Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

On the differentiability of solutions of stochastic evolution equations with respect to their initial values

A. Andersson and A. Jentzen and R. Kurniawan and T. Welti

Research Report No. 2016-47
November 2016

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 200021_156603
Funding: ETH Research Grant ETH-47 15-2

On the differentiability of solutions of stochastic evolution equations with respect to their initial values

Adam Andersson, Arnulf Jentzen, Ryan Kurniawan, and Timo Welti

November 3, 2016

Abstract

In this article we study the differentiability of solutions of parabolic semilinear stochastic evolution equations (SEEs) with respect to their initial values. We prove that if the nonlinear drift coefficients and the nonlinear diffusion coefficients of the considered SEEs are n-times continuously Fréchet differentiable, then the solutions of the considered SEEs are also n-times continuously Fréchet differentiable with respect to their initial values. Moreover, a key contribution of this work is to establish suitable enhanced regularity properties of the derivative processes of the considered SEE in the sense that the dominating linear operator appearing in the SEE smoothes the higher order derivative processes.

1 Introduction

In this article we study the differentiability of solutions of parabolic semilinear stochastic evolution equations (SEEs) with respect to their initial values. (Semilinear) SEEs have been extensively studied in the last decades by means of several different approaches; see, e.g., the monographs by Rozovskiĭ [21], Prévôt & Röckner [19], and Liu & Röckner [18] for results on SEEs in the context of the so-called "variational approach" for SEEs, see, e.g., Da Prato & Zabczyk [8] for results on semilinear SEEs in the context of the so-called "semigroup approach" for SEEs, and see, e.g., Walsh [24] for results on semilinear SEEs in the context of the so-called "martingale measure approach". In this paper we employ the semigroup approach to establish differentiability of solutions of parabolic semilinear SEEs with respect to their initial values. More precisely, we prove that the smoothness of the coefficients of the considered SEEs transfers to the smoothness of the solutions of the SEEs with respect to their initial values. We demonstrate that if the nonlinear drift coefficients and the nonlinear diffusion coefficients of the considered SEEs are n-times continuously Fréchet differentiable, then the solutions of the considered SEEs are also n-times continuously Fréchet differentiable with respect to their initial values. In addition, a key contribution of this work is to establish suitable enhanced regularity properties of the derivative processes of the considered SEE in the sense that the dominating linear operator appearing in the SEE smoothes the higher order derivative processes (see (3)–(6) below). In the following theorem we summarize some of the key findings of this article.

Theorem 1.1. *Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ be non-trivial separable $\mathbb{R}$-Hilbert spaces, let $n \in \mathbb{N} = \{1,2,\ldots\}$, $T \in (0,\infty)$, $\eta \in \mathbb{R}$, let $F\colon H \to H$ and $B\colon H \to HS(U,H)$ be n-times continuously Fréchet differentiable functions with globally bounded derivatives, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $A\colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with* $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C}\colon \operatorname{Re}(z) < \eta\}$, *let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$ (cf., e.g., [22, Section 3.7]), and for every $\mathcal{F}/\mathcal{B}(H)$-measurable function $X\colon \Omega \to H$ let $[\![X]\!]$ be the set given by $[\![X]\!] = \big\{Y\colon \Omega \to H\colon (Y$ is $\mathcal{F}/\mathcal{B}(H)$-measurable and $\mathbb{P}(X = Y) = 1)\big\}$. Then*

(i) there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{0,x}\colon [0,T]\times\Omega \to H$, $x \in H$, which fulfill for all $p \in [2,\infty)$, $x \in H$, $t \in [0,T]$ that $\int_0^t \|e^{(t-s)A}F(X_s^{0,x})\|_H + \|e^{(t-s)A}B(X_s^{0,x})\|^2_{HS(U,H)}\,ds < \infty$, $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s^{0,x}\|_H^p\big] < \infty$, and

$$[\![X_t^{0,x}]\!] = \left[\!\!\left[e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^{0,x})\,ds \right]\!\!\right] + \int_0^t e^{(t-s)A}B(X_s^{0,x})\,dW_s, \tag{1}$$

(ii) it holds for all $p \in [2,\infty)$, $t \in [0,T]$ that $H \ni x \mapsto [\![X_t^{0,x}]\!] \in L^p(\mathbb{P};H)$ is n-times continuously Fréchet differentiable with globally bounded derivatives,

(iii) there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{k,\mathbf{u}}\colon [0,T]\times\Omega \to H$, $\mathbf{u} \in H^{k+1}$, $k \in \{1,2,\ldots,n\}$, which fulfill for all $p \in [2,\infty)$, $k \in \{1,2,\ldots,n\}$, $x, u_1, u_2, \ldots, u_k \in H$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s^{k,(x,u_1,u_2,\ldots,u_k)}\|_H^p\big] < \infty$ and

$$\big(\tfrac{d^k}{dx^k}[\![X_t^{0,x}]\!]\big)(u_1,u_2,\ldots,u_k) = [\![X_t^{k,(x,u_1,u_2,\ldots,u_k)}]\!], \tag{2}$$

(iv) it holds for all $p \in (0,\infty)$, $k \in \{1,2,\ldots,n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, 1/2)$ with $\sum_{i=1}^k \delta_i < 1/2$ that

$$\sup_{\mathbf{u}=(u_0,u_1,\ldots,u_k)\in H\times(H\setminus\{0\})^k}\ \sup_{t\in(0,T]} \left[\frac{t^{(\sum_{i=1}^k \delta_i)-1/2\,\mathbb{1}_{[2,\infty)}(k)}\,\|X_t^{k,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] < \infty, \tag{3}$$

and

(v) it holds for all $p \in (0,\infty)$, $k \in \{1,2,\ldots,n\}$, $\delta_1, \delta_2, \ldots, \delta_k \in [0, 1/2)$ with $\sum_{i=1}^k \delta_i < 1/2$, $|F|_{\operatorname{Lip}^k(H,H)} < \infty$, and $|B|_{\operatorname{Lip}^k(H,HS(U,H))} < \infty$ that

$$\sup_{\substack{x,y\in H,\\ x\neq y}}\ \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k}\ \sup_{t\in(0,T]} \left[\frac{t^{(\sum_{i=1}^k \delta_i)-1/2}\,\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] < \infty. \tag{4}$$

In Theorem 1.1 we denote for non-trivial $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, a natural number $k \in \mathbb{N}$, and a k-times continuously differentiable function $f\colon V \to W$ by $|f|_{\operatorname{Lip}^k(V,W)}$ the k-Lipschitz semi-norm associated to f (see (8) in Subsection 1.1 below for details). Theorem 1.1 is an immediate consequence of items (i), (ii), (iv), (ix), and (x) of Theorem 2.1 below. In Theorem 2.1 below we also specify explicitly for every natural number $k \in \mathbb{N}$ the SEEs which the k-th derivative processes in (2) above are solutions of (see item (i) of Theorem 2.1 below for details). Moreover, Theorem 2.1 below provides explicit bounds for the left hand sides of (3) and (4) (see items (ii) and (iv) of Theorem 2.1 below) in a more general framework than in Theorem 1.1 above and establishes several further regularity properties for the derivative processes in item (iii) of Theorem 1.1. Next we would like to emphasize that Theorem 1.1 and Theorem 2.1, respectively, prove finiteness of (3) and (4) even though the denominators in (3) and (4) contain rather weak norms from negative Sobolev-type spaces for the multilinear arguments of the derivative processes. In particular, item (iv) of Theorem 1.1 and item (ii) of Theorem 2.1 below, respectively, reveal for every $p \in [1, \infty)$, $k \in \{1, 2, \dots, n\}$, $\delta_1, \delta_2, \dots, \delta_k \in [0, 1/2)$, $x \in H$ that the derivative processes $\big(H^k \ni (u_1, u_2, \dots, u_k) \mapsto [\![X_t^{k,(x,u_1,u_2,\dots,u_k)}]\!] \in L^p(\mathbb{P}; H)\big) \in L(H^{\otimes k}, L^p(\mathbb{P}; H))$, $t \in (0, T]$, even take values in the continuously embedded subspace

$$L(\otimes_{i=1}^k H_{-\delta_i}, L^p(\mathbb{P}; H)) \tag{5}$$

of $L(H^{\otimes k}, L^p(\mathbb{P}; H))$ provided that the hypothesis

$$\textstyle\sum_{i=1}^k \delta_i < 1/2 \tag{6}$$

is satisfied. Items (iv)–(v) of Theorem 1.1 and items (ii) and (iv) of Theorem 2.1 below, respectively, are of major importance for establishing essentially sharp probabilistically *weak convergence rates* for numerical approximation processes as the analytically weak norms for the multilinear arguments of the derivative processes (see the denominators in (3) and (4) above) translate in analytically weak norms for the approximation errors in the probabilistically weak error analysis which, in turn, result in essentially sharp probabilistically weak convergence rates for the numerical approximation processes (cf., e.g., Theorem 2.2 in Debussche [10], Theorem 2.1 in Wang & Gan [26], Theorem 1.1 in Andersson & Larsson [2], Theorem 1.1 in Bréhier [3], Theorem 5.1 in Bréhier & Kopec [4], Corollary 1 in Wang [25], Corollary 5.2 in Conus et al. [7], Theorem 6.1 in Kopec [17], and Corollary 8.2 in [14]). In the following we briefly relate items (i)–(v) of Theorem 1.1 and Theorem 2.1 below with results from the literature. Item (i) of Theorem 1.1 is well-known and can, e.g., be found in Theorem 7.4 in Da Prato & Zabczyk [8] (cf., e.g., Theorem 4.3 in Brzeźniak [5], Theorem 7.3.5 in Da Prato & Zabczyk [9], Theorem 6.2 in Van Neerven et al. [23], and Theorem 6.2.3 in Liu & Röckner [18]). Items (ii)–(iii) of Theorem 1.1 and items (i), (vii), and (viii) of Theorem 2.1 below are generalizations and enhancements of Theorem 7.3.6 in Da Prato & Zabczyk [9]. In particular, we allow F and B to grow linearly (cf. (7) in Subsection 1.1 below), we prove continuous Fréchet differentiability (cf. item (ii) of Theorem 1.1), and we develop the combinatorics (cf., e.g., Theorem 2 in Clark & Houssineau [6]) to explicitly specify the SEEs to which the derivative processes of any order are solutions of (cf. item (i) of Theorem 2.1 below). Nonetheless, the main contribution of this paper is to establish that the derivative processes even take values in the space (5) provided that the assumption (6) is fulfilled.

1.1 Notation

In this section we introduce some of the notation which we employ throughout this article (cf., e.g., Section 1.1 in [1]). For two measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of $\mathcal{A}/\mathcal{B}$-measurable functions. For a set A we denote by $\mathcal{P}(A)$ the power set of A and we denote by

$\#_A \in \mathbb{N}_0 \cup \{\infty\}$ the number of elements of A. For an $\mathbb{R}$-vector space V we denote by $V^{[k]} \subseteq V$, $k \in \mathbb{N}_0$, the sets which satisfy for all $k \in \mathbb{N}$ that $V^{[0]} = V$ and $V^{[k]} = V \backslash \{0\}$. For a real number $T \in (0, \infty)$, a set Ω, and a family $(\mathcal{F}_t)_{t\in[0,T]} \subseteq \mathcal{P}(\mathcal{P}(\Omega))$ of sigma-algebras on Ω we denote by $\operatorname{Pred}((\mathcal{F}_t)_{t\in[0,T]})$ the sigma-algebra given by $\operatorname{Pred}((\mathcal{F}_t)_{t\in[0,T]}) = \sigma_{[0,T]\times\Omega}\big(\big\{(s,t] \times A \colon s \in [0,T), t \in (s,T], A \in \mathcal{F}_s\big\} \cup \big\{\{0\} \times A \colon A \in \mathcal{F}_0\big\}\big)$ (the predictable sigma-algebra associated to $(\mathcal{F}_t)_{t\in[0,T]}$). For $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $\#_V > 1$ and a natural number $n \in \mathbb{N}$ we denote by $|\cdot|_{\mathcal{C}^n_{\mathrm{b}}(V,W)} \colon \mathcal{C}^n(V,W) \to [0,\infty]$ and $\|\cdot\|_{\mathcal{C}^n_{\mathrm{b}}(V,W)} \colon \mathcal{C}^n(V,W) \to [0,\infty]$ the functions which satisfy for all $f \in \mathcal{C}^n(V,W)$ that

$$|f|_{\mathcal{C}^n_{\mathrm{b}}(V,W)} = \sup_{x\in V} \big\|f^{(n)}(x)\big\|_{L^{(n)}(V,W)}, \qquad \|f\|_{\mathcal{C}^n_{\mathrm{b}}(V,W)} = \|f(0)\|_W + \sum_{k=1}^{n} |f|_{\mathcal{C}^k_{\mathrm{b}}(V,W)} \tag{7}$$

and we denote by $\mathcal{C}^n_{\mathrm{b}}(V,W)$ the set given by $\mathcal{C}^n_{\mathrm{b}}(V,W) = \{f \in \mathcal{C}^n(V,W) \colon \|f\|_{\mathcal{C}^n_{\mathrm{b}}(V,W)} < \infty\}$. For $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $\#_V > 1$ and a nonnegative integer $n \in \mathbb{N}_0$ we denote by $|\cdot|_{\operatorname{Lip}^n(V,W)} \colon \mathcal{C}^n(V,W) \to [0,\infty]$ and $\|\cdot\|_{\operatorname{Lip}^n(V,W)} \colon \mathcal{C}^n(V,W) \to [0,\infty]$ the functions which satisfy for all $f \in \mathcal{C}^n(V,W)$ that

$$\begin{aligned} |f|_{\operatorname{Lip}^n(V,W)} &= \begin{cases} \sup_{x,y\in V,\, x\neq y} \left(\frac{\|f(x)-f(y)\|_W}{\|x-y\|_V}\right) & \colon n = 0 \\ \sup_{x,y\in V,\, x\neq y} \left(\frac{\|f^{(n)}(x)-f^{(n)}(y)\|_{L^{(n)}(V,W)}}{\|x-y\|_V}\right) & \colon n \in \mathbb{N} \end{cases}, \\ \|f\|_{\operatorname{Lip}^n(V,W)} &= \|f(0)\|_W + \sum_{k=0}^{n} |f|_{\operatorname{Lip}^k(V,W)} \end{aligned} \tag{8}$$

and we denote by $\operatorname{Lip}^n(V,W)$ the set given by $\operatorname{Lip}^n(V,W) = \{f \in \mathcal{C}^n(V,W) \colon \|f\|_{\operatorname{Lip}^n(V,W)} < \infty\}$. For an $\mathbb{R}$-Hilbert space $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, real numbers $r \in [0,1]$, $\eta \in \mathbb{R}$, $T \in (0,\infty)$, and a generator of a strongly continuous analytic semigroup $A \colon D(A) \subseteq H \to H$ with $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C} \colon \operatorname{Re}(z) < \eta\}$ we denote by $\chi^{r,T}_{A,\eta} \in [0,\infty)$ the real number given by $\chi^{r,T}_{A,\eta} = \sup_{t\in(0,T]} t^r \,\|(\eta - A)^r e^{tA}\|_{L(H)}$ (cf., e.g., [20, Lemma 11.36]). We denote by $\mathbb{B} \colon (0,\infty)^2 \to (0,\infty)$ the function which satisfies for all $x,y \in (0,\infty)$ that $\mathbb{B}(x,y) = \int_0^1 t^{(x-1)}\,(1-t)^{(y-1)}\,dt$ (Beta function). We denote by $\mathrm{E}_{\alpha,\beta} \colon [0,\infty) \to [0,\infty)$, $\alpha,\beta \in (-\infty,1)$, the functions which satisfy for all $\alpha,\beta \in (-\infty,1)$, $x \in [0,\infty)$ that $\mathrm{E}_{\alpha,\beta}[x] = 1 + \sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1} \mathbb{B}\big(1-\beta, k(1-\beta)+1-\alpha\big)$ (generalized exponential function; cf. Exercise 3 in Chapter 7 in Henry [12], (1.0.3) in Chapter 1 in Gorenflo et al. [11], and (16) in [1]). For real numbers $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $p \in [1,\infty)$, $a \in [0,1)$, $b \in [0, 1/2)$, $\lambda \in (-\infty,1)$, an $\mathbb{R}$-Hilbert space $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, and a generator $A \colon D(A) \subseteq H \to H$ of a strongly continuous analytic semigroup with $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C} \colon \operatorname{Re}(z) < \eta\}$ we denote by $\Theta^{a,b,\lambda}_{A,\eta,p,T} \colon [0,\infty)^2 \to [0,\infty]$ the function which satisfies for all $L, \hat{L} \in [0,\infty)$ that

$$\Theta^{a,b,\lambda}_{A,\eta,p,T}(L,\hat{L}) = \begin{cases} \sqrt{2}\left|E_{2\lambda,\max\{a,2b\}}\left[\left|\frac{\chi^{a,T}_{A,\eta}\, L\, \sqrt{2}\, T^{(1-a)}}{\sqrt{1-a}} + \chi^{b,T}_{A,\eta}\, \hat{L}\, \sqrt{p\,(p-1)\,T^{(1-2b)}}\right|^2\right]\right|^{1/2} & \colon (\lambda,\hat{L}) \in (-\infty,\tfrac{1}{2}) \times (0,\infty) \\ E_{\lambda,a}\left[\chi^{a,T}_{A,\eta}\, L\, T^{(1-a)}\right] & \colon \hat{L} = 0 \\ \infty & \colon \text{otherwise} \end{cases}. \tag{9}$$

We denote by $\Pi_k, \Pi_k^* \in \mathcal{P}\big(\mathcal{P}\big(\mathcal{P}(\mathbb{N})\big)\big)$, $k \in \mathbb{N}_0$, the sets which satisfy for all $k \in \mathbb{N}$ that $\Pi_0 = \Pi_0^* = \emptyset$, $\Pi_k^* = \Pi_k \backslash \big\{\{\{1,2,\ldots,k\}\}\big\}$, and

$$\Pi_k = \big\{ A \subseteq \mathcal{P}(\mathbb{N})\colon\ [\emptyset \notin A] \wedge [\cup_{a\in A} a = \{1,2,\ldots,k\}] \wedge [\forall\, a,b \in A\colon\ (a \neq b \Rightarrow a \cap b = \emptyset)] \big\} \tag{10}$$

(cf., e.g., [6, Theorem 2]). Observe, for example, that $\Pi_0 = \emptyset$, $\Pi_1 = \big\{\{\{1\}\}\big\}$, $\Pi_2 = \big\{\{\{1,2\}\},\{\{1\}, \{2\}\}\big\}$, and $\Pi_3 = \big\{\{\{1,2,3\}\},\{\{1,2\},\{3\}\},\{\{1,3\},\{2\}\},\{\{1\},\{2,3\}\},\{\{1\},\ \{2\},\{3\}\}\big\}$ and note that for every $k \in \mathbb{N}$ it holds that Π_k is the set of all partitions of $\{1,2,\ldots,k\}$. For a natural number $k \in \mathbb{N}$ and a set $\varpi \in \Pi_k$ we denote by $I_1^\varpi, I_2^\varpi, \ldots, I_{\#_\varpi}^\varpi \in \varpi$ the sets which satisfy that $\min\big(I_1^\varpi\big) < \min\big(I_2^\varpi\big) < \cdots < \min\big(I_{\#_\varpi}^\varpi\big)$. For a natural number $k \in \mathbb{N}$, a set $\varpi \in \Pi_k$, and a natural number $i \in \{1,2,\ldots,\#_\varpi\}$ we denote by $I_{i,1}^\varpi, I_{i,2}^\varpi, \ldots, I_{i,\#_{I_i^\varpi}}^\varpi \in I_i^\varpi$ the natural numbers which satisfy that $I_{i,1}^\varpi < I_{i,2}^\varpi < \cdots < I_{i,\#_{I_i^\varpi}}^\varpi$. For a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable space $(S, \mathcal{S})$, a set R, and a function $f\colon \Omega \to R$ we denote by $[f]_{\mu,\mathcal{S}}$ the set given by

$$[f]_{\mu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F},\mathcal{S})\colon (\exists\, A \in \mathcal{F}\colon \mu(A) = 0 \text{ and } \{\omega \in \Omega\colon f(\omega) \neq g(\omega)\} \subseteq A)\}\,. \tag{11}$$

2 Stochastic evolution equations with smooth coefficients

2.1 Setting

Let $T \in (0,\infty)$, $\eta \in \mathbb{R}$, let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H > 1$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$-Wiener process, let $A\colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with $\mathrm{spectrum}(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, for every $k \in \mathbb{N}$, $\varpi \in \Pi_k$, $i \in \{1,2,\ldots,\#_\varpi\}$ let $[\cdot]_i^\varpi\colon H^{k+1} \to H^{\#_{I_i^\varpi}+1}$ be the function which satisfies for all $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in H^{k+1}$ that $[\mathbf{u}]_i^\varpi = (u_0, u_{I_{i,1}^\varpi}, u_{I_{i,2}^\varpi}, \ldots, u_{I_{i,\#_{I_i^\varpi}}^\varpi})$, let $[[\cdot]]\colon \mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}), \mathcal{B}(H)) \to \mathcal{P}\big(\mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}), \mathcal{B}(H))\big)$ be the function which satisfies for all $X \in \mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}), \mathcal{B}(H))$ that $\big[[X]\big] = \big\{Y \in \mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}), \mathcal{B}(H))\colon\ \inf_{t\in[0,T]} \mathbb{P}\big(Y_t = X_t\big) = 1\big\}$, for every $p \in (0,\infty)$ let $\mathcal{L}^p$ and $\mathbb{L}^p$ be the sets given by $\mathcal{L}^p = \big\{X \in \mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}), \mathcal{B}(H))\colon\ \sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};H)} < \infty\big\}$ and $\mathbb{L}^p = \big\{\big[[X]\big] : X \in \mathcal{L}^p\big\}$ and let $\|\cdot\|_{\mathbb{L}^p} : \mathbb{L}^p \to [0,\infty)$ be the function which satisfies for all $X \in \mathcal{L}^p$ that $\big\|\big[[X]\big]\big\|_{\mathbb{L}^p} = \sup_{t\in[0,T]} \|X_t\|_{\mathcal{L}^p(\mathbb{P};H)}$, and for every separable $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ and every $a \in \mathbb{R}$, $b \in (a,\infty)$, $A \in \mathcal{B}(\mathbb{R})$, $X \in \mathcal{M}(\mathcal{B}(A) \otimes \mathcal{F}, \mathcal{B}(V))$ with $(a,b) \subseteq A$ let $\int_a^b X_s\,\mathbf{ds} \in \{[Y]_{\mathbb{P},\mathcal{B}(V)}\colon Y \in \mathcal{M}(\mathcal{F},\mathcal{B}(V))\}$ be the set given by $\int_a^b X_s\,\mathbf{ds} = \big[\int_a^b \mathbb{1}_{\{\int_a^b \|X_u\|_V\,du < \infty\}} X_s\,ds\big]_{\mathbb{P},\mathcal{B}(V)}$.

