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QMC integration for lognormal-parametric, elliptic PDEs: local supports and product weights *

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Abstract

We analyze convergence rates of quasi-Monte Carlo (QMC) quadratures for countably-parametric solutions of linear, elliptic partial differential equations (PDE) in divergence form with log-Gaussian diffusion coefficient, based on the error bounds in [James A. Nichols and Frances Y. Kuo: Fast CBC construction of randomly shifted lattice rules achieving $\mathcal{O}(N^{-1+\delta})$ convergence for unbounded integrands over $\mathbb{R}^s$ in weighted spaces with POD weights. J. Complexity, 30(4):444-468, 2014].

We prove, for representations of the Gaussian random field PDE input with locally supported basis functions, and for continuous, piecewise polynomial Finite Element discretizations in the physical domain novel QMC error bounds in weighted spaces with *product weights* that exploit localization of supports of the basis elements representing the input Gaussian random field . In this case, the cost of the fast component-by-component algorithm for constructing the QMC points scales linearly in terms of the integration dimension. The QMC convergence rate $\mathcal{O}(N^{-1+\delta})$ (independent of the parameter space dimension s) is achieved under weak summability conditions on the expansion coefficients.

1 Introduction

A particular quasi-Monte Carlo (QMC) quadrature for the approximation of the mean field of (output functions of) the solution of lognormal diffusion problems is analyzed. The lognormal diffusion problem under consideration is an elliptic partial differential equation (PDE) with lognormal stochastic diffusion coefficient a and with deterministic right hand side f. For a bounded Lipschitz domain $D \subset \mathbb{R}^d$, we thus consider

$$-\nabla \cdot (a \nabla u) = f \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{1}$$

Let $\Omega := \mathbb{R}^{\mathbb{N}}$ and define a Gaussian product measure on Ω by

$$\mu(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j\geq 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_j^2}{2}} \mathrm{d}y_j, \quad \boldsymbol{y} \in \Omega.$$

The triplet $(\Omega, \bigotimes_{j\geq 1} \mathcal{B}(\mathbb{R}), \mu)$ is a probability space, cp. for example [3, Example 2.3.5]. We suppose that the Gaussian random field $Z = \log(a) : \Omega \to L^\infty(D)$ is (formally) represented in the following way

$$Z := \sum_{j\geq 1} y_j \psi_j, \tag{2}$$

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where $(\psi_j)_{j\geq 1}$ is a function system of real-valued, bounded, and measurable functions. In particular, with respect to μ the sequence $\boldsymbol{y} = (y_j)_{j\geq 1}$ has independent and identically distributed (i.i.d.) components and for every $j \geq 1$, y_j is standard normally distributed. That is to say, $y_j \sim \mathcal{N}(0,1)$, i.i.d. for $j \in \mathbb{N}$. The lognormal coefficient a in (1) is formally given by

$$a := \exp(Z) \; . \tag{3}$$

For a Banach space B and a strongly measurable mapping $F : \Omega \to B$ that is μ-integrable, denote the expectation with respect to μ by the Bochner integral

$$\mathbb{E}(F) = \int_\Omega F(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}). \tag{4}$$

The lognormal-parametric PDE in (1) is a prominent example for a class of elliptic PDEs with unbounded random coefficients, which was considered in [4, 14, 20, 17, 25]. Specifically, we are interested in approximations of (4) with QMC quadrature. The integrands $F := G(u)$ are linear, continuous functionals $G : H_0^1(D) \to \mathbb{R}$ of the parametric solution u of (1). The evaluation of integrands F for many parameter instances given by QMC points requires solutions of the PDE (1) for multiple realizations of the input Gaussian random field Z in (2) which in general are to be solved numerically. Approximate integrand evaluation through Galerkin Finite Element (FE) discretization introduces a discretization error which is controlled by dimension-independent error bounds.

The assumptions for the QMC convergence theory in [14] on the functions $(\psi_j)_{j\geq 1}$ relied on the p-summability of their $L^\infty(D)$-norms: in [14], it was assumed that for some $p \in (0,1]$

$$\sum_{j\geq 1} \|\psi_j\|^p_{L^\infty(D)} < \infty. \tag{5}$$

The resulting QMC error analysis in [14] requires so called *product and order dependent* (POD) QMC weights. These weights are practically relevant for the construction of the generating vector for the QMC points as they are taken as an input for the construction algorithm. The computational cost with POD weights of the QMC points is $\mathcal{O}(sN\log(N) + s^2N)$, cp. [28, Section 5.2], where s denotes the dimension of the integration domain and where N is the number of QMC points. In the present paper, we extend the QMC convergence theory of [14] by analyzing consequences for the QMC weights due to accounting for possible locality of the supports of the functions $(\psi_j)_{j\geq 1}$ in the representation (2) of the Gaussian random field in the physical domain D. In the analysis in [14, 23], POD QMC weights were essential. As one principal new contribution, we prove a convergence rate of QMC with randomly shifted lattice rules from [28] and *product weights* which is independent of the dimension of integration and which is (essentially) first order with respect to the number N of QMC points, under weak summability conditions on the norms of ψ_j in (2). The computational cost of the construction of QMC points with product weights is $\mathcal{O}(sN\log(N))$, cp. [29, 30], where s is the dimension of integration and N is the number of QMC points. Another main result of this paper is that with the use of Gaussian weight functions in the construction of the QMC points a convergence rate can be shown under considerably weaker assumptions as compared to exponential weight functions. Randomly shifted lattice rules with POD weights and exponential weight functions have been analyzed in [14] and numerical experiments have been reported in [14] as well. Since exponential weight functions have been analyzed before and public domain software for computational QMC rule generation is available, we present the error analysis for them as well; besides being of interest in its own right, this part of the present paper also serves as point of comparison to the (asymptotically stronger) error estimates with Gaussian weight functions. Theoretical analysis of QMC integration with product weights is relevant due to the smaller computational cost and simpler implementation on a computer compared to QMC with POD weights. In [21, Section 5], the authors analyzed the error of multilevel QMC with POD weights in the case of globally supported $(\psi_j)_{j\geq 1}$ for a problem of the type of (1). However, the numerical experiments in [21, Section 4] were performed using QMC with product weights. Similar to what has been shown for N-term convergence rates in [1], in certain cases this can imply significant gains in the convergence rate. Here we analyze, as in the case of affine-parametric coefficients, cp. [12], convergence rates of first order, randomly shifted lattice rules from [28] for function systems $(\psi_j)_{j\geq 1}$ used in the representation (2) which have local supports in D. This is motivated, on the one hand, by the complexity of QMC rules according to product weights scaling linearly with the dimension of integration, cp. [29, 30, 13]. On the other hand, systems of locally supported ψ_j may afford better local resolution of the Gaussian random field Z in D. In particular, if the Gaussian random field is only numerically available, and is neither stationary nor periodic as, e.g., in [9].

Convergence in $L^q(\Omega; L^\infty(D))$, $q \in [1,\infty)$ of the series in (2) will be shown under the assumption that there exists a positive sequence $(b_j)_{j\geq 1}$ such that

$$K := \left\| \sum_{j\geq 1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} < \infty \tag{\textbf{A1}}$$

and that $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (0,\infty)$. The sequence $(b_j)_{j\geq 1}$ will enter the construction of QMC integration rules via the product weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$. These are defined by $\gamma_\emptyset = 1$ and

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} b_j^\rho, \quad \emptyset \neq \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty, \tag{6}$$

where $\rho > 0$ is a constant. With weight sequence $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ and for $p \in (2/3, 2)$, we obtain a convergence rate of $\mathcal{O}(N^{-1/4-1/(2p)+\varepsilon})$ for sufficiently small $\varepsilon > 0$ [1] with a randomly shifted lattice QMC quadrature rule with product weights (6) and Gaussian weight functions. In the case that $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for $p \in (2/3, 1]$, a randomly shifted lattice QMC quadrature rule with exponential weight functions and product weights (6) has a convergence rate of $\mathcal{O}(N^{-1/p+1/2})$. In either case, the implied constants are independent of N, the number of sample points, and of s, the QMC integration dimension.

In Section 2, we review results from [28] on QMC quadrature required in the following. In Section 3, we show integrability and approximation of the lognormal diffusion coefficient, which is applied in Section 5 to estimate the error that is introduced by truncating the expansion of the Gaussian random field. Existence and uniqueness is shown in Section 4. The main parametric regularity estimates are discussed in Section 6, which result in convergence rates of the exact solution in Section 7. Section 8 addresses the impact of a FE discretization in D. Section 9 discusses a particular choice of basis for representation of Gaussian random fields in D, and verifies that this representation satisfies the conditions in our QMC convergence rate analysis. Section 10 specializes the foregoing, general QMC-FE error bounds in terms of the widely used models with Gaussian random fields that have Matérn covariance. Numerical experiments are given in Section 11. Finally, Section 12 presents some conclusions and generalizations.

2 QMC integration of Gaussian random fields

We recapitulate elements from randomly shifted lattice rules and weighted Sobolev spaces that are necessary for the QMC convergence theory, cp. [28, Theorem 8].

[1] Here and throughout, all constants implied in $\mathcal{O}(\cdot)$ are independent of the integration dimension s.

We seek to approximate with a QMC quadrature s-dimensional integrals of the form

$$I_s(F) := \int_{\mathbb{R}^s} F(\boldsymbol{y}) \prod_{j=1}^{s} \phi(y_j) \mathrm{d}\boldsymbol{y}, \tag{7}$$

where $s \in \mathbb{N}$ and ϕ is the standard normal density function, i.e.,

$$\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}.$$

These integrals arise by truncation of expansions of random quantities in particular function systems that are random inputs for PDEs. The integrand F will be the composition of a linear functional $G(\cdot)$ with the solution u. An N-point QMC quadrature for the s-dimensional integral in (7) is an equal-weight quadrature rule and denoted by $Q^{\boldsymbol{\Delta}}_{s,N}$. Here, $\boldsymbol{\Delta} \sim \mathcal{U}((0,1)^s)$ denotes a random shift (see, e.g., [8] and the references there). For every $s, N \in \mathbb{N}$, let us define

$$Q^{\boldsymbol{\Delta}}_{s,N}(F) := \frac{1}{N} \sum_{i=0}^{N-1} F(\boldsymbol{y}^{(i)}),$$

with judiciously chosen points $\{\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(N-1)}\} \subset \mathbb{R}^s$.

The s-dimensional cumulative distribution function Φ_s corresponding to the probability density ϕ is defined by

$$\Phi_s(\boldsymbol{y}) := \int_{\boldsymbol{y}' \leq \boldsymbol{y}} \prod_{j=1}^{s} \phi(y'_j) \, \mathrm{d}\boldsymbol{y}', \quad \boldsymbol{y} \in \mathbb{R}^s,$$

where $\boldsymbol{y}' \leq \boldsymbol{y}$ is understood as $y'_j \leq y_j$ for $j = 1, \ldots, s$. In the case that $s = 1$ we omit the subscript. For *randomly shifted lattice rules* the QMC points are obtained by

$$\boldsymbol{y}^{(i)} := \Phi_s^{-1}\left(\left\{\frac{(i+1)\boldsymbol{z}}{N} + \boldsymbol{\Delta}\right\}\right), \quad i = 0, \ldots, N-1, \tag{8}$$

where $\boldsymbol{z}$ is a generating vector, and for every $c \in (0, \infty)$, $\{c\} \in [0,1)$ denotes the fractional part of c. We refer to the surveys [8, 22] for further details and references.

The integrands in (7) that are under consideration in this paper belong to weighted, unanchored Sobolev spaces. The error analysis of randomly shifted lattice rules involves spaces of type $\mathcal{W}_{\boldsymbol{\gamma}}$, which require *weight functions* for their definition. In this paper we consider two particular kinds of coordinate weight functions. Specifically, let us define the Gaussian weight functions with "variance" $\alpha_{\mathrm{g}} > 1$

$$w^2_{\mathrm{g},j}(y) := e^{-\frac{y^2}{2\alpha_{\mathrm{g}}}}, \quad y \in \mathbb{R}, j \in \mathbb{N}, \tag{9}$$

and the exponential weight functions with variance $\alpha_{\exp}{}^{-2} > 0$

$$w^2_{\exp,j}(y) := e^{-\alpha_{\exp}|y|}, \quad y \in \mathbb{R}, j \in \mathbb{N}\,. \tag{10}$$

For a Hilbert space H and for a collection of positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, define the weighted Sobolev space $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$ as a Bochner space of strongly measurable functions from $\mathbb{R}^s$ taking values in the separable Hilbert space H that have finite $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$-norm. Here, for finite parameter dimension $s \in \mathbb{N}$, the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$-norm of *unanchored, mixed first order partial derivatives* is defined by

$$\|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;H)} := \left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left\| \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \partial^{\mathfrak{u}} F(\boldsymbol{y}) \prod_{j \in \{1:s\} \setminus \mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}_{\{1:s\} \setminus \mathfrak{u}} \right\|_H^2 \prod_{j \in \mathfrak{u}} w_j^2(y_j) \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2}, \tag{11}$$

where $(w_j)_{j\geq 1}$ denotes either of the weight functions defined in (9) or in (10) and where the inner integral is understood as a Bochner integral (cp. for example [35, Chapter V.5]). In the case that $H = \mathbb{R}$, we simply write $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$. We used the notation that $\{1:s\}$ denotes the set of integers $\{1,\ldots,s\}$ and for $\boldsymbol{y} \in \mathbb{R}^s$ and $\mathfrak{u} \subset \{1:s\}$, $\boldsymbol{y}_{\mathfrak{u}}$ denotes the coordinates $(y_j)_{j\in\mathfrak{u}}$ of $\boldsymbol{y}$. We recall a version of [28, Theorem 8] for our choices of weight functions in (9) and in (10).

