Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

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Research Report No. 2016-41
September 2016

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
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Funding SNF: 200021_156603

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September 26, 2016

Abstract

Stochastic partial differential equations (SPDEs) have become a crucial ingredient in a number of models from economics and the natural sciences. Many SPDEs that appear in such applications include non-globally monotone nonlinearities. Solutions of SPDEs with non-globally monotone nonlinearities are in nearly all cases not known explicitly. Such SPDEs can thus only be solved approximatively and it is an important research problem to construct and analyze discrete numerical approximation schemes which converge with positive strong convergence rates to the solutions of such infinite dimensional SPDEs. In the case of finite dimensional stochastic ordinary differential equations (SODEs) with non-globally monotone nonlinearities it has recently been revealed that exponential integrability properties of the discrete numerical approximation scheme are a key instrument to establish positive strong convergence rates for the considered approximation scheme. Exponential integrability properties for appropriate approximation schemes have been established in the literature in the case of a large class of finite dimensional SODEs with non-globally monotone nonlinearities. To the best of our knowledge, there exists no result in the scientific literature which proves exponential integrability properties for a time discrete approximation scheme in the case of an infinite dimensional SPDE. In particular, to the best of our knowledge, there exists no result in the scientific literature which establishes strong convergence rates for a time discrete approximation scheme in the case of a SPDE with a non-globally monotone nonlinearity. In this paper we propose a new class of tamed space-time-noise discrete exponential Euler approximation schemes that admit exponential integrability properties in the case of infinite dimensional SPDEs. More specifically, the main result of this article proves that these approximation schemes enjoy exponential integrability properties for a large class of SPDEs with possibly non-globally monotone nonlinearities. In particular, we establish exponential moment bounds for the proposed approximation schemes in the case of stochastic Burgers equations, stochastic Kuramoto-Sivashinsky equations, and two-dimensional stochastic Navier-Stokes equations.

Contents

1 Introduction

Stochastic partial differential equations (SPDEs) have become a crucial ingredient in a number of models from economics and the natural sciences. For example, SPDEs frequently appear in models for the approximative pricing of interest-rate based financial derivatives (cf., e.g., Theorem 2.5 in Harms et al. [23] and (1.2) in Filipović et al. [19]), for the approximative description of random surfaces in surface growth models (cf., e.g., (1) in Blömker & Romito [6] and (3) in Hairer [21]), for describing the temporal dynamics associated to Euclidean quantum field theories (cf., e.g., (1.1) in Mourrat & Weber [35]), for the approximative description of velocity fields in fully developed turbulent flows (cf., e.g., (7) in Birnir [4] and (1.5) in Birnir [5]), and for the approximative description of the temporal evolution of the concentration of an undesired (chemical or biological) contaminant in water (e.g., in a water basin, the groundwater system, or a river; cf., e.g., (1.1) in Kouritzin & Long [33] and also (1.1) in Kallianpur & Xiong [31]). Many SPDEs that appear in such applications include non-globally monotone nonlinearities. Solutions of SPDEs with non-globally monotone nonlinearities are in nearly all cases not known explicitly. Such SPDEs can thus only be solved approximatively and it is an important research problem to construct and analyze discrete numerical approximation schemes which converge with positive strong convergence rates to the solutions of such infinite dimensional SPDEs. In the case of finite dimensional stochastic ordinary differential equations (SODEs) with non-globally monotone nonlinearities it has recently been revealed in the literature that exponential integrability properties of the discrete numerical approximation scheme are a key ingredient to establish positive strong convergence rates for the considered approximation scheme; cf., e.g., Hutzenthaler et al. [27], Hutzenthaler & Jentzen [24], and Cozma & Reisinger [12]. In particular, e.g., Corollary 3.8 in Hutzenthaler et al. [27] and Proposition 3.3 in Cozma & Reisinger [12] (cf. also Lemma 3.6 in Bou-Rabee & Hairer [8]) establish exponential integrability properties for appropriate stopped/tamed/truncated approximation schemes in the case of a large class of finite dimensional SODEs with non-globally monotone nonlinearities. To the best of our knowledge, there exists no result in the scientific literature which proves exponential integrability properties for a time discrete approximation scheme in the case of an infinite dimensional SPDE. In particular, to the best of our knowledge, there exists no result in the scientific literature which establishes strong convergence rates for a time discrete approximation scheme in the case of a SPDE with a non-globally monotone nonlinearity (cf., e.g., Dörsek [18] and Hutzenthaler & Jentzen [24]). In this paper we propose a new class of tamed space-time-noise discrete exponential Euler approximation schemes that admit exponential integrability properties in the case of infinite dimensional SPDEs. More specifically, the main result of this article (see Theorem 3.3 in Section 3 below) proves that these approximation schemes enjoy exponential integrability properties for a large class of SPDEs with possibly non-globally monotone nonlinearities. In particular, we establish exponential moment bounds for the proposed approximation schemes in the case of stochastic Burgers equations (see Corollary 4.11 in Subsection 4.3 below), stochastic Kuramoto-Sivashinsky equations (see Corollary 4.13 in Subsection 4.4 below), and two-dimensional stochastic Navier-Stokes equations (see Corollary 4.15 in Subsection 4.5 below).

In this introductory section we now illustrate the proposed approximation schemes and our main result (see Theorem 3.3) in the case of a stochastic Burgers equation (cf., e.g., Section 1 in Da Prato et al. [14] and Section 2 in Hairer & Voss [22]). Let $T \in (0, \infty)$, $\delta \in (0, 1/18)$, $H = L^2((0,1);\mathbb{R})$, let $Q \in L_1(H)$ be non-negative and symmetric, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical $\mathbb{P}$-Wiener process, let $A \colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H, let $(e_n)_{n\in\mathbb{N}} \subseteq H$, $(P_n)_{n\in\mathbb{N}} \subseteq L(H)$, $F \colon W_0^{1,2}((0,1),\mathbb{R}) \to H$, $\xi \in W_0^{1,2}((0,1),\mathbb{R})$ satisfy for all $n \in \mathbb{N}$, $u \in H$, $v \in W_0^{1,2}((0,1),\mathbb{R})$ that $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot))$, $P_n(u) = \sum_{k=1}^{n} \langle e_k, u\rangle_H e_k$, $F(v) = -v' \cdot v$, let $W^n \colon [0,T] \times \Omega \to P_n(H)$, $n \in \mathbb{N}$, be stochastic processes with continuous sample paths which satisfy for all $n \in \mathbb{N}$,

$t \in [0,T]$ that $\mathbb{P}(W_t^n = \int_0^t P_n \, dW_s) = 1$, and let $Y^{N,M} \colon [0,T] \times \Omega \to P_N(H)$, $N, M \in \mathbb{N}$, be stochastic processes which satisfy for all $N, M \in \mathbb{N}$, $m \in \{0, 1, \ldots, M-1\}$, $t \in [\frac{mT}{M}, \frac{(m+1)T}{M}]$ that $Y_0^{N,M} = P_N(\xi)$ and

$$Y_t^{N,M} = e^{(t - mT/M)A}\Big(Y_{mT/M}^{N,M} + \mathbb{1}_{\{\|(-A)^{1/2} Y_{mT/M}^{N,M}\|_H^2 + 1 \leq M^\delta / T^\delta\}} P_N \Big[F(Y_{mT/M}^{N,M})\,(t - \tfrac{mT}{M}) + \tfrac{Q^{1/2}(W_t^N - W_{mT/M}^N)}{1 + \|P_N Q^{1/2}(W_t^N - W_{mT/M}^N)\|_H^2}\Big]\Big) \quad (1)$$

(cf., e.g., [26, 25, 37, 38, 27, 34, 20, 30, 2, 29] for related schemes). In Corollary 4.11 in Subsection 4.3 below we demonstrate that the approximation scheme (1) enjoys finite exponential moments. More precisely, Corollary 4.11 in Subsection 4.3 proves[1] that for all $\varepsilon \in [1, \infty)$ it holds that

$$\sup_{N,M \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\Big[\exp\Big(\tfrac{\varepsilon \|Y_t^{N,M}\|_H^2}{e^{2\varepsilon \, \mathrm{trace}_H(Q) t}}\Big)\Big] < \infty. \quad (2)$$

Corollary 4.11 follows from an application of Corollary 3.4 below (see Subsection 4.3 below for details). Corollary 3.4, in turn, is a direct consequence of Theorem 3.3, which is the main result of this article. Theorem 3.3 establishes exponential integrability properties for a more general class of SPDEs (such as stochastic Burgers equations with non-additive noise, stochastic Kuramoto-Sivashinsky equations, and two-dimensional stochastic Navier-Stokes equations on a torus) as well as for a more general type of approximation schemes. Exponential integrability properties such as (2) are a key instrument to establish strong convergence rates for SPDEs with non-globally monotone nonlinearities (cf. [24]). In particular we intend to use (2) and Theorem 3.3, respectively, in succeeding articles to establish strong convergence rates for numerical approximations of stochastic Burgers equations and other SPDEs with non-globally monotone nonlinearities.

While polynomial moment bounds for numerical approximations of infinite dimensional SPDEs and exponential moment bounds for numerical approximations of finite dimensional SODEs have been established in the scientific literature, Theorem 3.3 is – to the best of our knowledge – the first result in the literature which establishes exponential moment bounds for time discrete numerical approximations in the case of infinite dimensional SPDEs. In particular, Theorem 3.3 and its consequences in Corollaries 3.4, 4.11, 4.13, and 4.15, respectively, are – to the best of our knowledge – the first results in the literature that establish exponential integrability properties for time discrete numerical approximations of stochastic Burgers equations, stochastic Kuramoto Sivashinsky equations, and two-dimensional stochastic Navier Stokes equations.

1.1 Notation

Throughout this article the following notation is used. For sets A and B we denote by $\mathbb{M}(A, B)$ the set of all mappings from A to B. For a topological space (X, τ) and a set $D \subseteq X$ we denote by $\mathring{D} \subseteq X$ the interior of D. For a natural number $k \in \mathbb{N}$ and normed $\mathbb{R}$-vector spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ we denote by $L^{(k)}(U, V)$ the set of all continuous k-linear mappings from U^k to V, we denote by $\|\cdot\|_{L^{(k)}(U,V)} \colon L^{(k)}(U, V) \to [0, \infty)$ the mapping which satisfies for all $A \in L^{(k)}(U, V)$

[1] (with $d = 1$, $\mathcal{D} = (0,1)$, $\eta = 0$, $\gamma = 1/2$, $T = T$, $\varepsilon = \varepsilon - 1/\sqrt{3}$, $\delta = \delta$, $U = H$, $H = H$, $\mathbb{H} = \{e_n \colon n \in \mathbb{N}\}$, $\mathbb{U} = \{e_n \colon n \in \mathbb{N}\}$, $\lambda_{e_N} = -\pi^2 N^2$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) = (\Omega, \mathcal{F}, \mathbb{P}, (\sigma_\Omega((W_s)_{s \in [0,t]}))_{t \in [0,T]})$, $W = W$, $Q = Q$, $A = A$, $r = (H_{1/2} \ni v \mapsto 2\varepsilon \max\{1, \sqrt{\mathrm{trace}_H(Q)}\} + 2\varepsilon \max\{1, \sqrt{\mathrm{trace}_H(Q)}\} \|(-A)^{1/2} v\|_H^2 \in [0, \infty))$, $b = ((0,1) \times \mathbb{R} \ni (x, y) \mapsto 1 \in \mathbb{R})$, $\vartheta = \mathrm{trace}_H(Q)$, $c = 2\varepsilon \max\{1, \sqrt{\mathrm{trace}_H(Q)}\}$, $R = \mathrm{Id}_H$, $F = F$, $\xi = (\Omega \ni \omega \mapsto \xi \in W_0^{1,2}((0,1), \mathbb{R}))$, $Y^{\{0, T/M, \ldots, T\}, \{e_1, \ldots, e_N\}, \{e_1, \ldots, e_N\}} = Y^{N,M}$ for $N, M \in \mathbb{N}$, $\varepsilon \in [1, \infty)$ in the notation of Corollary 4.11)

that $\|A\|_{L^{(k)}(U,V)} = \sup_{u_1,u_2,\dots,u_k\in U\setminus\{0\}} \big(\frac{\|A(u_1,u_2,\dots,u_k)\|_V}{\|u_1\|_U\cdot\|u_2\|_U\cdots\|u_k\|_U}\big)$, we denote by $L^{(0)}(U,V)$ the set given by $L^{(0)}(U,V) = V$, and we denote by $\|\cdot\|_{L^{(0)}(U,V)} \colon V \to [0,\infty)$ the mapping which satisfies for all $v \in V$ that $\|v\|_{L^{(0)}(U,V)} = \|v\|_V$. For measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ we denote by $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable functions. For a normed $\mathbb{R}$-vector space $(V, \|\cdot\|_V)$, a measure space $(\Omega, \mathcal{F}, \mu)$, a real number $p \in (0,\infty)$, and a measurable function $f \in \mathcal{M}(\mathcal{F}, \mathcal{B}(V))$ we denote by $\|f\|_{\mathcal{L}^p(\mu;V)} \in [0,\infty]$ and $\|f\|_{\mathcal{L}^\infty(\mu;V)} \in [0,\infty]$ the extended real numbers given by $\|f\|_{\mathcal{L}^p(\mu;V)} = \big(\int_\Omega \|f(\omega)\|_V^p \,\mu(d\omega)\big)^{1/p}$ and $\|f\|_{\mathcal{L}^\infty(\mu;V)} = \inf\{c \in [0,\infty) \colon \mu(\{v \in V \colon |f(v)| > c\}) = 0\}$. For a topological space (X,τ) we denote by $\mathcal{B}(X)$ the sigma-algebra of all Borel measurable sets in X. For a natural number $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\mu_A \colon \mathcal{B}(A) \to [0,\infty]$ the Lebesgue-Borel measure on $A \subseteq \mathbb{R}^d$. For $\mathbb{R}$-Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, a set $\mathcal{H} \in \mathcal{P}(H)$, and functions $F \colon \mathcal{H} \to H$ and $B \colon \mathcal{H} \to \mathrm{HS}(U,H)$ we denote by $\mathcal{G}_{F,B} \colon \mathcal{C}^2(H,\mathbb{R}) \to \mathbb{M}(\mathcal{H},\mathbb{R})$ the function which satisfies for all $x \in \mathcal{H}$, $\phi \in \mathcal{C}^2(H,\mathbb{R})$ that

$$(\mathcal{G}_{F,B}\phi)(x) = \langle F(x), (\nabla\phi)(x)\rangle_H + \tfrac{1}{2}\,\mathrm{trace}_H\big(B(x)B(x)^*(\mathrm{Hess}\ \phi)(x)\big). \tag{3}$$

For sets x and A we denote by $\mathbb{1}_A(x)$ the real number given by

$$\mathbb{1}_A(x) = \begin{cases} 1 \colon x \in A \\ 0 \colon x \notin A \end{cases}. \tag{4}$$

For sets Ω and A we denote by $\mathbb{1}_A^\Omega \colon \Omega \to \{0,1\}$ the function which satisfies for all $x \in \Omega$ that $\mathbb{1}_A^\Omega(x) = \mathbb{1}_A(x)$. For a set X we denote by $\mathcal{P}(X)$ the power set of X, we denote by $\#_X \in \mathbb{N}_0 \cup \{\infty\}$ the number of elements of X, and we denote by $\mathcal{P}_0(X)$ the set given by $\mathcal{P}_0(X) = \{\theta \in \mathcal{P}(X) \colon \#_\theta < \infty\}$. For a normed $\mathbb{R}$-vector space $(V, \|\cdot\|_V)$ with $\#_V > 1$, real numbers $n \in \mathbb{N}$, $c \in [1,\infty)$, a set $B \subseteq \mathbb{R}$, and an open and convex set $A \subseteq V$ we denote by $\mathcal{C}_c^n(A,B)$ the set given by

$$\mathcal{C}_c^n(A,B) = \left\{ f \in \mathcal{C}^{n-1}(A,B) \colon \begin{array}{l} \forall\, x,y \in A, i \in \mathbb{N}_0 \cap [0,n) \colon \|f^{(i)}(x) - f^{(i)}(y)\|_{L^{(i)}(H,\mathbb{R})} \\ \leq c\|x-y\|_H(1 + \sup_{r\in[0,1]} |f(rx + (1-r)y)|)^{1-1/c} \end{array} \right\} \tag{5}$$

(cf., e.g., (1.12) in Hutzenthaler & Jentzen [25]). We denote by $(\cdot) \wedge (\cdot) \colon \mathbb{R}^2 \to \mathbb{R}$ the function which satisfies for all $x,y \in \mathbb{R}$ that $x \wedge y = \min\{x,y\}$. For a real number $T \in (0,\infty)$ we denote by ϖ_T the set given by $\varpi_T = \{\theta \subseteq [0,T] \colon \{0,T\} \subseteq \theta \text{ and } \#_\theta < \infty\}$. For a real number $T \in (0,\infty)$ we denote by $|\cdot|_T \colon \varpi_T \to [0,T]$ the mapping which satisfies for all $\theta \in \varpi_T$ that

$$|\theta|_T = \max\Big\{x \in (0,\infty) \colon \big(\exists\, a,b \in \theta \colon \big[x = b - a \text{ and } \theta \cap (a,b) = \emptyset\big]\big)\Big\} \in (0,T]. \tag{6}$$

Let us note for every $T \in (0,\infty)$, $\theta \in \varpi_T$ that $|\theta|_T \in [0,T]$ is the maximum step size of the partition θ. We denote by $\lfloor\cdot\rfloor_\theta \colon [0,\infty) \to [0,\infty)$, $\theta \in (\cup_{T\in(0,\infty)}\varpi_T)$, and $\llcorner\cdot\lrcorner_\theta \colon [0,\infty) \to [0,\infty)$, $\theta \in (\cup_{T\in(0,\infty)}\varpi_T)$, the mappings which satisfy for all $\theta \in (\cup_{T\in(0,\infty)}\varpi_T)$, $t \in (0,\infty)$ that $\lfloor t\rfloor_\theta = \max([0,t] \cap \theta)$, $\llcorner t\lrcorner_\theta = \max([0,t) \cap \theta)$, and $\lfloor 0\rfloor_\theta = \llcorner 0\lrcorner_\theta = 0$. For a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable space $(S, \mathcal{S})$, a set R, and a function $f \colon \Omega \to R$ we denote by $[f]_{\mu,\mathcal{S}}$ the set given by $[f]_{\mu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F},\mathcal{S}) \colon (\exists\, A \in \mathcal{F} \colon \mu(A) = 0 \text{ and } \{\omega \in \Omega \colon f(\omega) \neq g(\omega)\} \subseteq A)\}$.

2 Exponential moments for time discrete approximation schemes

2.1 Factorization lemma for conditional expectations

In this subsection we recall in Definitions 2.1–2.3, Lemma 2.4, Theorem 2.5, and Lemmas 2.6–2.9 some well known concepts and facts from measure and probability theory. In particular, we recall in Lemma 2.9 below a well-known factorization property for conditional expectations. We use this factorization property in the proofs of our later results. Definitions 2.1–2.3, Lemma 2.4, Theorem 2.5, and Lemma 2.6 can, e.g., in a very similar form be found in Section 1 in Klenke [32] (see Definition 1.1, Definition 1.10, Theorem 1.16, Theorem 1.18, Theorem 1.19, and Theorem 1.96 in Klenke [32]). Lemmas 2.7–2.9 can, e.g., in a very similar form be found in Chapter 1 in Da Prato & Zabczyk [16] (see Proposition 1.12 in Da Prato & Zabczyk [16]).

Definition 2.1 ($\cap$-Stability)**.** *Let $\mathcal{E}$ be a set. Then we say that $\mathcal{E}$ is $\cap$-stable if and only if for all $a, b \in \mathcal{E}$ it holds that $a \cap b \in \mathcal{E}$.*

Definition 2.2 (Dynkin system)**.** *Let Ω and $\mathcal{A}$ be sets. Then we say that $\mathcal{A}$ is a Dynkin system on Ω if and only if*

(i) it holds that $\Omega \in \mathcal{A} \subseteq \mathcal{P}(\Omega)$,

(ii) it holds for all $A \in \mathcal{A}$ that $\Omega\backslash A \in \mathcal{A}$, and

(iii) it holds for all pairwise disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Definition 2.3. *Let Ω and $\mathcal{A}$ be sets with $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Then we denote by $\delta_\Omega(\mathcal{A})$ the set given by*

$$\delta_\Omega(\mathcal{A}) = \cap_{\mathcal{B} \in \left\{ \substack{\mathcal{C} \text{ is a Dynkin system} \\ \text{on } \Omega \text{ with } \mathcal{C} \supseteq \mathcal{A}} \right\}} \mathcal{B}. \tag{7}$$

Lemma 2.4. *Let Ω be a set and let $\mathcal{A}$ be a Dynkin system on Ω. Then it holds that $\mathcal{A}$ is $\cap$-stable if and only if $\mathcal{A}$ is a sigma-algebra on Ω.*

Proof of Lemma 2.4. Throughout this proof assume w.l.o.g. that $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is a $\cap$-stable Dynkin system on Ω (otherwise the statement of Lemma 2.4 is clear). Note that the assumption that $\mathcal{A}$ is a Dynkin system on Ω ensures for all $A \in \mathcal{A}$ that

$$(\Omega\backslash A) \in \mathcal{A}. \tag{8}$$

This and the fact that $\forall\, A, B \in \mathcal{A}\colon A \cap B \in \mathcal{A}$ imply that for all $A, B \in \mathcal{A}$ it holds that $A\backslash B = A \cap (\Omega\backslash B) \in \mathcal{A}$. Hence, we obtain that for all $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ it holds that

$$\cup_{n\in\mathbb{N}} A_n = A_1 \cup \big[\cup_{n\in\mathbb{N}} \left((\cdots ((A_{n+1}\backslash A_n)\backslash A_{n-1}) \cdots)\backslash A_1 \right) \big] \in \mathcal{A}. \tag{9}$$

Combining this, the fact that $\Omega \in \mathcal{A}$, and (8) proves that $\mathcal{A}$ is a sigma-algebra on Ω. The proof of Lemma 2.4 is thus completed. □

Theorem 2.5. *Let Ω be a set and let $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\Omega))$ be $\cap$-stable. Then $\sigma_\Omega(\mathcal{A}) = \delta_\Omega(\mathcal{A})$.*

Proof of Theorem 2.5. Throughout this proof let $\mathcal{D}_A \subseteq \mathcal{P}(\Omega)$, $A \in \delta_\Omega(\mathcal{A})$, be the sets which satisfy for all $A \in \delta_\Omega(\mathcal{A})$ that $\mathcal{D}_A = \{B \in \delta_\Omega(\mathcal{A})\colon A \cap B \in \delta_\Omega(\mathcal{A})\}$. Note that for all $A \in \delta_\Omega(\mathcal{A})$ it holds that $A \cap \Omega = A \in \delta_\Omega(\mathcal{A})$. This proves that for all $A \in \delta_\Omega(\mathcal{A})$ it holds that

$$\Omega \in \mathcal{D}_A. \tag{10}$$

In the next step we observe that for all $A \in \delta_\Omega(\mathcal{A})$, $B \in \mathcal{D}_A$ it holds that

$$A \cap (\Omega\backslash B) = A\backslash(A \cap B) = A \cap [\Omega\backslash(A \cap B)] = \Omega\backslash[(\Omega\backslash A) \cup (A \cap B)] \in \delta_\Omega(\mathcal{A}). \tag{11}$$

Moreover, note that for all $A \in \delta_\Omega(\mathcal{A})$ and all pairwise disjoint sets $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_A$ it holds that

$$A \cap (\cup_{n\in\mathbb{N}} B_n) = \cup_{n\in\mathbb{N}} (A \cap B_n) \in \delta_\Omega(\mathcal{A}). \tag{12}$$

Combining (10), (11), and (12) proves that for every $A \in \delta_\Omega(\mathcal{A})$ it holds that $\mathcal{D}_A$ is a Dynkin system on Ω. Next note that the assumption that $\mathcal{A}$ is $\cap$-stable implies that for all $A \in \mathcal{A}$ it holds that $\mathcal{A} \subseteq \mathcal{D}_A$. This and the fact that for every $A \in \delta_\Omega(\mathcal{A})$ it holds that $\mathcal{D}_A$ is a Dynkin system on Ω proves that for all $A \in \mathcal{A}$ it holds that $\delta_\Omega(\mathcal{A}) \subseteq \delta_\Omega(\mathcal{D}_A) = \mathcal{D}_A$. This implies that for all $A \in \mathcal{A}$, $B \in \delta_\Omega(\mathcal{A})$ it holds that $A \cap B \in \delta_\Omega(\mathcal{A})$. This ensures that for all $A \in \mathcal{A}$, $B \in \delta_\Omega(\mathcal{A})$ it holds that $A \in \mathcal{D}_B$. Hence, we obtain that for all $B \in \delta_\Omega(\mathcal{A})$ it holds that $\mathcal{A} \subseteq \mathcal{D}_B$. In particular, we obtain that for all $A \in \delta_\Omega(\mathcal{A})$ it holds that $\mathcal{A} \subseteq \mathcal{D}_A$. Combining this with the fact that for every $A \in \delta_\Omega(\mathcal{A})$ it holds that $\mathcal{D}_A$ is a Dynkin system on Ω assures that for all $A \in \delta_\Omega(\mathcal{A})$ it holds that $\delta_\Omega(\mathcal{A}) \subseteq \delta_\Omega(\mathcal{D}_A) = \mathcal{D}_A \subseteq \delta_\Omega(\mathcal{A})$. Therefore, we obtain that for all $A, B \in \delta_\Omega(\mathcal{A})$ it holds that $A \cap B \in \delta_\Omega(\mathcal{A})$. Combining this with Lemma 2.4 completes the proof of Theorem 2.5. □

Lemma 2.6. *Let $(\Omega, \mathcal{F})$ be a measurable space and let $f \in \mathcal{M}(\mathcal{F}, \mathcal{B}([0,\infty]))$. Then there exists a sequence $f_n \in \mathcal{M}(\mathcal{F}, \mathcal{B}([0,\infty)))$, $n \in \mathbb{N}$, which satisfies for all $n \in \mathbb{N}$, $\omega \in \Omega$ that $\#_{f_n(\Omega)} < \infty$, $f_n(\omega) \leq f_{n+1}(\omega)$, and $\lim_{m\to\infty} f_m(\omega) = f(\omega)$.*

Proof of Lemma 2.6. Throughout this proof let $A_n \in \mathcal{P}(\Omega)$, $n \in \mathbb{N}$, and $B_{n,j} \in \mathcal{P}(\Omega)$, $j \in \mathbb{N} \cap [1, n2^n]$, $n \in \mathbb{N}$, be the sets which satisfy for all $n \in \mathbb{N}$, $j \in \mathbb{N} \cap [1, n2^n]$ that $A_n = \{\omega \in \Omega \colon f(\omega) \in [n, \infty]\}$ and $B_{n,j} = \{\omega \in \Omega \colon f(\omega) \in [(j-1)/2^n, j/2^n)\}$ and let $f_n \colon \Omega \to [0, \infty)$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$, $\omega \in \Omega$ that

$$f_n(\omega) = n\, \mathbb{1}_{A_n}(\omega) + \sum_{j=1}^{n2^n} \tfrac{j-1}{2^n}\, \mathbb{1}_{B_{n,j}}(\omega). \tag{13}$$

Note that for every $\omega \in \Omega$ with $f(\omega) \in [0, \infty)$ it holds that there exist $n \in \mathbb{N}$, $j \in \mathbb{N} \cap [1, n2^n]$ such that $f(\omega) \in [(j-1)/2^n, j/2^n)$. This and (13) imply that for every $\omega \in \Omega$ with $f(\omega) \in [0, \infty)$ it holds that there exists $n \in \mathbb{N}$ such that $0 \leq f(\omega) - f_n(\omega) < 2^{-n}$. Hence, we obtain that for all $\omega \in \Omega$ with $f(\omega) \in [0, \infty)$ it holds that

$$\limsup_{n\to\infty} |f_n(\omega) - f(\omega)| = 0. \tag{14}$$

In addition, note that for all $m \in \mathbb{N}$, $\omega \in \Omega$ with $f(\omega) = \infty$ it holds that $f_m(\omega) = m$. This proves that for all $\omega \in \Omega$ with $f(\omega) = \infty$ it holds that $\liminf_{m\to\infty} f_m(\omega) = \infty$. Combining this and (14) completes the proof of Lemma 2.6. □

Lemma 2.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(D, \mathcal{D})$ and $(E, \mathcal{E})$ be measurable spaces, let $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(\mathcal{F})$ be $\mathbb{P}$-independent sigma-algebras, let $X \in \mathcal{M}(\mathcal{X}, \mathcal{D})$, $Y \in \mathcal{M}(\mathcal{Y}, \mathcal{E})$, $A \in \mathcal{D} \otimes \mathcal{E}$, $\Psi \in \mathbb{M}(D, [0,\infty])$, and assume for all $x \in D$ that $\Psi(x) = \mathbb{E}[\mathbb{1}_A^{D\times E}(x, Y)]$. Then it holds that $\Psi \in \mathcal{M}(\mathcal{D}, \mathcal{B}([0,\infty]))$ and*

$$\mathbb{E}[\mathbb{1}_A^{D\times E}(X, Y)|\mathcal{X}] = [\Psi(X)]_{\mathbb{P}|_{\mathcal{X}}, \mathcal{B}([0,\infty])}. \tag{15}$$

Proof of Lemma 2.7. Throughout this proof let $\gamma_C \colon D \to [0, \infty]$, $C \in \mathcal{D} \otimes \mathcal{E}$, be the functions which satisfy for all $C \in \mathcal{D} \otimes \mathcal{E}$, $x \in D$ that $\gamma_C(x) = \mathbb{E}[\mathbb{1}_C^{D\times E}(x, Y)]$ and let $\mathcal{C} \subseteq \mathcal{D} \otimes \mathcal{E}$ be the set given by $\mathcal{C} = \{C \in \mathcal{D} \otimes \mathcal{E} \colon \mathbb{E}[\mathbb{1}_C^{D\times E}(X, Y)|\mathcal{X}] = [\gamma_C(X)]_{\mathbb{P}|_{\mathcal{X}}, \mathcal{B}([0,\infty])}\}$. Note that Tonelli's theorem and the fact that $D \times E \ni (x, y) \mapsto \mathbb{1}_A^{D\times E}(x, y) \in [0, \infty]$ is $(\mathcal{D} \otimes \mathcal{E})/\mathcal{B}([0,\infty])$-measurable show that

$$\Psi \in \mathcal{M}(\mathcal{D}, \mathcal{B}([0,\infty])). \tag{16}$$

Moreover, observe that for all $x \in D$, $C \in \mathcal{D} \otimes \mathcal{E}$ it holds that

$$\begin{aligned} \gamma_{(D\times E)\setminus C}(x) &= \mathbb{E}[\mathbb{1}_{D\times E}^{D\times E}(x, Y) - \mathbb{1}_C^{D\times E}(x, Y)] \\ &= \mathbb{E}[\mathbb{1}_{D\times E}^{D\times E}(x, Y)] - \mathbb{E}[\mathbb{1}_C^{D\times E}(x, Y)] = \gamma_{D\times E}(x) - \gamma_C(x). \end{aligned} \tag{17}$$

