ETH Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Multilevel QMC with Product Weights for Affine-Parametric, Elliptic PDEs

R. N. Gantner and L. Herrmann and Ch. Schwab

Research Report No. 2016-54

December 2016
Latest revision: July 2017

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 159940, 149819

Multilevel QMC with Product Weights for Affine-Parametric, Elliptic PDEs

Robert N. Gantner, Lukas Herrmann, and Christoph Schwab

Abstract We present an error analysis of higher order Quasi-Monte Carlo (QMC) integration and of randomly shifted QMC lattice rules for parametric operator equations with uncertain input data taking values in Banach spaces. Parametric expansions of these input data in locally supported bases such as splines or wavelets was shown in [R.N. Gantner, L. Herrmann, and Ch. Schwab, Quasi-Monte Carlo integration for affine-parametric, elliptic PDEs: local supports and product weights, Report 2016-32, Seminar for Applied Mathematics, ETH Zürich] to allow for dimension independent convergence rates of combined QMC-Galerkin approximations. In the present work, we review and refine the results in that reference to the multilevel setting, along the lines of [F.Y. Kuo, Ch. Schwab, and I.H. Sloan: Multi-level Quasi-Monte Carlo Finite Element Methods for a Class of Elliptic PDEs with Random Coefficients, Journ. Found. Comp. Math. **15**(2) 441-449 (2015)] where randomly shifted lattice rules and globally supported representations were considered, and also the results of [J. Dick, F.Y. Kuo, Q.T. LeGia, and Ch. Schwab: Multi-level higher order QMC Galerkin discretization for affine parametric operator equations, SIAM J. Numer. Anal., 54/4 (2016), pp. 2541-2568] in the particular situation of locally supported bases in the parametrization of uncertain input data. In particular, we show that locally supported basis functions allow for multilevel QMC quadrature with product weights, and prove new error vs. work estimates superior to those in these references (albeit at stronger, mixed regularity assumptions on the parametric integrand functions than what was required in the single-level QMC error analysis in the first reference above). Numerical experiments on a model affine parametric elliptic problem confirm the analysis.

Robert N. Gantner · Lukas Herrmann · Christoph Schwab
Seminar for Applied Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zurich, Switzerland
e-mail: robert.gantner@sam.math.ethz.ch, e-mail: lukas.herrmann@sam.math.ethz.ch, e-mail: christoph.schwab@sam.math.ethz.ch

1 Introduction

A core task in computational uncertainty quantification (computational UQ for short) is to approximate the statistics of (functionals of) solutions to partial differential equations (PDEs for short) which depend on parameters describing uncertain input data. Upon placing measures on the set of admissible parameters, the computation of mathematical expectations in so-called *forward UQ* and in *Bayesian inverse UQ* of such PDEs on possibly large sets of data amounts to a problem of *high dimensional numerical integration*; we refer to the surveys [26, 5, 4] and the references there for details.

In the present note, we address the numerical analysis of high-dimensional numerical integrations methods of *Quasi Monte-Carlo* (QMC for short) type for the efficient numerical approximation of expectations of solutions of parametric PDEs over high-dimensional parameter spaces. Pioneering contributions to the mathematical foundation of dimension-independent convergence rates for QMC quadrature methods for such problems, that also the present note will draw on, have been made by I. H. Sloan and H. Woźniakowski in [30], after earlier, foundational work by I. H. Sloan and S. Joe in [27].

Specifically, we consider the linear, affine-parametric elliptic PDE

$$-\nabla\cdot(a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y})) = f(x) \text{ in } D, \qquad u(x,\boldsymbol{y})\Big|_{\Gamma_1} = 0, \qquad a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y})\cdot n(x)\Big|_{\Gamma_2} = 0, \tag{1}$$

with input $a(x,\boldsymbol{y})$ parametrized by $\boldsymbol{y}=(y_j)_{j\geq 1}$, fixed right hand side $f(x)$, and mixed boundary conditions. The domain $D\subset\mathbb{R}^d$, $d=1,2$, is a polygon with straight sides or an interval. The set $\Gamma_1\neq\emptyset$ is the union of some of the closed edges of ∂D, $\Gamma_2 := D\backslash\Gamma_1$, and n is the unit outward pointing normal vector of D. Specifically, QMC rules with *product weights* are considered which are known to have linear complexity in the integration dimension, cp. [24, 25]. The purpose of the present paper is to prove error versus work bounds of these algorithms, with explicit estimation of constants on the dimension s of integration, and of the form $\mathscr{O}(\varepsilon^{-\theta})$, $\theta>0$, for a given accuracy $\varepsilon>0$.

Convergence analysis of QMC methods with randomly shifted lattice rules applied to a parametric PDE of the type of (1) was first established in [22] together with the survey [21]. Randomly shifted lattice rules were first proposed in [28]. A multilevel version for parametric PDEs was first analyzed in [23]. This theory was extended in [6, 7] with interlaced polynomial lattice rules, which achieve higher order convergence rates. These convergence rates are independent of the number of scalar variables that is the dimension of the domain of intergration. Conditions for such dimension independent error bounds of QMC algorithms were first shown in the seminal work [30] for integrand functions belonging to certain weighted function spaces with so-called *product weights*. In [21], analogous results were shown to hold for randomly shifted lattice rules, and for input parametrizations in terms of globally supported basis functions (as, e.g., Karhunen-Loève expansions) with so-called *product and order (POD) dependent weights*. General references for QMC integration are [21, 8]; see also the survey [13] for multilevel Monte Carlo methods and [9, 20] for available software implementations.

As in the mentioned references, we admit parameter vectors $\boldsymbol{y} = (y_j)_{j\geq 1}$ whose components take values in the closed interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, i.e., we will consider

$$\boldsymbol{y} \in U := \left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathbb{N}}.$$

We model uncertainty in diffusion coefficients $a(x,\boldsymbol{y})$ to the PDE (1) by assuming the parameter vectors to be independent, identically distributed (i.i.d.) with respect to the uniform product probability measure

$$\mu(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j\geq 1} \mathrm{d}y_j.$$

The triplet $(U, \bigotimes_{j\geq 1} \mathscr{B}([-1/2, 1/2]), \mu)$ is a probability space. For any Banach space B, the mathematical expectation of F with respect to the probability measure μ is a Bochner integral of the strongly measurable, integrable map $F: U \to B$ which will be denoted by

$$\mathbb{E}(F) := \int_U F(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}). \tag{2}$$

The parametric input $a(x,\boldsymbol{y})$ of (1) is assumed to be of the form

$$a(x,\boldsymbol{y}) = \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x), \quad \text{a.e. } x \in D, \boldsymbol{y} \in U, \tag{3}$$

where $\{\bar{a}, \psi_j : j \geq 1\} \subset L^\infty(D)$ and $\bar{a}$ is such that $0 < \bar{a}_{\min} \leq \bar{a}_{\max}$ exist and satisfy

$$\bar{a}_{\min} \leq \operatorname{ess\,inf}_{x\in D}\{\bar{a}(x)\} \leq \operatorname{ess\,sup}_{x\in D}\{\bar{a}(x)\} \leq \bar{a}_{\max}.$$

Convergence analysis for QMC with product weights was recently carried out in [10] under the assumption that there exists $\kappa \in (0,1)$ and a sequence $(b_j)_{j\geq 1} \in (0,1]^{\mathbb{N}}$ such that

$$\left\| \frac{\sum_{j\geq 1} |\psi_j|/b_j}{2\bar{a}} \right\|_{L^\infty(D)} \leq \kappa < 1. \tag{\textbf{A1}}$$

A (dimension independent) convergence rate of $1/p$ in terms of the number of QMC points for the approximate evaluation of (2) was shown in [10, Section 6] if $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for the range $p \in (0,2]$. These rates coincide, in the mentioned range of summability exponents, with the convergence rates of best N-term approximation rates of generalized polynomial chaos expansions obtained in [3, Theorem 1.2 and Equation (1.11)]. As in [3], the assumption in (**A1**) can accomodate possible localization in D of the supports of the function system $(\psi_j)_{j\geq 1}$.

For every parameter instance $\boldsymbol{y} \in U$, in the physical domain D a standard, first order accurate Galerkin Finite Element (FE for short) discretization of the parametric

PDE (1) will be applied. In the polygonal domain D, first order FEM based on sequences of uniformly refined, regular simplicial meshes are well known to converge at suboptimal rates due to *corner singularities* in the solution, even if $a(x,\boldsymbol{y})$ and f in (1) are smooth. To establish full FE convergence rates on locally refined meshes coupled with the QMC error estimates, parametric regularity estimates in *Kondrat'ev* spaces will be demonstrated.

In Section 2 well-posedness of the parametric solution and approximation by dimension truncation and FE is discussed. Particular weighted Sobolev spaces of parametric regularity that are required for the error analysis of multilevel QMC and general error estimates are reviewed in Section 3. Parametric regularity estimates of the dimension truncation and FE error are proven in Section 4. These estimates yield error bounds of multilevel QMC algorithms that are demonstrated in Section 5. In Section 6, parameter choices are derived that minimize the needed work for a certain error threshold. In the numerical experiments, we analyze piecewise (bi)linear wavelet basies to expand the diffusion coefficient in one and two spatial dimensions. The experiment confirms the theory and also shows that the multilevel QMC algorithm beats the single-level version comparing the needed work against the achieved accuracy.

2 Well-posedness and spatial approximation

The parametric problem in (1) admits a symmetric variational formulation with trial and test space $V := \{v \in H^1(D) : v|_{\Gamma_1} = 0\}$, with dual space denoted by $V^* = H^{-1}(D)$, where $v|_{\Gamma_1} = 0$ is to be understood as a trace in $H^{1/2}(\Gamma_1)$. Let $f \in V^*$ and let the assumption in (**A1**) be satisfied. Then the *parametric weak formulation of* (1) reads: For every $\boldsymbol{y} \in U$ find $u(\cdot,\boldsymbol{y}) \in V$ such that

$$\int_D a(\cdot,\boldsymbol{y})\nabla u(\cdot,\boldsymbol{y}) \cdot \nabla v \mathrm{d}x = \langle f, v\rangle_{V^*,V}, \quad \forall v \in V, \tag{4}$$

where $\langle\cdot,\cdot\rangle_{V^*,V}$ denotes the dual pairing between V and V^*. Since the assumption in (**A1**) implies that

$$0 < (1-\kappa)\bar{a}_{\min} \leq \operatorname{ess\,inf}_{x\in D}\{a(x,\boldsymbol{y})\}, \quad \boldsymbol{y} \in U,$$

and

$$\operatorname{ess\,sup}_{x\in D}\{a(x,\boldsymbol{y})\} \leq (1+\kappa)\bar{a}_{\max}, \quad \boldsymbol{y} \in U,$$

the parametric bilinear form $(w,v) \mapsto \int_D a(\boldsymbol{y})\nabla w \cdot \nabla v \mathrm{d}x$ is continuous and coercive on $V \times V$, uniformly with respect to the parameter $\boldsymbol{y} \in U$. By the Lax–Milgram lemma the unique solution $u(\cdot,\boldsymbol{y}) \in V$ to (4) exists, is a strongly measurable mapping from U to V (by the second Strang lemma), and satisfies the a priori estimate

$$\|u(\cdot,\boldsymbol{y})\|_V \leq \frac{\|f\|_{V^*}}{(1-\kappa)\bar{a}_{\min}}, \quad \boldsymbol{y} \in U.$$

A finite dimensional domain of integration which is required for the use of QMC is achieved by truncating the expansion of $a(x,\boldsymbol{y})$ to a finite number of $s \in \mathbb{N}$ terms. We introduce the notation that for every $\boldsymbol{y} \in U$, $\boldsymbol{y}_{\{1:s\}}$ is such that $(\boldsymbol{y}_{\{1:s\}})_j = y_j$ if $j \leq s$ and 0 otherwise, where $\{1:s\}$ denotes the set $\{1,\dots,s\}$. Specifically, for every $s \in \mathbb{N}$ define

$$u^s(\cdot,\boldsymbol{y}) := u(\cdot,\boldsymbol{y}_{\{1:s\}}), \quad \boldsymbol{y} \in U.$$

Proposition 1 ([10, Proposition 8]). *Let the assumption in* (**A1**) *be satisfied for some* $\kappa \in (0,1)$ *and recall the right hand side* $f \in V^*$ *in* (4). *If for some* $s \in \mathbb{N}$

$$\frac{\kappa \bar{a}_{\max}}{(1-\kappa)\bar{a}_{\min}} \max_{j \geq s+1} \{b_j\} < 1,$$

then there exists a constant $C > 0$ *such that for every* $G(\cdot) \in V^*$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s))| \leq C \|G(\cdot)\|_{V^*} \|f\|_{V^*} \left(\max_{j \geq s+1} \{b_j\} \right)^2.$$

For the study of the spatial regularity of $u(\cdot,\boldsymbol{y})$, we consider weighted Sobolev spaces of *Kondrat'ev* type, which allow for full regularity shifts in polygonal domains $D \subset \mathbb{R}^2$ cp. [2]. In our setting the domain D is either a polygon in $\mathbb{R}^2$ with corners $\{c_1,\dots,c_J\}$ or an interval. To introduce weighted Sobolev spaces, we define the functions $r_i(x) := |x - c_i|$, $x \in D$, $i = 1,\dots,J$, where $|\cdot|$ denotes the Euclidean norm. For a J-tuple $\boldsymbol{\beta} = (\beta_1,\dots,\beta_J)$ with $\beta_i \in [0,1)$, $i = 1,\dots,J$, we define the weight function $\Phi_{\boldsymbol{\beta}}$ by

$$\Phi_{\boldsymbol{\beta}}(x) := \prod_{i=1}^{J} r_i^{\beta_i}(x), \quad x \in D.$$

For multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^2$, define the notation $\partial_x^{\boldsymbol{\alpha}} := \partial^{|\boldsymbol{\alpha}|}/(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2})$. We define the weighted spaces $L^2_{\boldsymbol{\beta}}(D)$ and $H^2_{\boldsymbol{\beta}}(D)$ as the completion of $C^\infty(\overline{D})$ with respect to the following norms

$$\|v\|_{L^2_{\boldsymbol{\beta}}(D)} := \|v \Phi_{\boldsymbol{\beta}}\|_{L^2(D)} \tag{5}$$

and $\|v\|_{H^2_{\boldsymbol{\beta}}(D)}$ is given by

$$\|v\|^2_{H^2_{\boldsymbol{\beta}}(D)} := \|v\|^2_{H^1(D)} + \sum_{|\boldsymbol{\alpha}|=2} \||\partial_x^{\boldsymbol{\alpha}} v| \Phi_{\boldsymbol{\beta}}\|^2_{L^2(D)}.$$