Theorem 1 *Let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subseteq\{1:s\}}$ be some product weights, $s \in \mathbb{N}$ the truncation level, and $(w_j)_{j\geq 1}$ be either of the weight functions defined in (9) or in (10). Then, a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN\log N)$ operations using the fast CBC algorithm of [30, 29] such that for every $F \in \mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$ and for every $\lambda \in (1/(2r), 1]$ there holds the error bound*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(F) - Q^{\boldsymbol{\Delta}}_{s,N}(F)|^2)} \leq \left((\varphi(N))^{-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \prod_{j\in\mathfrak{u}} \rho(\lambda) \right)^{1/(2\lambda)} \|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)},$$

where Euler's totient function is denoted by $\varphi(\cdot)$. For weight functions $(w_{\mathrm{g},j})_{j\geq 1}$ defined in (9),

$$\rho(\lambda) = 2\left(\frac{4\sqrt{2\pi}\alpha_{\mathrm{g}}^2}{\pi^{2-1/\alpha_{\mathrm{g}}}(2\alpha_{\mathrm{g}}-1)} \right)^{\lambda} \zeta(2r\lambda) \quad \text{and} \quad r = 1 - \frac{1}{2\alpha_{\mathrm{g}}}$$

and for weight functions $(w_{\exp,j})_{j\geq 1}$ defined in (10)

$$\rho(\lambda) = 2\left(\frac{\sqrt{2\pi}\exp(\alpha_{\exp}^2/(4\delta))}{\pi^{2-2\delta}(1-\delta)\delta} \right)^{\lambda} \zeta\,(2r\lambda) \quad \text{and} \quad r = 1-\delta \quad \text{for any } \delta \in \left(0, \frac{1}{2}\right).$$

This result is [28, Theorem 8]. The value of the first factor in the expression $\rho(\lambda)$ and the values of r that correspond to the weight functions $(w_{\mathrm{g},j})_{j\geq 1}$ and $(w_{\exp,j})_{j\geq 1}$ are derived in [24, Example 4 and Example 5], respectively.

3 Lognormal random fields

The Gaussian random field Z is in (2) formally defined as the limit of a series expansion with i.i.d. standard normally distributed coefficients. For any Banach space $(B, \|\cdot\|_B)$ and every $q \in [1,\infty)$, let $L^q(\Omega; B)$ denote the space of all strongly measurable mappings $X : \Omega \to B$ such that $\|X\|_B^q$ is μ-integrable. We note that the measurability of $\|X\|_B$ follows from the strong measurability of X in B. We investigate convergence of the series in (2) in the following theorem. In its proof and in what follows, for $s \in \mathbb{N}$, we define partial sums of (2)

$$Z^s := \sum_{j=1}^{s} y_j \psi_j \,. \tag{12}$$

Theorem 2 *Let the assumption in (**A1**) be satisfied for some $p_0 \in (0,\infty)$ and for some $K \in (0,\infty)$. Then, the Gaussian random field Z is well defined and for every $q \in [1,\infty)$, $Z \in L^q(\Omega; L^\infty(D))$. Moreover, for every $\varepsilon \in (0,1)$ there exists $C_{q,\varepsilon} > 0$ such that*

$$\|Z - Z^s\|_{L^q(\Omega;L^\infty(D))} \leq C_{q,\varepsilon} \sup_{j>s} \left\{ b_j^{1-\varepsilon} \right\},$$

where for $r \in \mathbb{N}$ such that $r \geq \max\{p_0/(2\varepsilon), \lceil q/2 \rceil\}$, the constant $C_{q,\varepsilon}$ is given by

$$C_{q,\varepsilon} := K\|(b_j)_{j\geq 1}\|_{\ell^{p_0}(\mathbb{N})}^{\varepsilon} \frac{\sqrt{2}}{\pi^{1/4}} \sqrt{r}.$$

Proof. For a sequence of i.i.d. standard normally distributed variables $(y_j)_{j\geq 1}$ and for every $s\in\mathbb{N}$, the finite sum $Z^s = \sum_{j=1}^s y_j\psi_j$ is weakly measurable, i.e., for every $\ell\in(L^\infty(D))^*$, where $(L^\infty(D))^*$ denotes the dual space of $L^\infty(D)$, $\ell(Z^s)=\sum_{j=1}^s y_j\ell(\psi_j)$ is measurable as a finite sum of real valued random variables.

Since the span$\{\psi_j : j\in\{1:s\}\}$ is finite dimensional, Z^s is separably valued in span$\{\psi_j : j\in\{1:s\}\}\subset L^\infty(D)$. Pettis' theorem (e.g. cp. [35, Theorem V.4]) implies that Z^s is also strongly measurable in the space $L^\infty(D)$. Let $s<s_1\in\mathbb{N}$ be arbitrary. We observe that

$$\|Z^{s_1}-Z^s\|_{L^q(\Omega;L^\infty(D))} \leq \left\|\sum_{s<j\leq s_1}\frac{|\psi_j|}{b_j}\right\|_{L^\infty(D)} \left\|\sup_{s<j\leq s_1}\{b_j|y_j|\}\right\|_{L^q(\Omega)} .$$

We set $q' := \lceil q/2\rceil$ such that $2q' = 2\lceil q/2\rceil$ is the smallest even natural number that is greater or equal than q. We pick $r\in\mathbb{N}$ such that $2\varepsilon r\geq p_0$ and such that $r\geq q'$ and conclude with the Jensen inequality for concave functions and with the norm estimate $\|\cdot\|_{\ell^\infty(\{s+1:s_1\})}\leq \|\cdot\|_{\ell^{2r}(\{s+1:s_1\})}$

$$\begin{aligned}
\mathbb{E}\left(\left(\sup_{s<j\leq s_1}\{b_j|y_j|\}\right)^{2q'}\right) &\leq \left(\sup_{s<j\leq s_1}\left\{b_j^{1-\varepsilon}\right\}\right)^{2q'}\mathbb{E}\left(\left(\sum_{s<j\leq s_1} b_j^{2\varepsilon r}|y_j|^{2r}\right)^{q'/r}\right)\\
&\leq \left(\sup_{s<j\leq s_1}\left\{b_j^{1-\varepsilon}\right\}\right)^{2q'}\left(\mathbb{E}\left(\sum_{s<j\leq s_1} b_j^{2\varepsilon r}|y_j|^{2r}\right)\right)^{q'/r}\\
&\leq \left(\sup_{s<j\leq s_1}\left\{b_j^{1-\varepsilon}\right\}\right)^{2q'}\left(\sum_{s<j\leq s_1} b_j^{2\varepsilon r}\frac{(2r)!}{2^r r!}\right)^{q'/r}\\
&\leq \left(\sup_{s<j\leq s_1}\left\{b_j^{1-\varepsilon}\right\}\right)^{2q'}\|(b_j)_{j\geq 1}\|_{\ell^{p_0}(\mathbb{N})}^{2\varepsilon q'}\left(\frac{(2r)!}{2^r r!}\right)^{q'/r},
\end{aligned}$$

where we used the fact that for a random variable $X\sim\mathcal{N}(0,1)$, $\mathbb{E}(X^{2r}) = (2r)!/(2^r r!)$. The assumption $(b_j)_{j\geq1}\in\ell^{p_0}(\mathbb{N})$ implies that $(Z^{s'})_{s'\geq 1}$ is a Cauchy sequence in $L^q(\Omega;L^\infty(D))$, which is a Banach space (cp. [10, Theorem III.6.6]). Define $\widetilde{Z}$ to be the unique limit of $(Z^{s'})_{s'\geq1}$ in $L^q(\Omega;L^\infty(D))$. The continuous embedding $L^{q_2}(\Omega;L^\infty(D))\subset L^{q_1}(\Omega;L^\infty(D))$, for every $q_1\leq q_2\in[1,\infty)$, implies that the limit $\widetilde{Z}$ does not depend on q. We denote this limit by Z. As an element of $L^q(\Omega;L^\infty(D))$, Z is a $L^\infty(D)$-valued μ-equivalence class.

The Stirling bounds $\sqrt{2\pi}n^{n+1/2}e^{-n}\leq n!\leq en^{n+1/2}e^{-n}$, for every $n\in\mathbb{N}$, cp. [11, Equations (9.5) and (9.8)], imply the assertion of the proposition with

$$\left(\frac{(2r)!}{2^r r!}\right)^{1/(2r)} \leq 2^{1/2}\left(\frac{e}{\sqrt{\pi}}\right)^{1/(2r)}\sqrt{\frac{r}{e}} .$$

□

For the partial sum Z^s in (12), we define

$$a^s := \exp(Z^s) , \quad \text{for every} \quad s\in\mathbb{N} .$$

Proposition 3 *Let the assumption in* (**A1**) *be satisfied for some* $p_0\in(0,\infty)$ *and for* $K\in(0,\infty)$. *Then, for every* $q\in[1,\infty)$, $a\in L^q(\Omega;L^\infty(D))$ *and there exists a constant* $C>0$ *such that for every* $s\in\mathbb{N}$

$$\|a^s\|_{L^q(\Omega;L^\infty(D))}\leq C .$$

Proof. While the space $L^\infty(D)$ is not separable, the strong measurability of $a = \exp(Z)$ in $L^\infty(D)$ follows because the composition with the exponential function is a continuous mapping from $L^\infty(D)$ to $L^\infty(D)$. The proof of this proposition is based on an application of Fernique's theorem (e.g. cp. [3, Theorem 2.8.5] or [7, Theorem 2.7]). We verify the conditions in order to apply Fernique's theorem. Our approach is similar to the proof of [17, Proposition B.1]. We detail the argument for the convenience of the reader. We claim that for every $\ell \in (L^\infty(D))^*$, $\ell(Z)$ is centered, normally distributed, i.e., the law of Z is a centered Gaussian measure on $L^\infty(D)$. Indeed, for arbitrary $s' \in \mathbb{N}$, $\ell(Z^{s'}) \sim \mathcal{N}(0, \sum_{j=1}^{s'} \ell(\psi_j)^2)$. Since

$$\sum_{j=1}^{s'} \ell(\psi_j)^2 \leq \left(\sum_{j=1}^{s'} \ell(\psi_j) \right)^2 = \left(\ell \left(\sum_{j=1}^{s'} \psi_j \right) \right)^2 \leq \|\ell\|^2_{(L^\infty(D))^*} K^2 \sup_{j \geq 1} \left\{ b_j^2 \right\},$$

the monotone sequence $\sum_{j=1}^{s'} \ell(\psi_j)^2$ indexed by $s' \in \mathbb{N}$ is bounded and hence has a finite limit that we denote by σ_ℓ^2. This implies that for fixed $\ell \in (L^\infty(D))^*$, the characteristic functions of the random variables $(\ell(Z^{s'}) : s' \in \mathbb{N})$ converge pointwise to the characteristic function of a $\mathcal{N}(0, \sigma_\ell^2)$ distributed random variable as $s' \to \infty$. Since $\ell(Z^{s'})$ converges to $\ell(Z)$ as $s' \to \infty$ in particular in the L^2-sense by Theorem 2 and thus also in distribution, Lévy's continuity theorem (e.g. cp. [26, Theorem IV.13.2.B]) implies that $\ell(Z) \sim \mathcal{N}(0, \sigma_\ell^2)$ and we conclude that the law of Z is a Gaussian measure on $L^\infty(D)$, which is one of the conditions of Fernique's theorem.

We will treat the case $s < \infty$ first. By Theorem 2, there exists an upper bound C of the $L^2(\Omega; L^\infty)$-norm of the Gaussian random fields Z and Z^s, that is independent of s. The existence of this uniform upper bound C is the main ingredient of the remaining argument. Let in the following $X \in \{Z, Z^s\}$ be arbitrary. Let $\kappa_1 \in (1/(1+\exp(-2)), 1)$ and set $\kappa_2 := C/\sqrt{1-\kappa_1}$ and conclude with the Chebychev inequality that

$$1 - \mu(\|X\|_{L^\infty(D)} \leq \kappa_2) = \mu(\|X\|_{L^\infty(D)} > \kappa_2) \leq \frac{\mathbb{E}(\|X\|^2_{L^\infty(D)})}{\kappa_2^2} \leq \frac{C^2}{\kappa_2^2} = 1 - \kappa_1.$$

Hence, $\mu(\|X\|_{L^\infty(D)} \leq \kappa_2) \geq \kappa_1 > 1/(1+\exp(-2)) > 1/2$. Let us set $\lambda := (1-\kappa_1)/(32C^2)$, which implies that $32\lambda\kappa_2^2 \leq 1$. Thus, by the monotonicity of the logarithm

$$\log \left(\frac{1 - \mu(\|X\|_{L^\infty(D)} \leq \kappa_2)}{\mu(\|X\|_{L^\infty(D)} \leq \kappa_2)} \right) + 32\lambda\kappa_2^2 \leq \log \left(\frac{1-\kappa_1}{\kappa_1} \right) \leq -1.$$

This is the second requirement in order to apply [7, Theorem 2.7]. Since Z^s is in particular a Gaussian measure on the separable Banach space $\text{span}\{\psi_j : j \in \{1 : s\}\}$ with respect to the $L^\infty(D)$-norm, [7, Theorem 2.7] implies that

$$\mathbb{E}(\exp(\lambda \|Z^s\|^2_{L^\infty(D)})) \leq \exp(16\lambda\kappa_2^2) + \frac{\exp(2)}{\exp(2)-1}. \tag{13}$$

Since κ_2 and λ do not depend on s (because C does not), the upper bound in (13) is uniform with respect to s. For every $x \in \mathbb{R}$, $qx \leq \lambda x^2 + q^2/(4\lambda)$ is concluded from $0 \leq (\sqrt{\lambda}x - q/(2\sqrt{\lambda}))^2$, which yields the second assertion of the proposition, i.e.,

$$\mathbb{E}(\|\exp(Z^s)\|^q_{L^\infty(D)}) \leq \mathbb{E}(\exp(\lambda \|Z^s\|_{L^\infty(D)})) \exp\left(\frac{q^2}{4\lambda} \right). \tag{14}$$

The case of Z (corresponding formally to the case of $s = \infty$) is treated separately. Since $L^\infty(D)$ is not separable, [7, Theorem 2.7] is not applicable. We argue with [3, Theorem 2.8.5] instead. To this end, we define

$$\widehat{\lambda} := \frac{1}{24\kappa_2^2} \log \left(\frac{\mu(\|Z\|_{L^\infty(D)} \leq \kappa_2)}{1 - \mu(\|Z\|_{L^\infty(D)} \leq \kappa_2)} \right),$$

which is strictly positive because $\kappa_1 > 1/2$. Then, [3, Theorem 2.8.5] is applicable and we obtain

$$\mathbb{E}(\exp(\widehat{\lambda}\|Z\|^2_{L^\infty(D)})) < \infty,$$

which implies as above, cp. (14), that $\mathbb{E}(\|\exp(Z)\|^q_{L^\infty(D)}) < \infty$. □

We note that the line of argument in the second paragraph of the proof originates from the proof of [4, Proposition 3.10].