This ensures for all $C \in \mathcal{C}$ that

$$\begin{aligned}\mathbb{E}[\mathbb{1}^{D\times E}_{(D\times E)\setminus C}(X,Y)|\mathcal{X}] &= \mathbb{E}[\mathbb{1}^{D\times E}_{D\times E}(X,Y) - \mathbb{1}^{D\times E}_{C}(X,Y)|\mathcal{X}] \\ &= \mathbb{E}[\mathbb{1}^{D\times E}_{D\times E}(X,Y)|\mathcal{X}] - \mathbb{E}[\mathbb{1}^{D\times E}_{C}(X,Y)|\mathcal{X}] \\ &= [\gamma_{D\times E}(X) - \gamma_C(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])} = [\gamma_{(D\times E)\setminus C}(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{18}$$

Next observe that the monotone convergence theorem proves that for all $x \in D$ and all pairwise disjoint sets $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}\otimes\mathcal{E}$ it holds that

$$\begin{aligned}\gamma_{\cup_{n=1}^\infty C_n}(x) &= \mathbb{E}\big[\mathbb{1}^{D\times E}_{\cup_{n=1}^\infty C_n}(x,Y)\big] = \mathbb{E}\big[\textstyle\sum_{n=1}^\infty \mathbb{1}^{D\times E}_{C_n}(x,Y)\big] \\ &= \textstyle\sum_{n=1}^\infty \mathbb{E}[\mathbb{1}^{D\times E}_{C_n}(x,Y)] = \sum_{n=1}^\infty \gamma_{C_n}(x).\end{aligned} \tag{19}$$

The monotone convergence theorem for conditional expectations hence shows that for all pairwise disjoint sets $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ it holds that

$$\begin{aligned}\mathbb{E}\big[\mathbb{1}^{D\times E}_{\cup_{n=1}^\infty C_n}(X,Y)|\mathcal{X}\big] &= \mathbb{E}\big[\textstyle\sum_{n=1}^\infty \mathbb{1}^{D\times E}_{C_n}(X,Y)|\mathcal{X}\big] = \sum_{n=1}^\infty \mathbb{E}[\mathbb{1}^{D\times E}_{C_n}(X,Y)|\mathcal{X}] \\ &= \textstyle\sum_{n=1}^\infty [\gamma_{C_n}(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])} = [\gamma_{\cup_{n=1}^\infty C_n}(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{20}$$

Combining (18), (20), and the fact that $(D\times E) \in \mathcal{C}$ implies that $\mathcal{C}$ is a Dynkin system on $D\times E$. Moreover, note that for all $\mathbf{D}\in\mathcal{D}$, $\mathbf{E}\in\mathcal{E}$ it holds that

$$\begin{aligned}\mathbb{E}[\mathbb{1}^{D\times E}_{\mathbf{D}\times\mathbf{E}}(X,Y)|\mathcal{X}] &= \mathbb{E}[\mathbb{1}^{D}_{\mathbf{D}}(X)\,\mathbb{1}^{E}_{\mathbf{E}}(Y)|\mathcal{X}] = \mathbb{1}^{D}_{\mathbf{D}}(X)\,\mathbb{E}[\mathbb{1}^{E}_{\mathbf{E}}(Y)|\mathcal{X}] \\ &= [\mathbb{1}^{D}_{\mathbf{D}}(X)\,\mathbb{E}[\mathbb{1}^{E}_{\mathbf{E}}(Y)]]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{21}$$

This ensures that $\{\mathbf{D}\times\mathbf{E}\in\mathcal{P}(D\times E)\colon \mathbf{D}\in\mathcal{D}, \mathbf{E}\in\mathcal{E}\} \subseteq \mathcal{C}$. Combining this, the fact that the set $\{\mathbf{D}\times\mathbf{E}\in\mathcal{P}(D\times E)\colon \mathbf{D}\in\mathcal{D}, \mathbf{E}\in\mathcal{E}\}$ is $\cap$-stable, and Theorem 2.5 (with $\Omega = D\times E$, $\mathcal{A} = \{\mathbf{D}\times\mathbf{E}\in\mathcal{P}(D\times E)\colon \mathbf{D}\in\mathcal{D}, \mathbf{E}\in\mathcal{E}\}$ in the notation of Theorem 2.5) proves that

$$\begin{aligned}&\mathcal{D}\otimes\mathcal{E} = \sigma_{D\times E}(\{\mathbf{D}\times\mathbf{E}\in\mathcal{P}(D\times E)\colon \mathbf{D}\in\mathcal{D}, \mathbf{E}\in\mathcal{E}\}) \\ &\quad= \delta_{D\times E}(\{\mathbf{D}\times\mathbf{E}\in\mathcal{P}(D\times E)\colon \mathbf{D}\in\mathcal{D}, \mathbf{D}\in\mathcal{E}\}) \subseteq \delta_{D\times E}(\mathcal{C}) \subseteq \delta_{D\times E}(\mathcal{D}\otimes\mathcal{E}) = \mathcal{D}\otimes\mathcal{E}.\end{aligned} \tag{22}$$

The fact that the set $\mathcal{C}$ is a Dynkin system on $D\times E$ hence assures that $\mathcal{D}\otimes\mathcal{E} = \delta_{D\times E}(\mathcal{C}) = \mathcal{C}$. Therefore, we obtain that for all $C \in \mathcal{D}\otimes\mathcal{E}$ it holds that

$$\mathbb{E}[\mathbb{1}^{D\times E}_{C}(X,Y)|\mathcal{X}] = [\gamma_C(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}. \tag{23}$$

This and (16) complete the proof of Lemma 2.7. □

Lemma 2.8. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(D,\mathcal{D})$ and $(E,\mathcal{E})$ be measurable spaces, let $\mathcal{X},\mathcal{Y}\in\mathcal{P}(\mathcal{F})$ be $\mathbb{P}$-independent sigma-algebras, let $X\in\mathcal{M}(\mathcal{X},\mathcal{D})$, $Y\in\mathcal{M}(\mathcal{Y},\mathcal{E})$, $\Phi\in\mathcal{M}(\mathcal{D}\otimes\mathcal{E},\mathcal{B}([0,\infty]))$, $\Psi\in\mathbb{M}(D,[0,\infty])$, and assume for all $x\in D$ that $\#_{\Phi(D\times E)}<\infty$, $\Psi(x)=\mathbb{E}[\Phi(x,Y)]$. Then it holds that $\Psi\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and*

$$\mathbb{E}[\Phi(X,Y)|\mathcal{X}] = [\Psi(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}. \tag{24}$$

Proof of Lemma 2.8. Throughout this proof let $\gamma_C\colon D\to[0,\infty]$, $C\in\mathcal{D}\otimes\mathcal{E}$, be the functions which satisfy for all $C\in\mathcal{D}\otimes\mathcal{E}$, $x\in D$ that $\gamma_C(x)=\mathbb{E}[\mathbb{1}^{D\times E}_{C}(x,Y)]$. Note that for all $x\in D$, $y\in E$ it holds that

$$\Phi(x,y) = \sum_{z\in\Phi(D\times E)} z\,\mathbb{1}^{D\times E}_{\Phi^{-1}(\{z\})}(x,y). \tag{25}$$

The assumption that $\Phi\in\mathcal{M}(\mathcal{D}\otimes\mathcal{E},[0,\infty])$ implies that for all $z\in\Phi(D\times E)$ it holds that $\Phi^{-1}(\{z\})\in\mathcal{D}\otimes\mathcal{E}$. Combining this and Lemma 2.7 (with $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(D,\mathcal{D})=$

$(D,\mathcal{D})$, $(E,\mathcal{E})=(E,\mathcal{E})$, $\mathcal{X}=\mathcal{X}$, $\mathcal{Y}=\mathcal{Y}$, $X=X$, $Y=Y$, $A=\Phi^{-1}(\{z\})$ for $z\in\Phi(D\times E)$ in the notation of Lemma 2.7) proves that for all $z\in\Phi(D\times E)$ it holds that $(D\ni x\mapsto \mathbb{E}[\mathbb{1}^{D\times E}_{\Phi^{-1}(\{z\})}(x,Y)]\in[0,\infty])\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and

$$\mathbb{E}[\mathbb{1}^{D\times E}_{\Phi^{-1}(\{z\})}(X,Y)|\mathcal{X}]=[\gamma_{\Phi^{-1}(\{z\})}(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}. \tag{26}$$

This and (25) show that $\Psi\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and

$$\begin{aligned}&\mathbb{E}[\Phi(X,Y)|\mathcal{X}]=\mathbb{E}\big[\textstyle\sum_{z\in\Phi(D\times E)} z\,\mathbb{1}^{D\times E}_{\Phi^{-1}(\{z\})}(X,Y)\big|\mathcal{X}\big]=\textstyle\sum_{z\in\Phi(D\times E)} z\,\mathbb{E}\big[\mathbb{1}^{D\times E}_{\Phi^{-1}(\{z\})}(X,Y)\big|\mathcal{X}\big]\\&=\textstyle\sum_{z\in\Phi(D\times E)} z\,[\gamma_{\Phi^{-1}(\{z\})}(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}=\big[\textstyle\sum_{z\in\Phi(D\times E)} z\,\gamma_{\Phi^{-1}(\{z\})}(X)\big]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}\\&=[\Psi(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{27}$$

This completes the proof of Lemma 2.8. □

Lemma 2.9. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(D,\mathcal{D})$ and $(E,\mathcal{E})$ be measurable spaces, let $\mathcal{X},\mathcal{Y}\in\mathcal{P}(\mathcal{F})$ be $\mathbb{P}$-independent sigma-algebras, let $X\in\mathcal{M}(\mathcal{X},\mathcal{D})$, $Y\in\mathcal{M}(\mathcal{Y},\mathcal{E})$, $\Phi\in\mathcal{M}(\mathcal{D}\otimes\mathcal{E},\mathcal{B}([0,\infty]))$, $\Psi\in\mathbb{M}(D,[0,\infty])$, and assume for all $x\in D$ that $\Psi(x)=\mathbb{E}[\Phi(x,Y)]$. Then it holds that $\Psi\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and*

$$\mathbb{E}[\Phi(X,Y)|X]=[\Psi(X)]_{\mathbb{P}|_{\sigma_\Omega(X)},\mathcal{B}([0,\infty])}\subseteq\mathbb{E}[\Phi(X,Y)|\mathcal{X}]=[\Psi(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}. \tag{28}$$

Proof of Lemma 2.9. Throughout this proof let $\phi_n\in\mathcal{M}(\mathcal{D}\otimes\mathcal{E},\mathcal{B}([0,\infty]))$, $n\in\mathbb{N}$, be functions which satisfy for all $m\in\mathbb{N}$, $z\in D\times E$ that $\#_{\phi_m(D\times E)}<\infty$, $\phi_m(z)\leq\phi_{m+1}(z)$, and $\lim_{n\to\infty}\phi_n(z)=\Phi(z)$ and let $\psi_n\colon D\to[0,\infty]$, $n\in\mathbb{N}$, be the functions which satisfy for all $n\in\mathbb{N}$, $x\in D$ that $\psi_n(x)=\mathbb{E}[\phi_n(x,Y)]$. Note that the monotone convergence theorem ensures for all $x\in D$ that

$$\lim_{n\to\infty}\psi_n(x)=\lim_{n\to\infty}\mathbb{E}[\phi_n(x,Y)]=\mathbb{E}[\lim_{n\to\infty}\phi_n(x,Y)]=\mathbb{E}[\Phi(x,Y)]=\Psi(x). \tag{29}$$

Combining the monotone convergence theorem for conditional expectations and Lemma 2.8 (with $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $(D,\mathcal{D})=(D,\mathcal{D})$, $(E,\mathcal{E})=(E,\mathcal{E})$, $\mathcal{X}=\mathcal{X}$, $\mathcal{Y}=\mathcal{Y}$, $X=X$, $Y=Y$, $\Phi=\phi_n$, $\Psi=\psi_n$ for $n\in\mathbb{N}$ in the notation of Lemma 2.8) hence shows

(i) that $\forall\, n\in\mathbb{N}\colon \psi_n\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and

(ii) that

$$\begin{aligned}&\mathbb{E}[\Phi(X,Y)|\mathcal{X}]=\mathbb{E}[\lim_{n\to\infty}\phi_n(X,Y)|\mathcal{X}]=\lim_{n\to\infty}\mathbb{E}[\phi_n(X,Y)|\mathcal{X}]\\&=\lim_{n\to\infty}[\psi_n(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}=[\lim_{n\to\infty}\psi_n(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}=[\Psi(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{30}$$

Combining this and (29) proves that $\Psi\in\mathcal{M}(\mathcal{D},\mathcal{B}([0,\infty]))$ and

$$\begin{aligned}\mathbb{E}[\Phi(X,Y)|X]&=\mathbb{E}\big[\mathbb{E}[\Phi(X,Y)|\mathcal{X}]|X\big]=\mathbb{E}[\Psi(X)|X]\\&=[\Psi(X)]_{\mathbb{P}|_{\sigma_\Omega(X)},\mathcal{B}([0,\infty])}\subseteq[\Psi(X)]_{\mathbb{P}|_{\mathcal{X}},\mathcal{B}([0,\infty])}.\end{aligned} \tag{31}$$

The proof of Lemma 2.9 is thus completed. □

2.2 From one-step estimates to exponential moments

In this subsection we establish in Corollary 2.10 below exponential integral properties for approximation schemes (see (34) in Corollary 2.10) under a general one-step condition on the considered approximation scheme (see (33) in Corollary 2.10 below). We will verify this one-step condition for a specific class of approximation schemes in Subsection 2.3 below. Corollary 2.10 is an extension of Corollary 2.3 in Hutzenthaler et al. [27].

Corollary 2.10. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, $\theta \in \varpi_T$, $\rho, c \in [0,\infty)$, $V \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}([0,\infty)))$, $\bar{V} \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}(\mathbb{R}))$, $\Phi \in \mathcal{M}(\mathcal{B}(H \times [0,T] \times U), \mathcal{B}(H))$, $E \in \mathcal{B}(H)$, $S \in \mathbb{M}((0,T], L(H))$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $W\colon [0,T]\times\Omega \to U$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process with continuous sample paths, let $Y \in \mathcal{M}(\mathcal{B}([0,T])\otimes\mathcal{F}, \mathcal{B}(H))$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process, assume for all $t \in (0,T]$, $x \in H$ that $V(S_t x) \le V(x)$, $\bar{V}(S_t x) \le \bar{V}(x)$, and*

$$Y_t = S_{t-\llcorner t\lrcorner_\theta}\left[\mathbb{1}_{H\setminus E}(Y_{\llcorner t\lrcorner_\theta})\cdot Y_{\llcorner t\lrcorner_\theta} + \mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\cdot \Phi(Y_{\llcorner t\lrcorner_\theta}, t-\llcorner t\lrcorner_\theta, W_t - W_{\llcorner t\lrcorner_\theta})\right], \tag{32}$$

and assume for all $x \in E$, $t \in (0, |\theta|_T]$ that $\int_0^T \mathbb{1}_E(Y_{\lfloor s\rfloor_\theta})\,|\bar{V}(Y_s)|\,ds + \int_0^{|\theta|_T} |\bar{V}(\Phi(x,s,W_s))|\,ds < \infty$ and

$$\mathbb{E}\left[\exp\left(\tfrac{V(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Phi(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \le e^{ct+V(x)}. \tag{33}$$

Then it holds for all $t \in [0,T]$ that

$$\mathbb{E}\left[\exp\left(\tfrac{V(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_E(Y_{\lfloor s\rfloor_\theta})\,\bar{V}(Y_s)}{e^{\rho s}}\,ds\right)\right] \le e^{ct}\,\mathbb{E}\big[e^{V(Y_0)}\big]. \tag{34}$$

Proof of Corollary 2.10. We prove Corollary 2.10 through an application of Lemma 2.2 in Hutzenthaler et al. [27]. Assumption (33) implies that for all $t \in (0, |\theta|_T]$, $x \in H$ it holds that

$$\mathbb{E}\left[\exp\left(\tfrac{\mathbb{1}_E(x)\,V(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_E(x)\,\bar{V}(\Phi(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \le e^{ct+\mathbb{1}_E(x)\,V(x)}. \tag{35}$$

Next note that (32) ensures that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(Y_t)}{e^{\rho t}} + \int_{\llcorner t\lrcorner_\theta}^t \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,\bar{V}(Y_s)}{e^{\rho s}}\,ds\right)\,\middle|\,(Y_s)_{s\in[0,\llcorner t\lrcorner_\theta]}\right]\\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(S_{t-\llcorner t\lrcorner_\theta}\Phi(Y_{\llcorner t\lrcorner_\theta}, t-\llcorner t\lrcorner_\theta, W_t - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho t}}\\
&\quad + \int_{\llcorner t\lrcorner_\theta}^t \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,\bar{V}(S_{s-\llcorner t\lrcorner_\theta}\Phi(Y_{\llcorner t\lrcorner_\theta}, s-\llcorner t\lrcorner_\theta, W_s - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg)\,\Bigg|\,(Y_s)_{s\in[0,\llcorner t\lrcorner_\theta]}\Bigg]\\
&\le \mathbb{E}\Bigg[\Bigg\{\exp\Bigg(\tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(\Phi(Y_{\llcorner t\lrcorner_\theta}, t-\llcorner t\lrcorner_\theta, W_t - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho(t-\llcorner t\lrcorner_\theta)}}\\
&\quad + \int_0^{t-\llcorner t\lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,\bar{V}(\Phi(Y_{\llcorner t\lrcorner_\theta}, s, W_{\llcorner t\lrcorner_\theta + s} - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg)\Bigg\}^{\exp(-\rho\llcorner t\lrcorner_\theta)}\,\Bigg|\,(Y_s)_{s\in[0,\llcorner t\lrcorner_\theta]}\Bigg].
\end{aligned} \tag{36}$$

Jensen's inequality, (35), and, e.g., Lemma 2.9 hence imply for all $t \in (0,T]$ that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(Y_t)}{e^{\rho t}} + \int_{\llcorner t\lrcorner_\theta}^t \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,\bar{V}(Y_s)}{e^{\rho s}}\,ds\right)\,\middle|\,(Y_s)_{s\in[0,\llcorner t\lrcorner_\theta]}\right]\\
&\le \Bigg|\mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(\Phi(Y_{\llcorner t\lrcorner_\theta}, t-\llcorner t\lrcorner_\theta, W_t - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho(t-\llcorner t\lrcorner_\theta)}}\\
&\quad + \int_0^{t-\llcorner t\lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,\bar{V}(\Phi(Y_{\llcorner t\lrcorner_\theta}, s, W_{\llcorner t\lrcorner_\theta + s} - W_{\llcorner t\lrcorner_\theta}))}{e^{\rho s}}\,ds\Bigg)\,\Bigg|\,(Y_s)_{s\in[0,\llcorner t\lrcorner_\theta]}\Bigg]\Bigg|^{\exp(-\rho\llcorner t\lrcorner_\theta)}\\
&\le \left|\left[e^{c(t-\llcorner t\lrcorner_\theta)+\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(Y_{\llcorner t\lrcorner_\theta})}\right]_{\mathbb{P},\mathcal{B}([0,\infty])}\right|^{e^{-\rho\llcorner t\lrcorner_\theta}}\\
&\le \left[\exp\left(c(t-\llcorner t\lrcorner_\theta) + \tfrac{\mathbb{1}_E(Y_{\llcorner t\lrcorner_\theta})\,V(Y_{\llcorner t\lrcorner_\theta})}{e^{\rho\llcorner t\lrcorner_\theta}}\right)\right]_{\mathbb{P},\mathcal{B}([0,\infty])}.
\end{aligned} \tag{37}$$

This and (32) show for all $t \in (0,T]$ that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\exp\Bigg(-ct + \tfrac{V(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_E(Y_{\lfloor r \rfloor_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Bigg|\,(Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_E(Y_{\llcorner t \lrcorner_\theta})\,V(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{H\setminus E}(Y_{\llcorner t \lrcorner_\theta})\,V(Y_t)}{e^{\rho t}} + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_E(Y_{\llcorner t \lrcorner_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Bigg|\,(Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&\quad\cdot \exp\Bigg(-ct + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\lfloor r \rfloor_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg) \\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_E(Y_{\llcorner t \lrcorner_\theta})\,V(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{H\setminus E}(Y_{\llcorner t \lrcorner_\theta})\,V(S_{t-\llcorner t \lrcorner_\theta}Y_{\llcorner t \lrcorner_\theta})}{e^{\rho t}} + \int_{\llcorner t \lrcorner_\theta}^t \tfrac{\mathbb{1}_E(Y_{\llcorner t \lrcorner_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Bigg|\,(Y_r)_{r\in[0,\llcorner t \lrcorner_\theta]}\Bigg] \\
&\quad\cdot \exp\Bigg(-ct + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\lfloor r \rfloor_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg) \\
&\le \Bigg[\exp\Bigg(c(t-\llcorner t \lrcorner_\theta) + \tfrac{\mathbb{1}_E(Y_{\llcorner t \lrcorner_\theta})\,V(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho \llcorner t \lrcorner_\theta}} + \tfrac{\mathbb{1}_{H\setminus E}(Y_{\llcorner t \lrcorner_\theta})\,V(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho t}} - ct + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\lfloor r \rfloor_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg)\Bigg]_{\mathbb{P},\mathcal{B}([0,\infty])} \\
&\le \Bigg[\exp\Bigg(-c\llcorner t \lrcorner_\theta + \tfrac{V(Y_{\llcorner t \lrcorner_\theta})}{e^{\rho \llcorner t \lrcorner_\theta}} + \int_0^{\llcorner t \lrcorner_\theta} \tfrac{\mathbb{1}_E(Y_{\lfloor r \rfloor_\theta})\,\bar{V}(Y_r)}{e^{\rho r}}\,dr\Bigg)\Bigg]_{\mathbb{P},\mathcal{B}([0,\infty])}.
\end{aligned}
\tag{38}
$$

Lemma 2.2 in Hutzenthaler et al. [27] and (38) establish (34). The proof of Corollary 2.10 is thus completed. □

Remark 2.11. *Let $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon [0,T]\times\Omega \to U$ be an Id_U-cylindrical $\mathbb{P}$-Wiener process. Then $\dim(U) < \infty$.*

2.3 A one-step estimate for exponential moments

In this subsection we establish in (63) in Lemma 2.21 below an appropriate exponential one-step estimate for a general class of one-step approximation schemes. This exponential one-step estimate and Corollary 2.10 above (cf. (63) in Lemma 2.21 below with (33) in Corollary 2.10 above) will allow us to establish exponential integrability properties for some tamed approximation schemes in Subsection 2.4 below. Lemma 2.21 below extends Lemma 2.7 in Hutzenthaler et al. [27] from finite dimensional stochastic ordinary differential equations to infinite dimensional stochastic partial differential equations. Our proof of Lemma 2.21 exploits several elementary/well known auxiliary lemmas (see Lemmas 2.12–2.20 below). Lemma 2.12 below is a straightforward extension of Lemma 2.5 in Hutzenthaler et al. [27]. Lemma 2.15 below follows, e.g., from Theorem 5.8.12 in Bogachev [7].

Lemma 2.12. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, $B \in \mathrm{HS}(U,H)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $\mathbb{P}$-Wiener process. Then it holds for all $t \in [0,T]$ that*

$$\mathbb{E}\big[\exp\big(\|\textstyle\int_0^t B\,dW_s\|_H\big)\big] \le 2\exp\big(\tfrac{t}{2}\|B\|^2_{\mathrm{HS}(U,H)}\big). \tag{39}$$

Lemma 2.13. *Let (E,d_E), (F,d_F), and (G,d_G) be metric spaces and let $f\colon E \to F$ and $g\colon F \to G$ be locally Lipschitz continuous functions. Then it holds that $g \circ f\colon E \to G$ is a locally Lipschitz continuous function.*

Proof of Lemma 2.13. The assumption that $f\colon E \to F$ is locally Lipschitz continuous implies that for every $x \in E$ there exist real numbers $\delta_x, L_x \in (0,\infty)$ such that for all $x_1, x_2 \in E$ with $\max\{d_E(x,x_1), d_E(x,x_2)\} < \delta_x$ it holds that

$$d_F(f(x_1), f(x_2)) \le L_x d_E(x_1,x_2). \tag{40}$$

Moreover, the assumption that $g\colon F \to G$ is locally Lipschitz continuous implies that for every $y \in F$ there exist real numbers $\hat{\delta}_y, \hat{L}_y \in (0,\infty)$ such that for all $y_1, y_2 \in F$ with $\max\{d_F(y,y_1), d_F(y,y_2)\} < \hat{\delta}_y$ it holds that

$$d_G(g(y_1), g(y_2)) \leq \hat{L}_y d_F(y_1, y_2). \tag{41}$$

Next note that (40) proves for all $x, x_1, x_2 \in E$ with $\max\{d_E(x,x_1), d_E(x,x_2)\} < \min\{\delta_x, {}^{\hat{\delta}_{f(x)}}/_{L_x}\}$ that $\max\{d_F(f(x), f(x_1)), d_F(f(x), f(x_2))\} \leq L_x \max\{d_E(x,x_1), d_E(x,x_2)\} < \hat{\delta}_{f(x)}$. Combining this, (40), and (41) ensures that for all $x, x_1, x_2 \in E$ with $\max\{d_E(x,x_1), d_E(x,x_2)\} < \min\{\delta_x, {}^{\hat{\delta}_{f(x)}}/_{L_x}\}$ it holds that

$$d_G(g(f(x_1)), g(f(x_2))) \leq \hat{L}_{f(x)} d_F(f(x_1), f(x_2)) \leq \hat{L}_{f(x)} L_x d_E(x_1, x_2). \tag{42}$$

The proof of Lemma 2.13 is thus completed. □

Lemma 2.14. *Let $(V, \|\cdot\|_V)$ be a normed $\mathbb{R}$-vector space with $\#_V > 1$ and let $c \in [1,\infty)$, $n \in \mathbb{N}_0$, $U \in \mathcal{C}_c^{n+1}(V,\mathbb{R})$, $i \in \mathbb{N}_0 \cap [0,n]$. Then it holds that $U^{(i)}$ is locally Lipschitz continuous.*

Proof of Lemma 2.14. The fact that U is continuous proves that for every $(x,\varepsilon) \in V \times (0,\infty)$ there exists a real number $\delta_{x,\varepsilon} \in (0,\infty)$ such that for all $v \in V$ with $\|x - v\|_V < \delta_{x,\varepsilon}$ it holds that $|U(x) - U(v)| < \varepsilon$. This and the triangle inequality prove that for all $x, x_1, x_2 \in V$ with $\max\{\|x - x_1\|_V, \|x - x_2\|_V\} < \delta_{x,1}$ it holds that

$$\begin{aligned} &\|U^{(i)}(x_1) - U^{(i)}(x_2)\|_{L^{(i)}(V,\mathbb{R})} \leq c\|x_1 - x_2\|_V (1 + \sup\nolimits_{r\in[0,1]} |U(rx_1 + (1-r)x_2)|) \\ &\leq c\|x_1 - x_2\|_V (1 + |U(x)| + \sup\nolimits_{r\in[0,1]} |U(rx_1 + (1-r)x_2) - U(x)|) \\ &\leq c\|x_1 - x_2\|_V (2 + |U(x)|). \end{aligned} \tag{43}$$

The proof of Lemma 2.14 is thus completed. □

Lemma 2.15. *Let $a \in \mathbb{R}$, $b \in (a,\infty)$ and let $f \in \mathcal{C}([a,b],\mathbb{R})$ be a locally Lipschitz continuous function. Then*

(i) it holds that $\{s \in [a,b]\colon f \text{ is differentiable at } s\} \in \mathcal{B}(\mathbb{R})$,

(ii) it holds that $\mu_{\mathbb{R}}([a,b]\backslash\{s \in [a,b]\colon f \text{ is differentiable at } s\}) = 0$, and

(iii) it holds that f is absolutely continuous.

