))}\, ds \right]^{1/2} \\ &\le \chi\, \|x-y\|_{V_1} + T\, \chi\, |F|_{\mathrm{Lip}^0(V,V_1)} \left[\sup_{s\in[0,T]} \|X_s^{0,x} - X_s^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V)} \right] \\ &\quad + \Upsilon_p \sqrt{T}\, \chi\, |B|_{\mathrm{Lip}^0(V,\gamma(U,V_1))} \left[\sup_{s\in[0,T]} \|X_s^{0,x} - X_s^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V)} \right]. \end{aligned} \tag{141}$$

Moreover, item (viii) and the fact that $V_1 \subseteq V$ continuously imply that for all $p \in [2,\infty)$ it holds that

$$\sup_{\substack{x,y\in V_1,\\ x\neq y}} \sup_{t\in[0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|x-y\|_{V_1}} \right] < \infty. \tag{142}$$

This, (141), and Jensen's inequality prove that for all $p \in (0,\infty)$ it holds that

$$\sup_{\substack{x,y\in V_1,\\ x\neq y}} \sup_{t\in[0,T]} \left[\frac{\|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};V_1)}}{\|x-y\|_{V_1}} \right] < \infty. \tag{143}$$

In the next step observe that (126) implies that for all $k \in \{1,2\}$, $x \in V$, $\mathbf{v} = (v_1,\dots,v_k) \in V^k$,

$s, t \in [0,T]$ with $s \leq t$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &X_t^{k,(x,\mathbf{v})} - X_s^{k,(x,\mathbf{v})} = \mathbb{1}_{\{1\}}(k)\, e^{sA}(e^{(t-s)A} - \mathrm{Id}_V)v_k \\ &+ \int_s^t e^{(t-u)A} \sum_{\varpi \in \Pi_k} F^{(\#_\varpi)}(X_u^{0,x})\big(X_u^{\#_{I_1^\varpi},[(x,\mathbf{v})]_1^\varpi}, \ldots, X_u^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{v})]_{\#_\varpi}^\varpi}\big)\, du \\ &+ \int_s^t e^{(t-u)A} \sum_{\varpi \in \Pi_k} B^{(\#_\varpi)}(X_u^{0,x})\big(X_u^{\#_{I_1^\varpi},[(x,\mathbf{v})]_1^\varpi}, \ldots, X_u^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{v})]_{\#_\varpi}^\varpi}\big)\, dW_u \\ &+ \int_0^s e^{(s-u)A}(e^{(t-s)A} - \mathrm{Id}_V) \sum_{\varpi \in \Pi_k} F^{(\#_\varpi)}(X_u^{0,x})\big(X_u^{\#_{I_1^\varpi},[(x,\mathbf{v})]_1^\varpi}, \ldots, X_u^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{v})]_{\#_\varpi}^\varpi}\big)\, du \\ &+ \int_0^s e^{(s-u)A}(e^{(t-s)A} - \mathrm{Id}_V) \sum_{\varpi \in \Pi_k} B^{(\#_\varpi)}(X_u^{0,x})\big(X_u^{\#_{I_1^\varpi},[(x,\mathbf{v})]_1^\varpi}, \ldots, X_u^{\#_{I_{\#_\varpi}^\varpi},[(x,\mathbf{v})]_{\#_\varpi}^\varpi}\big)\, dW_u. \end{aligned} \tag{144}$$

Combining Jensen's inequality and the Burkholder-Davis-Gundy type inequality in, e.g., [46, Corollary 3.10] with item (ix), Hölder's inequality, the fact that $F \in C_b^2(V,V)$, and the fact that $B \in C_b^2(V, \gamma(U,V))$ therefore establish item (xiv) and prove that for all $k \in \{1,2\}$, $p \in (0,\infty)$, $x \in V$, $t \in (0,T]$ it holds that

$$\limsup_{[0,T] \ni s \to t} \sup_{v_1,\ldots,v_k \in V \setminus \{0\}} \left[\frac{\|X_s^{k,(x,v_1,\ldots,v_k)} - X_t^{k,(x,v_1,\ldots,v_k)}\|_{\mathcal{L}^p(\mathbb{P};V)}}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V} \right] = 0. \tag{145}$$

In the next step we combine items (i) & (xi)–(xiii), the fact that $\varphi \in C_b^2(V,\mathcal{V})$, the fact that $F \in \mathrm{Lip}^0(V,V)$, and the fact that $B \in \mathrm{Lip}^0(V,\gamma(U,V))$ with the standard Itô formula in Theorem 2.4 in Brzeźniak et al. [11] to obtain that for all $x \in V_1$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} \varphi(X_t^{0,x}) &= \varphi(x) + \int_0^t \varphi'(X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\, ds + \int_0^t \varphi'(X_s^{0,x})B(X_s^{0,x})\, dW_s \\ &\quad + \int_0^t \frac{1}{2} \sum_{b \in \mathbb{U}} \varphi''(X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b)\, ds. \end{aligned} \tag{146}$$

Lemma 1.2, items (xi)–(xiii), the fact that $\varphi \in C_b^2(V,\mathcal{V})$, the fact that $F \in \mathrm{Lip}^0(V,V)$, and the fact that $B \in \mathrm{Lip}^0(V,\gamma(U,V))$ show that for all $x \in V_1$, $t \in [0,T]$ it holds that

$$\begin{aligned} &\int_0^t \mathbb{E}\bigg[\|\varphi'(X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\|_{\mathcal{V}} + \Big\| \sum_{b \in \mathbb{U}} \varphi''(X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b) \Big\|_{\mathcal{V}} \\ &\quad + \|\varphi'(X_s^{0,x})B(X_s^{0,x})\|_{\gamma(U,\mathcal{V})}^2 \bigg]\, ds < \infty. \end{aligned} \tag{147}$$

This and (146) imply that for all $x \in V_1$, $t \in [0,T]$ it holds that

$$\begin{aligned} \phi(t,x) &= \mathbb{E}[\varphi(X_t^{0,x})] \\ &= \varphi(x) + \mathbb{E}\bigg[\int_0^t \varphi'(X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\, ds \bigg] + \mathbb{E}\bigg[\int_0^t \varphi'(X_s^{0,x})B(X_s^{0,x})\, dW_s \bigg] \\ &\quad + \mathbb{E}\bigg[\int_0^t \frac{1}{2} \sum_{b \in \mathbb{U}} \varphi''(X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b)\, ds \bigg] \\ &= \phi(0,x) + \int_0^t \mathbb{E}\big[\varphi'(X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\big]\, ds \\ &\quad + \int_0^t \frac{1}{2}\, \mathbb{E}\bigg[\sum_{b \in \mathbb{U}} \varphi''(X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b) \bigg]\, ds. \end{aligned} \tag{148}$$

Furthermore, note that Hölder's inequality, Lemma 1.2, items (viii) & (xi)–(xiii), (143), the fact that $\varphi \in \operatorname{Lip}^2(V, \mathcal{V})$, the fact that $F \in \operatorname{Lip}^0(V, V)$, and the fact that $B \in \operatorname{Lip}^0(V, \gamma(U, V))$ show that

$$\begin{aligned}\Bigg([0,T] \times V_1 \ni (t,x) \mapsto \mathbb{E}\big[\varphi'(X_t^{0,x})(AX_t^{0,x} + F(X_t^{0,x}))\big] \\ + \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} \varphi''(X_t^{0,x})(B(X_t^{0,x})b, B(X_t^{0,x})b)\Bigg] \in \mathcal{V}\Bigg) \in C([0,T] \times V_1, \mathcal{V}).\end{aligned} \tag{149}$$

This, (148), and the fact that

$$\forall\, g \in C([0,T], \mathcal{V}),\ t \in [0,T]\colon \limsup_{[-t,T-t]\setminus\{0\}\ni h\to 0} \left\| \frac{1}{h} \int_t^{t+h} g(s)\, ds - g(t) \right\|_{\mathcal{V}} = 0 \tag{150}$$

prove items (xv)–(xvi). Moreover, Hölder's inequality, (145), items (iv)–(v) & (viii)–(xiv), the fact that $\varphi \in \operatorname{Lip}^2(V, \mathcal{V})$, and the fact that $\forall\, r \in (0, \infty)\colon V_r \subseteq V$ continuously establish items (xvii)–(xviii) and prove that for all $k \in \{1, 2\}$ it holds that

$$\big((0,T] \times V \ni (t,x) \mapsto \big(\tfrac{\partial^k}{\partial x^k}\phi\big)(t,x) \in L^{(k)}(V, \mathcal{V})\big) \in C((0,T] \times V, L^{(k)}(V, \mathcal{V})). \tag{151}$$

It thus remains to prove item (xix). For this observe that Lemma 6.1 and the fact that for every $x \in V$ it holds that $X^{0,x}$ has a continuous modification imply that for all $x \in V$, $t, h \in [0,T]$ with $t + h \leq T$ it holds that

$$\psi(t,x) = \mathbb{E}[\varphi(X_{T-t}^{0,x})] = \mathbb{E}[\psi(t+h, X_h^{0,x})]. \tag{152}$$

Moreover, Lemma 1.2, items (iii) & (xi)–(xiii), the fact that $F \in \operatorname{Lip}^0(V, V)$, and the fact that $B \in \operatorname{Lip}^0(V, \gamma(U, V))$ ensure that for all $x \in V_1$, $t \in [0,T]$, $h \in [0, T-t]$ it holds that

$$\begin{aligned}\int_0^h \mathbb{E}\Bigg[\Big\|\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\Big\|_{\mathcal{V}} + \Big\|\sum_{b\in\mathbb{U}} \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b)\Big\|_{\mathcal{V}} \\ + \Big\|\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x})B(X_s^{0,x})\Big\|_{\gamma(U,\mathcal{V})}^2\Bigg]\, ds < \infty.\end{aligned} \tag{153}$$

This, (152), item (iii), and the standard Itô formula in Theorem 2.4 in Brzeźniak et al. [11] yield that for all $x \in V_1$, $t \in [0,T)$, $h \in [0, T-t]$ it holds that

$$\begin{aligned}&\psi(t+h, x) - \psi(t,x) \\ &= \psi(t+h, x) - \mathbb{E}[\psi(t+h, X_h^{0,x})] \\ &= -\mathbb{E}\Bigg[\int_0^h \big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\, ds\Bigg] \\ &\quad - \mathbb{E}\Bigg[\int_0^h \big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x})B(X_s^{0,x})\, dW_s\Bigg] \\ &\quad - \mathbb{E}\Bigg[\int_0^h \frac{1}{2}\sum_{b\in\mathbb{U}} \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b)\, ds\Bigg] \\ &= -\int_0^h \mathbb{E}\big[\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x})(AX_s^{0,x} + F(X_s^{0,x}))\big]\, ds \\ &\quad - \int_0^h \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x})(B(X_s^{0,x})b, B(X_s^{0,x})b)\Bigg]\, ds.\end{aligned} \tag{154}$$

Next observe that Hölder's inequality and Lemma 1.2 show that for all $x \in V_1$, $t \in [0,T)$ it holds that

$$
\begin{aligned}
&\limsup_{(0,T-t]\ni h\to 0} \frac{1}{h}\bigg\| \int_0^h \mathbb{E}\Big[\big[\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial}{\partial x}\psi\big)(t, X_s^{0,x})\big](AX_s^{0,x} + F(X_s^{0,x}))\Big]\, ds \\
&\quad + \int_0^h \frac{1}{2}\,\mathbb{E}\Big[\sum_{b\in\mathbb{U}} \big[\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t, X_s^{0,x})\big](B(X_s^{0,x})b, B(X_s^{0,x})b)\Big]\, ds\bigg\|_{\mathcal{V}} \\
&\leq \limsup_{(0,T-t]\ni h\to 0} \frac{1}{h}\bigg[\int_0^h \Big(\mathbb{E}\Big[\big\|\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial}{\partial x}\psi\big)(t, X_s^{0,x})\big\|^2_{L(V,\mathcal{V})}\Big]\Big)^{1/2} \\
&\quad \cdot \|AX_s^{0,x} + F(X_s^{0,x})\|_{\mathcal{L}^2(\mathbb{P};V)}\, ds \\
&\quad + \int_0^h \frac{1}{2}\Big(\mathbb{E}\Big[\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t, X_s^{0,x})\big\|^2_{L^{(2)}(V,\mathcal{V})}\Big]\Big)^{1/2} \\
&\quad \cdot \|B(X_s^{0,x})\|^2_{\mathcal{L}^4(\mathbb{P};\gamma(U,V))}\, ds\bigg] \\
&\leq \bigg[\limsup_{[0,T-t]\ni h\to 0}\, \sup_{s\in[0,h]} \Big(\mathbb{E}\Big[\big\|\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial}{\partial x}\psi\big)(t, X_s^{0,x})\big\|^2_{L(V,\mathcal{V})}\Big]\Big)^{1/2}\bigg] \\
&\quad \cdot \bigg[\sup_{s\in[0,T]} \|AX_s^{0,x} + F(X_s^{0,x})\|_{\mathcal{L}^2(\mathbb{P};V)}\bigg] \\
&\quad + \bigg[\limsup_{[0,T-t]\ni h\to 0}\, \sup_{s\in[0,h]} \Big(\mathbb{E}\Big[\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t, X_s^{0,x})\big\|^2_{L^{(2)}(V,\mathcal{V})}\Big]\Big)^{1/2}\bigg] \\
&\quad \cdot \bigg[\sup_{s\in[0,T]} \|B(X_s^{0,x})\|^2_{\mathcal{L}^4(\mathbb{P};\gamma(U,V))}\bigg].
\end{aligned} \tag{155}
$$

Note that (151) and item (xiii) imply that for all $k \in \{1,2\}$, $x \in V$, $\varepsilon \in (0,\infty)$, $t_0 \in [0,T-t)$, $s_0 \in [0,T]$ it holds that

$$
\limsup_{[0,T-t)\times[0,T]\ni(t,s)\to(t_0,s_0)} \mathbb{P}\Big(\big\|\big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_s^{0,x}) - \big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t_0, X_{s_0}^{0,x})\big\|_{L^{(k)}(V,\mathcal{V})} \geq \varepsilon\Big) = 0. \tag{156}
$$

Item (xviii) and the Vitali convergence theorem in, e.g., Proposition 4.5 in Hutzenthaler et al. [28] therefore imply that for all $k \in \{1,2\}$, $x \in V$, $t_0 \in [0,T-t)$, $s_0 \in [0,T]$ it holds that

$$
\limsup_{[0,T-t)\times[0,T]\ni(t,s)\to(t_0,s_0)} \mathbb{E}\Big[\big\|\big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_s^{0,x}) - \big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t_0, X_{s_0}^{0,x})\big\|^2_{L^{(k)}(V,\mathcal{V})}\Big] = 0. \tag{157}
$$