Remark 4 *The property that $a = \exp(Z) \in L^q(\Omega; L^\infty(D))$, for every $q \in [1, \infty)$, also holds under weaker summability assumptions on $(b_j)_{j\geq 1}$, cp. [1, Theorem 2.2] that was proven with a different approach. However the membership of $(b_j)_{j\geq 1}$ in $\ell^p(\mathbb{N})$ for a certain range of p (as assumed in Proposition 3) seems indispensible for the considered QMC rules to be applicable, cp. Section 7. Also, our argument yields bounds of truncated expansions of Gaussian random fields that are uniform in s.*

Proposition 5 *Let the assumption in* (**A1**) *be satisfied for some $p_0 \in (0, \infty)$ and for $K \in (0, \infty)$. Then, for every $q \in [1, \infty)$ and every $\varepsilon \in (0, 1)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\|a - a^s\|_{L^q(\Omega;L^\infty(D))} \leq C \sup_{j>s} \left\{ b_j^{1-\varepsilon} \right\},$$

Proof. The fundamental theorem of calculus implies that for every $t_1, t_2 \in \mathbb{R}$, $|e^{t_2} - e^{t_1}| \leq (e^{t_2} + e^{t_1})|t_2 - t_1|$. Thus, by the Cauchy–Schwarz inequality

$$\|a - a^s\|_{L^q(\Omega;L^\infty(D))} \leq \|a + a^s\|_{L^{2q}(\Omega;L^\infty(D))}\|Z - Z^s\|_{L^{2q}(\Omega;L^\infty(D))}.$$

The assertion follows with the triangle inequality, Theorem 2, and Proposition 3. □

Our ensuing analysis of the solution to (1) will require the following random variables:

$$a_{\min} := \operatorname*{ess\,inf}_{x\in D}\{a(x)\},\ a_{\max} := \|a\|_{L^\infty(D)},\ a^s_{\min} := \operatorname*{ess\,inf}_{x\in D}\{a^s(x)\},\ a^s_{\max} := \|a^s\|_{L^\infty(D)}\ .$$

Here, $s \in \mathbb{N}$ is arbitrary.

Corollary 6 *Let the assumption of Proposition 3 be satisfied. Then, for every $q \in [1, \infty)$, $a^{-1}_{\min} \in L^q(\Omega; L^\infty(D))$ and there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\left\| \frac{1}{a^s_{\min}} \right\|_{L^q(\Omega;L^\infty(D))} \leq C.$$

4 Existence and uniqueness

In this paper we are interested in mean field approximations. We consider the solution to (1) as a μ-equivalence class taking values in $V := H^1_0(D)$. The existence and uniqueness of the solution to (1) is well known, cp. [4, Proposition 2.4]; we review the basic results, following the presentation in [17, Section 3.1].

Since the right hand side in (1) is deterministic, we are interested in the data-to-solution map $\mathcal{S}_f$ that maps a (realization of the) diffusion coefficient $\widehat{a} \in L^\infty(D)$ to the solution $\widehat{u} \in V$ for fixed right hand side $f \in V^*$, where V^* denotes the dual space of V. In what follows, we fix $f \in V^*$

unless explicitly stated otherwise. For every $\widehat{a} \in L^\infty_+(D) := \{\widetilde{a} \in L^\infty(D) : \operatorname{ess\,inf}_{x\in D}\{\widetilde{a}(x)\} > 0\}$, consider the deterministic diffusion equation problem: find a unique $\widehat{u} \in V$ such that

$$\int_D \widehat{a}\, \nabla \widehat{u} \cdot \nabla v \,\mathrm{d}x = f(v), \quad \forall v \in V. \tag{15}$$

For such $\widehat{a}$, the bilinear form $(w, v) \mapsto \int_D \widehat{a}\, \nabla w \cdot \nabla v \,\mathrm{d}x$ is continuous and coercive on $V \times V$, since by $\widehat{a} \in L^\infty_+(D)$

$$\left| \int_D \widehat{a}\, \nabla w \cdot \nabla v \,\mathrm{d}x \right| \le \|\widehat{a}\|_{L^\infty(D)} \|w\|_V \|v\|_V, \quad \forall w, v \in V,$$

and

$$\int_D \widehat{a}\, \nabla w \cdot \nabla w \,\mathrm{d}x \ge \operatorname*{ess\,inf}_{x\in D}\{\widehat{a}(x)\} \|w\|_V^2, \quad \forall w \in V.$$

The Lax–Milgram lemma implies that the problem in (15) is well posed. Thus, for every fixed $f \in V^*$ the mapping

$$\mathcal{S}_f : L^\infty_+(D) \to V : \widehat{a} \mapsto \widehat{u}$$

is well defined. Moreover, the Lax–Milgram lemma implies that for every $\widehat{a} \in L^\infty_+(D)$

$$\|\mathcal{S}_f(\widehat{a})\|_V \le \frac{1}{\operatorname{ess\,inf}_{x\in D}\{\widehat{a}(x)\}} \|f\|_{V^*}. \tag{16}$$

Also it is well known that $\mathcal{S}_f : L^\infty_+(D) \to V$ is locally Lipschitz continuous, which can be shown by the second Strang lemma (see (18) ahead or [17] for details).

Let in the following a denotes the lognormal random field in Proposition 3. The weak (or variational) formulation of the parametric, elliptic PDE (1) for fixed, deterministic $f \in V^*$ reads: find a strongly measurable V-valued mapping $u : \Omega \to V$ such that

$$\int_D a\, \nabla u \cdot \nabla v \,\mathrm{d}x = f(v), \quad \forall v \in V\ . \tag{17}$$

Due to Proposition 3, a is strongly measurable in $L^\infty(D)$ and by Corolloary 6, $a_{\min} > 0$ μ-almost surely (a.s.). Hence, a takes values in $L^\infty_+(D)$ and $u := \mathcal{S}_f(a)$ is the unique solution to (17), where we recall that uniqueness is meant as V-valued μ-equivalence class. The strong measurability in V of u is deduced from the strong measurability of a and the continuity of $\mathcal{S}_f$. By (16) and Corollary 6, for every $q \in [1, \infty)$ there holds

$$\|u\|_{L^q(\Omega;V)} \le \left\| \frac{1}{a_{\min}} \right\|_{L^q(\Omega)} \|f\|_{V^*} < \infty\ .$$

5 Dimension truncation

In applications of QMC integration a finite dimensional integration domain is required, which in our case will be $\mathbb{R}^s$ for $s \in \mathbb{N}$. Truncation of the series in (2) will introduce a truncation error. For every $s \in \mathbb{N}$, $u^s := \mathcal{S}_f(a^s)$ uniquely solves

$$\int_D a^s\, \nabla u^s \cdot \nabla v \,\mathrm{d}x = f(v), \quad \forall v \in V.$$

Proposition 7 *Let the assumption in* (**A1**) *be satisfied for* $(b_j)_{j\ge1} \in \ell^{p_0}$ *for some* $p_0 \in (0, \infty)$ *and for some* $K > 0$. *Let further* $G(\cdot) \in V^*$ *and* $\varepsilon \in (0, 1)$ *be arbitrary. For every* $q \in [1, \infty)$ *there exists a positive constant* C_ε *that is independent of* f *and such that for every* $s \in \mathbb{N}$

$$\|u - u^s\|_{L^q(\Omega;V)} \le C_\varepsilon \|f\|_{V^*} \sup_{j>s} \left\{ b_j^{1-\varepsilon} \right\}.$$

Suppose further that the sequence $(b_j)_{j\geq 1}$ in assumption (**A1**) *belongs to $\ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0,4]$. Then, there exists a constant $C > 0$ independent of f and of $G(\cdot)$ such that for every $s \in \mathbb{N}$*

$$|\mathbb{E}(G(u)) - I_s(G(u^s))| \leq C\|G(\cdot)\|_{V^*}\|f\|_{V^*} \sup_{j>s}\left\{b_j^{2-p_0/2}\right\}.$$

Proof. The second Strang lemma implies that

$$\|u - u^s\|_V \leq \frac{1}{a_{\min}\, a^s_{\min}}\|a - a^s\|_{L^\infty(D)}\|f\|_{V^*}\ . \tag{18}$$

By the Hölder inequality

$$\|u - u^s\|_{L^q(\Omega;V)} \leq \left\|\frac{1}{a_{\min}}\right\|_{L^{3q}(\Omega)} \left\|\frac{1}{a^s_{\min}}\right\|_{L^{3q}(\Omega)} \|a - a^s\|_{L^{3q}(\Omega;L^\infty(D))}\|f\|_{V^*}.$$

The first assertion follows with Proposition 5 and Corollary 6.

For the proof of the second estimate, we introduce the Wiener–Hermite polynomial chaos expansion of the solution u

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\tau}\in\mathcal{F}} u_{\boldsymbol{\tau}} H_{\boldsymbol{\tau}}(\boldsymbol{y}), \quad u_{\boldsymbol{\tau}} := \int_\Omega u(\boldsymbol{y})H_{\boldsymbol{\tau}}(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}), \quad H_{\boldsymbol{\tau}}(\boldsymbol{y}) := \prod_{j\geq 1} H_{\tau_j}(y_j), \tag{19}$$

where $H_k(y)$ is the k-th order univariate Hermite polynomial such that H_k, $k \in \mathbb{N}_0$, are normalized in $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\mathrm{d}y)$. Note that $\mathcal{F} := \{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}} : \sum_{j\geq 1}\tau_j < \infty\}$. As a consequence, $(H_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\mathcal{F}}$ is an orthonormal basis of $L^2(\Omega, \mu; V)$. This implies that the polynomial chaos expansion in (19) converges in $L^2(\Omega, \mu; V)$. Since $H_0(y) = 1$ and $H_{2k-1}(0) = 0$, $k \in \mathbb{N}$,

$$\mathbb{E}(u) - \mathbb{E}(u^s) = -\sum_{\boldsymbol{0}\neq\boldsymbol{\tau}\in\mathcal{F}} u_{\boldsymbol{\tau}} \prod_{j=1}^{s}\left(\int_{\mathbb{R}} H_{\tau_j}(y_j)\frac{1}{\sqrt{2\pi}}e^{-y_j^2/2}\mathrm{d}y_j\right)\prod_{j>s} H_{\tau_j}(0) = -\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} u_{\boldsymbol{\tau}} H_{\boldsymbol{\tau}}(\boldsymbol{0}), \tag{20}$$

where

$$\mathcal{F}_2^s := \{\boldsymbol{\tau} \in \mathcal{F} : \boldsymbol{\tau} \neq \boldsymbol{0}, \tau_j = 0, j = 1, \ldots, s, \tau_j \in 2\mathbb{N}_0, j > s\}$$

and $2\mathbb{N}_0 = \{0, 2, 4, \ldots\}$ denotes the set of even, nonnegative integers. Estimates of Hermite polynomials are given in [32, Chapter 1]. There, the author uses a convention that $\widetilde{H}_k(y) = (-1)^k e^{y^2}\frac{\mathrm{d}^k}{\mathrm{d}y^k}e^{-y^2}$ (in [32, Chapter 1] also denoted by $(H_k)_{k\geq 0}$). Note that $H_k(y) = \widetilde{H}_k(y/\sqrt{2})/\sqrt{2^k k!}$, $k \in \mathbb{N}_0$, $y \in \mathbb{R}$. By [32, Equations (1.1.2), (1.1.18), and (1.1.21)],

$$(H_{2k}(0))^2 = \frac{1}{\sqrt{\pi}}\frac{\Gamma(k + 1/2)}{\Gamma(k + 1)} \leq 1, \quad k \in \mathbb{N}_0,$$

where $\Gamma(\cdot)$ denotes the Gamma function. Weighted $\ell^2(\mathcal{F})$-summability of the polynomial chaos coefficients in this setting has been analyzed in [1, Section 5]. By [1, Theorems 3.3 and 4.1], there exists a constant $C > 0$ such that

$$\sum_{\boldsymbol{\tau}\in\mathcal{F}} B_{\boldsymbol{\tau}}\|u_{\boldsymbol{\tau}}\|_V^2 \leq C\mathbb{E}\left(\frac{a_{\max}}{a_{\min}}\|u\|_V^2\right), \tag{21}$$

where for arbitrary integer $r \in \mathbb{N}$ and some $\kappa \in (0, \log(2)/(\sqrt{r}K))$ with K as in (**A1**)

$$B_{\boldsymbol{\tau}} := \prod_{j\geq 1}\left(\sum_{\ell=0}^{r\wedge\tau_j}\binom{\tau_j}{\ell}\rho_j^{2\ell}\right) \quad \text{and} \quad \rho_j := \frac{\kappa}{b_j}, j \geq 1.$$