Lemma 2.16. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be an $\mathbb{R}$-Hilbert space with $\#_H > 1$ and let $c \in [1,\infty)$, $n \in \mathbb{N}_0$, $x, y \in H$, $V \in \mathcal{C}_c^{n+1}(H,[0,\infty))$. Then*

(i) it holds for all $t \in \{s \in [0,1]\colon \mathbb{R} \ni u \mapsto V(x+uy) \in \mathbb{R} \text{ is differentiable at } s\}$ that $|\frac{\partial}{\partial t} V(x+ty)| \leq c\|y\|_H (1 + V(x+ty))^{1-1/c}$ and

(ii) it holds for all $i \in \mathbb{N} \cap [0,n]$, $z_1, \ldots, z_i \in H$, $t \in \{s \in [0,1]\colon \mathbb{R} \ni u \mapsto V^{(i)}(x+uy)(z_1,\ldots,z_i) \in \mathbb{R} \text{ is differentiable at } s\}$ that

$$\left|\tfrac{\partial}{\partial t}\big(V^{(i)}(x+ty)(z_1,\ldots,z_i)\big)\right| \leq c\|z_1\|_H \cdots \|z_i\|_H \|y\|_H (1 + V(x+ty))^{1-1/c}. \tag{44}$$

Proof of Lemma 2.16. First of all, note that the assumption that $V \in \mathcal{C}_c^{n+1}(H,[0,\infty))$ ensures that for all $t \in [0,1]$, $h \in \mathbb{R}$ it holds that

$$\begin{aligned} |V(x+ty) - V(x+(t+h)y)| &\leq c|h|\|y\|_H [1 + \sup\nolimits_{r\in[0,1]} V(x + (t+(1-r)h)y)]^{1-1/c} \\ &= c|h|\|y\|_H [1 + \sup\nolimits_{r\in[0,1]} V(x + (t+rh)y)]^{1-1/c}. \end{aligned} \tag{45}$$

Next observe that Lemma 2.14 ensures for all $t \in [0,1]$ that

$$\limsup_{(\mathbb{R}\setminus\{0\})\ni h\to 0} |\sup_{r\in[0,1]} V(x+(t+rh)y) - V(x+ty)| = 0. \tag{46}$$

Combining this with (45) proves (i). In the next step observe that for all $i \in \mathbb{N}\cap[0,n]$, $z_1,\dots,z_i \in H\setminus\{0\}$, $t\in[0,1]$, $h\in\mathbb{R}$ it holds that

$$\begin{aligned} &\tfrac{|V^{(i)}(x+ty)(z_1,\dots,z_i)-V^{(i)}(x+(t+h)y)(z_1,\dots,z_i)|}{\|z_1\|_H\cdots\|z_i\|_H} \le \|V^{(i)}(x+ty)-V^{(i)}(x+(t+h)y)\|_{L^{(i)}(H,\mathbb{R})} \\ &\le c|h|\|y\|_H[1+\sup_{r\in[0,1]} V(x+(t+(1-r)h)y)]^{1-1/c} \\ &= c|h|\|y\|_H[1+\sup_{r\in[0,1]} V(x+(t+rh)y)]^{1-1/c}. \end{aligned} \tag{47}$$

This and (46) establish (ii). The proof of Lemma 2.16 is thus completed. □

Lemma 2.17. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be an $\mathbb{R}$-Hilbert space with $\#_H > 1$ and let $c\in[1,\infty)$, $x,y\in H$, $V\in\mathcal{C}^1_c(H,[0,\infty))$. Then $1+V(x+y)\le 2^{c-1}(1+V(x)+\|y\|_H^c)$.*

Proof of Lemma 2.17. Throughout this proof let $f\colon\mathbb{R}\to\mathbb{R}$ be the function which satisfies for all $t\in\mathbb{R}$ that $f(t)=V(x+ty)$. Next observe that item (i) in Lemma 2.16 implies for all $t\in\{s\in[0,1]\colon \mathbb{R}\ni u\mapsto f(u)\in\mathbb{R}$ is differentiable at $s\}$ that

$$|\tfrac{\partial}{\partial t}(1+f(t))| \le c\|y\|_H(1+f(t))^{1-1/c}. \tag{48}$$

Lemma 2.11 in Hutzenthaler & Jentzen [25], Lemma 2.13, Lemma 2.14, and Lemma 2.15 hence prove for all $t\in[0,1]$ that

$$1+f(t)\le 2^{c-1}\left[1+f(0)+t^c\|y\|_H^c\right]. \tag{49}$$

This implies that $1+V(x+y)\le 2^{c-1}\left[1+V(x)+\|y\|_H^c\right]$. The proof of Lemma 2.17 is thus completed. □

Lemma 2.18. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be an $\mathbb{R}$-Hilbert space with $\#_H > 1$ and let $c\in[1,\infty)$, $n\in\mathbb{N}_0$, $x,y\in H$, $V\in\mathcal{C}^{n+1}_c(H,[0,\infty))$. Then*

$$\max_{i\in\{0,1,\dots,n\}} \|V^{(i)}(x)-V^{(i)}(y)\|_{L^{(i)}(H,\mathbb{R})} \le c\,2^{c-1}\|x-y\|_H\big(1+V(x)+\|x-y\|_H^{c-1}\big). \tag{50}$$

Proof of Lemma 2.18. Note that Lemma 2.13, Lemma 2.14, Lemma 2.15, item (i) in Lemma 2.16, Lemma 2.17, and the fact that $\forall\, r\in[0,1]$, $a\in[1,\infty)$, $b\in[0,\infty)\colon (a+b)^r\le a+b^r$ prove that

$$\begin{aligned} |V(y)-V(x)| &\le \int_{\{s\in[0,1]\colon \mathbb{R}\ni u\mapsto V(x+u(y-x))\in\mathbb{R} \text{ is differentiable at } s\}} \big|\tfrac{\partial}{\partial r}V(x+r(y-x))\big|\,dr \\ &\le c\|x-y\|_H\int_0^1[1+V(x+r(y-x))]^{1-1/c}\,dr \le c\,2^{c-1}\|x-y\|_H\big(1+V(x)+\|y-x\|_H^{c-1}\big). \end{aligned} \tag{51}$$

Moreover, Lemma 2.13, Lemma 2.14, Lemma 2.15, item (ii) in Lemma 2.16, Lemma 2.17, and the fact that $\forall\, r\in[0,1]$, $a\in[1,\infty)$, $b\in[0,\infty)\colon (a+b)^r\le a+b^r$ ensure that for all $i\in\mathbb{N}\cap[0,n]$, $z_1,\dots,z_i\in H\setminus\{0\}$ it holds that

$$\begin{aligned} &\tfrac{|(V^{(i)}(y)-V^{(i)}(x))(z_1,\dots,z_i)|}{\|z_1\|_H\cdots\|z_i\|_H} \le \tfrac{1}{\|z_1\|_H\cdots\|z_i\|_H} \\ &\cdot\int_{\{s\in[0,1]\colon \mathbb{R}\ni u\mapsto V^{(i)}(x+u(y-x))(z_1,\dots,z_i)\in\mathbb{R} \text{ is differentiable at } s\}} \big|\tfrac{\partial}{\partial r}\big(V^{(i)}(x+r(y-x))(z_1,\dots,z_i)\big)\big|\,dr \\ &\le c\|x-y\|_H\int_0^1[1+V(x+r(y-x))]^{1-1/c}\,dr \le c\,2^{c-1}\|x-y\|_H\big(1+V(x)+\|y-x\|_H^{c-1}\big). \end{aligned} \tag{52}$$

Combining this with (51) completes the proof of Lemma 2.18. □

Lemma 2.19. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be an $\mathbb{R}$-Hilbert space with $\#_H > 1$ and let $c \in [1,\infty)$, $n \in \mathbb{N}_0$, $x \in H$, $V \in \mathcal{C}_c^{n+1}(H,[0,\infty))$. Then $\max_{i\in\{0,1,\ldots,n\}} \|V^{(i)}(x)\|_{L^{(i)}(H,\mathbb{R})} \leq c\,(1+V(x))$.*

Proof of Lemma 2.19. Lemma 2.18 proves for all $y \in H$, $t \in [0,1]$ that

$$|V(tx+(1-t)y) - V(x)| \leq c\,2^{c-1}\|x-y\|_H\big(1+V(x)+\|x-y\|_H^{c-1}\big)\,. \tag{53}$$

This implies for all $\varepsilon \in (0,\infty)$, $y \in H$ with $\|x-y\|_H < \varepsilon$ that

$$|\sup\nolimits_{r\in[0,1]} V(rx+(1-r)y) - V(x)| \leq c\,2^{c-1}\varepsilon\left(1+V(x)+\varepsilon^{c-1}\right). \tag{54}$$

Hence, we obtain that

$$\limsup\nolimits_{y\to x}\left|\sup\nolimits_{r\in[0,1]} V(rx+(1-r)y) - V(x)\right| = 0. \tag{55}$$

Moreover, the assumption that $V \in \mathcal{C}_c^{n+1}(H,[0,\infty))$ assures for all $i \in \mathbb{N}_0 \cap [0,n]$, $y \in H$ that

$$\|V^{(i)}(x) - V^{(i)}(y)\|_{L^{(i)}(H,\mathbb{R})} \leq c\|x-y\|_H(1+\sup\nolimits_{r\in[0,1]} V(rx+(1-r)y))^{1-1/c}. \tag{56}$$

Combining this with (55) completes the proof of Lemma 2.19. □

Lemma 2.20. *Let $(V, \|\cdot\|_V)$ be a normed $\mathbb{R}$-vector space, let $(\Omega,\mathcal{F},\mu)$ be a finite measure space, and let $X \in \mathcal{M}(\mathcal{F},\mathcal{B}(V))$. Then $\liminf_{p\to\infty}\|X\|_{\mathcal{L}^p(\mu;V)} = \limsup_{p\to\infty}\|X\|_{\mathcal{L}^p(\mu;V)} = \|X\|_{\mathcal{L}^\infty(\mu;V)}$.*

Proof of Lemma 2.20. Throughout this proof assume w.l.o.g. that $\|X\|_{\mathcal{L}^\infty(\mu;V)} > 0$ and let $A_\delta \subseteq \Omega$, $\delta \in (0,\infty)$, be the sets with the property that for all $\delta \in (0,\infty)$ it holds that $A_\delta = \{\omega \in \Omega \colon \|X(\omega)\|_V \geq \delta\}$. Next observe that for all $p \in (0,\infty)$, $\delta \in (0, \|X\|_{\mathcal{L}^\infty(\mu;V)})$ it holds that

$$\|X\|_{\mathcal{L}^p(\mu;V)} \geq \|X\mathbb{1}_{A_\delta}\|_{\mathcal{L}^p(\mu;V)} \geq \|\delta\,\mathbb{1}_{A_\delta}\|_{\mathcal{L}^p(\mu;\mathbb{R})} = \delta\,[\mu(A_\delta)]^{1/p}. \tag{57}$$

Hence, we obtain for all $\delta \in (0,\|X\|_{\mathcal{L}^\infty(\mu;V)})$ that $\liminf_{p\to\infty}\|X\|_{\mathcal{L}^p(\mu;V)} \geq \delta$. This shows that $\liminf_{p\to\infty}\|X\|_{\mathcal{L}^p(\mu;V)} \geq \|X\|_{\mathcal{L}^\infty(\mu;V)}$. Moreover, note that for all $p \in (0,\infty)$, $q \in (0,p)$ it holds that

$$\|X\|_{\mathcal{L}^p(\mu;V)} = \left(\int_\Omega \|X(\omega)\|_V^q\|X(\omega)\|_V^{p-q}\,\mu(d\omega)\right)^{\frac{1}{p}} \leq \|X\|_{\mathcal{L}^\infty(\mu;V)}^{(p-q)/p}\|X\|_{\mathcal{L}^q(\mu;V)}^{q/p}. \tag{58}$$

This implies that $\limsup_{p\to\infty}\|X\|_{\mathcal{L}^p(\mu;V)} \leq \|X\|_{\mathcal{L}^\infty(\mu;V)}$. The proof of Lemma 2.20 is thus completed. □

Lemma 2.21. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H > 1 < \#_U$, let $\varsigma, h \in (0,\infty)$, $c,\gamma_0,\gamma_1 \in [1,\infty)$, $\rho,\delta \in [0,\infty)$, $\gamma_2 \in [0,1/2]$, $x \in H$, $F \in \mathcal{M}(\mathcal{B}(H),\mathcal{B}(H))$, $B \in \mathcal{M}(\mathcal{B}(H),\mathcal{B}(\mathrm{HS}(U,H)))$, $\bar{V} \in \mathcal{C}(H,\mathbb{R})$, $V \in \mathcal{C}_c^3(H,[0,\infty))$, $\Phi \in \mathcal{C}^{1,2}([0,h]\times U,H)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,h]\times\Omega \to U$ be an Id_U-cylindrical $\mathbb{P}$-Wiener process with continuous sample paths, assume for all $r \in [1,\infty)$, $s \in (0,h]$, $y,z \in H$ that $|\bar{V}(y)-\bar{V}(z)| \leq c\,(1+|V(y)|^{\gamma_0}+|V(z)|^{\gamma_0})\,\|y-z\|_H$, $|\bar{V}(y)| \leq c\,(1+|V(y)|^{\gamma_1})$, $V(x) \leq ch^{-\varsigma}$, $\max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta}$, $\Phi(0,0) = x$, and*

$$(\mathcal{G}_{F,B}V)(x) + \tfrac{1}{2}\,\|B(x)^*(\nabla V)(x)\|_U^2 + \bar{V}(x) \leq \rho V(x), \tag{59}$$

$$\max\left\{\|(\tfrac{\partial}{\partial s}\Phi)(s,W_s) - F(x)\|_{\mathcal{L}^4(\mathbb{P};H)}, \|(\tfrac{\partial}{\partial y}\Phi)(s,W_s) - B(x)\|_{\mathcal{L}^8(\mathbb{P};\mathrm{HS}(U,H))}\right\} \leq cs^{\gamma_2}, \tag{60}$$

$$\left\|\sum\nolimits_{u\in\mathbb{U}}\left((\tfrac{\partial^2}{\partial y^2}\Phi)(s,W_s)\right)(u,u)\right\|_{\mathcal{L}^4(\mathbb{P};H)} \leq cs^{\gamma_2}, \tag{61}$$

$$\|\Phi(s,W_s) - x\|_{\mathcal{L}^r(\mathbb{P};H)} \leq c\min\left\{1, \left\|\|F(x)\|_H s + \|B(x)W_s\|_H\right\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})}\right\}. \tag{62}$$

Then it holds for all $t \in (0,h]$ that

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\tfrac{V(\Phi(t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Phi(s,W_s))}{e^{\rho s}}\,ds\right)\right]\\ &\leq e^{V(x)}\left[1+\int_0^t \exp\left(\frac{[2c]^{(2c+6)\gamma_1}s}{\min\{s^{2\delta+\max\{2,\gamma_1\}\varsigma},1\}}\right)\frac{[2c]^{(2c+13)\gamma_0}\,[\max\{s,\rho,1\}]^4}{[\min\{s,1\}]^{\varsigma+\varsigma\gamma_0+4\delta-\gamma_2}}\,ds\right].\end{aligned} \tag{63}$$

Proof of Lemma 2.21. Throughout this proof let $\mathbb{U} \in \mathcal{P}(U)$ be an orthonormal basis of U, let $Y\colon [0,h] \times \Omega \to H$ be the function which satisfies for all $s \in [0,h]$ that $Y_s = \Phi(s, W_s)$, and let $\tau_n\colon \Omega \to [0,h]$, $n \in \mathbb{N}$, be the functions which satisfy for all $n \in \mathbb{N}$ that $\tau_n = \inf(\{s \in [0,h]\colon \|W_s\|_U > n\} \cup \{h\})$. Next observe that $\mathbb{P}(Y_0 = x) = \mathbb{P}(\Phi(0, W_0) = x) = \mathbb{P}(\Phi(0,0) = x) = 1$. Itô's formula hence implies for all $t \in [0,h]$ that

$$
\begin{split}
&\left[\exp\!\left(e^{-\rho t} V(Y_t) + \int_0^t e^{-\rho r} \bar{V}(Y_r)\, dr\right) - e^{V(x)}\right]_{\mathbb{P},\mathcal{B}(\mathbb{R})} \\
&= \int_0^t \left[\exp\!\left(e^{-\rho s} V(Y_s) + \int_0^s e^{-\rho r} \bar{V}(Y_r)\, dr\right) e^{-\rho s} V'(Y_s) \big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\right] dW_s \\
&+ \Bigg[\int_0^t \exp\!\left(e^{-\rho s} V(Y_s) + \int_0^s e^{-\rho r} \bar{V}(Y_r)\, dr\right) e^{-\rho s} \Big[\bar{V}(Y_s) - \rho V(Y_s) \\
&+ V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) + \tfrac{1}{2}\operatorname{trace}_U\!\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\operatorname{Hess} V)(Y_s) \big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big) \\
&+ \tfrac{1}{2e^{\rho s}} \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\nabla V)(Y_s)\big\|_U^2 + \tfrac{1}{2}\sum_{u \in \mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u,u)\Big]\, ds\Bigg]_{\mathbb{P},\mathcal{B}(\mathbb{R})}.
\end{split}
\tag{64}
$$

Therefore, we obtain for all $t \in [0,h]$, $n \in \mathbb{N}$ that

$$
\begin{split}
&\mathbb{E}\left[\exp\!\left(e^{-\rho(t\wedge\tau_n)} V(Y_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r} \bar{V}(Y_r)\, dr\right)\right] - e^{V(x)} \\
&= \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n} \exp\!\left(e^{-\rho s} V(Y_s) + \int_0^s e^{-\rho r} \bar{V}(Y_r)\, dr\right) e^{-\rho s} \Big(\bar{V}(Y_s) - \rho V(Y_s) \\
&+ V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) + \tfrac{1}{2}\operatorname{trace}_U\!\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\operatorname{Hess} V)(Y_s) \big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big) \\
&+ \tfrac{1}{2e^{\rho s}} \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\nabla V)(Y_s)\big\|_U^2 + \tfrac{1}{2}\sum_{u \in \mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u,u)\Big)\, ds\Bigg].
\end{split}
\tag{65}
$$

This implies for all $t \in [0,h]$, $n \in \mathbb{N}$ that

$$
\begin{split}
&\mathbb{E}\left[\exp\!\left(e^{-\rho(t\wedge\tau_n)} V(Y_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r} \bar{V}(Y_r)\, dr\right)\right] - e^{V(x)} \\
&= \mathbb{E}\Bigg[\int_0^{t\wedge\tau_n} \exp\!\left(e^{-\rho s} V(Y_s) + \int_0^s e^{-\rho r} \bar{V}(Y_r)\, dr\right) e^{-\rho s} \Big((\mathcal{G}_{F,B} V)(x) \\
&\quad + \tfrac{1}{2e^{\rho s}} \|B(x)^* (\nabla V)(x)\|_U^2 + \bar{V}(x) - \rho V(x) + \bar{V}(Y_s) - \bar{V}(x) \\
&\quad - \rho(V(Y_s) - V(x)) + \tfrac{1}{2}\operatorname{trace}_U\!\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\operatorname{Hess} V)(Y_s) \big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big) \\
&\quad + V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) - V'(x)F(x) - \tfrac{1}{2}\operatorname{trace}_U\!\big(B(x)^*(\operatorname{Hess} V)(x) B(x)\big) \\
&\quad + \tfrac{1}{2e^{\rho s}} \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^* (\nabla V)(Y_s)\big\|_U^2 - \tfrac{1}{2e^{\rho s}} \|B(x)^* (\nabla V)(x)\|_U^2 \\
&\quad + \tfrac{1}{2}\textstyle\sum_{u \in \mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u,u)\Big)\, ds\Bigg].
\end{split}
\tag{66}
$$

Assumption (59) hence proves for all $t \in [0, h]$, $n \in \mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}V(Y_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg] - e^{V(x)} \\
&\leq \mathbb{E}\bigg[\int_0^{t\wedge\tau_n} \exp\bigg(e^{-\rho s}V(Y_s) + \int_0^s e^{-\rho r}\bar{V}(Y_r)\,dr\bigg) \\
&\cdot\bigg(\big|\bar{V}(Y_s) - \bar{V}(x)\big| + \rho|V(Y_s) - V(x)| + \big|V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) - V'(x)F(x)\big| \\
&+ \tfrac{1}{2}\bigg|\operatorname{trace}_U\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\operatorname{Hess} V)(Y_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big) \\
&- \operatorname{trace}_U\big(B(x)^*(\operatorname{Hess} V)(x)B(x)\big)\bigg| + \tfrac{1}{2}\Big|\sum\nolimits_{u\in\mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u, u)\Big| \\
&+ \tfrac{1}{2e^{\rho s}}\Big|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\nabla V)(Y_s)\big\|_U^2 - \|B(x)^*(\nabla V)(x)\|_U^2\Big|\bigg)\,ds\bigg].
\end{aligned} \tag{67}$$

Fatou's lemma therefore shows for all $t \in [0, h]$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho t}V(Y_t) + \int_0^t e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg] - e^{V(x)} \\
&\leq \liminf_{n\to\infty} \mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}V(Y_{t\wedge\tau_n}) + \int_0^{t\wedge\tau_n} e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg] - e^{V(x)} \\
&\leq \mathbb{E}\bigg[\int_0^t \exp\bigg(V(Y_s) + \int_0^s e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg(\big|\bar{V}(Y_s) - \bar{V}(x)\big| + \rho|V(Y_s) - V(x)| \\
&+ \big|V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) - V'(x)F(x)\big| + \tfrac{1}{2}\bigg|\operatorname{trace}_U\big(B(x)^*(\operatorname{Hess} V)(x)B(x)\big) \\
&- \operatorname{trace}_U\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\operatorname{Hess} V)(Y_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big)\bigg| \\
&+ \tfrac{1}{2e^{\rho s}}\Big|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\nabla V)(Y_s)\big\|_U^2 - \|B(x)^*(\nabla V)(x)\|_U^2\Big| \\
&+ \tfrac{1}{2}\Big|\sum\nolimits_{u\in\mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u, u)\Big|\bigg)\,ds\bigg].
\end{aligned} \tag{68}$$

Tonelli's theorem and Hölder's inequality hence imply for all $t \in [0, h]$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho t}V(Y_t) + \int_0^t e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg] - e^{V(x)} \\
&\leq \int_0^t \bigg\|\exp\bigg(V(Y_s) + \int_0^s e^{-\rho r}\bar{V}(Y_r)\,dr\bigg)\bigg\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})}\Big(\rho\|V(Y_s) - V(x)\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&+ \big\|V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s, W_s) - V'(x)F(x)\big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} + \tfrac{1}{2}\Big\|\operatorname{trace}_U\big(B(x)^*(\operatorname{Hess} V)(x)B(x)\big) \\
&- \operatorname{trace}_U\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\operatorname{Hess} V)(Y_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\Big)\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{2e^{\rho s}}\Big\|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s, W_s)\big]^*(\nabla V)(Y_s)\big\|_U^2 - \|B(x)^*(\nabla V)(x)\|_U^2\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&+ \tfrac{1}{2}\Big\|\sum\nolimits_{u\in\mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s, W_s)\big)(u, u)\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} + \big\|\bar{V}(Y_s) - \bar{V}(x)\big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})}\Big)\,ds.
\end{aligned} \tag{69}$$

Next we estimate the $\mathcal{L}^2(\mathbb{P};\mathbb{R})$-semi-norms on the right-hand side of (69) separately. Lemma 2.18 implies for all $y, z \in H$, $i \in \{0, 1, 2\}$ that

$$\|V^{(i)}(y) - V^{(i)}(z)\|_{L^{(i)}(H,\mathbb{R})} \leq c\, 2^{c-1}\|y - z\|_H \left(1 + V(y) + \|y - z\|_H^{c-1}\right). \tag{70}$$

The assumption that $\forall\, y \in H\colon |\bar{V}(y)| \leq c\,(1+|V(y)|^{\gamma_1})$ and Hölder's inequality hence prove for all $s \in (0,h]$ that

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\!\bigg(2V(Y_s) + 2\int_0^s e^{-\rho r}\bar{V}(Y_r)\,dr - 2V(x)\bigg)\bigg] \\
&\leq \mathbb{E}\bigg[\exp\!\bigg(2\,|V(Y_s)-V(x)| + 2\int_0^s \big|\bar{V}(Y_r)\big|\,dr\bigg)\bigg] \\
&\leq \mathbb{E}\bigg[\exp\!\bigg(c\,2^c\big[1+V(x)+\|Y_s-x\|_H^{c-1}\big]\|Y_s-x\|_H + 2c\int_0^s (1+|V(Y_r)|^{\gamma_1})\,dr\bigg)\bigg] \\
&\leq \big\|\exp\!\big(c\,2^c\big[1+V(x)+\|Y_s-x\|_H^{c-1}\big]\|Y_s-x\|_H\big)\big\|_{\mathcal{L}^1(\mathbb{P};\mathbb{R})} \\
&\quad\cdot \bigg\|\exp\!\bigg(2c\int_0^s (1+|V(Y_r)|^{\gamma_1})\,dr\bigg)\bigg\|_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})} \\
&\leq \mathbb{E}\big[\exp\!\big(c\,2^c\big[1+V(x)+\|Y_s-x\|_H^{c-1}\big]\,\|Y_s-x\|_H\big)\big]\exp\!\bigg(2c\int_0^s \big[1+\|V(Y_r)\|^{\gamma_1}_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})}\big]\,dr\bigg).
\end{aligned}
\tag{71}
$$

Next we estimate the two factors on the right-hand side of (71) separately. Observe that (62) ensures that for all $r \in [1,\infty)$, $s \in (0,h]$ it holds that $\|\Phi(s,W_s)-x\|_{\mathcal{L}^r(\mathbb{P};H)} \leq c$. Combining this with Lemma 2.20 establishes that for all $s \in (0,h]$ it holds that $\|\Phi(s,W_s)-x\|_{\mathcal{L}^\infty(\mathbb{P};H)} \leq c$. Hölder's inequality, Tonelli's theorem, and (62) therefore show that for all $s \in (0,h]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\big[\exp\!\big(c\,2^c\big[1+V(x)+\|Y_s-x\|_H^{c-1}\big]\,\|Y_s-x\|_H\big)\big] \\
&= \mathbb{E}\Bigg[\sum_{n=0}^\infty \tfrac{(2^c c)^n}{n!}\big[1+V(x)+\|\Phi(s,W_s)-x\|_H^{c-1}\big]^n\,\|\Phi(s,W_s)-x\|_H^n\Bigg] \\
&= \sum_{n=0}^\infty \Big\|\tfrac{(2^c c)^n}{n!}\big[1+V(x)+\|\Phi(s,W_s)-x\|_H^{c-1}\big]^n\,\|\Phi(s,W_s)-x\|_H^n\Big\|_{\mathcal{L}^1(\mathbb{P};\mathbb{R})} \\
&\leq \sum_{n=0}^\infty \tfrac{(2^c c)^n}{n!}\big[1+V(x)+\|\Phi(s,W_s)-x\|_{\mathcal{L}^\infty(\mathbb{P};H)}^{c-1}\big]^n\,\|\Phi(s,W_s)-x\|_{\mathcal{L}^n(\mathbb{P};H)}^n \\
&\leq \sum_{n=0}^\infty \tfrac{(2^c c)^n}{n!}\big[1+V(x)+c^{c-1}\big]^n\,\|\Phi(s,W_s)-x\|_{\mathcal{L}^n(\mathbb{P};H)}^n \\
&\leq \sum_{n=0}^\infty \tfrac{(2^c c)^n}{n!}\,\big[c\big(1+V(x)\big)+c^c\big]^n\,\mathbb{E}\big[(\|F(x)\|_H s + \|B(x)W_s\|_H)^n\big] \\
&= \mathbb{E}\big[\exp\!\big(c\,2^c\big[c(1+V(x))+c^c\big](\|F(x)\|_H s+\|B(x)W_s\|_H)\big)\big].
\end{aligned}
\tag{72}
$$

The assumption that $\forall\, s \in (0,h]\colon \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta} \leq cs^{-\delta}$, the fact that $\forall\, a,b \in [0,\infty)\colon (a+b)^c \leq 2^{c-1}(a^c+b^c)$, and Lemma 2.12 hence show that for all $s \in (0,h]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\big[\exp\!\big(c\,2^c\big[1+V(x)+\|Y_s-x\|_H^{c-1}\big]\|Y_s-x\|_H\big)\big] \\
&\leq \mathbb{E}\big[\exp\!\big(c^2\,2^c\,\big[1+V(x)+c^{c-1}\big]\,(\|F(x)\|_H s+\|B(x)W_s\|_H)\,\big)\big] \\
&\leq \mathbb{E}\big[\exp\!\big(c^2\,2^c\,\big[1+c[\min\{s,1\}]^{-\varsigma}+c^{c-1}\big]\,(\|F(x)\|_H s+\|B(x)W_s\|_H)\,\big)\big] \\
&\leq \mathbb{E}\big[\exp\!\big(3\cdot 2^c c^{c+2}[\min\{s,1\}]^{-\varsigma}\,(\|F(x)\|_H s+\|B(x)W_s\|_H)\big)\big] \\
&= \exp\!\big(3\cdot 2^c c^{c+2}[\min\{s,1\}]^{-\varsigma}\|F(x)\|_H s\big)\,\mathbb{E}\big[\exp\!\big(3\cdot 2^c c^{c+2}[\min\{s,1\}]^{-\varsigma}\|B(x)W_s\|_H\big)\big] \\
&\leq \exp\!\big(3\cdot 2^c c^{c+2}[\min\{s,1\}]^{-\varsigma}\|F(x)\|_H s\big)\,2\exp\!\bigg(\frac{9\cdot 2^{2c}c^{2c+4}\|B(x)\|^2_{\mathrm{HS}(U,H)}s}{2\,[\min\{s,1\}]^{2\varsigma}}\bigg) \\
&\leq 2\exp\!\big(9\cdot 2^{2c-1}c^{2c+4}s[\min\{s,1\}]^{-2\varsigma}\,\big[\|F(x)\|_H+\|B(x)\|^2_{\mathrm{HS}(U,H)}\big]\big) \\
&\leq 2\exp\!\big(9\cdot 2^{2c-1}c^{2c+4}s[\min\{s,1\}]^{-2\varsigma}\,\big[cs^{-\delta}+c^2 s^{-2\delta}\big]\big) \\
&\leq 2\exp\!\big(9\cdot 2^{2c}c^{2c+6}s[\min\{s,1\}]^{-2\delta-2\varsigma}\big)\,.
\end{aligned}
\tag{73}
$$

In the next step we combine (62), (70), and Lemma 2.20 to obtain for all $s \in (0, h]$ that

$$
\begin{aligned}
&\|V(Y_s)\|_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})} \le V(x) + \|V(Y_s) - V(x)\|_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})} \\
&\le V(x) + c\, 2^{c-1} \left\| \left(1 + V(x) + \|Y_s - x\|_H^{c-1}\right) \|Y_s - x\|_H \right\|_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})} \\
&\le V(x) + c\, 2^{c-1} \left[c + c^2 [\min\{s, 1\}]^{-\varsigma} + c^c\right] \le c s^{-\varsigma} + c\, 2^{c-1} \left[2c^2 [\min\{s, 1\}]^{-\varsigma} + c^c\right] \\
&\le 2^{c+1} c^{c+2} [\min\{s, 1\}]^{-\varsigma}.
\end{aligned} \tag{74}
$$

Therefore, we obtain for all $s \in (0, h]$ that

$$
\begin{aligned}
2c \int_0^s \left[1 + \|V(Y_r)\|_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})}^{\gamma_1}\right] dr &\le 2cs + 2cs\left(2^{c+1} c^{c+2} [\min\{s, 1\}]^{-\varsigma}\right)^{\gamma_1} \\
&\le 2^{2+(c+1)\gamma_1} c^{1+(c+2)\gamma_1} [\min\{s, 1\}]^{-\gamma_1 \varsigma} s.
\end{aligned} \tag{75}
$$