2.2 Differentiability with respect to the initial values

Theorem 2.1 (Differentiability with respect to the initial value)**.** *Assume the setting in Section 2.1, let $n \in \mathbb{N}$, $F \in \mathcal{C}_{\mathrm{b}}^n(H,H)$, $B \in \mathcal{C}_{\mathrm{b}}^n(H, HS(U,H))$, $\alpha \in [0,1)$, $\beta \in [0,1/2)$, and for every $k \in \mathbb{N}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{R}^k$, $J \in \mathcal{P}(\mathbb{R})$ let $\iota_J^{\boldsymbol{\delta}} \in \mathbb{R}$ be the real number given by $\iota_J^{\boldsymbol{\delta}} = \sum_{i\in J\cap\{1,2,\ldots,k\}} \delta_i - \mathbb{1}_{[2,\infty)}(\#_{J\cap\{1,2,\ldots,k\}}) \min\{1-\alpha, 1/2 - \beta\}$. Then*

(i) *there exist up-to-modifications unique* $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$*-predictable stochastic processes* $X^{k,\mathbf{u}}\colon [0,T]\times\Omega\to H$, $\mathbf{u}\in H^{k+1}$, $k\in\{0,1,\ldots,n\}$, *which fulfill for all* $k\in\{0,1,\ldots,n\}$, $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1,\ldots,u_k)\in H^{k+1}$, $t\in[0,T]$ *that* $\sup_{s\in[0,T]}\mathbb{E}[\|X_s^{k,\mathbf{u}}\|_H^p]<\infty$ *and*

$$\begin{aligned}
&[X_t^{k,\mathbf{u}} - e^{tA}\,\mathbb{1}_{\{0,1\}}(k)\,u_k]_{\mathbb{P},\mathcal{B}(H)}\\
&= \int_0^t e^{(t-s)A}\Bigg[\mathbb{1}_{\{0\}}(k)\,F(X_s^{0,u_0}) + \sum_{\varpi\in\Pi_k} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\ldots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\Bigg]\,\mathbf{ds}\\
&+ \int_0^t e^{(t-s)A}\Bigg[\mathbb{1}_{\{0\}}(k)\,B(X_s^{0,u_0}) + \sum_{\varpi\in\Pi_k} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\ldots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\Bigg]\,dW_s,
\end{aligned}\tag{12}$$

(ii) *for all* $k\in\{1,2,\ldots,n\}$, $p\in[2,\infty)$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in[0,1/2)^k$ *with* $\sum_{i=1}^k\delta_i<1/2$ *it holds that*

$$\begin{aligned}
&\sup_{\mathbf{u}=(u_0,u_1,\ldots,u_k)\in(\times_{i=0}^k H^{[i]})}\sup_{t\in(0,T]}\left[\frac{t^{\iota_{\mathbb{N}}^{\boldsymbol{\delta}}}\,\|X_t^{k,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right]\\
&\leq \Theta_{A,\eta,p,T}^{\alpha,\beta,\iota_{\mathbb{N}}^{\boldsymbol{\delta}}}(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})},|B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))})\Bigg[\chi_{A,\eta}^{\delta_1,T}\,\mathbb{1}_{\{1\}}(k)\\
&\quad+\max\{T^k,1\}\Bigg[\chi_{A,\eta}^{\alpha,T}\,\mathbb{B}\big(1-\alpha,1-\textstyle\sum_{i=1}^k\delta_i\big)\|F\|_{\mathcal{C}_{\mathrm{b}}^k(H,H_{-\alpha})}\\
&\quad+\chi_{A,\eta}^{\beta,T}\sqrt{\tfrac{p\,(p-1)}{2}\,\mathbb{B}\big(1-2\beta,1-2\textstyle\sum_{i=1}^k\delta_i\big)}\,\|B\|_{\mathcal{C}_{\mathrm{b}}^k(H,HS(U,H_{-\beta}))}\Bigg]\\
&\quad\cdot\sum_{\varpi\in\Pi_k^*}\prod_{I\in\varpi}\sup_{\mathbf{u}=(u_i)_{i\in I\cup\{0\}}\in(\times_{i\in I\cup\{0\}}H^{[i]})}\sup_{t\in(0,T]}\left[\frac{t^{\iota_I^{\boldsymbol{\delta}}}\,\|X_t^{\#_I,\mathbf{u}}\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right]\Bigg]<\infty,
\end{aligned}\tag{13}$$

(iii) *for all* $k\in\{1,2,\ldots,n\}$, $p\in[2,\infty)$, $x\in H$ *it holds that* $\big(H^k\ni\mathbf{u}\mapsto\big[[X^{k,(x,\mathbf{u})}]\big]\in\mathbb{L}^p\big)\in L^{(k)}(H,\mathbb{L}^p)$,

(iv) *for all* $k\in\{1,2,\ldots,n\}$, $p\in[2,\infty)$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in[0,1/2)^k$ *with* $\sum_{i=1}^k\delta_i<1/2$, $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})}<\infty$, *and* $|B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))}<\infty$ *it holds that*

$$\begin{aligned}
&\sup_{\substack{x,y\in H,\\ x\neq y}}\sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k}\sup_{t\in(0,T]}\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}}\,\|X_t^{k,(x,\mathbf{u})}-X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\\
&\leq\max\{T^k,1\}\,\Theta_{A,\eta,p,T}^{\alpha,\beta,\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})},|B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big)\\
&\quad\cdot\Bigg(\chi_{A,\eta}^{0,T}\,\Theta_{A,\eta,p,T}^{\alpha,\beta,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})},|B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big)\\
&\quad\cdot\sum_{\varpi\in\Pi_k}\prod_{I\in\varpi}\sup_{\mathbf{u}=(u_i)_{i\in I\cup\{0\}}\in(\times_{i\in I\cup\{0\}}H^{[i]})}\sup_{t\in(0,T]}\left[\frac{t^{\iota_I^{\boldsymbol{\delta}}}\,\|X_t^{\#_I,\mathbf{u}}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right]
\end{aligned}$$

$$+\sum_{\varpi\in\Pi_k^*}\sum_{I\in\varpi}\sup_{\substack{x,y\in H,\\ x\neq y}}\sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\backslash\{0\})^{\#_I}}\sup_{t\in(0,T]}\left[\frac{t^{\iota^{(\boldsymbol{\delta},0)}_{I\cup\{k+1\}}}\|X_t^{\#_I,(x,\mathbf{u})}-X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right] \tag{14}$$

$$\cdot\prod_{J\in\varpi\backslash\{I\}}\sup_{\mathbf{u}=(u_i)_{i\in J\cup\{0\}}\in(\times_{i\in J\cup\{0\}}H^{[i]})}\sup_{t\in(0,T]}\left[\frac{t^{\iota_J^{\boldsymbol{\delta}}}\|X_t^{\#_J,\mathbf{u}}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in J}\|u_i\|_{H_{-\delta_i}}}\right]\Bigg)$$

$$\cdot\Bigg[\chi_{A,\eta}^{\alpha,T}\,\mathbb{B}\big(1-\alpha,1-\textstyle\sum_{i=1}^k\delta_i\big)\,\|F\|_{\mathrm{Lip}^k(H,H_{-\alpha})}$$

$$+\chi_{A,\eta}^{\beta,T}\sqrt{\tfrac{p\,(p-1)}{2}\,\mathbb{B}\big(1-2\beta,1-2\textstyle\sum_{i=1}^k\delta_i\big)}\,\|B\|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))}\Bigg]<\infty,$$

(v) for all $k\in\{1,2,\dots,n\}$, $p\in[2,\infty)$ *it holds that* $\big(H\ni x\mapsto\big[H^k\ni\mathbf{u}\mapsto\big[[X^{k,(x,\mathbf{u})}]\big]\in\mathbb{L}^p\big]\in L^{(k)}(H,\mathbb{L}^p)\big)\in\mathcal{C}(H,L^{(k)}(H,\mathbb{L}^p))$,

(vi) for all $k\in\{1,2,\dots,n\}$, $p\in[2,\infty)$, $x\in H$ *it holds that*

$$\begin{cases}\limsup\limits_{H\backslash\{0\}\ni u_k\to 0}\sup\limits_{t\in[0,T]}\frac{\|X_t^{0,x+u_k}-X_t^{0,x}-X_t^{1,(x,u_k)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|u_k\|_H}=0 & : k=1\\ \limsup\limits_{H\backslash\{0\}\ni u_k\to 0}\sup\limits_{\mathbf{u}=(u_1,u_2,\dots,u_{k-1})\in(H\backslash\{0\})^{k-1}}\sup\limits_{t\in[0,T]}\frac{\|X_t^{k-1,(x+u_k,\mathbf{u})}-X_t^{k-1,(x,\mathbf{u})}-X_t^{k,(x,\mathbf{u},u_k)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k\|u_i\|_H}=0 & : k>1\end{cases}, \tag{15}$$

(vii) for all $p\in[2,\infty)$ *it holds that* $\big(H\ni x\mapsto\big[[X^{0,x}]\big]\in\mathbb{L}^p\big)\in\mathcal{C}_{\mathrm{b}}^n(H,\mathbb{L}^p)$,

(viii) for all $k\in\{1,2,\dots,n\}$, $p\in[2,\infty)$, $x,u_1,u_2,\dots,u_k\in H$ *it holds that*

$$\begin{aligned}\big(\tfrac{d^k}{dx^k}\big[[X^{0,x}]\big]\big)(u_1,u_2,\dots,u_k)&=\big(H\ni y\mapsto\big[[X^{0,y}]\big]\in\mathbb{L}^p\big)^{(k)}(x)(u_1,u_2,\dots,u_k)\\&=\big[[X^{k,(x,u_1,u_2,\dots,u_k)}]\big],\end{aligned} \tag{16}$$

(ix) for all $p\in[2,\infty)$, $t\in[0,T]$ *it holds that* $\big(H\ni x\mapsto[X_t^{0,x}]_{\mathbb{P},\mathcal{B}(H)}\in L^p(\mathbb{P};H)\big)\in\mathcal{C}_{\mathrm{b}}^n(H,L^p(\mathbb{P};H))$, *and*

(x) for all $k\in\{1,2,\dots,n\}$, $p\in[2,\infty)$, $x,u_1,u_2,\dots,u_k\in H$, $t\in[0,T]$ *it holds that*

$$\begin{aligned}&\big(\tfrac{d^k}{dx^k}[X_t^{0,x}]_{\mathbb{P},\mathcal{B}(H)}\big)(u_1,u_2,\dots,u_k)\\&=\big(H\ni y\mapsto[X_t^{0,y}]_{\mathbb{P},\mathcal{B}(H)}\in L^p(\mathbb{P};H)\big)^{(k)}(x)(u_1,u_2,\dots,u_k)=[X_t^{k,(x,u_1,u_2,\dots,u_k)}]_{\mathbb{P},\mathcal{B}(H)}.\end{aligned} \tag{17}$$

Proof. Throughout this proof let $r_0,r_1\in[0,1)$ be the real numbers given by $r_0=\alpha$ and $r_1=\beta$, let $\mathbf{0}_k\in\mathbb{R}^k$, $k\in\mathbb{N}$, be the vectors which satisfy for all $k\in\mathbb{N}$ that $\mathbf{0}_k=(0,0,\dots,0)$, let $(V_{l,r},\|\cdot\|_{V_{l,r}},\langle\cdot,\cdot\rangle_{V_{l,r}})$, $l\in\{0,1\}$, $r\in[0,\infty)$, be the $\mathbb{R}$-Hilbert spaces which satisfy for all $r\in[0,\infty)$ that

$$(V_{0,r},\|\cdot\|_{V_{0,r}},\langle\cdot,\cdot\rangle_{V_{0,r}})=(H_{-r},\|\cdot\|_{H_{-r}},\langle\cdot,\cdot\rangle_{H_{-r}}) \tag{18}$$

and

$$(V_{1,r},\|\cdot\|_{V_{1,r}},\langle\cdot,\cdot\rangle_{V_{1,r}})=(HS(U,H_{-r}),\|\cdot\|_{HS(U,H_{-r})},\langle\cdot,\cdot\rangle_{HS(U,H_{-r})}), \tag{19}$$

let $G_l\colon H \to V_{l,0}$, $l \in \{0,1\}$, be the functions given by $G_0 = F$ and $G_1 = B$, let $\lfloor \cdot \rfloor\colon \mathbb{R} \to \mathbb{R}$ and $\lceil \cdot \rceil\colon \mathbb{R} \to \mathbb{R}$ be the functions which satisfy for all $t \in \mathbb{R}$ that

$$\lfloor t \rfloor = \max((-\infty, t] \cap \{0, 1, -1, 2, -2, \dots\}) \tag{20}$$

and

$$\lceil t \rceil = \min([t, \infty) \cap \{0, 1, -1, 2, -2, \dots\}), \tag{21}$$

let $\theta_l^m\colon H^{m+1} \to H^m$, $m \in \mathbb{N}$, $l \in \{0,1\}$, be the functions which satisfy for all $l \in \{0,1\}$, $m \in \mathbb{N}$, $\mathbf{u} = (u_0, u_1, \dots, u_m) \in H^{m+1}$ that

$$\theta_l^m(\mathbf{u}) = \begin{cases} u_0 + l u_1 & : m = 1 \\ (u_0 + l u_m, u_1, u_2, \dots, u_{m-1}) & : m > 1 \end{cases}, \tag{22}$$

and let $\mathbb{D}_k \in \mathcal{P}(\mathbb{R}^k)$, $k \in \mathbb{N}$, be the sets which satisfy for all $k \in \mathbb{N}$ that

$$\mathbb{D}_k = \big\{(\delta_1, \delta_2, \dots, \delta_k) \in [0, 1/2)^k \colon \textstyle\sum_{i=1}^k \delta_i < 1/2\big\}. \tag{23}$$

Next we claim that for every $k \in \{1, 2, \dots, n\}$ there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{l,\mathbf{u}}\colon [0,T] \times \Omega \to H$, $\mathbf{u} \in H^{l+1}$, $l \in \{0, 1, \dots, k\}$, which fulfill for all $l \in \{0, 1, \dots, k\}$, $p \in [2, \infty)$, $\mathbf{u} = (u_0, u_1, \dots, u_l) \in H^{l+1}$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}[\|X_s^{l,\mathbf{u}}\|_H^p] < \infty$ and

$$\begin{aligned} &[X_t^{l,\mathbf{u}} - e^{tA}\, \mathbb{1}_{\{0,1\}}(l)\, u_l]_{\mathbb{P},\mathcal{B}(H)} \\ &= \int_0^t e^{(t-s)A} \bigg[\mathbb{1}_{\{0\}}(l)\, F(X_s^{0,u_0}) + \sum_{\varpi \in \Pi_l} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \dots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\bigg]\, \mathbf{ds} \\ &+ \int_0^t e^{(t-s)A} \bigg[\mathbb{1}_{\{0\}}(l)\, B(X_s^{0,u_0}) + \sum_{\varpi \in \Pi_l} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \dots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\bigg]\, dW_s. \end{aligned} \tag{24}$$

We now prove (24) by induction on $k \in \{1, 2, \dots, n\}$. For the base case $k = 1$ note that, e.g., item (i) of Corollary 2.10 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $\alpha = 0$, $\beta = 0$, $W = W$, $A = A$, $F = F$, $B = B$, $\delta = 0$ in the notation of Corollary 2.10 in [1]) ensures the existence of up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{0,x}\colon [0,T] \times \Omega \to H$, $x \in H$, which fulfill for all $p \in [2, \infty)$, $x \in H$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}[\|X_s^{0,x}\|_H^p] < \infty$ and

$$[X_t^{0,x} - e^{tA}x]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A} F(X_s^{0,x})\, \mathbf{ds} + \int_0^t e^{(t-s)A} B(X_s^{0,x})\, dW_s. \tag{25}$$

Next we note that for all $l \in \{0,1\}$, $p \in [2, \infty)$, $u \in H$, $Y, Z \in \mathcal{L}^p(\mathbb{P}; H)$, $t \in (0,T]$ it holds that

$$\begin{aligned} \|G_l'(X_t^u)Y - G_l'(X_t^u)Z\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})} &\le |G_l|_{\mathcal{C}_b^1(H,V_{l,0})}\, \|Y - Z\|_{\mathcal{L}^p(\mathbb{P};H)} \qquad \text{and} \\ \|G_l'(X_t^u)0\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})} &= 0. \end{aligned} \tag{26}$$

This allows us to apply item (i) of Theorem 2.9 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = 0$, $\hat{\alpha} = 0$, $\beta = 0$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}_b^1(H,H)}$, $\hat{L}_0 = 0$, $L_1 = |B|_{\mathcal{C}_b^1(H,HS(U,H))}$, $\hat{L}_1 = 0$,

$W=W$, $A=A$, $\mathbf{F}=\big([0,T]\times\Omega\times H\ni(t,\omega,x)\mapsto F'(X_t^{0,u_0}(\omega))x\in H\big)$, $\mathbf{B}=\big([0,T]\times\Omega\times H\ni(t,\omega,x)\mapsto B'(X_t^{0,u_0}(\omega))x\in HS(U,H)\big)$, $\delta=0$, $\lambda=0$, $\xi=(\Omega\ni\omega\mapsto u_1\in H)$ for $u_0,u_1\in H$, $p\in[2,\infty)$ in the notation of Theorem 2.9 in [1]) to obtain that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{1,\mathbf{u}}\colon[0,T]\times\Omega\to H$, $\mathbf{u}\in H^2$, which fulfill for all $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1)\in H^2$, $t\in[0,T]$ that $\sup_{s\in[0,T]}\mathbb{E}[\|X_s^{1,\mathbf{u}}\|_H^p]<\infty$ and

$$[X_t^{1,\mathbf{u}}-e^{tA}u_1]_{\mathbb{P},\mathcal{B}(H)}=\int_0^t e^{(t-s)A}F'(X_s^{0,u_0})\,X_s^{1,\mathbf{u}}\,\mathbf{ds}+\int_0^t e^{(t-s)A}B'(X_s^{0,u_0})\,X_s^{1,\mathbf{u}}\,dW_s. \tag{27}$$

This and (25) prove (24) in the base case $k=1$. For the induction step $\{1,2,\dots,n-1\}\ni k\to k+1\in\{2,3,\dots,n\}$ we introduce more notation. Assume that there exists a natural number $k\in\{1,2,\dots,n-1\}$ such that (24) holds for $k=k$, let $X^{l,\mathbf{u}}\colon[0,T]\times\Omega\to H$, $\mathbf{u}\in H^{l+1}$, $l\in\{2,3,\dots,k\}$, be up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes which fulfill for all $l\in\{2,3,\dots,k\}$, $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1,\dots,u_l)\in H^{l+1}$, $t\in[0,T]$ that $\sup_{s\in[0,T]}\mathbb{E}[\|X_s^{l,\mathbf{u}}\|_H^p]<\infty$ and

$$\begin{aligned}[X_t^{l,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H)}&=\int_0^t e^{(t-s)A}\sum_{\varpi\in\Pi_l}F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi},X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\,\mathbf{ds}\\&\quad+\int_0^t e^{(t-s)A}\sum_{\varpi\in\Pi_l}B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi},X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\,dW_s,\end{aligned} \tag{28}$$

let $\mathcal{G}_l^{\mathbf{u}}\colon[0,T]\times\Omega\times H\to V_{l,0}$, $\mathbf{u}\in H^{k+2}$, $l\in\{0,1\}$, be the functions which satisfy for all $l\in\{0,1\}$, $\mathbf{u}=(u_0,u_1,\dots,u_{k+1})\in H^{k+2}$, $t\in[0,T]$, $x\in H$ that

$$\mathcal{G}_l^{\mathbf{u}}(t,x)=G_l'(X_t^{0,u_0})\,x+\textstyle\sum_{\varpi\in\Pi_{k+1}^*}G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi},X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big), \tag{29}$$

and let $\bar{L}_l^{\mathbf{u},p}\in[0,\infty)$, $\mathbf{u}\in H^{k+2}$, $p\in[2,\infty)$, $l\in\{0,1\}$, be the real numbers which satisfy for all $l\in\{0,1\}$, $p\in[2,\infty)$, $\mathbf{u}\in H^{k+2}$ that

$$\bar{L}_l^{\mathbf{u},p}=\textstyle\sum_{\varpi\in\Pi_{k+1}^*}|G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})}\prod_{i=1}^{\#_\varpi}\big\|\big[[X^{\#_{I_i^\varpi},[\mathbf{u}]_i^\varpi}]\big]\big\|_{\mathbb{L}^{p\#_\varpi}}. \tag{30}$$

Next we note that Hölder's inequality implies for all $l\in\{0,1\}$, $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1,\dots,u_{k+1})\in H^{k+2}$, $Y,Z\in\mathcal{L}^p(\mathbb{P};H)$, $t\in(0,T]$ that

$$\|\mathcal{G}_l^{\mathbf{u}}(t,Y)-\mathcal{G}_l^{\mathbf{u}}(t,Z)\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\le|G_l|_{\mathcal{C}_{\mathrm{b}}^1(H,V_{l,0})}\,\|Y-Z\|_{\mathcal{L}^p(\mathbb{P};H)} \tag{31}$$

and

$$\begin{aligned}&\big\|\mathcal{G}_l^{\mathbf{u}}(t,0)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\\&\le\textstyle\sum_{\varpi\in\Pi_{k+1}^*}\big\|G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi},X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi},\dots,X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\\&\le\textstyle\sum_{\varpi\in\Pi_{k+1}^*}|G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})}\prod_{i=1}^{\#_\varpi}\big\|X_t^{\#_{I_i^\varpi},[\mathbf{u}]_i^\varpi}\big\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}\le\bar{L}_l^{\mathbf{u},p}.\end{aligned} \tag{32}$$

We can hence apply item (i) of Theorem 2.9 in [1] (with $H=H$, $U=U$, $T=T$, $\eta=\eta$, $p=p$, $\alpha=0$, $\hat\alpha=0$, $\beta=0$, $\hat\beta=0$, $L_0=|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}$, $\hat L_0=\bar L_0^{\mathbf{u},p}$, $L_1=|B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}$, $\hat L_1=\bar L_1^{\mathbf{u},p}$,

$W = W$, $A = A$, $\mathbf{F} = \mathcal{G}_0^{\mathbf{u}}$, $\mathbf{B} = \mathcal{G}_1^{\mathbf{u}}$, $\delta = 0$, $\lambda = 0$, $\xi = (\Omega \ni \omega \mapsto 0 \in H)$ for $\mathbf{u} \in H^{k+2}$, $p \in [2, \infty)$ in the notation of Theorem 2.9 in [1]) to obtain that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes $X^{k+1,\mathbf{u}} \colon [0,T] \times \Omega \to H$, $\mathbf{u} \in H^{k+2}$, which fulfill for all $p \in [2, \infty)$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}[\|X_s^{k+1,\mathbf{u}}\|_H^p] < \infty$ and

$$\begin{aligned}
[X_t^{k+1,\mathbf{u}}]_{\mathbb{P},\mathcal{B}(H)} &= \int_0^t e^{(t-s)A} \mathcal{G}_0^{\mathbf{u}}(s, X_s^{k+1,\mathbf{u}})\, \mathbf{ds} + \int_0^t e^{(t-s)A} \mathcal{G}_1^{\mathbf{u}}(s, X_s^{k+1,\mathbf{u}})\, dW_s \\
&= \int_0^t e^{(t-s)A} \sum_{\varpi\in\Pi_{k+1}} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\, \mathbf{ds} \\
&\quad + \int_0^t e^{(t-s)A} \sum_{\varpi\in\Pi_{k+1}} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)\, dW_s.
\end{aligned} \tag{33}$$

This proves (24) in the case $k + 1$. Induction hence establishes (24). The proof of item (i) is thus completed.