In the corresponding weighted Sobolev spaces, there is a full regularity shift of the Laplacean, cp. [2, Theorem 3.2 and Equation (3.2)], i.e., there exists a constant $C > 0$ such that for every $w \in V$ satisfying $\Delta w \in L^2_{\boldsymbol{\beta}}(D)$ there holds

$$\|w\|_{H^2_{\boldsymbol{\beta}}(D)} \leq C \|\Delta w\|_{L^2_{\boldsymbol{\beta}}(D)}, \tag{6}$$

where $\beta_i > 1-\pi/\omega_i$ and $\beta_i \geq 0$, where ω_i denotes the interior angle of the corner c_i for $i=1,\dots,J$ such that both edges that have c_i as an endpoint are both in Γ_1 or Γ_2. Otherwise (change of the boundary conditions at c_i), we require $\beta_i > 1-\pi/(2\omega_i)$ and $\beta_i \geq 0$. Note that we allow the case $\omega_i = \pi$, which facilitates the case that the boundary conditions change within an edge of ∂D. Hence, in the case that the domain D is convex and $\Gamma_1 = \partial D$, we may choose $\boldsymbol{\beta} = (0,\dots,0)$. There holds an approximation property in FE spaces on D with local mesh refinement towards the corners of D. To state it, let $\{\mathscr{T}_\ell\}_{\ell\geq 0}$ be a sequence of regular, simplicial triangulations of the polygon D, which can be generated either by judicious mesh grading in a vicinity of each corner of D cp. [2, Section 4] or by *newest vertex bisection*, cp. [12]. Let $V_\ell := \{v \in V : v|_K \in \mathbb{P}_1(K), K \in \mathscr{T}_\ell\}$, $\ell \geq 0$, where $\mathbb{P}_1(K)$ denotes the affine functions on K. The FE space V_ℓ is of finite dimension $M_\ell := \dim(V_\ell)$, $\ell \geq 0$. Then, there exists a constant $C>0$ such that for every $w \in H^2_{\boldsymbol{\beta}}(D)$ and every $\ell \geq 0$ there exists $w_\ell \in V_\ell$

$$\|w - w_\ell\|_V \leq C M_\ell^{-1/d} \|w\|_{H^2_{\boldsymbol{\beta}}(D)}, \tag{7}$$

where $d=1,2$ is the dimension of the domain D. For $d=2$, and in the case of graded meshes, this follows, for example, from [2, Lemmas 4.1 and 4.5]. An approximation property of this kind for nearest vertex bisection is shown in [12]. The regularity shift in (6) and the approximation property in (7) also hold if D is an interval (for $d=1$).

Assume that the right hand side $f \in L^2_{\boldsymbol{\beta}}(D)$, and that $\{|\nabla \bar{a}|\Phi_{\boldsymbol{\beta}}, |\nabla \psi_j|\Phi_{\boldsymbol{\beta}} : j \geq 1\} \subset L^\infty(D)$, and that there exists a bounded, positive sequence $(\bar{b}_j)_{j\geq 1}$, which satisfies

$$\left\| \left(|\nabla \bar{a}| + \sum_{j\geq 1} \frac{|\nabla \psi_j|}{\bar{b}_j} \right) \Phi_{\boldsymbol{\beta}} \right\|_{L^\infty(D)} =: K < \infty. \tag{A2}$$

This assumption readily implies that $\sup_{\boldsymbol{y}\in U}\{\||\nabla a(\cdot,\boldsymbol{y})|\Phi_{\boldsymbol{\beta}}\|_{L^\infty(D)}\} < \infty$. As a consequence, $|\nabla a(\cdot,\boldsymbol{y})| \in L^\infty(\widetilde{D})$ and $\Delta u(\cdot,\boldsymbol{y}) \in L^2(\widetilde{D})$ for every compactly included subset $\widetilde{D} \subset\subset D$ and $\boldsymbol{y} \in U$. Then, by the divergence theorem and by the product rule for every $v \in C_0^\infty(D) \subset V$

$$\begin{aligned}\int_D a(\cdot,\boldsymbol{y})\nabla u(\cdot,\boldsymbol{y})\cdot\nabla v \,\mathrm{d}x &= -\int_D [\nabla\cdot(a(\cdot,\boldsymbol{y})\nabla u(\cdot,\boldsymbol{y}))] v \,\mathrm{d}x \\ &= -\int_D (a(\cdot,\boldsymbol{y})\Delta u(\cdot,\boldsymbol{y}) + \nabla a(\cdot,\boldsymbol{y})\cdot\nabla u(\cdot,\boldsymbol{y})) v \,\mathrm{d}x.\end{aligned}$$

We have to show that $\Delta u(\boldsymbol{y}) \in L^2_{\boldsymbol{\beta}}(D)$. By duality of the space $L^2_{\boldsymbol{\beta}}(D)$, the previous identity, and by the Cauchy–Schwarz inequality,

$$\begin{aligned}\|a(\cdot,\boldsymbol{y})\Delta u(\cdot,\boldsymbol{y})\|_{L^2_{\boldsymbol{\beta}}(D)} &= \sup_{v\in L^2_{\boldsymbol{\beta}}(D),\|v\|_{L^2_{\boldsymbol{\beta}}(D)}\leq 1}\int_D a(\cdot,\boldsymbol{y})\Delta u(\cdot,\boldsymbol{y})v\Phi^2_{\boldsymbol{\beta}}\,\mathrm{d}x\\ &= \sup_{v\in C_0^\infty(D),\|v\|_{L^2(D)}\leq 1}\int_D a(\cdot,\boldsymbol{y})\Delta u(\cdot,\boldsymbol{y})v\Phi_{\boldsymbol{\beta}}\,\mathrm{d}x\\ &= \sup_{v\in C_0^\infty(D),\|v\|_{L^2(D)}\leq 1}\int_D (f+\nabla a(\cdot,\boldsymbol{y})\cdot\nabla u(\cdot,\boldsymbol{y}))v\Phi_{\boldsymbol{\beta}}\,\mathrm{d}x\\ &\leq \|f\|_{L^2_{\boldsymbol{\beta}}(D)} + \||\nabla a(\cdot,\boldsymbol{y})|\Phi_{\boldsymbol{\beta}}\|_{L^\infty(D)}\|u(\cdot,\boldsymbol{y})\|_V.\end{aligned} \tag{8}$$

Since $\operatorname{ess\,inf}_{x\in D}\{a(x,\boldsymbol{y})\}\geq(1-\kappa)\bar{a}_{\min}$, $\Delta u(\boldsymbol{y})\in L^2_{\boldsymbol{\beta}}(D)$. In (8), we applied that $C_0^\infty(D)$ is dense in $L^2(D)$ and used that the operator of pointwise multiplication $w\mapsto w\Phi_{-\boldsymbol{\beta}}$ is an isometry from $L^2(D)$ to $L^2_{\boldsymbol{\beta}}(D)$.

The parametric FE solution is defined as the unique solution of the variational problem: for $\boldsymbol{y}\in U$ and $\ell\geq 0$, find $u^{\mathscr{T}_\ell}(\cdot,\boldsymbol{y})\in V_\ell$ such that

$$\int_D a(\cdot,\boldsymbol{y})\nabla u^{\mathscr{T}_\ell}(\cdot,\boldsymbol{y})\cdot\nabla v\,\mathrm{d}x = \langle f,v\rangle_{V^*,V},\quad \forall v\in V_\ell. \tag{9}$$

Well-posedness of the parametric FE solution also follows by the Lax–Milgram lemma. As above, we define for every truncation level $s\in\mathbb{N}$ and $\ell\in\mathbb{N}_0$

$$u^{s,\mathscr{T}_\ell}(\cdot,\boldsymbol{y}) := u^{\mathscr{T}_\ell}(\cdot,\boldsymbol{y}_{\{1:s\}}),\quad \boldsymbol{y}\in U.$$

By Céa's lemma, an Aubin–Nitsche argument, Proposition 1, (6), (7), and (8), there exists a constant $C>0$ such that for every $s\in\mathbb{N}$, $\ell\geq 0$, and every $G(\cdot)\in L^2_{\boldsymbol{\beta}}(D)$,

$$\begin{aligned}|\mathbb{E}(G(u))-\mathbb{E}(G(u^{s,\mathscr{T}_\ell}))| &\leq C\|G(\cdot)\|_{V^*}\|f\|_{V^*}\left(\sup_{j\geq s+1}\{b_j\}\right)^2\\ &\quad + C\|G(\cdot)\|_{L^2_{\boldsymbol{\beta}}(D)}\|f\|_{L^2_{\boldsymbol{\beta}}(D)}(M_\ell)^{-2/d}.\end{aligned} \tag{10}$$

Remark 1. If f and $G(\cdot)$ have less regularity, say $f\in(V^*,L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ and $G(\cdot)\in(V^*,L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$, $t,t'\in[0,1]$, the estimate in (10) holds with $M_\ell^{-(t+t')/d}$. This follows by interpolation. The interpolation spaces are in the sense of the *K-method*, cp. [31]. Since $L^2_{\boldsymbol{\beta}}(D)\subset V^*$ continuously which follows by [2, Equation (3.2)], V^* and $L^2_{\boldsymbol{\beta}}(D)$ is an interpolation couple. Naturally the embedding $H^{-1+t'}(D)=(V^*,L^2(D))_{t',2}\subset(V^*,L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$ is continuous, since $L^2(D)$ is continuously embedded in $L^2_{\boldsymbol{\beta}}(D)$.

3 Multilevel QMC integration

Randomly shifted lattice rules and interlaced polynomial lattice rule are QMC rules that have well known *worst case error estimates* in particular weighted Sobolev spaces of regularity with respect to the dimensionally truncated parameter vectors $\boldsymbol{y}_{\{1:s\}}$, $s \in \mathbb{N}$. Generally, these QMC rules approximate dimensionally truncated integrals

$$I_s(F) := \int_{[-\frac{1}{2},\frac{1}{2}]^s} F(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}.$$

Denote by $Q^{\mathrm{RS}}_{s,N}(\cdot)$ and $Q^{\mathrm{IP}}_{s,N}(\cdot)$ randomly shifted lattice rules and interlaced polynomial lattice rules in s dimension with N points, respectively. If the superscript is omitted both of the QMC rules are meant. For sequences $(s_\ell)_{\ell\geq 0}$ of truncation dimensions and numbers of QMC points $(N_\ell)_{\ell\geq 0}$, and for the meshes $\{\mathscr{T}_\ell\}_{\ell\geq 0}$ from Section 2, the multilevel QMC quadrature for $L \in \mathbb{N}$ levels is, for every $G(\cdot) \in V^*$, defined by

$$Q_L(G(u^L)) := \sum_{\ell=0}^{L} Q_{s_\ell,N_\ell}(G(u^\ell - u^{\ell-1})),$$

where we introduced the notation $u^\ell := u^{s_\ell,\mathscr{T}_\ell}$, $\ell \in \mathbb{N}_0$, and have set $u^{-1} := 0$. For the error analysis, we introduce for a sequence of QMC weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ the weighted Sobolev spaces $\mathscr{W}_{s,\boldsymbol{\gamma}}$ and $\mathscr{W}_{s,\alpha,\boldsymbol{\gamma},q,r}$ as closures of $C^\infty([-1/2,1/2]^s)$ with respect to the norms

$$\|F\|_{\mathscr{W}_{s,\boldsymbol{\gamma}}} := \left(\sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \partial^{\mathfrak{u}}_{\boldsymbol{y}} F(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right|^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2}$$

and for $2 \leq \alpha \in \mathbb{N}$, $q, r \in [1,\infty]$

$$\|F\|_{\mathscr{W}_{s,\alpha,\boldsymbol{\gamma},q,r}} := \left(\sum_{\mathfrak{u}\subset\{1:s\}} \left(\gamma_{\mathfrak{u}}^{-q} \sum_{\mathfrak{v}\subset\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{v}|}} \left| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{v}|}} \partial^{(\boldsymbol{\alpha},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\boldsymbol{0})}_{\boldsymbol{y}} F(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{v}} \right|^q \mathrm{d}\boldsymbol{y}_{\mathfrak{v}} \right)^{r/q} \right)^{1/r}$$

with the obvious modifications if q or r is infinite. Here, $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0}) \in \{0:\alpha\}^s$ denotes a multi-index such that $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = \alpha$ for $j \in \mathfrak{v}$, $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = \boldsymbol{\tau}_j$ for $j \in \mathfrak{u}\setminus\mathfrak{v}$, and $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = 0$ for $j \notin \mathfrak{u}$, for every $\mathfrak{u} \subseteq \{1:s\}$, $\mathfrak{v} \subseteq \mathfrak{u}$, $\boldsymbol{\tau} \in \{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}$. Note that the integer $\alpha \geq 2$ is the interlacing factor. For every $\mathfrak{u} \subset \{1:s\}$, $\mathrm{d}\boldsymbol{y}_{\mathfrak{u}}$ denotes the product measure $\bigotimes_{j\in\mathfrak{u}} \mathrm{d}y_j$. The following two estimates follow essentially from the worst case error estimates in [22, Theorem 2.1] and [6, Theorem 3.10] for $Q^{\mathrm{RS}}_{s,N}(\cdot)$ and $Q^{\mathrm{IP}}_{s,N}(\cdot)$, respectively. For every $\lambda \in (1/2,1]$,

$$\begin{aligned}\mathbb{E}^{\boldsymbol{\Delta}}(|I_{s_L}(G(u^L)) - Q^{\mathrm{RS}}_L(G(u^L))|^2) \\ \leq \sum_{\ell=0}^{L} \left(\sum_{\emptyset\neq\mathfrak{u}\subset\{1:s_\ell\}} \gamma_{\mathfrak{u}}^{\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^\lambda} \right)^{|\mathfrak{u}|} \right)^{1/\lambda} (\varphi(N_\ell))^{-1/\lambda} \|G(u^\ell - u^{\ell-1})\|^2_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}},\end{aligned} \tag{11}$$

cp. [23, Equation (25)], where $\boldsymbol{\Delta}$ denotes the random shift and φ denotes the Euler totient function. For every $\lambda \in (1/\alpha, 1]$,

$$\begin{aligned} &|I_{s_L}(G(u^L)) - Q_L^{\mathrm{IP}}(G(u^L))| \\ &\quad \leq \sum_{\ell=0}^{L} \left(\frac{2}{N_\ell - 1} \sum_{\emptyset \neq \mathrm{u} \subset \{1:s_\ell\}} \gamma_{\mathrm{u}}^{\lambda} (\rho_\alpha(\lambda))^{|\mathrm{u}|} \right)^{1/\lambda} \|G(u^\ell - u^{\ell-1})\|_{\mathscr{W}_{s_\ell,\alpha,\boldsymbol{\gamma},\infty,\infty}}, \end{aligned} \tag{12}$$

cp. [7, Equation (42)], where the constant $\rho_\alpha(\lambda)$ is finite if $\lambda > 1/\alpha$ as stated in [6, Equation (3.37)]. We remark that the choice of λ, α, $\boldsymbol{\gamma}$ in (11) and (12) may also depend on the level $\ell = 0, \dots, L$, which is not explicit in the notation.