Combining this with (71) and (73) ensures that for all $s \in (0, h]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(2V(Y_s) + 2\int_0^s e^{-\rho r} \bar{V}(Y_r)\, dr - 2V(x)\right)\right] \\
&\le 2\exp\left(9 \cdot 2^{2c} c^{2c+6} s [\min\{s, 1\}]^{-2\delta - 2\varsigma}\right) \exp\left(2^{2+(c+1)\gamma_1} c^{1+(c+2)\gamma_1} [\min\{s, 1\}]^{-\gamma_1 \varsigma} s\right) \\
&\le 2\exp\left(\left[9 \cdot 2^{2c} c^{2c+6} + 2^{2+(c+1)\gamma_1} c^{1+(c+2)\gamma_1}\right] s [\min\{s, 1\}]^{-2\delta - \max\{2, \gamma_1\}\varsigma}\right) \\
&\le 2\exp\left(2^{2c\gamma_1+4} c^{2c\gamma_1+\gamma_1+5} s [\min\{s, 1\}]^{-2\delta - \max\{2, \gamma_1\}\varsigma}\right).
\end{aligned} \tag{76}
$$

Hence, we obtain for all $s \in (0, h]$ that

$$
\left\| \exp\left(V(Y_s) + \int_0^s \tfrac{\bar{V}(Y_r)}{e^{\rho r}}\, dr\right) \right\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \le \sqrt{2} \exp\left(\frac{2^{2c\gamma_1+3} c^{2c\gamma_1+\gamma_1+5} s}{[\min\{s, 1\}]^{2\delta + \max\{2, \gamma_1\}\varsigma}}\right) e^{V(x)}. \tag{77}
$$

Moreover, (62), the assumption that $\forall\, s \in (0, h]\colon \max\{\|F(x)\|_H,\ \|B(x)\|_{\mathrm{HS}(U,H)}\} \le c h^{-\delta} \le c s^{-\delta}$, the triangle inequality, and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [16] assure that for all $r \in [2, \infty)$, $s \in (0, h]$ it holds that

$$
\begin{aligned}
&\|Y_s - x\|_{\mathcal{L}^r(\mathbb{P};H)} = \|\Phi(s, W_s) - x\|_{\mathcal{L}^r(\mathbb{P};H)} \le c \left\| \|F(x)\|_H s + \|B(x) W_s\|_H \right\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \\
&\le c\left(\|F(x)\|_H s + \sqrt{sr(r-1)/2}\, \|B(x)\|_{\mathrm{HS}(U,H)}\right) \le c\left(c s^{1-\delta} + c\sqrt{r(r-1)/2}\, s^{1/2-\delta}\right) \\
&\le c^2 r s^{1/2-\delta} \max\{\sqrt{s}, 1\} \le c^2 r [\min\{s, 1\}]^{1/2-\delta} \max\{s, 1\}.
\end{aligned} \tag{78}
$$

Combining this with (62), (70), and Hölder's inequality implies that for all $r \in [2, \infty)$, $i \in \{0, 1, 2\}$, $s \in (0, h]$ it holds that

$$
\begin{aligned}
&\left\|V^{(i)}(Y_s) - V^{(i)}(x)\right\|_{\mathcal{L}^r(\mathbb{P};L^{(i)}(H,\mathbb{R}))} \\
&\le \left\| c\, 2^{c-1} \left(1 + V(x) + \|Y_s - x\|_H^{c-1}\right) \|Y_s - x\|_H \right\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \\
&\le c\, 2^{c-1} \left(\|Y_s - x\|_{\mathcal{L}^r(\mathbb{P};H)} + V(x) \|Y_s - x\|_{\mathcal{L}^r(\mathbb{P};H)} + \|Y_s - x\|_{\mathcal{L}^{rc}(\mathbb{P};H)}^c\right) \\
&\le c\, 2^{c-1} \left(1 + c s^{-\varsigma} + \|Y_s - x\|_{\mathcal{L}^{rc}(\mathbb{P};H)}^{c-1}\right) \|Y_s - x\|_{\mathcal{L}^{rc}(\mathbb{P};H)} \\
&\le c^4\, 2^{c-1} \left[2c [\min\{s, 1\}]^{-\varsigma} + c^{c-1}\right] r [\min\{s, 1\}]^{1/2-\delta} \max\{s, 1\} \\
&\le r\, c^{c+4}\, 2^{c+1} [\min\{s, 1\}]^{1/2-\delta-\varsigma} \max\{s, 1\}.
\end{aligned} \tag{79}
$$

Hölder's inequality, the assumption that $\forall\, s \in (0, h]\colon \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \le c h^{-\delta} \le$

$cs^{-\delta}$, (60), and Lemma 2.19 hence show for all $s \in (0,h]$ that

$$
\begin{aligned}
&\big\|V'(Y_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(s,W_s) - V'(x)F(x)\big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \leq \|V'(Y_s) - V'(x)\|_{\mathcal{L}^2(\mathbb{P};L(H,\mathbb{R}))}\, \|F(x)\|_H \\
&\quad + \Big\| \|V'(Y_s)\|_{L(H,\mathbb{R})} \, \big\|\big(\tfrac{\partial}{\partial s}\Phi\big)(s,W_s) - F(x)\big\|_H \Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&\leq \|V'(x)\|_{L(H,\mathbb{R})} \, \big\|\big(\tfrac{\partial}{\partial s}\Phi\big)(s,W_s) - F(x)\big\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&\quad + \|V'(Y_s) - V'(x)\|_{\mathcal{L}^4(\mathbb{P};L(H,\mathbb{R}))} \Big[\|F(x)\|_H + \big\|\big(\tfrac{\partial}{\partial s}\Phi\big)(s,W_s) - F(x)\big\|_{\mathcal{L}^4(\mathbb{P};H)}\Big] \\
&\leq c^2\,[1+V(x)]\, s^{\gamma_2} + 2^{c+3}\, c^{c+4}\, [\min\{s,1\}]^{1/2-\delta-\varsigma} \big[cs^{-\delta} + cs^{\gamma_2}\big] \max\{s,1\} \\
&\leq c^2 \big[1 + cs^{-\varsigma}\big]\, s^{\gamma_2} + 2^{c+4}\, c^{c+5}\, [\min\{s,1\}]^{1/2-2\delta-\varsigma}\, [\max\{s,1\}]^{1+\gamma_2} \\
&\leq 2^{c+5}\, c^{c+5}\, [\min\{s,1\}]^{\gamma_2-2\delta-\varsigma}\, [\max\{s,1\}]^{1+\gamma_2}.
\end{aligned}
\tag{80}
$$

In addition, observe that (79), Hölder's inequality, (60), Lemma 2.19, and the assumption that $\forall\, s \in (0,h]\colon \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta} \leq cs^{-\delta}$ ensure that for all $s \in (0,h]$ it holds that

$$
\begin{aligned}
&\big\|V'(Y_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - V'(x)B(x)\big\|_{\mathcal{L}^4(\mathbb{P};L(U,\mathbb{R}))} \\
&\leq \Big\| \|V'(Y_s)\|_{L(H,\mathbb{R})} \, \big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)\big\|_{\mathrm{HS}(U,H)} \Big\|_{\mathcal{L}^4(\mathbb{P};\mathbb{R})} \\
&+ \|V'(Y_s) - V'(x)\|_{\mathcal{L}^4(\mathbb{P};L(H,\mathbb{R}))}\, \|B(x)\|_{\mathrm{HS}(U,H)} \\
&\leq \|V'(x)\|_{L(H,\mathbb{R})} \, \big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)\big\|_{\mathcal{L}^4(\mathbb{P};\mathrm{HS}(U,H))} \\
&+ \|V'(Y_s) - V'(x)\|_{\mathcal{L}^8(\mathbb{P};L(H,\mathbb{R}))} \Big[\|B(x)\|_{\mathrm{HS}(U,H)} + \big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)\big\|_{\mathcal{L}^8(\mathbb{P};\mathrm{HS}(U,H))}\Big] \\
&\leq c\,[1+V(x)]\, cs^{\gamma_2} + 2^{c+4} c^{c+4}\, [\min\{s,1\}]^{1/2-\delta-\varsigma}\, [cs^{-\delta} + cs^{\gamma_2}] \max\{s,1\} \\
&\leq c^2 s^{\gamma_2} \big[1 + cs^{-\varsigma}\big] + 2^{c+5} c^{c+5}\, [\min\{s,1\}]^{1/2-2\delta-\varsigma}\, [\max\{s,1\}]^{1+\gamma_2} \\
&\leq 2^{c+6} c^{c+5}\, [\min\{s,1\}]^{\gamma_2-2\delta-\varsigma}\, [\max\{s,1\}]^{1+\gamma_2}.
\end{aligned}
\tag{81}
$$

Moreover, note that for all $A_1, A_2 \in \mathrm{HS}(U,H)$, $B_1, B_2 \in L(H)$ it holds that

$$
\begin{aligned}
&|\operatorname{trace}_U(A_1^* B_1 A_1 - A_2^* B_2 A_2)| = \Bigg| \sum_{u \in \mathbb{U}} \langle (A_1^* B_1 A_1 - A_2^* B_2 A_2)\, u, u \rangle_U \Bigg| \\
&= \Bigg| \sum_{u \in \mathbb{U}} \langle B_1 A_1 u, A_1 u \rangle_H - \sum_{u \in \mathbb{U}} \langle B_2 A_2 u, A_2 u \rangle_H \Bigg| = |\, \langle A_1, B_1 A_1 \rangle_{\mathrm{HS}(U,H)} - \langle A_2, B_2 A_2 \rangle_{\mathrm{HS}(U,H)} \,| \\
&= |\, \langle A_1 - A_2, B_1 A_1 \rangle_{\mathrm{HS}(U,H)} + \langle A_2, B_1(A_1 - A_2) \rangle_{\mathrm{HS}(U,H)} + \langle A_2, (B_1 - B_2) A_2 \rangle_{\mathrm{HS}(U,H)} \,| \\
&\leq \|A_1 - A_2\|_{\mathrm{HS}(U,H)}\, \|B_1 A_1\|_{\mathrm{HS}(U,H)} + \|A_2\|_{\mathrm{HS}(U,H)}\, \|B_1(A_1 - A_2)\|_{\mathrm{HS}(U,H)} \\
&\quad + \|A_2\|_{\mathrm{HS}(U,H)}\, \|(B_1 - B_2) A_2\|_{\mathrm{HS}(U,H)} \\
&\leq \|A_1 - A_2\|_{\mathrm{HS}(U,H)}\, \|B_1\|_{L(H)} \Big[\|A_1\|_{\mathrm{HS}(U,H)} + \|A_2\|_{\mathrm{HS}(U,H)}\Big] + \|B_1 - B_2\|_{L(H)}\, \|A_2\|^2_{\mathrm{HS}(U,H)} \\
&\leq \Big[\|A_1 - A_2\|^2_{\mathrm{HS}(U,H)} + 2\, \|A_1 - A_2\|_{\mathrm{HS}(U,H)}\, \|A_2\|_{\mathrm{HS}(U,H)}\Big] \Big[\|B_1 - B_2\|_{L(H)} + \|B_2\|_{L(H)}\Big] \\
&\quad + \|B_1 - B_2\|_{L(H)}\, \|A_2\|^2_{\mathrm{HS}(U,H)}
\end{aligned}
\tag{82}
$$

(cf., e.g., (62) in Hutzenthaler et al. [27]). Next we apply (82) (with $A_1 = (\frac{\partial}{\partial y}\Phi)(s,W_s)$, $A_2 = B(x)$, $B_1 = (\operatorname{Hess} V)(Y_s)$, and $B_2 = (\operatorname{Hess} V)(x)$ for $s \in [0,h]$ in the notation of (82)), we take expectations, we apply Hölder's inequality, we apply Lemma 2.19, we use the assumption that $\forall\, s \in (0,h]\colon \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta} \leq cs^{-\delta}$, and we apply (60) and (79) to obtain that for all $s \in (0, h]$ it holds that

$$
\begin{aligned}
&\Big\|\text{trace}_U\Big(\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s)\big]^*(\text{Hess}\,V)(Y_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)^*(\text{Hess}\,V)(x)B(x)\Big)\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&\le \Big\|\big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)\big\|^2_{\text{HS}(U,H)} + 2\big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s) - B(x)\big\|_{\text{HS}(U,H)}\|B(x)\|_{\text{HS}(U,H)}\Big\|_{\mathcal{L}^4(\mathbb{P};\mathbb{R})} \\
&\quad\cdot\big[\|(\text{Hess}\,V)(x)\|_{L(H)} + \|(\text{Hess}\,V)(Y_s) - (\text{Hess}\,V)(x)\|_{\mathcal{L}^4(\mathbb{P};L(H))}\big] \\
&\quad + \|(\text{Hess}\,V)(Y_s) - (\text{Hess}\,V)(x)\|_{\mathcal{L}^2(\mathbb{P};L(H))}\,\|B(x)\|^2_{\text{HS}(U,H)} \\
&\le \big[c^2 s^{2\gamma_2} + 2cs^{\gamma_2}cs^{-\delta}\big]\big[c(1+V(x)) + 2^{c+3}c^{c+4}[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\}\big] \\
&\quad + 2^{c+2}c^{c+4}[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\}c^2s^{-2\delta} \\
&\le 3c^2[\min\{s,1\}]^{\gamma_2-\delta}[\max\{s,1\}]^{2\gamma_2}\big[2c^2[\min\{s,1\}]^{-\varsigma} + 2^{c+3}c^{c+4}[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\}\big] \\
&\quad + 2^{c+2}c^{c+6}[\min\{s,1\}]^{1/2-3\delta-\varsigma}\max\{s,1\} \\
&\le 3c^2[\max\{s,1\}]^{1+2\gamma_2}\big[2c^2[\min\{s,1\}]^{\gamma_2-\delta-\varsigma} + 2^{c+3}c^{c+4}[\min\{s,1\}]^{1/2+\gamma_2-2\delta-\varsigma}\big] \\
&\quad + 2^{c+2}c^{c+6}[\min\{s,1\}]^{1/2-3\delta-\varsigma}\max\{s,1\} \\
&\le 2^{c+5}c^{c+6}[\max\{s,1\}]^{1+2\gamma_2}[\min\{s,1\}]^{\gamma_2-3\delta-\varsigma}.
\end{aligned}
\tag{83}
$$

Furthermore, the fact that $\forall\, a,b \in U\colon \big|\|a\|_U^2 - \|b\|_U^2\big| \le \|a-b\|_U\,(2\,\|b\|_U + \|a-b\|_U)$, Hölder's inequality, (81), Lemma 2.19, and the assumption that $\forall\, s \in (0,h]\colon \max\{\|F(x)\|_H, \|B(x)\|_{\text{HS}(U,H)}\} \le ch^{-\delta} \le cs^{-\delta}$ show that for all $s \in (0,h]$ it holds that

$$
\begin{aligned}
&\Big\|\big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s)\big]^*(\nabla V)(Y_s)\big\|_U^2 - \|B(x)^*(\nabla V)(x)\|_U^2\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&\le \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s)\big]^*(\nabla V)(Y_s) - B(x)^*(\nabla V)(x)\big\|_{\mathcal{L}^4(\mathbb{P};U)}\Big\|2\|B(x)^*\|_{\text{HS}(H,U)}\|(\nabla V)(x)\|_H \\
&\qquad + \big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(s,W_s)\big]^*(\nabla V)(Y_s) - B(x)^*(\nabla V)(x)\big\|_U\Big\|_{\mathcal{L}^4(\mathbb{P};\mathbb{R})} \\
&\le 2^{c+6}c^{c+5}[\min\{s,1\}]^{\gamma_2-2\delta-\varsigma}[\max\{s,1\}]^{1+\gamma_2} \\
&\quad\cdot\big[2c^2s^{-\delta}[1+cs^{-\varsigma}] + 2^{c+6}c^{c+5}[\min\{s,1\}]^{\gamma_2-2\delta-\varsigma}[\max\{s,1\}]^{1+\gamma_2}\big] \\
&\le 2^{2c+13}c^{2c+10}[\min\{s,1\}]^{\gamma_2-4\delta-2\varsigma}[\max\{s,1\}]^{2+2\gamma_2}.
\end{aligned}
\tag{84}
$$

In addition, note that Hölder's inequality, Lemma 2.19, (61), and (79) imply for all $s \in (0,h]$ that

$$
\begin{aligned}
&\Big\|\textstyle\sum_{u\in\mathbb{U}} V'(Y_s)\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s,W_s)\big)(u,u)\Big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \\
&\le \|V'(Y_s)\|_{\mathcal{L}^4(\mathbb{P};L(H,\mathbb{R}))}\Big\|\textstyle\sum_{u\in\mathbb{U}}\Big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s,W_s)\Big)(u,u)\Big\|_{\mathcal{L}^4(\mathbb{P};H)} \\
&\le \Big(\|V'(Y_s) - V'(x)\|_{\mathcal{L}^4(\mathbb{P};L(H,\mathbb{R}))} + \|V'(x)\|_{L(H,\mathbb{R})}\Big)\Bigg\|\sum_{u\in\mathbb{U}}\Big(\big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(s,W_s)\Big)(u,u)\Bigg\|_{\mathcal{L}^4(\mathbb{P};H)} \\
&\le \big(2^{c+3}c^{c+4}[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\} + c[1+V(x)]\big)\,cs^{\gamma_2} \\
&\le \big(2^{c+3}c^{c+4}[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\} + 2c^2[\min\{s,1\}]^{-\varsigma}\big)\,cs^{\gamma_2} \\
&\le 2^{c+4}c^{c+5}[\min\{s,1\}]^{\gamma_2-\delta-\varsigma}[\max\{s,1\}]^{1+\gamma_2}.
\end{aligned}
\tag{85}
$$

Moreover, the assumption that $\forall\, y,z \in H\colon |\bar{V}(y) - \bar{V}(z)| \le c\,(1 + |V(y)|^{\gamma_0} + |V(z)|^{\gamma_0})\,\|y-z\|_H$, (74), and (78) show for all $s \in (0,h]$ that

$$\begin{aligned}
&\big\|\bar V(Y_s)-\bar V(x)\big\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})} \leq \|c\,(1+|V(x)|^{\gamma_0}+|V(Y_s)|^{\gamma_0})\,\|Y_s-x\|_H\|_{\mathcal{L}^2(\mathbb{P};\mathbb{R})}\\
&\leq c\left(1+|V(x)|^{\gamma_0}+\|V(Y_s)\|^{\gamma_0}_{\mathcal{L}^\infty(\mathbb{P};\mathbb{R})}\right)\|Y_s-x\|_{\mathcal{L}^2(\mathbb{P};H)}\\
&\leq c\left(1+|V(x)|^{\gamma_0}+\left[2^{c+1}c^{c+2}[\min\{s,1\}]^{-\varsigma}\right]^{\gamma_0}\right)2c^2[\min\{s,1\}]^{1/2-\delta}\max\{s,1\}\\
&\leq 2c^3[\min\{s,1\}]^{1/2-\delta}\max\{s,1\}\left[1+c^{\gamma_0}s^{-\varsigma\gamma_0}+2^{\gamma_0(c+1)}c^{\gamma_0(c+2)}[\min\{s,1\}]^{-\varsigma\gamma_0}\right]\\
&\leq 2^{\gamma_0(c+1)+2}c^{\gamma_0(c+2)+3}\max\{s,1\}[\min\{s,1\}]^{1/2-\delta-\varsigma\gamma_0}.
\end{aligned}\tag{86}$$

In the next step we insert (77), (79), (80), (83), (84), (85), and (86) into (69) to obtain for all $t \in (0,h]$ that

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(e^{-\rho t}V(Y_t)+\textstyle\int_0^t e^{-\rho r}\bar V(Y_r)\,dr\big)\big]-e^{V(x)}\\
&\leq \int_0^t \sqrt{2}\exp\Big(\tfrac{2^{(2c+3)\gamma_1}c^{(2c+6)\gamma_1}s}{[\min\{s,1\}]^{2\delta+\max\{2,\gamma_1\}\varsigma}}\Big)\,e^{V(x)}\Big[\rho\,2^{c+2}\,c^{c+4}\,[\min\{s,1\}]^{1/2-\delta-\varsigma}\max\{s,1\}\\
&\quad+2^{c+5}\,c^{c+5}\,[\min\{s,1\}]^{\gamma_2-2\delta-\varsigma}\,[\max\{s,1\}]^{1+\gamma_2}\\
&\quad+\tfrac{1}{2}\cdot 2^{c+5}\,c^{c+6}\,[\max\{s,1\}]^{1+2\gamma_2}\,[\min\{s,1\}]^{\gamma_2-3\delta-\varsigma}\\
&\quad+\tfrac{1}{2}\cdot 2^{2c+13}\,c^{2c+10}\,[\min\{s,1\}]^{\gamma_2-4\delta-2\varsigma}\,[\max\{s,1\}]^{2+2\gamma_2}\\
&\quad+\tfrac{1}{2}\cdot 2^{c+4}\,c^{c+5}\,[\min\{s,1\}]^{\gamma_2-\delta-\varsigma}\,[\max\{s,1\}]^{1+\gamma_2}\\
&\quad+2^{\gamma_0(c+1)+2}\,c^{\gamma_0(c+2)+3}\max\{s,1\}\,[\min\{s,1\}]^{1/2-\delta-\varsigma\gamma_0}\Big]\,ds\\
&\leq e^{V(x)}\int_0^t \sqrt{2}\exp\Big(\tfrac{2^{(2c+3)\gamma_1}c^{(2c+6)\gamma_1}s}{[\min\{s,1\}]^{2\delta+\max\{2,\gamma_1\}\varsigma}}\Big)\max\{\rho,1\}\,c^{(2c+10)\gamma_0}\,2^{(2c+25/2)\gamma_0}\\
&\quad\cdot[\max\{s,1\}]^{2+2\gamma_2}\,[\min\{s,1\}]^{\gamma_2-4\delta-\varsigma-\varsigma\gamma_0}\,ds.
\end{aligned}\tag{87}$$

This implies for all $t \in (0,h]$ that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\tfrac{V(Y_t)}{e^{\rho t}}+\int_0^t\tfrac{\bar V(Y_r)}{e^{\rho r}}\,dr\right)\right]\\
&\leq e^{V(x)}\left[1+\max\{\rho,1\}\,2^{(2c+13)\gamma_0}\int_0^t\frac{\exp\Big(\tfrac{2^{(2c+3)\gamma_1}c^{(2c+6)\gamma_1}s}{[\min\{s,1\}]^{2\delta+\max\{2,\gamma_1\}\varsigma}}\Big)c^{(2c+10)\gamma_0}[\max\{s,1\}]^{2+2\gamma_2}}{[\min\{s,1\}]^{\varsigma+\varsigma\gamma_0+4\delta-\gamma_2}}\,ds\right].
\end{aligned}\tag{88}$$

The proof of Lemma 2.21 is thus completed. □

2.4 Exponential moments for tamed approximation schemes

In this subsection we apply Lemma 2.10 and Lemma 2.21 above to establish in Proposition 2.22 below exponential moment bounds for an appropriate tamed exponential Euler-type approximation scheme (cf., e.g., [26, 25, 37, 38, 27] for related schemes in the case of finite dimensional SODEs and, e.g., [34, 20, 30, 2, 29] for related schemes in the case of infinite dimensional SPDEs).

Proposition 2.22. *Let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ and $(U,\langle\cdot,\cdot\rangle_U,\|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H > 1 < \#_U$, let $T\in(0,\infty)$, $\rho\in[0,\infty)$, $\delta\in[0,1/14)$, $c,\gamma\in[1,\infty)$, $\varsigma\in\big(0,\tfrac{1-14\delta}{2+2\gamma}\big)$, $F\in\mathcal{M}\big(\mathcal{B}(H),\mathcal{B}(H)\big)$, $B\in\mathcal{M}\big(\mathcal{B}(H),\mathcal{B}(\mathrm{HS}(U,H))\big)$, $V\in\mathcal{C}^3_c(H,[0,\infty))$, $\bar V\in\mathcal{C}(H,\mathbb{R})$, $S\in\mathbb{M}((0,T],L(H))$, $D\in\mathbb{M}((0,T],\mathcal{B}(H))$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $W\colon[0,T]\times\Omega\to U$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process with continuous sample paths, assume for all $h\in(0,T]$, $x,y\in H$ that $V(S_hx)\leq V(x)$, $\bar V(S_hx)\leq\bar V(x)$, $|\bar V(x)-\bar V(y)|\leq c\,(1+|V(x)|^\gamma+|V(y)|^\gamma)\,\|x-y\|_H$, $|\bar V(x)|\leq c\,(1+|V(x)|^\gamma)$, and $D_h\subseteq\{v\in H\colon V(v)\leq ch^{-\varsigma}\}$,* *assume for all $h \in (0,T]$, $x \in D_h$ that $\max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta}$ and $(\mathcal{G}_{F,B}V)(x) + \frac{1}{2}\|B(x)^*(\nabla V)(x)\|_U^2 + \bar{V}(x) \leq \rho V(x)$, and let $Y^\theta\colon [0,T]\times\Omega \to H$, $\theta \in \varpi_T$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for all $\theta \in \varpi_T$, $t \in (0,T]$ that*

$$Y_t^\theta = S_{t-\llcorner t \lrcorner_\theta}\left(Y^\theta_{\llcorner t \lrcorner_\theta} + \mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\llcorner t \lrcorner_\theta})\left[F(Y^\theta_{\llcorner t \lrcorner_\theta})\,(t - \llcorner t \lrcorner_\theta) + \tfrac{B(Y^\theta_{\llcorner t \lrcorner_\theta})(W_t - W_{\llcorner t \lrcorner_\theta})}{1+\|B(Y^\theta_{\llcorner t \lrcorner_\theta})(W_t - W_{\llcorner t \lrcorner_\theta})\|_H^2}\right]\right). \tag{89}$$

Then

(i) it holds that

$$\limsup_{|\theta|_T \searrow 0} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{V(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s \rfloor_\theta})\,\bar{V}(Y_s^\theta)}{e^{\rho s}}\,ds\right)\right] \leq \limsup_{|\theta|_T \searrow 0} \mathbb{E}\big[e^{V(Y_0^\theta)}\big] \tag{90}$$

and

(ii) it holds for all $\theta \in \varpi_T$ that

$$\begin{split}&\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{V(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s \rfloor_\theta})\,\bar{V}(Y_s^\theta)}{e^{\rho s}}\,ds\right)\right]\\ &\leq \exp\left(\tfrac{\exp\left(2[720\max\{T,\rho,1\}c^3]^{(720c^3\max\{T,1\}+7)\gamma}\right)}{[\min\{|\theta|_T,1\}]^{\varsigma+\varsigma\gamma+7\delta-1/2}}\right)\mathbb{E}\big[e^{V(Y_0^\theta)}\big].\end{split} \tag{91}$$

Proof of Proposition 2.22. Throughout this proof let $\hat{c} \in [1,\infty)$ and $\varrho_h \in (0,\infty)$, $h \in (0,T]$, be the real numbers which satisfy for all $s \in (0,T]$ that $\hat{c} = 360c^3\max\{T,1\}$ and

$$\varrho_s = \exp\left(\tfrac{[2\hat{c}]^{(2\hat{c}+6)\gamma}\,s}{\min\{s^{2\delta+\max\{2,\gamma\}\varsigma},1\}}\right)\tfrac{[2\hat{c}]^{(2\hat{c}+13)\gamma}[\max\{s,\rho,1\}]^4}{[\min\{s,1\}]^{\varsigma+\varsigma\gamma+7\delta-1/2}}, \tag{92}$$

let $\psi\colon H \to H$ be the mapping which satisfies for all $x \in H$ that $\psi(x) = \frac{x}{1+\|x\|_H^2}$, and let $\Psi\colon H\times[0,T]\times U \to H$ and $\Phi_h^x\colon [0,h]\times U \to H$, $(x,h) \in H\times(0,T]$, be the mappings which satisfy for all $h \in (0,T]$, $x \in H$, $s \in [0,h]$, $y \in U$ that

$$\Psi(x,s,y) = x + F(x)s + \tfrac{B(x)y}{1+\|B(x)y\|_H^2} \qquad \text{and} \qquad \Phi_h^x(s,y) = \Psi(x,s,y). \tag{93}$$

We now verify step by step the assumptions of Lemma 2.21. First, note that for all $h \in (0,T]$, $x \in H$ it holds that

$$\Phi_h^x(0,0) = x. \tag{94}$$

Furthermore, observe that for all $h \in (0,T]$, $x \in H$, $s \in (0,h]$, $y \in U$ it holds that

$$\left(\tfrac{\partial}{\partial s}\Phi_h^x\right)(s,y) = F(x). \tag{95}$$

Next we note that $\psi \in \mathcal{C}^2(H,H)$ and we observe that for all $z,u,v \in H$ it holds that

$$\psi'(z)u = \tfrac{u}{1+\|z\|_H^2} - \tfrac{2z\langle z,u\rangle_H}{(1+\|z\|_H^2)^2} \tag{96}$$

and

$$\psi''(z)(u,v) = -\tfrac{2[u\langle z,v\rangle_H + v\langle z,u\rangle_H + z\langle u,v\rangle_H]}{(1+\|z\|_H^2)^2} + \tfrac{8z\langle z,u\rangle_H\langle z,v\rangle_H}{(1+\|z\|_H^2)^3}. \tag{97}$$

Moreover, note that (96) ensures for all $h \in (0,T]$, $x \in H$, $s \in (0,h]$, $y,u \in U$ that

$$\left(\tfrac{\partial}{\partial y}\Phi_h^x\right)(s,y)u = \tfrac{B(x)u}{1+\|B(x)y\|_H^2} - \tfrac{2B(x)y\langle B(x)y,B(x)u\rangle_H}{(1+\|B(x)y\|_H^2)^2}. \tag{98}$$

The Cauchy-Schwarz inequality hence implies for all $h \in (0,T]$, $x \in H$, $s \in (0,h]$, $y \in U$ that

$$\begin{aligned} &\big\|\big(\tfrac{\partial}{\partial y}\Phi_h^x\big)(s,y) - B(x)\big\|_{\mathrm{HS}(U,H)} \\ &\leq \Big\|\tfrac{B(x)}{1+\|B(x)y\|_H^2} - B(x)\Big\|_{\mathrm{HS}(U,H)} + \Big\|\tfrac{2B(x)y\langle B(x)y,B(x)(\cdot)\rangle_H}{(1+\|B(x)y\|_H^2)^2}\Big\|_{\mathrm{HS}(U,H)} \\ &\leq \Big\|\Big(\tfrac{1}{1+\|B(x)y\|_H^2} - 1\Big)\, B(x)\Big\|_{\mathrm{HS}(U,H)} + \tfrac{\|2B(x)y\|_H\|B(x)y\|_H\|B(x)\|_{\mathrm{HS}(U,H)}}{(1+\|B(x)y\|_H^2)^2} \\ &\leq \Big|\tfrac{1}{1+\|B(x)y\|_H^2} - 1\Big|\, \|B(x)\|_{\mathrm{HS}(U,H)} + 2\|B(x)y\|_H^2\|B(x)\|_{\mathrm{HS}(U,H)}. \end{aligned} \tag{99}$$