This shows that for all $k \in \{1,2\}$, $x \in V$, $t \in [0,T)$ it holds that

$$
\begin{aligned}
&\limsup_{[0,T-t]\ni h\to 0}\, \sup_{s\in[0,h]} \mathbb{E}\Big[\big\|\big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_s^{0,x})\big\|^2_{L^{(k)}(V,\mathcal{V})}\Big] \\
&\leq \limsup_{[0,T-t]\ni h\to 0}\, \sup_{s\in[0,h]} \mathbb{E}\Big[\big\|\big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_0^{0,x})\big\|^2_{L^{(k)}(V,\mathcal{V})}\Big] \\
&\quad + \limsup_{[0,T-t]\ni h\to 0}\, \sup_{s\in[0,h]} \mathbb{E}\Big[\big\|\big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_0^{0,x}) - \big(\tfrac{\partial^k}{\partial x^k}\psi\big)(t, X_s^{0,x})\big\|^2_{L^{(k)}(V,\mathcal{V})}\Big] = 0.
\end{aligned} \tag{158}
$$

Combining this with (155), item (xiii), the fact that $F \in \mathrm{Lip}^0(V,V)$, and the fact that $B \in \mathrm{Lip}^0(V,\gamma(U,V))$ yields that for all $x \in V_1$, $t \in [0,T)$ it holds that

$$
\begin{aligned}
&\limsup_{(0,T-t]\ni h\to 0} \frac{1}{h}\bigg\| \int_0^h \mathbb{E}\Big[\big[\big(\tfrac{\partial}{\partial x}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial}{\partial x}\psi\big)(t, X_s^{0,x})\big](AX_s^{0,x} + F(X_s^{0,x}))\Big]\, ds \\
&\quad + \int_0^h \frac{1}{2}\,\mathbb{E}\Big[\sum_{b\in\mathbb{U}} \big[\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t+h, X_s^{0,x}) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t, X_s^{0,x})\big](B(X_s^{0,x})b, B(X_s^{0,x})b)\Big]\, ds\bigg\|_{\mathcal{V}} = 0.
\end{aligned} \tag{159}
$$

In the next step note that Hölder's inequality, Lemma 1.2, items (iii), (vii), & (xiii), the fact that $F \in \operatorname{Lip}^0(V,V)$, and the fact that $B \in \operatorname{Lip}^0(V,\gamma(U,V))$ show that for all $x \in V_1$, $t \in [0,T]$ it holds that

$$[0,T] \ni s \mapsto \mathbb{E}\big[\big(\tfrac{\partial}{\partial x}\psi\big)(t,X_s^{0,x})(AX_s^{0,x}+F(X_s^{0,x}))\big] \\ + \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}}\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t,X_s^{0,x})(B(X_s^{0,x})b,B(X_s^{0,x})b)\Bigg] \in \mathcal{V} \quad (160)$$

is continuous. This, (150), and the fact that $\forall\, x \in V\colon \mathbb{P}(X_0^{0,x}=x)=1$ ensure that for all $x \in V_1$, $t \in [0,T)$ it holds that

$$\limsup_{(0,T-t]\ni h\to 0}\Bigg\|\frac{1}{h}\int_0^h \mathbb{E}\big[\big(\tfrac{\partial}{\partial x}\psi\big)(t,X_s^{0,x})(AX_s^{0,x}+F(X_s^{0,x}))\big]\,ds \\ + \frac{1}{h}\int_0^h \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}}\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t,X_s^{0,x})(B(X_s^{0,x})b,B(X_s^{0,x})b)\Bigg]\,ds \\ - \big(\tfrac{\partial}{\partial x}\psi\big)(t,x)(Ax+F(x)) - \frac{1}{2}\sum_{b\in\mathbb{U}}\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t,x)(B(x)b,B(x)b)\Bigg\|_{\mathcal{V}} = 0. \quad (161)$$

Combining this with (154), (159), and the triangle inequality assures that for all $x \in V_1$, $t \in [0,T)$ it holds that

$$\limsup_{(0,T-t]\ni h\to 0}\Bigg\|\frac{\psi(t+h,x)-\psi(t,x)}{h} + \big(\tfrac{\partial}{\partial x}\psi\big)(t,x)(Ax+F(x)) \\ + \frac{1}{2}\sum_{b\in\mathbb{U}}\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(t,x)(B(x)b,B(x)b)\Bigg\|_{\mathcal{V}} = 0. \quad (162)$$

This, item (xv), and the fact that $\forall\, k \in \{1,2\}$, $x \in V_1$, $t \in [0,T]\colon (\frac{\partial}{\partial t}\phi)(t,x) = -(\frac{\partial}{\partial t}\psi)(t,x)$ and $(\frac{\partial^k}{\partial x^k}\phi)(t,x) = (\frac{\partial^k}{\partial x^k}\psi)(t,x)$ establish item (xix). The proof of Lemma 6.2 is thus completed. □

6.2 Setting

Assume the setting in Section 1.2, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C}\colon \operatorname{Re}(z) < \eta\}$, let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $h \in (0,\infty)$, $\vartheta \in [0,\frac{1}{2})$, $F \in \operatorname{Lip}^4(V,V_2)$, $B \in \operatorname{Lip}^4(V,\gamma(U,V_2))$, $\varphi \in \operatorname{Lip}^4(V,\mathcal{V})$, let $(B^b)_{b\in\mathbb{U}} \subseteq C(V,V)$ be the functions which satisfy for all $b \in \mathbb{U}$, $v \in V$ that $B^b(v) = B(v)\,b$, let $\varsigma_{F,B} \in \mathbb{R}$ be the real number given by $\varsigma_{F,B} = \max\big\{1, \|F\|^3_{C_b^3(V,V_{-\vartheta})}, \|B\|^6_{C_b^3(V,\gamma(U,V_{-\vartheta/2}))}\big\}$, let $\chi_r \in [1,\infty)$, $r \in [0,1]$, be the real numbers which satisfy for all $r \in [0,1]$ that $\chi_r = \max\{1, \sup_{t\in(0,T]} t^r\,\|(\eta-A)^r e^{tA}\|_{L(V)}, \sup_{t\in(0,T]} t^{-r}\|(\eta-A)^{-r}(e^{tA}-\mathrm{Id}_V)\|_{L(V)}\}$, let $X, Y\colon [0,T]\times\Omega \to V$, $\bar{Y}\colon [0,T]\times\Omega \to V_2$, and $X^x\colon [0,T]\times\Omega \to V$, $x \in V$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for all $x \in V$ that $\sup_{t\in[0,T]}\big[\|X_t\|_{\mathcal{L}^5(\mathbb{P};V)} + \|X_t^x\|_{\mathcal{L}^5(\mathbb{P};V)}\big] < \infty$, $X_0^x = x$, $\bar{Y}_0 \in \mathcal{L}^5(\mathbb{P};V_2)$, and $Y_0 = X_0 = \bar{Y}_0$ and which satisfy that for all $x \in V$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\,X_0 + \int_0^t e^{(t-s)A}\,F(X_s)\,ds + \int_0^t e^{(t-s)A}\,B(X_s)\,dW_s, \quad (163)$$

$$X_t^x = e^{tA}\,x + \int_0^t e^{(t-s)A}\,F(X_s^x)\,ds + \int_0^t e^{(t-s)A}\,B(X_s^x)\,dW_s, \quad (164)$$

$$Y_t = e^{tA}\,Y_0 + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,F(Y_{\lfloor s\rfloor_h})\,ds + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,B(Y_{\lfloor s\rfloor_h})\,dW_s, \quad (165)$$

$$\bar{Y}_t = e^{tA}\,\bar{Y}_0 + \int_0^t e^{(t-s)A}\,F(Y_{\lfloor s \rfloor_h})\,ds + \int_0^t e^{(t-s)A}\,B(Y_{\lfloor s \rfloor_h})\,dW_s \tag{166}$$

(cf., e.g., Theorem 4.3 in Brzeźniak [10] and Theorem 6.2 in Van Neerven et al. [47]), let $(K_r)_{r\in[0,\infty)} \subseteq [0,\infty]$ be the extended real numbers which satisfy for all $r \in [0,\infty)$ that $K_r = \sup_{s,t\in[0,T]} \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_V^r, \|Y_t\|_V^r\}\big]$, let $u\colon [0,T]\times V \to \mathcal{V}$ be the function which satisfies for all $x \in V$, $t \in [0,T]$ that $u(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^x)\big]$, let $c_{\delta_1,\dots,\delta_k} \in [0,\infty]$, $\delta_1,\dots,\delta_k \in (-1/2, 0]$, $k \in \{1,2,3,4\}$, be the extended real numbers which satisfy for all $k \in \{1,2,3,4\}$, $\delta_1,\dots,\delta_k \in (-1/2,0]$ that

$$c_{\delta_1,\dots,\delta_k} = \sup_{t\in[0,T)} \sup_{x\in V} \sup_{v_1,\dots,v_k\in V\setminus\{0\}} \left[\frac{\big\|\big(\frac{\partial^k}{\partial x^k} u\big)(t,x)(v_1,\dots,v_k)\big\|_{\mathcal{V}}}{(T-t)^{(\delta_1+\ldots+\delta_k)}\,\|v_1\|_{V_{\delta_1}} \cdot \ldots \cdot \|v_k\|_{V_{\delta_k}}} \right], \tag{167}$$

let $\tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} \in [0,\infty]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in (-1/2,0]$, be the extended real numbers which satisfy for all $\delta_1,\dots,\delta_4 \in (-1/2,0]$ that

$$\begin{aligned} &\tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} \\ &= \sup_{t\in[0,T)} \sup_{\substack{x_1,x_2\in V,\\ x_1\neq x_2}} \sup_{v_1,\dots,v_4\in V\setminus\{0\}} \left[\frac{\big\|\big(\big(\frac{\partial^4}{\partial x^4}u\big)(t,x_1) - \big(\frac{\partial^4}{\partial x^4}u\big)(t,x_2)\big)(v_1,\dots,v_4)\big\|_{\mathcal{V}}}{(T-t)^{(\delta_1+\ldots+\delta_4)}\,\|x_1-x_2\|_V\,\|v_1\|_{V_{\delta_1}} \cdot \ldots \cdot \|v_4\|_{V_{\delta_4}}} \right], \end{aligned} \tag{168}$$

and let $u_{1,0}\colon [0,T]\times V_1 \to \mathcal{V}$ and $u_{0,k}\colon [0,T]\times V \to L^{(k)}(V,\mathcal{V})$, $k \in \{1,2,3,4\}$, be the functions which satisfy for all $k \in \{1,2,3,4\}$, $x, v_1,\dots,v_k \in V_1$, $t \in [0,T]$ that $u_{1,0}(t,x) = \big(\frac{\partial}{\partial t}u\big)(t,x)$ and $u_{0,k}(t,x)(v_1,\dots,v_k) = \big(\frac{\partial^k}{\partial x^k}u\big)(t,x)(v_1,\dots,v_k)$ (cf. Lemma 6.2).

6.3 Weak convergence rates for semilinear integrated exponential Euler approximations of SPDEs with mollified nonlinearities

Lemma 6.3. *Assume the setting in Section 6.2. Then*

(i) it holds for all $k \in \{1,2,3,4\}$, $\delta_1,\dots,\delta_k \in (-1/2,0]$ with $\sum_{i=1}^k \delta_i > -1/2$ that $c_{\delta_1,\dots,\delta_k} < \infty$ and

(ii) it holds for all $\delta_1,\dots,\delta_4 \in (-1/2,0]$ with $\sum_{i=1}^4 \delta_i > -1/2$ that $\tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} < \infty$.

Proof. Items (i)–(ii) are an immediate consequence of (164), of items (vi)–(vii) of Lemma 6.2, and of the fact that $\forall\, x \in V, t \in [0,T]\colon u(T-t,x) = \mathbb{E}[\varphi(X_t^x)]$. The proof of Lemma 6.3 is thus completed. □

Lemma 6.4. *Assume the setting in Section 6.2 and let $t \in [0,T)$, $\psi = (\psi(x,y))_{x,y\in V} \in \mathbb{M}(V \times V, \mathcal{V})$, $\phi \in \mathbb{M}(V,\mathcal{V})$ satisfy for all $x,y \in V$ that $\psi(x,y) = u_{0,1}(t,x)F(y)$ and $\phi(x) = \psi(x,x)$. Then it holds for all $x,x_1,x_2,y,y_1,y_2 \in V$ that $\psi \in C^3(V\times V,\mathcal{V})$, $\phi \in C^3(V,\mathcal{V})$, and*

$$\begin{aligned} &\max_{i,j\in\mathbb{N}_0,\, i+j\le 2} \big\|\big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_1,y) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_2,y)\big\|_{L^{(i+j)}(V,\mathcal{V})} \\ &\le \tfrac{\|x_1-x_2\|_V}{(T-t)^{\vartheta}}\,\|F\|_{C_b^2(V,V_{-\vartheta})}\,\big[c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0}\big] \max\{1,\|y\|_V\}, \end{aligned} \tag{169}$$

$$\begin{aligned} &\max_{i,j\in\mathbb{N}_0,\, i+j\le 2} \big\|\big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_1) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_2)\big\|_{L^{(i+j)}(V,\mathcal{V})} \\ &\le \tfrac{\|y_1-y_2\|_V}{(T-t)^{\vartheta}}\,\|F\|_{C_b^3(V,V_{-\vartheta})}\,\big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0}\big], \end{aligned} \tag{170}$$

$$\begin{aligned} &\max_{i\in\{0,1,2\}} \big\|\phi^{(i)}(x_1) - \phi^{(i)}(x_2)\big\|_{L^{(i)}(V,\mathcal{V})} \\ &\le \tfrac{3\,\|x_1-x_2\|_V}{(T-t)^{\vartheta}}\,\|F\|_{C_b^3(V,V_{-\vartheta})}\,\big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0}\big] \max\{1,\|x_1\|_V,\|x_2\|_V\}. \end{aligned} \tag{171}$$