For our proof of items (ii)–(x) we introduce further notation. Let $X^{k,\mathbf{u}} \colon [0,T] \times \Omega \to H$, $\mathbf{u} \in H^{k+1}$, $k \in \{0, 1, \ldots, n\}$, be $(\mathcal{F}_t)_{t\in[0,T]}/\mathcal{B}(H)$-predictable stochastic processes which fulfill for all $k \in \{0, 1, \ldots, n\}$, $p \in [2, \infty)$, $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in H^{k+1}$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \mathbb{E}[\|X_s^{k,\mathbf{u}}\|_H^p] < \infty$ and

$$\begin{aligned}
&[X_t^{k,\mathbf{u}} - e^{tA}\, \mathbb{1}_{\{0,1\}}(k)\, u_k]_{\mathbb{P},\mathcal{B}(H)} \\
&= \int_0^t e^{(t-s)A} \Big[\mathbb{1}_{\{0\}}(k)\, F(X_s^{0,u_0}) + \sum_{\varpi\in\Pi_k} F^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big) \Big]\, \mathbf{ds} \\
&+ \int_0^t e^{(t-s)A} \Big[\mathbb{1}_{\{0\}}(k)\, B(X_s^{0,u_0}) + \sum_{\varpi\in\Pi_k} B^{(\#_\varpi)}(X_s^{0,u_0})\big(X_s^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_s^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_s^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big) \Big]\, dW_s,
\end{aligned} \tag{34}$$

let $L_{\varpi,p}^{\boldsymbol{\delta}} \in [0, \infty]$, $\varpi \in \mathcal{P}\big(\mathcal{P}(\{1, 2, \ldots, k\}) \setminus \{\emptyset\}\big)$, $\boldsymbol{\delta} \in \mathbb{D}_k$, $p \in (0, \infty)$, $k \in \{1, 2, \ldots, n\}$, be the extended real numbers which satisfy for all $k \in \{1, 2, \ldots, n\}$, $p \in (0, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$, $\varpi \in \mathcal{P}\big(\mathcal{P}(\{1, 2, \ldots, k\}) \setminus \{\emptyset\}\big) \setminus \{\emptyset\}$ that $L_{\emptyset,p}^{\boldsymbol{\delta}} = 1$ and

$$L_{\varpi,p}^{\boldsymbol{\delta}} = \prod_{I\in\varpi} \sup_{\mathbf{u}=(u_i)_{i\in I\cup\{0\}}\in(\times_{i\in I\cup\{0\}} H^{[i]})} \sup_{t\in(0,T]} \left[\frac{t^{\iota_I^{\boldsymbol{\delta}}}\, \|X_t^{\#_I,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \right], \tag{35}$$

let $\tilde{L}_p \in [0, \infty]$, $p \in (0, \infty)$, be the extended real numbers which satisfy for all $p \in (0, \infty)$ that

$$\tilde{L}_p = \sup_{u_0\in H} \sup_{u_1\in H\setminus\{0\}} \sup_{t\in(0,T]} \left[\frac{\|X_t^{0,u_0+u_1} - X_t^{0,u_0}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|u_1\|_H} \right], \tag{36}$$

let $\hat{L}_{k,l}^{\boldsymbol{\delta},\mathbf{u},p} \in [0, \infty]$, $\mathbf{u} \in H^{k+1}$, $\boldsymbol{\delta} \in \mathbb{D}_k$, $p \in (0, \infty)$, $l \in \{0, 1\}$, $k \in \{1, 2, \ldots, n\}$, be the extended real numbers which satisfy for all $k \in \{1, 2, \ldots, n\}$, $l \in \{0, 1\}$, $p \in (0, \infty)$, $\boldsymbol{\delta} \in \mathbb{D}_k$, $\mathbf{u} = (u_0, u_1, \ldots, u_k) \in$

H^{k+1} that

$$\hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k,l} = |T \vee 1|^{\lfloor k/2 \rfloor \min\{1-\alpha, 1/2-\beta\}} \cdot \sum_{\varpi \in \Pi_k^*} |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,r_l})} \prod_{i=1}^{\#_\varpi} \sup_{t\in(0,T]} \Big[t^{\iota^{\boldsymbol{\delta}}_{I_i^\varpi}} \| X_t^{\#_{I_i^\varpi}, [\mathbf{u}]_i^\varpi} \|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \Big], \tag{37}$$

for every $k \in \{1,2,\ldots,n\}$, $l \in \{0,1\}$, $\mathbf{u} = (u_0,u_1,\ldots,u_k) \in H^{k+1}$ let $\mathbf{G}^{\mathbf{u}}_{k,l} \colon [0,T]\times\Omega\times H \to V_{l,0}$ and $\bar{\mathbf{G}}^{\mathbf{u}}_{k,l} \colon [0,T]\times\Omega\times H \to V_{l,0}$ be the functions which satisfy for all $x \in H$, $t \in [0,T]$ that

$$\mathbf{G}^{\mathbf{u}}_{k,l}(t,x) = G_l'(X_t^{0,u_0})x + \textstyle\sum_{\varpi\in\Pi_k^*} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big) \tag{38}$$

and

$$\bar{\mathbf{G}}^{\mathbf{u}}_{k,l}(t,x) = \begin{cases} \int_0^1 G_l'\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_1} - X_t^{0,u_0}]\big)\, x\, d\rho & : k = 1 \\ G_l'(X_t^{0,u_0})\, x \\ \quad + \int_0^1 G_l''\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_k} - X_t^{0,u_0}]\big)\big(X_t^{k-1,\theta_1^k(\mathbf{u})}, X_t^{0,u_0+u_k} - X_t^{0,u_0}\big)\, d\rho \\ \quad + \sum_{\varpi\in\Pi_{k-1}^*} \Big[\int_0^1 G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_k} - X_t^{0,u_0}]\big)\big(X_t^{\#_{I_1^\varpi},[\theta_1^k(\mathbf{u})]_1^\varpi}, \\ \qquad X_t^{\#_{I_2^\varpi},[\theta_1^k(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^k(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_k} - X_t^{0,u_0}\big)\, d\rho & : k > 1 \\ \quad + G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_1^k(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_1^k(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^k(\mathbf{u})]_{\#_\varpi}^\varpi}\big) \\ \quad - G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^k(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^k(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^k(\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big] \end{cases}, \tag{39}$$

and for every $k \in \{1,2,\ldots,n\}$, $p \in (0,\infty)$ let $d_{k,p} \colon H^2 \to [0,\infty]$ and $\tilde{d}_{k,p} \colon H \times (H\setminus\{0\}) \to [0,\infty]$ be the functions which satisfy for all $x,y \in H$, $v \in H\setminus\{0\}$ that

$$d_{k,p}(x,y) = \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_H} \right] \tag{40}$$

and

$$\tilde{d}_{k,p}(x,v) = \begin{cases} \sup_{t\in(0,T]} \left[\frac{\|X_t^{0,x+v} - X_t^{0,x} - X_t^{1,(x,v)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|v\|_H} \right] & : k = 1 \\ \sup_{\mathbf{u}=(u_1,u_2,\ldots,u_{k-1})\in(H\setminus\{0\})^{k-1}} \sup_{t\in(0,T]} \left[\frac{\|X_t^{k-1,(x+v,\mathbf{u})} - X_t^{k-1,(x,\mathbf{u})} - X_t^{k,(x,\mathbf{u},v)}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|v\|_H \prod_{i=1}^{k-1}\|u_i\|_H} \right] & : k > 1 \end{cases}. \tag{41}$$

In the next step we prove item (ii) and the fact that for all $k \in \{1,2,\ldots,n\}$, $p \in [2,\infty)$, $x \in H$, $t \in [0,T]$ it holds that

$$H^k \ni \mathbf{u} \mapsto [X_t^{k,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P};H) \tag{42}$$

is a k-linear function.

We prove item (ii) and (42) by induction on $k \in \{1, 2, \ldots, n\}$. Note that for all $l \in \{0, 1\}$, $p \in [2, \infty)$, $\mathbf{u} \in H^2$, $Y, Z \in \mathcal{L}^p(\mathbb{P}; H)$, $t \in (0, T]$ it holds that

$$\begin{aligned} \|\mathbf{G}^{\mathbf{u}}_{1,l}(t, Y) - \mathbf{G}^{\mathbf{u}}_{1,l}(t, Z)\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})} \leq |G_l|_{\mathcal{C}^1_{\mathrm{b}}(H, V_{l,r_l})} \|Y - Z\|_{\mathcal{L}^p(\mathbb{P}; H)} \qquad \text{and} \\ \|\mathbf{G}^{\mathbf{u}}_{1,l}(t, 0)\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})} = 0. \end{aligned} \tag{43}$$

Moreover, observe that (34) and (38) ensure that for all $\mathbf{u} = (u_0, u_1) \in H^2$, $t \in [0, T]$ it holds that

$$[X^{1,\mathbf{u}}_t]_{\mathbb{P}, \mathcal{B}(H)} = [e^{tA} u_1]_{\mathbb{P}, \mathcal{B}(H)} + \int_0^t e^{(t-s)A} \mathbf{G}^{\mathbf{u}}_{1,0}(s, X^{1,\mathbf{u}}_s) \, \mathbf{ds} + \int_0^t e^{(t-s)A} \mathbf{G}^{\mathbf{u}}_{1,1}(s, X^{1,\mathbf{u}}_s) \, dW_s. \tag{44}$$

Combining (43)–(44) with items (i)–(ii) of Theorem 2.9 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = \alpha$, $\hat{\alpha} = 0$, $\beta = \beta$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}^1_{\mathrm{b}}(H, H_{-\alpha})}$, $\hat{L}_0 = 0$, $L_1 = |B|_{\mathcal{C}^1_{\mathrm{b}}(H, HS(U, H_{-\beta}))}$, $\hat{L}_1 = 0$, $W = W$, $A = A$, $\mathbf{F} = \big([0, T] \times \Omega \times H \ni (t, \omega, x) \mapsto \mathbf{G}^{\mathbf{u}}_{1,0}(t, \omega, x) \in H_{-\alpha}\big)$, $\mathbf{B} = \big([0, T] \times \Omega \times H \ni (t, \omega, x) \mapsto (U \ni u \mapsto \mathbf{G}^{\mathbf{u}}_{1,1}(t, \omega, x) u \in H_{-\beta}) \in HS(U, H_{-\beta})\big)$, $\delta = \delta$, $\lambda = \delta$, $\xi = (\Omega \ni \omega \mapsto u_1 \in H_{-\delta})$ for $\mathbf{u} = (u_0, u_1) \in H^2$, $\delta \in [0, 1/2)$, $p \in [2, \infty)$ in the notation of Theorem 2.9 in [1]) implies that for all $p \in [2, \infty)$, $\delta \in [0, 1/2)$ it holds that

$$\begin{aligned} &\sup_{u_0 \in H} \sup_{u_1 \in H \setminus \{0\}} \sup_{t \in (0, T]} \left[\frac{t^\delta \|X^{1,(u_0,u_1)}_t\|_{\mathcal{L}^p(\mathbb{P}; H)}}{\|u_1\|_{H_{-\delta}}} \right] \\ &\leq \Theta^{\alpha,\beta,\delta}_{A,\eta,p,T}(|F|_{\mathcal{C}^1_{\mathrm{b}}(H, H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H, HS(U, H_{-\beta}))}) \sup_{u_0 \in H} \sup_{u_1 \in H \setminus \{0\}} \left[\frac{\sup_{t \in (0,T]} (t^\delta \, \|e^{tA} u_1\|_H)}{\|u_1\|_{H_{-\delta}}} \right] \\ &\leq \left[\sup_{t \in (0, T]} t^\delta \, \|(\eta - A)^\delta e^{tA}\|_{L(H)} \right] \Theta^{\alpha,\beta,\delta}_{A,\eta,p,T}(|F|_{\mathcal{C}^1_{\mathrm{b}}(H, H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H, HS(U, H_{-\beta}))}) \\ &= \chi^{\delta,T}_{A,\eta} \, \Theta^{\alpha,\beta,\delta}_{A,\eta,p,T}(|F|_{\mathcal{C}^1_{\mathrm{b}}(H, H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H, HS(U, H_{-\beta}))}) < \infty. \end{aligned} \tag{45}$$

This proves item (ii) in the base case $k = 1$. Next we observe that (34) shows that for all $p \in [2, \infty)$, $x, u, \tilde{u} \in H$, $\lambda \in \mathbb{R}$, $t \in [0, T]$ it holds that $\sup_{s \in [0,T]} \mathbb{E}\big[\|X^{1,(x,u)}_s\|^p_H + \|X^{1,(x,\tilde{u})}_s\|^p_H\big] < \infty$ and

$$\begin{aligned} &[X^{1,(x,u)}_t + \lambda X^{1,(x,\tilde{u})}_t]_{\mathbb{P}, \mathcal{B}(H)} = [e^{tA}(u + \lambda \tilde{u})]_{\mathbb{P}, \mathcal{B}(H)} \\ &+ \int_0^t e^{(t-s)A} F'(X^{0,x}_s)(X^{1,(x,u)}_s + \lambda X^{1,(x,\tilde{u})}_s) \, \mathbf{ds} + \int_0^t e^{(t-s)A} B'(X^{0,x}_s)(X^{1,(x,u)}_s + \lambda X^{1,(x,\tilde{u})}_s) \, dW_s. \end{aligned} \tag{46}$$

Item (i) therefore ensures for all $x, u, \tilde{u} \in H$, $\lambda \in \mathbb{R}$, $t \in [0, T]$ that

$$[X^{1,(x,u+\lambda\tilde{u})}_t]_{\mathbb{P}, \mathcal{B}(H)} = [X^{1,(x,u)}_t + \lambda X^{1,(x,\tilde{u})}_t]_{\mathbb{P}, \mathcal{B}(H)}. \tag{47}$$

This proves (42) in the base case $k = 1$. For the induction step $\{1, 2, \ldots, n-1\} \ni k \to k+1 \in \{2, 3, \ldots, n\}$ of item (ii) and (42) assume that there exists a natural number $k \in \{1, 2, \ldots, n-1\}$ such that item (ii) and (42) hold for $k = 1$, $k = 2$, $\ldots$, $k = k$. This ensures that for all $l \in \{0, 1\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} \in \mathbb{D}_{k+1}$, $\mathbf{u} \in H^{k+2}$ it holds that

$$\hat{L}^{\boldsymbol{\delta}, \mathbf{u}, p}_{k+1, l} < \infty. \tag{48}$$

This and Hölder's inequality imply that for all $l \in \{0,1\}$, $p \in [2,\infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $Y, Z \in \mathcal{L}^p(\mathbb{P}; H)$, $t \in (0,T]$ it holds that

$$\big\|\mathbf{G}^{\mathbf{u}}_{k+1,l}(t,Y) - \mathbf{G}^{\mathbf{u}}_{k+1,l}(t,Z)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,r_l})} \leq |G_l|_{\mathcal{C}^1_{\mathrm{b}}(H,V_{l,r_l})} \, \|Y-Z\|_{\mathcal{L}^p(\mathbb{P};H)} \tag{49}$$

and

$$\begin{aligned}
&\big\|\mathbf{G}^{\mathbf{u}}_{k+1,l}(t,0)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,r_l})} \\
&\leq \sum_{\varpi \in \Pi^*_{k+1}} \big\|G^{(\#_\varpi)}_l(X^{0,u_0}_t)\big(X^{\#_{I^\varpi_1},[\mathbf{u}]^\varpi_1}_t, X^{\#_{I^\varpi_2},[\mathbf{u}]^\varpi_2}_t, \ldots, X^{\#_{I^\varpi_{\#_\varpi}},[\mathbf{u}]^\varpi_{\#_\varpi}}_t\big)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,r_l})} \\
&\leq \sum_{\varpi \in \Pi^*_{k+1}} |G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,V_{l,r_l})} \prod_{i=1}^{\#_\varpi} \big\|X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_t\big\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \\
&= \sum_{\varpi \in \Pi^*_{k+1}} \frac{|G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,V_{l,r_l})}}{t^{(\delta_1+\delta_2+\ldots+\delta_{k+1})}} \left(\prod_{i=1}^{\#_\varpi} \frac{t^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i}} \big\|X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_t\big\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)}}{t^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i} - (\delta_{I^\varpi_{i,1}} + \delta_{I^\varpi_{i,2}} + \ldots + \delta_{I^\varpi_{i,\#_{I^\varpi_i}}})}} \right) \\
&= \sum_{\varpi \in \Pi^*_{k+1}} \frac{|G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,V_{l,r_l})}}{t^{(\delta_1+\delta_2+\ldots+\delta_{k+1})}} \left(\prod_{i=1}^{\#_\varpi} t^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i}} \big\|X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_t\big\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \, t^{\mathbb{1}_{[2,\infty)}(\#_{I^\varpi_i}) \min\{1-\alpha, 1/2-\beta\}} \right) \\
&\leq \sum_{\varpi \in \Pi^*_{k+1}} \frac{|G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,V_{l,r_l})}}{t^{(\delta_1+\delta_2+\ldots+\delta_{k+1})}} \prod_{i=1}^{\#_\varpi} \left[t^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i}} \big\|X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_t\big\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \, |T \vee 1|^{\mathbb{1}_{[2,\infty)}(\#_{I^\varpi_i}) \min\{1-\alpha, 1/2-\beta\}} \right] \\
&\leq \left[\sum_{\varpi \in \Pi^*_{k+1}} |G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,V_{l,r_l})} \prod_{i=1}^{\#_\varpi} \sup_{s \in (0,T]} \left[s^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i}} \big\|X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_s\big\|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \right] \right] \\
&\quad \cdot |T \vee 1|^{\lfloor (k+1)/2 \rfloor \min\{1-\alpha, 1/2-\beta\}} \, t^{-(\delta_1+\delta_2+\ldots+\delta_{k+1})} = \hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k+1,l} \, t^{-(\delta_1+\delta_2+\ldots+\delta_{k+1})} < \infty.
\end{aligned} \tag{50}$$

In addition, note that (34) and (38) ensure that for all $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0,T]$ it holds that

$$[X^{k+1,\mathbf{u}}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A} \mathbf{G}^{\mathbf{u}}_{k+1,0}(s, X^{k+1,\mathbf{u}}_s) \, \mathbf{ds} + \int_0^t e^{(t-s)A} \mathbf{G}^{\mathbf{u}}_{k+1,1}(s, X^{k+1,\mathbf{u}}_s) \, dW_s. \tag{51}$$

Combining (48)–(51) with items (i)–(ii) of Theorem 2.9 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = \alpha$, $\hat{\alpha} = \sum_{i=1}^{k+1} \delta_i$, $\beta = \beta$, $\hat{\beta} = \sum_{i=1}^{k+1} \delta_i$, $L_0 = |F|_{\mathcal{C}^1_{\mathrm{b}}(H,H_{-\alpha})}$, $\hat{L}_0 = \hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k+1,0}$, $L_1 = |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H_{-\beta}))}$, $\hat{L}_1 = \hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k+1,1}$, $W = W$, $A = A$, $\mathbf{F} = \big([0,T] \times \Omega \times H \ni (t,\omega,x) \mapsto \mathbf{G}^{\mathbf{u}}_{k+1,0}(t,\omega,x) \in H_{-\alpha}\big)$, $\mathbf{B} = \big([0,T] \times \Omega \times H \ni (t,\omega,x) \mapsto (U \ni u \mapsto \mathbf{G}^{\mathbf{u}}_{k+1,1}(t,\omega,x)u \in H_{-\beta}) \in HS(U,H_{-\beta})\big)$, $\delta = -1/2$, $\lambda = \iota^{\boldsymbol{\delta}}_{\mathbb{N}}$, $\xi = (\Omega \ni \omega \mapsto 0 \in H)$ for $\mathbf{u} \in H^{k+2}$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$, $p \in [2,\infty)$ in the notation of Theorem 2.9 in [1]) ensures that for all $p \in [2,\infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$,

$\mathbf{u} \in H^{k+2}$ it holds that

$$
\begin{aligned}
&\sup_{t \in (0,T]} \big[t^{\iota^{\boldsymbol{\delta}}_{\mathbb{N}}} \, \| X^{k+1,\mathbf{u}}_t \|_{\mathcal{L}^p(\mathbb{P};H)} \big] \leq \Theta^{\alpha,\beta,\iota^{\boldsymbol{\delta}}_{\mathbb{N}}}_{A,\eta,p,T} \big(|F|_{\mathcal{C}^1_{\mathrm{b}}(H,H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H_{-\beta}))} \big) \\
&\quad \cdot \bigg[\chi^{\alpha,T}_{A,\eta} \, \hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k+1,0} \, T^{\max\{\beta-\alpha+1/2,0\}} \, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \\
&\quad + \chi^{\beta,T}_{A,\eta} \, \hat{L}^{\boldsymbol{\delta},\mathbf{u},p}_{k+1,1} \, T^{\max\{\alpha-\beta-1/2,0\}} \sqrt{\tfrac{p\,(p-1)}{2}} \, \big| \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big) \big|^{1/2} \bigg] \\
&\leq |T \vee 1|^{(k+1)} \, \Theta^{\alpha,\beta,\iota^{\boldsymbol{\delta}}_{\mathbb{N}}}_{A,\eta,p,T} \big(|F|_{\mathcal{C}^1_{\mathrm{b}}(H,H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H_{-\beta}))} \big) \\
&\quad \cdot \sum_{\varpi \in \Pi^*_{k+1}} \bigg[\chi^{\alpha,T}_{A,\eta} \, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \, |F|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,H_{-\alpha})} \\
&\quad + \chi^{\beta,T}_{A,\eta} \sqrt{\tfrac{p\,(p-1)}{2} \, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big)} \, |B|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,HS(U,H_{-\beta}))} \bigg] \\
&\quad \cdot \prod_{i=1}^{\#_\varpi} \sup_{t \in (0,T]} \Big[t^{\iota^{\boldsymbol{\delta}}_{I^\varpi_i}} \, \| X^{\#_{I^\varpi_i},[\mathbf{u}]^\varpi_i}_t \|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)} \Big].
\end{aligned}
\tag{52}
$$

This implies that for all $p \in [2,\infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_{k+1}) \in \mathbb{D}_{k+1}$ it holds that

$$
\begin{aligned}
&\sup_{\mathbf{u}=(u_0,u_1,\ldots,u_{k+1}) \in (\times_{i=0}^{k+1} H^{[i]})} \; \sup_{t \in (0,T]} \left[\frac{t^{\iota^{\boldsymbol{\delta}}_{\mathbb{N}}} \, \| X^{k+1,\mathbf{u}}_t \|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{k+1} \|u_i\|_{H_{-\delta_i}}} \right] \\
&\leq \max\{T^{(k+1)}, 1\} \, \Theta^{\alpha,\beta,\iota^{\boldsymbol{\delta}}_{\mathbb{N}}}_{A,\eta,p,T} \big(|F|_{\mathcal{C}^1_{\mathrm{b}}(H,H_{-\alpha})}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H_{-\beta}))} \big) \\
&\quad \cdot \sum_{\varpi \in \Pi^*_{k+1}} \bigg[\chi^{\alpha,T}_{A,\eta} \, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k+1} \delta_i\big) \, |F|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,H_{-\alpha})} \\
&\quad + \chi^{\beta,T}_{A,\eta} \sqrt{\tfrac{p\,(p-1)}{2} \, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k+1} \delta_i\big)} \, |B|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H,HS(U,H_{-\beta}))} \bigg] \\
&\quad \cdot \prod_{I \in \varpi} \sup_{\mathbf{u}=(u_i)_{i \in I \cup \{0\}} \in (\times_{i \in I \cup \{0\}} H^{[i]})} \; \sup_{t \in (0,T]} \left[\frac{t^{\iota^{\boldsymbol{\delta}}_I} \, \| X^{\#_I,\mathbf{u}}_t \|_{\mathcal{L}^{p\,\#_\varpi}(\mathbb{P};H)}}{\prod_{i \in I} \|u_i\|_{H_{-\delta_i}}} \right].
\end{aligned}
\tag{53}
$$