4 Parametric regularity

As in the single level QMC analysis in [10], we introduce the auxiliary parameter set $\widetilde{U} = [-1,1]^{\mathbb{N}}$ with elements $\boldsymbol{z} \in \widetilde{U}$. Set $\eta \in (\kappa, 1)$. We split the sparsity of the sequence $(b_j)_{j\geq 1}$ between spatial approximation and QMC approximation rates, which naturally couple in multilevel integration methods. For a sequence $(\widehat{b}_j)_{j\geq 1}$ (to be specified in the following) which satisfies the assumption (**A1**), and for every $\boldsymbol{y} \in U$ define

$$\bar{a}_{\boldsymbol{y}}(x) = \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x) \quad \text{and} \quad \psi_{\boldsymbol{y},j}(x) = \frac{\eta^{-1} - 2|y_j|}{2\widehat{b}_j} \psi_j(x), \quad \text{a.e. } x \in D, j \in \mathbb{N}, \tag{13}$$

which are used to construct

$$\widetilde{a}_{\boldsymbol{y}}(x, \boldsymbol{z}) := \bar{a}_{\boldsymbol{y}}(x) + \sum_{j\geq 1} z_j \psi_{\boldsymbol{y},j}(x), \quad \text{a.e. } x \in D, \boldsymbol{z} \in \widetilde{U}.$$

We recall that for every $\boldsymbol{y} \in U$

$$\left\| \frac{\sum_{j\geq 1} |\psi_{\boldsymbol{y}.j}|}{\bar{a}_{\boldsymbol{y}}} \right\|_{L^\infty(D)} \leq \frac{\kappa}{\eta} < 1, \tag{14}$$

which implies that the problem for arbitrary $\boldsymbol{y} \in U$ and $\boldsymbol{z} \in \widetilde{U}$ to find $\widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) \in V$ such that

$$\int_D \widetilde{a}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) \nabla \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) \cdot \nabla v \mathrm{d}x = f(v), \quad \forall v \in V,$$

is well-posed, cp. [10, Section 4]. Then, the affine mapping $T_{\boldsymbol{y}} : \widetilde{U} \to T_{\boldsymbol{y}}(\widetilde{U}) \subset \mathbb{R}^{\mathbb{N}}$, which is given by

$$(T_{\boldsymbol{y}}(\boldsymbol{z}))_j := y_j + \frac{\eta^{-1} + 2|y_j|}{2\widehat{b}_j} z_j, \quad j \geq 1, \boldsymbol{z} \in \widetilde{U}, \tag{15}$$

yields by construction, cp. [10, Section 4], a connection of $u(\cdot,\boldsymbol{y})$ and $\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})$, i.e.,

$$\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) = u(\cdot,T_{\boldsymbol{y}}(\boldsymbol{z})) \quad \text{in } V.$$

Finally, by the chain rule for every $\boldsymbol{\tau} \in \mathscr{F} := \{\boldsymbol{\tau}' \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\tau}'| < \infty\}$

$$\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} = \left(\prod_{j\geq 1}\left(\frac{\eta^{-1}-2|y_j|}{2\widehat{b}_j}\right)^{\tau_j}\right)\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} u(\cdot,\boldsymbol{y}). \tag{16}$$

A transformation of this type has been introduced in [3]. The dilated coordinate is analogously applied to dimensionally truncated solutions and the FE approximations, which are denoted by $\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})$, $\widetilde{u}^{\mathscr{T}_\ell}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})$, and $\widetilde{u}^{s,\mathscr{T}_\ell}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})$. As observed in [10, Theorems 10 and 12], this sequence $(\widehat{b}_j)_{j\geq 1}$ will be the input for the product weights of the considered QMC rules and its summability properties will be a sufficient condition to achieve a certain dimension-independent convergence rate of either type of QMC rule. Note that the parametric regularity results of [10] also hold for homogenous mixed boundary conditions, since the proof of [10, Lemma 3] relied on the variational formulation, which is the same and $v \mapsto (\int_D |\nabla v|^2 \mathrm{d}v)^{1/2}$ is also a norm on V.

4.1 Dimensionally truncated differences

Let $s \in \mathbb{N}$ be the truncation of the series expansion of $a(\cdot,\boldsymbol{y})$ and also of $\widetilde{a}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})$. The difference of solutions with respect to the full resp. to the truncated expansion of the parametric coefficient satisfies

$$\int_D \widetilde{a}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\nabla(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z}))\cdot\nabla v \mathrm{d}x = -\int_D \sum_{j>s} z_j \psi_{\boldsymbol{y},j}\nabla\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\cdot\nabla v\mathrm{d}x, \quad \forall v\in V.$$

In this section we split the sequence $(b_j)_{j\geq 1}$ into two sequences by $b_j = b_j^{1-\theta} b_j^{\theta}$, $j\in\mathbb{N}$, $\theta\in[0,1]$, and consider the dilated coordinate in (13) and (15) with respect to the sequence $(b_j^{1-\theta})_{j\geq 1}$, i.e., here $(\widehat{b}_j)_{j\geq 1} = (b_j^{1-\theta})_{j\geq 1}$, which satisfies (**A1**) by the condition $b_j \in (0,1]$, $j\in\mathbb{N}$. By the assumption in (**A1**) and (14), for every $\boldsymbol{y}\in U$,

$$\left\| \frac{\sum_{j\geq 1}|\psi_{\boldsymbol{y},j}|/b_j^{\theta}}{\bar{a}_{\boldsymbol{y}}} \right\|_{L^\infty(D)} \leq \frac{\kappa}{\eta} < 1. \tag{17}$$

Theorem 1. *Let the assumption in* (**A1**) *be satisfied. There exists a constant $C>0$ such that for every $\boldsymbol{y}\in U$ and for every $s\in\mathbb{N}$ and every $\theta\in[0,1]$*

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \frac{1}{(\boldsymbol{\tau}!)^2}\left\| \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\left(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\right)\Big|_{\boldsymbol{z}=\boldsymbol{0}}\right\|_V^2 \leq C\|f\|_{V^*}^2 \sup_{j>s}\left\{b_j^{2\theta}\right\}.$$

Proof. As in the proof of [10, Lemma 3], we will consider the Taylor coefficients

$$t_{\boldsymbol{y},\boldsymbol{\tau}} := \frac{1}{\boldsymbol{\tau}!}\partial_z^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \quad \text{and} \quad t^s_{\boldsymbol{y},\boldsymbol{\tau}} := \frac{1}{\boldsymbol{\tau}!}\partial_z^{\boldsymbol{\tau}} \widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \quad \forall \boldsymbol{\tau} \in \mathscr{F}. \tag{18}$$

We introduce a parametric energy norm $\|\cdot\|_{\bar{a}_{\boldsymbol{y}}}$ for every $\boldsymbol{y} \in U$ by

$$\|v\|^2_{\bar{a}_y} := \int_D \bar{a}_{\boldsymbol{y}} |\nabla v|^2 \mathrm{d}x, \quad \forall v \in V.$$

Evidently, $t^s_{\boldsymbol{y},\boldsymbol{\tau}} = 0$ in case that $\tau_j > 0$ for some $j > s$. For every $\boldsymbol{\tau} \in \mathscr{F}$,

$$\begin{aligned}\int_D \bar{a}_{\boldsymbol{y}} \nabla (t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}) \cdot \nabla v \mathrm{d}x = &- \sum_{j(\boldsymbol{\tau})} \int_D \psi_{\boldsymbol{y},j} \nabla (t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j} - t^s_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j}) \cdot \nabla v \mathrm{d}x \\ &- \sum_{j(\boldsymbol{\tau}),j>s} \int_D \psi_{\boldsymbol{y},j} \nabla t^s_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j} \cdot \nabla v \mathrm{d}x, \qquad \forall v \in V,\end{aligned}$$

where we used the notation $j(\boldsymbol{\tau}) := \{j \in \mathbb{N} : \tau_j > 0\}$. Testing with $v = t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}$, we find for $\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathscr{F}$,

$$\begin{aligned}\|t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_y} \le &\int_D \sum_{j(\boldsymbol{\tau})} |\psi_{\boldsymbol{y},j}| |\nabla (t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} - t^s_{\boldsymbol{y},\boldsymbol{\tau}-e_j})| |t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}| \mathrm{d}x \\ &+ \int_D \sum_{j(\boldsymbol{\tau}),j>s} |\psi_{\boldsymbol{y},j}| |t^s_{\boldsymbol{y},\boldsymbol{\tau}-e_j}| |t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}| \mathrm{d}x,\end{aligned}$$

where $e_j \in \mathscr{F}$ is such that $(e_j)_i = 1$ if $j = i$ and zero otherwise. We obtain with a twofold application of the Cauchy–Schwarz inequality using (**A1**) and (17)

$$\begin{aligned}\|t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_y} \le &\left(\frac{\kappa}{\eta} \int_D \sum_{j(\boldsymbol{\tau})} |\psi_{\boldsymbol{y},j}| |\nabla (t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} - t^s_{\boldsymbol{y},\boldsymbol{\tau}-e_j})|^2 \mathrm{d}x\right)^{1/2} \|t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_y} \\ &+ \left(\frac{\kappa}{\eta} \sup_{j>s} \left\{b_j^\theta\right\} \int_D \sum_{j(\boldsymbol{\tau}),j>s} |\psi_{\boldsymbol{y},j}| |t^s_{\boldsymbol{y},\boldsymbol{\tau}-e_j}|^2 \mathrm{d}x\right)^{1/2} \|t_{\boldsymbol{y},\boldsymbol{\tau}} - t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_y}.\end{aligned}$$

Hence, by the Young inequality with $\varepsilon > 0$ and by (**A1**)

$$\begin{aligned}\sum_{k\geq 1}\sum_{|\boldsymbol{\tau}|=k}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} &\leq (1+\varepsilon)\frac{\kappa}{\eta}\sum_{k\geq 1}\int_D\sum_{|\boldsymbol{\tau}|=k-1}\sum_{j\geq 1}|\psi_{\boldsymbol{y},j}||\nabla(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^s_{\boldsymbol{y},\boldsymbol{\tau}})|^2\mathrm{d}x\\ &\quad+\left(1+\frac{1}{\varepsilon}\right)\frac{\kappa}{\eta}\sup_{j>s}\left\{b_j^{\theta}\right\}\sum_{k\geq 1}\int_D\sum_{|\boldsymbol{\tau}|=k-1}\sum_{j>s}|\psi_{\boldsymbol{y},j}||t^s_{\boldsymbol{y},\boldsymbol{\tau}}|^2\mathrm{d}x\\ &\leq (1+\varepsilon)\left(\frac{\kappa}{\eta}\right)^2\sum_{k\geq 0}\sum_{|\boldsymbol{\tau}|=k}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}\\ &\quad+\left(1+\frac{1}{\varepsilon}\right)\left(\frac{\kappa}{\eta}\right)^2\sup_{j>s}\left\{b_j^{2\theta}\right\}\sum_{k\geq 0}\sum_{|\boldsymbol{\tau}|=k}\|t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}.\end{aligned}$$

Since $\kappa<\eta$, we can choose ε such that $(1+\varepsilon)(\kappa/\eta)^2<1$ and conclude that

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}\leq 2\|t_{\boldsymbol{y},\boldsymbol{0}}-t^s_{\boldsymbol{y},\boldsymbol{0}}\|^2_{\bar{a}_{\boldsymbol{y}}}+\frac{1+1/\varepsilon}{1-(1+\varepsilon)(\kappa/\eta)^2}\sup_{j>s}\left\{b_j^{2\theta}\right\}\sum_{\boldsymbol{\tau}\in\mathscr{F}}\|t^s_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}},$$

which implies the assertion with [10, Lemma 3 and Proposition 8] using that $b_j\in(0,1]$, $j\in\mathbb{N}$. Note that [10, Lemma 3] gives an upper bound, since $\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})=\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})$ if $\tau_j=0$ for every $j>s$ and $\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})=0$ otherwise. □

Remark 2. The estimate in Theorem 1 also holds when the differences $\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}(\widetilde{u}_{\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z})-\widetilde{u}_{\boldsymbol{y}}^{s,\mathscr{T}_\ell}(\cdot,\boldsymbol{z}))|_{\boldsymbol{z}=\boldsymbol{0}}$, $\boldsymbol{\tau}\in\mathscr{F}$, $\ell\geq 0$, are considered and the constant is independent of $\{\mathscr{T}_\ell\}_{\ell\geq 0}$. Since only the varitional formulation was used in the proof, the corresponding variational formulation with trial and test space V_ℓ can be used instead.

4.2 FE differences

We assume now that the sequence $(\bar{b}_j)_{j\geq 1}$ satisfies the assumptions in (**A1**) and in (**A2**). We consider the dilated coordinate in (13) and (15) with respect to this sequence $(\bar{b}_j)_{j\geq 1}$, i.e., here $(\widehat{b}_j)_{j\geq 1}=(\bar{b}_j)_{j\geq 1}$.