Proof. We first note that item (iii) of Lemma 6.2 ensures that $\big(V \ni x \mapsto u_{0,1}(t,x) \in L(V,\mathcal{V})\big) \in C^3(V, L(V,\mathcal{V}))$. The assumption that $F \in \operatorname{Lip}^4(V, V_2)$ therefore assures that $\psi \in C^3(V \times V, \mathcal{V})$ and $\phi \in C^3(V, \mathcal{V})$. Next we observe that for all $x, y, v_1, v_2, v_3 \in V$ with $\max\{\|v_1\|_V, \|v_2\|_V, \|v_3\|_V\} \leq 1$ it holds that

$$\big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,y)\, v_1\big\|_{\mathcal{V}} = \big\|u_{0,2}(t,x)\big(F(y), v_1\big)\big\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta,0}}{(T-t)^{\vartheta}}\, \|F(y)\|_{V_{-\vartheta}}, \tag{172}$$

$$\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,y)\, (v_1, v_2)\big\|_{\mathcal{V}} = \big\|u_{0,3}(t,x)\big(F(y), v_1, v_2\big)\big\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta,0,0}}{(T-t)^{\vartheta}}\, \|F(y)\|_{V_{-\vartheta}}, \tag{173}$$

$$\big\|\big(\tfrac{\partial^3}{\partial x^3}\psi\big)(x,y)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} = \big\|u_{0,4}(t,x)\big(F(y), v_1, v_2, v_3\big)\big\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta,0,0,0}}{(T-t)^{\vartheta}}\, \|F(y)\|_{V_{-\vartheta}}, \tag{174}$$

$$\big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,y)\, v_1\big\|_{\mathcal{V}} = \|u_{0,1}(t,x)\, F'(y)\, v_1\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta}}{(T-t)^{\vartheta}}\, \|F'(y)\|_{L(V,V_{-\vartheta})}, \tag{175}$$

$$\big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,y)(v_1, v_2)\big\|_{\mathcal{V}} = \big\|u_{0,1}(t,x)\big(F''(y)(v_1, v_2)\big)\big\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta}}{(T-t)^{\vartheta}}\, \|F''(y)\|_{L^{(2)}(V,V_{-\vartheta})}, \tag{176}$$

$$\begin{split}\big\|\big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,y)(v_1, v_2, v_3)\big\|_{\mathcal{V}} &= \big\|u_{0,1}(t,x)\big(F^{(3)}(y)(v_1, v_2, v_3)\big)\big\|_{\mathcal{V}} \\ &\leq \tfrac{c_{-\vartheta}}{(T-t)^{\vartheta}}\, \|F^{(3)}(y)\|_{L^{(3)}(V,V_{-\vartheta})},\end{split} \tag{177}$$

$$\big\|\big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,y)(v_1, v_2)\big\|_{\mathcal{V}} = \big\|u_{0,2}(t,x)\big(F'(y)\, v_1, v_2\big)\big\|_{\mathcal{V}} \leq \tfrac{c_{-\vartheta,0}}{(T-t)^{\vartheta}}\, \|F'(y)\|_{L(V,V_{-\vartheta})}, \tag{178}$$

$$\begin{split}\big\|\big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,y)(v_1, v_2, v_3)\big\|_{\mathcal{V}} &= \big\|u_{0,3}(t,x)\big(F'(y)\, v_1, v_2, v_3\big)\big\|_{\mathcal{V}} \\ &\leq \tfrac{c_{-\vartheta,0,0}}{(T-t)^{\vartheta}}\, \|F'(y)\|_{L(V,V_{-\vartheta})},\end{split} \tag{179}$$

$$\begin{split}\big\|\big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,y)(v_1, v_2, v_3)\big\|_{\mathcal{V}} &= \big\|u_{0,2}(t,x)\big(F''(y)(v_1, v_2), v_3\big)\big\|_{\mathcal{V}} \\ &\leq \tfrac{c_{-\vartheta,0}}{(T-t)^{\vartheta}}\, \|F''(y)\|_{L^{(2)}(V,V_{-\vartheta})}.\end{split} \tag{180}$$

Combining (172)–(174) and (178)–(180) with item (i) of Lemma 6.3 and the fundamental theorem of calculus in Banach spaces proves (169). Moreover, combining (175)–(180) with item (i) of Lemma 6.3 and the fundamental theorem of calculus in Banach spaces shows (170). It thus remains to prove (171). For this we observe that (172)–(180) ensure that for all $x, v_1, v_2, v_3 \in V$ with $\max\{\|v_1\|_V, \|v_2\|_V, \|v_3\|_V\} \leq 1$ it holds that

$$\begin{split}\big\|\phi'(x)\, v_1\big\|_{\mathcal{V}} &\leq \big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,x)\, v_1\big\|_{\mathcal{V}} + \big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,x)\, v_1\big\|_{\mathcal{V}} \\ &\leq \tfrac{c_{-\vartheta,0}\, \|F(x)\|_{V_{-\vartheta}} + c_{-\vartheta}\, \|F'(x)\|_{L(V,V_{-\vartheta})}}{(T-t)^{\vartheta}} \leq \tfrac{[c_{-\vartheta} + c_{-\vartheta,0}]}{(T-t)^{\vartheta}}\, \|F\|_{C^1_b(V,V_{-\vartheta})}\, \max\{1, \|x\|_V\},\end{split} \tag{181}$$

$$\begin{split}&\big\|\phi''(x)\, (v_1, v_2)\big\|_{\mathcal{V}} \\ &\leq \big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\, (v_1, v_2)\big\|_{\mathcal{V}} + 2\,\big\|\big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,x)\, (v_1, v_2)\big\|_{\mathcal{V}} + \big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,x)\, (v_1, v_2)\big\|_{\mathcal{V}} \\ &\leq \frac{c_{-\vartheta,0,0}\, \|F(x)\|_{V_{-\vartheta}} + 2\, c_{-\vartheta,0}\, \|F'(x)\|_{L(V,V_{-\vartheta})} + c_{-\vartheta}\, \|F''(x)\|_{L^{(2)}(V,V_{-\vartheta})}}{(T-t)^{\vartheta}} \\ &\leq \tfrac{2\, [c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0}]}{(T-t)^{\vartheta}}\, \|F\|_{C^2_b(V,V_{-\vartheta})}\, \max\{1, \|x\|_V\},\end{split} \tag{182}$$

and

$$\begin{split}&\big\|\phi^{(3)}(x)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} \leq \big\|\big(\tfrac{\partial^3}{\partial x^3}\psi\big)(x,x)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} + 3\,\big\|\big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,x)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} \\ &+ 3\,\big\|\big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,x)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} + \big\|\big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,x)\, (v_1, v_2, v_3)\big\|_{\mathcal{V}} \\ &\leq \frac{c_{-\vartheta,0,0,0}\, \|F(x)\|_{V_{-\vartheta}} + 3\, c_{-\vartheta,0,0}\, \|F'(x)\|_{L(V,V_{-\vartheta})} + 3\, c_{-\vartheta,0}\, \|F''(x)\|_{L^{(2)}(V,V_{-\vartheta})} + c_{-\vartheta}\, \|F^{(3)}(x)\|_{L^{(3)}(V,V_{-\vartheta})}}{(T-t)^{\vartheta}} \\ &\leq \tfrac{3\, [c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0}]}{(T-t)^{\vartheta}}\, \|F\|_{C^3_b(V,V_{-\vartheta})}\, \max\{1, \|x\|_V\}.\end{split} \tag{183}$$

Combining (181)–(183) with item (i) of Lemma 6.3 and the fundamental theorem of calculus in Banach spaces establishes (171). The proof of Lemma 6.4 is thus completed. □

Lemma 6.5. *Assume the setting in Section 6.2 and let* $t \in [0,T)$, $\psi = (\psi(x,y))_{x,y\in V} \in \mathbb{M}(V \times V, \mathcal{V})$, $\phi \in \mathbb{M}(V,\mathcal{V})$ *satisfy for all* $x, y \in V$ *that* $\psi(x,y) = \sum_{b\in\mathbb{U}} u_{0,2}(t,x)\big(B^b(y), B^b(y)\big)$ *and* $\phi(x) = \psi(x,x)$. *Then it holds for all* $x, x_1, x_2, y, y_1, y_2 \in V$ *that* $\psi \in C^2(V\times V, \mathcal{V})$, $\phi \in C^2(V,\mathcal{V})$, *and*

$$\begin{split}\max_{i,j\in\mathbb{N}_0,\, i+j\le 2} \big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_1,y) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_2,y)\big\|_{L^{(i+j)}(V,\mathcal{V})} \le \tfrac{2\,\|x_1-x_2\|_V}{(T-t)^{\vartheta}} \\ \cdot \|B\|^2_{C^2_b(V,\gamma(U,V_{-\vartheta/2}))} \big[c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|y\|^2_V\},\end{split} \tag{184}$$

$$\begin{split}\max_{i,j\in\mathbb{N}_0,\, i+j\le 2} \big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_1) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_2)\big\|_{L^{(i+j)}(V,\mathcal{V})} \le \tfrac{6\,\|y_1-y_2\|_V}{(T-t)^{\vartheta}} \\ \cdot \|B\|^2_{C^3_b(V,\gamma(U,V_{-\vartheta/2}))} \big[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|y_1\|_V, \|y_2\|_V\},\end{split} \tag{185}$$

$$\begin{split}\max_{i\in\{0,1,2\}} \big\|\phi^{(i)}(x_1) - \phi^{(i)}(x_2)\big\|_{L^{(i)}(V,\mathcal{V})} \le \tfrac{8\,\|x_1-x_2\|_V}{(T-t)^{\vartheta}} \|B\|^2_{C^3_b(V,\gamma(U,V_{-\vartheta/2}))} \\ \cdot \big[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|x_1\|^2_V, \|x_2\|^2_V\}.\end{split} \tag{186}$$

Proof. We first note that item (iii) of Lemma 6.2 ensures that $\big(V \ni x \mapsto u_{0,2}(t,x) \in L^{(2)}(V,\mathcal{V})\big) \in C^2(V, L^{(2)}(V,\mathcal{V}))$. Lemma 1.2 and the assumption that $B \in \operatorname{Lip}^4(V, \gamma(U,V_2))$ therefore assure that

$$\psi \in C^2(V\times V, \mathcal{V}), \qquad \phi \in C^2(V,\mathcal{V}), \tag{187}$$

$$\big(V\times V \ni (x,y) \mapsto \big(\tfrac{\partial}{\partial y}\psi\big)(x,y) \in L(V,\mathcal{V})\big) \in C^2(V\times V, L(V,\mathcal{V})), \tag{188}$$

and

$$\forall\, x\in V\colon \big(V \ni y \mapsto \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,y) \in L^{(2)}(V,\mathcal{V})\big) \in C^1(V, L^{(2)}(V,\mathcal{V})). \tag{189}$$

Next we use Lemma 1.2 and Lemma 6.3 to obtain that for all $x, x_1, x_2, y, v_1, v_2, v_3 \in V$ with $\max\{\|v_1\|_V, \|v_2\|_V, \|v_3\|_V\} \le 1$ it holds that

$$\begin{split}\big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,y)\, v_1\big\|_{\mathcal{V}} &= \Big\|\sum_{b\in\mathbb{U}} u_{0,3}(t,x)\big(B^b(y), B^b(y), v_1\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^{\vartheta}} \|B(y)\|^2_{\gamma(U,V_{-\vartheta/2})},\end{split} \tag{190}$$

$$\begin{split}\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,y)\,(v_1,v_2)\big\|_{\mathcal{V}} &= \Big\|\sum_{b\in\mathbb{U}} u_{0,4}(t,x)\big(B^b(y), B^b(y), v_1, v_2\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{c_{-\vartheta/2,-\vartheta/2,0,0}}{(T-t)^{\vartheta}} \|B(y)\|^2_{\gamma(U,V_{-\vartheta/2})},\end{split} \tag{191}$$

$$\begin{split}&\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,y)\,(v_1,v_2) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,y)\,(v_1,v_2)\big\|_{\mathcal{V}} \\ &= \Big\|\sum_{b\in\mathbb{U}} \big(u_{0,4}(t,x_1) - u_{0,4}(t,x_2)\big)\big(B^b(y), B^b(y), v_1, v_2\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\,\|x_1-x_2\|_{\mathcal{V}}}{(T-t)^{\vartheta}} \|B(y)\|^2_{\gamma(U,V_{-\vartheta/2})},\end{split} \tag{192}$$

$$\begin{split}\big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,y)\, v_1\big\|_{\mathcal{V}} &= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,2}(t,x)\big(B^b(y), (B^b)'(y)\, v_1\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^{\vartheta}} \|B(y)\|_{\gamma(U,V_{-\vartheta/2})} \|B'(y)\|_{L(V,\gamma(U,V_{-\vartheta/2}))},\end{split} \tag{193}$$

$$\begin{split}&\big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,y)(v_1,v_2)\big\|_{\mathcal{V}} \\ &= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,2}(t,x)\big((B^b)'(y)\, v_1, (B^b)'(y)\, v_2\big) + u_{0,2}(t,x)\big(B^b(y), (B^b)''(y)(v_1,v_2)\big)\Big\|_{\mathcal{V}} \\ &\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^{\vartheta}} \big(\|B'(y)\|^2_{L(V,\gamma(U,V_{-\vartheta/2}))} + \|B(y)\|_{\gamma(U,V_{-\vartheta/2})} \|B''(y)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))}\big),\end{split} \tag{194}$$

$$
\begin{aligned}
&\big\|\big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,2}(t,x)\big((B^b)'(y)\,v_2,(B^b)''(y)(v_1,v_3)\big) \\
&\quad + u_{0,2}(t,x)\big((B^b)'(y)\,v_1,(B^b)''(y)(v_2,v_3)\big) \\
&\quad + u_{0,2}(t,x)\big((B^b)'(y)\,v_3,(B^b)''(y)(v_1,v_2)\big) \\
&\quad + u_{0,2}(t,x)\big(B^b(y),(B^b)^{(3)}(y)(v_1,v_2,v_3)\big)\Big\|_{\mathcal{V}} \\
&\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^{\vartheta}}\,\big(3\,\|B'(y)\|_{L(V,\gamma(U,V_{-\vartheta/2}))}\,\|B''(y)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))} \\
&\quad + \|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B^{(3)}(y)\|_{L^{(3)}(V,\gamma(U,V_{-\vartheta/2}))}\big),
\end{aligned}
\tag{195}
$$

$$
\begin{aligned}
\big\|\big(\tfrac{\partial^2}{\partial x\partial y}\psi\big)(x,y)(v_1,v_2)\big\|_{\mathcal{V}} &= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,3}(t,x)\big(B^b(y),(B^b)'(y)\,v_1,v_2\big)\Big\|_{\mathcal{V}} \\
&\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^{\vartheta}}\,\|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B'(y)\|_{L(V,\gamma(U,V_{-\vartheta/2}))},
\end{aligned}
\tag{196}
$$

$$
\begin{aligned}
\big\|\big(\tfrac{\partial^3}{\partial x^2\partial y}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_{\mathcal{V}} &= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,4}(t,x)\big(B^b(y),(B^b)'(y)\,v_1,v_2,v_3\big)\Big\|_{\mathcal{V}} \\
&\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0,0}}{(T-t)^{\vartheta}}\,\|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B'(y)\|_{L(V,\gamma(U,V_{-\vartheta/2}))},
\end{aligned}
\tag{197}
$$

$$
\big\|\big(\tfrac{\partial^3}{\partial y\partial x^2}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_{\mathcal{V}} \le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0,0}}{(T-t)^{\vartheta}}\,\|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B'(y)\|_{L(V,\gamma(U,V_{-\vartheta/2}))},
\tag{198}
$$

$$
\begin{aligned}
&\big\|\big(\tfrac{\partial^3}{\partial x\partial y^2}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&= 2\,\Big\|\sum_{b\in\mathbb{U}} u_{0,3}(t,x)\big((B^b)'(y)\,v_1,(B^b)'(y)\,v_2,v_3\big) + u_{0,3}(t,x)\big(B^b(y),(B^b)''(y)(v_1,v_2),v_3\big)\Big\|_{\mathcal{V}} \\
&\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^{\vartheta}}\,\big(\|B'(y)\|^2_{L(V,\gamma(U,V_{-\vartheta/2}))} + \|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B''(y)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))}\big),
\end{aligned}
\tag{199}
$$

and

$$
\begin{aligned}
&\big\|\big(\tfrac{\partial^3}{\partial y\partial x\partial y}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&\le \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^{\vartheta}}\,\big(\|B'(y)\|^2_{L(V,\gamma(U,V_{-\vartheta/2}))} + \|B(y)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B''(y)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))}\big).
\end{aligned}
\tag{200}
$$