This and the induction hypothesis imply item (ii) in the case $k+1$ and thus complete the induction step for item (ii). In the next step we note that for all $\lambda \in \mathbb{R}$, $i \in \{1, 2, \ldots, k+1\}$, $\varpi \in \Pi^*_{k+1}$ and all $\mathbf{u}^{(m)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(m)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1}) \in H^{k+2}$, $m \in \{1, 2, 3\}$, with $u_i^{(3)} = u_i^{(1)} + \lambda u_i^{(2)}$ it holds that there exists a unique natural number $j \in \{1, 2, \ldots, \#_\varpi\}$ such that there exists a natural number $q \in \{1, 2, \ldots, \#_{I^\varpi_j}\}$ such that for all $l \in \{1, 2, \ldots, \#_\varpi\} \setminus \{j\}$ it holds that

$$
I^\varpi_{j,q} = i, \qquad [\mathbf{u}^{(1)}]^\varpi_l = [\mathbf{u}^{(2)}]^\varpi_l = [\mathbf{u}^{(3)}]^\varpi_l,
\tag{54}
$$

and

$$
[\mathbf{u}^{(3)}]^\varpi_j = (u_0, u_{I^\varpi_{j,1}}, u_{I^\varpi_{j,2}}, \ldots, u_{I^\varpi_{j,q-1}}, u_i^{(1)} + \lambda u_i^{(2)}, u_{I^\varpi_{j,q+1}}, u_{I^\varpi_{j,q+2}}, \ldots, u_{I^\varpi_{j,\#_{I^\varpi_j}}}).
\tag{55}
$$

In addition, observe that for all $\varpi \in \Pi^*_{k+1}$, $j \in \{1, 2, \ldots, \#_\varpi\}$ it holds that

$$
\#_{I^\varpi_j} \in \{1, 2, \ldots, k\}.
\tag{56}
$$

Moreover, observe that the induction hypothesis establishes that for all $m \in \{1, 2, \ldots, k\}$, $p \in [2, \infty)$, $x \in H$, $t \in [0, T]$ it holds that

$$H^m \ni \mathbf{u} \mapsto [X_t^{m,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P}; H) \tag{57}$$

is an m-linear function. Combining (54) and (55) with (56) hence assures that for all $\lambda \in \mathbb{R}$, $i \in \{1, 2, \ldots, k+1\}$, $\varpi \in \Pi_{k+1}^*$, $t \in [0, T]$ and all $\mathbf{u}^{(m)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(m)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1}) \in H^{k+2}$, $m \in \{1, 2, 3\}$, with $u_i^{(3)} = u_i^{(1)} + \lambda u_i^{(2)}$ it holds that there exists a unique natural number $j \in \{1, 2, \ldots, \#_\varpi\}$ such that for all $l \in \{1, 2, \ldots, \#_\varpi\} \setminus \{j\}$ it holds that

$$i \in I_j^\varpi, \qquad X_t^{\#_{I_l^\varpi}, [\mathbf{u}^{(1)}]_l^\varpi} = X_t^{\#_{I_l^\varpi}, [\mathbf{u}^{(2)}]_l^\varpi} = X_t^{\#_{I_l^\varpi}, [\mathbf{u}^{(3)}]_l^\varpi}, \tag{58}$$

and

$$[X_t^{\#_{I_j^\varpi}, [\mathbf{u}^{(1)}]_j^\varpi} + \lambda X_t^{\#_{I_j^\varpi}, [\mathbf{u}^{(2)}]_j^\varpi}]_{\mathbb{P},\mathcal{B}(H)} = [X_t^{\#_{I_j^\varpi}, [\mathbf{u}^{(3)}]_j^\varpi}]_{\mathbb{P},\mathcal{B}(H)}. \tag{59}$$

This shows that for all $\lambda \in \mathbb{R}$, $l \in \{0, 1\}$, $i \in \{1, 2, \ldots, k+1\}$, $t \in [0, T]$ and all $\mathbf{u}^{(m)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(m)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1}) \in H^{k+2}$, $m \in \{1, 2, 3\}$, with $u_i^{(3)} = u_i^{(1)} + \lambda u_i^{(2)}$ it holds that there exist $j_\varpi \in \{1, 2, \ldots, \#_\varpi\}$, $\varpi \in \Pi_{k+1}^*$, such that

$$\begin{aligned}
&\big[\mathbf{G}_{k+1,l}^{\mathbf{u}^{(3)}}(t, X_t^{k+1,\mathbf{u}^{(1)}} + \lambda X_t^{k+1,\mathbf{u}^{(2)}})\big]_{\mathbb{P},\mathcal{B}(V_{l,0})} \\
&= \Big[G_l'(X_t^{0,u_0})(X_t^{k+1,\mathbf{u}^{(1)}} + \lambda X_t^{k+1,\mathbf{u}^{(2)}}) + \textstyle\sum_{\varpi \in \Pi_{k+1}^*} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi}, [\mathbf{u}^{(1)}]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\mathbf{u}^{(1)}]_2^\varpi}, \ldots, \\
&X_t^{\#_{I_{j_\varpi-1}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi-1}^\varpi}, X_t^{\#_{I_{j_\varpi}^\varpi}, [\mathbf{u}^{(3)}]_{j_\varpi}^\varpi}, X_t^{\#_{I_{j_\varpi+1}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi+1}^\varpi}, X_t^{\#_{I_{j_\varpi+2}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{u}^{(1)}]_{\#_\varpi}^\varpi}\big)\Big]_{\mathbb{P},\mathcal{B}(V_{l,0})} \\
&= \Big[G_l'(X_t^{0,u_0})(X_t^{k+1,\mathbf{u}^{(1)}} + \lambda X_t^{k+1,\mathbf{u}^{(2)}}) + \textstyle\sum_{\varpi \in \Pi_{k+1}^*} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi}, [\mathbf{u}^{(1)}]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\mathbf{u}^{(1)}]_2^\varpi}, \ldots, \\
&X_t^{\#_{I_{j_\varpi-1}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi-1}^\varpi}, X_t^{\#_{I_{j_\varpi}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi}^\varpi} + \lambda X_t^{\#_{I_{j_\varpi}^\varpi}, [\mathbf{u}^{(2)}]_{j_\varpi}^\varpi}, X_t^{\#_{I_{j_\varpi+1}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi+1}^\varpi}, X_t^{\#_{I_{j_\varpi+2}^\varpi}, [\mathbf{u}^{(1)}]_{j_\varpi+2}^\varpi}, \ldots, \\
&X_t^{\#_{I_{\#_\varpi}^\varpi}, [\mathbf{u}^{(1)}]_{\#_\varpi}^\varpi}\big)\Big]_{\mathbb{P},\mathcal{B}(V_{l,0})} \\
&= \big[\mathbf{G}_{k+1,l}^{\mathbf{u}^{(1)}}(t, X_t^{k+1,\mathbf{u}^{(1)}}) + \lambda\, \mathbf{G}_{k+1,l}^{\mathbf{u}^{(2)}}(t, X_t^{k+1,\mathbf{u}^{(2)}})\big]_{\mathbb{P},\mathcal{B}(V_{l,0})}.
\end{aligned} \tag{60}$$

This, (51), and Lemma 3.1 in Jentzen & Pušnik [15] (with $(\Omega, \mathcal{F}, \mu) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = t$, $Y(\omega, s) = \big\|e^{(t-s)A}\mathbf{G}_{k+1,l}^{\mathbf{u}^{(3)}}(s, \omega, X_s^{k+1,\mathbf{u}^{(1)}}(\omega) + \lambda X_s^{k+1,\mathbf{u}^{(2)}}(\omega)) - e^{(t-s)A}\mathbf{G}_{k+1,l}^{\mathbf{u}^{(1)}}(s, \omega, X_s^{k+1,\mathbf{u}^{(1)}}(\omega)) - \lambda e^{(t-s)A}\mathbf{G}_{k+1,l}^{\mathbf{u}^{(2)}} (s, \omega, X_s^{k+1,\mathbf{u}^{(2)}}(\omega))\big\|_{V_{l,0}}^{(l+1)}$, $Z(\omega, s) = 0$ for $s \in [0, t]$, $t \in (0, T]$, $\omega \in \Omega$, $l \in \{0, 1\}$, $\mathbf{u}^{(3)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(1)} + \lambda u_i^{(2)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1})$, $\mathbf{u}^{(2)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(2)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1})$, $\mathbf{u}^{(1)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(1)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1}) \in H^{k+2}$, $i \in \{1, 2, \ldots, k+1\}$, $\lambda \in \mathbb{R}$ in the notation of Lemma 3.1 in Jentzen & Pušnik [15]) prove that for all $\lambda \in \mathbb{R}$, $i \in \{1, 2, \ldots, k+1\}$, $t \in [0, T]$ and all $\mathbf{u}^{(m)} = (u_0, u_1, \ldots, u_{i-1}, u_i^{(m)}, u_{i+1}, u_{i+2}, \ldots, u_{k+1}) \in H^{k+2}$, $m \in \{1, 2, 3\}$, with $u_i^{(3)} = u_i^{(1)} + \lambda u_i^{(2)}$ it holds that

$$\begin{aligned}
[X_t^{k+1,\mathbf{u}^{(1)}} + \lambda X_t^{k+1,\mathbf{u}^{(2)}}]_{\mathbb{P},\mathcal{B}(H)} &= \int_0^t e^{(t-s)A}\mathbf{G}_{k+1,0}^{\mathbf{u}^{(3)}}(s, X_s^{k+1,\mathbf{u}^{(1)}} + \lambda X_s^{k+1,\mathbf{u}^{(2)}})\, \mathbf{ds} \\
&\quad + \int_0^t e^{(t-s)A}\mathbf{G}_{k+1,1}^{\mathbf{u}^{(3)}}(s, X_s^{k+1,\mathbf{u}^{(1)}} + \lambda X_s^{k+1,\mathbf{u}^{(2)}})\, dW_s.
\end{aligned} \tag{61}$$

This and item (i) imply for all $\lambda \in \mathbb{R}$, $i \in \{1, 2, \dots, k+1\}$, $t \in [0, T]$ and all $\mathbf{u}^{(m)} = (u_0, u_1, \dots, u_{i-1}, u_i^{(m)}, u_{i+1}, u_{i+2}, \dots, u_{k+1}) \in H^{k+2}$, $m \in \{1, 2, 3\}$, with $u_i^{(3)} = u_i^{(1)} + \lambda u_i^{(2)}$ that

$$[X_t^{k+1,\mathbf{u}^{(3)}}]_{\mathbb{P},\mathcal{B}(H)} = [X_t^{k+1,\mathbf{u}^{(1)}} + \lambda X_t^{k+1,\mathbf{u}^{(2)}}]_{\mathbb{P},\mathcal{B}(H)}. \tag{62}$$

This proves (42) in the case $k+1$ and hence completes the induction step for (42). Induction thus completes the proof of item (ii) and (42).

Combining (42) with item (ii) establishes item (iii). Next we prove item (iv). We first note that item (ii) implies that for all $k \in \{1, 2, \dots, n\}$, $l \in \{0, 1\}$, $p \in (0, \infty)$, $\boldsymbol{\delta} \in \mathbb{D}_k$, $\varpi \in \mathcal{P}\big(\mathcal{P}(\{1, 2, \dots, k\}) \setminus \{\emptyset\}\big)$, $\mathbf{u} \in H^{k+1}$ it holds that

$$L_{\varpi,p}^{\boldsymbol{\delta}} + \hat{L}_{k,l}^{\boldsymbol{\delta},\mathbf{u},p} < \infty. \tag{63}$$

We next apply the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8], (43), (44), (49), (50), (51), and Proposition 2.7 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = \alpha$, $\hat{\alpha} = 0$, $\beta = \beta$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}_b^1(H,H_{-\alpha})}$, $\hat{L}_0 = \hat{L}_{k,0}^{\mathbf{0}_k,\mathbf{y},p}$, $L_1 = |B|_{\mathcal{C}_b^1(H,HS(U,H_{-\beta}))}$, $\hat{L}_1 = \hat{L}_{k,1}^{\mathbf{0}_k,\mathbf{y},p}$, $W = W$, $A = A$, $\mathbf{F} = \big([0, T] \times \Omega \times H \ni (t, \omega, z) \mapsto \mathbf{G}_{k,0}^{\mathbf{y}}(t, \omega, z) \in H_{-\alpha}\big)$, $\mathbf{B} = \big([0, T] \times \Omega \times H \ni (t, \omega, z) \mapsto (U \ni u \mapsto \mathbf{G}_{k,1}^{\mathbf{y}}(t, \omega, z)u \in H_{-\beta}) \in HS(U, H_{-\beta})\big)$, $\delta = 0$, $Y^1 = X^{k,\mathbf{x}}$, $Y^2 = X^{k,\mathbf{y}}$, $\lambda = \lambda$ for $\mathbf{x} = (x, u_1, u_2, \dots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \dots, u_k) \in H^{k+1}$, $\lambda \in (-\infty, 1/2)$, $p \in [2, \infty)$, $k \in \{1, 2, \dots, n\}$ in the notation of Proposition 2.7 in [1]) to obtain that for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $\lambda \in (-\infty, 1/2)$, $\mathbf{x} = (x, u_1, u_2, \dots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \dots, u_k) \in H^{k+1}$ it holds that

$$\begin{aligned}
&\sup_{t\in(0,T]} t^{\lambda} \left\|X_t^{k,\mathbf{x}} - X_t^{k,\mathbf{y}}\right\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \Theta_{A,\eta,p,T}^{\alpha,\beta,\lambda}\big(|F|_{\mathcal{C}_b^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_b^1(H,HS(U,H_{-\beta}))}\big) \\
&\quad\cdot \sup_{t\in(0,T]} \Bigg[t^{\lambda} \Bigg\| \int_0^t e^{(t-s)A} \left(\mathbf{G}_{k,0}^{\mathbf{x}}(s, X_s^{k,\mathbf{x}}) - \mathbf{G}_{k,0}^{\mathbf{y}}(s, X_s^{k,\mathbf{x}})\right) \mathbf{ds} \\
&\quad + \int_0^t e^{(t-s)A} \left(\mathbf{G}_{k,1}^{\mathbf{x}}(s, X_s^{k,\mathbf{x}}) - \mathbf{G}_{k,1}^{\mathbf{y}}(s, X_s^{k,\mathbf{x}})\right) dW_s \Bigg\|_{L^p(\mathbb{P};H)} \Bigg] \\
&\leq \Theta_{A,\eta,p,T}^{\alpha,\beta,\lambda}\big(|F|_{\mathcal{C}_b^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_b^1(H,HS(U,H_{-\beta}))}\big) \\
&\quad\cdot \Bigg[\sup_{t\in(0,T]} \left\{ t^{\lambda}\, \chi_{A,\eta}^{\alpha,T} \int_0^t \tfrac{\|\mathbf{G}_{k,0}^{\mathbf{x}}(s,X_s^{k,\mathbf{x}}) - \mathbf{G}_{k,0}^{\mathbf{y}}(s,X_s^{k,\mathbf{x}})\|_{\mathcal{L}^p(\mathbb{P};H_{-\alpha})}}{(t-s)^{\alpha}}\, ds \right\} \\
&\quad + \sup_{t\in(0,T]} \left\{ t^{\lambda}\, \chi_{A,\eta}^{\beta,T} \left[\tfrac{p\,(p-1)}{2} \int_0^t \tfrac{\|\mathbf{G}_{k,1}^{\mathbf{x}}(s,X_s^{k,\mathbf{x}}) - \mathbf{G}_{k,1}^{\mathbf{y}}(s,X_s^{k,\mathbf{x}})\|^2_{\mathcal{L}^p(\mathbb{P};HS(U,H_{-\beta}))}}{(t-s)^{2\beta}}\, ds \right]^{1/2} \right\} \Bigg].
\end{aligned} \tag{64}$$

Moreover, observe that (38) ensures that for all $k \in \{1, 2, \dots, n\}$, $l \in \{0, 1\}$, $\mathbf{x} = (x, u_1, u_2, \dots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \dots, u_k) \in H^{k+1}$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\mathbf{G}_{k,l}^{\mathbf{x}}(t, X_t^{k,\mathbf{x}}) - \mathbf{G}_{k,l}^{\mathbf{y}}(t, X_t^{k,\mathbf{x}}) = \left(G_l'(X_t^{0,x}) - G_l'(X_t^{0,y})\right) X_t^{k,\mathbf{x}} \\
&+ \textstyle\sum_{\varpi\in\Pi_k^*} \Big[\left(G_l^{(\#_\varpi)}(X_t^{0,x}) - G_l^{(\#_\varpi)}(X_t^{0,y})\right) \big(X_t^{\#_{I_1^\varpi},[\mathbf{y}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{y}]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{y}]_{\#_\varpi}^\varpi}\big) \\
&+ \textstyle\sum_{i=1}^{\#_\varpi} G_l^{(\#_\varpi)}(X_t^{0,x}) \big(X_t^{\#_{I_1^\varpi},[\mathbf{x}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{x}]_2^\varpi}, \dots, X_t^{\#_{I_{i-1}^\varpi},[\mathbf{x}]_{i-1}^\varpi}, X_t^{\#_{I_i^\varpi},[\mathbf{x}]_i^\varpi} - X_t^{\#_{I_i^\varpi},[\mathbf{y}]_i^\varpi}, \\
&\quad X_t^{\#_{I_{i+1}^\varpi},[\mathbf{y}]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\mathbf{y}]_{i+2}^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{y}]_{\#_\varpi}^\varpi}\big) \Big].
\end{aligned} \tag{65}$$

Next note that Hölder's inequality ensures that for all $k \in \{1, 2, \dots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{D}_k$, $\varpi \in \Pi_k$, $x \in H$, $\mathbf{y} = (y, u_1, u_2, \dots, u_k) \in \times_{i=0}^k H^{[i]}$, $t \in (0, T]$ it holds that

$$\begin{split}
&\frac{\big\|\big(G_l^{(\#_\varpi)}(X_t^{0,x}) - G_l^{(\#_\varpi)}(X_t^{0,y})\big)\big(X_t^{\#_{I_1^\varpi},[\mathbf{y}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{y}]_2^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{y}]_{\#_\varpi}^\varpi}\big)\big\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \\
&\leq \big\|G_l^{(\#_\varpi)}(X_t^{0,x}) - G_l^{(\#_\varpi)}(X_t^{0,y})\big\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))} \prod_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\mathbf{y}]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\left[\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}\right]} \\
&= \big\|G_l^{(\#_\varpi)}(X_t^{0,x}) - G_l^{(\#_\varpi)}(X_t^{0,y})\big\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))} \left[\prod_{I \in \varpi} \frac{1}{t^{\iota_I^{\boldsymbol{\delta}}}}\right] \prod_{i=1}^{\#_\varpi} \frac{t^{\iota_{I_i^\varpi}^{\boldsymbol{\delta}}} \|X_t^{\#_{I_i^\varpi},[\mathbf{y}]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\left[\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}\right]} \\
&\leq \frac{|T \vee 1|^{\lfloor k/2 \rfloor \min\{1-\alpha, 1/2-\beta\}}}{t^{(\delta_1+\delta_2+\ldots+\delta_k)}} L_{\varpi, p(\#_\varpi+1)}^{\boldsymbol{\delta}} \big\|G_l^{(\#_\varpi)}(X_t^{0,x}) - G_l^{(\#_\varpi)}(X_t^{0,y})\big\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))}.
\end{split} \tag{66}$$

In addition, Hölder's inequality establishes that for all $k \in \{1, 2, \dots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $\gamma \in [0, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{D}_k$, $\varpi \in \Pi_k^*$, $j \in \{1, 2, \dots, \#_\varpi\}$, $\mathbf{x} = (x, u_1, u_2, \dots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \dots, u_k) \in \times_{i=0}^k H^{[i]}$, $t \in (0, T]$ it holds that

$$\begin{split}
&\tfrac{1}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \big\|G_l^{(\#_\varpi)}(X_t^{0,x})\big(X_t^{\#_{I_1^\varpi},[\mathbf{x}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{x}]_2^\varpi}, \dots, X_t^{\#_{I_{j-1}^\varpi},[\mathbf{x}]_{j-1}^\varpi}, X_t^{\#_{I_j^\varpi},[\mathbf{x}]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\mathbf{y}]_j^\varpi}, \\
&\quad X_t^{\#_{I_{j+1}^\varpi},[\mathbf{y}]_{j+1}^\varpi}, X_t^{\#_{I_{j+2}^\varpi},[\mathbf{y}]_{j+2}^\varpi}, \dots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{y}]_{\#_\varpi}^\varpi}\big)\big\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})} \\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi}(H, V_{l,r_l})} \left[\prod_{i=1}^{j-1} \frac{\|X_t^{\#_{I_i^\varpi},[\mathbf{x}]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}}\right] \left[\prod_{i=j+1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\mathbf{y}]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}}\right] \\
&\quad \cdot \frac{\|X_t^{\#_{I_j^\varpi},[\mathbf{x}]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\mathbf{y}]_j^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_j^\varpi}} \|u_{I_{j,m}^\varpi}\|_{H_{-\delta_{I_{j,m}^\varpi}}}} \\
&= |G_l|_{\mathcal{C}_b^{\#_\varpi}(H, V_{l,r_l})} \left[\frac{1}{t^{\gamma + \iota_{I_j^\varpi \cup \{k+1\}}^{(\boldsymbol{\delta},0)}}} \prod_{I \in \varpi \setminus \{I_j^\varpi\}} \frac{1}{t^{\iota_I^{\boldsymbol{\delta}}}}\right] \left[\prod_{i=1}^{j-1} \frac{t^{\iota_{I_i^\varpi}^{\boldsymbol{\delta}}} \|X_t^{\#_{I_i^\varpi},[\mathbf{x}]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}}\right] \\
&\quad \cdot \left[\prod_{i=j+1}^{\#_\varpi} \frac{t^{\iota_{I_i^\varpi}^{\boldsymbol{\delta}}} \|X_t^{\#_{I_i^\varpi},[\mathbf{y}]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_i^\varpi}} \|u_{I_{i,m}^\varpi}\|_{H_{-\delta_{I_{i,m}^\varpi}}}}\right] \frac{t^{\gamma + \iota_{I_j^\varpi \cup \{k+1\}}^{(\boldsymbol{\delta},0)}} \|X_t^{\#_{I_j^\varpi},[\mathbf{x}]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\mathbf{y}]_j^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{m=1}^{\#_{I_j^\varpi}} \|u_{I_{j,m}^\varpi}\|_{H_{-\delta_{I_{j,m}^\varpi}}}} \\
&\leq \frac{|T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}}}{t^{(\gamma+\delta_1+\delta_2+\ldots+\delta_k)}} \sup_{\mathbf{v}=(v_i)_{i \in I_j^\varpi} \in (H \setminus \{0\})^{\#_{I_j^\varpi}}} \sup_{s \in (0,T]} \Bigg\{ |G_l|_{\mathcal{C}_b^{\#_\varpi}(H, V_{l,r_l})} L_{\varpi \setminus \{I_j^\varpi\}, p\#_\varpi}^{\boldsymbol{\delta}} \\
&\quad \cdot \frac{s^{\gamma + \iota_{I_j^\varpi \cup \{k+1\}}^{(\boldsymbol{\delta},0)}} \|X_s^{\#_{I_j^\varpi},(x,\mathbf{v})} - X_s^{\#_{I_j^\varpi},(y,\mathbf{v})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i \in I_j^\varpi} \|v_i\|_{H_{-\delta_i}}} \Bigg\}.
\end{split} \tag{67}$$