Proposition 2. *Let the assumption in* (**A1**) *and* (**A2**) *be satisfied for* $(\bar{b}_j)_{j\geq 1}$. *Then, there exists a constant* $C>0$ *(independent of* f*) such that for every* $\boldsymbol{y}\in U$

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}}\frac{1}{(\boldsymbol{\tau}!)^2}\left\|\Delta\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}\right\|^2_{L^2_{\boldsymbol{\beta}}(D)}\leq C\|f\|^2_{L^2_{\boldsymbol{\beta}}(D)}.$$

Proof. Recall that the Taylor coefficients $\{t_{\boldsymbol{y},\boldsymbol{\tau}}:\boldsymbol{\tau}\in\mathscr{F}\}$ have been defined in (18). We also recall that for any $\boldsymbol{0}\neq\boldsymbol{\tau}\in\mathscr{F}$,

$$\int_D\bar{a}_{\boldsymbol{y}}\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}\cdot\nabla v\mathrm{d}x=-\sum_{j(\boldsymbol{\tau})}\int_D\psi_{\boldsymbol{y},j}\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}\cdot\nabla v\mathrm{d}x,\quad\forall v\in V.$$

Similarly as in Section 2, by the divergence theorem for every $v\in C_0^\infty(D)$

$$-\int_D \bar{a}_{\boldsymbol{y}} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}} v \mathrm{d}x = \int_D \left(\nabla \bar{a}_{\boldsymbol{y}} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}} + \sum_{j(\boldsymbol{\tau})} \left(\nabla \psi_{\boldsymbol{y},j} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} + \psi_{\boldsymbol{y},j} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} \right) \right) v \mathrm{d}x.$$

Since $\Delta t_{\boldsymbol{y},\boldsymbol{\tau}} \Phi_{\boldsymbol{\beta}} \in L^2(D)$, cp. (8), we may use $-\Delta t_{\boldsymbol{y},\boldsymbol{\tau}} \Phi_{\boldsymbol{\beta}}^2$ as a test function and obtain with the Young inequality for any $\varepsilon > 0$

$$\begin{aligned}
\int_D \bar{a}_{\boldsymbol{y}} |\Delta t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x &= -\int_D \left(\nabla \bar{a}_{\boldsymbol{y}} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}} + \sum_{j(\boldsymbol{\tau})} \nabla \psi_{\boldsymbol{y},j} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} \right) \Delta t_{\boldsymbol{y},\boldsymbol{\tau}} \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x \\
&\quad - \int_D \sum_{j(\boldsymbol{\tau})} \psi_{\boldsymbol{y},j} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}} \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x \\
&\le \varepsilon \int_D \bar{a}_{\boldsymbol{y}} |\Delta t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x \\
&\quad + \frac{1}{4\varepsilon} \int_D \frac{\Phi_{\boldsymbol{\beta}}^2}{\bar{a}_{\boldsymbol{y}}} \left(\nabla \bar{a}_{\boldsymbol{y}} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}} + \sum_{j(\boldsymbol{\tau})} \nabla \psi_{\boldsymbol{y},j} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} \right)^2 \mathrm{d}x \\
&\quad + \frac{1}{2} \int_D \sum_{j(\boldsymbol{\tau})} |\psi_{\boldsymbol{y},j}| (|\Delta t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}|^2 + |\Delta t_{\boldsymbol{y},\boldsymbol{\tau}}|^2) \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x.
\end{aligned}$$

For $k \ge 1$, by a twofold application of the Cauchy–Schwarz inequality (applied to the sum) and (**A2**)

$$\begin{aligned}
&\sum_{|\boldsymbol{\tau}|=k} \int_D \frac{\Phi_{\boldsymbol{\beta}}^2}{\bar{a}_{\boldsymbol{y}}} \left(\nabla \bar{a}_{\boldsymbol{y}} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}} + \sum_{j(\boldsymbol{\tau})} \nabla \psi_{\boldsymbol{y},j} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} \right)^2 \mathrm{d}x \\
&\quad \le 2K \int_D \frac{\Phi_{\boldsymbol{\beta}}}{\bar{a}_{\boldsymbol{y}}} \left(|\nabla \bar{a}_{\boldsymbol{y}}| \sum_{|\boldsymbol{\tau}|=k} |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 + \sum_{|\boldsymbol{\tau}|=k-1} \sum_{j \ge 1} |\nabla \psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \right) \mathrm{d}x \\
&\quad \le \frac{4K^2}{(\bar{a}_{\boldsymbol{y},\min})^2} \left(\sum_{|\boldsymbol{\tau}|=k} \|\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_{\boldsymbol{y}}}^2 + \sum_{|\boldsymbol{\tau}|=k-1} \|\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_{\boldsymbol{y}}}^2 \right).
\end{aligned}$$

Note that also by (**A1**) for every $k \ge 1$,

$$\begin{aligned}
&\sum_{|\boldsymbol{\tau}|=k} \int_D \sum_{j(\boldsymbol{\tau})} |\psi_{\boldsymbol{y},j}| (|\Delta t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}|^2 + |\Delta t_{\boldsymbol{y},\boldsymbol{\tau}}|^2) \Phi_{\boldsymbol{\beta}}^2 \mathrm{d}x \\
&\quad \le \frac{\kappa}{\eta} \left(\sum_{|\boldsymbol{\tau}|=k-1} \|\sqrt{\bar{a}_{\boldsymbol{y}}} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{L^2_{\boldsymbol{\beta}}(D)}^2 + \sum_{|\boldsymbol{\tau}|=k} \|\sqrt{\bar{a}_{\boldsymbol{y}}} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{L^2_{\boldsymbol{\beta}}(D)}^2 \right).
\end{aligned}$$

We now choose $\varepsilon > 0$ such that $\varepsilon < 1/2(1-\kappa/\eta)$, which implies $\varepsilon + \kappa/(2\eta) < 1/2$. Then, we sum over $k \ge 1$ to obtain

$$\sum_{k \ge 1} \sum_{|\boldsymbol{\tau}|=k} \|\sqrt{\bar{a}_{\boldsymbol{y}}} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{L^2_{\boldsymbol{\beta}}(D)}^2 \le C \sum_{\boldsymbol{\tau} \in \mathscr{F}} \|\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_{\boldsymbol{y}}}^2 + C_\varepsilon \sum_{\boldsymbol{\tau} \in \mathscr{F}} \|\sqrt{\bar{a}_{\boldsymbol{y}}} \Delta t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{L^2_{\boldsymbol{\beta}}(D)}^2,$$

where $C_\varepsilon = 1/2(1-\kappa/(2\eta)-\varepsilon)^{-1} < 1$ and $C = K^2/(\varepsilon(\bar{a}_{\boldsymbol{y},\min})^2)(1-\kappa/(2\eta)-\varepsilon)^{-1}$. It follows

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \|\sqrt{\bar{a}_{\boldsymbol{y}}}\Delta t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{L^2_{\boldsymbol{\beta}}(D)} \leq \frac{C}{1-C_\varepsilon} \sum_{\boldsymbol{\tau}\in\mathscr{F}} \|t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} + 2\|\sqrt{\bar{a}_{\boldsymbol{y}}}\Delta t_{\boldsymbol{y},\boldsymbol{0}}\|^2_{L^2_{\boldsymbol{\beta}}(D)},$$

which implies the assertion with [10, Lemma 3] and (8). ☐

Remark 3. For every truncation dimension $s \in \mathbb{N}$, the estimate in Proposition 2 also holds when $\Delta\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z})|_{\boldsymbol{z}=\boldsymbol{0}}$, $\boldsymbol{\tau}\in\mathscr{F}$, are considered and the constant is independent of s. This follows from the observation that $\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})$ if $\tau_j = 0$ for every $j > s$ and $\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) = 0$ otherwise. Then, the sum of the estimate in Proposition 2 only consists of more terms and is an upper bound.

Proposition 3. *Let the assumptions in* (**A1**) *and* (**A2**) *be satisfied and let* $\boldsymbol{\beta}$ *satisfy* $\beta_i > 1-\pi/\omega_i$ *and if the boundary conditions change at* c_i *also* $\beta_i > 1-\pi/(2\omega_i)$ *,* $i = 1,\dots,J$*. Then, there exists a constant* $C > 0$ *such that for every* $\boldsymbol{y}\in U$ *and for every integer* $\ell \geq 0$

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \frac{1}{(\boldsymbol{\tau}!)^2} \left\| \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \left(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}^{\mathscr{T}_\ell}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) \right) \Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|^2_V \leq C M_\ell^{-2/d} \|f\|^2_{L^2_{\boldsymbol{\beta}}(D)}.$$

Proof. We argue similarly as in the proof of [10, Lemma 3] and consider the Taylor coefficients for fixed $\boldsymbol{y}\in U$

$$t_{\boldsymbol{y},\boldsymbol{\tau}} := \frac{1}{\boldsymbol{\tau}!}\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \quad \text{and} \quad t^\ell_{\boldsymbol{y},\boldsymbol{\tau}} := \frac{1}{\boldsymbol{\tau}!}\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}}\widetilde{u}^{\mathscr{T}_\ell}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \quad \boldsymbol{\tau}\in\mathscr{F}.$$

We observe that

$$\int_D \bar{a}_{\boldsymbol{y}} \nabla(t_{\boldsymbol{y},\boldsymbol{\tau}} - t^\ell_{\boldsymbol{y},\boldsymbol{\tau}}) \cdot \nabla v \mathrm{d}x = -\sum_{j(\boldsymbol{\tau})} \psi_{\boldsymbol{y},j} \nabla(t_{\boldsymbol{y},\boldsymbol{\tau}-e_j} - t^\ell_{\boldsymbol{y},\boldsymbol{\tau}-e_j}) \cdot \nabla v \mathrm{d}x, \quad \forall v \in V_\ell.$$

For every $\boldsymbol{y}\in U$ and for every $\ell\in\mathbb{N}_0$, let $\mathscr{P}_{\boldsymbol{y},\ell} : V \to V_\ell$ denote the dilated Galerkin projection, i.e., for every $w \in V$, $\mathscr{P}_{\boldsymbol{y},\ell} w$ is defined by

$$\int_D \bar{a}_{\boldsymbol{y}} \nabla(w - \mathscr{P}_{\boldsymbol{y},\ell} w) \cdot \nabla v \, \mathrm{d}x = 0, \quad \forall v \in V_\ell. \tag{19}$$

By the definition of $\mathscr{P}_{\boldsymbol{y},\ell}$ in (19) and by testing with $v = \mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}} - t_{\boldsymbol{y},\boldsymbol{\tau}}) \in V_\ell$,

$$\begin{aligned}
&\sum_{|\boldsymbol{\tau}|=k}\int_D \bar{a}_{\boldsymbol{y}}|\nabla\mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2\mathrm{d}x\\
&\quad\le \int_D\sum_{|\boldsymbol{\tau}|=k}\sum_{j(\boldsymbol{\tau})}|\psi_{\boldsymbol{y},j}|\frac{1}{2}(|\nabla(t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}-e_j})|^2+|\nabla\mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2)\mathrm{d}x\\
&\quad\le \frac{1}{2}\int_D\sum_{|\boldsymbol{\tau}|=k-1}\sum_{j\ge 1}|\psi_{\boldsymbol{y},j}||\nabla(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2\mathrm{d}x\\
&\qquad+\frac{1}{2}\int_D\sum_{|\boldsymbol{\tau}|=k-1}\sum_{j\ge 1}|\psi_{\boldsymbol{y},j}||\nabla\mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2\mathrm{d}x,
\end{aligned}$$

which implies with (**A1**)

$$\begin{aligned}
\sum_{|\boldsymbol{\tau}|=k}\|\mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})\|^2_{\bar{a}_{\boldsymbol{y}}} &\le \frac{1}{2-\kappa/\eta}\int_D\sum_{|\boldsymbol{\tau}|=k}\sum_{j(\boldsymbol{\tau})}|\psi_{\boldsymbol{y},j}||\nabla(t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}-e_j})|^2\mathrm{d}x\\
&\le \frac{1}{2-\kappa/\eta}\frac{\kappa}{\eta}\sum_{|\boldsymbol{\tau}|=k-1}\|t_{\boldsymbol{y},\boldsymbol{\tau}-e_j}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}-e_j}\|^2_{\bar{a}_{\boldsymbol{y}}}.
\end{aligned}\tag{20}$$

Note that by the triangle inequality

$$\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_{\boldsymbol{y}}}\le\|\mathscr{P}_{\boldsymbol{y},\ell}(t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})\|_{\bar{a}_{\boldsymbol{y}}}+\|(\mathscr{I}-\mathscr{P}_{\boldsymbol{y},\ell})t_{\boldsymbol{y},\boldsymbol{\tau}}\|_{\bar{a}_{\boldsymbol{y}}},$$

where $\mathscr{I}:V\to V$ denotes the identity. With the Young inequality and the previous two inequalities we obtain for any $\varepsilon>0$

$$\begin{aligned}
&\sum_{|\boldsymbol{\tau}|=k}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}\\
&\quad\le\frac{(1+\varepsilon)\kappa}{2\eta-\kappa}\sum_{|\boldsymbol{\tau}|=k-1}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}+\left(1+\frac{1}{\varepsilon}\right)\sum_{|\boldsymbol{\tau}|=k}\|(\mathscr{I}-\mathscr{P}_{\boldsymbol{y},\ell})t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}.
\end{aligned}$$

Since $\kappa<\eta<1$, $2\eta-\kappa>1$ and so we choose $\varepsilon>0$ such that $(1+\varepsilon)\kappa<1$ and conclude by subtracting the first sum in the previous inequality that

$$\sum_{k\ge1}\sum_{|\boldsymbol{\tau}|=k}\|t_{\boldsymbol{y},\boldsymbol{\tau}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}\le\|t_{\boldsymbol{y},\boldsymbol{0}}-t^{\ell}_{\boldsymbol{y},\boldsymbol{0}}\|^2_{\bar{a}_{\boldsymbol{y}}}+\frac{2\eta-\kappa}{\varepsilon\kappa}\sum_{k\ge1}\sum_{|\boldsymbol{\tau}|=k}\|(\mathscr{I}-\mathscr{P}_{\boldsymbol{y},\ell})t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}},$$

which implies the assertion with (7), (6), and Proposition 2. □

Remark 4. The estimate in Proposition 3 holds if $f\in(V^*,L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ with the error being controlled by $M_\ell^{-2t/d}$, $t\in[0,1]$. This can be seen by interpolation applied in the last step of the proof of Proposition 3, where (7), (6), and Proposition 2 were used (see also Remark 1).

Lemma 1. *For every $c\in(1,\infty)$ and every $\boldsymbol{\tau}\in\mathscr{F}$,*

$$|\boldsymbol{\tau}| \leq \frac{1}{\log(c)\,e}\,c^{\boldsymbol{\tau}} = \frac{1}{\log(c)\,e}\prod_{j\geq 1} c^{\tau_j}.$$

Proof. Let $\alpha > 0$ be arbitrary. By elementary calculus, we observe that the function $x \mapsto \log(x)/x^\alpha$ has a maximum at $x_0 = e^{1/\alpha}$. Since $\lim_{x\to\infty}\log(x)/x^\alpha = 0$, this maximum is global on $[1,\infty)$. Hence, $\log(x) \leq x^\alpha/(\alpha e)$ for every $x \in [1,\infty)$.