In the following we choose $r=2$ and obtain the lower bound $B_{\boldsymbol{\tau}} \geq \prod_{j\in\operatorname{supp}(\boldsymbol{\tau})} \rho_j^{2(2\wedge\tau_j)}$. We note that for real numbers $y_1, y_2 \in \mathbb{R}$, $y_1 \wedge y_2 := \min\{y_1, y_2\}$. Let $(A_{\boldsymbol{\tau}})_{\boldsymbol{\tau}\in\mathcal{F}_2^s} \in \ell^2(\mathcal{F}_2^s)$. By (20) and (21) and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
|\mathbb{E}(G(u)) - I_s(G(u^s))| &\leq \|G(\cdot)\|_{V^*} \left(\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} \|u_{\boldsymbol{\tau}}\|_V^2 A_{\boldsymbol{\tau}}^{-2}\right)^{1/2} \left(\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} A_{\boldsymbol{\tau}}^2\right)^{1/2} \\
&\leq \|G(\cdot)\|_{V^*} \sup_{\boldsymbol{\tau}\in\mathcal{F}_2^s} \{B_{\boldsymbol{\tau}}^{-1/2} A_{\boldsymbol{\tau}}^{-1}\} \left(\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} B_{\boldsymbol{\tau}} \|u_{\boldsymbol{\tau}}\|_V^2\right)^{1/2} \left(\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} A_{\boldsymbol{\tau}}^2\right)^{1/2}.
\end{aligned}
$$

We choose for some $c \in (0, 1/\|(b_j)_{j\geq 1}\|_{\ell^\infty(\mathbb{N})})$

$$
A_{\boldsymbol{\tau}} := \prod_{j\geq 1} (cb_j)^{\tau_j p_0/4}, \quad \boldsymbol{\tau} \in \mathcal{F}_2^s,
$$

and observe that

$$
\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} A_{\boldsymbol{\tau}}^2 = \prod_{j>s} \sum_{k\geq 0} (cb_j)^{(p_0/4)4k} = \prod_{j>s} \frac{1}{1-(cb_j)^{(p_0/4)4}} \leq \exp\left(\sum_{j>s} \frac{(cb_j)^{p_0}}{1-(cb_j)^{p_0}}\right) < \infty, \ .
$$

Here, we used the estimate $1 + x \leq \exp(x)$, $x \geq 0$, and the fact that the multiindices in $\mathcal{F}_2^s$ only have even entries. We conculde now that there exists a constant $C > 0$ such that

$$
|\mathbb{E}(G(u)) - I_s(G(u^s))| \leq C \sup_{j>s} \left\{ b_j^{2-p_0/2} \right\} \|G(\cdot)\|_{V^*} \|f\|_{V^*},
$$

where we crucially used that multiindices in $\mathcal{F}_2^s$ satisfy that $\tau_j = 0$, $j = 1, \ldots, s$, and $\tau_j \neq 1$, $j > s$. □

Realizations of the Gaussian random field Z^s can be obtained from Gaussian vectors $\boldsymbol{y} \in \mathbb{R}^s$. Specifically, one realization of Z^s requires s draws of independent, standard normally distributed random variables which results in a vector $(y_1, \ldots, y_s)^\top \in \mathbb{R}^s$. Since the support of the s-dimensional multivariate Gaussian measure on $\mathbb{R}^s$ with covariance equal to the identity is $\mathbb{R}^s$, the whole of $\mathbb{R}^s$ is the parameter set. We denote realizations of Z^s by $Z^s(\boldsymbol{y}) := \sum_{j=1}^s y_j \psi_j$, where $\boldsymbol{y} = (y_1, \ldots, y_s)^\top \in \mathbb{R}^s$ is the particular realization of the i.i.d. standard normally distributed coefficient sequence $(y_j)_{1\leq j\leq s}$. Moreover, for every $s \in \mathbb{R}^s$, Z^s also denotes the respective mapping from $\mathbb{R}^s$ to $L^\infty(D)$. Similarly, for every $s \in \mathbb{R}^s$, a^s also denotes the respective mapping from $\mathbb{R}^s$ to $L^\infty_+(D)$, $a^s_{\min}$ and $a^s_{\max}$ also denote the respective mappings from $\mathbb{R}^s$ to $(0, \infty)$, and u^s also denotes the respective mapping from $\mathbb{R}^s$ to V.

6 Parametric regularity

By the definition of the weighted Sobolev norm in (11), it is crucial for the QMC convergence analysis to derive estimates of the mixed partial derivatives $\partial^{\mathfrak{u}} u^s$, $\mathfrak{u} \subset \{1:s\}$, in order to bound the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; V)$-norm of u^s uniformly in the parameter dimension s.

Bounds on the parametric partial derivatives of the solution u^s have been proven in [20, 14, 1]. It is well known that for every $\mathbf{0} \neq \boldsymbol{\tau} \in \mathbb{N}_0^s$ and for every $\boldsymbol{y} \in \mathbb{R}^s$ there holds

$$
\int_D a^s(\boldsymbol{y}) \nabla \partial^{\boldsymbol{\tau}} u^s(\boldsymbol{y}) \cdot \nabla v \,\mathrm{d}x = - \int_D \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}, \boldsymbol{\nu}\neq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \prod_{j\in\operatorname{supp}(\boldsymbol{\tau})} \psi_j^{\tau_j - \nu_j} a^s(\boldsymbol{y}) \nabla \partial^{\boldsymbol{\nu}} u^s(\boldsymbol{y}) \cdot \nabla v \,\mathrm{d}x, \quad \forall v \in V, \tag{22}
$$

cp. for example [1, Lemma 3.1] and see also [20, Equation (3.6)]. The arguments in [20, 14] rely on global bounds of the functions $(\psi_j)_{j\geq 1}$. Specifically, the $L^\infty(D)$-norm of the functions $(\psi_j)_{j\geq 1}$ was (in these references) taken inside the summation over multiindices in (22). This way information of locality of the support of the functions $(\psi_j)_{j\geq 1}$ is lost. For the quantitative analysis of parametric regularity, we introduce for every $s\in\mathbb{N}$, for every $\boldsymbol{y}\in\mathbb{R}^s$ and every $v\in V$ the parametrized energy norm $\|v\|_{a^s(\boldsymbol{y})}$ by

$$\|v\|_{a^s(\boldsymbol{y})} := \sqrt{\int_D a^s(\boldsymbol{y})\,|\nabla v|^2\,\mathrm{d}x}\;.$$

For every $\boldsymbol{y}\in\mathbb{R}^s$ and every $v\in V$ there holds

$$(a^s_{\min}(\boldsymbol{y}))^{1/2}\|v\|_V \leq \|v\|_{a^s(\boldsymbol{y})} \leq (a^s_{\max}(\boldsymbol{y}))^{1/2}\|v\|_V\;. \tag{23}$$

The following proposition was proven with an approach that accounts for possible locality of the supports. We state a version of first order mixed partial derivatives and truncated dimension.

Proposition 8 *[1, Theorem 4.1] Assume that there exists a sequence $(\rho_j)_{j\geq 1}$ of positive reals such that*

$$\left\|\sum_{j\geq 1}\rho_j|\psi_j|\right\|_{L^\infty(D)} < \log(2)\;.$$

Then, there exists a constant $C>0$ such that for every $s\in\mathbb{N}$ and every $\boldsymbol{y}\in\mathbb{R}^s$

$$\sum_{\mathfrak{u}\subset\{1:s\}}\|\partial^{\mathfrak{u}}u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})}\prod_{j\in\mathfrak{u}}\rho_j^2 \leq C\|u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})}\;.$$

We extend the parametric regularity estimates that are given in Proposition 8 in order to obtain estimates that are suitable to yield dimension independent convergence rates of randomly shifted lattice rules.

Theorem 9 *Let the assumption in* (**A1**) *be satisfied for some $K>0$. Let $(w_j)_{j\geq 1}$ be either of the weight functions defined in* (9) *and* (10)*. Let $\kappa\in(0,\log(2)/K)$ be fixed and $p'\in(0,1)$. There exists a constant $C>0$ such that for every $s\in\mathbb{N}$, and for positive $\boldsymbol{\gamma}$,*

$$\|u^s\|^2_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;V)} \leq C\|f\|^2_{V^*}\int_{\mathbb{R}^s}\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\sup_{\mathfrak{u}\subset\{1:s\}}\left\{\prod_{j\in\mathfrak{u}}\left(\frac{b_j^{2(1-p')}}{\kappa^2}\right)w_j^2(y_j)\prod_{j\in\{1:s\}\setminus\mathfrak{u}}\phi(y_j)\right\}\mathrm{d}\boldsymbol{y}$$
$$\times\sup_{\mathfrak{u}\subset\{1:s\}}\gamma_{\mathfrak{u}}^{-1}\prod_{j\in\mathfrak{u}}b_j^{2p'}.$$

Proof. We obtain with the Jensen inequality, for any $s\in\mathbb{N}$,

$$\begin{aligned}
&\|u^s\|^2_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;V)}\\
&\quad\leq\sum_{\mathfrak{u}\subset\{1:s\}}\frac{1}{\gamma_{\mathfrak{u}}}\int_{\mathbb{R}^s}\|\partial^{\mathfrak{u}}u(\boldsymbol{y})\|^2_V\prod_{j\in\{1:s\}\setminus\mathfrak{u}}\phi_j(y_j)\prod_{j\in\mathfrak{u}}w_j^2(y_j)\mathrm{d}\boldsymbol{y}\\
&\quad\leq\int_{\mathbb{R}^s}\sum_{\mathfrak{u}\subset\{1:s\}}\frac{\kappa^{2|\mathfrak{u}|}}{\gamma_{\mathfrak{u}}\prod_{j\in\mathfrak{u}}b_j^{2(1-p')}}\|\partial^{\mathfrak{u}}u(\boldsymbol{y})\|^2_V\sup_{\mathfrak{u}\subset\{1:s\}}\left\{\prod_{j\in\mathfrak{u}}\left(\frac{b_j^{2(1-p')}}{\kappa^2}\right)w_j^2(y_j)\prod_{j\in\{1:s\}\setminus\mathfrak{u}}\phi(y_j)\right\}\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

In the present setting, the assumption of Proposition 8 is satisfied by the sequence $(\rho_j)_{j\geq 1} = (\kappa/b_j)_{j\geq 1}$. Hence, by the Hölder inequality and Proposition 8, and using (23) we obtain the following bound

$$
\begin{aligned}
\|u^s\|^2_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;V)} \leq& \int_{\mathbb{R}^s} \frac{1}{a^s_{\min}(\boldsymbol{y})} \left(\sum_{\mathfrak{u}\subset\{1:s\}} \|\partial^{\mathfrak{u}} u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})} \gamma_{\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} \frac{\kappa^2}{b_j^{2(1-p')}} \right) \\
&\times \sup_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \left\{ \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
\leq& \int_{\mathbb{R}^s} \frac{1}{a^s_{\min}(\boldsymbol{y})} C \|u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
&\times \sup_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} b_j^{2p'} \\
\leq& C\|f\|^2_{V^*} \int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
&\times \sup_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} b_j^{2p'},
\end{aligned}
$$

where we have used that $\|u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})} \leq \|f\|_{V^*}/\sqrt{a^s_{\min}(\boldsymbol{y})}$. □

Corollary 10 *Under the assumption of Theorem 9, there exists a finite constant C such that for every $s \in \mathbb{N}$ and for every $G(\cdot) \in V^*$ holds for $F = G(u^s)$*

$$
\begin{aligned}
\|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)} \leq& C\|G(\cdot)\|_{V^*}\|f\|_{V^*} \sqrt{\int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}} \\
&\times \sup_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1/2} \prod_{j\in\mathfrak{u}} b_j^{p'} .
\end{aligned}
$$

7 QMC analysis for the exact solution

In this section we show dimension-independent convergence rates for QMC integration of (functionals of) the parametric solution $u^s(\boldsymbol{y})$, which are obtained from the parametric regularity bounds shown in Section 6. The cases of Gaussian and exponential weight functions in the norm (11) will be treated separately, since the ensuing analysis suggests that the convergence rates hold under different summability assumptions on the sequence $(b_j)_{j\geq 1}$. In this section we assume that the integrand functions can be evaluated exactly. Ahead, in Section 8, the additional discretization error that arises by single-level Galerkin discretizations of the parametric PDE (17) is taken into account.

Theorem 11 *[Gaussian weight functions] Let assumption* (**A1**) *be satisfied for $K > 0$ and for $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$. For some $\varepsilon \in (0, 3/4 - 1/(2p))$ such that $\varepsilon \leq 1/(2p) - 1/4$ set $p' = p/4 + 1/2 - \varepsilon p$. Let $(w_{\mathrm{g},j})_{j\geq 1}$ be the weight functions defined in* (9) *with*

$$
\alpha_{\mathrm{g}} \in \left(\frac{p}{2(p-p')}, \frac{p}{p-2(1-p')} \right). \tag{24}
$$

Define the product weights

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} b_j^{2p'}, \quad \mathfrak{u}\subset\mathbb{N}, |\mathfrak{u}|<\infty\,. \tag{25}$$

Let $s\in\mathbb{N}$ and $G(\cdot)\in V^*$ be given. Then, for every $N\in\mathbb{N}$ a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN\log N)$ operations using the fast CBC algorithm of [30, 29] such that the root-mean square error over all random shifts can be estimated as follows: there exists a constant $C>0$ that is independent of s and N such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^s)) - Q^{\boldsymbol{\Delta}}_{s,N}(G(u^s))|^2)} \le C\,(\varphi(N))^{-(1/(2p)+1/4-\varepsilon)}.$$

Proof. The assertion of the theorem will follow by Theorem 1 once the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;V)$-norm of u^s has been bounded independently of s, which in turn will be deduced from the bound in Theorem 9 and in Corollary 10. To this end, fix $\kappa\in(0,\log(2)/K)$. Since $p>2(1-p')$ is implied by $(3/4-1/(2p))>\varepsilon$, thus $q:=p/(2(1-p'))>1$. From the Jensen inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^s}\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\sup_{\mathfrak{u}\subset\{1:s\}}\left\{\prod_{j\in\mathfrak{u}}\frac{b_j^{2(1-p')}}{\kappa^2}w^2_{\mathrm{g},j}(y_j)\prod_{j\in\{1:s\}\setminus\mathfrak{u}}\phi(y_j)\right\}\mathrm{d}\boldsymbol{y}\\
&\le\int_{\mathbb{R}^s}\left(\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^q\sum_{\mathfrak{u}\subset\{1:s\}}\prod_{j\in\mathfrak{u}}\frac{b_j^p}{\kappa^{2q}}w^{2q}_{\mathrm{g},j}(y_j)\prod_{j\in\{1:s\}\setminus\mathfrak{u}}\phi(y_j)^q\right)^{1/q}\mathrm{d}\boldsymbol{y}\\
&\le\left(\int_{\mathbb{R}^s}\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^q\sum_{\mathfrak{u}\subset\{1:s\}}\prod_{j\in\mathfrak{u}}\frac{b_j^p}{\kappa^{2q}}w^{2q}_{\mathrm{g},j}(y_j)\phi(y_j)^{-q}\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}\right)^{1/q}\\
&=\left(\sum_{\mathfrak{u}\subset\{1:s\}}\prod_{j\in\mathfrak{u}}\frac{b_j^p}{\kappa^{2q}}\int_{\mathbb{R}^s}\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^q\prod_{j\in\mathfrak{u}}w^{2q}_{\mathrm{g},j}(y_j)\phi(y_j)^{-q}\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}\right)^{1/q}.
\end{aligned}\tag{26}$$

Here, we inserted the factor $1=\prod_{j\in\{1:s\}}\phi(y_j)\phi(y_j)^{-1}$ and we moved factors under the exponent $1/q$ to move the exponent $1/q$ outside of the integral with the Jensen inequality.