Combining (190)–(192), (196), (197), and (199) with Lemma 6.3 and the fundamental theorem of calculus in Banach spaces proves (184). Moreover, combining (193)–(196), (198), and (200) with item (i) of Lemma 6.3 and the fundamental theorem of calculus in Banach spaces establishes (185). It thus remains to prove (186). For this we observe that (190)–(199) ensure that for all $x,v_1,v_2,v_3\in V$ with $\max\{\|v_1\|_V,\|v_2\|_V,\|v_3\|_V\}\le 1$ it holds that

$$
\begin{aligned}
\big\|\phi'(x)\,v_1\big\|_{\mathcal{V}} &\le \big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,x)\,v_1\big\|_{\mathcal{V}} + \big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,x)\,v_1\big\|_{\mathcal{V}} \\
&\le \frac{c_{-\vartheta/2,-\vartheta/2,0}\,\|B(x)\|^2_{\gamma(U,V_{-\vartheta/2})} + 2\,c_{-\vartheta/2,-\vartheta/2}\,\|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B'(x)\|_{L(V,\gamma(U,V_{-\vartheta/2}))}}{(T-t)^{\vartheta}} \\
&\le \tfrac{2\,[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}]}{(T-t)^{\vartheta}}\,\|B\|^2_{C^1_b(V,\gamma(U,V_{-\vartheta/2}))}\,\max\{1,\|x\|_V^2\}
\end{aligned}
\tag{201}
$$

and

$$
\begin{aligned}
&\big\|\phi''(x)\,(v_1,v_2)\big\|_{\mathcal{V}} \\
&\le \big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\,(v_1,v_2)\big\|_{\mathcal{V}} + 2\,\big\|\big(\tfrac{\partial^2}{\partial x\partial y}\psi\big)(x,x)\,(v_1,v_2)\big\|_{\mathcal{V}} + \big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,x)\,(v_1,v_2)\big\|_{\mathcal{V}} \\
&\le \frac{c_{-\vartheta/2,-\vartheta/2,0,0}\,\|B(x)\|^2_{\gamma(U,V_{-\vartheta/2})} + 4\,c_{-\vartheta/2,-\vartheta/2,0}\,\|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B'(x)\|_{L(V,\gamma(U,V_{-\vartheta/2}))}}{(T-t)^{\vartheta}} \\
&\quad + \frac{2\,c_{-\vartheta/2,-\vartheta/2}\,\big(\|B'(x)\|^2_{L(V,\gamma(U,V_{-\vartheta/2}))} + \|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\,\|B''(x)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))}\big)}{(T-t)^{\vartheta}} \\
&\le \tfrac{4\,[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}]}{(T-t)^{\vartheta}}\,\|B\|^2_{C^2_b(V,\gamma(U,V_{-\vartheta/2}))}\,\max\{1,\|x\|_V^2\}.
\end{aligned}
\tag{202}
$$

In the next step we observe that (195), (197), (199), (200), and the fact that $\big(V \ni x \mapsto \phi''(x) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x) \in L^{(2)}(V,\mathcal{V})\big) \in C^1(V, L^{(2)}(V,\mathcal{V}))$ show that for all $x, x_1, x_2, v_1, v_2, v_3 \in V$ with $\max\{\|v_1\|_V, \|v_2\|_V, \|v_3\|_V\} \leq 1$ it holds that

$$
\begin{aligned}
&\big\| \tfrac{\partial}{\partial x}\big(\phi''(x) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\big)(v_1,v_2,v_3)\big\|_{\mathcal{V}} \leq 2\, \big\| \big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&+ 2\, \big\| \big(\tfrac{\partial^3}{\partial y \partial x \partial y}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_{\mathcal{V}} + \big\| \big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&+ \big\| \big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_{\mathcal{V}} \\
&\leq \tfrac{4\, c_{-\vartheta/2,-\vartheta/2,0,0}\, \|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\, \|B'(x)\|_{L(V,\gamma(U,V_{-\vartheta/2}))}}{(T-t)^{\vartheta}} \\
&+ \tfrac{6\, c_{-\vartheta/2,-\vartheta/2,0}\, \left(\|B'(x)\|^2_{L(V,\gamma(U,V_{-\vartheta/2}))} + \|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\, \|B''(x)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))}\right)}{(T-t)^{\vartheta}} \\
&+ \tfrac{6\, c_{-\vartheta/2,-\vartheta/2}\, \left(\|B'(x)\|_{L(V,\gamma(U,V_{-\vartheta/2}))}\, \|B''(x)\|_{L^{(2)}(V,\gamma(U,V_{-\vartheta/2}))} + \|B(x)\|_{\gamma(U,V_{-\vartheta/2})}\, \|B^{(3)}(x)\|_{L^{(3)}(V,\gamma(U,V_{-\vartheta/2}))}\right)}{(T-t)^{\vartheta}} \\
&\leq \tfrac{6\,[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}]}{(T-t)^{\vartheta}}\, \|B\|^2_{C^3_b(V,\gamma(U,V_{-\vartheta/2}))}\, \max\{1, \|x\|_V\}.
\end{aligned}
\tag{203}
$$

In addition, we combine (192) and (198) with item (i) of Lemma 6.3 and the fundamental theorem of calculus in Banach spaces to obtain that for all $x_1, x_2, v_1, v_2 \in V$ with $\max\{\|v_1\|_V, \|v_2\|_V\} \leq 1$ it holds that

$$
\begin{aligned}
&\big\| \big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_2)\big)(v_1,v_2)\big\|_{\mathcal{V}} \\
&\leq \big\| \big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_1)\big)(v_1,v_2)\big\|_{\mathcal{V}} \\
&+ \big\| \big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_2)\big)(v_1,v_2)\big\|_{\mathcal{V}} \\
&\leq \tfrac{\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\, \|x_1 - x_2\|_V}{(T-t)^{\vartheta}}\, \|B(x_1)\|^2_{\gamma(U,V_{-\vartheta/2})} \\
&+ \tfrac{2\, c_{-\vartheta/2,-\vartheta/2,0,0}\, \|x_1 - x_2\|_V}{(T-t)^{\vartheta}}\, \|B\|^2_{C^1_b(V,\gamma(U,V_{-\vartheta/2}))}\, \max\{1, \|x_1\|_V, \|x_2\|_V\} \\
&\leq \tfrac{2\, \|x_1 - x_2\|_V}{(T-t)^{\vartheta}}\, \|B\|^2_{C^1_b(V,\gamma(U,V_{-\vartheta/2}))}\, \big[c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big]\, \max\{1, \|x_1\|^2_V, \|x_2\|^2_V\}.
\end{aligned}
\tag{204}
$$

Combining (201)–(204) with Lemma 6.3 and the fundamental theorem of calculus in Banach spaces finally yields (186). The proof of Lemma 6.5 is thus completed. ☐

Lemma 6.6 (Weak convergence of semilinear integrated exponential Euler approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 6.2 and let $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(\bar{Y}_T)\|_{\mathcal{V}}\big] < \infty$ and*

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}_T)\big]\big\|_{\mathcal{V}} \leq \tfrac{5\, |\chi_0|^3\, |\chi_\rho|^2\, T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\, \varsigma_{F,B}\, K_5\, h^{\rho} \\
&\cdot \Big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big] \\
&\cdot \Bigg[2^{\rho} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(3\,\chi_\vartheta + 2\,\chi_{\rho+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2} + 2\,(|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\,\chi_\rho \\
&\cdot \Big[\tfrac{\chi_\vartheta\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\Upsilon_4\, \chi_{\vartheta/2}\, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg] < \infty.
\end{aligned}
\tag{205}
$$

Proof. We first observe that the assumption that $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^5_V\big] < \infty$ implies that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}}\big] < \infty$. Moreover, combining the assumption that $Y_0 \in \mathcal{L}^5(\mathbb{P}; V_2)$ with Lemma 5.1 proves that $K_5 < \infty$. This shows, in particular, that

$$
\sup_{s\in[0,T]} \mathbb{E}\Big[\|\varphi(\bar{Y}_T)\|_{\mathcal{V}} + \|\bar{Y}_s\|_{V_2} + \int_0^T \|u_{0,1}(t,\bar{Y}_t)\, B(Y_{\lfloor t \rfloor_h})\|^2_{\gamma(U,\mathcal{V})}\, dt\Big] < \infty. \tag{206}
$$

In addition, note that Kolmogorov-Chentsov's theorem, the fact that $X_0 \in \mathcal{L}^5(\mathbb{P}; V)$, the fact that $F \in \mathrm{Lip}^0(V, V)$, and the fact that $B \in \mathrm{Lip}^0(V, \gamma(U, V))$ ensure that there exists a continuous modification of X. This, Lemma 6.1, and the assumption that $X_0 = \bar{Y}_0$ yield that

$$\mathbb{E}\big[\varphi(\bar{Y}_T)\big] - \mathbb{E}\big[\varphi(X_T)\big] = \mathbb{E}\big[u(T, \bar{Y}_T) - u(0, \bar{Y}_0)\big]\,. \tag{207}$$

Items (xvi)–(xvii) of Lemma 6.2, (206), and the standard Itô formula in Theorem 2.4 in Brzeźniak et al. [11] therefore prove that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(\bar{Y}_T)\big] - \mathbb{E}\big[\varphi(X_T)\big] = \mathbb{E}\big[u(T, \bar{Y}_T) - u(0, \bar{Y}_0)\big] \\
&= \int_0^T \mathbb{E}\big[u_{1,0}(t, \bar{Y}_t) + u_{0,1}(t, \bar{Y}_t)\big(A\bar{Y}_t + F(Y_{\lfloor t \rfloor_h})\big)\big]\, dt \\
&+ \frac{1}{2} \sum_{b \in \mathbb{U}} \int_0^T \mathbb{E}\big[u_{0,2}(t, \bar{Y}_t)\big(B^b(Y_{\lfloor t \rfloor_h}), B^b(Y_{\lfloor t \rfloor_h})\big)\big]\, dt.
\end{aligned} \tag{208}$$

Item (xix) of Lemma 6.2 hence shows that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(\bar{Y}_T)\big] - \mathbb{E}\big[\varphi(X_T)\big] \\
&= \int_0^T \mathbb{E}\big[u_{0,1}(t, \bar{Y}_t)\, F(Y_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_t)\, F(\bar{Y}_t)\big]\, dt \\
&+ \frac{1}{2} \sum_{b \in \mathbb{U}} \int_0^T \mathbb{E}\big[u_{0,2}(t, \bar{Y}_t)\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big) - u_{0,2}(t, \bar{Y}_t)\big(B^b\big(\bar{Y}_t\big), B^b\big(\bar{Y}_t\big)\big)\big]\, dt.
\end{aligned} \tag{209}$$

The triangle inequality hence shows that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}_T)\big]\big\|_{\mathcal{V}} \\
&\leq \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t, \bar{Y}_t)\, F(Y_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(Y_{\lfloor t \rfloor_h})\big]\big\|_{\mathcal{V}}\, dt \\
&+ \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(Y_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(\bar{Y}_{\lfloor t \rfloor_h})\big]\big\|_{\mathcal{V}}\, dt \\
&+ \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(\bar{Y}_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_t)\, F(\bar{Y}_t)\big]\big\|_{\mathcal{V}}\, dt \\
&+ \tfrac{1}{2} \int_0^T \Big\|\mathbb{E}\Big[\sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_t)\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big) - \sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big)\Big]\Big\|_{\mathcal{V}}\, dt \\
&+ \tfrac{1}{2} \int_0^T \Big\|\mathbb{E}\Big[\sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big) - \sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big), B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big)\big)\Big]\Big\|_{\mathcal{V}}\, dt \\
&+ \tfrac{1}{2} \int_0^T \Big\|\mathbb{E}\Big[\sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big), B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big)\big) - \sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_t)\big(B^b\big(\bar{Y}_t\big), B^b\big(\bar{Y}_t\big)\big)\Big]\Big\|_{\mathcal{V}}\, dt.
\end{aligned} \tag{210}$$

In the next step we combine Lemma 6.4 and Lemma 6.5 with Proposition 5.2 to obtain that for all $t \in (0, T)$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[u_{0,1}(t, \bar{Y}_t)\, F(Y_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(Y_{\lfloor t \rfloor_h})\big]\big\|_{\mathcal{V}} \\
&+ \big\|\mathbb{E}\big[u_{0,1}(t, \bar{Y}_{\lfloor t \rfloor_h})\, F(\bar{Y}_{\lfloor t \rfloor_h}) - u_{0,1}(t, \bar{Y}_t)\, F(\bar{Y}_t)\big]\big\|_{\mathcal{V}} \\
&+ \tfrac{1}{2} \Big\|\mathbb{E}\Big[\sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_t)\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big) - \sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(Y_{\lfloor t \rfloor_h}\big), B^b\big(Y_{\lfloor t \rfloor_h}\big)\big)\Big]\Big\|_{\mathcal{V}} \\
&+ \tfrac{1}{2} \Big\|\mathbb{E}\Big[\sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_{\lfloor t \rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big), B^b\big(\bar{Y}_{\lfloor t \rfloor_h}\big)\big) - \sum_{b \in \mathbb{U}} u_{0,2}(t, \bar{Y}_t)\big(B^b\big(\bar{Y}_t\big), B^b\big(\bar{Y}_t\big)\big)\Big]\Big\|_{\mathcal{V}} \\
&\leq \tfrac{|\chi_0|^3\, |\chi_\rho|^2}{(T-t)^{\vartheta}}\, K_5\, h^{\rho}\, \max\big\{1, \|F\|_{\mathrm{Lip}^0(V, V_{-\vartheta})}, \|B\|^2_{\mathrm{Lip}^0(V, \gamma(U, V_{-\vartheta/2}))}\big\}
\end{aligned} \tag{211}$$

$$\cdot \Bigg[4\left[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}+c_{-\vartheta,0,0,0}\right]\|F\|_{C_b^3(V,V_{-\vartheta})}$$

$$+5\left[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\right]\|B\|^2_{C_b^3(V,\gamma(U,V_{-\vartheta/2}))}\Bigg]$$

$$\cdot \left[\tfrac{2^\rho}{t^\rho}+\tfrac{\left(2\,\chi_\vartheta+\chi_{\rho+\vartheta}+2\,|\chi_{\vartheta/2}|^2+2\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2}\right)|\lfloor t\rfloor_h|^{(1-\vartheta-\rho)}+\left(\chi_\vartheta+\frac{1}{2}|\chi_{\vartheta/2}|^2\right)(t-\lfloor t\rfloor_h)^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right].$$