Then, $\sum_{j\geq 1}\tau'_j = \log(\prod_{j\geq 1} e^{\tau'_j}) \leq 1/(\alpha e)\prod_{j\geq 1} e^{\alpha\tau'_j} = 1/(\alpha e)c^{\boldsymbol{\tau}'}$, where α is chosen to be $\alpha = \log(c) > 0$. □

For any $G \in V^*$, we introduce $u_G(\cdot,\boldsymbol{y})$ and $u_G^{\mathscr{T}_\ell}(\cdot,\boldsymbol{y})$, $\ell \in \mathbb{N}_0$, as the parametric solution to the dual problem of (4) and the parametric FE solution to the dual problem of (9), respectively, with right hand side G. Consideration of the dilated coefficient resulting from (13) gives $\widetilde{u}_{G,\boldsymbol{y}}(\cdot,\boldsymbol{z})$ and $\widetilde{u}_{G,\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z})$, $\ell \in \mathbb{N}_0$. By an Aubin–Nitsche argument, for every $\boldsymbol{y} \in U$ and every $\boldsymbol{z} \in \widetilde{U}$,

$$G(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}_{\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z})) = \int_D \widetilde{a}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\nabla(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}_{\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z}))\cdot\nabla(\widetilde{u}_{G,\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}_{G,\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z}))\mathrm{d}x. \tag{21}$$

Theorem 2. *Let the assumptions in* (**A1**) *and* (**A2**) *be satisfied. Then, for every $c > 1$ there exists a constant $C > 0$ such that for every $G(\cdot) \in L^2_{\boldsymbol{\beta}}(D)$ and for every integer $\ell \geq 0$*

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \frac{1}{c^{\boldsymbol{\tau}}(\boldsymbol{\tau}!)^2}\left|\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} G\left(\widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}_{\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z})\right)\Big|_{\boldsymbol{z}=\boldsymbol{0}}\right|^2 \leq C M_\ell^{-4/d}\|f\|^2_{L^2_{\boldsymbol{\beta}}(D)}\|G(\cdot)\|^2_{L^2_{\boldsymbol{\beta}}(D)}.$$

Proof. The Taylor coefficients of $\widetilde{u}_{G,\boldsymbol{y}}(\cdot,\boldsymbol{z})$ and $\widetilde{u}_{G,\boldsymbol{y}}^{\mathscr{T}_\ell}(\cdot,\boldsymbol{z})$ will be denoted by $\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}}$ and $\widehat{t}^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}$, $\boldsymbol{\tau} \in \mathscr{F}$, respectively (see also (18)). By differentiating (21), for every $\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathscr{F}$,

$$G(t_{\boldsymbol{y},\boldsymbol{\tau}} - t^\ell_{\boldsymbol{y},\boldsymbol{\tau}}) = \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}}\int_D\left[\sum_{j(\boldsymbol{\nu})}\psi_{\boldsymbol{y},j}\nabla(t_{\boldsymbol{y},\boldsymbol{\nu}-e_j} - t^\ell_{\boldsymbol{y},\boldsymbol{\nu}-e_j})\right]\cdot\nabla(\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}} - \widehat{t}^\ell_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}})\mathrm{d}x.$$

Squaring the previous equality and applying the Cauchy–Schwarz inequality and Lemma 1 to obtain with $C = 1/(\log(c)e)$

$$|G(t_{\boldsymbol{y},\boldsymbol{\tau}} - t^\ell_{\boldsymbol{y},\boldsymbol{\tau}})|^2 \leq C c^{\boldsymbol{\tau}}\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}}\left\|\sqrt{1/\bar{a}_{\boldsymbol{y}}}[\ldots]\right\|^2_{L^2(D)}\|\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}} - \widehat{t}^\ell_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}}\|^2_{\bar{a}_{\boldsymbol{y}}},$$

where we used that $\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}} = |\boldsymbol{\tau}|$. The hidden term is $[\ldots] = \sum_{j(\boldsymbol{\nu})}\psi_{\boldsymbol{y},j}\nabla(t_{\boldsymbol{y},\boldsymbol{\nu}-e_j} - t^\ell_{\boldsymbol{y},\boldsymbol{\nu}-e_j})$. By changing the order of summation

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \frac{1}{c^{\boldsymbol{\tau}}} |G(t_{\boldsymbol{y},\boldsymbol{\tau}} - t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2 \le C \sum_{\boldsymbol{\nu}\in\mathscr{F}} \left\| \sqrt{1/\bar{a}_{\boldsymbol{y}}}[\dots] \right\|^2_{L^2(D)} \sum_{\boldsymbol{\tau}\in\mathscr{F},\boldsymbol{\tau}\ge\boldsymbol{\nu}} \|\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}} - \widehat{t}^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{\nu}}\|^2_{\bar{a}_{\boldsymbol{y}}}$$
$$= C \sum_{\boldsymbol{\nu}\in\mathscr{F}} \left\| \sqrt{1/\bar{a}_{\boldsymbol{y}}}[\dots] \right\|^2_{L^2(D)} \sum_{\boldsymbol{\tau}\in\mathscr{F}} \|\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}} - \widehat{t}^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}. \tag{22}$$

By the Cauchy–Schwarz inequality we obtain with (**A1**)

$$\left\| \sqrt{1/\bar{a}_{\boldsymbol{y}}}[\dots] \right\|^2_{L^2(D)} \le \frac{\kappa}{\eta} \int_D \sum_{j(\boldsymbol{\nu})} |\psi_{\boldsymbol{y},j}| |\nabla (t_{\boldsymbol{y},\boldsymbol{\nu}-e_j} - t^{\ell}_{\boldsymbol{y},\boldsymbol{\nu}-e_j})|^2 \mathrm{d}x.$$

By another application of the Cauchy–Schwarz inequality and (**A1**)

$$\sum_{k\ge1} \sum_{|\boldsymbol{\nu}|=k} \left\| \sqrt{1/\bar{a}_{\boldsymbol{y}}}[\dots] \right\|^2_{L^2(D)} \le \left(\frac{\kappa}{\eta}\right)^2 \sum_{k\ge1} \sum_{|\boldsymbol{\nu}|=k-1} \|t_{\boldsymbol{y},\boldsymbol{\nu}} - t^{\ell}_{\boldsymbol{y},\boldsymbol{\nu}}\|^2_{\bar{a}_{\boldsymbol{y}}},$$

which implies with (22)

$$\sum_{\boldsymbol{\tau}\in\mathscr{F}} \frac{1}{c^{\boldsymbol{\tau}}} |G(t_{\boldsymbol{y},\boldsymbol{\tau}} - t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}})|^2 \le C \left(\sum_{\boldsymbol{\tau}\in\mathscr{F}} \|t_{\boldsymbol{y},\boldsymbol{\tau}} - t^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} \right) \left(\sum_{\boldsymbol{\tau}\in\mathscr{F}} \|\widehat{t}_{\boldsymbol{y},\boldsymbol{\tau}} - \widehat{t}^{\ell}_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} \right).$$

The assertion now follows with Proposition 3. □

Remark 5. The estimate in Theorem 2 also holds if $f \in (V^*, L^2_{\boldsymbol{\beta}})_{t,\infty}$ and $G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}})_{t',\infty}$, $t, t' \in [0,1]$, with error control $M_\ell^{-2(t+t')/d}$, which follows by Remark 4.

Remark 6. For every truncation dimension $s \in \mathbb{N}$, the estimates in Proposition 3 and Theorem 2 also hold when the differences $\partial^{\boldsymbol{\tau}}_{\boldsymbol{z}} (\widetilde{u}^s_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) - \widetilde{u}^{s,\mathscr{T}_\ell}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}))|_{\boldsymbol{z}=\boldsymbol{0}}$, $\boldsymbol{\tau} \in \mathscr{F}$, are considered and the constant is independent of s. This follows by the same argument which is used to verify Remark 3.

5 Convergence of multilevel QMC

The parametric regularity estimates from Section 4 will result in explicit error estimates of multilevel QMC. Let the sequence $(\mathfrak{b}_j)_{j\ge1}$ be a generic input for the QMC weights. For interlaced polynomial lattice rules with interlacing factor $\alpha \ge 2$ we will consider the product weights $\boldsymbol{\gamma}^{\mathrm{IP}} = (\gamma^{\mathrm{IP}}_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ given by $\gamma^{\mathrm{IP}}_{\emptyset} := 1$ and

$$\gamma^{\mathrm{IP}}_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \left(\sum_{\nu=1}^{\alpha} \left(\frac{2\mathfrak{b}_j}{1-\eta} \right)^{\nu} \sqrt{2^{\delta(\nu,\alpha)}\nu!} \right), \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty, \tag{23}$$

and for randomly shifted lattice rules the product weights $\boldsymbol{\gamma}^{\mathrm{RS}} = (\gamma^{\mathrm{RS}}_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ given by $\gamma^{\mathrm{RS}}_{\emptyset} := 1$ and

$$\gamma_{\mathfrak{u}}^{\mathrm{RS}} := \prod_{j\in\mathfrak{u}} \left(\frac{2\mathfrak{b}_j}{1-\eta}\right)^2, \quad \mathfrak{u}\subset\mathbb{N}, |\mathfrak{u}|<\infty. \tag{24}$$

We will apply a QMC rule on levels $\ell = 1,\dots,L$ and in general a different version on the level $\ell = 0$. The parametric regularity estimates that were derived in Section 4 are based on a dilated coordinate, cp. (13) and (15), with respect to sequences $(b_j^{1-\theta})_{j\geq 1}$ for the truncation error and $(\bar{b}_j)_{j\geq 1}$ for the FE error. These sequences will be the input for the product weights. Their summability in terms of membership in $\ell^{\bar{p}}(\mathbb{N})$, $\bar{p}\in(0,2]$, will result in explicit bounds of the coupled errors between the levels. On the levels $\ell = 1,\dots,L$, we use $(c(b_j^{1-\theta}\vee\bar{b}_j))_{j\geq1} := (\max\{cb_j^{1-\theta}, c\bar{b}_j\})_{j\geq1}$ as input for the product weights in (23) and (24) for a constant $c>1$, i.e. here $(\mathfrak{b}_j)_{j\geq1} = (c(b_j^{1-\theta}\vee\bar{b}_j))_{j\geq1}$. On the level $\ell=0$ we use $(b_j)_{j\geq1}$ as an input for (23) and (24), which has potentially stronger summability properties.

Theorem 3. *Let the assumption in* (**A1**) *be satisfied by* $(b_j)_{j\geq1}$ *and by* $(\bar{b}_j)_{j\geq1}$. *Let the assumption in* (**A2**) *be satisfied by* $(\bar{b}_j)_{j\geq1}$. *Let* $(b_j)_{j\geq1}\in\ell^p(\mathbb{N})$ *for some* $p\in(0,2]$ *and assume that* $(b_j^{1-\theta}\vee\bar{b}_j)_{j\geq1}\in\ell^{\bar{p}}(\mathbb{N})$ *for some* $\bar{p}\in[p,2]$ *and any* $\theta\in[0,1)$ *admitting this summability. For* $p\in(0,1]$ *and* $\bar{p}\in[p,1]$, Q_L^{IP}, $L\in\mathbb{N}$, *satisfies with product weights* (23) *and order* $\alpha=\lfloor 1/p+1\rfloor$ *on level* $\ell=0$, *and of order* $\bar{\alpha}=\lfloor 1/\bar{p}+1\rfloor$ *on levels* $\ell=1,\dots,L$, *the error estimate*

$$|\mathbb{E}(G(u)) - Q_L^{\mathrm{IP}}(G(u^L))| \leq C\Bigg(\sup_{j>s_L}\{b_j^2\} + M_L^{-2/d} + N_0^{-1/p} + \sum_{\ell=1}^{L} N_\ell^{-1/\bar{p}}\left(\xi_{\ell,\ell-1}\sup_{j>s_{\ell-1}}\{b_j^{\theta}\} + M_{\ell-1}^{-2/d}\right)\Bigg),$$

where $\xi_{\ell,\ell-1}:=0$ *if* $s_\ell = s_{\ell-1}$ *and* $\xi_{\ell,\ell-1}:=1$ *otherwise. For* $p\in(1,2]$ *and* $\bar{p}\in[p,2]$, Q_L^{RS}, $L\in\mathbb{N}$, *satisfies with product weights* (24) *the error estimate*

$$\begin{aligned}&\sqrt{\mathbb{E}^{\Delta}(|\mathbb{E}(G(u)) - Q_L^{\mathrm{RS}}(G(u^L))|^2)}\\ &\quad\leq C\Bigg(\sup_{j>s_L}\{b_j^4\} + M_L^{-4/d} + (\varphi(N_0))^{-2/p}\\ &\quad\quad + \sum_{\ell=1}^{L}(\varphi(N_\ell))^{-2/\bar{p}}\left(\xi_{\ell,\ell-1}\sup_{j>s_{\ell-1}}\{b_j^{2\theta}\} + M_{\ell-1}^{-4/d}\right)\Bigg)^{1/2}.\end{aligned}$$

The constant C *is in particular independent of* L, $(N_\ell)_{\ell\geq0}$, $(M_\ell)_{\ell\geq0}$, *and* $(s_\ell)_{\ell\geq0}$.

Proof. By the error estimates in (12) and (11), we have to estimate the difference $G(u^\ell - u^{\ell-1}) = G(u^{s_\ell,\mathcal{T}_\ell} - u^{s_{\ell-1},\mathcal{T}_{\ell-1}})$ in the $\mathcal{W}_{s_\ell,\alpha,\boldsymbol{\gamma},\infty,\infty}$-norm and in the $\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}$-norm, $\ell=1,\dots,L$. We decompose by the triangle inequality

$$\begin{aligned}&\|G(u^{s_\ell,\mathcal{T}_\ell} - u^{s_{\ell-1},\mathcal{T}_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}}\\ &\quad\leq \|G(u^{s_\ell,\mathcal{T}_\ell} - u^{s_\ell,\mathcal{T}_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}} + \|G(u^{s_\ell,\mathcal{T}_{\ell-1}} - u^{s_{\ell-1},\mathcal{T}_{\ell-1}})\|_{\mathcal{W}_{s_\ell,\boldsymbol{\gamma}}},\end{aligned}$$

and

$$\|G(u^{s_\ell,\mathscr{T}_\ell}-u^{s_\ell,\mathscr{T}_{\ell-1}})\|_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}\le\|G(u^{s_\ell}-u^{s_\ell,\mathscr{T}_\ell})\|_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}+\|G(u^{s_\ell}-u^{s_\ell,\mathscr{T}_{\ell-1}})\|_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}.$$

The contributions from the dimension truncation and the FE error have been separated in the $\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}$-norm. For the dimension truncation error, we obtain by the Jensen inequality, the relation of higher order partial derivative in terms of the dilated coordinate in (16), Theorem 1, and Remark 2

$$\begin{aligned}
&\|G(u^{s_\ell,\mathscr{T}_{\ell-1}}-u^{s_{\ell-1},\mathscr{T}_{\ell-1}})\|^2_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}\\
&\quad\le\|G(\cdot)\|_{V^*}\int_{[-\frac12,\frac12]^s}\sum_{\mathrm{u}\subset\{1:s\}}(\gamma^{\mathrm{RS}}_{\mathrm{u}})^{-1}\|\partial^{\mathrm{u}}_{\boldsymbol{y}}(u^{s_\ell,\mathscr{T}_{\ell-1}}(\cdot,\boldsymbol{y})-u^{s_{\ell-1},\mathscr{T}_{\ell-1}}(\cdot,\boldsymbol{y}))\|^2_V\mathrm{d}\boldsymbol{y}\\
&\quad\le C\|G(\cdot)\|_{V^*}\|f\|_{V^*}\sup_{\mathrm{u}\subset\{1:s\}}(\gamma^{\mathrm{RS}}_{\mathrm{u}})^{-1}\prod_{j\in\mathrm{u}}\left(\frac{2b_j^{1-\theta}}{1-\eta}\right)^2\sup_{j>s_\ell}\{b_j^{2\theta}\}.
\end{aligned}$$

Due to the choice of the weights, there exists a constant $C>0$ independent of the sequences $(s_\ell)_{\ell\ge0}$ and $\{\mathscr{T}_\ell\}_{\ell\ge0}$ such that

$$\|G(u^{s_\ell,\mathscr{T}_{\ell-1}}-u^{s_{\ell-1},\mathscr{T}_{\ell-1}})\|_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}\le C\|G(\cdot)\|_{V^*}\|f\|_{V^*}\sup_{j>s_\ell}\{b_j^\theta\}.$$

Note that if $s_\ell=s_{\ell-1}$ this difference is zero. Similarly, we obtain with Theorem 2 and Remark 4 that there exists a constant $C>0$ which is independent of $(s_\ell)_{\ell\ge0}$ and $(M_\ell)_{\ell\ge0}$ such that for every $\ell\in\mathbb{N}$

$$\|G(u^{s_\ell}-u^{s_\ell,\mathscr{T}_{\ell-1}})\|_{\mathscr{W}_{s_\ell,\boldsymbol{\gamma}}}\le C\|G(\cdot)\|_{L^2_{\boldsymbol{\beta}}(D)}\|f\|_{L^2_{\boldsymbol{\beta}}(D)}M_{\ell-1}^{-2/d}.$$

Note that here the constant $c>1$ in the weight sequence is necessary to compensate c in the estimate of Theorem 2. The corresponding estimate on the level $\ell=0$ of $\|G(u^{s_0,\mathscr{T}_0})\|_{\mathscr{W}_{s_0,\boldsymbol{\gamma}}}$ is due to [10, Corollary 5], which is also applicable in the case of a dimensionally truncated FE solution, cp. Remarks 2 and 4. The estimate for the randomly shifted lattice rules follows then with (10) and (11) with $\lambda=p/2$.