This ensures for all $h \in (0,T]$, $x \in H$, $s \in (0,h]$, $y \in U$ that

$$\begin{aligned} &\big\|\big(\tfrac{\partial}{\partial y}\Phi_h^x\big)(s,y) - B(x)\big\|_{\mathrm{HS}(U,H)} \\ &\leq \Big|1 - \tfrac{1}{1+\|B(x)y\|_H^2}\Big|\, \|B(x)\|_{\mathrm{HS}(U,H)} + 2\|B(x)y\|_H^2\|B(x)\|_{\mathrm{HS}(U,H)} \\ &= \tfrac{\|B(x)y\|_H^2\|B(x)\|_{\mathrm{HS}(U,H)}}{1+\|B(x)y\|_H^2} + 2\|B(x)y\|_H^2\|B(x)\|_{\mathrm{HS}(U,H)} \leq 3\|B(x)y\|_H^2\|B(x)\|_{\mathrm{HS}(U,H)}. \end{aligned} \tag{100}$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [16] therefore proves that for all $h \in (0,T]$, $x \in D_h$, $s \in (0,h]$ it holds that

$$\begin{aligned} &\big\|\big(\tfrac{\partial}{\partial y}\Phi_h^x\big)(s,W_s) - B(x)\big\|_{\mathcal{L}^8(\mathbb{P};\mathrm{HS}(U,H))} \leq 3\|B(x)\|_{\mathrm{HS}(U,H)}\big\|\|B(x)W_s\|_H^2\big\|_{\mathcal{L}^8(\mathbb{P};\mathbb{R})} \\ &= 3\|B(x)\|_{\mathrm{HS}(U,H)}\|B(x)W_s\|_{\mathcal{L}^{16}(\mathbb{P};H)}^2 \leq 3\|B(x)\|_{\mathrm{HS}(U,H)}\big(\tfrac{16\cdot 15}{2}\big)\|B(x)\|_{\mathrm{HS}(U,H)}^2 s \\ &= 360\|B(x)\|_{\mathrm{HS}(U,H)}^3 s \leq 360(ch^{-\delta})^3 s \leq 360c^3 s^{1-3\delta}. \end{aligned} \tag{101}$$

This and (95) show that for all $h \in (0,T]$, $x \in D_h$, $s \in (0,h]$ it holds that

$$\begin{aligned} &\max\Big\{\big\|\big(\tfrac{\partial}{\partial s}\Phi_h^x\big)(s,W_s) - F(x)\big\|_{\mathcal{L}^4(\mathbb{P};H)}, \big\|\big(\tfrac{\partial}{\partial y}\Phi_h^x\big)(s,W_s) - B(x)\big\|_{\mathcal{L}^8(\mathbb{P};\mathrm{HS}(U,H))}\Big\} \\ &\leq 360c^3 s^{1-3\delta} \leq \hat{c}s^{1/2-3\delta}. \end{aligned} \tag{102}$$

Next observe that (97) implies that for all $z,u \in H$ it holds that

$$\psi''(z)(u,u) = \tfrac{8z|\langle z,u\rangle_H|^2}{\big(1+\|z\|_H^2\big)^3} - \tfrac{2\big[2u\langle z,u\rangle_H + z\|u\|_H^2\big]}{\big(1+\|z\|_H^2\big)^2}. \tag{103}$$

Therefore, we obtain that for all $x \in H$, $y,u \in U$ it holds that

$$\begin{aligned} \big(\tfrac{\partial^2}{\partial y^2}\psi(B(x)y)\big)(u,u) &= \psi''\big(B(x)y\big)\big(B(x)u,B(x)u\big) \\ &= \tfrac{8B(x)y|\langle B(x)y,B(x)u\rangle_H|^2}{\big(1+\|B(x)y\|_H^2\big)^3} - \tfrac{2\big[2B(x)u\langle B(x)y,B(x)u\rangle_H + B(x)y\|B(x)u\|_H^2\big]}{\big(1+\|B(x)y\|_H^2\big)^2}. \end{aligned} \tag{104}$$

The Cauchy-Schwarz inequality hence shows for all $x \in H$, $y,u \in U$ that

$$\begin{aligned} \Big\|\big(\tfrac{\partial^2}{\partial y^2}\psi(B(x)y)\big)(u,u)\Big\|_H &\leq \tfrac{8\|B(x)y\|_H^3\|B(x)u\|_H^2}{\big(1+\|B(x)y\|_H^2\big)^3} + \tfrac{6\|B(x)y\|_H\|B(x)u\|_H^2}{\big(1+\|B(x)y\|_H^2\big)^2} \\ &= \Bigg[\tfrac{8\|B(x)y\|_H^2 + 6(1+\|B(x)y\|_H^2)}{\big(1+\|B(x)y\|_H^2\big)^3}\Bigg]\|B(x)y\|_H\|B(x)u\|_H^2 \\ &\leq 6\|B(x)y\|_H\|B(x)u\|_H^2. \end{aligned} \tag{105}$$

This, the triangle inequality, and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [16] imply that for all $h \in (0,T]$, $x \in D_h$, $s \in (0,h]$ it holds that

$$\begin{aligned} &\Big\|\textstyle\sum_{u\in\mathbb{U}}\big(\big(\tfrac{\partial^2}{\partial y^2}\Phi_h^x\big)(s,W_s)\big)(u,u)\Big\|_{\mathcal{L}^4(\mathbb{P};H)} \leq 6\|B(x)W_s\|_{\mathcal{L}^4(\mathbb{P};H)}\textstyle\sum_{u\in\mathbb{U}}\|B(x)u\|_H^2 \\ &\leq 6\|B(x)\|_{\mathrm{HS}(U,H)}^3\sqrt{6s} \leq 6\sqrt{6}c^3 s^{1/2-3\delta} \leq \hat{c}s^{1/2-3\delta}. \end{aligned} \tag{106}$$

Next observe that for all $h \in (0,T]$, $x \in D_h$, $s \in (0,h]$, $r \in [1,\infty)$ it holds that

$$\begin{aligned}
&\|\Phi_h^x(s,W_s) - x\|_{\mathcal{L}^r(\mathbb{P};H)} \leq \Big\| \|F(x)\|_H s + \tfrac{\|B(x)W_s\|_H}{1+\|B(x)W_s\|_H^2} \Big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \\
&\leq \min\big\{ \big\| \|F(x)\|_H s + \tfrac{1}{2} \big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})}, \big\| \|F(x)\|_H s + \|B(x)W_s\|_H \big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \big\} \\
&\leq \min\big\{ \tfrac{1}{2} + ch^{-\delta}s, \big\| \|F(x)\|_H s + \|B(x)W_s\|_H \big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \big\} \\
&\leq \min\big\{ \tfrac{1}{2} + cT^{1-\delta}, \big\| \|F(x)\|_H s + \|B(x)W_s\|_H \big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \big\} \\
&\leq \hat{c} \min\big\{ 1, \big\| \|F(x)\|_H s + \|B(x)W_s\|_H \big\|_{\mathcal{L}^r(\mathbb{P};\mathbb{R})} \big\}.
\end{aligned} \tag{107}$$

Moreover, note that the fact that $\psi \in \mathcal{C}^2(H,H)$ implies that for all $h \in (0,T]$, $x \in H$ it holds that $\Phi_h^x \in \mathcal{C}^{1,2}([0,h] \times U, H)$. Combining this, (94), (102), (106), and (107) allows us to apply Lemma 2.21 (with $\varsigma = \varsigma$, $h = h$, $c = \hat{c}$, $\gamma_0 = \gamma$, $\gamma_1 = \gamma$, $\rho = \rho$, $\delta = \delta$, $\gamma_2 = 1/2 - 3\delta$, $x = x$, $F = F$, $B = B$, $\bar{V} = \bar{V}$, $V = V$, $\Phi = \Phi_h^x$ for $x \in D_h$, $h \in (0,T]$ in the notation of Lemma 2.21) to obtain that for all $h \in (0,T]$, $x \in D_h$, $t \in (0,h]$ it holds that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\Phi_h^x(t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Phi_h^x(s,W_s))}{e^{\rho s}}\,ds\bigg)\bigg] \leq \big(1 + \textstyle\int_0^t \varrho_s\,ds\big)\, e^{V(x)}. \tag{108}$$

Next note that the estimates $1 - 2\delta - \max\{2,\gamma\}\varsigma \geq 0$ and $1/2 - \varsigma - \varsigma\gamma - 7\delta > 0$ ensure that the function $(0,T] \ni h \mapsto \varrho_h \in (0,\infty)$ is non-decreasing and that $\limsup_{h \searrow 0} \varrho_h = 0$. Combining this with (108) implies that for all $h \in (0,T]$, $x \in D_h$, $t \in (0,h]$ it holds that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\Phi_h^x(t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Phi_h^x(s,W_s))}{e^{\rho s}}\,ds\bigg)\bigg] \leq \big(1 + \textstyle\int_0^t \varrho_s\,ds\big)\, e^{V(x)} \leq (1 + \varrho_h t)\, e^{V(x)}. \tag{109}$$

This ensures for all $\theta \in \varpi_T$, $x \in D_{|\theta|_T}$, $t \in (0, |\theta|_T]$ that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\Phi_{|\theta|_T}^x(t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Phi_{|\theta|_T}^x(s,W_s))}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{\varrho_{|\theta|_T} t + V(x)}. \tag{110}$$

Hence, we obtain for all $\theta \in \varpi_T$, $x \in D_{|\theta|_T}$, $t \in (0, |\theta|_T]$ that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\Psi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\bar{V}(\Psi(x,s,W_s))}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{\varrho_{|\theta|_T} t + V(x)}. \tag{111}$$

Corollary 2.10 (with $T = T$, $\theta = \theta$, $\rho = \rho$, $c = \varrho_{|\theta|_T}$, $V = V$, $\bar{V} = \bar{V}$, $\Phi = \Psi$, $E = D_h$, $S = S$, $W = W$, $Y = Y^\theta$ for $\theta \in \varpi_T$, $h \in (0,T]$ in the notation of Corollary 2.10) therefore yields that for all $\theta \in \varpi_T$, $t \in [0,T]$ it holds that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{V(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s \rfloor_\theta})\,\bar{V}(Y_s^\theta)}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{\varrho_{|\theta|_T} t}\, \mathbb{E}\Big[e^{V(Y_0^\theta)}\Big]. \tag{112}$$

This assures that for all $\theta \in \varpi_T$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{V(Y_t^\theta)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_{|\theta|_T}}(Y^\theta_{\lfloor s \rfloor_\theta})\,\bar{V}(Y_s^\theta)}{e^{\rho s}}\,ds\bigg)\bigg] \leq e^{\varrho_{|\theta|_T} T}\, \mathbb{E}\Big[e^{V(Y_0^\theta)}\Big]. \tag{113}$$

This and the fact that $\limsup_{h \searrow 0} \varrho_h = 0$ establish (90). It thus remains to prove (91). For this observe that the fact that $\forall\, x \in [2^{20},\infty)\colon x \leq \exp(x^{1/4})$ and the fact that $\forall\, \theta \in \varpi_T\colon (|\theta|_T)^{1-2\delta-\max\{2,\gamma\}\varsigma} \leq \max\{1,T\}$ show that for all $\theta \in \varpi_T$ it holds that

$$\begin{aligned}
\varrho_{|\theta|_T} T &= \exp\bigg(\tfrac{[720c^3\max\{T,1\}]^{(720c^3\max\{T,1\}+6)\gamma}|\theta|_T}{[\min\{|\theta|_T,1\}]^{2\delta+\max\{2,\gamma\}\varsigma}}\bigg)\, \tfrac{[720c^3\max\{T,1\}]^{(720c^3\max\{T,1\}+13)\gamma}[\max\{|\theta|_T,\rho,1\}]^4 T}{[\min\{|\theta|_T,1\}]^{\varsigma+\varsigma\gamma+7\delta-1/2}} \\
&\leq \exp\Big([720c^3\max\{T,1\}]^{(720c^3\max\{T,1\}+7)\gamma}\Big)\, \tfrac{[720c^3\max\{T,\rho,1\}]^{(720c^3\max\{T,1\}+18)\gamma}}{[\min\{|\theta|_T,1\}]^{\varsigma+\varsigma\gamma+7\delta-1/2}} \\
&\leq \exp\Big(2[720c^3\max\{T,\rho,1\}]^{(720c^3\max\{T,1\}+7)\gamma}\Big)\, \tfrac{1}{[\min\{|\theta|_T,1\}]^{\varsigma+\varsigma\gamma+7\delta-1/2}}.
\end{aligned} \tag{114}$$

Combining (113) with (114) establishes (91). The proof of Proposition 2.22 is thus completed. □

3 Exponential moments for space-time-noise discrete approximation schemes

In Proposition 2.22 in Section 2 above we established exponential moment bounds for a class of time discrete approximation schemes. In this section we extend this result in Theorem 3.3 and Corollary 3.4 below to obtain exponential moments for a class of space-time-noise discrete approximation schemes. Theorem 3.3 below proves exponential moment bounds for numerical approximations of SPDEs whose coefficients satisfy a general Lyapunov-type condition. Corollary 3.4 below specialises Theorem 3.3 to the case where the considered Lyapunov-type function is an affine linear transformation of the squared Hilbert space norm. Our proof of Theorem 3.3 uses two well-known auxiliary lemmas (see Lemma 3.1 and Lemma 3.2 below).

3.1 Setting

Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\mathbb{U} \subseteq U$ be a non-empty orthonormal basis of U, let $T \in (0, \infty)$, $\gamma \in [0, \infty)$, $\delta \in [0, 1/14)$, $\lambda \in \mathbb{M}(\mathbb{H}, \mathbb{R})$ satisfy that $\sup(\operatorname{im}(\lambda)) < 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $A \colon D(A) \subseteq H \to H$ be the linear operator which satisfies for all $v \in D(A)$ that $D(A) = \{ w \in H \colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, w \rangle_H|^2 < \infty \}$ and $Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Definition 3.6.30 in [28]), let $\xi \in \mathcal{M}(\mathcal{F}_0, \mathcal{B}(H_\gamma))$, $F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H))$, $B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U, H)))$, $D = (D_h^I)_{(I,h)\in\mathcal{P}(\mathbb{H})\times(0,T]} \in \mathbb{M}(\mathcal{P}(\mathbb{H}) \times (0, T], \mathcal{B}(H_\gamma))$, and let $P_I \in L(H)$, $I \in \mathcal{P}(\mathbb{H})$, and $\hat{P}_J \in L(U)$, $J \in \mathcal{P}(\mathbb{U})$, be the linear operators which satisfy for all $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $x \in H$, $y \in U$ that $P_I(x) = \sum_{h\in I} \langle h, x \rangle_H h$ and $\hat{P}_J(y) = \sum_{u\in J} \langle u, y \rangle_U u$.

3.2 Exponential moments for tamed approximation schemes

Lemma 3.1 (cf., e.g., Lemma 1 in Da Prato et al. [15]). *Let $(\Omega, \mathcal{F}, \mu)$ be a sigma-finite measure space and let $T \in (0, \infty)$, $Y, Z \in \mathcal{M}(\mathcal{F} \otimes \mathcal{B}([0, T]), \mathcal{B}(\mathbb{R}))$ satisfy for all $t \in [0, T]$ that $\mu(Y_t \neq Z_t) = \mu\big(\int_0^T |Y_s| \, ds = \infty\big) = 0$. Then $\mu\big(\Omega \backslash \{\omega \in \Omega \colon \int_0^T |Y_s(\omega)| + |Z_s(\omega)| \, ds < \infty$ and $\int_0^T Y_s(\omega) \, ds = \int_0^T Z_s(\omega) \, ds\}\big) = 0$.*

Proof of Lemma 3.1. First, note that the Tonelli theorem implies that

$$\int_\Omega \left(\int_0^T |Y_s - Z_s| \, ds \right) d\mu = \int_0^T \left(\int_\Omega |Y_s - Z_s| \, d\mu \right) ds = 0. \tag{115}$$

This shows that $\mu\big(\int_0^T |Y_s - Z_s| \, ds > 0\big) = 0$. Therefore, we obtain that $\mu\big(\int_0^T |Y_s - Z_s| \, ds = \infty\big) = 0$. This and the assumption that $\mu\big(\int_0^T |Y_s| \, ds = \infty\big) = 0$ proves that

$$\begin{aligned} &\mu\Big(\int_0^T |Y_s| \, ds + \int_0^T |Y_s - Z_s| \, ds = \infty \Big) = \mu\Big(\Big\{ \int_0^T |Y_s| \, ds = \infty \Big\} \cup \Big\{ \int_0^T |Y_s - Z_s| \, ds = \infty \Big\} \Big) \\ &\leq \mu\Big(\int_0^T |Y_s| \, ds = \infty \Big) + \mu\Big(\int_0^T |Y_s - Z_s| \, ds = \infty \Big) = 0. \end{aligned} \tag{116}$$

The triangle inequality hence proves that

$$\mu\Big(\int_0^T |Z_s| \, ds = \infty \Big) \leq \mu\Big(\int_0^T |Z_s - Y_s| \, ds + \int_0^T |Y_s| \, ds = \infty \Big) = 0. \tag{117}$$

Next note that (115) ensures that

$$\int_\Omega \left| \int_0^T \mathbb{1}_{\{\int_0^T |Y_u - Z_u|\, du < \infty\}} (Y_s - Z_s)\, ds \right| d\mu \leq \int_\Omega \left(\int_0^T |Y_s - Z_s|\, ds \right) d\mu = 0. \tag{118}$$

Hence, we obtain that

$$\mu\Big(\int_0^T \mathbb{1}_{\{\int_0^T |Y_u - Z_u|\, du < \infty\}} (Y_s - Z_s)\, ds \neq 0 \Big) = 0. \tag{119}$$

This shows that

$$\mu\Big(\int_0^T \mathbb{1}_{\{\int_0^T |Y_u| + |Z_u|\, du < \infty\}} (Y_s - Z_s)\, ds \neq 0 \Big) = 0. \tag{120}$$

Combining (117) and (120) completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0, \infty)$, let $Q \in L(U)$ be a non-negative and symmetric linear operator, let $R \in \mathrm{HS}(Q^{1/2}(U), H)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $(W_t)_{t\in[0,T]}$ be a Q-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $\mathcal{G}_t \subseteq \mathcal{F}$, $t \in [0, T]$, satisfy for all $t \in [0, T]$ that $\mathcal{G}_t = \sigma_\Omega(\mathcal{F}_t \cup \{C \in \mathcal{F} \colon \mathbb{P}(C) = 0\})$, and let $\tilde{W} \colon [0, T] \times \Omega \to H$ be a stochastic process with continuous sample paths which satisfies for all $t \in [0, T]$ that $[\tilde{W}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t R\, dW_s$. Then it holds that $\tilde{W}$ is an RR^*-standard $(\mathcal{G}_t^+)_{t\in[0,T]}$-Wiener process.*

Proof of Lemma 3.2. Throughout this proof let $\mathbb{U}_0 \subseteq U$ be an orthonormal basis of $\mathrm{Kern}(Q^{1/2})$ and let $\mathbb{U}_1 \subseteq U$ be an orthonormal basis of $\mathrm{Kern}(Q^{1/2})^\perp$. Next note that for all $v, w \in H$, $s \in [0, T)$, $t \in (s, T]$ it holds that

$$\begin{aligned} &\mathbb{E}[\langle v, \tilde{W}_t - \tilde{W}_s \rangle_H \langle w, \tilde{W}_t - \tilde{W}_s \rangle_H] = \mathbb{E}\bigg[\Big\langle v, \int_s^t R\, dW_r \Big\rangle_H \Big\langle w, \int_s^t R\, dW_r \Big\rangle_H \bigg] \\ &= \mathbb{E}\bigg[\Big(\int_s^t \langle v, R\, dW_r \rangle_H \Big) \Big(\int_s^t \langle w, R\, dW_r \rangle_H \Big) \bigg] \\ &= \mathbb{E}\bigg[\Big(\int_s^t \langle R^* v, dW_r \rangle_{Q^{1/2}(U)} \Big) \Big(\int_s^t \langle R^* w, dW_r \rangle_{Q^{1/2}(U)} \Big) \bigg]. \end{aligned} \tag{121}$$

Itô's isometry hence shows for all $v, w \in H$, $s \in [0, T)$, $t \in (s, T]$ that

$$\begin{aligned} &\tfrac{1}{(t-s)}\, \mathbb{E}[\langle v, \tilde{W}_t - \tilde{W}_s \rangle_H \langle w, \tilde{W}_t - \tilde{W}_s \rangle_H] \\ &= \big\langle (Q^{1/2}(U) \ni z \mapsto \langle R^* v, z \rangle_{Q^{1/2}(U)} \in \mathbb{R}), (Q^{1/2}(U) \ni z \mapsto \langle R^* w, z \rangle_{Q^{1/2}(U)} \in \mathbb{R}) \big\rangle_{\mathrm{HS}(Q^{1/2}(U),\mathbb{R})} \\ &= \big\langle (U \ni z \mapsto \langle R^* v, Q^{1/2} z \rangle_{Q^{1/2}(U)} \in \mathbb{R}), (U \ni z \mapsto \langle R^* w, Q^{1/2} z \rangle_{Q^{1/2}(U)} \in \mathbb{R}) \big\rangle_{\mathrm{HS}(U,\mathbb{R})} \\ &= \textstyle\sum_{u \in \mathbb{U}_0 \cup \mathbb{U}_1} \langle R^* v, Q^{1/2} u \rangle_{Q^{1/2}(U)} \langle R^* w, Q^{1/2} u \rangle_{Q^{1/2}(U)} \\ &= \textstyle\sum_{u \in \mathbb{U}_1} \langle R^* v, Q^{1/2} u \rangle_{Q^{1/2}(U)} \langle R^* w, Q^{1/2} u \rangle_{Q^{1/2}(U)} \\ &= \textstyle\sum_{u \in \mathbb{U}_1} \langle Q^{-1/2}(R^* v), Q^{-1/2}(Q^{1/2} u) \rangle_U \langle Q^{-1/2}(R^* w), Q^{-1/2}(Q^{1/2} u) \rangle_U \\ &= \textstyle\sum_{u \in \mathbb{U}_1} \langle Q^{-1/2}(R^* v), u \rangle_U \langle Q^{-1/2}(R^* w), u \rangle_U \\ &= \textstyle\sum_{u \in \mathbb{U}_0 \cup \mathbb{U}_1} \langle Q^{-1/2}(R^* v), u \rangle_U \langle Q^{-1/2}(R^* w), u \rangle_U \\ &= \langle Q^{-1/2}(R^* v), Q^{-1/2}(R^* w) \rangle_U = \langle R^* v, R^* w \rangle_{Q^{1/2}(U)} = \langle v, RR^* w \rangle_H. \end{aligned} \tag{122}$$

Next observe that the assumption that W is a Q-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process and, e.g., Proposition 6.1.16 in [28] ensure that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{G}_t^+)_{t\in[0,T]})$ is a stochastic basis and that W is a

Q-cylindrical $(\mathcal{G}_t^+)_{t\in[0,T]}$-Wiener process. This implies that for all $s \in [0,T)$, $t \in (s,T]$ it holds that $\sigma_\Omega(\tilde{W}_t - \tilde{W}_s)$ and $\mathcal{G}_s^+$ are $\mathbb{P}$-independent and that $\tilde{W}$ is $(\mathcal{G}_r^+)_{r\in[0,T]}$-adapted. Combining this with (122) completes the proof of Lemma 3.2. □

Theorem 3.3. *Assume the setting in Subsection 3.1, let $\rho \in [0,\infty)$, $c, \iota \in [1,\infty)$, $\varsigma \in \big(0, \frac{1-14\delta}{2+2\iota}\big)$, $V \in \mathcal{C}_c^3(H,[0,\infty))$, $\bar{V} \in \mathcal{C}(H,\mathbb{R})$, assume for all $h \in (0,T]$, $x, y \in H$ that $V(e^{hA}x) \leq V(x)$, $\bar{V}(e^{hA}x) \leq \bar{V}(x)$, $|\bar{V}(x) - \bar{V}(y)| \leq c\,(1 + |V(x)|^\iota + |V(y)|^\iota)\,\|x-y\|_H$, $|\bar{V}(x)| \leq c\,(1+|V(x)|^\iota)$, and $\sup_{I\in\mathcal{P}_0(\mathbb{H})} \mathbb{E}\big[e^{V(P_I\xi)}\big] < \infty$, assume for all $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $h \in (0,T]$, $x \in D_h^I$ that $D_h^I \subseteq \{v \in H \colon V(v) \leq ch^{-\varsigma}\}$, $\max\{\|P_I F(x)\|_H, \|P_I B(x)\hat{P}_J\|_{\mathrm{HS}(U,H)}\} \leq ch^{-\delta}$, and $(\mathcal{G}_{P_I F, P_I B\hat{P}_J} V)(x) + \frac{1}{2}\,\|(P_I B(x)\hat{P}_J)^*(\nabla V)(x)\|_U^2 + \bar{V}(x) \leq \rho V(x)$, and let $Y^{\theta,I,J} \colon [0,T]\times\Omega \to P_I(H_\gamma)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for all $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $t \in (0,T]$ that $Y_0^{\theta,I,J} = P_I(\xi)$ and*

$$
\begin{split}
[Y_t^{\theta,I,J}]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))} = \Big[e^{(t-\llcorner t \lrcorner_\theta)A}\big(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J} + \mathbb{1}_{D^I_{|\theta|_T}}(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J}) P_I F(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\,(t - \llcorner t \lrcorner_\theta)\big)\Big]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))} \\
+ \frac{\int_{\llcorner t \lrcorner_\theta}^t e^{(t-\llcorner t \lrcorner_\theta)A}\,\mathbb{1}_{D^I_{|\theta|_T}}(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\,P_I B(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\hat{P}_J\,dW_s}{1+\|\int_{\llcorner t \lrcorner_\theta}^t P_I B(Y_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\hat{P}_J\,dW_s\|_H^2}.
\end{split}
\tag{123}
$$

Then it holds that

$$
\begin{split}
&\limsup_{|\theta|_T \searrow 0} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\frac{V(Y_t^{\theta,I,J})}{e^{\rho t}} + \int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\lfloor s \rfloor_\theta})\,\frac{\bar{V}(Y_s^{\theta,I,J})}{e^{\rho s}}\,ds\bigg)\bigg] \\
&\leq \sup_{I\in\mathcal{P}_0(\mathbb{H})} \mathbb{E}\big[e^{V(P_I\xi)}\big] \\
&\leq \sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\frac{V(Y_t^{\theta,I,J})}{e^{\rho t}} + \int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\lfloor s \rfloor_\theta})\,\frac{\bar{V}(Y_s^{\theta,I,J})}{e^{\rho s}}\,ds\bigg)\bigg] < \infty.
\end{split}
\tag{124}
$$

Proof of Theorem 3.3. Throughout this proof let $\mathcal{G}_t \subseteq \mathcal{F}$, $t \in [0,T]$, be the sets which satisfy for all $t \in [0,T]$ that

$$
\mathcal{G}_t = \sigma_\Omega\big(\mathcal{F}_t \cup \{C \in \mathcal{F} \colon \mathbb{P}(C) = 0\}\big), \tag{125}
$$

let $u_0 \in \mathbb{U}$, let $W^J \colon [0,T]\times\Omega \to \hat{P}_{J\cup\{u_0\}}(U)$, $J \in \mathcal{P}_0(\mathbb{U})$, be stochastic processes with continuous sample paths which satisfy for all $J \in \mathcal{P}_0(\mathbb{U})$, $t \in [0,T]$ that $[W_t^J]_{\mathbb{P},\mathcal{B}(U)} = \int_0^t \hat{P}_{J\cup\{u_0\}}\,dW_s$ and $W_0^J = 0$, let $\tilde{F}_I \colon H \to H$, $I \in \mathcal{P}_0(\mathbb{H})$, and $\tilde{B}_{I,J} \colon H \to \mathrm{HS}(\hat{P}_{J\cup\{u_0\}}(U), H)$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, be the functions which satisfy for all $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in H$, $u \in \hat{P}_{J\cup\{u_0\}}(U) \subseteq U$ that

$$
\tilde{F}_I(x) = \begin{cases} P_I(F(x)) & \colon x \in H_\gamma \\ 0 & \colon x \in H\backslash H_\gamma \end{cases} \quad \text{and} \quad \tilde{B}_{I,J}(x)u = \begin{cases} P_I(B(x)\hat{P}_J u) & \colon x \in H_\gamma \\ 0 & \colon x \in H\backslash H_\gamma \end{cases}, \tag{126}
$$

and let $\tilde{Y}^{\theta,I,J} \colon [0,T]\times\Omega \to H$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, be the functions which satisfy for all $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $t \in [0,T]$ that $\tilde{Y}_0^{\theta,I,J} = P_I(\xi)$ and

$$
\begin{split}
&\tilde{Y}_t^{\theta,I,J} \\
&= e^{(t-\llcorner t \lrcorner_\theta)A}\bigg(\tilde{Y}_{\llcorner t \lrcorner_\theta}^{\theta,I,J} + \mathbb{1}_{D^I_{|\theta|_T}}(\tilde{Y}_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\bigg[\tilde{F}_I(\tilde{Y}_{\llcorner t \lrcorner_\theta}^{\theta,I,J})\,(t - \llcorner t \lrcorner_\theta) + \frac{\tilde{B}_{I,J}(\tilde{Y}_{\llcorner t \lrcorner_\theta}^{\theta,I,J})(W_t^J - W_{\llcorner t \lrcorner_\theta}^J)}{1+\|\tilde{B}_{I,J}(\tilde{Y}_{\llcorner t \lrcorner_\theta}^{\theta,I,J})(W_t^J - W_{\llcorner t \lrcorner_\theta}^J)\|_H^2}\bigg]\bigg).
\end{split}
\tag{127}
$$