Combining (65)–(67) yields that for all $k \in \{1, 2, \ldots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $\gamma \in [0, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$, $\mathbf{x} = (x, u_1, u_2, \ldots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \ldots, u_k) \in \times_{i=0}^k H^{[i]}$, $t \in (0, T]$ it holds that

$$\begin{aligned} &\frac{\|\mathbf{G}^{\mathbf{x}}_{k,l}(t, X^{k,\mathbf{x}}_t) - \mathbf{G}^{\mathbf{y}}_{k,l}(t, X^{k,\mathbf{x}}_t)\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})}}{\prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \leq \frac{|T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}}}{t^{(\delta_1+\delta_2+\ldots+\delta_k)}} \\ &\quad \cdot \Bigg(\sum_{\varpi \in \Pi_k} \Big[L^{\boldsymbol{\delta}}_{\varpi, p(\#_\varpi+1)} \big\| G^{(\#_\varpi)}_l(X^{0,x}_t) - G^{(\#_\varpi)}_l(X^{0,y}_t) \big\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))} \Big] \\ &\quad + \frac{1}{t^\gamma} \sum_{\varpi \in \Pi^*_k} \sum_{I \in \varpi} \sup_{\mathbf{v}=(v_i)_{i\in I} \in (H \setminus \{0\})^{\#_I}} \sup_{s \in (0,T]} \Bigg\{ |G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H, V_{l,r_l})} L^{\boldsymbol{\delta}}_{\varpi \setminus \{I\}, p\#_\varpi} \\ &\quad \cdot \frac{s^{\gamma + \iota^{(\boldsymbol{\delta},0)}_{I \cup \{k+1\}}} \|X^{\#_I,(x,\mathbf{v})}_s - X^{\#_I,(y,\mathbf{v})}_s\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P}; H)}}{\prod_{i \in I} \|v_i\|_{H_{-\delta_i}}} \Bigg\} \Bigg). \end{aligned} \tag{68}$$

This and Minkowski's inequality imply that for all $k \in \{1, 2, \ldots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$, $\gamma \in [0, 1/2 - \sum_{i=1}^k \delta_i)$, $\mathbf{x} = (x, u_1, u_2, \ldots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \ldots, u_k) \in \times_{i=0}^k H^{[i]}$, $t \in (0, T]$ it holds that

$$\begin{aligned} &\left[\int_0^t \left(\frac{\|\mathbf{G}^{\mathbf{x}}_{k,l}(s, X^{k,\mathbf{x}}_s) - \mathbf{G}^{\mathbf{y}}_{k,l}(s, X^{k,\mathbf{x}}_s)\|_{\mathcal{L}^p(\mathbb{P}; V_{l,r_l})}}{(t-s)^{r_l} \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}} \right)^{(l+1)} ds \right]^{1/(l+1)} \leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}} \\ &\cdot \Bigg[\int_0^t \Bigg(\sum_{\varpi \in \Pi_k} \frac{L^{\boldsymbol{\delta}}_{\varpi, p(\#_\varpi+1)} \|G^{(\#_\varpi)}_l(X^{0,x}_s) - G^{(\#_\varpi)}_l(X^{0,y}_s)\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))}}{s^{(\delta_1+\delta_2+\ldots+\delta_k)} (t-s)^{r_l}} \\ &+ \sum_{\varpi \in \Pi^*_k} \sum_{I \in \varpi} \frac{1}{s^{(\gamma+\delta_1+\delta_2+\ldots+\delta_k)} (t-s)^{r_l}} \sup_{\mathbf{v}=(v_i)_{i\in I} \in (H \setminus \{0\})^{\#_I}} \sup_{w \in (0,T]} \Bigg\{ |G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H, V_{l,r_l})} L^{\boldsymbol{\delta}}_{\varpi \setminus \{I\}, p\#_\varpi} \\ &\cdot \frac{w^{\gamma + \iota^{(\boldsymbol{\delta},0)}_{I \cup \{k+1\}}} \|X^{\#_I,(x,\mathbf{v})}_w - X^{\#_I,(y,\mathbf{v})}_w\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P}; H)}}{\prod_{i \in I} \|v_i\|_{H_{-\delta_i}}} \Bigg\} \Bigg)^{(l+1)} ds \Bigg]^{1/(l+1)} \\ &\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}} \\ &\cdot \Bigg(\sum_{\varpi \in \Pi_k} L^{\boldsymbol{\delta}}_{\varpi, p(\#_\varpi+1)} \Bigg[\int_0^t \left(\frac{\|G^{(\#_\varpi)}_l(X^{0,x}_s) - G^{(\#_\varpi)}_l(X^{0,y}_s)\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P}; L^{(\#_\varpi)}(H, V_{l,r_l}))}}{s^{(\delta_1+\delta_2+\ldots+\delta_k)} (t-s)^{r_l}} \right)^{(l+1)} ds \Bigg]^{1/(l+1)} \\ &+ \sum_{\varpi \in \Pi^*_k} \sum_{I \in \varpi} t^{(1/(l+1) - r_l - \gamma - \sum_{i=1}^k \delta_i)} \Big[\mathbb{B}\big(1 - (l+1) r_l, 1 - (l+1)(\gamma + \textstyle\sum_{i=1}^k \delta_i)\big) \Big]^{1/(l+1)} \\ &\cdot \sup_{\mathbf{v}=(v_i)_{i\in I} \in (H \setminus \{0\})^{\#_I}} \sup_{w \in (0,T]} \Bigg\{ |G_l|_{\mathcal{C}^{\#_\varpi}_{\mathrm{b}}(H, V_{l,r_l})} L^{\boldsymbol{\delta}}_{\varpi \setminus \{I\}, p\#_\varpi} \frac{w^{\gamma + \iota^{(\boldsymbol{\delta},0)}_{I \cup \{k+1\}}} \|X^{\#_I,(x,\mathbf{v})}_w - X^{\#_I,(y,\mathbf{v})}_w\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P}; H)}}{\prod_{i \in I} \|v_i\|_{H_{-\delta_i}}} \Bigg\} \Bigg). \end{aligned} \tag{69}$$

Hence, we obtain that for all $k \in \{1, 2, \ldots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{D}_k$, $\lambda \in [\iota^{(\boldsymbol{\delta},0)}_{\mathbb{N}}, 1/2)$, $\gamma \in [0, \lambda - \iota^{(\boldsymbol{\delta},0)}_{\mathbb{N}}] \cap [0, 1/2 - \sum_{i=1}^k \delta_i)$, $\mathbf{x} = (x, u_1, u_2, \ldots, u_k)$, $\mathbf{y} = (y, u_1, u_2, \ldots, u_k) \in$

$\times_{i=0}^{k} H^{[i]}$ it holds that

$$\sup_{t\in(0,T]}\left\{t^{\lambda}\,\chi_{A,\eta}^{r_l,T}\left[\tfrac{p\,(p-1)}{2}\right]^{l/2}\left[\int_0^t\left(\frac{\|\mathbf{G}_{k,l}^{\mathbf{x}}(s,X_s^{k,\mathbf{x}})-\mathbf{G}_{k,l}^{\mathbf{y}}(s,X_s^{k,\mathbf{x}})\|_{\mathcal{L}^p(\mathbb{P};V_{l,r_l})}}{(t-s)^{r_l}\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right)^{(l+1)}ds\right]^{1/(l+1)}\right\}$$
$$\leq |T\vee 1|^{\lceil k/2\rceil\min\{1-\alpha,1/2-\beta\}}\Bigg(\sum_{\varpi\in\Pi_k}L_{\varpi,\,p(\#_\varpi+1)}^{\boldsymbol{\delta}}\,\chi_{A,\eta}^{r_l,T}$$
$$\cdot\sup_{t\in(0,T]}\left\{t^{\lambda}\left[\tfrac{p\,(p-1)}{2}\right]^{l/2}\left[\int_0^t\frac{\|G_l^{(\#_\varpi)}(X_s^{0,x})-G_l^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,V_{l,r_l}))}^{(l+1)}}{s^{(l+1)(\delta_1+\delta_2+\ldots+\delta_k)}\,(t-s)^{(l+1)r_l}}\,ds\right]^{1/(l+1)}\right\}$$
$$+\sum_{\varpi\in\Pi_k^*}\sum_{I\in\varpi}\chi_{A,\eta}^{r_l,T}\,T^{(\lambda+1/(l+1)-r_l-\gamma-\sum_{i=1}^k\delta_i)}\left[\tfrac{p\,(p-1)}{2}\right]^{l/2}\left[\mathbb{B}\big(1-(l+1)r_l,1-(l+1)(\gamma+\textstyle\sum_{i=1}^k\delta_i)\big)\right]^{1/(l+1)}$$
$$\cdot\sup_{\mathbf{v}=(v_i)_{i\in I}\in(H\backslash\{0\})^{\#_I}}\;\sup_{w\in(0,T]}\left\{|G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,r_l})}\,L_{\varpi\backslash\{I\},\,p\#_\varpi}^{\boldsymbol{\delta}}\,\frac{w^{\gamma+\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0)}}\|X_w^{\#_I,(x,\mathbf{v})}-X_w^{\#_I,(y,\mathbf{v})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|v_i\|_{H_{-\delta_i}}}\right\}\Bigg).\tag{70}$$

This shows that for all $k\in\{1,2,\ldots,n\}$, $p\in[2,\infty)$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in\mathbb{D}_k$, $\lambda\in[\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)},1/2)$, $\gamma\in[0,\lambda-\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}]\cap[0,1/2-\sum_{i=1}^k\delta_i)$, $x,y\in H$ it holds that

$$\sum_{l=0}^{1}\sup_{\mathbf{u}=(u_1,u_2,\ldots,u_k)\in(H\backslash\{0\})^k}\;\sup_{t\in(0,T]}\Bigg\{t^{\lambda}\,\chi_{A,\eta}^{r_l,T}\left[\tfrac{p\,(p-1)}{2}\right]^{l/2}$$
$$\cdot\left[\int_0^t\left(\frac{\|\mathbf{G}_{k,l}^{(x,\mathbf{u})}(s,X_s^{k,(x,\mathbf{u})})-\mathbf{G}_{k,l}^{(y,\mathbf{u})}(s,X_s^{k,(x,\mathbf{u})})\|_{\mathcal{L}^p(\mathbb{P};V_{l,r_l})}}{(t-s)^{r_l}\prod_{i=1}^k\|u_i\|_{H_{-\delta_i}}}\right)^{(l+1)}ds\right]^{1/(l+1)}\Bigg\}$$
$$\leq |T\vee 1|^{\lceil k/2\rceil\min\{1-\alpha,1/2-\beta\}}$$
$$\cdot\Bigg(\sum_{\varpi\in\Pi_k}L_{\varpi,p(\#_\varpi+1)}^{\boldsymbol{\delta}}\Bigg[\chi_{A,\eta}^{\alpha,T}\sup_{t\in(0,T]}\left\{t^{\lambda}\int_0^t\frac{\|F^{(\#_\varpi)}(X_s^{0,x})-F^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,H_{-\alpha}))}}{(t-s)^{\alpha}\,s^{(\delta_1+\delta_2+\ldots+\delta_k)}}\,ds\right\}$$
$$+\chi_{A,\eta}^{\beta,T}\sup_{t\in(0,T]}\left\{t^{\lambda}\left[\tfrac{p\,(p-1)}{2}\int_0^t\frac{\|B^{(\#_\varpi)}(X_s^{0,x})-B^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,HS(U,H_{-\beta})))}^2}{(t-s)^{2\beta}\,s^{2(\delta_1+\delta_2+\ldots+\delta_k)}}\,ds\right]^{1/2}\right\}\Bigg]\tag{71}$$
$$+\sum_{\varpi\in\Pi_k^*}\sum_{I\in\varpi}\sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\backslash\{0\})^{\#_I}}\;\sup_{t\in(0,T]}\Bigg\{L_{\varpi\backslash\{I\},p\,\#_\varpi}^{\boldsymbol{\delta}}\frac{t^{\gamma+\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0)}}\|X_t^{\#_I,(x,\mathbf{u})}-X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}$$
$$\cdot\Bigg[\chi_{A,\eta}^{\alpha,T}\,T^{(\lambda+1-\alpha-\gamma-\sum_{i=1}^k\delta_i)}\,|F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})}\,\mathbb{B}\big(1-\alpha,1-\gamma-\textstyle\sum_{i=1}^k\delta_i\big)$$
$$+\chi_{A,\eta}^{\beta,T}\,T^{(\lambda+1/2-\beta-\gamma-\sum_{i=1}^k\delta_i)}\,|B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))}\sqrt{\tfrac{p\,(p-1)}{2}\,\mathbb{B}\big(1-2\beta,1-2\gamma-2\textstyle\sum_{i=1}^k\delta_i\big)}\Bigg]\Bigg\}\Bigg).$$

Combining (64) with (71) yields that for all $k\in\{1,2,\ldots,n\}$, $p\in[2,\infty)$, $\boldsymbol{\delta}=(\delta_1,\delta_2,\ldots,\delta_k)\in\mathbb{D}_k$,

$\lambda \in [\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}, 1/2)$, $\gamma \in [0, \lambda - \iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}] \cap [0, 1/2 - \sum_{i=1}^{k} \delta_i)$, $x, y \in H$ it holds that

$$\begin{aligned}
&\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \frac{t^{\lambda}\, \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \\
&\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,\lambda}\big(|F|_{\mathcal{C}_{\mathrm{b}}^{1}(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^{1}(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Bigg(\sum_{\varpi\in\Pi_k} L_{\varpi,p(\#_\varpi+1)}^{\boldsymbol{\delta}} \Bigg[\chi_{A,\eta}^{\alpha,T} \sup_{t\in(0,T]} \Bigg\{ t^{\lambda} \int_0^t \frac{\|F^{(\#_\varpi)}(X_s^{0,x}) - F^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,H_{-\alpha}))}}{(t-s)^{\alpha}\, s^{(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds \Bigg\} \\
&+ \chi_{A,\eta}^{\beta,T} \sup_{t\in(0,T]} \Bigg\{ t^{\lambda} \Bigg[\frac{p\,(p-1)}{2} \int_0^t \frac{\|B^{(\#_\varpi)}(X_s^{0,x}) - B^{(\#_\varpi)}(X_s^{0,y})\|^2_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,HS(U,H_{-\beta})))}}{(t-s)^{2\beta}\, s^{2(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds \Bigg]^{1/2} \Bigg\} \Bigg] \\
&+ \sum_{\varpi\in\Pi_k^*} \sum_{I\in\varpi} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \Bigg\{ L_{\varpi\setminus\{I\},p\,\#_\varpi}^{\boldsymbol{\delta}} \frac{t^{\gamma+\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0)}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \\
&\cdot \Bigg[\chi_{A,\eta}^{\alpha,T}\, T^{(\lambda+1-\alpha-\gamma-\sum_{i=1}^{k}\delta_i)}\, |F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\gamma-\textstyle\sum_{i=1}^{k}\delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, T^{(\lambda+1/2-\beta-\gamma-\sum_{i=1}^{k}\delta_i)}\, |B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\gamma-2\textstyle\sum_{i=1}^{k}\delta_i\big)} \Bigg] \Bigg\} \Bigg).
\end{aligned} \tag{72}$$

In particular, this shows that for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_k) \in \mathbb{D}_k$, $x, y \in H$ it holds that

$$\begin{aligned}
&\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}}\, \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{k} \|u_i\|_{H_{-\delta_i}}} \\
&\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}}\big(|F|_{\mathcal{C}_{\mathrm{b}}^{1}(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^{1}(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Bigg(\sum_{\varpi\in\Pi_k} L_{\varpi,p(\#_\varpi+1)}^{\boldsymbol{\delta}} \Bigg[\chi_{A,\eta}^{\alpha,T} \sup_{t\in(0,T]} \Bigg\{ t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \int_0^t \frac{\|F^{(\#_\varpi)}(X_s^{0,x}) - F^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,H_{-\alpha}))}}{(t-s)^{\alpha}\, s^{(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds \Bigg\} \\
&+ \chi_{A,\eta}^{\beta,T} \sup_{t\in(0,T]} \Bigg\{ t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \Bigg[\frac{p\,(p-1)}{2} \int_0^t \frac{\|B^{(\#_\varpi)}(X_s^{0,x}) - B^{(\#_\varpi)}(X_s^{0,y})\|^2_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,HS(U,H_{-\beta})))}}{(t-s)^{2\beta}\, s^{2(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds \Bigg]^{1/2} \Bigg\} \Bigg] \\
&+ \sum_{\varpi\in\Pi_k^*} \sum_{I\in\varpi} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \Bigg\{ L_{\varpi\setminus\{I\},p\,\#_\varpi}^{\boldsymbol{\delta}} \frac{t^{\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0)}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{i\in I} \|u_i\|_{H_{-\delta_i}}} \\
&\cdot \Bigg[\chi_{A,\eta}^{\alpha,T}\, T^{(\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}+1-\alpha-\sum_{i=1}^{k}\delta_i)}\, |F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^{k}\delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, T^{(\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}+1/2-\beta-\sum_{i=1}^{k}\delta_i)}\, |B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^{k}\delta_i\big)} \Bigg] \Bigg\} \Bigg).
\end{aligned} \tag{73}$$

Furthermore, we note that Corollary 2.8 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = \alpha$, $\hat{\alpha} = 0$, $\beta = \beta$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}_{\mathrm{b}}^{1}(H,H_{-\alpha})}$, $\hat{L}_0 = \|F(0)\|_{H_{-\alpha}}$, $L_1 = |B|_{\mathcal{C}_{\mathrm{b}}^{1}(H,HS(U,H_{-\beta}))}$, $\hat{L}_1 = \|B(0)\|_{HS(U,H_{-\beta})}$, $W = W$, $A = A$, $\mathbf{F} = \big([0,T] \times \Omega \times H \ni (t,\omega,z) \mapsto F(z) \in H_{-\alpha}\big)$,

$\mathbf{B} = \big([0,T] \times \Omega \times H \ni (t,\omega,z) \mapsto (U \ni u \mapsto B(z)u \in H_{-\beta}) \in HS(U,H_{-\beta})\big)$, $\delta = 0$, $X^1 = X^{0,x}$, $X^2 = X^{0,y}$, $\lambda = 0$ for $x, y \in H$, $p \in [2,\infty)$ in the notation of Corollary 2.8 in [1]) and (34) show that for all $p \in [2,\infty)$, $x, y \in H$ it holds that

$$\sup_{t\in(0,T]} \|X_t^{0,x} - X_t^{0,y}\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \chi_{A,\eta}^{0,T} \|x-y\|_H \, \Theta_{A,\eta,p,T}^{\alpha,\beta,0}\big(|F|_{\mathcal{C}_b^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_b^1(H,HS(U,H_{-\beta}))}\big) < \infty. \tag{74}$$

This implies that for all $k \in \{1,2,\ldots,n\}$, $m \in \{1,2,\ldots,k\}$, $p \in [2,\infty)$, $l \in \{0,1\}$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\ldots,\delta_k) \in \mathbb{D}_k$, $x, y \in H$, $t \in (0,T]$ with $x \neq y$ it holds that

$$\begin{aligned}
& t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \left[\int_0^t \tfrac{\|G_l^{(m)}(X_s^{0,x}) - G_l^{(m)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(m)}(H,V_{l,r_l}))}^{(l+1)}}{(t-s)^{(l+1)r_l}\, s^{(l+1)(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds\right]^{1/(l+1)} \\
& \leq t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \left[\int_0^t \tfrac{1}{(t-s)^{(l+1)r_l}\, s^{(l+1)(\delta_1+\delta_2+\ldots+\delta_k)}}\, ds\right]^{1/(l+1)} \\
& \quad \cdot \sup_{s\in(0,T]} \|G_l^{(m)}(X_s^{0,x}) - G_l^{(m)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(m)}(H,V_{l,r_l}))} \\
& \leq T^{(1/(l+1) - r_l - \min\{1-\alpha, 1/2-\beta\})} \left|\mathbb{B}\big(1-(l+1)r_l, 1-(l+1)\textstyle\sum_{i=1}^k \delta_i\big)\right|^{1/(l+1)} \\
& \quad \cdot \sup_{s\in(0,T]} \|G_l^{(m)}(X_s^{0,x}) - G_l^{(m)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(m)}(H,V_{l,r_l}))} \\
& \leq T^{(1/(l+1) - r_l - \min\{1-\alpha, 1/2-\beta\})} \left|\mathbb{B}\big(1-(l+1)r_l, 1-(l+1)\textstyle\sum_{i=1}^k \delta_i\big)\right|^{1/(l+1)} \\
& \quad \cdot |G_l|_{\operatorname{Lip}^m(H,V_{l,r_l})} \sup_{s\in(0,T]} \|X_s^{0,x} - X_s^{0,y}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
& \leq T^{(1/(l+1) - r_l - \min\{1-\alpha, 1/2-\beta\})}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,0}\big(|F|_{\mathcal{C}_b^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_b^1(H,HS(U,H_{-\beta}))}\big) \\
& \quad \cdot \left|\mathbb{B}\big(1-(l+1)r_l, 1-(l+1)\textstyle\sum_{i=1}^k \delta_i\big)\right|^{1/(l+1)} \chi_{A,\eta}^{0,T}\, |G_l|_{\operatorname{Lip}^m(H,V_{l,r_l})} \, \|x-y\|_H.
\end{aligned} \tag{75}$$

Combining this with (73) establishes that for all $k \in \{1,2,\ldots,n\}$, $p \in [2,\infty)$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\ldots,\delta_k) \in$