The parameter $\alpha_{\mathrm{g}}>1$ of the weight functions $(w_{\mathrm{g},j})_{j\ge1}$ is chosen such that $\alpha_{\mathrm{g}}<q/(q-1)$, which implies that $1>(1-1/\alpha_{\mathrm{g}})q$. The function $x\mapsto x/(x-1)$ is strictly decreasing on $(1,\infty)$. Thus, there exists $q'>q$ such that $\alpha_{\mathrm{g}}<q'/(q'-1)$ and therefore also $1>(1-1/\alpha_{\mathrm{g}})q'$. Since $\int_{\mathbb{R}}\exp(-y^2/(2\sigma^2))\mathrm{d}y=\sqrt{2\pi}\sigma$ for every $\sigma>0$, it holds that

$$\int_{\mathbb{R}}w^{2q'}_{\mathrm{g},j}(y)\phi(y)^{-q'}\phi(y)\mathrm{d}y=(\sqrt{2\pi})^{q'-1}\int_{\mathbb{R}}e^{-\frac{y^2}{2}\left(1-\left(1-\frac{1}{\alpha_{\mathrm{g}}}\right)q'\right)}\mathrm{d}y=\sqrt{(2\pi)^{q'}\frac{\alpha_{\mathrm{g}}}{\alpha_{\mathrm{g}}-(\alpha_{\mathrm{g}}-1)q'}}=:C'.$$

The Hölder inequality applied with $q'/q>1$ and conjugate $q'/(q'-q)$ results in

$$\begin{aligned}
&\int_{\mathbb{R}^s}\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^q\prod_{j\in\mathfrak{u}}w^{2q}_{\mathrm{g},j}(y_j)\phi(y_j)^{-q}\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}\\
&\le\left(\int_{\mathbb{R}^s}\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^{qq'/(q'-q)}\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}\right)^{(q'-q)/q'}\left(\int_{\mathbb{R}^{|\mathfrak{u}|}}\prod_{j\in\mathfrak{u}}w^{2q'}_{\mathrm{g},j}(y_j)\phi(y_j)^{-q'}\phi(y_j)\mathrm{d}\boldsymbol{y}\right)^{q/q'}\\
&=\left(\int_{\mathbb{R}^s}\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right)^{qq'/(q'-q)}\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}\right)^{(q'-q)/q'}(C')^{|\mathfrak{u}|q/q'}\\
&=\left(\mathbb{E}\left(\left(\frac{1}{(a^s_{\min})^2}\right)^{qq'/(q'-q)}\right)\right)^{(q'-q)/q'}(C')^{|\mathfrak{u}|q/q'}=:C''(C')^{|\mathfrak{u}|q/q'},
\end{aligned}$$

where C'' can be bounded independently of s by Corollary 6 and by the Cauchy–Schwarz inequality. Together with (26) and [23, Lemma 6.3], we obtain that

$$\int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w^2_{\mathrm{g},j}(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \leq \left(C'' \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p (C')^{q/q'}}{\kappa^{2q}} \right)^{1/q}$$

$$\leq (C'')^{1/q} \exp\left(\frac{(C')^{q/q'}}{q\kappa^{2q}} \sum_{j\geq 1} b_j^p \right),$$

which bound is independent of s and finite by the assumption $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$. Then, by Corollary 10, there exists a constant C independently of s such that for our chosen weights

$$\|G(u^s)\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(\mathbb{R}^s)} \leq C \|G(\cdot)\|_{V^*} \|f\|_{V^*}.$$

The parameter α_{g} of the weight functions $(w_{\mathrm{g},j})_{j\geq1}$ is chosen such that $\alpha_{\mathrm{g}} > p/(2(p-p'))$, which implies that $\lambda > 1/(2r)$, where $\lambda := p/(2p')$ and $r := 1 - 1/(2\alpha_{\mathrm{g}})$. Also note that $\varepsilon \leq 1/(2p) - 1/4$ implies $\lambda \leq 1$. We recall from Theorem 1

$$\rho(\lambda) := 2 \left(\frac{4\sqrt{2\pi}\alpha_{\mathrm{g}}^2}{\pi^{2-1/\alpha_{\mathrm{g}}}(2\alpha_{\mathrm{g}} - 1)} \right)^{\lambda} \zeta\left(2r\lambda\right) .$$

The two conditions on the parameter α_{g} of the weight functions, that $\alpha_{\mathrm{g}} < q/(q-1)$ and that $\alpha_{\mathrm{g}} > p/(2(p-p'))$, are compatible, since

$$\frac{p}{2(p-p')} < \frac{q}{q-1} = \frac{p}{p-2(1-p')}$$

is implied by

$$p' < \frac{p}{4} + \frac{1}{2}.$$

Note that $p > p/4 + 1/2 > p'$ implies that α_{g} is well defined. Since product weights are considered, [23, Lemma 6.3] implies with the assumption $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$ that

$$\sum_{\emptyset\neq\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \rho(\lambda)^{|\mathfrak{u}|} \leq \sum_{\mathfrak{u}\subset\mathbb{N}, |\mathfrak{u}|<\infty} \prod_{j\in\mathfrak{u}} b_j^p \rho(\lambda)^{|\mathfrak{u}|} \leq \exp\left(\sum_{j\geq1} b_j^p \rho(\lambda) \right) < \infty,$$

which bound is uniform in s. The assertion of the theorem follows with Theorem 1 applied with the choices $\lambda = p/(2p')$ and $p/(2(p-p')) < \alpha_{\mathrm{g}} < p/(p-2(1-p'))$. The convergence rate resulting from Theorem 1 is $1/(2\lambda) = p'/p = 1/(2p) + 1/4 - \varepsilon$. □

Remark 12 *In Theorem 11, the case $p = 2$ does not seem accessible with the present argument, since in Theorem 1 neither of the choices $\lambda > 1$ nor $\alpha_{\mathrm{g}} = 1$ are permitted.*

Theorem 13 *[Exponential weight functions] Let assumption* (**A1**) *be satisfied for $K > 0$ and for $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$ for $p \in (2/3, 1]$. Let $(w_{\exp,j})_{j\geq1}$ be the weight functions defined in* (10) *with $\alpha_{\exp} > 2K \sup_{j\geq1}\{b_j\}$. Define $p' := 1 - p/2 \in [1/2, 2/3)$. Let $s \in \mathbb{N}$ and $G(\cdot) \in V^*$ be given and define product weights*

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} b_j^{2p'}, \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty . \tag{27}$$

Then, for every $N \in \mathbb{N}$ a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN \log N)$ operations using the fast CBC algorithm of [30, 29] such that the root-mean square *error over all random shifts can be estimated independently of s and N, i.e., there exists a constant $C > 0$ that is independent of s and N such that*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^s)) - Q^{\boldsymbol{\Delta}}_{s,N}(G(u^s))|^2)} \leq C\,(\varphi(N))^{-1/p+1/2}\,.$$

Proof. The assertion of the theorem will follow from Theorem 1 once the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;V)$-norm of u^s has been bounded independently of s. This, in turn, will be shown using Theorem 9. Let $\kappa \in (0, \log(2)/K)$ be fixed. The choice $p' = 1 - p/2$ implies that $2(1-p') = p$ and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w^2_{\exp,j}(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
&\quad\leq \int_{\mathbb{R}^s} \left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right) \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p}{\kappa^2} w^2_{\exp,j}(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y} \\
&\quad= \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p}{\kappa^2} \int_{\mathbb{R}^s} \left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right) \prod_{j\in\mathfrak{u}} w^2_{\exp,j}(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}.
\end{aligned}$$

We observe that for every $\boldsymbol{y} \in \mathbb{R}^s$,

$$\left(\frac{1}{(a^s_{\min}(\boldsymbol{y}))^2}\right) \leq e^{2\|Z(\boldsymbol{y})\|_{L^\infty(D)}} \leq e^{2K\sup_{j\in\{1:s\}}\{|y_j|b_j\}} \leq e^{2K\sum_{j\in\{1:s\}}|y_j|b_j},$$

which allows for an upper bound of the integrand that is in product form to separate the integrals. Since the parameter $\alpha_{\exp}$ of the weight functions satisfies that $\alpha_{\exp} > 2K\|(b_j)_{j\geq1}\|_{\ell^\infty(\mathbb{N})}$, we obtain that for every $j \in \{1:s\}$

$$\int_{\mathbb{R}} e^{2K|y_j|b_j} w_j^2(y_j)\mathrm{d}y_j = \frac{1}{\alpha_{\exp} - 2Kb_j}$$

and (as in [14, Equation (4.15)])

$$1 \leq \int_{\mathbb{R}} e^{2K|y_j|b_j} \phi(y_j)\mathrm{d}y_j = 2\exp\left(\frac{(2Kb_j)^2}{2}\right)\Phi(2Kb_j) \leq \exp\left(\frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}}\right)\,.$$

Here, we used the bound $\Phi(y) \leq 1/2\exp(2y/\sqrt{2\pi})$ for every $y \geq 0$, which can be shown by an affine approximation of Φ and the elementary bound $1+x \leq e^x$ for every $x \in [0,\infty)$ (we refer to [14, p. 355] for details). By the assumption that $(b_j)_{j\geq1} \in \ell^p(\mathbb{N}) \subset \ell^1(\mathbb{N})$, for every $\mathfrak{u} \subset \{1:s\}$ holds

$$\prod_{j\in\mathfrak{u}} \exp\left(\frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}}\right) \leq \exp\left(\sum_{j\geq1} \frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}}\right) =: C < \infty\,.$$

We conclude with [23, Lemma 6.3] and the assumption $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$ that

$$\begin{aligned}
&\int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w^2_{\exp,j}(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
&\qquad\leq C \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j} \\
&\qquad\leq C\exp\left(\sum_{j\geq1} \frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j}\right) < \infty.
\end{aligned}$$

By Theorem 9 we obtain for our choice (27) of product weights $\boldsymbol{\gamma}$

$$\|G(u^s)\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)} \leq \sqrt{C} \exp\left(\frac{1}{2}\sum_{j\geq 1}\frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j}\right) < \infty.$$

Here, the constant $C > 0$ is independent of the integration dimension s. The assertion now follows similarly as in the proof of Theorem 11 from Theorem 1. We have chosen the weight functions defined in (10) with $\lambda = p/(2p')$ and $\delta < 1 - 1/(2\lambda)$. We note that by the assumption $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ and [23, Lemma 6.3], for every $s \in \mathbb{N}$,

$$\sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \rho(\lambda)^{|\mathfrak{u}|} = \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \left(b_j^p \rho(\lambda)\right) \leq \exp\left(\sum_{j\geq 1} b_j^p \rho(\lambda)\right) < \infty.$$

□

The QMC convergence rate bounds in Theorems 11 and 13 are also applicable for globally supported functions $(\psi_j)_{j\geq 1}$ as studied in [14]. The product structure of the QMC weight sequences $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N},|\mathfrak{u}|<\infty}$ considered here entails stronger summability conditions than those in these references on the sequence $(b_j)_{j\in\mathbb{N}}$ to achieve a prescribed, dimension-independent convergence rate.

Corollary 14 *Suppose that the assumption that* (5) *is satisfied for some* $p \in (2/5, 2/3)$. *Define the sequence* $(b_j)_{j\geq 1}$ *by* $b_j := \|\psi_j\|_{L^\infty(D)}^{1-p}$, $j \geq 1$. *Then,*

1. *a randomly shifted lattice QMC rule based on Gaussian weight functions with product weights converges with rate* $1/(2p) - 1/4 - \varepsilon$ *for* $\varepsilon > 0$ *sufficiently small.*
2. *a randomly shifted lattice QMC rule based on exponential weight functions with product weights for* $p \in (2/5, 1/2]$ *converges with rate* $1/p - 3/2$.