In the next step observe that, e.g., Theorem 2.4 in Chapter V in Parthasarathy [36] ensures that $\mathcal{B}(H_\gamma) \subseteq \mathcal{B}(H)$. This implies that for all $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $h \in (0,T]$ it holds that

$$
D_h^I \in \mathcal{B}(H), \quad \tilde{F}_I \in \mathcal{M}(\mathcal{B}(H),\mathcal{B}(H)), \quad \text{and} \quad \tilde{B}_{I,J} \in \mathcal{M}(\mathcal{B}(H),\mathcal{B}(\mathrm{HS}(\hat{P}_{J\cup\{u_0\}}(U),H))). \tag{128}
$$

In addition, note that, e.g., Proposition 6.1.16 in [28] ensures that $(\mathcal{G}^+_t)_{t\in[0,T]}$ is a normal filtration on $(\Omega,\mathcal{F},\mathbb{P})$ and that W is an Id_U-cylindrical $(\mathcal{G}^+_t)_{t\in[0,T]}$-Wiener process. Lemma 3.2 (with $H=\hat{P}_{J\cup\{u_0\}}(U)$, $U=U$, $R=(U\ni u\mapsto \hat{P}_{J\cup\{u_0\}}(u)\in\hat{P}_{J\cup\{u_0\}}(U))$, $Q=\mathrm{Id}_U$, $(\mathcal{F}_t)_{t\in[0,T]}=(\mathcal{F}_t)_{t\in[0,T]}$, $W=W$, $\tilde{W}=W^J$ for $J\in\mathcal{P}_0(\mathbb{U})$ in the notation of Lemma 3.2) hence assures that for all $J\in\mathcal{P}_0(\mathbb{U})$ it holds that W^J is an $((U\ni u\mapsto \hat{P}_{J\cup\{u_0\}}(u)\in\hat{P}_{J\cup\{u_0\}}(U))(U\ni u\mapsto \hat{P}_{J\cup\{u_0\}}(u)\in\hat{P}_{J\cup\{u_0\}}(U))^*)$-standard $(\mathcal{G}^+_t)_{t\in[0,T]}$-Wiener process. This shows that for all $J\in\mathcal{P}_0(\mathbb{U})$ it holds that W^J is an $\mathrm{Id}_{\hat{P}_{J\cup\{u_0\}}(U)}$-standard $(\mathcal{G}^+_t)_{t\in[0,T]}$-Wiener process with continuous sample paths. Combining the fact that for all $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$ it holds that $\tilde{Y}^{\theta,I,J}$ is a $(\mathcal{G}^+_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths, the fact that $\forall\, I\in\mathcal{P}_0(\mathbb{H})$, $h\in(0,T]\colon D^I_h\subseteq\{v\in H\colon V(v)\leq ch^{-\varsigma}\}$, (128), and item (ii) of Proposition 2.22 (with $H=H$, $U=\hat{P}_{J\cup\{u_0\}}(U)$, $T=T$, $\rho=\rho$, $\delta=\delta$, $c=c$, $\gamma=\iota$, $\varsigma=\varsigma$, $F=\tilde{F}_I$, $B=\tilde{B}_{I,J}$, $V=V$, $\bar{V}=\bar{V}$, $S=((0,T]\ni t\mapsto(H\ni x\mapsto e^{tA}x\in H)\in L(H))$, $D_h=D^I_h$, $(\mathcal{F}_t)_{t\in[0,T]}=(\mathcal{G}^+_t)_{t\in[0,T]}$, $W=W^J$, $Y^\theta=\tilde{Y}^{\theta,I,J}$ for $h\in(0,T]$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$ in the notation of Proposition 2.22) hence proves that for all $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$ it holds that

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}\Big[\exp\Big(\tfrac{V(\tilde{Y}^{\theta,I,J}_t)}{e^{\rho t}}+\int_0^t\mathbb{1}_{D^I_{|\theta|_T}}(\tilde{Y}^{\theta,I,J}_{\lfloor s\rfloor_\theta})\,\tfrac{\bar{V}(\tilde{Y}^{\theta,I,J}_s)}{e^{\rho s}}\,ds\Big)\Big]\\&\leq\exp\bigg(\frac{\exp\!\big(2[720\max\{T,\rho,1\}c^3]^{(720c^3\max\{T,1\}+7)\iota}\big)}{[\min\{|\theta|_T,1\}]^{\varsigma+\varsigma\iota+7\delta-1/2}}\bigg)\,\mathbb{E}\big[e^{V(Y^{\theta,I,J}_0)}\big].\end{aligned}\tag{129}$$

This, the fact that $1/2-\varsigma-\varsigma\iota-7\delta>0$, and the assumption that $\sup_{I\in\mathcal{P}_0(\mathbb{H})}\mathbb{E}[e^{V(P_I(\xi))}]<\infty$ imply that

$$\begin{aligned}&\limsup_{|\theta|_T\searrow 0}\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{J\in\mathcal{P}_0(\mathbb{U})}\sup_{t\in[0,T]}\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\tilde{Y}^{\theta,I,J}_t)}{e^{\rho t}}+\int_0^t\mathbb{1}_{D^I_{|\theta|_T}}(\tilde{Y}^{\theta,I,J}_{\lfloor s\rfloor_\theta})\,\tfrac{\bar{V}(\tilde{Y}^{\theta,I,J}_s)}{e^{\rho s}}\,ds\bigg)\bigg]\\&\leq\sup_{I\in\mathcal{P}_0(\mathbb{H})}\mathbb{E}\big[e^{V(P_I\xi)}\big]\\&\leq\sup_{\theta\in\varpi_T}\sup_{I\in\mathcal{P}_0(\mathbb{H})}\sup_{J\in\mathcal{P}_0(\mathbb{U})}\sup_{t\in[0,T]}\mathbb{E}\bigg[\exp\bigg(\tfrac{V(\tilde{Y}^{\theta,I,J}_t)}{e^{\rho t}}+\int_0^t\mathbb{1}_{D^I_{|\theta|_T}}(\tilde{Y}^{\theta,I,J}_{\lfloor s\rfloor_\theta})\,\tfrac{\bar{V}(\tilde{Y}^{\theta,I,J}_s)}{e^{\rho s}}\,ds\bigg)\bigg]<\infty.\end{aligned}\tag{130}$$

Furthermore, note that (123) and (127) ensure that for all $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$, $t\in[0,T]$ it holds that $[Y^{\theta,I,J}_t]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))}=[\tilde{Y}^{\theta,I,J}_t]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))}$. Combining this, Lemma 3.1, and (130) establishes (124). The proof of Theorem 3.3 is thus completed. □

Corollary 3.4. *Assume the setting in Subsection 3.1, let* $\vartheta\in[\sup_{x\in H_\gamma}\|B(x)\|^2_{\mathrm{HS}(U,H)},\infty]\cap\mathbb{R}$, $b_1,b_2\in[0,\infty)$, $\varepsilon\in(0,\infty)$, $\varsigma\in\big(0,\frac{1-14\delta}{4}\big)$, $c\in[2\max\{1,\varepsilon b_1,\varepsilon\sqrt{\vartheta},\varepsilon\},\infty)$, *assume that* $\mathbb{E}[e^{\varepsilon\|\xi\|^2_H}]<\infty$, *assume for all* $h\in(0,T]$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$, $x\in D^I_h$ *that* $D^I_h\subseteq\{v\in H\colon\sqrt{\vartheta}+\varepsilon\|v\|^2_H\leq ch^{-\varsigma}\}$, $\max\{\|P_IF(x)\|_H,\|P_IB(x)\hat{P}_J\|_{\mathrm{HS}(U,H)}\}\leq ch^{-\delta}$, *and* $\langle x,P_IF(x)\rangle_H\leq b_1+b_2\|x\|^2_H$, *and let* $Y^{\theta,I,J}\colon[0,T]\times\Omega\to P_I(H_\gamma)$, $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$, *be* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted stochastic processes with continuous sample paths which satisfy for all* $\theta\in\varpi_T$, $I\in\mathcal{P}_0(\mathbb{H})$, $J\in\mathcal{P}_0(\mathbb{U})$, $t\in(0,T]$ *that* $Y^{\theta,I,J}_0=P_I(\xi)$ *and*

$$\begin{aligned}[Y^{\theta,I,J}_t]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))}&=\Big[e^{(t-\llcorner t\lrcorner_\theta)A}\big(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta}+\mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta})P_IF(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta})\,(t-\llcorner t\lrcorner_\theta)\big)\Big]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))}\\&\quad+\frac{\int_{\llcorner t\lrcorner_\theta}^t e^{(t-\llcorner t\lrcorner_\theta)A}\,\mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta})\,P_IB(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta})\hat{P}_J\,dW_s}{1+\|\int_{\llcorner t\lrcorner_\theta}^t P_IB(Y^{\theta,I,J}_{\llcorner t\lrcorner_\theta})\hat{P}_J\,dW_s\|^2_H}.\end{aligned}\tag{131}$$

Then it holds that

$$
\begin{aligned}
&\limsup_{|\theta|_T \searrow 0} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\!\left(\tfrac{\sqrt{\vartheta}+\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2(b_2+\varepsilon\vartheta)t}} - \int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\lfloor s\rfloor_\theta})\,\tfrac{\varepsilon(2b_1+\vartheta)}{e^{2(b_2+\varepsilon\vartheta)s}}\,ds\right)\right] \\
&\le \mathbb{E}\big[e^{\sqrt{\vartheta}+\varepsilon\|\xi\|_H^2}\big] \\
&\le \sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\!\left(\tfrac{\sqrt{\vartheta}+\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2(b_2+\varepsilon\vartheta)t}} - \int_0^t \mathbb{1}_{D^I_{|\theta|_T}}(Y^{\theta,I,J}_{\lfloor s\rfloor_\theta})\,\tfrac{\varepsilon(2b_1+\vartheta)}{e^{2(b_2+\varepsilon\vartheta)s}}\,ds\right)\right] \\
&\le e^{\sqrt{\vartheta}} \sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\!\left(\tfrac{\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2(b_2+\varepsilon\vartheta)t}}\right)\right] < \infty.
\end{aligned}
\tag{132}
$$

Proof of Corollary 3.4. Throughout this proof let $V\colon H \to [0,\infty)$ and $\bar{V}\colon H \to \mathbb{R}$ be the functions with the property that for all $x \in H$ it holds that $V(x) = \sqrt{\vartheta} + \varepsilon\|x\|_H^2$ and $\bar{V}(x) = -2\varepsilon b_1 - \varepsilon\vartheta$. First of all, observe that for all $x, y \in H$ it holds that $|V(x) - V(y)| \le 2\sqrt{\varepsilon}\|x - y\|_H(1 + \sup_{r\in[0,1]} |V(rx + (1-r)y)|)^{1/2}$, $\|V'(x) - V'(y)\|_{L^{(1)}(H,\mathbb{R})} \le 2\varepsilon\|x-y\|_H$, and $|V''(x) - V''(y)\|_{L^{(2)}(H,\mathbb{R})} = 0$. Hence, we obtain that

$$
V \in \mathcal{C}^3_{2\max\{1,\varepsilon\}}(H, [0,\infty)). \tag{133}
$$

Next note that the assumption that $\forall\, h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in D_h^I\colon \langle x, P_I F(x)\rangle_H \le b_1 + b_2\|x\|_H^2$ shows that for all $h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in D_h^I$ it holds that

$$
\begin{aligned}
&(\mathcal{G}_{P_I F, P_I B \hat{P}_J} V)(x) + \tfrac{1}{2}\big\|(P_I B(x)\hat{P}_J)^*(\nabla V)(x)\big\|_U^2 + \bar{V}(x) \\
&= 2\varepsilon\,\langle x, P_I F(x)\rangle_H + \varepsilon \sum\nolimits_{u\in\mathbb{U}} \langle P_I B(x)\hat{P}_J u, P_I B(x)\hat{P}_J u\rangle_U \\
&\quad + 2\varepsilon^2\|(P_I B(x)\hat{P}_J)^* x\|_U^2 - 2\varepsilon b_1 - \varepsilon\vartheta \\
&= 2\varepsilon\,\langle x, P_I F(x)\rangle_H + \varepsilon\|P_I B(x)\hat{P}_J\|^2_{\mathrm{HS}(U,H)} + 2\varepsilon^2\|(P_I B(x)\hat{P}_J)^* x\|_U^2 - 2\varepsilon b_1 - \varepsilon\vartheta \\
&\le 2\varepsilon(b_2 + \varepsilon\vartheta)\|x\|_H^2 \le 2(b_2 + \varepsilon\vartheta)V(x).
\end{aligned}
\tag{134}
$$

Combining this, (133), the fact that $\sup_{I\in\mathcal{P}_0(\mathbb{H})} \mathbb{E}[e^{V(P_I\xi)}] \le e^{\sqrt{\vartheta}}\,\mathbb{E}[e^{\varepsilon\|\xi\|_H^2}]$, the assumption that $\mathbb{E}[e^{\varepsilon\|\xi\|_H^2}] < \infty$, the fact that $\forall\, x \in H\colon |\bar{V}(x)| \le c(1 + |V(x)|)$, and Theorem 3.3 (with $\rho = 2(b_2 + \varepsilon\vartheta)$, $c = c$, $\iota = 1$, $\delta = \delta$, $\varsigma = \varsigma$, $V = V$, $\bar{V} = \bar{V}$, $Y^{\theta,I,J} = Y^{\theta,I,J}$ for $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$ in the notation of Theorem 3.3) establishes (132). The proof of Corollary 3.4 is thus completed. □

4 Examples

In this section we illustrate Corollary 3.4 by some examples. In particular, we prove in the case of a class of stochastic Burgers equations (see Subsection 4.3), stochastic Kuramoto-Sivashinsky equations (see Subsection 4.4), and two-dimensional stochastic Navier-Stokes equations (see Subsection 4.5) that a certain tamed and space-time-noise discrete approximation scheme (see (135) below) has bounded exponential moments.

4.1 Setting

Let $d \in \mathbb{N}$, $\mathcal{D} = (0,1)^d$, $\eta, \gamma \in [0,\infty)$, $T, \varepsilon \in (0,\infty)$, $\delta \in (0, 1/18)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U) = (L^2(\mu_{\mathcal{D}};\mathbb{R}^d), \langle\cdot,\cdot\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}, \|\cdot\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)})$, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $H \subseteq U$ be a closed subvector space of U, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\lambda \in \mathbb{M}(\mathbb{H},\mathbb{R})$ satisfy that $\sup(\mathrm{im}(\lambda)) < 0$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $Q \in L(U)$ be a non-negative symmetric trace class operator, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies for all $v \in D(A)$ that $D(A) = \{w \in H\colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, w\rangle_H|^2 < \infty\}$ and $Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Definition 3.6.30 in [28]), let $r \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}([0,\infty)))$, $b \in \mathcal{M}(\mathcal{B}(\mathcal{D}\times\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times d}))$ satisfy $\sup_{x\in\mathcal{D}, y\in\mathbb{R}^d, z\in\mathbb{R}^d\setminus\{y\}} \big(\|b(x,y)\|_{\mathbb{R}^{d\times d}} + \frac{\|b(x,y)-b(x,z)\|_{\mathbb{R}^{d\times d}}}{\|y-z\|_{\mathbb{R}^d}}\big) < \infty$, let $\vartheta = \operatorname{trace}_U(Q)(\sup_{x\in\mathcal{D}, y\in\mathbb{R}^d} \|b(x,y)\|^2_{\mathbb{R}^{d\times d}})$, $c \in [2\max\{1, \varepsilon\sqrt{\vartheta}, \varepsilon\}, \infty)$, let $P_I \in L(H)$, $I \in \mathcal{P}(\mathbb{H})$, and $\hat{P}_J \in L(U)$, $J \in \mathcal{P}(\mathbb{U})$, be the linear operators which satisfy for all $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}(\mathbb{U})$, $v \in H$, $w \in U$ that $P_I(v) = \sum_{h\in I} \langle h, v\rangle_H h$ and $\hat{P}_J(w) = \sum_{u\in\mathbb{U}} \langle u, w\rangle_U u$, for every $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ let $D_h^I \in \mathcal{P}(H_\gamma)$ be the set given by $D_h^I = \{x \in P_I(H_\gamma)\colon r(x) \leq ch^{-\delta}\}$, let $R \in L(U)$ be the orthogonal projection of U on H, for every $n \in \mathbb{N}$, $v \in W^{1,2}(\mathcal{D}, \mathbb{R}^n)$ let $\partial v = (\partial_1 v, \ldots, \partial_d v) \in L^2(\mu_{\mathcal{D}}; \mathbb{R}^{n\times d})$ be the vector which satisfies for all $i \in \{1,\ldots,d\}$, $\phi \in \mathcal{C}^\infty_{cpt}(\mathcal{D}, \mathbb{R}^n)$ that $\langle \partial_i v, [\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^n)}\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R}^n)} = -\langle v, [\frac{\partial}{\partial x_i}\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^n)}\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R}^n)}$, let $F \in \mathbb{M}(H_\gamma, H)$, $B \in \mathbb{M}(H_\gamma, \mathrm{HS}(U,H))$, $\xi \in \mathcal{M}(\mathcal{F}_0, \mathcal{B}(H_\gamma))$ satisfy for all $u \in U$, $v \in \mathcal{M}(\mathcal{B}(\mathcal{D}), \mathcal{B}(\mathbb{R}^d))$, $w \in [H_\gamma \cap W^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d)]$ with $[v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)} \in H_\gamma$ that $\mathbb{E}[e^{\varepsilon\|\xi\|^2_H}] < \infty$, $B([v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)})u = R\big([\{b(x, v(x))\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^{d\times d})}(\sqrt{Q}u)\big)$, and $F(w) = R(\eta w - \sum_{i=1}^d w_i \partial_i w)$, and let $Y^{\theta,I,J}\colon [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths which satisfy for all $t \in (0,T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$ that $Y_0^{\theta,I,J} = P_I(\xi)$ and

$$
\begin{split}
[Y_t^{\theta,I,J}]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))} = \Big[e^{(t-\llcorner t \lrcorner_\theta)A}\Big(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta} + \mathbb{1}_{\{r(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta}) \leq c[|\theta|_T]^{-\delta}\}} P_I F(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta})\,(t - \llcorner t \lrcorner_\theta)\Big)\Big]_{\mathbb{P},\mathcal{B}(P_I(H_\gamma))} \\
+ \frac{\int_{\llcorner t \lrcorner_\theta}^t e^{(t-\llcorner t \lrcorner_\theta)A}\, \mathbb{1}_{\{r(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta}) \leq c[|\theta|_T]^{-\delta}\}} P_I B(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta}) \hat{P}_J \, dW_s}{1 + \|\int_{\llcorner t \lrcorner_\theta}^t P_I B(Y^{\theta,I,J}_{\llcorner t \lrcorner_\theta}) \hat{P}_J \, dW_s\|_H^2}.
\end{split}
\tag{135}
$$

4.2 Properties of the nonlinearities

In this subsection we establish a few elementary properties for the nonlinearities F and B in Subsection 4.1 (see Lemma 4.1, Lemma 4.2, Lemma 4.8, Lemma 4.9, and Corollary 4.10 below). To do so, we also recall in this subsection some well-known properties of the involved Sobolev and interpolation spaces (see Lemmas 4.3–4.8 below).

Lemma 4.1. *Assume the setting in Subsection 4.1 and let $v, w \in [H_\gamma \cap W^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d)]$. Then it holds that*

$$
\|F(v)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \leq \eta\|v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + d\|v\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\|\partial v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})} < \infty \tag{136}
$$

and

$$
\begin{split}
&\|F(v) - F(w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \leq \eta\|v - w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&+ d\big(\|\partial v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\|v - w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} + \|w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\|\partial(v - w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\big) < \infty.
\end{split}
\tag{137}
$$

Proof of Lemma 4.1. Note that the triangle inequality and Hölder's inequality imply that

$$
\begin{split}
\|F(v)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} &\leq \eta\|v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \big\|\textstyle\sum_{j=1}^d v_j \partial_j v\big\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq \eta\|v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \textstyle\sum_{j=1}^d \|v_j\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}\|\partial_j v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq \eta\|v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \sqrt{\textstyle\sum_{j=1}^d \|v_j\|^2_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}}\sqrt{\textstyle\sum_{j=1}^d \|\partial_j v\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}} \\
&\leq \eta\|v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + d\|v\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\|\partial v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}.
\end{split}
\tag{138}
$$

In addition, observe that

$$\begin{aligned}
&\|F(v)-F(w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}-\eta\|v-w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \leq \big\| \textstyle\sum_{j=1}^d (\partial_j v) v_j - \sum_{j=1}^d (\partial_j w) w_j \big\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq \big\| \textstyle\sum_{j=1}^d (\partial_j v)(v_j - w_j) \big\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \big\| \textstyle\sum_{j=1}^d (\partial_j v - \partial_j w) w_j \big\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq \textstyle\sum_{j=1}^d \|(\partial_j v)(v_j - w_j)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \sum_{j=1}^d \|(\partial_j v - \partial_j w) w_j\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq \textstyle\sum_{j=1}^d \|\partial_j v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \|v_j - w_j\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} + \sum_{j=1}^d \|\partial_j v - \partial_j w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \|w_j\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}.
\end{aligned} \tag{139}$$

Hölder's inequality hence proves that

$$\begin{aligned}
&\|F(v)-F(w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \leq \sqrt{\textstyle\sum_{j=1}^d \|\partial_j v\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}} \sqrt{\textstyle\sum_{j=1}^d \|v_j - w_j\|^2_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}} \\
&\quad + \sqrt{\textstyle\sum_{j=1}^d \|\partial_j v - \partial_j w\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}} \sqrt{\textstyle\sum_{j=1}^d \|w_j\|^2_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}} + \eta\|v-w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\
&\leq d\big(\|\partial v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\|v-w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} + \|w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\|\partial(v-w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\big) \\
&\quad + \eta\|v-w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}.
\end{aligned} \tag{140}$$

The proof of Lemma 4.1 is thus completed. □

Lemma 4.2. *Assume the setting in Subsection 4.1. Then it holds for all $v, w \in H_\gamma$ that*

$$\|B(v)\|_{\mathrm{HS}(U,H)} \leq \big(\sup\nolimits_{x\in\mathcal{D},y\in\mathbb{R}^d} \|b(x,y)\|_{\mathbb{R}^{d\times d}}\big) \sqrt{\mathrm{trace}_U(Q)} = \sqrt{\vartheta} < \infty \tag{141}$$

and

$$\begin{aligned}
&\|B(v)-B(w)\|_{\mathrm{HS}(U,H)} \\
&\leq \Big(\sup\nolimits_{x\in\mathcal{D},y\in\mathbb{R}^d,z\in\mathbb{R}^d\setminus\{y\}} \tfrac{\|b(x,y)-b(x,z)\|_{\mathbb{R}^{d\times d}}}{\|y-z\|_{\mathbb{R}^d}}\Big)\|v-w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\sqrt{\mathrm{trace}_U(Q)}.
\end{aligned} \tag{142}$$

Proof of Lemma 4.2. First of all, note for all $v \in \mathcal{M}(\mathcal{B}(\mathcal{D}),\mathcal{B}(\mathbb{R}^d))$ with $[v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)} \in H_\gamma$ that

$$\begin{aligned}
&\|B([v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)})\|^2_{\mathrm{HS}(U,H)} = \sum\nolimits_{u\in\mathbb{U}} \|B([v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)})u\|^2_H \\
&\leq \sum\nolimits_{u\in\mathbb{U}} \big\|[\{b(x,v(x))\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^{d\times d})}(Q^{1/2}u)\big\|^2_U \\
&\leq \big(\sup\nolimits_{x\in\mathcal{D},y\in\mathbb{R}^d} \|b(x,y)\|^2_{\mathbb{R}^{d\times d}}\big) \sum\nolimits_{u\in\mathbb{U}} \|Q^{1/2}u\|^2_U = \big(\sup\nolimits_{x\in\mathcal{D},y\in\mathbb{R}^d} \|b(x,y)\|^2_{\mathbb{R}^{d\times d}}\big)\,\mathrm{trace}_U(Q).
\end{aligned} \tag{143}$$

Next observe for all $v, w \in \mathcal{M}(\mathcal{B}(\mathcal{D}),\mathcal{B}(\mathbb{R}^d))$ with $[v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}, [w]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)} \in H_\gamma$ that

$$\begin{aligned}
&\|B([v]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}) - B([w]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)})\|^2_{\mathrm{HS}(U,H)} \\
&\leq \sum\nolimits_{u\in\mathbb{U}} \big\|[\{b(x,v(x))-b(x,w(x))\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^{d\times d})}(Q^{1/2}u)\big\|^2_U \\
&\leq \|\{b(x,v(x))-b(x,w(x))\}_{x\in\mathcal{D}}\|^2_{\mathcal{L}^\infty(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})} \sum\nolimits_{u\in\mathbb{U}} \|Q^{1/2}u\|^2_U \\
&\leq \Big[\sup\nolimits_{x\in\mathcal{D},y\in\mathbb{R}^d,z\in\mathbb{R}^d\setminus\{y\}} \tfrac{\|b(x,y)-b(x,z)\|_{\mathbb{R}^{d\times d}}}{\|y-z\|_{\mathbb{R}^d}}\Big]^2 \|v-w\|^2_{\mathcal{L}^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\,\mathrm{trace}_U(Q).
\end{aligned} \tag{144}$$

Combining (143) and (144) completes the proof of Lemma 4.2. □

Lemma 4.3. *Assume the setting in Subsection 4.1 and let $\rho \in [0,\infty)$, $v \in H_\rho$. Then*

$$\|v\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} \leq \|v\|_{H_\rho}\bigg(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\bigg)\bigg[\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\bigg]^{1/2}. \tag{145}$$

Proof of Lemma 4.3. Note that Hölder's inequality proves that

$$\begin{aligned}\sum_{h\in\mathbb{H}} \|\langle h,v\rangle_H\, h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} &\le \left(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\right)\sum_{h\in\mathbb{H}} |\langle h,v\rangle_H| \\ &\le \left(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\right)\sum_{h\in\mathbb{H}} |\lambda_h|^\rho\,|\langle h,v\rangle_H|\,|\lambda_h|^{-\rho} \\ &\le \|v\|_{H_\rho}\left(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)}\right)\left[\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\right]^{1/2}.\end{aligned} \tag{146}$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *Assume the setting in Subsection 4.1, let* $\rho\in[0,\infty)$*, and assume for all* $j\in\{1,\dots,d\}$*,* $v,w\in\mathbb{H}$ *that* $\mathbb{H}\subseteq W^{1,2}(\mathcal{D},\mathbb{R}^d)$*,* $\sup_{h\in\mathbb{H}}\left(\|\partial_j h\|_U\,|\lambda_h|^{-\rho}\right)<\infty$*,* $\langle\partial_j v,\partial_j w\rangle_U\,\mathbb{1}_{\mathbb{H}\setminus\{v\}}(w)=0$*. Then*

(i) it holds that $H_\rho\subseteq W^{1,2}(\mathcal{D},\mathbb{R}^d)$*,*

(ii) it holds for all $u\in H_\rho$*,* $j\in\{1,\dots,d\}$ *that*

$$\|\partial_j u\|_U=\left(\textstyle\sum_{h\in\mathbb{H}}\|\langle h,u\rangle_H\partial_j h\|_U^2\right)^{1/2}\le\left(\sup_{h\in\mathbb{H}}\frac{\|\partial_j h\|_U}{|\lambda_h|^\rho}\right)\|u\|_{H_\rho}<\infty \tag{147}$$

and $\partial_j u=\sum_{h\in\mathbb{H}}\langle h,u\rangle_H\partial_j h$*, and*

(iii) it holds for all $u\in H_\rho$ *that*

$$\|\partial u\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\le\left[\sum_{j=1}^d\|\partial_j u\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}\right]^{1/2}\le\sqrt{d}\left[\sup_{h\in\mathbb{H}}\sup_{j\in\{1,\dots,d\}}\frac{\|\partial_j h\|_U}{|\lambda_h|^\rho}\right]\|u\|_{H_\rho}<\infty. \tag{148}$$

Proof of Lemma 4.4. Note that for all $u\in H_\rho$, $j\in\{1,\dots,d\}$ it holds that

$$\begin{aligned}\textstyle\sum_{h\in\mathbb{H}}\|\langle h,u\rangle_H\partial_j h\|_U^2&\le\left(\sup_{h\in\mathbb{H}}\frac{\|\partial_j h\|_U^2}{|\lambda_h|^{2\rho}}\right)\textstyle\sum_{h\in\mathbb{H}}|\lambda_h|^{2\rho}|\langle h,u\rangle_H|^2\\&=\left(\sup_{h\in\mathbb{H}}\frac{\|\partial_j h\|_U^2}{|\lambda_h|^{2\rho}}\right)\|u\|_{H_\rho}^2<\infty.\end{aligned} \tag{149}$$

The fact that for all $j\in\{1,\dots,d\}$, $v,w\in\mathbb{H}$ with $v\neq w$ it holds that $\langle\partial_j v,\partial_j w\rangle_U=0$ hence shows that for all $u\in H_\rho$, $\phi\in\mathcal{C}^\infty_{cpt}(\mathcal{D},\mathbb{R}^d)$, $j\in\{1,\dots,d\}$ it holds that

$$\begin{aligned}\langle u,[\tfrac{\partial}{\partial x_j}\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}\rangle_U&=\left\langle\sum_{h\in\mathbb{H}}\langle h,u\rangle_U h,[\tfrac{\partial}{\partial x_j}\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}\right\rangle_U=\sum_{h\in\mathbb{H}}\langle h,u\rangle_U\langle h,[\tfrac{\partial}{\partial x_j}\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}\rangle_U\\&=-\sum_{h\in\mathbb{H}}\langle h,u\rangle_U\langle\partial_j h,[\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}\rangle_U=-\left\langle\sum_{h\in\mathbb{H}}\langle h,u\rangle_U\partial_j h,[\phi]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^d)}\right\rangle_U.\end{aligned} \tag{150}$$

This and (149) complete the proof of Lemma 4.4. □

Lemma 4.5 (Weak product rule (cf., e.g., Proposition 7.1.11 in Atkinson & Han [1]))**.** *Let* $d\in\mathbb{N}$*,* $u,v\in[W^{1,2}((0,1)^d,\mathbb{R})\cap L^\infty(\mu_{(0,1)^d};\mathbb{R})]$*,* $j\in\{1,\dots,d\}$*. Then it holds that* $u\cdot v\in[W^{1,2}((0,1)^d,\mathbb{R})\cap L^\infty(\mu_{(0,1)^d};\mathbb{R})]$ *and* $\partial_j(uv)=u\,\partial_j v+v\,\partial_j u$*.*