$\mathbb{D}_k$ with $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} < \infty$ it holds that

$$\begin{aligned}
&\sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\backslash\{0\})^k} \sup_{t\in(0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] \\
&\leq |T\vee 1|^{\lceil k/2\rceil \min\{1-\alpha,1/2-\beta\}}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Bigg(\sum_{\varpi\in\Pi_k} L^{\boldsymbol{\delta}}_{\varpi,p(\#_\varpi+1)}\, \chi_{A,\eta}^{0,T}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Big[\chi_{A,\eta}^{\alpha,T}\, T^{(1-\alpha-\min\{1-\alpha,1/2-\beta\})}\, |F|_{\mathrm{Lip}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^k \delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, T^{(1/2-\beta-\min\{1-\alpha,1/2-\beta\})}\, |B|_{\mathrm{Lip}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^k \delta_i\big)}\Big] \\
&+ \sum_{\varpi\in\Pi_k^*} \sum_{I\in\varpi} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\backslash\{0\})^{\#_I}} \sup_{t\in(0,T]} \Bigg\{L^{\boldsymbol{\delta}}_{\varpi\backslash\{I\},p\,\#_\varpi} \frac{t^{\iota^{(\boldsymbol{\delta},0)}_{I\cup\{k+1\}}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i\in I}\|u_i\|_{H_{-\delta_i}}} \\
&\cdot \Big[\chi_{A,\eta}^{\alpha,T}\, T^{(1-\alpha-\min\{1-\alpha,1/2-\beta\})}\, |F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^k \delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, T^{(1/2-\beta-\min\{1-\alpha,1/2-\beta\})}\, |B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^k \delta_i\big)}\Big]\Bigg\}\Bigg).
\end{aligned} \tag{76}$$

Induction and (63) hence imply that for all $k \in \{1,2,\dots,n\}$, $p\in[2,\infty)$, $\boldsymbol{\delta} = (\delta_1,\delta_2,\dots,\delta_k)\in\mathbb{D}_k$

with $|F|_{\mathrm{Lip}^k(H,H_{-\alpha})} + |B|_{\mathrm{Lip}^k(H,HS(U,H_{-\beta}))} < \infty$ it holds that

$$\begin{aligned}
&\sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{t^{\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|x-y\|_H \prod_{i=1}^k \|u_i\|_{H_{-\delta_i}}}\right] \\
&\leq |T\vee 1|^k\, \Theta_{A,\eta,p,T}^{\alpha,\beta,\iota_{\mathbb{N}}^{(\boldsymbol{\delta},0)}} \big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Bigg(\sum_{\varpi\in\Pi_k} L^{\boldsymbol{\delta}}_{\varpi,p(\#_\varpi+1)}\, \chi_{A,\eta}^{0,T}\, \Theta_{A,\eta,p,T}^{\alpha,\beta,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))}\big) \\
&\cdot \Big[\chi_{A,\eta}^{\alpha,T}\, |F|_{\mathrm{Lip}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^k \delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, |B|_{\mathrm{Lip}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^k \delta_i\big)}\Big] \\
&+ \sum_{\varpi\in\Pi_k^*} \sum_{I\in\varpi} L^{\boldsymbol{\delta}}_{\varpi\setminus\{I\},p\,\#_\varpi} \sup_{\substack{x,y\in H,\\ x\neq y}} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \left[\frac{t^{\iota_{I\cup\{k+1\}}^{(\boldsymbol{\delta},0)}} \|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|x-y\|_H \prod_{i\in I}\|u_i\|_{H_{-\delta_i}}}\right] \\
&\cdot \Big[\chi_{A,\eta}^{\alpha,T}\, |F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})}\, \mathbb{B}\big(1-\alpha, 1-\textstyle\sum_{i=1}^k \delta_i\big) \\
&+ \chi_{A,\eta}^{\beta,T}\, |B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \sqrt{\tfrac{p\,(p-1)}{2}\, \mathbb{B}\big(1-2\beta, 1-2\textstyle\sum_{i=1}^k \delta_i\big)}\Big]\Bigg) < \infty.
\end{aligned} \tag{77}$$

This implies (14) and thus completes the proof of item (iv). To prove item (v) we first observe that (74) ensures that for all $x\in H$, $t\in[0,T]$ it holds that

$$\limsup_{H\ni y\to x} \mathbb{E}\big[\min\{1, \|X_t^{0,x} - X_t^{0,y}\|_H\}\big] = 0. \tag{78}$$

This implies for all $x\in H$, $\rho\in[0,1]$, $t\in[0,T]$ that

$$\limsup_{H\ni y\to x} \mathbb{E}\big[\min\{1, \|(X_t^{0,x} + \rho[X_t^{0,y} - X_t^{0,x}]) - X_t^{0,x}\|_H\}\big] = 0. \tag{79}$$

The fact that $\forall\, k\in\{1,2,\dots,n\},\ l\in\{0,1\}\colon G_l^{(k)}\in\mathcal{C}(H, L^{(k)}(H,V_{l,0}))$, e.g., Lemma 4.2 in Hutzenthaler et al. [13], and, e.g., item (ii) of Theorem 6.12 in Klenke [16] hence ensure that for all $k\in\{1,2,\dots,n\}$, $l\in\{0,1\}$, $x\in H$, $\rho\in[0,1]$, $t\in[0,T]$, $(x_m)_{m\in\mathbb{N}_0}\subseteq H$ with $\limsup_{m\to\infty}\|x_m - x_0\|_H = 0$ it holds that

$$\limsup_{m\to\infty} \mathbb{E}\big[\min\{1, \|G_l^{(k)}(X_t^{0,x_0} + \rho[X_t^{0,x_m} - X_t^{0,x_0}]) - G_l^{(k)}(X_t^{0,x_0})\|_{L^{(k)}(H,V_{l,0})}\}\big] = 0. \tag{80}$$

Combining this and, e.g., Lemma 4.2 in Hutzenthaler et al. [13] (with $I=\{\emptyset\}$, $c=1$, $X^m(\emptyset,\omega) = \|G_l^{(k)}(X_t^{0,x}(\omega) + \rho[X_t^{0,x_m}(\omega) - X_t^{0,x}(\omega)]) - G_l^{(k)}(X_t^{0,x}(\omega))\|_{L^{(k)}(H,V_{l,0})}$ for $\omega\in\Omega$, $t\in[0,T]$, $\rho\in[0,1]$, $l\in\{0,1\}$, $k\in\{1,2,\dots,n\}$, $(x_j)_{j\in\mathbb{N}}\in\{y\in\mathbb{M}(\mathbb{N},H)\colon \limsup_{j\to\infty}\|y_j - x\|_H = 0\}$, $m\in\mathbb{N}$, $x\in H$ in the notation of Lemma 4.2 in Hutzenthaler et al. [13]) establishes that for all $\varepsilon\in(0,\infty)$, $k\in\{1,2,\dots,n\}$, $l\in\{0,1\}$, $x\in H$, $\rho\in[0,1]$, $t\in[0,T]$, $(x_m)_{m\in\mathbb{N}}\subseteq H$ with $\limsup_{m\to\infty}\|x_m - x\|_H = 0$ it holds that

$$\begin{aligned}
\limsup_{m\to\infty} \mathbb{P}\big(\big\{\omega\in\Omega\colon \|G_l^{(k)}(X_t^{0,x}(\omega) + \rho[X_t^{0,x_m}(\omega) - X_t^{0,x}(\omega)]) \\
- G_l^{(k)}(X_t^{0,x}(\omega))\|_{L^{(k)}(H,V_{l,0})} \geq \varepsilon\big\}\big) = 0.
\end{aligned} \tag{81}$$

This, the fact that $\forall\, k \in \{1,2,\dots,n\}$, $l \in \{0,1\}$: $\sup_{x\in H} \|G_l^{(k)}(x)\|_{L^{(k)}(H,V_{l,0})} < \infty$, and, e.g., Proposition 4.5 in Hutzenthaler et al. [13] (with $I = \{\emptyset\}$, $p = p$, $V = \mathbb{R}$, $X^m(\emptyset, \omega) = \|G_l^{(k)}(X_t^{0,x_0}(\omega) + \rho[X_t^{0,x_m}(\omega) - X_t^{0,x_0}(\omega)]) - G_l^{(k)}(X_t^{0,x_0}(\omega))\|_{L^{(k)}(H,V_{l,0})}$ for $\omega \in \Omega$, $t \in [0,T]$, $\rho \in [0,1]$, $p \in (0,\infty)$, $l \in \{0,1\}$, $k \in \{1,2,\dots,n\}$, $(x_j)_{j\in\mathbb{N}_0} \in \{y \in \mathbb{M}(\mathbb{N}_0, H)\colon \limsup_{j\to\infty} \|y_j - y_0\|_H = 0\}$, $m \in \mathbb{N}_0$ in the notation of Proposition 4.5 in Hutzenthaler et al. [13]) ensure that for all $k \in \{1,2,\dots,n\}$, $l \in \{0,1\}$, $p \in (0,\infty)$, $\rho \in [0,1]$, $t \in [0,T]$, $(x_m)_{m\in\mathbb{N}_0} \subseteq H$ with $\limsup_{m\to\infty} \|x_m - x_0\|_H = 0$ it holds that

$$\limsup\nolimits_{m\to\infty} \mathbb{E}\Big[\big\|G_l^{(k)}(X_t^{0,x_0} + \rho[X_t^{0,x_m} - X_t^{0,x_0}]) - G_l^{(k)}(X_t^{0,x_0})\big\|_{L^{(k)}(H,V_{l,0})}^p\Big] = 0. \tag{82}$$

Combining Hölder's inequality and Lebesgue's theorem of dominated convergence with (82) (with $\rho = 1$ in the notation of (82)) yields that for all $k \in \{1,2,\dots,n\}$, $l \in \{0,1\}$, $p \in [2,\infty)$, $q \in (1, \frac{1}{\max\{\alpha, 2\beta, 1/2\}})$, $\lambda \in [\max\{\alpha - 1/q, \beta - 1/(2q)\}, \infty)$, $x \in H$ it holds that

$$\begin{aligned}
&\limsup_{H\ni y\to x} \sup_{t\in(0,T]} \left\{ t^\lambda \left[\int_0^t \frac{\|G_l^{(k)}(X_s^{0,x}) - G_l^{(k)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(k)}(H,V_{l,r_l}))}^{(l+1)}}{(t-s)^{(l+1)r_l}}\, ds \right]^{1/(l+1)} \right\} \\
&\leq \limsup_{H\ni y\to x} \sup_{t\in(0,T]} \left\{ t^\lambda \left[\int_0^t \frac{1}{(t-s)^{q(l+1)r_l}}\, ds \right]^{1/[q(l+1)]} \right. \\
&\quad \left. \cdot \left[\int_0^t \|G_l^{(k)}(X_s^{0,x}) - G_l^{(k)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(k)}(H,V_{l,r_l}))}^{q(l+1)/(q-1)}\, ds \right]^{(q-1)/[q(l+1)]} \right\} \\
&= \limsup_{H\ni y\to x} \sup_{t\in(0,T]} \left\{ \frac{t^{(\lambda + 1/[q(l+1)] - r_l)}}{[1 - q(l+1)r_l]^{1/[q(l+1)]}} \right. \\
&\quad \left. \cdot \left[\int_0^t \|G_l^{(k)}(X_s^{0,x}) - G_l^{(k)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(k)}(H,V_{l,r_l}))}^{q(l+1)/(q-1)}\, ds \right]^{(q-1)/[q(l+1)]} \right\} \\
&= \frac{T^{(\lambda + 1/[q(l+1)] - r_l)}}{[1 - q(l+1)r_l]^{1/[q(l+1)]}} \\
&\quad \cdot \left[\limsup_{H\ni y\to x} \int_0^T \|G_l^{(k)}(X_s^{0,x}) - G_l^{(k)}(X_s^{0,y})\|_{\mathcal{L}^p(\mathbb{P};L^{(k)}(H,V_{l,r_l}))}^{q(l+1)/(q-1)}\, ds \right]^{(q-1)/[q(l+1)]} = 0.
\end{aligned} \tag{83}$$

Moreover, observe that the fact that $\forall\, q \in (1, \frac{1}{\max\{\alpha,2\beta,1/2\}})$: $0 < \min\{1/q - \alpha, 1/(2q) - \beta\} < \min\{1 - \alpha, 1/2 - \beta\} \leq 1/2$ and (72) (with $k = k$, $p = p$, $\boldsymbol{\delta} = \mathbf{0}_k$, $\lambda = -\min\{1/q - \alpha, 1/(2q) - \beta\}$, $\gamma = \min\{1-\alpha, 1/2 - \beta\} - \min\{1/q - \alpha, 1/(2q) - \beta\}$, $x = x$, $y = y$ for $x, y \in H$, $q \in (1, \frac{1}{\max\{\alpha,2\beta,1/2\}})$, $p \in [2,\infty)$, $k \in \{1,2,\dots,n\}$ in the notation of (72)) imply that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$,

$q \in (1, \frac{1}{\max\{\alpha, 2\beta, 1/2\}})$, $x, y \in H$ it holds that

$$\begin{aligned}
&\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{t^{\min\{1/q-\alpha, 1/(2q)-\beta\}} \prod_{i=1}^k \|u_i\|_H} \right] \\
&\leq |T \vee 1|^{\lceil k/2 \rceil \min\{1-\alpha, 1/2-\beta\}} \, \Theta_{A,\eta,p,T}^{\alpha,\beta,-\min\{1/q-\alpha,1/(2q)-\beta\}} \big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H_{-\alpha})}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H_{-\beta}))} \big) \\
&\cdot \Bigg(\sum_{\varpi\in\Pi_k} L_{\varpi,p(\#_\varpi+1)}^{\mathbf{0}_k} \Bigg[\chi_{A,\eta}^{\alpha,T} \sup_{t\in(0,T]} \left\{ \frac{1}{t^{\min\{1/q-\alpha,1/(2q)-\beta\}}} \int_0^t \frac{\|F^{(\#_\varpi)}(X_s^{0,x}) - F^{(\#_\varpi)}(X_s^{0,y})\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,H_{-\alpha}))}}{(t-s)^\alpha} \, ds \right\} \\
&+ \chi_{A,\eta}^{\beta,T} \sup_{t\in(0,T]} \left\{ \frac{1}{t^{\min\{1/q-\alpha,1/(2q)-\beta\}}} \left[\frac{p\,(p-1)}{2} \int_0^t \frac{\|B^{(\#_\varpi)}(X_s^{0,x}) - B^{(\#_\varpi)}(X_s^{0,y})\|^2_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};L^{(\#_\varpi)}(H,HS(U,H_{-\beta})))}}{(t-s)^{2\beta}} \, ds \right]^{1/2} \right\} \Bigg] \\
&+ \sum_{\varpi\in\Pi_k^*} \sum_{I\in\varpi} \sup_{\mathbf{u}=(u_i)_{i\in I}\in(H\setminus\{0\})^{\#_I}} \sup_{t\in(0,T]} \Bigg\{ L_{\varpi\setminus\{I\},p\,\#_\varpi}^{\mathbf{0}_k} \frac{\|X_t^{\#_I,(x,\mathbf{u})} - X_t^{\#_I,(y,\mathbf{u})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{t^{\min\{1/q-\alpha,1/(2q)-\beta\}} \prod_{i\in I} \|u_i\|_H} \\
&\cdot \Big[\chi_{A,\eta}^{\alpha,T} \, T^{(1-\alpha-\min\{1-\alpha,1/2-\beta\})} \, |F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H_{-\alpha})} \, \mathbb{B}\big(1-\alpha, 1-\min\{1-\alpha, 1/2-\beta\} + \min\{1/q-\alpha, 1/(2q)-\beta\}\big) \\
&+ \chi_{A,\eta}^{\beta,T} \, T^{(1/2-\beta-\min\{1-\alpha,1/2-\beta\})} \, |B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H_{-\beta}))} \\
&\cdot \left[\tfrac{p\,(p-1)}{2} \, \mathbb{B}\big(1-2\beta, 1-2\min\{1-\alpha, 1/2-\beta\} + 2\min\{1/q-\alpha, 1/(2q)-\beta\}\big) \right]^{1/2} \Big] \Bigg\} \Bigg).
\end{aligned} \tag{84}$$

Induction and (83)–(84) hence ensure that for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $q \in (1, \frac{1}{\max\{\alpha, 2\beta, 1/2\}})$, $x \in H$ it holds that

$$\limsup_{H\ni y\to x} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \left[\frac{\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{t^{\min\{1/q-\alpha,1/(2q)-\beta\}} \prod_{i=1}^k \|u_i\|_H} \right] = 0. \tag{85}$$

This and (34) show that for all $k \in \{1, 2, \dots, n\}$, $p \in [2, \infty)$, $q \in (1, \frac{1}{\max\{\alpha, 2\beta, 1/2\}})$, $x \in H$ it holds that

$$\begin{aligned}
&\limsup_{H\ni y\to x} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in[0,T]} \left[\frac{\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^k \|u_i\|_H} \right] \\
&\leq T^{\min\{1/q-\alpha,1/(2q)-\beta\}} \limsup_{H\ni y\to x} \sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in(0,T]} \frac{\|X_t^{k,(x,\mathbf{u})} - X_t^{k,(y,\mathbf{u})}\|_{\mathcal{L}^p(\mathbb{P};H)}}{t^{\min\{1/q-\alpha,1/(2q)-\beta\}} \prod_{i=1}^k \|u_i\|_H} = 0.
\end{aligned} \tag{86}$$

Combining (86) with item (iii) proves item (v).

We now prove item (vi) by induction on $k \in \{1, 2, \dots, n\}$. Note that (74) ensures that for all $p \in (0, \infty)$ it holds that

$$\tilde{L}_p < \infty. \tag{87}$$

Furthermore, observe that for all $l \in \{0, 1\}$, $\mathbf{u} = (u_0, u_1) \in H^2$, $t \in [0, T]$ it holds that

$$G_l(X_t^{0,u_0+u_1}) - G_l(X_t^{0,u_0}) = \bar{\mathbf{G}}_{1,l}^{\mathbf{u}}(t, X_t^{0,u_0+u_1} - X_t^{0,u_0}). \tag{88}$$

This and (34) imply that for all $\mathbf{u} = (u_0, u_1) \in H^2$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
&[X_t^{0,u_0+u_1} - X_t^{0,u_0}]_{\mathbb{P},\mathcal{B}(H)} = [e^{tA} u_1]_{\mathbb{P},\mathcal{B}(H)} \\
&+ \int_0^t e^{(t-s)A}\, \bar{\mathbf{G}}_{1,0}^{\mathbf{u}}(s, X_s^{0,u_0+u_1} - X_s^{0,u_0})\, \mathbf{ds} + \int_0^t e^{(t-s)A}\, \bar{\mathbf{G}}_{1,1}^{\mathbf{u}}(s, X_s^{0,u_0+u_1} - X_s^{0,u_0})\, dW_s.
\end{aligned}
\tag{89}
$$

Combining this with the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8], (26), (34), and Proposition 2.7 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = 0$, $\hat{\alpha} = 0$, $\beta = 0$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}$, $\hat{L}_0 = 0$, $L_1 = |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}$, $\hat{L}_1 = 0$, $W = W$, $A = A$, $\mathbf{F} = \mathbf{G}_{1,0}^{\mathbf{u}}$, $\mathbf{B} = \mathbf{G}_{1,1}^{\mathbf{u}}$, $\delta = 0$, $Y^1 = X^{0,\theta_1^1(\mathbf{u})} - X^{0,\theta_0^1(\mathbf{u})}$, $Y^2 = X^{1,\mathbf{u}}$, $\lambda = 0$ for $\mathbf{u} = (u_0, u_1) \in H^2$, $p \in [2,\infty)$ in the notation of Proposition 2.7 in [1]) ensures that for all $p \in [2,\infty)$, $\mathbf{u} \in H^2$ it holds that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \big\| X_t^{0,\theta_1^1(\mathbf{u})} - X_t^{0,\theta_0^1(\mathbf{u})} - X_t^{1,\mathbf{u}} \big\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \Theta_{A,\eta,p,T}^{0,0,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}\big) \\
&\cdot \sup_{t\in(0,T]} \Bigg[\bigg\| \int_0^t e^{(t-s)A} \big(\bar{\mathbf{G}}_{1,0}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) - \mathbf{G}_{1,0}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) \big)\, \mathbf{ds} \\
&+ \int_0^t e^{(t-s)A} \big(\bar{\mathbf{G}}_{1,1}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) - \mathbf{G}_{1,1}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) \big)\, dW_s \bigg\|_{L^p(\mathbb{P};H)} \Bigg] \\
&\leq \chi_{A,\eta}^{0,T}\, \Theta_{A,\eta,p,T}^{0,0,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}\big) \\
&\cdot \Bigg[\int_0^T \| \bar{\mathbf{G}}_{1,0}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) - \mathbf{G}_{1,0}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) \|_{\mathcal{L}^p(\mathbb{P};H)}\, ds \\
&+ \bigg[\tfrac{p\,(p-1)}{2} \int_0^T \| \bar{\mathbf{G}}_{1,1}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) - \mathbf{G}_{1,1}^{\mathbf{u}}(s, X_s^{0,\theta_1^1(\mathbf{u})} - X_s^{0,\theta_0^1(\mathbf{u})}) \|_{\mathcal{L}^p(\mathbb{P};HS(U,H))}^2\, ds \bigg]^{1/2} \Bigg].
\end{aligned}
\tag{90}
$$

In addition, Hölder's inequality yields that for all $p \in [2,\infty)$, $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1) \in H \times (H \setminus \{0\})$, $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\frac{\| \bar{\mathbf{G}}_{1,l}^{\mathbf{u}}(t, X_t^{0,u_0+u_1} - X_t^{0,u_0}) - \mathbf{G}_{1,l}^{\mathbf{u}}(t, X_t^{0,u_0+u_1} - X_t^{0,u_0}) \|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}}{\|u_1\|_H} \\
&= \frac{1}{\|u_1\|_H} \bigg\| \int_0^1 [G_l'(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_1} - X_t^{0,u_0}]) - G_l'(X_t^{0,u_0})](X_t^{0,u_0+u_1} - X_t^{0,u_0})\, d\rho \bigg\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})} \\
&\leq \tilde{L}_{2p} \int_0^1 \| G_l'(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_1} - X_t^{0,u_0}]) - G_l'(X_t^{0,u_0}) \|_{\mathcal{L}^{2p}(\mathbb{P};L(H,V_{l,0}))}\, d\rho.
\end{aligned}
\tag{91}
$$

In the next step we combine (90) with (91) and Jensen's inequality to obtain that for all $p \in [2,\infty)$,

$\mathbf{u} = (u_0, u_1) \in H \times (H \setminus \{0\})$ it holds that

$$\sup_{t\in[0,T]} \frac{\|X_t^{0,\theta_1^1(\mathbf{u})} - X_t^{0,\theta_0^1(\mathbf{u})} - X_t^{1,\mathbf{u}}\|_{\mathcal{L}^p(\mathbb{P};H)}}{\|u_1\|_H} \leq \tilde{L}_{2p}\, \chi_{A,\eta}^{0,T}\, \Theta_{A,\eta,p,T}^{0,0,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}\big)$$
$$\cdot \Bigg[\int_0^T \int_0^1 \|F'(X_s^{0,u_0} + \rho[X_s^{0,u_0+u_1} - X_s^{0,u_0}]) - F'(X_s^{0,u_0})\|_{\mathcal{L}^{2p}(\mathbb{P};L(H,H))}\, d\rho\, ds \tag{92}$$
$$+ \left[\tfrac{p\,(p-1)}{2} \int_0^T \int_0^1 \|B'(X_s^{0,u_0} + \rho[X_s^{0,u_0+u_1} - X_s^{0,u_0}]) - B'(X_s^{0,u_0})\|^2_{\mathcal{L}^{2p}(\mathbb{P};L(H,HS(U,H)))}\, d\rho\, ds \right]^{1/2} \Bigg].$$