The proof for interlaced polynomial lattice rules follows along the same lines, where the estimate $\|F\|_{\mathscr{W}_{s_\ell,\alpha,\boldsymbol{\gamma},\infty,\infty}}\le\|F\|_{\mathscr{W}_{s_\ell,\alpha,\boldsymbol{\gamma},2,2}}$, $F\in\mathscr{W}_{s_\ell,\alpha,\boldsymbol{\gamma},2,2}$, is used and [10, Corollary 7] is used for the level $\ell=0$ (see also the proof of [10, Proposition 6]). □

Remark 7. The estimate in Theorem 3 also holds if $f\in(V^*,L^2_{\boldsymbol{\beta}})_{t,\infty}$ and $G(\cdot)\in(V^*,L^2_{\boldsymbol{\beta}})_{t',\infty}$, $t,t'\in[0,1]$, with an error contribution of $M_\ell^{-(t+t')/d}$ and $M_\ell^{-2(t+t')/d}$ in the estimates for Q^{IP} and Q^{RS}, respectively. This follows by Remark 5.

Remark 8. The factor $2/(1-\eta)$ in the weights in (23) and (24) as well as the constant $c>1$ in the sequence $(c(b_j^{1-\theta}\vee\bar{b}_j))_{j\ge1}$ can be omitted. Then, the error estimates in Theorem 3 hold under the same assumptions with QMC convergence rates $1/p-\varepsilon$ and $1/\bar{p}-\varepsilon$ in the mulitlevel error estimates for every sufficiently small $\varepsilon>0$. This can be seen by the same argument that we used to show [10, Corollary 11] (see also [10, Corollary 13]).

6 Error vs. work analysis

The error estimates in Theorem 3 are the key ingredient to calibrate and choose the parameters $(s_\ell)_{\ell\geq 0}$, $(M_\ell)_{\ell\geq 0}$, $\theta\in[0,1)$, and $(N_\ell)_{\ell\geq 0}$ of either considered type of the multilevel QMC algorithm. We seek to derive choices that optimize the work for a given error threshold. The analysis will be demonstrated for a class of multiresolution analyses (MRA for short), which will serve as the function system $(\psi_\lambda)_{\lambda\in\bigtriangledown}$, here indexed by $\lambda\in\bigtriangledown$. We will use notation that is standard for wavelets and MRA. Assume that $(\psi_\lambda)_{\lambda\in\bigtriangledown}$ is a MRA that is obtained by scaling and translation from a finite number of mother wavelets, i.e.,

$$\psi_\lambda(x)=\psi(2^{|\lambda|}x-k),\quad k\in\bigtriangledown_{|\lambda|}, x\in D.$$

The index set $\bigtriangledown_{|\lambda|}$ has cardinality $|\bigtriangledown_{|\lambda|}|=\mathscr{O}(2^{|\lambda|d})$ and $|\mathrm{supp}(\psi_\lambda)|=\mathscr{O}(2^{-|\lambda|d})$. Let $j:\bigtriangledown\to\mathbb{N}$ be a suitable bijective enumeration. We also assume that on every level $|\lambda|$ there is a finite overlap, i.e., there exists *support overlap constant* $K>0$ such that for every $i\in\mathbb{N}_0$ and for every $x\in D$

$$|\{\lambda\in\bigtriangledown:|\lambda|=i,\psi_\lambda(x)\neq 0\}|\leq K\;.$$

The work needed to assemble the stiffness matrix for a generic parameter instance $\boldsymbol{y}\in[-1/2,1/2]^s$ is therefore $\mathscr{O}(M_\ell|j^{-1}(s_\ell)|)=\mathscr{O}(M_\ell\log(s_\ell))$. Assuming at hand a linear complexity solver, the *overall work for either multilevel QMC algorithm* with the number of levels $L\in\mathbb{N}_0$ levels satisfies

$$\text{work}=\mathscr{O}\left(\sum_{\ell=0}^{L}N_\ell M_\ell\log(s_\ell)\right).$$

Note that error vs. work estimates for general function systems have been derived in [23, 7].

The parameter θ in the coupled estimates of Theorem 3 allows to discuss two possible strategies in the choices of the truncation levels $(s_\ell)_{\ell\geq 0}$. We recall from [10, Section 8] that if $\|\psi_{j(\lambda)}\|_{L^\infty(D)}\leq\sigma 2^{-\widehat{\alpha}|\lambda|}$ the sequence

$$b_{j(\lambda)}=\left(1+\frac{\bar{a}_{\min}(1-\kappa)(1-2^{\widehat{\beta}-\widehat{\alpha}})}{\sigma 2K}2^{\widehat{\beta}|\lambda|}\right)^{-1},\quad j\in\mathbb{N},\tag{25}$$

satisfies (**A1**) for $\widehat{\alpha}>\widehat{\beta}>1$ and $b_j\sim j^{-\widehat{\beta}/d}$, $j\geq 1$ holds. The sequence

$$\bar{b}_j = b_j^{(\widehat{\beta}-1)/\widehat{\beta}}, \quad j \in \mathbb{N},$$

satisfies (**A2**) and (**A1**) and $\bar{b}_j \sim j^{-(\widehat{\beta}-1)/d}$, $j \geq 1$ holds. Note that $\|\nabla \psi_{j(\lambda)}\|_{L^\infty(D)} \leq C\sigma 2^{-(\widehat{\alpha}-1)|\lambda|}$ assuming $\|\nabla \psi\|_{L^\infty(D)} \leq C\|\psi\|_{L^\infty(D)}$ for some $C > 0$. The truncation levels $(s_\ell)_{\ell\geq 0}$ are chosen so as to cover entire levels of the MRA expansion of the uncertain PDE input, so that we choose $s_\ell \in \{\sum_{i=0}^{I} |\nabla_i| : I \in \mathbb{N}_0\}$, $\ell \geq 0$. We also assume that

$$M_\ell \sim 2^{d\ell}, \quad \ell \geq 0. \qquad \textbf{(A3)}$$

In this section we assume for simplicity that only one version of the QMC rule is applied with convergence rate $\bar{p}$. We remark that in some cases the application of two different weight sequences with different sparsity may be beneficial. Also we assume that

$$f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty} \quad \text{and} \quad G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}, \quad t,t' \in [0,1], \qquad \textbf{(A4)}$$

which yields a FE convergence rate of $\tau := t + t' \in [0,2]$, cp. Remark 7.

Strategy 1: We equilibrate the decay of the sequences $(b_j^{1-\theta})_{j\geq 1}$ and $(\bar{b}_j)_{j\geq 1}$, which determines the estimate of the QMC error in Theorem 3. The parameter $\theta \in [0,1)$ is chosen to be $\theta = 1/\widehat{\beta}$, which implies $b_j^{1-\theta} = \bar{b}_j$, $j \in \mathbb{N}$, and $(\bar{b}_j)_{j\geq 0} \in \ell^{\bar{p}}(\mathbb{N})$ for every $\bar{p} > d/(\widehat{\beta}-1)$. We equilibrate the error contributions on the highest level L. Since $M_L \sim 2^{dL}$, we choose

$$s_L \sim 2^{d\lceil L\tau/(2\widehat{\beta})\rceil}.$$

On the different levels of the coupled error terms, we either increase the truncation levels or leave it constant, which is reflected in the choice

$$s_\ell \sim \min\left\{2^{d\lceil \ell\tau/(\theta\widehat{\beta})\rceil}, s_L\right\}, \quad \ell = 0, \ldots, L-1.$$

Strategy 2: For certain function systems $(\psi_\lambda)_{\lambda\in\nabla}$ and meshes $\{\mathcal{T}_\ell\}_{\ell\geq 0}$ it may be interesting (also for implementation purposes) to couple their discretizations, i.e., we choose

$$s_\ell \sim M_\ell, \quad \ell = 0, \ldots, L.$$

To equilibrate the truncation and FE error on the levels we choose $\theta = \tau/\widehat{\beta}$, which imposes the constraint $\widehat{\beta} > \tau$ and implies that $b_j^{1-\theta} \sim j^{-(\widehat{\beta}-\tau)/d}$. Hence, $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for every $\bar{p} > d/(\min\{\widehat{\beta}-\tau, \widehat{\beta}-1\})$.

We will discuss interlaced polynomial lattice rules first and follow [7, Section 3.3]. In either of our parameter choices, the error estimate

$$\text{error} = \mathscr{O}\left(M_L^{-\tau/d} + \sum_{\ell=0}^{L} N_\ell^{-1/\bar{p}} M_\ell^{-\tau/d}\right)$$

holds, where we used that $M_\ell = \mathscr{O}(2^{d\ell})$. The QMC sample numbers $(N_\ell)_{\ell=0,\dots,L}$ are chosen to optimize the error versus the required work. Optimizing error (bound) vs. cost as in [23, 7], we search the stationary point of the function

$$g(\xi) = M_L^{-\tau/d} + \sum_{\ell=0}^{L} N_\ell^{-1/\bar{p}} M_\ell^{-\tau/d} + \xi \sum_{\ell=0}^{L} N_\ell M_\ell \log(s_\ell)$$

with respect to N_ℓ, i.e., choose N_ℓ such that $\partial g(\xi)/\partial N_\ell = 0$. We thus obtain

$$N_\ell = \left\lceil N_0 \left(M_\ell^{-1-\tau/d} \log(s_\ell)^{-1} \right)^{\bar{p}/(1+\bar{p})} \right\rceil, \quad \ell = 1,\dots,L, \tag{26}$$

and for $E_\ell := (M_\ell^{1-\bar{p}\tau/d} \log(s_\ell))^{1/(\bar{p}+1)}$,

$$\text{error} = \mathscr{O}\left(M_L^{-\tau/d} + N_0^{-1/\bar{p}} \sum_{\ell=0}^{L} E_\ell \right) \quad \text{and} \quad \text{work} = \mathscr{O}\left(N_0 \sum_{\ell=0}^{L} E_\ell \right).$$

Since for every $0 \neq r_1 \in \mathbb{R}$ and $r_2 > 0$,

$$\sum_{\ell=0}^{L} 2^{r_1 \ell} \ell^{r_2} \leq \frac{2^{r_1(L+1)} - 1}{2^{r_1} - 1} L^{r_2},$$

$\log(s_\ell) = \mathscr{O}(\ell)$, which holds in the considered cases, implies that

$$\sum_{\ell=0}^{L} E_\ell = \begin{cases} \mathscr{O}(1) & \text{if } d < \bar{p}\tau, \\ \mathscr{O}(L^{(\bar{p}+2)/(\bar{p}+1)}) & \text{if } d = \bar{p}\tau, \\ \mathscr{O}(2^{(d-\bar{p}\tau)L/(\bar{p}+1)} L^{1/(\bar{p}+1)}) & \text{if } d > \bar{p}\tau. \end{cases}$$

We choose N_0 to equilibrate the error, i.e.,

$$N_0^{-1/\bar{p}} \sum_{\ell=0}^{L} E_\ell = \mathscr{O}\left(M_L^{-\tau/d} \right),$$

which yields

$$N_0 := \begin{cases} \lceil 2^{\tau\bar{p}L} \rceil & \text{if } d < \bar{p}\tau, \\ \lceil 2^{\tau\bar{p}L} L^{\bar{p}(\bar{p}+2)/(\bar{p}+1)} \rceil & \text{if } d = \bar{p}\tau, \\ \lceil 2^{\bar{p}(d+\tau)L/(\bar{p}+1)} L^{\bar{p}/(\bar{p}+1)} \rceil & \text{if } d > \bar{p}\tau. \end{cases} \tag{27}$$

This implies that an error $= \mathscr{O}(M_L^{-\tau/d})$ requires

$$\text{work} = \begin{cases} \mathscr{O}(2^{\bar{p}\tau L}) & \text{if } d < \bar{p}\tau, \\ \mathscr{O}(2^{\tau\bar{p}L} L^{\bar{p}+2}) & \text{if } d = \bar{p}\tau, \\ \mathscr{O}(2^{dL} L) & \text{if } d > \bar{p}\tau. \end{cases}$$

In the other case of randomly shifted lattice rules sample numbers $(N_\ell)_{\ell=0,\dots,L}$ are derived in [23, Section 3.7]. There, also the work functional from a MRA is considered, cp. [23, Equations (74) and (77)] with $\lambda = \bar{p}/2$ and $K_\ell = M_\ell \log(M_\ell) \sim 2^{d\ell}\ell$. Specifically, for randomly shifted lattice rules we choose

$$N_\ell = \left\lceil N_0 \left(M_\ell^{-1-2\tau/d} \log(s_\ell)^{-1} \right)^{\bar{p}/(2+\bar{p})} \right\rceil, \quad \ell = 1,\dots,L, \tag{28}$$

and

$$N_0 := \begin{cases} \lceil 2^{\tau\bar{p}L} \rceil & \text{if } d < \bar{p}\tau, \\ \lceil 2^{\tau\bar{p}L} L^{\bar{p}(\bar{p}+4)/(2\bar{p}+4)} \rceil & \text{if } d = \bar{p}\tau, \\ \lceil 2^{\bar{p}(d+2\tau)/(\bar{p}+2)L} L^{\bar{p}/(\bar{p}+2)} \rceil & \text{if } d > \bar{p}\tau. \end{cases} \tag{29}$$

The work estimates for these choices in the case of randomly shifted lattice rules are stated on [23, p. 443]. We collect the foregoing estimates in the following theorem.