We remark that in [14], for exponential weight functions, globally supported $(\psi_j)_{j\geq 1}$ and for summability exponent $p \in (2/3, 1)$, the dimension-independent convergence rate $1/p - 1/2$ in terms of N was established for a randomly shifted lattice rule with *product and order dependent weights*, from [28]. For such weights, however, the fast CBC construction of QMC rules has cost which increases quadratically w.r. to the quadrature dimension s, whereas fast CBC constructions for product weights scale linearly w.r. to s. A trivial case where QMC with product weights is beneficial also for globally supported ψ_j's is for $p \approx 2/5$. Since QMC by randomly shifted lattice rules converges at most at first order, QMC with POD and product weights achieve essentially the same convergence rate, whereas QMC with product weights has a significantly smaller computational cost to construct N QMC points in s dimensions: $\mathcal{O}(sN\log(N))$ vs. $\mathcal{O}(sN\log(N) + s^2N)$.

8 Combined QMC Finite Element discretization

In general, the exact evaluation of the solution of (1) is not possible as required for the computation of $Q_{s,N}^{\boldsymbol{\Delta}}(G(u^s))$ for a functional $G(\cdot) \in V^*$. We approximate the solution by a Galerkin FE method. For simplicity we introduce the assumption that

$$D \subset \mathbb{R}^d \text{ is a bounded polyhedron with plane faces.} \qquad \textbf{(A2)}$$

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape regular, simplicial triangulations of the polygonal resp. polyhedral domain D, where h is the maximal diameter of all elements in $\mathcal{T}_h$. Let V_h denote all continuous piecewise polynomal functions of total degree $r \geq 1$, that vanish on ∂D. Thus, $V_h \subset V$ is a subspace such that $\dim(V_h) = \mathcal{O}(h^{-d})$ as $h \to 0$. The deterministic Galerkin discretization reads: for every $\widehat{a} \in L_+^\infty(D)$ find $\widehat{u}^h \in V_h$ such that

$$\int_D \widehat{a}\, \nabla \widehat{u}^h \cdot \nabla v^h \, \mathrm{d}x = f(v), \quad \forall v^h \in V_h. \tag{28}$$

From the discussion in Section 4 we know that the problem in (28) is well posed. Similar to Section 4, we denote by $\mathcal{S}_f^h$ the discretized data-to-solution map that maps a (realization of the) diffusion coefficient $\widehat{a} \in L_+^\infty(D)$ to the FE solution $\widehat{u}^h \in V_h$ for fixed right hand side $f \in V^*$. We note that $\mathcal{S}_f^h : L_+^\infty(D) \to V_h \subset V$ is continuous. This implies that the FE solution

$$u^{s,h} := \mathcal{S}_f^h(a^s)$$

is strongly measurable in V for every $h > 0$ and $s \geq 1$. It is the unique solution to the s-parametric, deterministic variational problem

$$\int_D a^s \, \nabla u^{s,h} \cdot \nabla v^h \, \mathrm{d}x = f(v), \quad \forall v^h \in V_h$$

as a V_h-valued μ-equivalence class; see also [17, Section 4.1] for details.

Let $C^t(\overline{D})$, $t \in [0,\infty)$, denote the Hölder spaces such that for $k \in \mathbb{N}$, $C^k(\overline{D})$ is the space of k-times continuously differentiable functions on $\overline{D}$ with bounded derivatives on $\overline{D}$. Regularity of solutions to (15) in Sobolev scales accounting for singularities due to re-entrant corners has been studied for $d = 2$ in [31, 15], where in [31, Lemma 5.2] the explicit dependence of the constant in the error bound has been tracked: let $t \in (0,1)$, $\tau \in (0, \max\{t, \pi/\beta_{\max}\}) \backslash \{1/2\}$, and assume that $f \in H^{-1+\tau}(D)$ and $\widehat{a} \in C^t(\overline{D}) \cap L_+^\infty(D)$, then $\mathcal{S}_f(\widehat{a}) \in H^{1+\tau}(D)$ and there exists a constant C such that for every $f \in H^{-1+\tau}(D)$ and for every $\widehat{a} \in C^t(\overline{D}) \cap L_+^\infty(D)$

$$\|\mathcal{S}_f(\widehat{a})\|_{H^{1+\tau}(D)} \leq C \frac{\|\widehat{a}\|_{L^\infty(D)}}{(\operatorname{ess\,inf}_{x \in D}\{\widehat{a}(x)\})^4} \|\widehat{a}\|^2_{C^t(\overline{D})} \|f\|_{H^{-1+\tau}(D)}, \tag{29}$$

where $\beta_{\max}$ is the maximal opening angle of the interior tangent cones to ∂D with vertex in the corner points of D. Under (**A2**), for $d = 2$, in polygons D with straight sides the regularity of the inverse of the Dirichlet Laplacean $(-\Delta)^{-1} : V^* \to V$ is, in Sobolev scales, limited by the maximal interior angle $\beta_{\max}$ of D such that $(-\Delta)^{-1} : H^{-1+\tau}(D) \to H^{1+\tau}(D) \cap V$ is bounded for every $\tau \in [0, \pi/\beta_{\max})$, cp. [15, Section 5]. Let to the end of this section $d = 2$.

We impose the hypothesis (see Proposition 18 ahead for a class of instances) that for some $t > 0$, a and a^s are strongly measurable in $C^t(\overline{D})$, for every $s \in \mathbb{N}$. Moreover, we assume that for every $q \in [1,\infty)$ there exists a constant C such that for every $s \in \mathbb{N}$

$$\|a^s\|_{L^q(\Omega; C^t(\overline{D}))} \leq C. \tag{\textbf{A3}}$$

Proposition 15 *Let the assumption in* (**A1**) *be satisfied for some* $p_0 \in (0,\infty)$ *and let the assumption in* (**A2**) *and in* (**A3**) *hold for* $d = 2$ *and for some* $t > 0$*. Let* $f \in H^{-1+\tau}(D)$ *and let* $G(\cdot) \in H^{-1+\tau'}(D)$ *for* $\tau, \tau' \in (0, \max\{t, \pi/\beta_{\max}\}) \backslash \{1/2 + \mathbb{N}_0\}$*. For every* $q \in [1,\infty)$ *there exists a constant* C *independent of* $h > 0$ *such that for every* $s \geq 1$

$$\|G(u^s) - G(u^{s,h})\|_{L^q(\Omega)} \leq C h^{\min\{\tau,r\} + \min\{\tau',r\}} .$$

Proof. The first part of the proof follows similarly as respective arguments that resulted in [17, Theorem 3.7]. We decompose $\tau = \lfloor \tau \rfloor + \{\tau\}$, where $\{\tau\}$ is the fractional part, and show by induction on $n \in \{0, \ldots, \lfloor \tau \rfloor\}$ that the $L^q(\Omega; H^{1+n+\{\tau\}}(D))$-norm of $\mathcal{S}_f(a^s)$ can be uniformly bounded in s for every $q \in [1, \infty)$. The base case, i.e., $n = 0$, follows by (29) and (**A3**) with a twofold application of the Cauchy–Schwarz inequality. For $n \in \{1, \dots, \lfloor \tau \rfloor\}$, $t > 1$ and thus a^s takes values in $C^1(\overline{D})$. Let us assume the statement holds for $n - 1$ as induction hypothesis. Hence, the equation (1) can be reformulated as

$$-\Delta \mathcal{S}_f(a^s) = \frac{1}{a^s} \left(f + \nabla a^s \cdot \nabla \mathcal{S}_f(a^s) \right) =: \widetilde{f}$$

with equality in V^*. Since for a constant C that is independent of a^s and f

$$\|\widetilde{f}\|_{H^{-1+n+\{\tau\}}(D)} \leq C \left(\|1/a^s\|_{C^t(\overline{D})} \left(\|f\|_{H^{-1+n+\{\tau\}}(D)} + \|a^s\|_{C^t(\overline{D})} \|\mathcal{S}_f(a^s)\|_{H^{n+\{\tau\}}(D)} \right) \right),$$

where we used that the pointwise product of functions in $C^{\widetilde{t}}(\overline{D})$ with functions in $H^{\widetilde{\tau}}(D)$ is continuous for all $0 \leq \widetilde{\tau} < \widetilde{t}$, cp. [15, Theorem 1.4.1.1]. This implies with the induction hypothesis and a twofold application of the Cauchy–Schwarz inequality that the $L^q(\Omega; H^{-1+n+\{\tau\}}(D))$-norm of $\widetilde{f}$ is bounded uniformly in s for every $q \in [1, \infty)$. Since $(-\Delta)^{-1} : H^{-1+n+\{\tau\}}(D) \to H^{1+n+\{\tau\}}(D) \cap V$ is bounded the induction step is completed and thus the $L^q(\Omega; H^{1+\tau}(D))$-norm of $\mathcal{S}_f(a^s)$ is bounded uniformly in s for every $q \in [1, \infty)$. Note that the strong measurability of $\mathcal{S}_f(a^s)$ in $H^{1+\tau}(D)$ follows, since $\mathcal{S}_f : C^t(\overline{D}) \cap L^\infty_+(D) \to H^{1+\tau}(D)$ is continuous, which can be shown with the estimate in (29) and a perturbation argument with respect to the diffusion coefficient; see the proof of [17, Proposition 3.6] for details. Verbatim, it holds that for every $q \in [1, \infty)$, the $L^q(\Omega; H^{1+\tau'}(D))$-norm of $\mathcal{S}_G(a^s)$ can be bounded by a constant which is independent of s. By the Aubin–Nitsche lemma, cp. [5, Theorem 3.2.4 and Equation (3.2.23)],

$$|G(u^s) - G(u^{s,h})| \leq \|a^s\|_{L^\infty(D)} \|\mathcal{S}_f(a^s) - \mathcal{S}_f^h(a^s)\|_V \|\mathcal{S}_G(a^s) - \mathcal{S}_G^h(a^s)\|_V,$$

which implies with the approximation property of V_h in V, cp. [5, Theorem 3.2.1] (which can be interpolated to non-integer Sobolev scales), Céa's lemma, and the Hölder inequality that for every $q \in [1, \infty)$

$$\begin{aligned} \|G(u^s) - G(u^{s,h})\|_{L^q(\Omega)} \leq C^2 \left\| \frac{(a^s_{\max})^3}{(a^s_{\min})^2} \right\|_{L^{3q}(\Omega)} & \|\mathcal{S}_f(a^s)\|_{L^{3q}(\Omega; H^{1+\tau}(D))} \|\mathcal{S}_G(a^s)\|_{L^{3q}(\Omega; H^{1+\tau'}(D))} \\ & \times h^{\min\{\tau, r\} + \min\{\tau', r\}}, \end{aligned}$$

where the constant C is due to the approximation property. The assertion of the proposition follows by Proposition 3 and Corollary 6 with the Cauchy–Schwarz inequality and by the fact shown above that the $L^{3q}(\Omega; H^{1+\tau}(D))$-norm and the $L^{3q}(\Omega; H^{1+\tau'}(D))$-norm of $\mathcal{S}_f(a^s)$ and respectively of $\mathcal{S}_G(a^s)$ can be bounded uniformly with respect to s. □

Remark 16 *In Proposition 15, the cases $\tau, \tau' \in \{1/2 + \mathbb{N}_0\}$ are permitted if $f \in H^{-1+\tau+\varepsilon}(D)$, respectively if $G(\cdot) \in H^{-1+\tau'+\varepsilon}(D)$, for some $\varepsilon > 0$.*

Theorem 17 *Let the assumption in (**A1**) be satisfied with $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$ and let the assumption in (**A2**) and in (**A3**) be satisfied for $d = 2$ and for some $t > 0$. Let $f \in H^{-1+\tau}(D)$ and let $G \in H^{-1+\tau'}(D)$ for $\tau, \tau' \in (0, \max\{t, \pi/\beta_{\max}\}) \setminus \{1/2 + \mathbb{N}_0\}$ such that $\max\{\tau, \tau'\} \leq r$. The error incurred in the approximation $Q^{\mathbf{\Delta}}_{s,n}(G(u^{s,h}))$ with the N-point randomly shifted lattice rule $Q^{\mathbf{\Delta}}_{s,N}$ applied to the s-variate, dimensionally truncated integral $I_s(G(u^{s,h}))$ satisfies:*

1. *For $p \in (2/3, 2)$ and $\varepsilon \in (0, 3/4 - 1/(2p))$ such that $\varepsilon \leq 1/(2p) - 1/4$, with Gaussian weight functions $(w_{\mathrm{g},j})_{j \geq 1}$ defined in (9) with α_{g} as in (24) the error is bounded by*

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}(G(u)) - Q^{\mathbf{\Delta}}_{s,N}(G(u^{s,h}))|^2)} \leq C \left((\varphi(N))^{-1/4 - 1/(2p) + \varepsilon} + \sup_{j > s} \left\{ b_j^{1-\varepsilon} \right\} + h^{\tau + \tau'} \right). \tag{30}$$

2. For $p \in (2/3, 1]$ and $\varepsilon \in (0,1)$, with exponential weight functions $(\boldsymbol{w}_{\exp,j})_{j\geq 1}$ defined in (10) with $\alpha_{\exp} > 2K \sup_{j\geq 1}\{b_j\}$ the error is bounded by

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q^{\boldsymbol{\Delta}}_{s,N}(G(u^{s,h}))|^2)} \leq C \left((\varphi(N))^{-1/p+1/2} + \sup_{j>s} \left\{ b_j^{1-\varepsilon} \right\} + h^{\tau+\tau'} \right) . \quad (31)$$

The constant C in the error bounds (30) *and* (31) *is independent of N, s, and h.*

Note that $(\varphi(N))^{-1} \leq N^{-1} \cdot (e^{\widehat{\gamma}} \log\log N + 3/\log\log N)$, for every $N \geq 3$, where $\widehat{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant.