In addition, we combine Lemma 6.4 and Lemma 6.5 with Proposition 5.4 to obtain that for all $t\in(0,T)$ it holds that

$$\begin{aligned}
&\left\|\mathbb{E}\left[u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(Y_{\lfloor t\rfloor_h})-u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(\bar{Y}_{\lfloor t\rfloor_h})\right]\right\|_{\mathcal{V}}\\
&+\tfrac{1}{2}\left\|\mathbb{E}\left[\sum_{b\in\mathbb{U}}u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b(Y_{\lfloor t\rfloor_h}),B^b(Y_{\lfloor t\rfloor_h})\big)-\sum_{b\in\mathbb{U}}u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b(\bar{Y}_{\lfloor t\rfloor_h}),B^b(\bar{Y}_{\lfloor t\rfloor_h})\big)\right]\right\|_{\mathcal{V}}\\
&\leq \tfrac{\chi_0\,\chi_\rho}{(T-t)^\vartheta}\,K_4\,h^\rho\,\max\big\{1,\|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})},\|B\|^2_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\big\}\\
&\cdot\max\big\{1,\|F\|_{\mathrm{Lip}^0(V,V_{-\vartheta})},\|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\vartheta/2}))}\big\}\Big(\left[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}\right]\|F\|_{C_b^3(V,V_{-\vartheta})}\\
&+3\left[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}\right]\|B\|^2_{C_b^3(V,\gamma(U,V_{-\vartheta/2}))}\Big)\\
&\cdot\tfrac{|\lfloor t\rfloor_h|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\left(\chi_{\rho+\vartheta}+2\,\chi_{\vartheta/2}\,\chi_{\rho+\vartheta/2}+2\,(|\chi_{\vartheta/2}|^2+\chi_\vartheta)\,\chi_\rho\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\Upsilon_4\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\right).
\end{aligned} \tag{212}$$

Combining (210)–(212) proves that

$$\begin{aligned}
&\left\|\mathbb{E}\left[\varphi(X_T)\right]-\mathbb{E}\left[\varphi(\bar{Y}_T)\right]\right\|_{\mathcal{V}}\leq 5\,|\chi_0|^3\,|\chi_\rho|^2\,\varsigma_{F,B}\,K_5\,h^\rho\int_0^T\tfrac{1}{(T-t)^\vartheta\,t^\rho}\,dt\\
&\cdot\left[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}+c_{-\vartheta,0,0,0}+c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\right]\\
&\cdot\Bigg[2^\rho+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\Bigg(3\,\chi_\vartheta+2\,\chi_{\rho+\vartheta}+3\,|\chi_{\vartheta/2}|^2+4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2}+2\,(|\chi_{\vartheta/2}|^2+\chi_\vartheta)\,\chi_\rho\\
&\cdot\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\Upsilon_4\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\Bigg)\Bigg].
\end{aligned} \tag{213}$$

This and the fact that for all $x,y\in(0,\infty)$ with $(x-1)(y-1)\geq 0$ and $x+y>1$ it holds that

$$\int_0^1(1-t)^{(x-1)}\,t^{(y-1)}\,dt\leq\frac{1}{(x+y-1)} \tag{214}$$

establish the first inequality in (205). The second inequality in (205) follows from Lemma 6.3. The proof of Lemma 6.6 is thus completed. □

6.4 Weak convergence rates for exponential Euler approximations of SPDEs with mollified nonlinearities

The next result, Corollary 6.7, provides a bound for the weak distance of the numerical approximation and its semilinear integrated counterpart. Corollary 6.7 is an immediate consequence of Proposition 5.4 and of Lemma 6.6.

Corollary 6.7 (Weak distance between exponential Euler approximations of SPDEs with mollified nonlinearities and their semilinear integrated counterparts)**.** *Assume the setting in Section 6.2 and let $\rho\in[0,1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(\bar{Y}_T)\|_{\mathcal{V}}+\|\varphi(Y_T)\|_{\mathcal{V}}\big]<\infty$ and*

$$\begin{aligned}
&\left\|\mathbb{E}\left[\varphi(\bar{Y}_T)\right]-\mathbb{E}\left[\varphi(Y_T)\right]\right\|_{\mathcal{V}}\leq\chi_\rho\,\|\varphi\|_{\mathrm{Lip}^2(V,\mathcal{V})}\,K_3\,h^\rho\,\varsigma_{F,B}\\
&\cdot\tfrac{T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\left(\chi_{\rho+\vartheta}+2\,\chi_{\vartheta/2}\,\chi_{\rho+\vartheta/2}+2\,(|\chi_{\vartheta/2}|^2+\chi_\vartheta)\,\chi_\rho\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\Upsilon_3\,\chi_{\vartheta/2}\sqrt{T^{(1-\vartheta)}}}{\sqrt{1-\vartheta}}\right]\right).
\end{aligned} \tag{215}$$

The next result is a direct consequence of the triangle inequality, of Corollary 6.7, and of Lemma 6.6.

Corollary 6.8 (Weak convergence of exponential Euler approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 6.2 and let $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y_T)\|_{\mathcal{V}}\big] < \infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_{\mathcal{V}} \leq \tfrac{5\,|\chi_0|^3\,|\chi_\rho|^2\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-\rho)\,T^\rho}\,\varsigma_{F,B}\,K_5\,h^\rho \\
&\cdot\Bigg[2^\rho + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(3\,\chi_\vartheta + 2\,\chi_{\rho+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2} \\
&+ 2\,(|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\,\chi_\rho\Big[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg] \\
&\cdot\Big[\|\varphi\|_{\mathrm{Lip}^2(V,\mathcal{V})} + c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} \\
&+ c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big] < \infty.
\end{aligned} \tag{216}$$

Corollary 6.9 (Weak convergence of exponential Euler approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 6.2 and let $\theta \in [0,1)$, $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y_T)\|_{\mathcal{V}}\big] < \infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_{\mathcal{V}} \leq \tfrac{57\,|\chi_0|^3\,|\chi_\rho|^2\,\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-\rho)\,T^\rho}\,\varsigma_{F,B}\,h^\rho \\
&\cdot\left[\chi_0\max\{1,\|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\} + \frac{\chi_\theta\,T^{(1-\theta)}\,\|F\|_{\mathrm{Lip}^0(V,V_{-\theta})}}{(1-\theta)} + \Upsilon_5\,\chi_{\theta/2}\sqrt{\frac{T^{(1-\theta)}}{(1-\theta)}}\,\|B\|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\theta/2}))}\right]^{10} \\
&\cdot\left|\mathcal{E}_{(1-\theta)}\left[\frac{\sqrt{2}\,\chi_\theta\,T^{(1-\theta)}\,|F|_{\mathrm{Lip}^0(V,V_{-\theta})}}{\sqrt{1-\theta}} + \Upsilon_5\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|B|_{\mathrm{Lip}^0(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5 \\
&\cdot\Bigg[2^\rho + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(3\,\chi_\vartheta + 2\,\chi_{\rho+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2} \\
&+ 2\,(|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\,\chi_\rho\Big[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg] \\
&\cdot\Big[\|\varphi\|_{\mathrm{Lip}^2(V,\mathcal{V})} + c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} \\
&+ c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big] < \infty.
\end{aligned} \tag{217}$$

7 Weak error estimates for exponential Euler approximations of SPDEs

7.1 Setting

Assume the setting in Section 1.2, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with $\mathrm{spectrum}(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$, let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $h \in (0,T]$, $\theta \in [0,1)$, $\vartheta \in [0, 1/2) \cap [0,\theta]$, $F \in \mathrm{Lip}^4(V, V_{-\theta})$, $B \in \mathrm{Lip}^4(V, \gamma(U, V_{-\theta/2}))$, $\varphi \in \mathrm{Lip}^4(V,\mathcal{V})$, let $\varsigma_{F,B} \in \mathbb{R}$ be the real number given by $\varsigma_{F,B} = \max\big\{1, \|F\|^3_{C^3_b(V,V_{-\theta})}, \|B\|^6_{C^3_b(V,\gamma(U,V_{-\theta/2}))}\big\}$, let $\chi_r \in [1,\infty)$, $r \in [0,1]$, be the real numbers which satisfy for all $r \in [0,1]$ that $\chi_r = \max\{1, \sup_{t\in(0,T]} t^r\,\|(\eta - A)^r e^{tA}\|_{L(V)}, \sup_{t\in(0,T]} t^{-r}\,\|(\eta - A)^{-r}(e^{tA} - \mathrm{Id}_V)\|_{L(V)}\}$, let $X, Y\colon [0,T]\times\Omega \to V$ and $X^{\kappa,x}\colon [0,T]\times\Omega \to V$, $x \in V$, $\kappa \in [0,T]$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for all $\kappa \in [0,T]$, $x \in V$ that

$\sup_{t\in[0,T]}\big[\|X_t\|_{\mathcal{L}^5(\mathbb{P};V)} + \|X_t^{\kappa,x}\|_{\mathcal{L}^5(\mathbb{P};V)}\big] < \infty$, $X_0^{\kappa,x} = x$, and $Y_0 = X_0$ and which satisfy that for all $\kappa \in [0,T]$, $x \in V$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA} X_0 + \int_0^t e^{(t-s)A}\, F(X_s)\, ds + \int_0^t e^{(t-s)A}\, B(X_s)\, dW_s, \tag{218}$$

$$X_t^{\kappa,x} = e^{tA}\, x + \int_0^t e^{(\kappa+t-s)A}\, F(X_s^{\kappa,x})\, ds + \int_0^t e^{(\kappa+t-s)A}\, B(X_s^{\kappa,x})\, dW_s, \tag{219}$$

$$Y_t = e^{tA}\, Y_0 + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\, F(Y_{\lfloor s\rfloor_h})\, ds + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\, B(Y_{\lfloor s\rfloor_h})\, dW_s, \tag{220}$$

let $u^{(\kappa)} \colon [0,T]\times V \to \mathcal{V}$, $\kappa \in (0,T]$, be the functions which satisfy for all $\kappa \in (0,T]$, $x \in V$, $t \in [0,T]$ that $u^{(\kappa)}(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^{\kappa,x})\big]$, let $c^{(\kappa)}_{\delta_1,\dots,\delta_k} \in [0,\infty]$, $\delta_1,\dots,\delta_k \in (-1/2, 0]$, $k \in \{1,2,3,4\}$, $\kappa \in (0,T]$, be the extended real numbers which satisfy for all $\kappa \in (0,T]$, $k \in \{1,2,3,4\}$, $\delta_1,\dots,\delta_k \in (-1/2,0]$ that

$$c^{(\kappa)}_{\delta_1,\dots,\delta_k} = \sup_{t\in[0,T)} \sup_{x\in V} \sup_{v_1,\dots,v_k\in V\setminus\{0\}} \left[\frac{\big\|\big(\tfrac{\partial^k}{\partial x^k} u^{(\kappa)}\big)(t,x)(v_1,\dots,v_k)\big\|_{\mathcal{V}}}{(T-t)^{(\delta_1+\ldots+\delta_k)}\, \|v_1\|_{V_{\delta_1}} \cdot \ldots \cdot \|v_k\|_{V_{\delta_k}}}\right], \tag{221}$$

and let $\tilde{c}^{(\kappa)}_{\delta_1,\delta_2,\delta_3,\delta_4} \in [0,\infty]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in (-1/2,0]$, $\kappa \in (0,T]$, be the extended real numbers which satisfy for all $\kappa \in (0,T]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in (-1/2,0]$ that

$$\begin{aligned} &\tilde{c}^{(\kappa)}_{\delta_1,\delta_2,\delta_3,\delta_4} \\ &= \sup_{t\in[0,T)} \sup_{\substack{x_1,x_2\in V,\\ x_1\neq x_2}} \sup_{v_1,\dots,v_4\in V\setminus\{0\}} \left[\frac{\big\|\big(\big(\tfrac{\partial^4}{\partial x^4} u^{(\kappa)}\big)(t,x_1) - \big(\tfrac{\partial^4}{\partial x^4} u^{(\kappa)}\big)(t,x_2)\big)(v_1,\dots,v_4)\big\|_{\mathcal{V}}}{(T-t)^{(\delta_1+\ldots+\delta_4)}\, \|x_1-x_2\|_V\, \|v_1\|_{V_{\delta_1}} \cdot \ldots \cdot \|v_4\|_{V_{\delta_4}}}\right] \end{aligned} \tag{222}$$

(cf. Lemma 6.2).

7.2 Weak convergence result

Lemma 7.1. *Assume the setting in Section 7.1. Then*

(i) it holds for all $k \in \{1,2,3,4\}$, $\delta_1,\dots,\delta_k \in (-1/2,0]$ with $\sum_{i=1}^k \delta_i > -1/2$ that $\sup_{\kappa\in(0,T]} c^{(\kappa)}_{\delta_1,\dots,\delta_k} < \infty$ and

(ii) it holds for all $\delta_1,\dots,\delta_4 \in (-1/2,0]$ with $\sum_{i=1}^4 \delta_i > -1/2$ that $\sup_{\kappa\in(0,T]} \tilde{c}^{(\kappa)}_{\delta_1,\delta_2,\delta_3,\delta_4} < \infty$.