Furthermore, Lebesgue's theorem of dominated convergence and (82) yield that for all $m \in \{1, 2, \ldots, n\}$, $l \in \{0, 1\}$, $p \in [2, \infty)$, $u_0 \in H$ it holds that

$$\limsup_{H\ni u_1\to 0} \int_0^T \int_0^1 \|G_l^{(m)}(X_s^{0,u_0} + \rho[X_s^{0,u_0+u_1} - X_s^{0,u_0}]) - G_l^{(m)}(X_s^{0,u_0})\|^{(l+1)}_{\mathcal{L}^p(\mathbb{P};L^{(m)}(H,V_{l,0}))}\, d\rho\, ds = 0. \tag{93}$$

Combining (92) with (87) and (93) establishes item (vi) in the base case $k = 1$. For the induction step $\{1, 2, \ldots, n-1\} \ni k \to k+1 \in \{2, 3, \ldots, n\}$ assume that there exists a natural number $k \in \{1, 2, \ldots, n-1\}$ such that item (vi) holds for $k = 1$, $k = 2$, $\ldots$, $k = k$. Note that item (ii) ensures that for all $m \in \{1, 2, \ldots, n\}$, $p \in (0, \infty)$, $x, y \in H$, $v \in H \setminus \{0\}$ it holds that $d_{m,p}(x, y) + \tilde{d}_{m,p}(x, v) < \infty$. We also note that item (v) and the induction hypothesis assure that for all $m \in \{1, 2, \ldots, k\}$, $p \in (0, \infty)$, $x \in H$ it holds that

$$\limsup_{H\ni y\to x} d_{m,p}(x, y) = 0 \qquad \text{and} \qquad \limsup_{H\setminus\{0\}\ni v\to 0} \tilde{d}_{m,p}(x, v) = 0. \tag{94}$$

Next observe that (38) shows that for all $l \in \{0, 1\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0, T]$ it holds that

$$\mathbf{G}_{k,l}^{\theta_1^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})}) = G_l'(X_t^{0,u_0+u_{k+1}})X_t^{k,\theta_1^{k+1}(\mathbf{u})}$$
$$+ \sum_{\varpi\in\Pi_k^*} G_l^{(\#_\varpi)}(X_t^{0,u_0+u_{k+1}})\big(X_t^{\#_{I_1^\varpi},[\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big) \tag{95}$$

and

$$\mathbf{G}_{k,l}^{\theta_0^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_0^{k+1}(\mathbf{u})}) = G_l'(X_t^{0,u_0})X_t^{k,\theta_0^{k+1}(\mathbf{u})}$$
$$+ \sum_{\varpi\in\Pi_k^*} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big). \tag{96}$$

This implies that for all $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\mathbf{G}_{k,l}^{\theta_1^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})}) - \mathbf{G}_{k,l}^{\theta_0^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_0^{k+1}(\mathbf{u})}) \\
&= G_l'(X_t^{0,u_0+u_{k+1}}) X_t^{k,\theta_1^{k+1}(\mathbf{u})} - G_l'(X_t^{0,u_0}) X_t^{k,\theta_0^{k+1}(\mathbf{u})} \\
&\quad + \sum_{\varpi \in \Pi_k^*} \Big[G_l^{(\#_\varpi)}(X_t^{0,u_0+u_{k+1}}) \big(X_t^{\#_{I_1^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \\
&\quad - G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \Big] \\
&= G_l'(X_t^{0,u_0}) (X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) + [G_l'(X_t^{0,u_0+u_{k+1}}) - G_l'(X_t^{0,u_0})] X_t^{k,\theta_1^{k+1}(\mathbf{u})} \\
&\quad + \sum_{\varpi \in \Pi_k^*} \Big[[G_l^{(\#_\varpi)}(X_t^{0,u_0+u_{k+1}}) - G_l^{(\#_\varpi)}(X_t^{0,u_0})] \big(X_t^{\#_{I_1^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, \\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) + G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \\
&\quad - G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \Big].
\end{aligned} \tag{97}$$

The fundamental theorem of calculus and (39) hence yield that for all $l \in \{0,1\}$, $\mathbf{u} \in H^{k+2}$, $t \in [0,T]$ it holds that

$$\mathbf{G}_{k,l}^{\theta_1^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})}) - \mathbf{G}_{k,l}^{\theta_0^{k+1}(\mathbf{u})}(t, X_t^{k,\theta_0^{k+1}(\mathbf{u})}) = \bar{\mathbf{G}}_{k+1,l}^{\mathbf{u}}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}). \tag{98}$$

This, (34), and (38) imply that for all $\mathbf{u} \in H^{k+2}$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\big[X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}\big]_{\mathbb{P},\mathcal{B}(H)} \\
&= \int_0^t e^{(t-s)A} \big[\mathbf{G}_{k,0}^{\theta_1^{k+1}(\mathbf{u})}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})}) - \mathbf{G}_{k,0}^{\theta_0^{k+1}(\mathbf{u})}(s, X_s^{k,\theta_0^{k+1}(\mathbf{u})})\big] \, \mathbf{ds} \\
&\quad + \int_0^t e^{(t-s)A} \big[\mathbf{G}_{k,1}^{\theta_1^{k+1}(\mathbf{u})}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})}) - \mathbf{G}_{k,1}^{\theta_0^{k+1}(\mathbf{u})}(s, X_s^{k,\theta_0^{k+1}(\mathbf{u})})\big] \, dW_s \\
&= \int_0^t e^{(t-s)A} \, \bar{\mathbf{G}}_{k+1,0}^{\mathbf{u}}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})}) \, \mathbf{ds} \\
&\quad + \int_0^t e^{(t-s)A} \, \bar{\mathbf{G}}_{k+1,1}^{\mathbf{u}}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})}) \, dW_s.
\end{aligned} \tag{99}$$

Combining this with (31), (32), (34), and Proposition 2.7 in [1] (with $H = H$, $U = U$, $T = T$, $\eta = \eta$, $p = p$, $\alpha = 0$, $\hat{\alpha} = 0$, $\beta = 0$, $\hat{\beta} = 0$, $L_0 = |F|_{\mathcal{C}_b^1(H,H)}$, $\hat{L}_0 = \sum_{\varpi \in \Pi_{k+1}^*} |F|_{\mathcal{C}_b^{\#_\varpi}(H,H)} \prod_{i=1}^{\#_\varpi} \big\| \big[[X^{\#_{I_i^\varpi}, [\mathbf{u}]_i^\varpi}]\big] \big\|_{\mathbb{L}^{p\#_\varpi}}$, $L_1 = |B|_{\mathcal{C}_b^1(H,HS(U,H))}$, $\hat{L}_1 = \sum_{\varpi \in \Pi_{k+1}^*} |B|_{\mathcal{C}_b^{\#_\varpi}(H,HS(U,H))} \prod_{i=1}^{\#_\varpi} \big\| \big[[X^{\#_{I_i^\varpi}, [\mathbf{u}]_i^\varpi}]\big] \big\|_{\mathbb{L}^{p\#_\varpi}}$, $W = W$, $A = A$, $\mathbf{F} = \mathbf{G}_{k+1,0}^{\mathbf{u}}$, $\mathbf{B} = \mathbf{G}_{k+1,1}^{\mathbf{u}}$, $\delta = 0$, $Y^1 = X^{k,\theta_1^{k+1}(\mathbf{u})} - X^{k,\theta_0^{k+1}(\mathbf{u})}$, $Y^2 = X^{k+1,\mathbf{u}}$, $\lambda = 0$ for $\mathbf{u} \in H^{k+2}$, $p \in [2,\infty)$ in the notation of Proposition 2.7 in [1]) implies that for all $p \in [2,\infty)$, $\mathbf{u} \in H^{k+2}$ it holds that

$$\sup_{t\in[0,T]} \big\|X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})} - X_t^{k+1,\mathbf{u}}\big\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \Theta^{0,0,0}_{A,\eta,p,T}\big(|F|_{\mathcal{C}^1_{\mathrm{b}}(H,H)}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H))}\big)$$
$$\cdot \sup_{t\in(0,T]} \Bigg[\bigg\| \int_0^t e^{(t-s)A}\big(\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,0}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,0}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})})\big)\,\mathbf{ds}$$
$$+ \int_0^t e^{(t-s)A}\big(\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,1}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,1}(s, X_s^{k,\theta_1^{k+1}(\mathbf{u})} - X_s^{k,\theta_0^{k+1}(\mathbf{u})})\big)\,dW_s\bigg\|_{L^p(\mathbb{P};H)}\Bigg]. \tag{100}$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [8] hence shows that for all $p \in [2,\infty)$, $\mathbf{u} \in H^{k+2}$ it holds that

$$\sup_{t\in[0,T]} \big\|X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})} - X_t^{k+1,\mathbf{u}}\big\|_{\mathcal{L}^p(\mathbb{P};H)} \leq \chi^{0,T}_{A,\eta}\,\Theta^{0,0,0}_{A,\eta,p,T}\big(|F|_{\mathcal{C}^1_{\mathrm{b}}(H,H)}, |B|_{\mathcal{C}^1_{\mathrm{b}}(H,HS(U,H))}\big)$$
$$\cdot \Bigg[\int_0^T \|\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,0}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,0}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})})\|_{\mathcal{L}^p(\mathbb{P};H)}\,dt + \Big[\tfrac{p\,(p-1)}{2}\Big]^{1/2}$$
$$\cdot \bigg[\int_0^T \|\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,1}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,1}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})})\|^2_{\mathcal{L}^p(\mathbb{P};HS(U,H))}\,dt\bigg]^{1/2}\Bigg]. \tag{101}$$

Next observe that for all $m \in \mathbb{N}$ it holds that

$$\begin{aligned}&\Pi_{m+1} = \Big\{\varpi \cup \big\{\{m+1\}\big\} \colon \varpi \in \Pi_m\Big\}\\ &\biguplus \Big\{\big\{I_1^\varpi, I_2^\varpi, \dots, I_{i-1}^\varpi, I_i^\varpi \cup \{m+1\}, I_{i+1}^\varpi, I_{i+2}^\varpi, \dots, I_{\#_\varpi}^\varpi\big\} \colon i \in \{1,2,\dots,\#_\varpi\},\ \varpi \in \Pi_m\Big\}.\end{aligned} \tag{102}$$

This implies that for all $m \in \mathbb{N}$ it holds that

$$\begin{aligned}&\Pi^*_{m+1} = \Big\{\varpi \cup \big\{\{m+1\}\big\} \colon \varpi \in \Pi_m\Big\}\\ &\biguplus \Big\{\big\{I_1^\varpi, I_2^\varpi, \dots, I_{i-1}^\varpi, I_i^\varpi \cup \{m+1\}, I_{i+1}^\varpi, I_{i+2}^\varpi, \dots, I_{\#_\varpi}^\varpi\big\} \colon i \in \{1,2,\dots,\#_\varpi\},\ \varpi \in \Pi^*_m\Big\}\\ &= \Big\{\big\{\{1,2,\dots,m\},\{m+1\}\big\}\Big\} \biguplus \Bigg[\bigcup_{\varpi\in\Pi^*_m} \bigg(\Big\{\varpi \cup \big\{\{m+1\}\big\}\Big\}\\ &\biguplus \Big\{\big\{I_1^\varpi, I_2^\varpi, \dots, I_{i-1}^\varpi, I_i^\varpi \cup \{m+1\}, I_{i+1}^\varpi, I_{i+2}^\varpi, \dots, I_{\#_\varpi}^\varpi\big\} \colon i \in \{1,2,\dots,\#_\varpi\}\Big\}\bigg)\Bigg].\end{aligned} \tag{103}$$

This and (38) prove that for all $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1, \dots, u_{k+1}) \in H^{k+2}$, $x \in H$, $t \in [0,T]$ it holds that

$$\mathbf{G}^{\mathbf{u}}_{k+1,l}(t,x) = G_l'(X_t^{0,u_0})x + \sum_{\varpi\in\Pi^*_{k+1}} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\mathbf{u}]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\mathbf{u}]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\mathbf{u}]_{\#_\varpi}^\varpi}\big)$$

$$\begin{aligned}
&= G_l'(X_t^{0,u_0})\,x + G_l''(X_t^{0,u_0})\big(X_t^{k,\theta_0^{k+1}(\mathbf{u})}, X_t^{1,(u_0,u_{k+1})}\big)\\
&+ \sum_{\varpi\in\Pi_k^*}\Bigg[G_l^{(\#_\varpi+1)}(X_t^{0,u_0})\Big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{1,(u_0,u_{k+1})}\Big)\\
&+ \sum_{i=1}^{\#_\varpi} G_l^{(\#_\varpi)}(X_t^{0,u_0})\Big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi},\\
&\quad X_t^{\#_{I_i^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_i^\varpi,u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\Big)\Bigg].
\end{aligned} \tag{104}$$

Moreover, observe that (39) shows that for all $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $x \in H$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t,x) = G_l'(X_t^{0,u_0})\,x\\
&+ \int_0^1 G_l''\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big)\big(X_t^{k,\theta_1^{k+1}(\mathbf{u})}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\,d\rho\\
&+ \sum_{\varpi\in\Pi_k^*}\Bigg[\int_0^1 G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big)\big(X_t^{\#_{I_1^\varpi},[\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots,\\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\,d\rho\\
&+ \sum_{i=1}^{\#_\varpi} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi}\\
&\quad - X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}, X_t^{\#_{I_{i+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Bigg].
\end{aligned} \tag{105}$$

This implies that for all $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) \\
&= \sum_{\varpi \in \Pi_k^*} \sum_{i=1}^{\#_\varpi} \Big[G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi}+1, ([\theta_0^{k+1}(\mathbf{u})]_i^\varpi, u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{i+1}^\varpi}, \\
&X_t^{\#_{I_{i+2}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \\
&+ G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi}+1, ([\theta_0^{k+1}(\mathbf{u})]_i^\varpi, u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \\
&- G_l^{(\#_\varpi)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi}+1, ([\theta_0^{k+1}(\mathbf{u})]_i^\varpi, u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi} \big) \Big] \\
&+ \sum_{\varpi \in \Pi_k} \Big[\int_0^1 \big[G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho [X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}] \big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} \big) \big] \big(X_t^{\#_{I_1^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, \\
&X_t^{\#_{I_2^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} \big) \, d\rho \\
&+ G_l^{(\#_\varpi+1)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, \\
&X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} - X_t^{1,(u_0,u_{k+1})} \big) \\
&+ G_l^{(\#_\varpi+1)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} \big) \\
&- G_l^{(\#_\varpi+1)}(X_t^{0,u_0}) \big(X_t^{\#_{I_1^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi}, [\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} \big) \Big].
\end{aligned}
\tag{106}
$$

This assures that for all $l \in \{0,1\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in H^{k+2}$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) \\
&= \sum_{\varpi \in \Pi_k^*} \sum_{i=1}^{\#_\varpi} \Bigg[G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_i^\varpi, u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+1}^\varpi}, \\
&X_t^{\#_{I_{i+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big) \\
&+ \sum_{j=i+1}^{\#_\varpi} G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_i^\varpi, u_{k+1})}, X_t^{\#_{I_{i+1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{j-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j-1}^\varpi}, \\
&X_t^{\#_{I_j^\varpi},[\theta_1^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\theta_0^{k+1}(\mathbf{u})]_j^\varpi}, X_t^{\#_{I_{j+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+1}^\varpi}, X_t^{\#_{I_{j+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Bigg] \\
&+ \sum_{\varpi \in \Pi_k} \Bigg[\int_0^1 \big[G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big]\big(X_t^{\#_{I_1^\varpi},[\theta_1^{k+1}(\mathbf{u})]_1^\varpi}, \\
&X_t^{\#_{I_2^\varpi},[\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\, d\rho \\
&+ G_l^{(\#_\varpi+1)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, \\
&X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} - X_t^{1,(u_0,u_{k+1})}\big) \\
&+ \sum_{i=1}^{\#_\varpi} G_l^{(\#_\varpi+1)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{i-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{i-1}^\varpi}, \\
&X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi} - X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}, X_t^{\#_{I_{i+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+1}^\varpi}, X_t^{\#_{I_{i+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{i+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, \\
&X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\Bigg].
\end{aligned}
\tag{107}
$$

Furthermore, Hölder's inequality shows that for all $l \in \{0,1\}$, $p \in [2,\infty)$, $\varpi \in \Pi_k^*$, $j \in \{1,2,\ldots,\#_\varpi\}$,

$m \in \{j+1, j+2, \ldots, \#_\varpi\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in \times_{i=0}^{k+1} H^{[i]}$, $t \in (0,T]$ it holds that

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^{k+1} \|u_i\|_H} \Big\| G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{j-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j-1}^\varpi}, \\
&\quad X_t^{\#_{I_j^\varpi},[\theta_1^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\theta_0^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_j^\varpi, u_{k+1})}, X_t^{\#_{I_{j+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+1}^\varpi}, \\
&\quad X_t^{\#_{I_{j+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+2}^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})} \\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} \left[\prod_{i=1}^{j-1} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right] \left[\prod_{i=j+1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right] \\
&\quad \cdot \frac{\|X_t^{\#_{I_j^\varpi},[\theta_1^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\theta_0^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_j^\varpi, u_{k+1})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|u_{k+1}\|_H \prod_{q=1}^{\#_{I_j^\varpi}} \|u_{I_{j,q}^\varpi}\|_H} \\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} L^{\mathbf{0}_k}_{\varpi\setminus\{I_j^\varpi\},p\,\#_\varpi} \tilde{d}_{\#_{I_j^\varpi}+1,p\,\#_\varpi}(u_0,u_{k+1}) \prod\nolimits_{I\in\varpi\setminus\{I_j^\varpi\}} t^{-\iota_I^{\mathbf{0}_k}} \\
&\leq |T\vee 1|^{\lfloor k/2\rfloor \min\{1-\alpha, 1/2-\beta\}} |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} L^{\mathbf{0}_k}_{\varpi\setminus\{I_j^\varpi\},p\,\#_\varpi} \tilde{d}_{\#_{I_j^\varpi}+1,p\,\#_\varpi}(u_0,u_{k+1})
\end{aligned} \tag{108}$$

and

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^{k+1} \|u_i\|_H} \Big\| G_l^{(\#_\varpi)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{j-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j-1}^\varpi}, \\
&\quad X_t^{\#_{I_j^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_j^\varpi, u_{k+1})}, X_t^{\#_{I_{j+1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j+1}^\varpi}, X_t^{\#_{I_{j+2}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j+2}^\varpi}, \ldots, X_t^{\#_{I_{m-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{m-1}^\varpi}, \\
&\quad X_t^{\#_{I_m^\varpi},[\theta_1^{k+1}(\mathbf{u})]_m^\varpi} - X_t^{\#_{I_m^\varpi},[\theta_0^{k+1}(\mathbf{u})]_m^\varpi}, X_t^{\#_{I_{m+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{m+1}^\varpi}, X_t^{\#_{I_{m+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{m+2}^\varpi}, \ldots, \\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}\big)\Big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})} \\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} \left[\prod_{i\in\{1,2,\ldots,m-1\}\setminus\{j\}} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right] \\
&\quad \cdot \left[\prod_{i=m+1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right] \left[\frac{\|X_t^{\#_{I_j^\varpi}+1,([\theta_0^{k+1}(\mathbf{u})]_j^\varpi, u_{k+1})}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\|u_{k+1}\|_H \prod_{q=1}^{\#_{I_j^\varpi}} \|u_{I_{j,q}^\varpi}\|_H}\right] \\
&\quad \cdot \frac{\|X_t^{\#_{I_m^\varpi},[\theta_1^{k+1}(\mathbf{u})]_m^\varpi} - X_t^{\#_{I_m^\varpi},[\theta_0^{k+1}(\mathbf{u})]_m^\varpi}\|_{\mathcal{L}^{p\#_\varpi}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_m^\varpi}} \|u_{I_{m,q}^\varpi}\|_H} \\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} L^{\mathbf{0}_{k+1}}_{\{I_j^\varpi\cup\{k+1\}\},p\,\#_\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I_j^\varpi, I_m^\varpi\},p\,\#_\varpi} d_{\#_{I_m^\varpi},p\,\#_\varpi}(u_0, u_0+u_{k+1}) \\
&\quad \cdot t^{-\iota^{\mathbf{0}_{k+1}}_{I_j^\varpi\cup\{k+1\}}} \prod\nolimits_{I\in\varpi\setminus\{I_j^\varpi, I_m^\varpi\}} t^{-\iota_I^{\mathbf{0}_k}} \\
&\leq |T\vee 1|^{\lfloor k/2\rfloor \min\{1-\alpha, 1/2-\beta\}} |G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})} L^{\mathbf{0}_{k+1}}_{\{I_j^\varpi\cup\{k+1\}\},p\,\#_\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I_j^\varpi, I_m^\varpi\},p\,\#_\varpi} \\
&\quad \cdot d_{\#_{I_m^\varpi},p\,\#_\varpi}(u_0, u_0+u_{k+1}).
\end{aligned} \tag{109}$$

In addition, Hölder's inequality also shows that for all $l \in \{0,1\}$, $p \in [2,\infty)$, $\varpi \in \Pi_k$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in \times_{i=0}^{k+1} H^{[i]}$, $t \in (0,T]$ it holds that

$$
\begin{split}
&\frac{1}{\prod_{i=1}^{k+1} \|u_i\|_H} \bigg\| \int_0^1 \big[G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big] \big(X_t^{\#_{I_1^\varpi},[\theta_1^{k+1}(\mathbf{u})]_1^\varpi},\\
&\quad X_t^{\#_{I_2^\varpi},[\theta_1^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\, d\rho \bigg\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\\
&\leq \int_0^1 \big\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\, d\rho\\
&\quad \cdot \left[\prod_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right] \frac{\|X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};H)}}{\|u_{k+1}\|_H}\\
&\leq \int_0^1 \big\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\, d\rho\\
&\quad \cdot L_{\varpi,p(\#_\varpi+2)}^{\mathbf{0}_k}\, \tilde{L}_{p(\#_\varpi+2)} \prod_{I\in\varpi} t^{-\iota_I^{\mathbf{0}_k}}\\
&\leq \int_0^1 \big\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\, d\rho\\
&\quad \cdot |T \vee 1|^{\lfloor k/2 \rfloor \min\{1-\alpha, 1/2-\beta\}}\, L_{\varpi,p(\#_\varpi+2)}^{\mathbf{0}_k}\, \tilde{L}_{p(\#_\varpi+2)}.
\end{split}
\tag{110}
$$