Proof. By the definition of the QMC points in (8), $\{\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(N-1)}\}$ are identically $\mathcal{N}(0, \mathrm{Id}_{\mathbb{R}^s})$-distributed. We observe that by the triangle inequality, for every square integrable function F with respect to the s-dimensional normal distribution with covariance being the identity,

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|Q^{\boldsymbol{\Delta}}_{s,N}(F)|^2} \leq \frac{1}{N} \sum_{i=0}^{N-1} \sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|F(\boldsymbol{y}^{(i)})|^2} = \sqrt{\int_{\mathbb{R}^s} |F(y)|^2 \prod_{j\in\{1:s\}} \phi(y_j) \mathrm{d}\boldsymbol{y}}.$$

Thus, by the triangle inequality,

$$\begin{aligned}\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q^{\boldsymbol{\Delta}}_{s,N}(G(u^{s,h}))|^2)} &\leq |\mathbb{E}(G(u)) - I_s(G(u^s))| \\ &+ \sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^s)) - Q^{\boldsymbol{\Delta}}_{s,N}(G(u^s))|^2)} \\ &+ \|G(u^s) - G(u^{s,h})\|_{L^2(\Omega)}.\end{aligned}$$

The assertion now follows with Proposition 7, Proposition 15, and by Theorem 11 for Gaussian weight functions and respectively by Theorem 13 for exponential weight functions. □

9 Multiresolution representation of Gaussian random fields

We investigate expansions of Gaussian random fields Z in particular function systems with local supports, for related recent work see [2]. In the polyhedral domain D, cp. the assumption (**A2**), consider an isotropic multiresolution analysis (MRA) $\boldsymbol{\Psi} = \{\psi_\lambda : \lambda \in \triangledown\}$ whose members ψ_λ are indexed by $\lambda \in \triangledown$, and are obtained from one or from a finite number of generating elements ψ by translation and scaling, i.e.,

$$\psi_\lambda(x) = 2^{d|\lambda|/2}\psi(2^{|\lambda|}x - k) \ , \quad k \in \triangledown_{|\lambda|} \ , \quad (32)$$

where the index set $\nabla_{|\lambda|}$ is of cardinality $\mathcal{O}(2^{d|\lambda|})$, and where $\operatorname{diam}\operatorname{supp}(\psi_\lambda) = \mathcal{O}(2^{-|\lambda|})$. The scaling in (32) by the factor $2^{d|\lambda|/2}$ refers to a normalization in $L^2(D)$, i.e., $\|\psi_\lambda\|_{L^2(D)} \sim \|\psi\|_{L^2(D)}$, $\lambda \in \triangledown$. For suitable, sufficiently smooth families of wavelets it can be shown that for every $q \in [1, \infty)$ and every $t \geq 0$, there exists a constant C such that

$$\left\| \sum_{\lambda\in\triangledown} c_\lambda \psi_\lambda \right\|_{B^t_{q,q}(D)} \leq C \left(\sum_{\ell\geq 0} 2^{tq\ell} 2^{(q/2-1)d\ell} \sum_{k\in\nabla_\ell} |c_{\ell,k}|^q \right)^{1/q} , \quad (33)$$

cp. for example for the case of orthonormal wavelets [34, Theorem 4.23], where $B^s_{p,q}(D)$ denote Besov spaces on D, $s \in [0, \infty)$, $p, q \in [1, \infty]$. However, in this manuscript we adopt for the $(\psi_\lambda)_{\lambda\in\triangledown}$ a pointwise normalization, such that for some $\widehat{\alpha} > 0$ and $\sigma > 0$ at our disposal,

$$\|\psi_\lambda\|_{L^\infty(D)} \simeq \sigma 2^{-\widehat{\alpha}|\lambda|}, \quad \lambda \in \triangledown. \quad (34)$$

With the scaling (34), the norm estimate in (33) then reads that for every $q \in [1, \infty)$ and every $t \geq 0$, there exists a constant C such that

$$\left\| \sum_{\lambda \in \bigtriangledown} c_\lambda \psi_\lambda \right\|_{B^t_{q,q}(D)} \leq C \left(\sum_{\ell \geq 0} 2^{tq\ell} 2^{-(d+\widehat{\alpha} q)\ell} \sum_{k \in \bigtriangledown_\ell} |c_{\ell,k}|^q \right)^{1/q} . \qquad (\mathbf{A4})$$

We assume that there exists a suitable enumeration of elements of the index set $\bigtriangledown$, i.e., a bijective mapping $j : \bigtriangledown \to \mathbb{N}$, which we denote by $j(\lambda)$, $\lambda \in \bigtriangledown$, such that $|j^{-1}(s_1)| \leq |j^{-1}(s_2)|$ for positive integers $s_1 \leq s_2$. The amount of overlap of the supports at refinement level $|\lambda|$ is assumed to be bounded by an absolute multiple M times $2^{-|\lambda|}$ such that

$$|\{\lambda \in \bigtriangledown : |\lambda| = \ell, \psi_\lambda(x) \neq 0\}| \leq M, \quad \text{for all} \quad x \in D, \ell \geq 0 .$$

For given $\widehat{\alpha} > 0$ we define the sequence $(b_j)_{j\geq 1}$ for $\widehat{\beta} < \widehat{\alpha}$ and for some $c > 0$ by

$$b_{j(\lambda)} = b_\lambda := c 2^{-\widehat{\beta}|\lambda|}, \quad \lambda \in \bigtriangledown. \qquad (35)$$

We observe that $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$. This sequence satisfies $(\mathbf{A1})$, i.e.,

$$\left\| \sum_{\lambda \in \bigtriangledown} \frac{|\psi_\lambda|}{b_\lambda} \right\|_{L^\infty(D)} \leq \left\| \sum_{\ell \geq 0} \sum_{k \in \bigtriangledown_\ell} \frac{|\psi_{\ell,k}|}{b_{\ell,k}} \right\|_{L^\infty(D)} \leq \sigma M / c \sum_{\ell \geq 0} 2^{-(\widehat{\alpha} - \widehat{\beta})\ell} = \frac{\sigma M / c}{1 - 2^{-(\widehat{\alpha} - \widehat{\beta})}} < \infty.$$

Proposition 18 *Let $(\psi_j)_{j\geq 1}$ satisfy the scaling in (34) for some $\widehat{\alpha} > 0$ and let $(\mathbf{A4})$ hold. For every $t \in (0, \widehat{\alpha})$ and every $q \in [1, \infty)$, $Z \in L^q(\Omega; C^t(\overline{D}))$ and for every $\varepsilon \in (0, \widehat{\alpha} - t)$ there exists a constant C such that for every $s \in \mathbb{N}$,*

$$\|Z - Z^s\|_{L^q(\Omega; C^t(\overline{D}))} \leq C \sup_{\ell \geq |j^{-1}(s)|} \left\{ 2^{-(\widehat{\alpha} - t - \varepsilon)\ell} \right\}.$$

Proof. A sequence $(b_j)_{j\geq 1}$ can be defined by (35) for some $0 < \widehat{\beta} < \widehat{\alpha}$. Since $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$, $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$, for every $p > d/\widehat{\beta}$. Hence, by Theorem 2, $Z = \lim_{s' \to \infty} Z^{s'}$ with convergence in the $L^q(\Omega; L^\infty(D))$-norm, which equals the $L^q(\Omega; C^0(\overline{D}))$-norm. Since $(\psi_j)_{j\geq 1}$ are continuous on $\overline{D}$, $Z \in L^q(\Omega; C^0(\overline{D}))$.

Let $t' \in (t, t + \varepsilon)$. We set $q' := \lceil q/2 \rceil$ such that $2q' = 2\lceil q/2 \rceil$ is the smallest even natural number that is greater or equal than q and pick $r \in \mathbb{N}$ such that $r \geq q'$ and such that $r > d/(2(t'-t))$, which implies that $t' - d/(2r) > t$. By the continuous embedding $B^{t'}_{2r,2r}(D) \subset C^t(\overline{D})$ using $t' - d/(2r) > t$, cp. [33, Theorem 1.107], $\psi_j \in C^t(\overline{D})$, $j \geq 1$. By [33, Theorem 1.122 and Remark 1.121], the spaces $C^{t'}(\overline{D})$ and $B^{t'}_{\infty,\infty}(D)$, $t' \in [0, \infty) \backslash \mathbb{N}_0$, are isomorphic with equivalent norms. Since $Z^{s'}$ is separably valued in $C^t(\overline{D})$, it is strongly measurable in $C^t(\overline{D})$ by Pettis' theorem (e.g. cp. [35, Theorem V.4]) for every $s' \geq 1$ (arguing as in the proof of Theorem 2). Also by the same embedding and $(\mathbf{A4})$ it follows similarly as in the proof of Theorem 2 that

$$\begin{aligned}
\|Z^{s'} - Z^s\|^{2r}_{L^{2r}(\Omega; C^t(\overline{D}))} &\leq C \sum_{j(\ell,k) \in \{s+1:s'\}} 2^{-(\widehat{\alpha} - t')2r\ell} 2^{-d\ell} \mathbb{E}(|y_{\ell,k}|^{2r}) \\
&\leq C' \sum_{\ell \geq |j^{-1}(s)|} 2^{-(\widehat{\alpha} - t')2r\ell} \frac{(2r)!}{2^r r!} \\
&\leq C' \frac{(2r)!}{2^r r!} \sup_{\ell \geq |j^{-1}(s)|} \left\{ 2^{-(\widehat{\alpha} - t - \varepsilon)2r\ell} \right\} \sum_{\ell \geq 0} 2^{-(t + \varepsilon - t')2r\ell} < \infty,
\end{aligned}$$

where we used that $\#(\nabla_\ell) = \mathcal{O}(2^{d\ell})$. Note that $(y_{\ell,k} : \ell \geq 0, k \in \nabla_\ell)$ is a sequence of i.i.d. $\mathcal{N}(0,1)$-distributed random variables. We observe that $(Z^{s'})_{s'\geq 1}$ is a Cauchy sequence in $L^q(\Omega; C^t(\overline{D}))$ with limit $\widetilde{Z}$. Since limits in $L^q(\Omega; C^0(\overline{D}))$ are unique and since the embedding $C^t(\overline{D}) \subset C^0(\overline{D})$ is continuous, $Z = \widetilde{Z}$ up to indistinguishability. □

The following proposition will give conditions for a class of systems $(\psi_j)_{j\geq 1}$ such that the resulting lognormal random fields satisfy the assumption in (**A3**).

Proposition 19 *Let $(\psi_j)_{j\geq 1}$ satisfy the scaling in (34) for some $\widehat{\alpha} > 0$ and let (**A4**) hold. For every $t \in (0, \widehat{\alpha})$ and every $q \in [1, \infty)$, $a \in L^q(\Omega; C^t(\overline{D}))$ and there exists a constant C such that for every $s \in \mathbb{N}$,*

$$\|a^s\|_{L^q(\Omega;C^t(\overline{D}))} \leq C.$$

Proof. Without loss of generality, let us assume that $t \notin \mathbb{Z}$. Hölder norms of compositions with the exponential function have been estimated in [17, Lemma A.1]. We recall from its proof the following estimate [17, Equation (28)]: there exists a constant C such that for every $v \in C^t(\overline{D})$,

$$\|\exp(v)\|_{C^t(\overline{D})} \leq C \|\exp(v)\|_{C^0(\overline{D})} \left(1 + \|v\|^{\lceil t \rceil}_{C^t(\overline{D})}\right).$$

This estimate follows by induction (cp. the proof of [17, Lemma A.1]) based on the facts that $\|\exp(v)\|_{C^{t'}(\overline{D})} \leq \|\exp(v)\|_{C^0(\overline{D})}(1 + \|v\|_{C^{t'}(\overline{D})})$ for every $v \in C^{t'}(\overline{D})$, $t' \in (0,1)$, and that there exists a constant C' such that for every $w, v \in C^t(\overline{D})$, $\|wv\|_{C^t(\overline{D})} \leq C' \|w\|_{C^t(\overline{D})} \|w\|_{C^t(\overline{D})}$. The first estimate is easily seen, the second estimate is for example due to [6, Theorem 16.28].

The assertion follows now with an application of the Cauchy–Schwarz inequality, Proposition 3, and Proposition 18, where the strong measurability of $a = \exp(Z)$ in $C^t(\overline{D})$ follows since the composition with the exponential function is a continuous mapping from $C^t(\overline{D})$ to $C^t(\overline{D})$. □

10 Application to Gaussian random fields with Matérn covariance

A frequently considered example is the case that the Gaussian random field Z is stationary. Then, the *covariance kernel* $k(x, x')$ of Z depends only on $x - x'$. A particular, parametric family of covariances for stationary Gaussian random fields which is widely used in applications is due to B. Matérn [27]. Its parametric covariance kernel in the isotropic case is given by

$$\mathbb{E}(Z(x)Z(x')) = k(x - x') = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}|x - x'|}{\lambda}\right)^\nu K_\nu \left(\frac{\sqrt{2\nu}|x - x'|}{\lambda}\right), \quad x, x' \in D, \tag{36}$$

where Γ denotes the gamma function and K_ν denotes the modifed Bessel function of the second kind. Gaussian random fields with Matérn covariance have become widely used in numerical modelling of problems of the type considered in this paper, cp. [14, 2, 27]. We will briefly recall known MRA representations of Gaussian random fields with Matérn covariance from [2] and compare our QMC convergence results with product weights to corresponding QMC convergence results with POD weights from [14], where also Gaussian random fields with Matérn covariance were analyzed using the Karhunen–Loève expansion. Function systems $(\psi_j)_{j\geq 1}$ result by applying the square root of the covariance operator $\mathcal{C}$ of Z that corresponds to the kernel k to a L^2-orthonormal basis $(\widetilde{\psi}_j)_{j\geq 1}$ in the case that Z is obtained by restricting an auxiliary Gaussian random field defined to an axiparallel cube containing the physical domain D, cp. [2, Sections 4 and 2]. The parameter $\nu > 0$ corresponds to the smoothness of (realisations of) Z.