Proof of Lemma 4.5. Throughout this proof let $\tilde{u}_n, \tilde{v}_n \in \mathcal{C}^\infty([0,1]^d, \mathbb{R})$, $n \in \mathbb{N}$, and $u_n, v_n \in W^{1,2}((0,1)^d, \mathbb{R})$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $u_n = [\tilde{u}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, $v_n = [\tilde{v}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, and $\limsup_{m\to\infty} \big(\|u - \tilde{u}_m\|_{W^{1,2}((0,1)^d,\mathbb{R})} + \|v - \tilde{v}_m\|_{W^{1,2}((0,1)^d,\mathbb{R})}\big) = 0$ (see, e.g., Theorem 7.3.2 in Atkinson & Han [1]). Observe that for all $f, g \in L^2(\mu_{(0,1)^d}; \mathbb{R})$, $h \in L^\infty(\mu_{(0,1)^d}; \mathbb{R})$ it holds that

$$\|(f-g)h\|_{L^2(\mu_{(0,1)^d};\mathbb{R})} \leq \|f-g\|_{L^2(\mu_{(0,1)^d};\mathbb{R})} \|h\|_{L^\infty(\mu_{(0,1)^d};\mathbb{R})} < \infty. \tag{151}$$

Hence, we obtain for all $\phi \in \mathcal{C}_{cpt}((0,1)^d, \mathbb{R})$, $i \in \{1, \ldots, d\}$ that

$$\begin{aligned}
&- \Big\langle uv, \big[\tfrac{\partial}{\partial x_i}\phi\big]_{\mu_{(0,1)^d},\mathcal{B}(\mathbb{R})} \Big\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = - \lim_{n\to\infty} \Big\langle u_n v, \big[\tfrac{\partial}{\partial x_i}\phi\big]_{\mu_{(0,1)^d},\mathcal{B}(\mathbb{R})} \Big\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \\
&= - \lim_{n\to\infty} \left[\lim_{m\to\infty} \Big\langle u_n v_m, \big[\tfrac{\partial}{\partial x_i}\phi\big]_{\mu_{(0,1)^d},\mathcal{B}(\mathbb{R})} \Big\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \right] \\
&= - \lim_{n\to\infty} \left[\lim_{m\to\infty} \int_{\mathcal{D}} \tilde{u}_n(x)\, \tilde{v}_m(x) \big(\tfrac{\partial}{\partial x_i}\phi\big)(x)\, dx \right] \\
&= \lim_{n\to\infty} \left[\lim_{m\to\infty} \int_{\mathcal{D}} \Big[\big(\tfrac{\partial}{\partial x_i}\tilde{u}_n\big)(x)\, \tilde{v}_m(x) + \tilde{u}_n(x) \big(\tfrac{\partial}{\partial x_i}\tilde{v}_m\big)(x) \Big] \phi(x)\, dx \right] \\
&= \lim_{n\to\infty} \left[\lim_{m\to\infty} \langle v_m\, \partial_i u_n + u_n\, \partial_i v_m, \phi \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \right] \\
&= \lim_{n\to\infty} \langle v\, \partial_i u_n + u_n\, \partial_i v, \phi \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = \langle v\, \partial_i u + u\, \partial_i v, \phi \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})}.
\end{aligned} \tag{152}$$

This completes the proof of Lemma 4.5. □

Lemma 4.6 (Weak integration by parts)**.** *Let $d \in \mathbb{N}$, $u, v \in W_P^{1,2}((0,1)^d, \mathbb{R})$, $j \in \{1, \ldots, d\}$. Then it holds that*

$$\langle \partial_j u, v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = - \langle u, \partial_j v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})}\,. \tag{153}$$

Proof of Lemma 4.6. Throughout this proof let $\tilde{u}_n, \tilde{v}_n \in \mathcal{C}_P^\infty([0,1]^d, \mathbb{R})$, $n \in \mathbb{N}$, and $u_n, v_n \in W_P^{1,2}((0,1)^d, \mathbb{R})$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $u_n = [\tilde{u}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, $v_n = [\tilde{v}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, and $\limsup_{m\to\infty} \big(\|u - [\tilde{u}_m|_{(0,1)^d}]_{\mu_{(0,1)^d},\mathcal{B}(\mathbb{R})}\|_{W^{1,2}((0,1)^d,\mathbb{R})} + \|v - [\tilde{v}_m|_{(0,1)^d}]_{\mu_{(0,1)^d},\mathcal{B}(\mathbb{R})}\|_{W^{1,2}((0,1)^d,\mathbb{R})}\big) = 0$. Observe that integration by parts and the fact that $\forall\, n \in \mathbb{N} \colon \tilde{u}_n, \tilde{v}_n \in \mathcal{C}_P^\infty([0,1]^d, \mathbb{R})$ prove that

$$\begin{aligned}
&\langle \partial_j u, v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = \lim_{n\to\infty} \langle \partial_j u_n, v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = \lim_{n\to\infty} \Big(\lim_{m\to\infty} \langle \partial_j u_n, v_m \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big) \\
&= \lim_{n\to\infty} \left(\lim_{m\to\infty} \int_{\mathcal{D}} \tilde{v}_m(x) \big(\tfrac{\partial}{\partial x_j}\tilde{u}_n\big)(x)\, dx \right) = - \lim_{n\to\infty} \left(\lim_{m\to\infty} \int_{\mathcal{D}} \tilde{u}_n(x) \big(\tfrac{\partial}{\partial x_j}\tilde{v}_m\big)(x)\, dx \right) \\
&= - \lim_{n\to\infty} \Big(\lim_{m\to\infty} \langle u_n, \partial_j v_m \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big) = - \lim_{n\to\infty} \langle u_n, \partial_j v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \\
&= - \langle u, \partial_j v \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})}.
\end{aligned} \tag{154}$$

The proof of Lemma 4.6 is thus completed. □

Lemma 4.7 (Weak integration by parts revisited)**.** *Let $d \in \mathbb{N}$, $u, v, w \in [W_P^{1,2}((0,1)^d, \mathbb{R}) \cap L^\infty(\mu_{(0,1)^d}; \mathbb{R})]$, $j \in \{1, \ldots, d\}$. Then it holds that $u \cdot v \in [W^{1,2}((0,1)^d, \mathbb{R}) \cap L^\infty(\mu_{(0,1)^d}; \mathbb{R})]$ and*

$$\langle \partial_j (uv), w \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = - \langle uv, \partial_j w \rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})}\,. \tag{155}$$

Proof of Lemma 4.7. Throughout this proof let $\tilde{u}_n, \tilde{v}_n, \tilde{w}_n \in \mathcal{C}_P^\infty([0,1]^d, \mathbb{R})$, $n \in \mathbb{N}$, and $u_n, v_n, w_n \in W_P^{1,2}((0,1)^d, \mathbb{R})$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $u_n = [\tilde{u}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, $v_n = [\tilde{v}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, $w_n = [\tilde{w}_n|_{(0,1)^d}]_{\mu_{(0,1)^d}, \mathcal{B}(\mathbb{R})}$, and $\limsup_{m\to\infty} \big(\|u - u_m\|_{W^{1,2}((0,1)^d,\mathbb{R})} + \|v - v_m\|_{W^{1,2}((0,1)^d,\mathbb{R})} + \|w - w_m\|_{W^{1,2}((0,1)^d,\mathbb{R})}\big) = 0$. Observe that Lemma 4.5 (with $d = d$, $u = u$, $v = v$, $j = j$ in the notation of Lemma 4.5) and the product rule for differentiation prove that $u \cdot v \in [W^{1,2}((0,1)^d, \mathbb{R}) \cap L^\infty(\mu_{(0,1)^d}; \mathbb{R})]$ and

$$\begin{aligned}
&\langle \partial_j(uv), w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = \langle u\, \partial_j v + v\, \partial_j u, w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \\
&= \lim_{l\to\infty} \Big(\langle u_l\, \partial_j v, w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} + \langle v\, \partial_j u_l, w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big) \\
&= \lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\langle u_l\, \partial_j v, w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} + \langle v\, \partial_j u_l, w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big)\Big) \\
&= \lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \Big(\langle u_l\, \partial_j v_m, w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} + \langle v_m\, \partial_j u_l, w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big)\Big)\Big) \\
&= \lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \langle u_l\, \partial_j v_m + v_m\, \partial_j u_l, w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big)\Big) \\
&= \lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \langle \partial_j(u_l v_m), w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big)\Big).
\end{aligned} \tag{156}$$

Integration by parts and the fact that $\forall\, n \in \mathbb{N}\colon \tilde{u}_n, \tilde{v}_n, \tilde{w}_n \in \mathcal{C}^\infty_P([0,1]^d, \mathbb{R})$ hence show that

$$\begin{aligned}
&\langle \partial_j(uv), w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = \lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \int_{(0,1)^d} \big[\tfrac{\partial}{\partial x_j}(\tilde{u}_l(x)\cdot \tilde{v}_m(x))\big]\, \tilde{w}_n(x)\, dx \Big)\Big) \\
&= -\lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \int_{(0,1)^d} \tilde{u}_l(x)\, \tilde{v}_m(x)\, \big(\tfrac{\partial}{\partial x_j}\tilde{w}_n\big)(x)\, dx \Big)\Big) \\
&= -\lim_{l\to\infty} \Big(\lim_{n\to\infty} \Big(\lim_{m\to\infty} \langle u_l v_m, \partial_j w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big)\Big) \\
&= -\lim_{l\to\infty} \Big(\lim_{n\to\infty} \langle u_l v, \partial_j w_n\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} \Big) \\
&= -\lim_{l\to\infty} \langle u_l v, \partial_j w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})} = -\langle uv, \partial_j w\rangle_{L^2(\mu_{(0,1)^d};\mathbb{R})}.
\end{aligned} \tag{157}$$

The proof of Lemma 4.7 is thus completed. □

Lemma 4.8. *Assume the setting in Subsection 4.1, let $\rho \in [\gamma, \infty)$, $u \in H_\rho$, and assume for all $j \in \{1, \ldots, d\}$, $v, w \in \mathbb{H}$ that $\mathbb{H} \subseteq W^{1,2}(\mathcal{D}, \mathbb{R}^d)$, $\big(\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\big) + \sup_{h\in\mathbb{H}} \big(\|h\|_{L^\infty(\mu_\mathcal{D};\mathbb{R}^d)} + \|\partial_j h\|_U |\lambda_h|^{-\rho}\big) < \infty$, $\langle \partial_j v, \partial_j w\rangle_U\, \mathbb{1}_{\mathbb{H}\setminus\{v\}}(w) = 0$. Then it holds that $u \in [W^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_\mathcal{D}; \mathbb{R}^d)]$ and*

$$\|\partial u\|_{L^2(\mu_\mathcal{D};\mathbb{R}^{d\times d})} \leq \Bigg[\sum_{j=1}^d \|\partial_j u\|^2_{L^2(\mu_\mathcal{D};\mathbb{R}^d)}\Bigg]^{1/2} \leq \sqrt{d}\Bigg[\sup_{h\in\mathbb{H}} \sup_{j\in\{1,\ldots,d\}} \tfrac{\|\partial_j h\|_U}{|\lambda_h|^\rho}\Bigg] \|u\|_{H_\rho} < \infty, \tag{158}$$

$$\|u\|_{L^\infty(\mu_\mathcal{D};\mathbb{R}^d)} \leq \|u\|_{H_\rho} \Bigg(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_\mathcal{D};\mathbb{R}^d)}\Bigg) \Bigg[\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\Bigg]^{1/2} < \infty, \tag{159}$$

$$\begin{aligned}
&\|F(u)\|_H \leq \eta \|u\|_H \\
&+ d\sqrt{d} \|u\|^2_{H_\rho} \Bigg[\sup_{h\in\mathbb{H}} \sup_{j\in\{1,\ldots,d\}} \tfrac{\|\partial_j h\|_U}{|\lambda_h|^\rho}\Bigg] \Bigg(\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_\mathcal{D};\mathbb{R}^d)}\Bigg) \Bigg[\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\Bigg]^{1/2} < \infty.
\end{aligned} \tag{160}$$

Proof of Lemma 4.8. First, note that Lemma 4.3 (with $\rho = \rho$, $v = u$ in the notation of Lemma 4.3) proves (159). Moreover, observe that Lemma 4.4 (with $\rho = \rho$ in the notation of Lemma 4.4) establishes that $u \in W^{1,2}(\mathcal{D}, \mathbb{R}^d)$ and (158). This and (159) ensure that $u \in [W^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_D; \mathbb{R}^d)]$. Combining Lemma 4.1 (with $v = u$, $w = u$ in the notation of Lemma 4.1), (158), and (159) hence proves (160). The proof of Lemma 4.8 is thus completed. □

Lemma 4.9. *Assume the setting in Subsection 4.1, let $\rho \in [\gamma, \infty)$, $u = (u_1, \ldots, u_d) \in H_\rho$, and assume for all $j \in \{1, \ldots, d\}$, $v, w \in \mathbb{H}$ that $\mathbb{H} \subseteq W_P^{1,2}(\mathcal{D}, \mathbb{R}^d)$, $\big(\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\rho}\big) + \sup_{h\in\mathbb{H}} \big(\|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} + \|\partial_j h\|_U |\lambda_h|^{-\rho}\big) < \infty$, $\langle \partial_j v, \partial_j w\rangle_U \, \mathbb{1}_{\mathbb{H}\setminus\{v\}}(w) = 0$. Then it holds that $u \in [W_P^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d)]$ and*

$$2\langle u, F(u)\rangle_H = 2\eta\|u\|_H^2 + \textstyle\sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U = 2\eta\|u\|_H^2 + \big\langle \sum_{i=1}^d (u_i)^2, \sum_{j=1}^d \partial_j u_j \big\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})}. \tag{161}$$

Proof of Lemma 4.9. First, note that Lemma 4.8 (with $\rho = \rho$, $u = u$ in the notation of Lemma 4.8) ensures that

$$u \in [W^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d)]. \tag{162}$$

Moreover, observe that

$$\limsup_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|u - \textstyle\sum_{h\in I} \langle h, u\rangle_H h\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} = 0. \tag{163}$$

In addition, note that item (ii) in Lemma 4.4 (with $\rho = \rho$, $u = u$, $j = j$ for $j \in \{1, \ldots, d\}$ in the notation of Lemma 4.4) proves that for all $j \in \{1, \ldots, d\}$ it holds that

$$\limsup_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_j u - \textstyle\sum_{h\in I} \langle h, u\rangle_H \partial_j h\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} = 0. \tag{164}$$

Combining (162)–(164) with the fact that $\forall\, v \in W^{1,2}(\mathcal{D}, \mathbb{R}^d)\colon \|v\|^2_{W^{1,2}(\mathcal{D},\mathbb{R}^d)} = \|v\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} + \sum_{j=1}^d \|\partial_j v\|^2_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)}$ proves that

$$\limsup_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|u - \textstyle\sum_{h\in I} \langle h, u\rangle_H h\|_{W^{1,2}(\mathcal{D};\mathbb{R}^d)} = 0. \tag{165}$$

The fact that $W_P^{1,2}((0,1)^d, \mathbb{R}^d)$ is a closed subspace of $W^{1,2}((0,1)^d, \mathbb{R}^d)$, (162), and the fact that $\forall\, I \in \mathcal{P}_0(\mathbb{H})\colon \sum_{h\in I} \langle h, u\rangle_H h \in W_P^{1,2}((0,1)^d, \mathbb{R}^d)$ hence show that

$$u \in [W_P^{1,2}(\mathcal{D}, \mathbb{R}^d) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d)]. \tag{166}$$

This and Lemma 4.5 (with $d = d$, $u = u_i$, $v = u_j$, $j = j$ for $i, j \in \{1, \ldots, d\}$ in the notation of Lemma 4.5) prove that for all $i, j \in \{1, \ldots, d\}$ it holds that $u_i u_j \in W^{1,2}(\mathcal{D}, \mathbb{R})$ and $\partial_j(u_i u_j) = u_i\,\partial_j u_j + u_j\,\partial_j u_i$. Combining this and the fact that $\forall\, i \in \{1, \ldots, d\}\colon u_i \in [W_P^{1,2}(\mathcal{D}, \mathbb{R}) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R}) \cap H]$ with Lemma 4.7 (with $d = d$, $u = u_i$, $v = u_j$, $w = u_i$, $j = j$ for $i, j \in \{1, \ldots, d\}$ in the notation of Lemma 4.7) ensures that

$$\begin{aligned}
&\langle u, F(u)\rangle_H = \langle u, R(\eta u - \textstyle\sum_{j=1}^d u_j\,\partial_j u)\rangle_H = \eta\|u\|_H^2 - \sum_{j=1}^d \langle Ru, u_j\,\partial_j u\rangle_U \\
&= \eta\|u\|_H^2 - \textstyle\sum_{j=1}^d \langle u, u_j\,\partial_j u\rangle_U = \eta\|u\|_H^2 - \sum_{j=1}^d \sum_{i=1}^d \langle u_i, u_j\,\partial_j u_i\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} \\
&= \eta\|u\|_H^2 - \textstyle\sum_{j=1}^d \sum_{i=1}^d \langle u_i u_j, \partial_j u_i\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} = \eta\|u\|_H^2 + \sum_{j=1}^d \sum_{i=1}^d \langle \partial_j(u_i u_j), u_i\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} \\
&= \eta\|u\|_H^2 + \textstyle\sum_{j=1}^d \sum_{i=1}^d \langle u_i\,\partial_j u_j + u_j\,\partial_j u_i, u_i\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})}.
\end{aligned} \tag{167}$$

Hence, we obtain that

$$\begin{aligned}
\langle u, F(u)\rangle_U &= \eta\|u\|_H^2 + \textstyle\sum_{j=1}^d \sum_{i=1}^d \langle u_i, u_i\,\partial_j u_j + u_j\,\partial_j u_i\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} \\
&= \eta\|u\|_H^2 + \textstyle\sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U + \sum_{j=1}^d \langle u, u_j\,\partial_j u\rangle_U \\
&= 2\eta\|u\|_H^2 - \big[\eta\|u\|_H^2 - \langle Ru, \textstyle\sum_{j=1}^d u_j\,\partial_j u\rangle_U\big] + \sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U \\
&= 2\eta\|u\|_H^2 - \big[\langle u, R(\eta u)\rangle_H - \langle u, R(\textstyle\sum_{i=1}^d u_i \partial_i u)\rangle_H\big] + \sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U \\
&= 2\eta\|u\|_H^2 - \langle u, F(u)\rangle_H + \textstyle\sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U.
\end{aligned} \tag{168}$$

In addition, note that

$$\begin{aligned}
&\textstyle\sum_{j=1}^d \langle u, u\,\partial_j u_j\rangle_U = \sum_{j=1}^d \sum_{i=1}^d \langle u_i, u_i\,\partial_j u_j\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} = \sum_{j=1}^d \sum_{i=1}^d \langle (u_i)^2, \partial_j u_j\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})} \\
&= \big\langle \textstyle\sum_{i=1}^d (u_i)^2, \sum_{j=1}^d \partial_j u_j \big\rangle_{L^2(\mu_{\mathcal{D}};\mathbb{R})}.
\end{aligned} \tag{169}$$

Combining this, (166), and (168) completes the proof of Lemma 4.9. ∎

Corollary 4.10. *Assume the setting in Subsection 4.1 and assume for all* $j \in \{1, \ldots, d\}$, $v, w \in \mathbb{H}$ *that* $\mathbb{H} \subseteq W^{1,2}(\mathcal{D}, \mathbb{R}^d)$, $\big(\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\gamma}\big) + \sup_{h\in\mathbb{H}} \big(\|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} + \|\partial_j h\|_U |\lambda_h|^{-\gamma}\big) < \infty$, $\langle \partial_j v, \partial_j w\rangle_U \, \mathbb{1}_{\mathbb{H}\setminus\{v\}}(w) = 0$. *Then it holds that* $F \in \mathcal{C}(H_\gamma, H)$ *and* $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U, H))$.

Proof of Corollary 4.10. First of all, note that Lemma 4.8 (with $\rho = \gamma$ in the notation of Lemma 4.8) assures that

$$H_\gamma \subseteq W^{1,2}(D, \mathbb{R}^d) \quad \text{continuously} \qquad \text{and} \qquad H_\gamma \subseteq L^\infty(\mu_{\mathcal{D}}; \mathbb{R}^d) \quad \text{continuously.} \tag{170}$$

This and Lemma 4.1 (with $v = v$, $w = w$ for $v, w \in H_\gamma$ in the notation of Lemma 4.1) show that for all $v, w \in H_\gamma$ it holds that

$$\begin{aligned} &\|F(v) - F(w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \le \eta \|v - w\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^d)} \\ &+ d\big(\|\partial v\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})} \|v - w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} + \|w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} \|\partial(v - w)\|_{L^2(\mu_{\mathcal{D}};\mathbb{R}^{d\times d})}\big) < \infty. \end{aligned} \tag{171}$$

In addition, Lemma 4.2 (with $v = v$, $w = w$ for $v, w \in H_\gamma$ in the notation of Lemma 4.2) proves that for all $v, w \in H_\gamma$ it holds that

$$\begin{aligned} &\|B(v) - B(w)\|_{\mathrm{HS}(U,H)} \\ &\le \Big(\sup_{x\in\mathcal{D}, y\in\mathbb{R}^d, z\in\mathbb{R}^d\setminus\{y\}} \tfrac{\|b(x,y) - b(x,z)\|_{\mathbb{R}^{d\times d}}}{\|y - z\|_{\mathbb{R}^d}}\Big) \|v - w\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^d)} \sqrt{\mathrm{trace}_U(Q)}. \end{aligned} \tag{172}$$

Combining this and (171) with (170) completes the proof of Corollary 4.10. □

4.3 Stochastic Burgers equations

Corollary 4.11. *Assume the setting in Subsection 4.1, let* $(e_n)_{n\in\mathbb{N}} \subseteq \mathbb{H}$, *and assume for all* $n \in \mathbb{N}$, $v \in H_\gamma$ *that* $\eta = 0$, $d = 1$, $\gamma \ge 1/2$, $e_n = [\{\sqrt{2}\sin(n\pi x)\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R})}$, $\lambda_{e_n} = -\pi^2 n^2$, $r(v) \ge \max\{\sqrt{\vartheta + \varepsilon\|v\|_H^2}, \tfrac{1}{\sqrt{3}}\|v\|_{H_\gamma}^2\}$. *Then*

$$\sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big(\tfrac{\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2\varepsilon\vartheta t}}\Big)\Big] < \infty. \tag{173}$$

Proof of Corollary 4.11. First of all, note that the fact that $\{e_n : n \in \mathbb{N}\} \subseteq \mathbb{H} \subseteq H \subseteq U$ shows that

$$\{e_n : n \in \mathbb{N}\} = \mathbb{H} \qquad \text{and} \qquad H = U. \tag{174}$$

In the next step we observe that

$$\sum\nolimits_{h\in\mathbb{H}} |\lambda_h|^{-2\gamma} = \sum\nolimits_{n\in\mathbb{N}} |\pi^2 n^2|^{-2\gamma} = \pi^{-4\gamma} \sum\nolimits_{n\in\mathbb{N}} n^{-4\gamma} \le \pi^{-2} \sum\nolimits_{n\in\mathbb{N}} n^{-2} < \infty. \tag{175}$$

Moreover, note that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} \|\partial e_n\|_U |\lambda_{e_n}|^{-\gamma} &= \|[\{\pi n\sqrt{2}\cos(n\pi x)\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R})}\|_U |\pi^2 n^2|^{-\gamma} \\ &= \pi n |\pi^2 n^2|^{-\gamma} = \tfrac{1}{(\pi n)^{2\gamma-1}} \le 1. \end{aligned} \tag{176}$$

Combining (174)–(176), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Lemma 4.8 (with $\rho = \gamma$, $u = v$ for $v \in H_\gamma$ in the notation of Lemma 4.8) proves that for all $v \in H_\gamma$ it holds that $H_\gamma \subseteq [W^{1,2}(\mathcal{D}, \mathbb{R}) \cap L^\infty(\mu_{\mathcal{D}}; \mathbb{R})]$ and

$$\|F(v)\|_H \le \tfrac{\sqrt{2}}{\pi}\big(\sum\nolimits_{n\in\mathbb{N}} n^{-2}\big)^{1/2} \|v\|_{H_\gamma}^2 = \tfrac{1}{\sqrt{3}}\|v\|_{H_\gamma}^2 < \infty. \tag{177}$$

Next note that

$$\begin{aligned} \mathbb{H} \subseteq D(A) &= [W^{2,2}(\mathcal{D}, \mathbb{R}) \cap W_0^{1,2}(\mathcal{D}, \mathbb{R})] \subseteq W_0^{1,2}(\mathcal{D}, \mathbb{R}) = \overline{\mathcal{C}_{cpt}^\infty(\mathcal{D}, \mathbb{R})}^{W^{1,2}(\mathcal{D},\mathbb{R})} \\ &\subseteq \overline{\mathcal{C}_P^\infty(\mathcal{D}, \mathbb{R})}^{W^{1,2}(\mathcal{D},\mathbb{R})} = W_P^{1,2}(\mathcal{D}, \mathbb{R}). \end{aligned} \tag{178}$$

This, (174)–(176), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Lemma 4.9 (with $\rho = \gamma$, $u = x$ for $x \in H_\gamma$ in the notation of Lemma 4.9) ensure that for all $x \in H_\gamma$ it holds that $H_\gamma \subseteq [W_P^{1,2}(\mathcal{D},\mathbb{R}) \cap L^\infty(\mu_{\mathcal{D}};\mathbb{R})]$ and

$$\begin{aligned} 2\langle x, F(x)\rangle_H &= 2\eta\|x\|_H^2 + \langle x, x\,\partial x\rangle_U = 2\eta\|x\|_H^2 + \langle x, R(x\,\partial x)\rangle_H \\ &= 2\eta\|x\|_H^2 + \langle x, F(x)\rangle_H = \langle x, F(x)\rangle_H. \end{aligned} \tag{179}$$

Hence, we obtain that for all $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\langle x, P_I F(x)\rangle_H = \langle P_I x, F(x)\rangle_H = \langle x, F(x)\rangle_H = 0. \tag{180}$$

In the next step we observe that (174)–(176), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Corollary 4.10 assure that $F \in \mathcal{C}(H_\gamma, H)$ and $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$. This proves that

$$F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H)) \qquad \text{and} \qquad B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))). \tag{181}$$

Moreover, (177) and Lemma 4.2 (with $v = x$, $w = x$ for $x \in \cup_{h\in(0,T]} \cup_{I\in\mathcal{P}_0(\mathbb{H})} D_h^I$ in the notation of Lemma 4.2) imply for all $h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in D_h^I$ that

$$\begin{aligned} \max\bigl\{\|P_I F(x)\|_H, \|P_I B(x)\hat{P}_J\|_{\mathrm{HS}(U,H)}\bigr\} &\le \max\bigl\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\bigr\} \\ &\le \max\bigl\{\tfrac{1}{\sqrt{3}}\|x\|_{H_\gamma}^2, \sqrt{\vartheta}\bigr\} \le r(x) \le ch^{-\delta}. \end{aligned} \tag{182}$$

Furthermore, we observe that the fact that $\forall\, v \in H_\gamma\colon \sqrt{\theta} + \varepsilon\|v\|_H^2 \le r(v)$ shows that for all $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$\begin{aligned} D_h^I = \{x \in P_I(H)\colon r(x) \le ch^{-\delta}\} &\subseteq \{x \in P_I(H)\colon \sqrt{\vartheta} + \varepsilon\|x\|_H^2 \le ch^{-\delta}\} \\ &\subseteq \{v \in H\colon \sqrt{\vartheta} + \varepsilon\|v\|_H^2 \le ch^{-\delta}\}. \end{aligned} \tag{183}$$

In addition, we note that Lemma 4.2 ensures that $\sup_{x\in H_\gamma} \|B(x)\|_{\mathrm{HS}(U,H)}^2 \le \vartheta < \infty$. Combining (180)–(183) and Corollary 3.4 (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathbb{U} = \mathbb{U}$, $T = T$, $\gamma = \gamma$, $\delta = \delta$, $\lambda = \lambda$, $A = A$, $\xi = \xi$, $F = F$, $B = B$, $D_h^I = D_h^I$, $\vartheta = \vartheta$, $b_1 = 0$, $b_2 = 0$, $\varepsilon = \varepsilon$, $\varsigma = \delta$, $c = c$, $Y^{\theta,I,J} = Y^{\theta,I,J}$ for $h \in (0,T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$ in the notation of Corollary 3.4) hence completes the proof of Corollary 4.11. □

Remark 4.12. *Consider the setting of Corollary 4.11. Then the stochastic processes $Y^{\theta,I,J}\colon [0,T] \times \Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, are space-time-noise discrete numerical approximation processes for the stochastic Burgers equation*

$$dX_t(x) = \left[\tfrac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \cdot \tfrac{\partial}{\partial x} X_t(x)\right] dt + b(x, X_t(x))\, d(\sqrt{Q}W)_t(x), \tag{184}$$

with $X_0(x) = \xi(x)$ and $X_t(0) = X_t(1) = 0$ for $t \in [0,T]$, $x \in (0,1)$ (cf., e.g., Section 1 in Da Prato et al. [14] and Section 2 in Hairer & Voss [22]).