Proof. The proof of items (i)–(ii) is entirely analogous to the proof of items (iv)–(v) of Corollary 4.2 in Andersson et al. [1]. The proof of Lemma 7.1 is thus completed. ☐

Proposition 7.2. *Assume the setting in Section 7.1 and let $r \in [0, 1-\vartheta)$, $\rho \in (0, 1-\theta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y_T)\|_{\mathcal{V}}\big] < \infty$ and*

$$\begin{aligned} &\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_{\mathcal{V}} \le \Big[57\,\big|\max\{T, \tfrac{1}{T}\}\big|^{(r+3(\theta-\vartheta))}\, |\chi_0|^{20}\Big]\, h^{\frac{\rho\, r}{(\rho+6(\theta-\vartheta))}} \\ &\cdot \left[\max\{1, \|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\} + \tfrac{\chi_\theta\, \chi_{\rho/2+\theta}\, T^{(1-\theta)}\, \|F\|_{C^1_b(V,V_{-\theta})}}{(1-\theta-\rho/2)} + \tfrac{\max\{\Upsilon_2,\Upsilon_5\}\, \chi_{\theta/2}\, \chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\, \|B\|_{C^1_b(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10} \\ &\cdot \left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,\chi_0\,\chi_\theta\, T^{(1-\theta)}\, |F|_{C^1_b(V,V_{-\theta})}}{\sqrt{1-\theta}} + \max\{\Upsilon_2,\Upsilon_5\}\, \chi_0\, \chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\, |B|_{C^1_b(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5 \\ &\cdot \left[2^r + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)}\Big(3\,\chi_\vartheta + 2\,\chi_{r+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{r+\vartheta/2}\,\chi_{\vartheta/2}\right. \end{aligned} \tag{223}$$

$$+ 2\left(|\chi_{\vartheta/2}|^2 + \chi_\vartheta\right)\chi_r \left[\tfrac{\chi_\vartheta\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\Bigg)\Bigg]$$

$$\cdot \left[\tfrac{|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C^1_b(V,\mathcal{V})} + \tfrac{|\chi_0|^3\,|\chi_r|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6 \max\{1,T^{(1-\vartheta)}\}\,\varsigma_{F,B}}{(1-\vartheta-r)\,T^r}\Bigg(\|\varphi\|_{C^3_b(V,\mathcal{V})} + \sup_{\kappa\in(0,T]}\Big[c^{(\kappa)}_{-\vartheta}\right.$$

$$\left. + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big]\Bigg)\right] < \infty.$$

Proof. We first note that, e.g., Theorem 2.7 in Cox & Van Neerven [16] and Theorem 6.2 in Van Neerven et al. [47] ensure that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $\hat{Y}^{\kappa,\delta}\colon [0,T]\times\Omega\to V$, $\kappa,\delta\in[0,T]$, and $\hat{X}^{\kappa,\delta}\colon [0,T]\times\Omega\to V$, $\kappa,\delta\in[0,T]$, which satisfy for all $\kappa,\delta\in[0,T]$ that $\sup_{t\in[0,T]}\|\hat{X}^{\kappa,\delta}_t\|_{\mathcal{L}^5(\mathbb{P};V)}<\infty$ and $\hat{X}^{\kappa,\delta}_0=\hat{Y}^{\kappa,\delta}_0=e^{\delta A}X_0$ and which satisfy that for all $\kappa,\delta\in[0,T]$, $t\in(0,T]$ it holds $\mathbb{P}$-a.s. that

$$\hat{X}^{\kappa,\delta}_t = e^{tA}\hat{X}^{\kappa,\delta}_0 + \int_0^t e^{(\kappa+t-s)A}F(\hat{X}^{\kappa,\delta}_s)\,ds + \int_0^t e^{(\kappa+t-s)A}B(\hat{X}^{\kappa,\delta}_s)\,dW_s, \tag{224}$$

$$\hat{Y}^{\kappa,\delta}_t = e^{tA}\,\hat{Y}^{\kappa,\delta}_0 + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,e^{\kappa A}F(\hat{Y}^{\kappa,\delta}_{\lfloor s\rfloor_h})\,ds + \int_0^t e^{(t-\lfloor s\rfloor_h)A}\,e^{\kappa A}B(\hat{Y}^{\kappa,\delta}_{\lfloor s\rfloor_h})\,dW_s. \tag{225}$$

In the next step we combine Lemma 5.1 with the fact that $\forall\,\kappa,\delta\in[0,T]\colon \|\hat{Y}^{\kappa,\delta}_0\|_{\mathcal{L}^5(\mathbb{P};V)}<\infty$ to obtain that for all $\kappa,\delta\in[0,T]$ it holds that $\sup_{t\in[0,T]}\|\hat{Y}^{\kappa,\delta}_t\|_{\mathcal{L}^5(\mathbb{P};V)}<\infty$. This, the fact that $\forall\,\kappa,\delta\in[0,T]\colon \sup_{t\in[0,T]}\|\hat{X}^{\kappa,\delta}_t\|_{\mathcal{L}^5(\mathbb{P};V)}<\infty$ and the assumption that $\varphi\in\operatorname{Lip}^4(V,\mathcal{V})$ ensure that for all $\kappa,\delta\in[0,T]$ it holds that

$$\mathbb{E}\big[\|\varphi(\hat{X}^{\kappa,\delta}_T)\|_{\mathcal{V}} + \|\varphi(\hat{Y}^{\kappa,\delta}_T)\|_{\mathcal{V}}\big] < \infty. \tag{226}$$

This proves, in particular, that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}}+\|\varphi(Y_T)\|_{\mathcal{V}}\big]<\infty$. It thus remains to show (223). For this we observe that the triangle inequality ensures that for all $\kappa,\delta\in[0,T]$ it holds that

$$\begin{aligned}\big\|\mathbb{E}\big[\varphi(\hat{X}^{0,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{Y}^{0,\delta}_T)\big]\big\|_{\mathcal{V}} &\le \big\|\mathbb{E}\big[\varphi(\hat{X}^{0,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{X}^{\kappa,\delta}_T)\big]\big\|_{\mathcal{V}}\\ &+\big\|\mathbb{E}\big[\varphi(\hat{X}^{\kappa,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{Y}^{\kappa,\delta}_T)\big]\big\|_{\mathcal{V}}+\big\|\mathbb{E}\big[\varphi(\hat{Y}^{\kappa,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{Y}^{0,\delta}_T)\big]\big\|_{\mathcal{V}}.\end{aligned} \tag{227}$$

In the following we provide suitable bounds for the three summands on the right hand side of (227). For the first and the third summand on the right hand side of (227) we observe that Proposition 4.3 together with Hölder's inequality and the fact that $\forall\,\kappa,\delta\in[0,T]\colon \sup_{t\in[0,T]}\|\hat{Y}^{\kappa,\delta}_{\lfloor t\rfloor_h}\|_{\mathcal{L}^2(\mathbb{P};V)}\le \sup_{t\in[0,T]}\|\hat{Y}^{\kappa,\delta}_t\|_{\mathcal{L}^2(\mathbb{P};V)}<\infty$ shows that for all $\kappa,\delta\in[0,T]$ it holds that

$$\begin{aligned}&\big\|\mathbb{E}\big[\varphi(\hat{X}^{0,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{X}^{\kappa,\delta}_T)\big]\big\|_{\mathcal{V}}+\big\|\mathbb{E}\big[\varphi(\hat{Y}^{\kappa,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{Y}^{0,\delta}_T)\big]\big\|_{\mathcal{V}} \le \tfrac{4\,|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C^1_b(V,\mathcal{V})}\,\kappa^{\frac{\rho}{2}}\\ &\cdot\left[\chi_0\max\{1,\|e^{\delta A}X_0\|_{\mathcal{L}^2(\mathbb{P};V)}\} + \tfrac{\chi_\theta\,\chi_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C^1_b(V,V_{-\theta})}}{(1-\theta-\rho/2)} + \tfrac{\Upsilon_2\,\chi_{\theta/2}\,\chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C^1_b(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^2\\ &\cdot\left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,T^{(1-\theta)}\,\chi_0\,\chi_\theta}{\sqrt{1-\theta}}|F|_{C^1_b(V,V_{-\theta})} + \Upsilon_2\sqrt{2\,T^{(1-\theta)}}\,\chi_0\,\chi_{\theta/2}\,|B|_{C^1_b(V,\gamma(U,V_{-\theta/2}))}\right]\right|^2.\end{aligned} \tag{228}$$

Next we bound the second summand on the right hand side of (227). For this we note that for all $\kappa\in(0,T]$ it holds that

$$\begin{aligned}&\max\big\{1,\|e^{\kappa A}F\|^3_{C^3_b(V,V_{-\vartheta})},\|e^{\kappa A}B\|^6_{C^3_b(V,\gamma(U,V_{-\vartheta/2}))}\big\}\\ &\le |\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6\,\varsigma_{F,B}\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}.\end{aligned} \tag{229}$$

This and Corollary 6.9 show that for all $\kappa,\delta\in(0,T]$ it holds that

$$\big\|\mathbb{E}\big[\varphi(\hat{X}^{\kappa,\delta}_T)\big]-\mathbb{E}\big[\varphi(\hat{Y}^{\kappa,\delta}_T)\big]\big\|_{\mathcal{V}}$$

$$
\begin{aligned}
&\le \tfrac{57\,|\chi_0|^3\,|\chi_r|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-r)\,T^r}\,\varsigma_{F,B}\max\{1,\kappa^{-3(\theta-\vartheta)}\}\,h^r\\
&\cdot\left[\chi_0\max\{1,\|e^{\delta A}X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\}+\tfrac{\chi_\theta\,T^{(1-\theta)}\,\|e^{\kappa A}F\|_{C_b^1(V,V_{-\theta})}}{(1-\theta)}+\tfrac{\Upsilon_5\,\chi_{\theta/2}\sqrt{T^{(1-\theta)}}\,\|e^{\kappa A}B\|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta}}\right]^{10}\\
&\cdot\left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,\chi_\theta\,T^{(1-\theta)}\,|e^{\kappa A}F|_{C_b^1(V,V_{-\theta})}}{\sqrt{1-\theta}}+\Upsilon_5\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|e^{\kappa A}B|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5\\
&\cdot\Bigg[2^r+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)}\bigg(3\,\chi_\vartheta+2\,\chi_{r+\vartheta}+3\,|\chi_{\vartheta/2}|^2+4\,\chi_{r+\vartheta/2}\,\chi_{\vartheta/2}\\
&+2\,(|\chi_{\vartheta/2}|^2+\chi_\vartheta)\,\chi_r\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\bigg)\Bigg]\\
&\cdot\Big[\|\varphi\|_{C_b^3(V,\mathcal{V})}+c^{(\kappa)}_{-\vartheta}+c^{(\kappa)}_{-\vartheta,0}+c^{(\kappa)}_{-\vartheta,0,0}+c^{(\kappa)}_{-\vartheta,0,0,0}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0}\\
&+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big].
\end{aligned}
\tag{230}
$$

Plugging (228) and (230) into (227) then shows that for all $\kappa,\delta\in(0,T]$ it holds that

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_{\mathcal{V}}\le\max\Big\{4\,\kappa^{\frac{\rho}{2}},57\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}\,h^r\Big\}\,|\chi_0|^{20}\\
&\cdot\left[\max\{1,\|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\}+\tfrac{\chi_\theta\,\chi_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C_b^1(V,V_{-\theta})}}{(1-\theta-\rho/2)}+\tfrac{\max\{\Upsilon_2,\Upsilon_5\}\,\chi_{\theta/2}\,\chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10}\\
&\cdot\left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,\chi_0\,\chi_\theta\,T^{(1-\theta)}\,|F|_{C_b^1(V,V_{-\theta})}}{\sqrt{1-\theta}}+\max\{\Upsilon_2,\Upsilon_5\}\,\chi_0\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|B|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5\\
&\cdot\Bigg[2^r+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)}\bigg(3\,\chi_\vartheta+2\,\chi_{r+\vartheta}+3\,|\chi_{\vartheta/2}|^2+4\,\chi_{r+\vartheta/2}\,\chi_{\vartheta/2}\\
&+2\,(|\chi_{\vartheta/2}|^2+\chi_\vartheta)\,\chi_r\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\bigg)\Bigg]\\
&\cdot\Bigg[\tfrac{|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C_b^1(V,\mathcal{V})}+\tfrac{|\chi_0|^3\,|\chi_r|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-r)\,T^r}\,\varsigma_{F,B}\,\Big[\|\varphi\|_{C_b^3(V,\mathcal{V})}+c^{(\kappa)}_{-\vartheta}+c^{(\kappa)}_{-\vartheta,0}\\
&+c^{(\kappa)}_{-\vartheta,0,0}+c^{(\kappa)}_{-\vartheta,0,0,0}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big]\Bigg].
\end{aligned}
\tag{231}
$$

In addition, we observe that

$$
\begin{aligned}
&\inf_{\kappa\in(0,T]}\max\Big\{4\,\kappa^{\frac{\rho}{2}},57\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}\,h^r\Big\}\\
&\le\max\left\{4\left[\min\{1,T\}\left|\tfrac{h}{T}\right|^{\frac{2r}{(\rho+6(\theta-\vartheta))}}\right]^{\frac{\rho}{2}},57\max\left\{1,\left[\min\{1,T\}\left|\tfrac{h}{T}\right|^{\frac{2r}{(\rho+6(\theta-\vartheta))}}\right]^{-3(\theta-\vartheta)}\right\}h^r\right\}\\
&=\max\left\{4\left[\min\{1,T\}\left|\tfrac{h}{T}\right|^{\frac{2r}{(\rho+6(\theta-\vartheta))}}\right]^{\frac{\rho}{2}},57\,h^r\left[\min\{1,T\}\left|\tfrac{h}{T}\right|^{\frac{2r}{(\rho+6(\theta-\vartheta))}}\right]^{-3(\theta-\vartheta)}\right\}\\
&=\max\left\{\tfrac{4\,|\min\{1,T\}|^{\frac{\rho}{2}}}{T^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}},\tfrac{57\,T^{\frac{6(\theta-\vartheta)r}{(\rho+6(\theta-\vartheta))}}}{|\min\{1,T\}|^{3(\theta-\vartheta)}}\right\}h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}\\
&\le 57\max\left\{\tfrac{1}{|\min\{1,T\}|^r},\tfrac{|\max\{1,T\}|^r}{|\min\{1,T\}|^{3(\theta-\vartheta)}}\right\}h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}\le\frac{57\,h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}}{\left|\min\{T,\frac{1}{T}\}\right|^{(r+3(\theta-\vartheta))}}.
\end{aligned}
\tag{232}
$$

Combining (231) and (232) yields that for all $\delta\in(0,T]$ it holds that

$$
\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_{\mathcal{V}}\le\Big[57\left|\max\{T,\tfrac{1}{T}\}\right|^{(r+3(\theta-\vartheta))}|\chi_0|^{20}\Big]\,h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}
$$

$$\cdot \left[\max\{1, \|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\} + \frac{\chi_\theta\,\chi_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C_b^1(V,V_{-\theta})}}{(1-\theta-\rho/2)} + \frac{\max\{\Upsilon_2,\Upsilon_5\}\,\chi_{\theta/2}\,\chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10}$$
$$\cdot \left|\mathcal{E}_{(1-\theta)}\left[\frac{\sqrt{2}\,\chi_0\,\chi_\theta\,T^{(1-\theta)}\,|F|_{C_b^1(V,V_{-\theta})}}{\sqrt{1-\theta}} + \max\{\Upsilon_2,\Upsilon_5\}\,\chi_0\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|B|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5$$
$$\cdot \left[2^r + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)}\bigg(3\,\chi_\vartheta + 2\,\chi_{r+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{r+\vartheta/2}\,\chi_{\vartheta/2} \right. \tag{233}$$
$$\left. + 2\,(|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\,\chi_r\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\bigg)\right]$$
$$\cdot \left[\tfrac{|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C_b^1(V,\mathcal{V})} + \tfrac{|\chi_0|^3\,|\chi_r|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6\,\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-r)\,T^r}\,\varsigma_{F,B}\left[\|\varphi\|_{C_b^3(V,\mathcal{V})} + \sup_{\kappa\in(0,T]}\Big[c^{(\kappa)}_{-\vartheta}\right.\right.$$
$$\left.\left. + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big]\right]\right].$$