Theorem 4. *Let the assumption in* (**A3**) *be satisfied and let for* $L \in \mathbb{N}$ *and* $Q_L^{\mathrm{IP}}(\cdot)$*, the sample numbers* $(N_\ell)_{\ell=0,\dots,L}$ *be given by* (26) *and* (27) *and for* Q_L^{RS}*, be given by* (28) *and* (29)*. Let the right hand side* f *and* $G(\cdot)$ *satisfy* (**A4**)*. For* $\bar{p} \in (d/(\widehat{\beta}-1), 1]$*, assuming* $d < \widehat{\beta} - 1$ *and error threshold* $\varepsilon > 0$*,*

$$|\mathbb{E}(G(u)) - Q_L^{\mathrm{IP}}(G(u^L))| = \mathscr{O}(\varepsilon)$$

and

$$\text{work} = \begin{cases} \mathscr{O}(\varepsilon^{-\bar{p}}) & \text{if } d < \bar{p}\tau, \\ \mathscr{O}(\varepsilon^{-\bar{p}} \log(\varepsilon^{-1})^{\bar{p}+2}) & \text{if } d = \bar{p}\tau, \\ \mathscr{O}(\varepsilon^{-d/\tau} \log(\varepsilon^{-1})) & \text{if } d > \bar{p}\tau. \end{cases}$$

For $\bar{p} \in (\max\{1, d/(\widehat{\beta}-1)\}, 2]$ *assuming* $d < 2(\widehat{\beta}-1)$ *and an error threshold* $\varepsilon > 0$*, we obtain*

$$\sqrt{\mathbb{E}^{\Delta}(|\mathbb{E}(G(u)) - Q_L^{\mathrm{RS}}(G(u^L))|^2)} = \mathscr{O}(\varepsilon)$$

and

$$\text{work} = \begin{cases} \mathscr{O}(\varepsilon^{-\bar{p}}) & \text{if } d < \bar{p}\tau, \\ \mathscr{O}(\varepsilon^{-\bar{p}} \log(\varepsilon^{-1})^{\bar{p}/2+2}) & \text{if } d = \bar{p}\tau, \\ \mathscr{O}(\varepsilon^{-d/\tau} \log(\varepsilon^{-1})) & \text{if } d > \bar{p}\tau. \end{cases}$$

Remark 9. The parameter choices for θ and $(s_\ell)_{\ell\geq 0}$ in Theorem 4 reflect *Strategy 1*. For *Strategy 2*, the assumptions $\bar{p} > d/(\min\{\widehat{\beta}-\tau, \widehat{\beta}-1\})$, $\widehat{\beta} > \tau$ are required, which is more restrictive if $\tau > 1$. However, aligning MRA and FE meshes might be useful in certain cases. Note that the truncation dimension in *Strategy 2* could also be capped as in *Strategy 1*, which may be beneficial in some cases. Adopting this strategy would affect the work measure only by a constant factor.

7 Numerical experiments

To illustrate the foregoing asymptotic error bounds, we present numerical experiments in dimensions $d = 1,2$ with parametric diffusion coefficient

$$a(x,\boldsymbol{y}) = \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x),$$

where $\bar{a}(x) \equiv 1$ and we assume $\psi_j = \psi_{j(\ell,k)}$ to be a system of continuous, piecewise (bi)linear spline wavelets for $\ell \geq 0$, $k \in \{0,\dots,2^\ell - 1\}^d$ with support overlap constant $K = 2d$, see e.g. [18, Ch. 12]. We assume in the following the scaling $\|\psi_{j(\ell,k)}\|_{L^\infty(D)} = \sigma 2^{-\widehat{\alpha}\ell}$. We pursue Strategy 2 from Section 6, which yields for $\widehat{\alpha} > \widehat{\beta} > \tau$ a QMC weight sequence of the form

$$b_{j(\ell,k)} = \left(1 + c_2 2^{\widehat{\beta}\ell}\right)^{-(\widehat{\beta}-\tau)/\widehat{\beta}},$$

for $0 < c_2 \in \mathbb{R}$ as specified in (25) (see also Remark 8). We use the implementation from [9] for applying the single-level and multilevel methods in parallel, and use the Walsh coefficient bound $C = 0.1$ in the component-by-component (CBC) construction, cp. [11] for details. For the multilevel method, we choose $N_\ell = 2^{m_\ell}$, where m_ℓ follows from (26). The resulting expression is given by

$$m_\ell = \left\lceil \bar{p}\tau L + \frac{\bar{p}(\bar{p}+2)}{\bar{p}+1}\log_2(L+1) + \frac{\bar{p}}{\bar{p}+1}\big(-\ell(d+\tau) + \log_2(s_\ell)\big)\right\rceil, \qquad (30)$$

with $m_\ell = 1$ if the expression is not positive. In the following examples, we consider the limiting case $d = \bar{p}\tau$ also with the limiting value $\bar{p}^{-1} = (\widehat{\beta} - \tau)/d$. This choice is based on the cost model

$$W_L^{\mathrm{ML}} = \sum_{\ell=0}^{L} N_\ell M_\ell \log_2(s_\ell),$$

which we use for computing the cost in the multilevel experiment below. We compare the multilevel computations to a single-level approach, where we equilibrate the QMC and FEM discretization errors, yielding on a fixed level L with $N_L^{-1/\bar{p}} \sim M_L^{-\tau/d}$ the choice $N_L = 2^{\bar{p}\tau(L+1)}$,

$$m_L = L + 1.$$

In the single-level case, the work is simply $W_L^{\mathrm{SL}} = N_L M_L \log_2(s_L)$.

7.1 Univariate model problem

We consider the domain $D = (0,1)$ and homogeneous Dirichlet boundary conditions, i.e., $\Gamma_1 = \partial D$, with right hand side $f(x) = 10x$, $x \in D$. As goal functional, we consider point evaluation of the solution at $\bar{x} = e^{-1}$ (which is not a node on any mesh used in our simulations), $G(u(\cdot,\boldsymbol{y})) = u(\bar{x},\boldsymbol{y})$, which implies the FE convergence rate $\tau = 1.5-\varepsilon$ for arbitrary $\varepsilon > 0$. The parameter calibration will be done under the formal case $\tau = 1.5$. For a given discretization level ℓ, we solve the parametric PDE (9) with the finite element method using piecewise linear basis functions on an equidistant mesh with meshwidth $h_\ell = 2^{-\ell-1}$ to approximate the solution of (1). Considering the wavelet basis for the coefficients on the same mesh, we obtain $s_\ell = h_\ell^{-1} - 1 = 2^{\ell+1} - 1$ parametric dimensions on level ℓ. We choose $\widehat{\alpha} = 3$, $\widehat{\beta} = 2.99$, $\sigma = 0.15$, yielding the expected QMC convergence rate $\widehat{\beta} - \tau = 1.49$. We use the same generating vectors as above for the single-level method; this is justified since the weight sequence used in the CBC construction majorizes the weight sequence for the single level quadrature, theoretically capping the rate at $N^{-1.5}$. With these generating vectors, as observed in Figure 2, the measured QMC convergence rate is independent of the parameter dimension, and equals $N^{-\alpha}$ for $\alpha = 2,3$ rather than the expected rate $N^{-1.5}$.

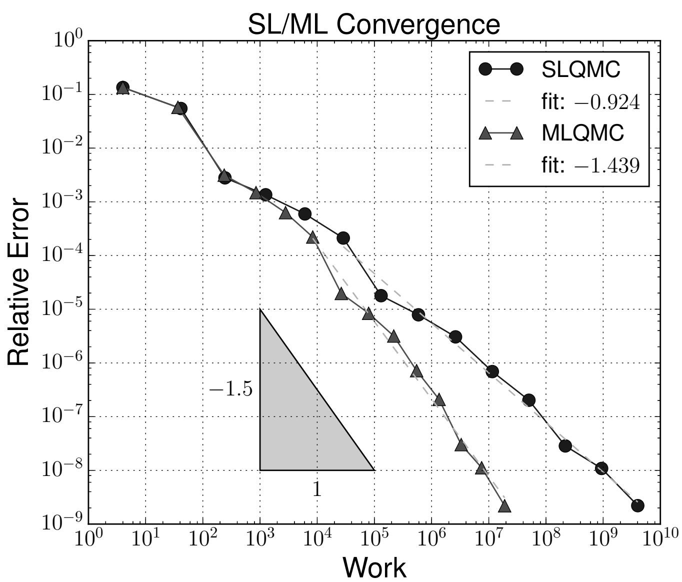

Fig. 1 Convergence of single-level and multilevel methods for a diffusion coefficient given in wavelet representation. As a reference solution, the multilevel approximation on the level $L = 14$ with a total of $s_L = 32767$ dimensions was used. The measured rates were obtained by a linear least squares fit on the last 9 points. The expected rates are 0.75 for SLQMC and 1.5 for MLQMC ignoring log factors. The work is $W_L^{\mathrm{ML}} = \sum_{\ell=0}^{L} N_\ell h_\ell^{-1}(1+\log_2(s_\ell))$ for multilevel and $W_L^{\mathrm{SL}} = N_L h_L^{-1}(1+\log_2(s_L))$ for single-level.

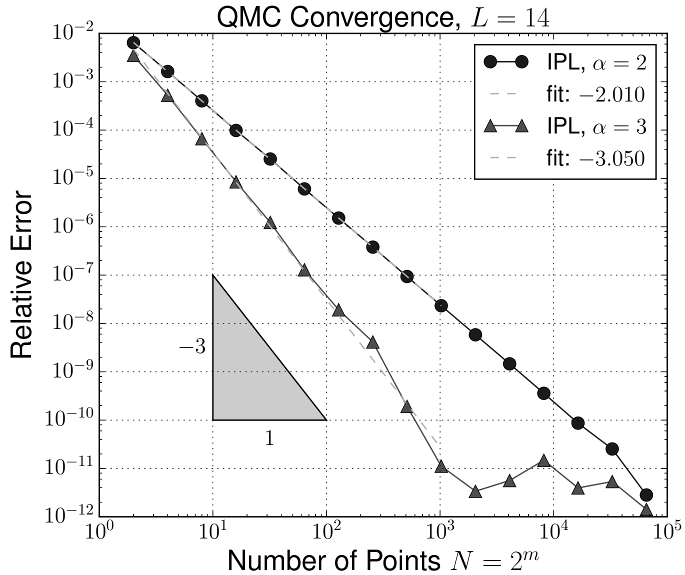

Fig. 2 Convergence of the QMC approximation using interlaced polynomial lattice (IPL) rules with $N = 2^m$ points, $m = 1,\dots,17$ and for digit interlacing factors $\alpha = 2,3$. We use the results with $m = 17$ as the reference value and keep the maximal discretization level $L = 14$ fixed, resulting in $s_L = 2^{15} - 1 = 32767$ parameter dimensions and smallest FEM meshwidth $h_L = 2^{-15}$.

7.2 Two spatial dimensions

For $d = 2$, we consider the domain $D = (-1,1)\times(0,1)$ with mixed boundary conditions. Specifically, the Neumann boundary is given by $\Gamma_2 = (-1,0)\times\{0\}$ anda the Dirichlet boundary is $\Gamma_1 = \partial D\backslash\Gamma_2$. Although the domain is convex, the change in boundary conditions at the origin induces a corner singularity in the parametric solutions corresponding to interior angle equal to π. Due to isotropy of the parametric diffusion coefficient, this leads to a non-$H^2(D)$ singularity of (y-independent) strength $O(\sqrt{r})$ of the parametric solution $u(\cdot,\boldsymbol{y})$ concentrated at the origin. The boundary conditions change also at the corner $(0,-1)^\top \in \partial D$, inducing weaker singularities there as well. The considered goal functional is here integration over the domain D, which is an element of $L^2(D)$. Since the parametric coefficients $a(x,\boldsymbol{y})$ are isotropic, i.e., scalar valued, the full regularity shift of the Laplacean in weighted Hilbert spaces is applicable as detailed in Section 2, we obtain $\tau = 2$. Analogous to the univariate problem considered in the previous subsection, we use continuous, bilinear FE on quadrilaterals on sequences of nested, locally refined meshes of the domain D which were obtained by a suitable bisection refinement, cp. [12].

Here, we have $J = 5$ singular points or corners and $\boldsymbol{\beta} \in [0,1)^J$ satisfies that $\beta_i > 1 - \pi/\omega_i$, $i = 1,2,3$, and $\beta_i > 1 - \pi/(2\omega_i)$, $i = 4,5$, cp. Section 2. Then, for the Laplacean with mixed boundary conditions in D there holds a full regularity shift in weighted Sobolev spaces, i.e. $(-\Delta)^{-1} : L^2_{\boldsymbol{\beta}}(D) \to H^1_0(D) \cap H^2_{\boldsymbol{\beta}}(D)$ is bounded with $\boldsymbol{\beta} = (0,0,0,\beta_4,\beta_5)$, $1 > \beta_4 > 0$, and $1 > \beta_5 > 1/2$, where singular points are enumerated counter clockwise, i.e., $c_1 = (1,0)^\top$, $c_2 = (1,1)^\top$, $c_3 = (-1,1)^\top$, $c_4 = (0,-1)^\top$, and $c_5 = (0,0)^\top$. We observe that solutions will in general have a weak non-$H^2(D)$ singularity at the corner c_4, i.e., $u(x,\boldsymbol{y}) \in H^{2-\varepsilon}(D_4)$ for every $\varepsilon > 0$, where $D_4 \subset D$ is a sufficiently small neighborhood of c_4. We use the values $\beta_1 = \beta_2 = \beta_3 = 0$, $\beta_4 = 0.05$, and $\beta_5 = 0.55$ as inputs for a bisection refinement algorithm, which results in 1-irregular quadrilateral meshes. In polar coordinates $(r,\phi) \in (0,\infty) \times (0,\pi)$, where $x = r(\cos(\phi),\sin(\phi)^\top$, the function $\bar{u}(r,\phi) = \sqrt{r}\sin(\phi/2)$ is harmonic, i.e., $\Delta\bar{u} = 0$, and satisfies the homogeneous Neumann boundary conditions. We solve the parametric boundary value problem

$$-\nabla \cdot (a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y})) = 0, \quad u(x,\boldsymbol{y})\Big|_{\Gamma_1} = \bar{u}(x)\Big|_{\Gamma_1}, \quad a(x,\boldsymbol{y})\nabla u(x,\boldsymbol{y}) \cdot n(x)\Big|_{\Gamma_2} = 0.$$

Clearly, $u(x,\mathbf{0}) = \bar{u}(x)$. The inhomogeneous Dirichlet boundary terms can be incorporated into the right hand side, for example by solving $-\nabla \cdot (a(\nabla u - \bar{u})) = \nabla \cdot (a\nabla\bar{u})$ and adding $\bar{u}$ to the solution afterwards. Instead of $\bar{u}$ one may use any other suitable extension of $\bar{u}|_{\partial D}$ to the domain D. The difference $u - \bar{u}$ satisfies the homogeneous mixed boundary conditions. The parametric right hand side is given by $f(x,\boldsymbol{y}) := \nabla \cdot (a(x,\boldsymbol{y})\nabla\bar{u}(x)) \in L^2_{\boldsymbol{\beta}}(D)$ for $\boldsymbol{\beta}$ stated above. This right hand side $f(x,\boldsymbol{y})$ depends linearly on the parameter vector $\boldsymbol{y}$. In previous sections, we assumed a fixed right hand side only for simplicity and conciseness of the presentation. A right hand side, which only depends linearly on the coefficient $a(x,\boldsymbol{y})$ is under the made assumptions a simple, admissible extension. The implementation of the spatial discretization in two space dimensions of bilinear FE uses `deal.II`, cp. [1].