Again Hölder's inequality assures that for all $l \in \{0,1\}$, $p \in [2,\infty)$, $\varpi \in \Pi_k$, $j \in \{1,2,\ldots,\#_\varpi\}$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in \times_{i=0}^{k+1} H^{[i]}$, $t \in (0,T]$ it holds that

$$
\begin{split}
&\frac{1}{\prod_{i=1}^{k+1} \|u_i\|_H} \big\|G_l^{(\#_\varpi+1)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots,\\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} - X_t^{1,(u_0,u_{k+1})}\big)\big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi+1}(H,V_{l,0})} \left[\prod_{i=1}^{\#_\varpi} \frac{\|X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}} \|u_{I_{i,q}^\varpi}\|_H}\right]\\
&\quad \cdot \frac{\|X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0} - X_t^{1,(u_0,u_{k+1})}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\|u_{k+1}\|_H}\\
&\leq |T \vee 1|^{\lfloor k/2 \rfloor \min\{1-\alpha, 1/2-\beta\}}\, |G_l|_{\mathcal{C}_b^{\#_\varpi+1}(H,V_{l,0})}\, L_{\varpi,p(\#_\varpi+1)}^{\mathbf{0}_k}\, \tilde{d}_{1,p(\#_\varpi+1)}(u_0, u_{k+1})
\end{split}
\tag{111}
$$

and

$$
\begin{aligned}
&\frac{1}{\prod_{i=1}^{k+1}\|u_i\|_H}\,\Big\|G_l^{(\#_\varpi+1)}(X_t^{0,u_0})\big(X_t^{\#_{I_1^\varpi},[\theta_0^{k+1}(\mathbf{u})]_1^\varpi}, X_t^{\#_{I_2^\varpi},[\theta_0^{k+1}(\mathbf{u})]_2^\varpi}, \ldots, X_t^{\#_{I_{j-1}^\varpi},[\theta_0^{k+1}(\mathbf{u})]_{j-1}^\varpi},\\
&\quad X_t^{\#_{I_j^\varpi},[\theta_1^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\theta_0^{k+1}(\mathbf{u})]_j^\varpi}, X_t^{\#_{I_{j+1}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+1}^\varpi}, X_t^{\#_{I_{j+2}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{j+2}^\varpi}, \ldots,\\
&\quad X_t^{\#_{I_{\#_\varpi}^\varpi},[\theta_1^{k+1}(\mathbf{u})]_{\#_\varpi}^\varpi}, X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\big)\Big\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}\\
&\leq |G_l|_{\mathcal{C}_b^{\#_\varpi+1}(H,V_{l,0})}\left[\prod_{i=1}^{j-1}\frac{\|X_t^{\#_{I_i^\varpi},[\theta_0^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}}\|u_{I_{i,q}^\varpi}\|_H}\right]\left[\prod_{i=j+1}^{\#_\varpi}\frac{\|X_t^{\#_{I_i^\varpi},[\theta_1^{k+1}(\mathbf{u})]_i^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_i^\varpi}}\|u_{I_{i,q}^\varpi}\|_H}\right]\\
&\quad\cdot\left[\frac{\|X_t^{\#_{I_j^\varpi},[\theta_1^{k+1}(\mathbf{u})]_j^\varpi} - X_t^{\#_{I_j^\varpi},[\theta_0^{k+1}(\mathbf{u})]_j^\varpi}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\prod_{q=1}^{\#_{I_j^\varpi}}\|u_{I_{j,q}^\varpi}\|_H}\right]\frac{\|X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}\|_{\mathcal{L}^{p(\#_\varpi+1)}(\mathbb{P};H)}}{\|u_{k+1}\|_H}\\
&\leq |T\vee 1|^{\lfloor k/2\rfloor\min\{1-\alpha,1/2-\beta\}}\,|G_l|_{\mathcal{C}_b^{\#_\varpi+1}(H,V_{l,0})}\,L^{\mathbf{0}_k}_{\varpi\setminus\{I_j^\varpi\},p(\#_\varpi+1)}\,\tilde{L}_{p(\#_\varpi+1)}\\
&\quad\cdot d_{\#_{I_j^\varpi},p(\#_\varpi+1)}(u_0,u_0+u_{k+1}).
\end{aligned}
\tag{112}
$$

Combining (107)–(112) yields that for all $l\in\{0,1\}$, $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1,\ldots,u_{k+1})\in\times_{i=0}^{k+1}H^{[i]}$, $t\in(0,T]$ it holds that

$$
\begin{aligned}
&\frac{\|\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t,X_t^{k,\theta_1^{k+1}(\mathbf{u})}-X_t^{k,\theta_0^{k+1}(\mathbf{u})})-\mathbf{G}^{\mathbf{u}}_{k+1,l}(t,X_t^{k,\theta_1^{k+1}(\mathbf{u})}-X_t^{k,\theta_0^{k+1}(\mathbf{u})})\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}}{\prod_{i=1}^{k+1}\|u_i\|_H}\leq|T\vee 1|^k\\
&\cdot\Bigg(\sum_{\varpi\in\Pi_k^*}|G_l|_{\mathcal{C}_b^{\#_\varpi}(H,V_{l,0})}\sum_{I\in\varpi}\Bigg[L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p\,\#_\varpi}\,\tilde{d}_{\#_I+1,p\,\#_\varpi}(u_0,u_{k+1})\\
&+L^{\mathbf{0}_{k+1}}_{\{I\cup\{k+1\}\},p\,\#_\varpi}\sum_{J\in\varpi\colon\,\min(J)>\min(I)}L^{\mathbf{0}_k}_{\varpi\setminus\{I,J\},p\,\#_\varpi}\,d_{\#_J,p\,\#_\varpi}(u_0,u_0+u_{k+1})\Bigg]\\
&+\sum_{\varpi\in\Pi_k}\Bigg[L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+2)}\,\tilde{L}_{p(\#_\varpi+2)}\\
&\cdot\int_0^1\Big\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}+\rho[X_t^{0,u_0+u_{k+1}}-X_t^{0,u_0}]\big)-G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\Big\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\,d\rho\\
&+|G_l|_{\mathcal{C}_b^{\#_\varpi+1}(H,V_{l,0})}\Bigg(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+1)}\,\tilde{d}_{1,p(\#_\varpi+1)}(u_0,u_{k+1})\\
&+\sum_{I\in\varpi}L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p(\#_\varpi+1)}\,\tilde{L}_{p(\#_\varpi+1)}\,d_{\#_I,p(\#_\varpi+1)}(u_0,u_0+u_{k+1})\Bigg)\Bigg]\Bigg).
\end{aligned}
\tag{113}
$$

This and Minkowski's inequality imply that for all $l\in\{0,1\}$, $p\in[2,\infty)$, $\mathbf{u}=(u_0,u_1,\ldots,u_{k+1})\in$

$\times_{i=0}^{k+1} H^{[i]}$ it holds that

$$
\begin{aligned}
&\left[\int_0^T \left(\frac{\|\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})})\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}}{\prod_{i=1}^{k+1}\|u_i\|_H}\right)^{(l+1)} dt\right]^{1/(l+1)} \\
&\leq |T \vee 1|^k \left[\int_0^T \left(\sum_{\varpi\in\Pi_k^*} |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})} \sum_{I\in\varpi}\left[L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p\,\#_\varpi}\, \tilde{d}_{\#_I+1,p\,\#_\varpi}(u_0,u_{k+1})\right.\right.\right. \\
&\quad \left. + L^{\mathbf{0}_{k+1}}_{\{I\cup\{k+1\}\},p\,\#_\varpi} \sum_{J\in\varpi\colon\ \min(J)>\min(I)} L^{\mathbf{0}_k}_{\varpi\setminus\{I,J\},p\,\#_\varpi}\, d_{\#_J,p\,\#_\varpi}(u_0,u_0+u_{k+1})\right] \\
&\quad + \sum_{\varpi\in\Pi_k}\left[L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+2)}\, \tilde{L}_{p(\#_\varpi+2)}\right. \\
&\quad \cdot \int_0^1 \left\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}+\rho[X_t^{0,u_0+u_{k+1}}-X_t^{0,u_0}]\big) - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\right\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\, d\rho \\
&\quad + |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,V_{l,0})}\left(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+1)}\, \tilde{d}_{1,p(\#_\varpi+1)}(u_0,u_{k+1})\right. \\
&\quad \left.\left.\left.\left. + \sum_{I\in\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p(\#_\varpi+1)}\, \tilde{L}_{p(\#_\varpi+1)}\ d_{\#_I,p(\#_\varpi+1)}(u_0,u_0+u_{k+1})\right)\right]\right)^{(l+1)} dt\right]^{1/(l+1)} \\
&\leq |T\vee 1|^k \left\{\left[\int_0^T \left(\sum_{\varpi\in\Pi_k^*} |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})} \sum_{I\in\varpi}\left[L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p\,\#_\varpi}\, \tilde{d}_{\#_I+1,p\,\#_\varpi}(u_0,u_{k+1})\right.\right.\right.\right. \\
&\quad \left.\left.\left. + L^{\mathbf{0}_{k+1}}_{\{I\cup\{k+1\}\},p\,\#_\varpi} \sum_{J\in\varpi\colon\ \min(J)>\min(I)} L^{\mathbf{0}_k}_{\varpi\setminus\{I,J\},p\,\#_\varpi}\, d_{\#_J,p\,\#_\varpi}(u_0,u_0+u_{k+1})\right]\right)^{(l+1)} dt\right]^{1/(l+1)} \\
&\quad + \sum_{\varpi\in\Pi_k}\left\{\left[\int_0^T \left(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+2)}\, \tilde{L}_{p(\#_\varpi+2)} \int_0^1 \left\|G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}+\rho[X_t^{0,u_0+u_{k+1}}-X_t^{0,u_0}]\big)\right.\right.\right.\right. \\
&\quad \left.\left. - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\right\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))}\, d\rho\right)^{(l+1)} dt\Bigg]^{1/(l+1)} \\
&\quad + \left[\int_0^T \left(|G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,V_{l,0})}\left(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+1)}\, \tilde{d}_{1,p(\#_\varpi+1)}(u_0,u_{k+1})\right.\right.\right. \\
&\quad \left.\left.\left.\left.\left. + \sum_{I\in\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p(\#_\varpi+1)}\, \tilde{L}_{p(\#_\varpi+1)}\ d_{\#_I,p(\#_\varpi+1)}(u_0,u_0+u_{k+1})\right)\right)^{(l+1)} dt\right]^{1/(l+1)}\right\}\right\}.
\end{aligned}
\tag{114}
$$

Jensen's inequality hence shows that for all $l \in \{0,1\}$, $p \in [2,\infty)$, $\mathbf{u} = (u_0, u_1, \ldots, u_{k+1}) \in \times_{i=0}^{k+1} H^{[i]}$

it holds that

$$
\begin{aligned}
&\left[\int_0^T \left(\frac{\|\bar{\mathbf{G}}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})}) - \mathbf{G}^{\mathbf{u}}_{k+1,l}(t, X_t^{k,\theta_1^{k+1}(\mathbf{u})} - X_t^{k,\theta_0^{k+1}(\mathbf{u})})\|_{\mathcal{L}^p(\mathbb{P};V_{l,0})}}{\prod_{i=1}^{k+1} \|u_i\|_H} \right)^{(l+1)} dt \right]^{1/(l+1)} \\
&\leq |T \vee 1|^k \Bigg\{ \sum_{\varpi \in \Pi_k^*} T^{1/(l+1)} \, |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})} \sum_{I \in \varpi} \Bigg[L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p\,\#_\varpi} \, \tilde{d}_{\#_I+1,p\,\#_\varpi}(u_0, u_{k+1}) \\
&\quad + L^{\mathbf{0}_{k+1}}_{\{I\cup\{k+1\}\},p\,\#_\varpi} \sum_{J\in\varpi\colon \min(J)>\min(I)} L^{\mathbf{0}_k}_{\varpi\setminus\{I,J\},p\,\#_\varpi} \, d_{\#_J,p\,\#_\varpi}(u_0, u_0+u_{k+1}) \Bigg] \\
&\quad + \sum_{\varpi\in\Pi_k} \Bigg\{ L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+2)} \, \tilde{L}_{p(\#_\varpi+2)} \Bigg[\int_0^T \Bigg(\int_0^1 \big\| G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) \\
&\quad - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))} \, d\rho \Bigg)^{(l+1)} dt \Bigg]^{1/(l+1)} \\
&\quad + T^{1/(l+1)} \, |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,V_{l,0})} \Bigg(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+1)} \, \tilde{d}_{1,p(\#_\varpi+1)}(u_0, u_{k+1}) \\
&\quad + \sum_{I\in\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p(\#_\varpi+1)} \, \tilde{L}_{p(\#_\varpi+1)} \; d_{\#_I,p(\#_\varpi+1)}(u_0, u_0+u_{k+1}) \Bigg) \Bigg\} \Bigg\} \\
&\leq |T \vee 1|^k \Bigg\{ \sum_{\varpi \in \Pi_k^*} T^{1/(l+1)} \, |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,V_{l,0})} \sum_{I \in \varpi} \Bigg[L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p\,\#_\varpi} \, \tilde{d}_{\#_I+1,p\,\#_\varpi}(u_0, u_{k+1}) \\
&\quad + L^{\mathbf{0}_{k+1}}_{\{I\cup\{k+1\}\},p\,\#_\varpi} \sum_{J\in\varpi\colon \min(J)>\min(I)} L^{\mathbf{0}_k}_{\varpi\setminus\{I,J\},p\,\#_\varpi} \, d_{\#_J,p\,\#_\varpi}(u_0, u_0+u_{k+1}) \Bigg] \\
&\quad + \sum_{\varpi\in\Pi_k} \Bigg\{ L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+2)} \, \tilde{L}_{p(\#_\varpi+2)} \Bigg[\int_0^T \int_0^1 \big\| G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0} + \rho[X_t^{0,u_0+u_{k+1}} - X_t^{0,u_0}]\big) \\
&\quad - G_l^{(\#_\varpi+1)}\big(X_t^{0,u_0}\big)\big\|^{(l+1)}_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,V_{l,0}))} \, d\rho \, dt \Bigg]^{1/(l+1)} \\
&\quad + T^{1/(l+1)} \, |G_l|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,V_{l,0})} \Bigg(L^{\mathbf{0}_k}_{\varpi,p(\#_\varpi+1)} \, \tilde{d}_{1,p(\#_\varpi+1)}(u_0, u_{k+1}) \\
&\quad + \sum_{I\in\varpi} L^{\mathbf{0}_k}_{\varpi\setminus\{I\},p(\#_\varpi+1)} \, \tilde{L}_{p(\#_\varpi+1)} \; d_{\#_I,p(\#_\varpi+1)}(u_0, u_0+u_{k+1}) \Bigg) \Bigg\} \Bigg\}.
\end{aligned}
\tag{115}
$$

Combining (101) with (115) ensures that for all $p \in [2,\infty)$, $x \in H$, $u_{k+1} \in H \setminus \{0\}$ it holds that

$$
\begin{aligned}
&\sup_{\mathbf{u}=(u_1,u_2,\dots,u_k)\in(H\setminus\{0\})^k} \sup_{t\in[0,T]} \frac{\big\|X_t^{k,(x+u_{k+1},\mathbf{u})} - X_t^{k,(x,\mathbf{u})} - X_t^{k+1,(x,\mathbf{u},u_{k+1})}\big\|_{\mathcal{L}^p(\mathbb{P};H)}}{\prod_{i=1}^{k+1}\|u_i\|_H} \\
&\leq |T\vee 1|^k \, \chi_{A,\eta}^{0,T} \, \Theta_{A,\eta,p,T}^{0,0,0}\big(|F|_{\mathcal{C}_{\mathrm{b}}^1(H,H)}, |B|_{\mathcal{C}_{\mathrm{b}}^1(H,HS(U,H))}\big) \\
&\quad\cdot \Bigg\{ \sum_{\varpi\in\Pi_k^*} \Big[T\,|F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,H)} + \sqrt{\tfrac{p\,(p-1)}{2}\,T}\,|B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi}(H,HS(U,H))} \Big] \\
&\quad\cdot \sum_{I\in\varpi} \Bigg[L_{\varpi\setminus\{I\},p\,\#_\varpi}^{\mathbf{0}_k} \, \tilde{d}_{\#_I+1,p\,\#_\varpi}(x,u_{k+1}) \\
&\quad + L_{\{I\cup\{k+1\}\},p\,\#_\varpi}^{\mathbf{0}_{k+1}} \sum_{J\in\varpi\colon\ \min(J)>\min(I)} L_{\varpi\setminus\{I,J\},p\,\#_\varpi}^{\mathbf{0}_k} \, d_{\#_J,p\,\#_\varpi}(x,x+u_{k+1}) \Bigg] \\
&\quad + \sum_{\varpi\in\Pi_k} \Bigg[L_{\varpi,p(\#_\varpi+2)}^{\mathbf{0}_k} \, \tilde{L}_{p(\#_\varpi+2)} \Bigg(\int_0^T \int_0^1 \|F^{(\#_\varpi+1)}(X_s^{0,x} + \rho[X_s^{0,x+u_{k+1}} - X_s^{0,x}]) \\
&\quad - F^{(\#_\varpi+1)}(X_s^{0,x})\|_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,H))} \, d\rho \, ds \\
&\quad + \Bigg[\tfrac{p\,(p-1)}{2} \int_0^T \int_0^1 \|B^{(\#_\varpi+1)}(X_s^{0,x} + \rho[X_s^{0,x+u_{k+1}} - X_s^{0,x}]) \\
&\quad - B^{(\#_\varpi+1)}(X_s^{0,x})\|^2_{\mathcal{L}^{p(\#_\varpi+2)}(\mathbb{P};L^{(\#_\varpi+1)}(H,HS(U,H)))} \, d\rho \, ds \Bigg]^{1/2} \Bigg) \\
&\quad + \Big[T\,|F|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,H)} + \sqrt{\tfrac{p\,(p-1)}{2}\,T}\;|B|_{\mathcal{C}_{\mathrm{b}}^{\#_\varpi+1}(H,HS(U,H))} \Big] \\
&\quad\cdot \Bigg(L_{\varpi,p(\#_\varpi+1)}^{\mathbf{0}_k} \, \tilde{d}_{1,p(\#_\varpi+1)}(x,u_{k+1}) \\
&\quad + \sum_{I\in\varpi} L_{\varpi\setminus\{I\},p(\#_\varpi+1)}^{\mathbf{0}_k} \, \tilde{L}_{p(\#_\varpi+1)} \, d_{\#_I,p(\#_\varpi+1)}(x,x+u_{k+1}) \Bigg) \Bigg] \Bigg\}.
\end{aligned} \tag{116}
$$

This, (93), and (94) establish item (vi) in the case $k+1$. Induction thus completes the proof of item (vi).

Combining item (iii), item (v), and item (vi) with item (ii) establishes item (vii) and item (viii). Next we note that (42) and item (ii) ensure that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$, $x \in H$, $t \in [0,T]$ it holds that

$$
\big(H^k \ni \mathbf{u} \mapsto [X_t^{k,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P};H)\big) \in L^{(k)}(H, L^p(\mathbb{P};H)). \tag{117}
$$

In addition, item (v) ensures that for all $k \in \{1,2,\dots,n\}$, $p \in [2,\infty)$, $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\big(H \ni x \mapsto \big[H^k \ni \mathbf{u} \mapsto [X_t^{k,(x,\mathbf{u})}]_{\mathbb{P},\mathcal{B}(H)} \in L^p(\mathbb{P};H)\big] \in L^{(k)}(H, L^p(\mathbb{P};H))\big) \\
&\qquad \in \mathcal{C}(H, L^{(k)}(H, L^p(\mathbb{P};H))).
\end{aligned} \tag{118}
$$

Combining (117) and (118) with item (ii) and item (vi) proves item (ix) and item (x). The proof of Theorem 2.1 is thus completed.

□

Acknowledgements

Stig Larsson and Christoph Schwab are gratefully acknowledged for a number of useful comments. This project has been supported through the SNSF-Research project 200021_156603 "Numerical approximations of nonlinear stochastic ordinary and partial differential equations" and the ETH Research Grant ETH-47 15-2 "Mild stochastic calculus and numerical approximations for nonlinear stochastic evolution equations with Lévy noise".

References

[1] Andersson, A., Jentzen, A., and Kurniawan, R. Existence, uniqueness, and regularity for stochastic evolution equations with irregular initial values. *arXiv:1512.06899v2* (2016), 35 pages.

[2] Andersson, A., and Larsson, S. Weak convergence for a spatial approximation of the nonlinear stochastic heat equation. *Math. Comp. 85*, 299 (2016), 1335–1358.

[3] Bréhier, C.-E. Approximation of the invariant measure with an Euler scheme for stochastic PDEs driven by space-time white noise. *Potential Anal. 40*, 1 (2014), 1–40.

[4] Bréhier, C.-E., and Kopec, M. Approximation of the invariant law of SPDEs: error analysis using a Poisson equation for a full-discretization scheme. *IMA Journal of Numerical Analysis* (2016).

[5] Brzeźniak, Z. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep. 61*, 3-4 (1997), 245–295.

[6] Clark, D. E., and Houssineau, J. Faa di Bruno's formula for chain differentials. *arXiv:1310.2833* (2013), 7 pages.

[7] Conus, D., Jentzen, A., and Kurniawan, R. Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. *arXiv:1408.1108* (2014), 29 pages.

[8] Da Prato, G., and Zabczyk, J. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[9] Da Prato, G., and Zabczyk, J. *Second order partial differential equations in Hilbert spaces*, vol. 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[10] Debussche, A. Weak approximation of stochastic partial differential equations: the nonlinear case. *Math. Comp. 80*, 273 (2011), 89–117.

[11] Gorenflo, R., Kilbas, A. A., Mainardi, F., and Rogosin, S. V. *Mittag-Leffler functions, related topics and applications.* Springer Monographs in Mathematics. Springer, Heidelberg, 2014.

[12] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1981.

[13] Hutzenthaler, M., Jentzen, A., and Salimova, D. Strong convergence of full-discrete nonlinearity-truncated accelerated exponential Euler-type approximations for stochastic Kuramoto-Sivashinsky equations. *arXiv:1604.02053* (2016), 43 pages.

[14] Jentzen, A., and Kurniawan, R. Weak convergence rates for Euler-type approximations of semilinear stochastic evolution equations with nonlinear diffusion coefficients. *arXiv:1501.03539* (2015), 51 pages.

[15] Jentzen, A., and Pušnik, P. Exponential moments for numerical approximations of stochastic partial differential equations. *arXiv:1609.07031* (2016), 44 pages.

[16] Klenke, A. *Probability theory.* Universitext. Springer-Verlag London Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original.

[17] Kopec, M. Quelques contributions à l'analyse numérique d'équations stochastiques. *PhD thesis, ENS Rennes* (2014), viii+189.

[18] Liu, W., and Röckner, M. *Stochastic partial differential equations: an introduction.* Universitext. Springer, Cham, 2015.

[19] Prévôt, C., and Röckner, M. *A concise course on stochastic partial differential equations*, vol. 1905 of *Lecture Notes in Mathematics.* Springer, Berlin, 2007.

[20] Renardy, M., and Rogers, R. C. *An introduction to partial differential equations*, vol. 13 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1993.

[21] Rozovskiĭ, B. L. *Stochastic evolution systems*, vol. 35 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1990. Linear theory and applications to nonlinear filtering, Translated from the Russian by A. Yarkho.

[22] Sell, G. R., and You, Y. *Dynamics of evolutionary equations*, vol. 143 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 2002.

[23] van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal. 255*, 4 (2008), 940–993.

[24] Walsh, J. B. *An introduction to stochastic partial differential equations*, vol. 1180 of *Lecture Notes in Mathematics.* Springer, Berlin, 1986.

[25] Wang, X. Weak error estimates of the exponential Euler scheme for semi-linear SPDEs without Malliavin calculus. *Discrete Contin. Dyn. Syst. 36*, 1 (2016), 481–497.

[26] Wang, X., and Gan, S. Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise. *J. Math. Anal. Appl. 398*, 1 (2013), 151–169.