In the next step we note that Corollary 3.2 yields that $\lim_{\delta\to 0}\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big] = \mathbb{E}\big[\varphi(X_T)\big]$ and $\lim_{\delta\to 0}\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big] = \mathbb{E}\big[\varphi(Y_T)\big]$. Combining this with inequality (233) proves the first inequality in (223). The second inequality in (223) follows from Lemma 7.1. The proof of Proposition 7.2 is thus completed. □

Corollary 7.3. *Assume the setting in Section 7.1 and let $\rho \in (0, 1-\theta) \cap (6(\theta-\vartheta), \infty)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y_T)\|_{\mathcal{V}}\big] < \infty$ and*

$$\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_{\mathcal{V}} \le \big[57\,|\max\{T, \tfrac{1}{T}\}|^{3(\rho+\theta)}\,|\chi_0|^{20}\big]\,h^{(\rho-6(\theta-\vartheta))}$$
$$\cdot \left[\max\{1, \|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\} + \frac{\chi_\theta\,\chi_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C_b^1(V,V_{-\theta})}}{(1-\theta-\rho/2)} + \frac{\max\{\Upsilon_2,\Upsilon_5\}\,\chi_{\theta/2}\,\chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10}$$
$$\cdot \left|\mathcal{E}_{(1-\theta)}\left[\frac{\sqrt{2}\,\chi_0\,\chi_\theta\,T^{(1-\theta)}\,|F|_{C_b^1(V,V_{-\theta})}}{\sqrt{1-\theta}} + \max\{\Upsilon_2,\Upsilon_5\}\,\chi_0\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|B|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5$$
$$\cdot \left[2^\rho + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(3\,\chi_\vartheta + 2\,\chi_{\rho+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2} \right. \tag{234}$$
$$\left. + 2\,(|\chi_{\vartheta/2}|^2 + \chi_\vartheta)\,\chi_\rho\left[\tfrac{\chi_\vartheta\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\bigg)\right]$$
$$\cdot \left[\tfrac{|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C_b^1(V,\mathcal{V})} + \tfrac{|\chi_0|^3\,|\chi_\rho|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6\,\max\{1,T^{(1-\vartheta)}\}\,\varsigma_{F,B}}{(1-\vartheta-\rho)\,T^\rho}\bigg(\|\varphi\|_{C_b^3(V,\mathcal{V})} + \sup_{\kappa\in(0,T]}\Big[c^{(\kappa)}_{-\vartheta}\right.$$
$$\left. + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big]\bigg)\right] < \infty.$$

Proof. We first apply[4] Proposition 7.2 to obtain that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y_T)\|_{\mathcal{V}}\big] < \infty$ and

$$\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_{\mathcal{V}} \le \big[57\,|\max\{T, \tfrac{1}{T}\}|^{(\rho+3(\theta-\vartheta))}\,|\chi_0|^{20}\big]\,h^{\frac{\rho^2}{(\rho+6(\theta-\vartheta))}}$$
$$\cdot \left[\max\{1, \|X_0\|_{\mathcal{L}^5(\mathbb{P};V)}\} + \frac{\chi_\theta\,\chi_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C_b^1(V,V_{-\theta})}}{(1-\theta-\rho/2)} + \frac{\max\{\Upsilon_2,\Upsilon_5\}\,\chi_{\theta/2}\,\chi_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10}$$
$$\cdot \left|\mathcal{E}_{(1-\theta)}\left[\frac{\sqrt{2}\,\chi_0\,\chi_\theta\,T^{(1-\theta)}\,|F|_{C_b^1(V,V_{-\theta})}}{\sqrt{1-\theta}} + \max\{\Upsilon_2,\Upsilon_5\}\,\chi_0\,\chi_{\theta/2}\sqrt{2\,T^{(1-\theta)}}\,|B|_{C_b^1(V,\gamma(U,V_{-\theta/2}))}\right]\right|^5$$

[4] with $r = \rho$ in the notation of Proposition 7.2

$$\cdot \Bigg[2^{\rho} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)} \Bigg(3\,\chi_{\vartheta} + 2\,\chi_{\rho+\vartheta} + 3\,|\chi_{\vartheta/2}|^2 + 4\,\chi_{\rho+\vartheta/2}\,\chi_{\vartheta/2}$$
$$+ 2\,(|\chi_{\vartheta/2}|^2 + \chi_{\vartheta})\,\chi_{\rho} \Big[\tfrac{\chi_{\vartheta}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\max\{\Upsilon_3,\Upsilon_4\}\,\chi_{\vartheta/2}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} \Big] \Bigg) \Bigg] \tag{235}$$
$$\cdot \Bigg[\tfrac{|\chi_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C^1_b(V,\mathcal{V})} + \tfrac{|\chi_0|^3\,|\chi_{\rho}|^2\,|\chi_{\theta-\vartheta}|^3\,|\chi_{(\theta-\vartheta)/2}|^6 \max\{1,T^{(1-\vartheta)}\}\,\varsigma_{F,B}}{(1-\vartheta-\rho)\,T^{\rho}} \Big(\|\varphi\|_{C^3_b(V,\mathcal{V})} + \sup_{\kappa\in(0,T]} \Big[c^{(\kappa)}_{-\vartheta}$$
$$+ c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} \Big] \Big) \Bigg] < \infty.$$

Next we note that

$$\begin{aligned} h^{\frac{\rho^2}{(\rho+6(\theta-\vartheta))}} &= h^{\rho\left[\frac{1}{1+6(\theta-\vartheta)/\rho} - 1 + \frac{6(\theta-\vartheta)}{\rho}\right]}\, h^{\rho\left[1-\frac{6(\theta-\vartheta)}{\rho}\right]} \\ &\leq |\max\{1,T\}|^{\rho\left[\frac{1}{1+6(\theta-\vartheta)/\rho} - 1 + \frac{6(\theta-\vartheta)}{\rho}\right]}\, h^{(\rho-6(\theta-\vartheta))} \leq |\max\{1,T\}|^{\rho}\, h^{(\rho-6(\theta-\vartheta))}. \end{aligned} \tag{236}$$

Plugging (236) into (235) establishes (234). This completes the proof of Corollary 7.3. □

8 Weak convergence rates for exponential Euler approximations of SPDEs

Corollary 8.1. *Consider the notation in Section 1.1, let $(V, \|\cdot\|_V)$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be separable UMD $\mathbb{R}$-Banach spaces with type 2, let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $\kappa \in [0, 4/7)$, $\xi \in V$, $\varphi \in \operatorname{Lip}^4(V,\mathcal{V})$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $A\colon D(A) \subseteq V \to V$ be a generator of a strongly continuous analytic semigroup with* $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C}\colon Re(z) < \eta\}$*, let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $F \in \operatorname{Lip}^4(V, V_{-\kappa})$, $B \in \operatorname{Lip}^4(V, \gamma(U, V_{-\kappa/2}))$, let $X\colon [0,T]\times\Omega \to V$ be a continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds + \int_0^t e^{(t-s)A}B(X_s)\,dW_s, \tag{237}$$

and let $Y^N\colon \{0,1,\ldots,N\}\times\Omega \to V$, $N \in \mathbb{N}$, be stochastic processes which satisfy that for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ it holds $\mathbb{P}$-a.s. that $Y_0^N = \xi$ and

$$Y^N_{n+1} = e^{\frac{T}{N}A}\Big(Y_n + F(Y_n)\tfrac{T}{N} + \textstyle\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\,dW_s \Big). \tag{238}$$

Then for every $\varepsilon \in (0,\infty)$ there exists a real number $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}} + \|\varphi(Y^N_N)\|_{\mathcal{V}}\big] < \infty$ and

$$\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y^N_N)\big]\big\|_{\mathcal{V}} \leq C \cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\varepsilon)}. \tag{239}$$

Proof. Throughout this proof let $\tilde{Y}^N\colon [0,T]\times\Omega \to V$, $N \in \mathbb{N}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy for all $N \in \mathbb{N}$ that $\tilde{Y}^N_0 = \xi$ and which satisfy that for all $N \in \mathbb{N}$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\tilde{Y}^N_t = e^{tA}\,\tilde{Y}^N_0 + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,F(\tilde{Y}^N_{\lfloor s\rfloor_{T/N}})\,ds + \int_0^t e^{(t-\lfloor s\rfloor_{T/N})A}\,B(\tilde{Y}^N_{\lfloor s\rfloor_{T/N}})\,dW_s. \tag{240}$$

Observe that for all $N \in \mathbb{N}$ it holds that

$$\mathbb{P}\big(\tilde{Y}^N_T = Y^N_N\big) = 1. \tag{241}$$

Next note that for all $\varepsilon \in (0, \min\{1-\kappa, 4-7\kappa\})$ it holds that

$$\begin{aligned}
&1-\kappa-(1+\mathbb{1}_{[0,1/2)}(\kappa))\tfrac{\varepsilon}{2}-6\big(\kappa-\big[\min\{\kappa,\tfrac{1}{2}\}-\tfrac{\varepsilon}{12}\mathbb{1}_{[1/2,1)}(\kappa)\big]\big)\\
&=1-\kappa-\tfrac{\varepsilon}{2}-\tfrac{\varepsilon}{2}\,\mathbb{1}_{[0,1/2)}(\kappa)-6\kappa+6\big[\min\{\kappa,\tfrac{1}{2}\}-\tfrac{\varepsilon}{12}\mathbb{1}_{[1/2,1)}(\kappa)\big]\\
&=1-7\kappa-\tfrac{\varepsilon}{2}-\tfrac{\varepsilon}{2}\,\mathbb{1}_{[0,1/2)}(\kappa)+6\min\{\kappa,\tfrac{1}{2}\}-\tfrac{\varepsilon}{2}\,\mathbb{1}_{[1/2,1)}(\kappa)\\
&=1-7\kappa-\varepsilon+6\min\{\kappa,\tfrac{1}{2}\}\\
&=1-\kappa-6\big[\kappa-\min\{\kappa,\tfrac{1}{2}\}\big]-\varepsilon\\
&=1-\kappa-6\max\{0,\kappa-\tfrac{1}{2}\}-\varepsilon\\
&=1-\kappa-6\max\{\kappa-\tfrac{1}{2},0\}-\varepsilon.
\end{aligned}\tag{242}$$

Corollary 7.3 (with $V=V$, $\mathcal{V}=\mathcal{V}$, $U=U$, $T=T$, $\eta=\eta$, $W=W$, $A=A$, $V_r=V_r$, $h={}^{T}\!/\!{}_{N}$, $\theta=\kappa$, $\vartheta=\min\{\kappa,1/2\}-{}^{\varepsilon}\!/\!{}_{12}\,\mathbb{1}_{[1/2,1)}(\kappa)$, $F=F$, $B=B$, $\varphi=\varphi$, $X=X$, $Y=\tilde{Y}^N$, $\rho=1-\kappa-(1+\mathbb{1}_{[0,1/2)}(\kappa))\,{}^{\varepsilon}\!/\!{}_{2}$ for $\varepsilon\in(0,\min\{1-\kappa,4-7\kappa\})$, $N\in\mathbb{N}$, $r\in\mathbb{R}$ in the notation of Corollary 7.3), (240), (241), items (i)–(vii) of Lemma 6.2, e.g., Kunze [36, Theorem 5.6], and, e.g., Van Neerven et al. [47, Theorem 6.2] hence ensure that for all $\varepsilon\in(0,\min\{1-\kappa,4-7\kappa\})$ there exists a real number $C\in[0,\infty)$ such that for all $N\in\mathbb{N}$ it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}}+\|\varphi(Y_N^N)\|_{\mathcal{V}}\big]<\infty$ and

$$\big\|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(Y_N^N)\big]\big\|_{\mathcal{V}}=\big\|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(\tilde{Y}_T^N)\big]\big\|_{\mathcal{V}}\leq C\cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\varepsilon)}.\tag{243}$$

This implies that for all $\varepsilon\in(0,\infty)$, $\epsilon\in(0,\min\{1-\kappa,4-7\kappa,\varepsilon\})$ there exists a real number $C\in[0,\infty)$ such that for all $N\in\mathbb{N}$ it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_{\mathcal{V}}+\|\varphi(Y_N^N)\|_{\mathcal{V}}\big]<\infty$ and

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(Y_N^N)\big]\big\|_{\mathcal{V}}\leq C\cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\epsilon)}\\
&=C\cdot N^{-(\varepsilon-\epsilon)}\cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\varepsilon)}=\frac{C}{N^{(\varepsilon-\epsilon)}}\cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\varepsilon)}\\
&\leq C\cdot N^{-(1-\kappa-6\max\{\kappa-1/2,0\}-\varepsilon)}.
\end{aligned}\tag{244}$$

The proof of Corollary 8.1 is thus completed. □

9 Weak convergence rates for nonlinear stochastic heat equations

Corollary 9.1. *Consider the notation in Section 1.1, let $n,d,k\in\mathbb{N}$, $\alpha\in(-\infty,0)$, $p\in(\max\{n,{}^{d(n-1)}\!/\!{}_{(2|\alpha|)}\},\infty)$, $(V,\|\cdot\|_V)=(L^p(\lambda_{(0,1)^d};\mathbb{R}^k),\|\cdot\|_{L^p(\lambda_{(0,1)^d};\mathbb{R}^k)})$, let $f\colon\mathbb{R}^k\to\mathbb{R}^k$ be an n-times continuously differentiable function with globally bounded derivatives, let $A\colon D(A)\subseteq V\to V$ be the Laplacian with Dirichlet boundary conditions on V, and let $(V_r,\|\cdot\|_{V_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [44, Section 3.7]). Then*

(i) there exists a unique function $F\colon V\to V_\alpha$ which satisfies for all $v\in\mathcal{L}^p(\lambda_{(0,1)^d};\mathbb{R}^k)$ that

$$F\big([v]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)}\big)=\big[\{f(v(x))\}_{x\in(0,1)^d}\big]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)}=[f\circ v]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)},\tag{245}$$

(ii) it holds that F is n-times continuously Fréchet differentiable with globally bounded derivatives,

(iii) it holds for all $m\in\{1,\ldots,n\}$, $v\in\mathcal{L}^p(\lambda_{(0,1)^d};\mathbb{R}^k)$, $u_1,\ldots,u_m\in\mathcal{L}^{pm}(\lambda_{(0,1)^d};\mathbb{R}^k)$ that

$$\begin{aligned}
&F^{(m)}([v]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)})([u_1]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)},\ldots,[u_m]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)})\\
&=[\{f^{(m)}(v(x))(u_1(x),\ldots,u_m(x))\}_{x\in(0,1)^d}]_{\lambda_{(0,1)^d},\mathcal{B}(\mathbb{R}^k)},
\end{aligned}\tag{246}$$