For the uncertain diffusion coefficient, we consider the parametrization obtained by tensorizing the univariate continuous, piecewise linear biorthogonal spline wavelets. Specifically, we choose

$$\widehat{\psi}_{\ell,k_1,k_2}(x_1,x_2) = \sigma 2^{-\widehat{\alpha}\ell}\psi_{\ell,k_1}(x_1)\psi_{\ell,k_2}(x_2), \quad k_1,k_2 \in \{0,\ldots,2^\ell - 1\}, \tag{31}$$

where $\psi_{\ell,k}(x)$ denotes the univariate continuous, piecewise linear wavelet function with scaling $\|\psi_{\ell,k}\|_{L^\infty(D)} = 1$ and $\sigma = 0.01$. Thus, $\|\widehat{\psi}_{\ell,k_1,k_2}\|_{L^\infty(D)} = \sigma 2^{-\widehat{\alpha}\ell}$ with $\widehat{\alpha} = 4$. This choice of parametrization results in $s_L = \sum_{\ell=0}^{L} 4^\ell = (4^{L+1} - 1)/3$ dimensions on level L. The generating vectors were constructed by the CBC algorithm based on a QMC weight sequence analogous to the univariate case, given here by $b_{j(\ell,k_1,k_2)} = \left(1 + c_2 2^{\widehat{\beta}\ell}\right)^{-(\widehat{\beta}-\tau)/\widehat{\beta}}$ where $\widehat{\beta} = 3.99$ and $\tau = 2$.

For the multilevel method, the number of samples per level is given by $N_\ell = 2^{m_\ell}$ where the exponent m_ℓ is given as in (30) with $d = 2$. To compare to a single-level approach, we equilibrate the finite element and QMC sampling error to obtain $N_L = 2^{L\tau/r} \sim M_L^{\tau/(dr)}$, where r is the QMC convergence rate, here $r \approx 2$ for interlacing factor $\alpha = 2$ and we take $r = 2$ to obtain the value of N_L.

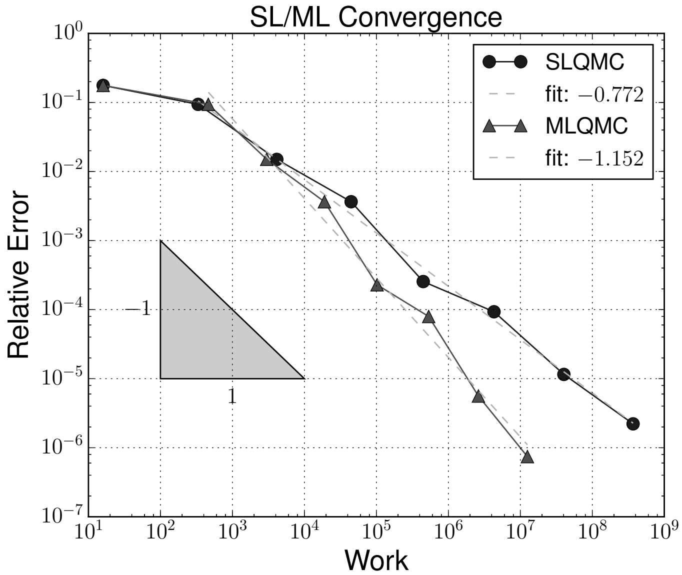

Fig. 3 Convergence of single-level and multilevel methods for a 2d diffusion equation with parametric coefficient given in wavelet representation. Continuous, piecewise bilinear biorthogonal spline wavelets (31) on uniform partitions of the domain D with meshwidth $O(2^{-\ell})$, $\ell = 0,\dots,L$, were used. As a reference solution, the multilevel approximation on the level $L = 8$ with a total of $s_L = 87381$ dimensions was used. The measured rates were obtained by a linear least squares fit on all points but the first and the two last ones. The rates expected from the theory for this problem are 0.67 for SLQMC and 1 for MLQMC ignoring log factors. The work is $W_L^{\mathrm{ML}} = \sum_{\ell=0}^{L} N_\ell 2^{2\ell}(1+\log_2(s_\ell))$ for multilevel and $W_L^{\mathrm{SL}} = N_L 2^{2L}(1+\log_2(s_L))$ for single-level.

8 Conclusions

We provided the convergence rate analysis of randomly shifted and high order, interlaced polynomial lattice rules for the numerical evaluation of linear functionals G of solutions of countably affine-parametric, linear second order elliptic partial differential equations. The spatially inhomogeneous diffusion coefficient was assumed to be represented by a multiresolution analysis (MRA) with local supports, rather than the globally supported Karhunen-Loève expansion considered, for example, in [22, 7, 6, 14] and the references there. As in the corresponding single level QMC Petrov Galerkin approaches considered in [10], we proved that QMC with product weights, originally proposed by I. H. Sloan and H. Woźniakowski in [30], can provide optimal QMC convergence rates which are independent of the parameter dimension, unlike the so-called *product and order dependent (POD) weights* which are mandated by globally supported representation systems of uncertain input data. This, in turn, results in linear w.r.t. to dimension scaling of fast CBC constructions from [25, 24], which originate in a dimension-wise, greedy strategy to minimize the worst case error, as proposed orginally in [29]. The present analysis addressed linear, affine parametric random input data where the supports of the parameters are bounded. The extension for log-Gaussian diffusion coefficients in the present setting, along the lines of [14, 19] (where the case of globally supported ψ_j were treated) and in the setting of the single level analysis in [15], is given in [16]. Numerical experiments were given for a model, linear elliptic problem in one and two space dimensions with local spatial mesh refinement. The present mathematical analysis holds, however, also for PDEs on polyhedra in three space dimensions. We refer to [16]. Analogous error bounds for product weight QMC also hold for log-Gaussian representations of uncertain PDE inputs. Details are presented in [16, 17].

Acknowledgements This work was supported in part by the Swiss National Science Foundation (SNSF) under grant SNF 159940 and SNF 149819. The authors thank Fabian Keller for his contribution to the implementation in two space dimensions and Fabian Müller for his help with the bisection meshes.

References

1. Arndt, D., Bangerth, W., Davydov, D., Heister, T., Heltai, L., Kronbichler, M., Maier, M., Pelteret, J.P., Turcksin, B., Wells, D.: The deal.II library, version 8.5. Journal of Numerical Mathematics (2017). DOI 10.1515/jnma-2016-1045
2. Babuška, I., Kellogg, R.B., Pitkäranta, J.: Direct and inverse error estimates for finite elements with mesh refinements. Numer. Math. **33**(4), 447–471 (1979). DOI 10.1007/BF01399326
3. Bachmayr, M., Cohen, A., Migliorati, G.: Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients. ESAIM: M2AN **51**(1), 321–339 (2017). DOI 10.1051/m2an/2016045
4. Chen, P., Schwab, Ch.: Model order reduction methods in computational uncertainty quantification. pp. 1–53. Springer International Publishing (2016). DOI 10.1007/978-3-319-11259-6_70-1
5. Dashti, M., Stuart, A.: The Bayesian approach to inverse problems. pp. 1–118. Springer International Publishing (2016). DOI 10.1007/978-3-319-11259-6_7-1
6. Dick, J., Kuo, F.Y., Le Gia, Q.T., Nuyens, D., Schwab, Ch.: Higher order QMC Petrov-Galerkin discretization for affine parametric operator equations with random field inputs. SIAM J. Numer. Anal. **52**(6), 2676–2702 (2014). DOI 10.1137/130943984
7. Dick, J., Kuo, F.Y., Le Gia, Q.T., Schwab, Ch.: Multilevel higher order QMC Petrov-Galerkin discretization for affine parametric operator equations. SIAM J. Numer. Anal. **54**(4), 2541–2568 (2016). DOI 10.1137/16M1078690
8. Dick, J., Kuo, F.Y., Sloan, I.H.: High-dimensional integration: the quasi-Monte Carlo way. Acta Numer. **22**, 133–288 (2013)
9. Gantner, R.N.: A generic C++ library for multilevel quasi-Monte Carlo. In: Proceedings of the Platform for Advanced Scientific Computing Conference, PASC '16, pp. 11:1–11:12. ACM, New York, NY, USA (2016). DOI 10.1145/2929908.2929915
10. Gantner, R.N., Herrmann, L., Schwab, Ch.: Quasi-Monte Carlo integration for affine-parametric, elliptic PDEs: local supports and product weights. Tech. Rep. 2016-32 (revised), Seminar for Applied Mathematics, ETH Zürich, Switzerland (2016). URL https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2016/2016-32_rev2.pdf
11. Gantner, R.N., Schwab, Ch.: Computational higher order quasi-Monte Carlo integration. In: Monte Carlo and Quasi-Monte Carlo Methods: MCQMC, Leuven, Belgium, April 2014, vol. 163, pp. 271–288 (2016). DOI 10.1007/978-3-319-33507-0_12

12. Gaspoz, F.D., Morin, P.: Convergence rates for adaptive finite elements. IMA J. Numer. Anal. **29**(4), 917–936 (2009). DOI 10.1093/imanum/drn039
13. Giles, M.B.: Multilevel Monte Carlo methods. Acta Numer. **24**, 259–328 (2015). DOI 10.1017/S096249291500001X
14. Graham, I.G., Kuo, F.Y., Nichols, J.A., Scheichl, R., Schwab, C., Sloan, I.H.: Quasi-Monte Carlo finite element methods for elliptic pdes with lognormal random coefficients. Numerische Mathematik **131**(2), 329–368 (2015). DOI 10.1007/s00211-014-0689-y
15. Herrmann, L., Schwab, C.: QMC integration for lognormal-parametric, elliptic PDEs: local supports imply product weights. Tech. Rep. 2016-39, Seminar for Applied Mathematics, ETH Zürich (2016). URL https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2016/2016-39.pdf
16. Herrmann, L., Schwab, C.: Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients. Tech. Rep. 2017-19, Seminar for Applied Mathematics, ETH Zürich, Switzerland (2017). URL https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2017/2017-19.pdf
17. Herrmann, L., Schwab, C.: QMC algorithms with product weights for lognormal-parametric, elliptic PDEs. Tech. Rep. 2017-04, Seminar for Applied Mathematics, ETH Zürich, Switzerland (2017). URL https://www.sam.math.ethz.ch/sam_reports/reports_final/reports2017/2017-04.pdf
18. Hilber, N., Reichmann, O., Schwab, C., Winter, C.: Computational methods for quantitative finance. Springer Finance. Springer, Heidelberg (2013). DOI 10.1007/978-3-642-35401-4. Finite element methods for derivative pricing
19. Kuo, F., Scheichl, R., Schwab, C., Sloan, I., Ullmann, E.: Multilevel quasi-Monte Carlo methods for lognormal diffusion problems. Math. Comp. **86**(308), 2827–2860 (2017). DOI 10.1090/mcom/3207
20. Kuo, F.Y., Nuyens, D.: Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients: a survey of analysis and implementation. Found. Comput. Math. **16**(6), 1631–1696 (2016). DOI 10.1007/s10208-016-9329-5
21. Kuo, F.Y., Schwab, Ch., Sloan, I.H.: Quasi-Monte Carlo methods for high-dimensional integration: the standard (weighted Hilbert space) setting and beyond. ANZIAM J. **53**(1), 1–37 (2011). DOI 10.1017/S1446181112000077
22. Kuo, F.Y., Schwab, Ch., Sloan, I.H.: Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM J. Numer. Anal. **50**(6), 3351–3374 (2012). DOI 10.1137/110845537
23. Kuo, F.Y., Schwab, Ch., Sloan, I.H.: Multi-level quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients. Journ. Found. Comp. Math. **15**(2), 411–449 (2015)
24. Nuyens, D., Cools, R.: Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. Math. Comp. **75**(254), 903–920 (electronic) (2006). DOI 10.1090/S0025-5718-06-01785-6
25. Nuyens, D., Cools, R.: Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points. J. Complexity **22**(1), 4–28 (2006). DOI 10.1016/j.jco.2005.07.002
26. Schwab, Ch., Gittelson, C.J.: Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs. Acta Numer. **20**, 291–467 (2011). DOI 10.1017/S0962492911000055
27. Sloan, I.H., Joe, S.: Lattice methods for multiple integration. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1994)
28. Sloan, I.H., Kuo, F.Y., Joe, S.: Constructing randomly shifted lattice rules in weighted Sobolev spaces. SIAM J. Numer. Anal. **40**(5), 1650–1665 (2002). DOI 10.1137/S0036142901393942
29. Sloan, I.H., Kuo, F.Y., Joe, S.: On the step-by-step construction of quasi-Monte Carlo integration rules that achieve strong tractability error bounds in weighted Sobolev spaces. Math. Comp. **71**(240), 1609–1640 (2002). DOI 10.1090/S0025-5718-02-01420-5
30. Sloan, I.H., Woźniakowski, H.: When are quasi-Monte Carlo algorithms efficient for high-dimensional integrals? J. Complexity **14**(1), 1–33 (1998). DOI 10.1006/jcom.1997.0463
31. Triebel, H.: Interpolation theory, function spaces, differential operators, second edn. Johann Ambrosius Barth, Heidelberg (1995)