4.4 Stochastic Kuramoto-Sivashinsky equations

Corollary 4.13. *Assume the setting in Subsection 4.1, let $(e_k)_{k\in\mathbb{Z}} \subseteq \mathbb{H}$, and assume for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $v \in H_\gamma$ that $\eta \in (0,\infty)$, $d = 1$, $\gamma \ge 1/4$, $e_0 = [\{1\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R})}$, $e_n = [\{\sqrt{2}\cos(2\pi n x)\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R})}$, $e_{-n} = [\{\sqrt{2}\sin(2\pi n x)\}_{x\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R})}$, $r(v) \ge \max\bigl\{\sqrt{\vartheta}+\varepsilon\|v\|_H^2, \eta\|v\|_H + 5\max\{1,\eta^{-\gamma}\}\|v\|_{H_\gamma}^2\bigr\}$, $\lambda_{e_k} = 4k^2\pi^2 - 16k^4\pi^4 - \eta$. Then*

$$\sup_{\theta\in\varpi_T} \sup_{I\in\mathcal{P}_0(\mathbb{H})} \sup_{J\in\mathcal{P}_0(\mathbb{U})} \sup_{t\in[0,T]} \mathbb{E}\Bigl[\exp\Bigl(\tfrac{\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2(\eta+\varepsilon\vartheta)t}}\Bigr)\Bigr] < \infty. \tag{185}$$

Proof of Corollary 4.13. First of all, note that the fact that $\{e_l \colon l \in \mathbb{Z}\} \subseteq \mathbb{H} \subseteq H \subseteq U$ shows that

$$\{e_l \colon l \in \mathbb{Z}\} = \mathbb{H} \qquad \text{and} \qquad H = U. \tag{186}$$

Hence, we obtain that

$$\begin{aligned}\textstyle\sum_{h\in\mathbb{H}} |\lambda_h|^{-2\gamma} &= \textstyle\sum_{k\in\mathbb{Z}} |\lambda_{e_k}|^{-2\gamma} = \sum_{k\in\mathbb{Z}} |16k^4\pi^4 - 4k^2\pi^2 + \eta|^{-2\gamma} \\ &= \eta^{-2\gamma} + 2\textstyle\sum_{k=1}^{\infty} |16k^4\pi^4 - 4k^2\pi^2 + \eta|^{-2\gamma} \le \eta^{-2\gamma} + 2\sum_{k=1}^{\infty} |12k^4\pi^4 + \eta|^{-2\gamma} \\ &\le \eta^{-2\gamma} + 2\textstyle\sum_{k=1}^{\infty} |12k^4\pi^4|^{-2\gamma} = \eta^{-2\gamma} + \frac{2}{|12\pi^4|^{2\gamma}} \sum_{k=1}^{\infty} \frac{1}{k^{8\gamma}} \\ &\le \eta^{-2\gamma} + \tfrac{2}{(12\pi^4)^{1/2}} \textstyle\sum_{k=1}^{\infty} \frac{1}{k^2} \le \eta^{-2\gamma} + \sum_{k=1}^{\infty} \frac{1}{k^2} = \eta^{-2\gamma} + \frac{\pi^2}{6} < \infty.\end{aligned} \tag{187}$$

Moreover, note that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}\|\partial e_n\|_U |\lambda_{e_n}|^{-\gamma} &= \tfrac{\|[\{2\pi n\sqrt{2}\sin(2\pi n x)\}_{x\in(0,1)}]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}\|_U}{|16n^4\pi^4-4n^2\pi^2+\eta|^{\gamma}} = \tfrac{2\pi n}{|16n^4\pi^4-4n^2\pi^2+\eta|^{\gamma}} \le \tfrac{2\pi n}{|12n^4\pi^4+\eta|^{\gamma}} \\ &\le \tfrac{2\pi n}{|12n^4\pi^2|^{\gamma}} \le \tfrac{2\pi n}{n(12)^{1/4}\sqrt{\pi}} = \tfrac{2\sqrt{\pi}}{(12)^{1/4}} \le 2.\end{aligned} \tag{188}$$

This shows that for all $n \in \mathbb{N}$ it holds that

$$\|\partial e_{-n}\|_U |\lambda_{e_{-n}}|^{-\gamma} = \tfrac{\|[\{2\pi n\sqrt{2}\cos(2\pi n x)\}_{x\in(0,1)}]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}\|_U}{|16n^4\pi^4-4n^2\pi^2+\eta|^{\gamma}} = \tfrac{2\pi n}{|16n^4\pi^4-4n^2\pi^2+\eta|^{\gamma}} \le 2. \tag{189}$$

Combining (186)–(189), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Lemma 4.8 (with $\rho = \gamma$, $u = v$ for $v \in H_\gamma$ in the notation of Lemma 4.8) proves that for all $v \in H_\gamma$ it holds that $H_\gamma \subseteq [W^{1,2}(\mathcal{D},\mathbb{R}) \cap L^\infty(\mu_{\mathcal{D}};\mathbb{R})]$ and

$$\begin{aligned}\|F(v)\|_H &\le \eta\|v\|_H + 2\sqrt{2}\big(\eta^{-2\gamma} + \tfrac{\pi^2}{6}\big)^{1/2}\|v\|^2_{H_\gamma} = \eta\|v\|_H + \big(8\eta^{-2\gamma} + \tfrac{4\pi^2}{3}\big)^{1/2}\|v\|^2_{H_\gamma} \\ &\le \eta\|v\|_H + \max\{1,\eta^{-\gamma}\}\big(8 + \tfrac{4\pi^2}{3}\big)^{1/2}\|v\|^2_{H_\gamma} \le \eta\|v\|_H + 5\max\{1,\eta^{-\gamma}\}\|v\|^2_{H_\gamma}.\end{aligned} \tag{190}$$

Next note that

$$\mathbb{H} \subseteq \overline{\mathcal{C}^\infty_P(\mathcal{D},\mathbb{R})}^{W^{1,2}(\mathcal{D},\mathbb{R})} = W^{1,2}_P(\mathcal{D},\mathbb{R}). \tag{191}$$

This, (186)–(189), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Lemma 4.9 (with $\rho = \gamma$, $u = x$ for $x \in H_\gamma$ in the notation of Lemma 4.9) ensure that for all $x \in H_\gamma$ it holds that $H_\gamma \subseteq [W^{1,2}_P(\mathcal{D},\mathbb{R}) \cap L^\infty(\mu_{\mathcal{D}};\mathbb{R})]$ and

$$\begin{aligned}2\langle x, F(x)\rangle_H &= 2\eta\|x\|^2_H + \langle x, x\,\partial x\rangle_U = 2\eta\|x\|^2_H + \langle x, R(x\,\partial x)\rangle_H \\ &= 3\eta\|x\|^2_H - [\langle x, R(\eta x)\rangle_H - \langle x, R(x\,\partial x)\rangle_H] = 3\eta\|x\|^2_H - \langle x, F(x)\rangle_H.\end{aligned} \tag{192}$$

Hence, we obtain that for all $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\langle x, P_I F(x)\rangle_H = \langle P_I x, F(x)\rangle_H = \langle x, F(x)\rangle_H = \eta\|x\|^2_H. \tag{193}$$

In the next step we observe that (186)–(189), the fact that $\sup_{h\in\mathbb{H}} \|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})} = \sqrt{2}$, and Corollary 4.10 assure that $F \in \mathcal{C}(H_\gamma, H)$ and $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$. This proves that

$$F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H)) \qquad \text{and} \qquad B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))). \tag{194}$$

Moreover, (190) and Lemma 4.2 (with $v = x$, $w = x$ for $x \in \cup_{h\in(0,T]} \cup_{I\in\mathcal{P}_0(\mathbb{H})} D^I_h$ in the notation of Lemma 4.2) imply that for all $h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in D^I_h$ it holds that

$$\begin{aligned}&\max\{\|P_I F(x)\|_H, \|P_I B(x)\hat{P}_J\|_{\mathrm{HS}(U,H)}\} \le \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \\ &\le \max\big\{\eta\|x\|_H + 5\max\{1,\eta^{-\gamma}\}\|x\|^2_{H_\gamma}, \sqrt{\vartheta}\big\} \le r(x) \le ch^{-\delta}.\end{aligned} \tag{195}$$

Furthermore, we observe that the fact that $\forall\, v \in H_\gamma\colon \sqrt{\theta} + \varepsilon\|v\|_H^2 \leq r(v)$ implies that for all $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$\begin{aligned} D_h^I = \{x \in P_I(H)\colon r(x) \leq ch^{-\delta}\} &\subseteq \{x \in P_I(H)\colon \sqrt{\vartheta} + \varepsilon\|x\|_H^2 \leq ch^{-\delta}\} \\ &\subseteq \{v \in H\colon \sqrt{\vartheta} + \varepsilon\|v\|_H^2 \leq ch^{-\delta}\}. \end{aligned} \tag{196}$$

In addition, we note that Lemma 4.2 ensures that $\sup_{x\in H_\gamma} \|B(x)\|^2_{\mathrm{HS}(U,H)} \leq \vartheta < \infty$. Combining (193)–(196) and Corollary 3.4 (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathbb{U} = \mathbb{U}$, $T = T$, $\gamma = \gamma$, $\delta = \delta$, $\lambda = \lambda$, $A = A$, $\xi = \xi$, $F = F$, $B = B$, $D_h^I = D_h^I$, $\vartheta = \vartheta$, $b_1 = 0$, $b_2 = \eta$, $\varepsilon = \varepsilon$, $\varsigma = \delta$, $c = c$, $Y^{\theta,I,J} = Y^{\theta,I,J}$ for $h \in (0,T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$ in the notation of Corollary 3.4) hence completes the proof of Corollary 4.13. □

Remark 4.14. *Consider the setting of Corollary 4.13. Then the stochastic processes $Y^{\theta,I,J}\colon [0,T] \times \Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, are space-time-noise discrete numerical approximation processes for the stochastic Kuramoto-Sivashinsky equation*

$$dX_t(x) = \big[-\tfrac{\partial^4}{\partial x^4} X_t(x) - \tfrac{\partial^2}{\partial x^2} X_t(x) - X_t(x) \cdot \tfrac{\partial}{\partial x} X_t(x)\big]\, dt + b(x, X_t(x))\, d(\sqrt{Q}W)_t(x), \tag{197}$$

with $X_t(0) = X_t(1)$, $X_t'(0) = X_t'(1)$, $X_t''(0) = X_t''(1)$, $X_t^{(3)}(0) = X_t^{(3)}(1)$, and $X_0(x) = \xi(x)$ for $t \in [0,T]$, $x \in (0,1)$ (cf., e.g, Duan & Ervin [17] and Section 1 in Hutzenthaler et al. [29]).

4.5 Two-dimensional stochastic Navier-Stokes equations

Corollary 4.15. *Assume the setting in Subsection 4.1, let $(\varphi_k)_{k\in\mathbb{Z}} \subseteq \mathcal{C}((0,1),\mathbb{R})$, $(\phi_{k,l})_{k,l\in\mathbb{Z}} \subseteq \mathcal{C}(\mathcal{D},\mathbb{R})$, $(e_{i,j,0})_{i,j\in\mathbb{Z}} \subseteq U$, $e_{0,0,1} \in U$, and assume for all $n \in \mathbb{N}$, $(k,l) \in \mathbb{Z}^2\backslash\{(0,0)\}$, $v \in H_\gamma$, $x, y \in (0,1)$ that $\mathbb{H} = \{e_{0,0,1}\} \cup \{e_{i,j,0}\colon i,j \in \mathbb{Z}\}$, $\eta \in (0,\infty)$, $d = 2$, $\gamma > 1/2$, $\varphi_0(x) = 1$, $\varphi_n(x) = \sqrt{2}\cos(2n\pi x)$, $\varphi_{-n}(x) = \sqrt{2}\sin(2n\pi x)$, $\phi_{k,l}(x,y) = \varphi_k(x)\varphi_l(y)$, $e_{0,0,0} = [\{(1,0)\}_{(x,y)\in\mathcal{D}}]_{\mu_\mathcal{D},\mathcal{B}(\mathbb{R}^2)}$, $e_{0,0,1} = [\{(0,1)\}_{(x,y)\in\mathcal{D}}]_{\mu_\mathcal{D},\mathcal{B}(\mathbb{R}^2)}$, $e_{k,l,0} = \big[\{1/\sqrt{k^2+l^2}(l\phi_{k,l}(x,y), k\phi_{-k,-l}(x,y))\}_{(x,y)\in\mathcal{D}}\big]_{\mu_\mathcal{D},\mathcal{B}(\mathbb{R}^2)}$, $\lambda_{e_{k,l,0}} = -\eta - 4\pi^2(k^2+l^2)$, $\lambda_{e_{0,0,0}} = \lambda_{e_{0,0,1}} = -\eta$, $r(v) \geq \max\big\{\sqrt{\vartheta} + \varepsilon\|v\|_H^2, \eta\|v\|_H + 6\big[\eta^{-2\gamma} + \sum_{i,j\in\mathbb{Z}}(\eta + 4\pi^2(i^2+j^2))^{-2\gamma}\big]^{1/2}\|v\|^2_{H_\gamma}\big\}$. Then*

$$\sup_{\theta\in\varpi_T}\ \sup_{I\in\mathcal{P}_0(\mathbb{H})}\ \sup_{J\in\mathcal{P}_0(\mathbb{U})}\ \sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big(\tfrac{\varepsilon\|Y_t^{\theta,I,J}\|_H^2}{e^{2(\eta+\varepsilon\vartheta)t}}\Big)\Big] < \infty. \tag{198}$$

Proof of Corollary 4.15. Observe that

$$\begin{aligned} \sum_{h\in\mathbb{H}} |\lambda_h|^{-2\gamma} &= \eta^{-2\gamma} + \sum_{(k,l)\in\mathbb{Z}^2} (\eta + 4\pi^2(k^2+l^2))^{-2\gamma} \\ &= \eta^{-2\gamma} + \eta^{-2\gamma} + 2\sum_{l=1}^\infty (\eta + 4\pi^2 l^2)^{-2\gamma} + 2\sum_{k=1}^\infty (\eta + 4\pi^2 k^2)^{-2\gamma} \\ &\quad + 4\sum_{k=1}^\infty\sum_{l=1}^\infty (\eta + 4\pi^2(k^2+l^2))^{-2\gamma} \\ &\leq 2\eta^{-2\gamma} + 4\sum_{k=1}^\infty (4\pi^2k^2)^{-2\gamma} + 4\sum_{k=1}^\infty\sum_{l=1}^\infty (4\pi^2(k^2+l^2))^{-2\gamma} \\ &= 2\eta^{-2\gamma} + 4^{1-2\gamma}\pi^{-4\gamma}\sum_{k=1}^\infty k^{-4\gamma} + 4^{1-2\gamma}\pi^{-4\gamma}\sum_{k=1}^\infty\sum_{l=1}^\infty (k^2+l^2)^{-2\gamma} \\ &\leq 2\eta^{-2\gamma} + \sum_{k=1}^\infty k^{-4\gamma} + \sum_{k,l=1}^\infty (k^2+l^2)^{-2\gamma}. \end{aligned} \tag{199}$$

Next note that the fact that $\forall\, k \in \mathbb{N}\colon k^{-4\gamma} \leq \int_{k-1}^k x^{-4\gamma}\, dx$ proves that

$$\sum_{k=1}^\infty k^{-4\gamma} = 1 + \sum_{k=2}^\infty k^{-4\gamma} \leq 1 + \sum_{k=2}^\infty \int_{k-1}^k x^{-4\gamma}\, dx = 1 + \int_1^\infty x^{-4\gamma}\, dx = 1 + \tfrac{1}{4\gamma-1}. \tag{200}$$

In addition, we observe that the fact that $\forall\, k,l \in \mathbb{N}\colon (k^2+l^2)^{-2\gamma} = \int_{k-1}^{k}\int_{l-1}^{l}(k^2+l^2)^{-2\gamma}\,dx\,dy \leq \int_{k-1}^{k}\int_{l-1}^{l}(y^2+x^2)^{-2\gamma}\,dx\,dy = \int_{k-1}^{k}\int_{l-1}^{l}(x^2+y^2)^{-2\gamma}\,dx\,dy$ proves that

$$\begin{aligned}
&\sum_{k,l=1}^{\infty}(k^2+l^2)^{-2\gamma} = \sum_{k=1}^{\infty}(k^2+1)^{-2\gamma} + \sum_{l=2}^{\infty}(1+l^2)^{-2\gamma} + \sum_{k=2}^{\infty}\sum_{l=2}^{\infty}(k^2+l^2)^{-2\gamma} \\
&\leq 2\sum_{k=1}^{\infty}k^{-4\gamma} + \sum_{k=2}^{\infty}\sum_{l=2}^{\infty}\int_{k-1}^{k}\int_{l-1}^{l}(x^2+y^2)^{-2\gamma}\,dx\,dy \\
&= 2\sum_{k=1}^{\infty}k^{-4\gamma} + \int_1^{\infty}\int_1^{\infty}(x^2+y^2)^{-2\gamma}\,dx\,dy = 2\sum_{k=1}^{\infty}k^{-4\gamma} + \int_0^{2\pi}\int_1^{\infty}s^{1-4\gamma}\,ds\,du \\
&= 2\pi\int_1^{\infty}s^{1-4\gamma}\,ds + 2\sum_{k=1}^{\infty}k^{-4\gamma} = \tfrac{2\pi}{4\gamma-2} + 2\sum_{k=1}^{\infty}k^{-4\gamma} = \tfrac{\pi}{2\gamma-1} + 2\sum_{k=1}^{\infty}k^{-4\gamma}.
\end{aligned} \tag{201}$$

Combining (199)–(201) proves that

$$\sum_{h\in\mathbb{H}}|\lambda_h|^{-2\gamma} < \infty. \tag{202}$$

Moreover, note that for all $(k,l)\in\mathbb{Z}^2\backslash\{(0,0)\}$ it holds that

$$\begin{aligned}
&\|\partial_1 e_{k,l,0}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= \|[\{(k^2+l^2)^{-1/2}(l\,\tfrac{\partial}{\partial x}\phi_{k,l}(x,y), k\,\tfrac{\partial}{\partial x}\phi_{-k,-l}(x,y))\}_{(x,y)\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^2)}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= \|[\{(k^2+l^2)^{-1/2}(-2\pi kl\phi_{-k,l}(x,y), 2\pi k^2\phi_{k,-l}(x,y))\}_{(x,y)\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^2)}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= (k^2+l^2)^{-1/2}2\pi k\sqrt{k^2+l^2}|\lambda_{e_{k,l,0}}|^{-\gamma} = 2\pi k|4\pi^2(k^2+l^2)+\eta|^{-\gamma} \\
&\leq \tfrac{2\pi k}{|4\pi^2(k^2+l^2)+\eta|^{1/2}} \leq \tfrac{2\pi k}{2\pi(k^2+l^2)^{1/2}} \leq 1
\end{aligned} \tag{203}$$

and

$$\begin{aligned}
&\|\partial_2 e_{k,l,0}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= \|[\{(k^2+l^2)^{-1/2}(l\,\tfrac{\partial}{\partial y}\phi_{k,l}(x,y), k\,\tfrac{\partial}{\partial y}\phi_{-k,-l}(x,y))\}_{(x,y)\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^2)}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= \|[\{(k^2+l^2)^{-1/2}(-2\pi l^2\phi_{k,-l}(x,y), 2\pi kl\phi_{-k,l}(x,y))\}_{(x,y)\in\mathcal{D}}]_{\mu_{\mathcal{D}},\mathcal{B}(\mathbb{R}^2)}\|_U|\lambda_{e_{k,l,0}}|^{-\gamma} \\
&= (k^2+l^2)^{-1/2}2\pi l\sqrt{k^2+l^2}|\lambda_{e_{k,l,0}}|^{-\gamma} = 2\pi l|4\pi^2(k^2+l^2)+\eta|^{-\gamma} \\
&\leq \tfrac{2\pi l}{|4\pi^2(k^2+l^2)+\eta|^{1/2}} \leq \tfrac{2\pi l}{2\pi(k^2+l^2)^{1/2}} \leq 1.
\end{aligned} \tag{204}$$

Furthermore, observe that for all $(k,l)\in\mathbb{Z}^2\backslash\{(0,0)\}$ it holds that

$$\begin{aligned}
\|e_{k,l,0}\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)} &= \tfrac{1}{\sqrt{k^2+l^2}}\|(l\phi_{k,l}, k\phi_{-k,-l})\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)} \\
&= \tfrac{1}{\sqrt{k^2+l^2}}\big\|l^2|\phi_{k,l}(\cdot)|^2 + k^2|\phi_{-k,-l}(\cdot)|^2\big\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R})}^{1/2} \\
&= \tfrac{1}{\sqrt{k^2+l^2}}\sup_{x_1,x_2\in(0,1)}\sqrt{l^2|\varphi_k(x_1)|^2|\varphi_l(x_2)|^2 + k^2|\varphi_{-k}(x_1)|^2|\varphi_{-l}(x_2)|^2} \\
&\leq \tfrac{1}{\sqrt{k^2+l^2}}\sqrt{l^2 4 + k^2 4} = \tfrac{2\sqrt{l^2+k^2}}{\sqrt{k^2+l^2}} = 2.
\end{aligned} \tag{205}$$

Hence, we obtain that

$$\begin{aligned}
&\sup\nolimits_{h\in\mathbb{H}}\|h\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)} \\
&= \max\big\{\|e_{0,0,0}\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)}, \|e_{0,0,1}\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)}, \sup\nolimits_{(k,l)\in\mathbb{Z}^2\backslash\{(0,0)\}}\|e_{k,l,0}\|_{L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)}\big\} \\
&\leq \max\{1,1,2\} = 2.
\end{aligned} \tag{206}$$

Combining (199), (202), (203), (204), (206), and Lemma 4.8 (with $\rho=\gamma$, $u=v$ for $v\in H_\gamma$ in the notation of Lemma 4.8) proves that for all $v\in H_\gamma$ it holds that $H_\gamma \subseteq [W^{1,2}(\mathcal{D},\mathbb{R}^2)\cap L^\infty(\mu_{\mathcal{D}};\mathbb{R}^2)]$ and

$$\|F(v)\|_H \leq \eta\|v\|_H + 6\big[\eta^{-2\gamma} + \textstyle\sum_{(k,l)\in\mathbb{Z}^2}(\eta+4\pi^2(k^2+l^2))^{-2\gamma}\big]^{1/2}\|v\|_{H_\gamma}^2. \tag{207}$$

Next note that

$$\mathbb{H} \subseteq \overline{\mathcal{C}_P^\infty(\mathcal{D};\mathbb{R}^2)}^{W^{1,2}(\mathcal{D},\mathbb{R}^2)} = W_P^{1,2}(\mathcal{D},\mathbb{R}). \tag{208}$$

This, (202), (203), (204), (206), and Lemma 4.9 (with $\rho = \gamma$, $u = u$ for $u = (u_1, u_2) \in H_\gamma$ in the notation of Lemma 4.9) ensure that for all $u = (u_1, u_2) \in H_\gamma$ it holds that $H_\gamma \subseteq [W_P^{1,2}(\mathcal{D},\mathbb{R}^2) \cap L^\infty(\mu_\mathcal{D};\mathbb{R}^2)]$ and

$$2\langle u, F(u)\rangle_H = 2\eta\|u\|_H^2 + \langle (u_1)^2 + (u_2)^2, \partial_1 u_1 + \partial_2 u_2\rangle_{L^2(\mu_\mathcal{D};\mathbb{R})}. \tag{209}$$

In addition, note that for all $(k,l) \in \mathbb{Z}^2$, $x, y \in (0,1)$ it holds that

$$\left|l\tfrac{\partial}{\partial x}\phi_{k,l}(x,y) + k\tfrac{\partial}{\partial y}\phi_{-k,-l}(x,y)\right| = \left|-2\pi kl\phi_{-k,l}(x,y) + 2\pi kl\phi_{-k,l}(x,y)\right| = 0. \tag{210}$$

This assures that for all $h = (h_1, h_2) \in \mathbb{H}$ it holds that

$$\partial_1 h_1 + \partial_2 h_2 = [\{0\}_{x\in\mathcal{D}}]_{\mu_\mathcal{D},\mathcal{B}(\mathbb{R})}. \tag{211}$$

Moreover, note that (203), (204), and item (ii) in Lemma 4.4 (with $\rho = \gamma$, $u = u$, $j = j$ for $u \in H_\gamma$, $j \in \{1,2\}$ in the notation of Lemma 4.4) prove that for all $u \in H_\gamma$, $j \in \{1,2\}$ it holds that

$$\limsup\nolimits_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_j u - \textstyle\sum_{h\in I}\langle h,u\rangle_H \partial_j h\|_{L^2(\mu_\mathcal{D};\mathbb{R}^2)} = 0. \tag{212}$$

This implies that for all $u = (u_1, u_2) \in H_\gamma$, $j \in \{1,2\}$ it holds that

$$\limsup\nolimits_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_j u_j - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H \partial_j h_j\|_{L^2(\mu_\mathcal{D};\mathbb{R})} = 0. \tag{213}$$

Next note that (211) ensures that for all $u = (u_1, u_2) \in W^{1,2}(\mathcal{D},\mathbb{R}^2)$, $I \in \mathcal{P}_0(\mathbb{H})$ it holds that

$$\begin{aligned}
&\|\partial_1 u_1 + \partial_2 u_2\|_{L^2(\mu_D;\mathbb{R})} \\
&= \|\partial_1 u_1 + \partial_2 u_2 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H(\partial_1 h_1 + \partial_2 h_2)\|_{L^2(\mu_\mathcal{D};\mathbb{R})} \\
&\le \|\partial_1 u_1 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H \partial_1 h_1\|_{L^2(\mu_\mathcal{D};\mathbb{R})} \\
&\quad + \|\partial_2 u_2 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H \partial_2 h_2\|_{L^2(\mu_\mathcal{D};\mathbb{R})}.
\end{aligned} \tag{214}$$

Combining (213) with the fact that $H_\gamma \subseteq W^{1,2}(\mathcal{D},\mathbb{R}^2)$ hence shows that for all $u = (u_1, u_2) \in H_\gamma$ it holds that

$$\begin{aligned}
&\|\partial_1 u_1 + \partial_2 u_2\|_{L^2(\mu_D;\mathbb{R})} \\
&= \limsup\nolimits_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_1 u_1 + \partial_2 u_2 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H(\partial_1 h_1 + \partial_2 h_2)\|_{L^2(\mu_\mathcal{D};\mathbb{R})} \\
&\le \limsup\nolimits_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_1 u_1 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H \partial_1 h_1\|_{L^2(\mu_\mathcal{D};\mathbb{R})} \\
&\quad + \limsup\nolimits_{\mathcal{P}_0(\mathbb{H})\ni I\to\mathbb{H}} \|\partial_2 u_2 - \textstyle\sum_{h=(h_1,h_2)\in I}\langle h,u\rangle_H \partial_2 h_2\|_{L^2(\mu_\mathcal{D};\mathbb{R})} = 0.
\end{aligned} \tag{215}$$

This assures that for all $u = (u_1, u_2) \in H_\gamma$ it holds that

$$\partial_1 u_1 + \partial_2 u_2 = [\{0\}_{x\in\mathcal{D}}]_{\mu_\mathcal{D},\mathcal{B}(\mathbb{R})}. \tag{216}$$

Equation (209) therefore proves that for all $I \in \mathcal{P}_0(\mathbb{H})$, $x \in P_I(H)$ it holds that

$$\langle x, P_I F(x)\rangle_H = \langle P_I x, F(x)\rangle_H = \langle x, F(x)\rangle_H = \eta\|x\|_H^2. \tag{217}$$

In the next step we observe that (202), (203), (204), (206), and Corollary 4.10 assure that $F \in \mathcal{C}(H_\gamma, H)$ and $B \in \mathcal{C}(H_\gamma, \mathrm{HS}(U,H))$. This proves that

$$F \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(H)) \qquad \text{and} \qquad B \in \mathcal{M}(\mathcal{B}(H_\gamma), \mathcal{B}(\mathrm{HS}(U,H))). \tag{218}$$

Moreover, note that (207) and Lemma 4.2 imply that for all $h \in (0,T]$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, $x \in D_h^I$ it holds that

$$\begin{aligned}
&\max\{\|P_I F(x)\|_H, \|P_I B(x)\hat{P}_J\|_{\mathrm{HS}(U,H)}\} \leq \max\{\|F(x)\|_H, \|B(x)\|_{\mathrm{HS}(U,H)}\} \\
&\leq \max\{\eta\|x\|_H + 6\big[\eta^{-2\gamma} + \textstyle\sum_{(k,l)\in\mathbb{Z}^2}(\eta + 4\pi^2(k^2+l^2))^{-2\gamma}\big]^{1/2}\|x\|_{H_\gamma}^2, \sqrt{\vartheta}\} \\
&\leq r(x) \leq ch^{-\delta}.
\end{aligned} \tag{219}$$

Furthermore, we observe that the fact that $\forall\, v \in H_\gamma\colon \sqrt{\theta} + \varepsilon\|v\|_H^2 \leq r(v)$ implies that for all $I \in \mathcal{P}_0(\mathbb{H})$, $h \in (0,T]$ it holds that

$$\begin{aligned}
D_h^I = \{x \in P_I(H)\colon r(x) \leq ch^{-\delta}\} &\subseteq \{x \in P_I(H)\colon \sqrt{\vartheta} + \varepsilon\|x\|_H^2 \leq ch^{-\delta}\} \\
&\subseteq \{v \in H\colon \sqrt{\vartheta} + \varepsilon\|v\|_H^2 \leq ch^{-\delta}\}.
\end{aligned} \tag{220}$$

In addition, we note that Lemma 4.2 ensures that $\sup_{x\in H_\gamma}\|B(x)\|_{\mathrm{HS}(U,H)}^2 \leq \vartheta < \infty$. Combining (217)–(220) and Corollary 3.4 (with $H = H$, $U = U$, $\mathbb{H} = \mathbb{H}$, $\mathbb{U} = \mathbb{U}$, $T = T$, $\gamma = \gamma$, $\delta = \delta$, $\lambda = \lambda$, $A = A$, $\xi = \xi$, $F = F$, $B = B$, $D_h^I = D_h^I$, $\vartheta = \vartheta$, $b_1 = 0$, $b_2 = \eta$, $\varepsilon = \varepsilon$, $\varsigma = \delta$, $c = c$, $Y^{\theta,I,J} = Y^{\theta,I,J}$ for $h \in (0,T]$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$ in the notation of Corollary 3.4) hence completes the proof of Corollary 4.15. □

Remark 4.16. *Consider the setting of Corollary 4.15. Then the stochastic processes* $Y^{\theta,I,J}\colon [0,T]\times\Omega \to P_I(H)$, $\theta \in \varpi_T$, $I \in \mathcal{P}_0(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{U})$, *are space-time-noise discrete numerical approximation processes for the two-dimensional stochastic Navier-Stokes equations*

$$dX_t(x) = \big[(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2})X_t(x) + (R((\tfrac{\partial}{\partial x}X_t)\cdot X_t))(x)\big]\,dt + b(x, X_t(x))\,d(\sqrt{Q}W)_t(x) \tag{221}$$

with periodic boundary conditions, $(\operatorname{div} X_t)(x) = 0$, *and* $X_0(x) = \xi(x)$ *for* $t \in [0,T]$, $x = (x_1,x_2) \in (0,1)^2$ *(cf., e.g., Section 2 in Da Prato & Debussche [13], Carelli & Prohl [11], Carelli et al. [10], Brzeźniak et al. [9], and Bessaih et al. [3]).*

Acknowledgements

We gratefully acknowledge Zdzisław Brzeźniak for several useful comments that helped to improve the presentation of the results. This project has been supported through the SNSF-Research project 200021_156603 "Numerical approximations of nonlinear stochastic ordinary and partial differential equations".

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