(iv) it holds for all $q \in [\max\{1, \frac{dp}{2p|\alpha|+d}\}, \frac{p}{n})$ that

$$\sup_{v \in V \setminus \{0\}} \left[\frac{\|v\|_{V_\alpha}}{\|v\|_{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] < \infty, \tag{247}$$

(v) it holds for all $m \in \{1, \dots, n\}$, $q \in [\max\{1, \frac{dp}{2p|\alpha|+d}\}, \frac{p}{n})$, $r \in [mq, \infty)$ that

$$\begin{split} &\sup_{v \in V} \sup_{u_1, \dots, u_m \in L^{\max\{r,p\}}(\lambda_{(0,1)^d}; \mathbb{R}^k) \setminus \{0\}} \left[\frac{\|F^{(m)}(v)(u_1, \dots, u_m)\|_{V_\alpha}}{\prod_{i=1}^m \|u_i\|_{L^r(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] \\ &\leq |f|_{C_b^m(\mathbb{R}^k, \mathbb{R}^k)} \left[\sup_{v \in V \setminus \{0\}} \frac{\|v\|_{V_\alpha}}{\|v\|_{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] < \infty, \end{split} \tag{248}$$

and

(vi) it holds for all $m \in \{1, \dots, n\}$, $q \in [\max\{1, \frac{dp}{2p|\alpha|+d}\}, \frac{p}{n})$, $r \in [(m+1)q, \infty)$ that

$$\begin{split} &\sup_{\substack{v, w \in L^{\max\{r,p\}}(\lambda_{(0,1)^d}; \mathbb{R}^k), \\ v \neq w}} \sup_{\substack{u_1, \dots, u_m \in \\ L^{\max\{r,p\}}(\lambda_{(0,1)^d}; \mathbb{R}^k) \setminus \{0\}}} \left[\frac{\|(F^{(m)}(v) - F^{(m)}(w))(u_1, \dots, u_m)\|_{V_\alpha}}{\|v - w\|_{L^r(\lambda_{(0,1)^d}; \mathbb{R}^k)} \cdot \prod_{i=1}^m \|u_i\|_{L^r(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] \\ &\leq |f|_{\mathrm{Lip}^m(\mathbb{R}^k, \mathbb{R}^k)} \left[\sup_{v \in V \setminus \{0\}} \frac{\|v\|_{V_\alpha}}{\|v\|_{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right]. \end{split} \tag{249}$$

Proof. Throughout this proof let $q \in [\max\{1, \frac{dp}{2p|\alpha|+d}\}, \frac{p}{n})$, let $G \colon V \to L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)$ be the function which satisfies for all $v \in \mathcal{L}^p(\lambda_{(0,1)^d}; \mathbb{R}^k)$ that

$$G\big([v]_{\lambda_{(0,1)^d}, \mathcal{B}(\mathbb{R}^k)}\big) = [f \circ v]_{\lambda_{(0,1)^d}, \mathcal{B}(\mathbb{R}^k)}, \tag{250}$$

and let $\iota \colon V \to V_\alpha$ be the function which satisfies for all $v \in V$ that $\iota(v) = v$. Observe that item (i) is an immediate consequence of the fact that $\forall\, v \in \mathcal{L}^p(\lambda_{(0,1)^d}; \mathbb{R}^k) \colon f \circ v \in \mathcal{L}^p(\lambda_{(0,1)^d}; \mathbb{R}^k)$. It thus remains to prove items (ii)–(vi). For this note that the Sobolev embedding theorem and the fact that

$$0 - 2\alpha = -2\alpha = 2|\alpha| \geq d \max\{0, {}^1\!/_q - {}^1\!/_p\} \tag{251}$$

show that

$$\sup_{v \in V \setminus \{0\}} \left[\frac{\|\iota(v)\|_{V_\alpha}}{\|v\|_{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] = \sup_{v \in V \setminus \{0\}} \left[\frac{\|v\|_{V_\alpha}}{\|v\|_{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)}} \right] < \infty. \tag{252}$$

Combining this with the fact that

$$\overline{V}^{L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)} = L^q(\lambda_{(0,1)^d}; \mathbb{R}^k) \tag{253}$$

proves that there exists a unique continuous linear function $\mathcal{I} \colon L^q(\lambda_{(0,1)^d}; \mathbb{R}^k) \to V_\alpha$ which satisfies for all $v \in V \subseteq L^q(\lambda_{(0,1)^d}; \mathbb{R}^k)$ that

$$\mathcal{I}(v) = \iota(v) = v. \tag{254}$$

Observe that (250) and (254) ensure that

$$F = \mathcal{I} \circ G. \tag{255}$$

Combining Proposition 2.6 in [15] (with $k = k$, $l = k$, $d = d$, $n = n$, $p = q$, $q = p$, $\mathcal{O} = (0,1)^d$, $f = f$, $F = G$ in the notation of Proposition 2.6 in [15]), (252), and the fact that $\mathcal{I} \in L(L^q(\lambda_{(0,1)^d}; \mathbb{R}^k), V_\alpha)$ hence establishes items (ii)–(vi). The proof of Corollary 9.1 is thus completed. □

Corollary 9.2. *Consider the notation in Section 1.1, let $n, d, k \in \mathbb{N}$, $\alpha \in (-\infty, 0)$, $p \in [\max\{n+1, {dn}/{(2|\alpha|)}\}, \infty)$, $(V, \|\cdot\|_V) = (L^p(\lambda_{(0,1)^d}; \mathbb{R}^k), \|\cdot\|_{L^p(\lambda_{(0,1)^d};\mathbb{R}^k)})$, let $f\colon \mathbb{R}^k \to \mathbb{R}^k$ be an n-times continuously differentiable function with globally Lipschitz continuous and globally bounded derivatives, let $A\colon D(A) \subseteq V \to V$ be the Laplacian with Dirichlet boundary conditions on V, and let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$. Then*

(i) there exists a unique function $F\colon V \to V_\alpha$ which satisfies for all $v \in \mathcal{L}^p(\lambda_{(0,1)^d}; \mathbb{R}^k)$ that

$$F\big([v]_{\lambda_{(0,1)^d}, \mathcal{B}(\mathbb{R}^k)}\big) = \big[\{f(v(x))\}_{x\in(0,1)^d}\big]_{\lambda_{(0,1)^d}, \mathcal{B}(\mathbb{R}^k)} = [f \circ v]_{\lambda_{(0,1)^d}, \mathcal{B}(\mathbb{R}^k)} \tag{256}$$

and

(ii) it holds that F is n-times continuously Fréchet differentiable with globally Lipschitz continuous and globally bounded derivatives.

Proof. First, we note that the assumption that $p \geq \max\{n+1, \frac{dn}{2|\alpha|}\}$ ensures that

$$1 \leq \tfrac{p}{(n+1)} \qquad \text{and} \qquad dn \leq 2p|\alpha|. \tag{257}$$

This implies that $d(n+1) \leq 2p|\alpha| + d$. Hence, we obtain that

$$\tfrac{d}{2p|\alpha|+d} \leq \tfrac{1}{(n+1)}. \tag{258}$$

Combining this with (257) assures that

$$\max\big\{1, \tfrac{dp}{2p|\alpha|+d}\big\} \leq \tfrac{p}{(n+1)}. \tag{259}$$

Therefore, we obtain that $\frac{p}{(n+1)} \in [\max\{1, \frac{dp}{2p|\alpha|+d}\}, \frac{p}{n})$. Items (i), (ii), & (vi) of Corollary 9.1 (with $n = n$, $d = d$, $k = k$, $\alpha = \alpha$, $p = p$, $f = f$, $A = A$, $m = m$, $q = \frac{p}{(n+1)}$, $r = p$ for $m \in \{1, \ldots, n\}$ in the notation of items (i), (ii), & (vi) Corollary 9.1) hence prove items (i)–(ii). The proof of Corollary 9.2 is thus completed. □

Theorem 9.3. *Consider the notation in Section 1.1, let $T, \varepsilon \in (0, \infty)$, $\beta \in ({1}/{4}, {1}/{4} + {\min\{\varepsilon,1\}}/{28})$, $p \in (\frac{5}{2(\beta - 1/4)}, \infty)$, $(V, \|\cdot\|_V) = (L^p(\lambda_{(0,1)}; \mathbb{R}), \|\cdot\|_{L^p(\lambda_{(0,1)};\mathbb{R})})$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U) = (L^2(\lambda_{(0,1)}; \mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})})$, $\xi \in V$, let $f, b\colon \mathbb{R} \to \mathbb{R}$ and $\varphi\colon V \to \mathbb{R}$ be four times continuously differentiable functions with globally Lipschitz continuous and globally bounded derivatives, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $A\colon D(A) \subseteq V \to V$ be the Laplacian with Dirichlet boundary conditions on V, and let $(V_r, \|\cdot\|_{V_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$. Then*

(i) there exists a unique continuous function $F\colon V \to V_{-2\beta}$ which satisfies for all $v \in \mathcal{L}^p(\lambda_{(0,1)}; \mathbb{R})$ that

$$F\big([v]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})}\big) = \big[\{f(v(x))\}_{x\in(0,1)}\big]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})} = [f \circ v]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})}, \tag{260}$$

(ii) there exists a unique continuous function $B\colon V \to \gamma(U, V_{-\beta})$ which satisfies for all $v, u \in \mathcal{L}^{2p}(\lambda_{(0,1)}; \mathbb{R})$ that

$$B\big([v]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})}\big)[u]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})} = \big[\{b(v(x)) \cdot u(x)\}_{x\in(0,1)}\big]_{\lambda_{(0,1)}, \mathcal{B}(\mathbb{R})}, \tag{261}$$

(iii) there exists an up-to-indistinguishability unique continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process $X\colon [0,T] \times \Omega \to V$ which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A} F(X_s)\, ds + \int_0^t e^{(t-s)A} B(X_s)\, dW_s, \tag{262}$$

(iv) there exist up-to-indistinguishability unique stochastic processes $Y^N\colon \{0,1,\ldots,N\}\times\Omega\to V$*,* $N\in\mathbb{N}$*, which satisfy that for all* $N\in\mathbb{N}$*,* $n\in\{0,1,\ldots,N-1\}$ *it holds* $\mathbb{P}$*-a.s. that* $Y_0^N=\xi$ *and*

$$Y_{n+1}^N = e^{\frac{T}{N}A}\Big(Y_n + F(Y_n)\tfrac{T}{N} + \textstyle\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\,dW_s\Big), \tag{263}$$

and

(v) there exists a real number $C\in\mathbb{R}$ *such that for all* $N\in\mathbb{N}$ *it holds that* $\mathbb{E}\big[|\varphi(X_T)|+|\varphi(Y_N^N)|\big]<\infty$ *and*

$$\big|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(Y_N^N)\big]\big| \le C\cdot N^{(\varepsilon-1/2)}. \tag{264}$$

Proof. First, note that the assumption that $p>\frac{5}{2(\beta-1/4)}$ and the assumption that $\beta\in(1/4,1/4+\min\{\varepsilon,1\}/28)$ ensure that

$$\begin{aligned} p &> \frac{5}{2(\beta-1/4)} > \frac{5}{2(\min\{\varepsilon,1\}/28)} = \frac{5}{(\min\{\varepsilon,1\}/14)} = \frac{70}{\min\{\varepsilon,1\}} \\ &\ge 70 \ge 5 = \max\{5,\beta\} = \max\big\{5,\tfrac{4}{2|2\beta|}\big\}. \end{aligned} \tag{265}$$

Corollary 9.2 (with $n=4$, $d=1$, $k=1$, $\alpha=-2\beta$, $p=p$, $f=f$, $A=A$ in the notation of Corollary 9.2) hence establishes that item (i) holds and that

$$F\in \operatorname{Lip}^4(V,V_{-2\beta}). \tag{266}$$

Moreover, observe that Corollary 4.9 in [15] (with $n=4$, $\beta=-\beta$, $p=p$, $b=b$, $A=A$ in the notation of Corollary 4.9 in [15]) together with (265) proves that item (ii) holds and that

$$B\in \operatorname{Lip}^4(V,\gamma(U,V_{-\beta})). \tag{267}$$

In addition, note that (266) and (267) assure that there exists an up-to-indistinguishability unique continuous $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process $X\colon[0,T]\times\Omega\to V$ which satisfies that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A}F(X_s)\,ds + \int_0^t e^{(t-s)A}B(X_s)\,dW_s \tag{268}$$

(cf., e.g., Theorem 6.2 in Van Neerven et al. [47]). This establishes item (iii). Next note that induction shows that there exist up-to-indistinguishability unique stochastic processes $Y^N\colon\{0,1,\ldots,N\}\times\Omega\to V$, $N\in\mathbb{N}$, which satisfy that for all $N\in\mathbb{N}$, $n\in\{0,1,\ldots,N-1\}$ it holds $\mathbb{P}$-a.s. that $Y_0^N=\xi$ and

$$Y_{n+1}^N = e^{\frac{T}{N}A}\Big(Y_n + F(Y_n)\tfrac{T}{N} + \textstyle\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\,dW_s\Big). \tag{269}$$

This demonstrates item (iv). It thus remains to prove item (v). For this observe that (266), (267), Corollary 8.1 (with $V=V$, $\mathcal{V}=\mathbb{R}$, $U=U$, $T=T$, $\eta=0$, $\kappa=2\beta$, $\xi=\xi$, $\varphi=\varphi$, $W=W$, $A=A$, $F=F$, $B=B$, $X=X$, $Y^N=Y^N$, $\varepsilon=\varepsilon/2$ for $N\in\mathbb{N}$ in the notation of Corollary 8.1), and the fact that $\beta\in(1/4,1/4+\varepsilon/28)$ show that there exists a real number $C\in[0,\infty)$ such that for all $N\in\mathbb{N}$ it holds that $\mathbb{E}\big[|\varphi(X_T)|+|\varphi(Y_N^N)|\big]<\infty$ and

$$\begin{aligned} \big|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(Y_N^N)\big]\big| &\le C\cdot N^{-(1-2\beta-6\max\{2\beta-1/2,0\}-\varepsilon/2)} = C\cdot N^{-(1-2\beta-6(2\beta-1/2)-\varepsilon/2)} \\ &= C\cdot N^{-(1-2\beta-12\beta+3-\varepsilon/2)} = C\cdot N^{(14\beta+\varepsilon/2-4)} \\ &\le C\cdot N^{(14(1/4+\varepsilon/28)+\varepsilon/2-4)} = C\cdot N^{(\varepsilon-1/2)}. \end{aligned} \tag{270}$$

This establishes item (v). The proof of Theorem 9.3 is thus completed. □

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