By a similar argument used to prove [17, Theorem 2.2], Z takes values in $C^t(\overline{D})$, μ-a.s., and $Z \in L^q(\Omega; C^t(\overline{D}))$, for every $t < \nu$ and every $q \in [1, \infty)$. The parameter $\lambda > 0$ is referred to as *correlation length*. It has been shown [2, Corollary 4.3] that for tensorized, orthonormal Meyer wavelets $(\psi_{\mathrm{ONB},j})_{j\geq 1}$, $\psi_{\mathrm{wav},j} = \sqrt{\mathcal{C}}\psi_{\mathrm{ONB},j}$, $j \geq 1$, satisfy Assumption (**A1**) with $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$, for every $\widehat{\beta} < \nu$ and that the Gaussian random field $Z = \sum_{j\geq 1} y_j \psi_{\mathrm{wav},j}$ has the desired Matérn covariance for i.i.d. $\boldsymbol{y} = (y_j)_{j\geq 1}$. Globally supported Karhunen–Loève bases denoted by $(\psi_{\mathrm{KL},j})_{j\geq 1}$ are also available, cp. [2, Sections 3 and 4], which are restrictions of trigonometric functions, cp. [2, Section 3], and $Z = \sum_{j\geq 1} y_j \sqrt{\lambda_j}\psi_{\mathrm{KL},j}$ for i.d.d., standard normally distributed $(y_j)_{j\geq 1}$, where $(\lambda_j)_{j\geq 1}$ are the eigenvalues of $\mathcal{C}$ in descending order. Let us denote by u^s_{wav} the dimensionally truncated solution w.r. to $(\psi_{\mathrm{wav},j})_{j\geq 1}$ and by u^s_{KL} the dimensionally truncated solution w.r. to $(\psi_{\mathrm{KL},j})_{j\geq 1}$.

The truncation error with respect to $(\psi_{\mathrm{wav},j})_{j\geq 1}$ is with right hand side $f \in V^*$, functional $G(\cdot) \in V^*$, and $d/\nu \leq 4$ by Proposition 7, for any arbitrarily small $\varepsilon > 0$,

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s_{\mathrm{wav}}))| = \mathcal{O}\left(s^{-(2\nu/d-1/2)+\varepsilon}\right)$$

Suppose that $f \in H^{-1+\tau}$ and $G(\cdot) \in H^{-1+\tau'}$ for $\max\{\tau, \tau'\} < \min\{\nu/d, \pi/\beta_{\max}\}$. The combined error of dimension truncation and QMC with product weights w.r. to Gaussian weight functions applied to $(\psi_{\mathrm{wav},j})_{j\geq 1}$ and FE discretization is by Theorem 17 for arbitrary $\varepsilon > 0$

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q_{s,N}(G(u^{s,h}_{\mathrm{wav}}))|^2)} = \mathcal{O}\left(s^{-(2\frac{\nu}{d}-\frac{1}{2})+\varepsilon} + (\varphi(N))^{-(\frac{1}{2}\min\left\{\frac{3}{2},\frac{\nu}{d}\right\}+\frac{1}{4})+\varepsilon} + h^{\tau+\tau'}\right). \tag{37}$$

The truncation error for u^s_{KL} is analyzed in [14, Proposition 9], where it was shown that

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s_{\mathrm{KL}}))| = \mathcal{O}\left(s^{-(\nu/d-1/2)+\varepsilon}\right).$$

With reference to a numerical experiment in [14, Section 6] it was conjectured on [14, p. 341] that an improved rate of $\approx \nu/d$ could hold. In the particular setting here (Z is the restriction of a Gaussian random field with Matérn covariance defined on a suitable product domain which contains the physical domain D), we prove this in the following proposition to be able to arrive at a comparison between the overall computational cost of QMC with product weights and $(\psi_{\mathrm{wav},j})_{j\geq 1}$ and of QMC with POD weights and $(\psi_{\mathrm{KL},j})_{j\geq 1}$. The difference in our proof technique will be not to rely on estimates of the gradients of the Karhunen–Loève eigenfunctions.

Proposition 20 *Suppose that $a \in L^q(\Omega; C^t(\overline{D}))$ for every $q \in [1, \infty)$ and some $t > 0$. Suppose also that $\|\psi_{\mathrm{KL},j}\|_{L^\infty(D)}$ is uniformly bounded in $j \geq 1$ and $f \in H^{-1+\tau}(D)$ for some $\tau > 0$. Then,*

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s_{\mathrm{KL}}))| = \mathcal{O}\left(s^{-\nu/d+\varepsilon}\right).$$

Proof. By [14, Corollary 5] or [2, Theorem 3.1 and Remark 2.2], there exists a constant $C > 0$ such that the eigenvalues of the covariance operator induced by the integration kernel $k(\cdot)$ in (36) satisfy

$$\lambda_j \leq C j^{-(1+2\nu/d)}, \quad j \geq 1. \tag{38}$$

Let $n \in \mathbb{N}$ be arbitrary. Since in the considered case the Karhunen–Loève eigenfunctions

$(\psi_{\mathrm{KL},j})_{j\geq 1}$ are uniformly bounded w.r. to $x \in D$ and $j \geq 1$,

$$\begin{aligned}\|Z - Z^s_{\mathrm{KL}}\|^{2n}_{L^{2n}(\Omega;L^{2n}(D))} &= \int_D \mathbb{E}\left(\left(\sum_{j>s} \sqrt{\lambda_j} y_j \psi_{\mathrm{KL},j}\right)^{2n}\right) \mathrm{d}x \\ &= \frac{(2n)!}{2^n n!} \int_D \left(\sum_{j>s} \lambda_j \psi^2_{\mathrm{KL},j}\right)^n \mathrm{d}x \\ &\leq \frac{(2n)!}{2^n n!} |D| C^{2n} \left(\sum_{j>s} \lambda_j\right)^n,\end{aligned}$$

where $C = \sup_{j\geq 1} \|\psi_{\mathrm{KL},j}\|_{L^\infty(D)}$. Here, we used $\sum_{j>s} \sqrt{\lambda_j} \psi_{\mathrm{KL},j}(x) \sim \mathcal{N}(0, \sum_{j>s} \lambda_j \psi^2_{\mathrm{KL},j}(x))$, a.e. $x \in D$, and the fact that the moments of a centered Gaussian random variable satisfy that $\mathbb{E}(X^{2n}) = (2n)!/(2^n n!)\mathbb{E}(X^2)^n$. By (38), for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exist $C, C' > 0$ such that

$$\|Z - Z^s_{\mathrm{KL}}\|_{L^{2n}(\Omega;L^{2n}(D))} \leq C \left(\sum_{j>s} \lambda_j\right)^{1/2} \leq C' s^{-\nu/d+\varepsilon}. \tag{39}$$

The solution map $\mathcal{S}_f$ is locally Lipschitz continuous, i.e., for $p \in (2,\infty)$ and $p' = 2p/(p-2)$,

$$\|u - u^s_{\mathrm{KL}}\|_V \leq \frac{1}{a^s_{\mathrm{KL,min}}} \|\nabla u\|_{L^p(D)} \|a - a^s_{\mathrm{KL}}\|_{L^{p'}(D)}. \tag{40}$$

Since $a \in L^q(\Omega; C^t(\overline{D}))$ for every $q \in [1,\infty)$ and some $t > 0$, by (29) there exists $\tau' \leq \tau$ such that $\tau' \in (0, \max\{t, \pi/\beta_{\max}\}) \setminus \{1/2\}$ such that $u \in L^q(\Omega; H^{1+\tau'})$; by assumption $f \in H^{-1+\tau}(D)$. The Sobolev embedding theorem (see for example [33, Theorem 1.107]) implies that the embedding $H^{1+\tau'}(D) \subset W^{1,p}(D)$ is continuous for $1+\tau'-2/d > 1-d/p$, which is satisfied by any $p \in \mathbb{R}$ such that $2 < p < 2d/(d-2\tau')$ for $d - 2\tau' \geq 0$ and $p \in (2,\infty)$ otherwise. We fix such a p and conclude with (40) (using $n = \max\{\lceil p'/2 \rceil, 4\}$), (39), and the simple estimate $|e^{b_1} - e^{b_2}| \leq (e^{b_1} + e^{b_2})|b_1 - b_2|$, for any $b_1, b_2 \in \mathbb{R}$,

$$\begin{aligned}|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s_{\mathrm{KL}}))| &\leq C \|G\|_{V^*} \|u\|_{L^2(\Omega;H^{1+\tau'}(D))} \|a - a^s_{\mathrm{KL}}\|_{L^2(\Omega;L^{p'}(D))} \\ &\leq C' \|u\|_{L^2(\Omega;H^{1+\tau'}(D))} \|a + a^s_{\mathrm{KL}}\|_{L^4(\Omega;L^{p'}(D))} \|Z - Z^s_{\mathrm{KL}}\|_{L^{2n}(\Omega;L^{2n}(D))} \\ &\leq C'' s^{-\nu/d+\varepsilon},\end{aligned}$$

where $\|u\|_{L^2(\Omega;H^{1+\tau'}(D))}$ is finite by a multiple application of the Cauchy–Schwarz inequality, (29), and by the assumption that $a \in L^q(\Omega; C^t(\overline{D}))$ for every $q \in [1,\infty)$ and $t > 0$. The term $\|a^s_{\mathrm{KL}}\|_{L^4(\Omega;L^{p'}(D))}$ can be bounded independently of s by a version of Proposition 3, where it is easy to see that the law of a is a Gaussian measure on $L^{p'}(D)$ using (38), since $L^{p'}(D)$ is a reflexive Banach space. □

Remark 21 *Generally, the uniform bound on $\|\psi_{\mathrm{KL},j}\|_{L^\infty(D)}$ with respect to $j \geq 1$ in Proposition 20 can be achieved by rescaling λ_j accordingly. Then, by the proof of Proposition 20, if $\lambda_j \sim j^{-(1+2\eta)}$ for some $\eta > 0$, the truncation error satisfies the asymptotic bound $\mathcal{O}(s^{-\eta+\varepsilon})$ for any $\varepsilon \in (0,\eta)$.*

Recall that $f \in H^{-1+\tau}(D)$ and $G(\cdot) \in H^{-1+\tau'}(D)$ for $\max\{\tau, \tau'\} < \min\{\nu/d, \pi/\beta_{\max}\}$. The overall error of QMC with POD weights with the Karhunen–Loève expansion with $(\psi_{\mathrm{KL},j})_{j\geq 1}$ is by [14, Theorem 20 and Corollary 5] and Proposition 20

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q^{\boldsymbol{\Delta}}_{s,N}(u^s_{\mathrm{KL}})|^2)} = \mathcal{O}\left(s^{-\nu/d+\varepsilon} + (\varphi(N))^{-\min\{1,\frac{\nu}{d}\}+\varepsilon} + h^{\tau+\tau'}\right). \tag{41}$$

The computational cost of either of the variants is the cost of constructing the QMC points by the fast CBC algorithm plus the cost of evaluating the QMC quadrature with approximate integrands obtained by FE discretization. We assume the cost of one integrand evaluation to equal the cost to assemble the stiffness matrix plus the cost to solve the linear system. Since for both representation systems FFT techniques are available, we suppose that the cost of assembling the stiffness matrix is $\mathcal{O}(M\log(M)+s\log(s))$, where $M=\dim(V_h)=\mathcal{O}(h^{-d})$ (this assumes that the FE basis functions have isotropic support as is customary in most engineering FE applications; for sparse grid FEM in D analogous considerations can be made). The solution of the linear system may be approximated by an iterative solver such as a multigrid method or preconditioned conjugate gradient methods. This has been analyzed with consistency in the $L^q(\Omega;V)$-norm, $q\in[1,\infty)$, in [16] with computational cost $\mathcal{O}(M^{1+\varepsilon})$ for any $\varepsilon>0$, cp. [16, Corollary 6.3]. Therefore, we may suppose that the cost of the approximate solution is $\mathcal{O}(M^{1+\varepsilon})$.

To compare the error versus the computational cost of both algorithms, we equilibrate the error contributions in (37) and in (41) of the FE discretization and the dimension truncation, respectively. Starting with the case of QMC-FE with product weights and multiresolution representation of the Gaussian random field input, this results in the choices $s=M^{(\tau+\tau')/(2\nu-d/2)}$ and $N=M^{4(\tau+\tau')/(2\nu+d)}$ for $d/\nu\in(2/3,2)$, where we have used the work bound with the (formal) limiting value $\varepsilon=0$. To calculate the computational cost we will continue to use $\varepsilon=0$ and, moreover, ignore log-factors for simplicity. The computational cost for QMC-FE with product weights and a multiresolution representation is then

$$\mathrm{work}_{\mathrm{PROD,\,wav}}=\mathcal{O}(sN+N(s+M))=\mathcal{O}(N(s+M))=\mathcal{O}\left(M^{\frac{4(\tau+\tau')}{2\nu+d}+\max\{\frac{\tau+\tau'}{2\nu-d/2},1\}}\right).$$

For a prescribed error $0<\delta=\mathcal{O}(M^{-(\tau+\tau')/d})$ and assuming $d/\nu\in(2/3,2)$,

$$\mathrm{error}=\mathcal{O}(\delta)\quad\text{is ensured with}\quad \mathrm{work}_{\mathrm{PROD,\,wav}}=\mathcal{O}\left((\delta^{-1})^{d(\frac{4}{2\nu+d}+1/\min\{2\nu-d/2,\tau+\tau'\})}\right).$$

In the case of QMC with POD weights and Karhunen–Loève expansion equilibrating the error contributions in (41) of the FE discretization and dimension truncation gives the choices $s=M^{(\tau+\tau')/\nu}$ and $N=M^{(\tau+\tau')/\nu}$ for $d/\nu\in(1,2)$. For $d/\nu\le1$, choose $N=M^{(\tau+\tau')/d}$. The computational cost for QMC-FE with POD weights and Karhunen–Loève expansion is then (again ignoring log-factors)

$$\mathrm{work}_{\mathrm{POD,\,KL}}=\mathcal{O}(s^2N+N(s+M))=\mathcal{O}(N(s^2+M))=\mathcal{O}\left(M^{\frac{\tau+\tau'}{\nu}+\max\{\frac{2(\tau+\tau')}{\nu},1\}}\right),$$

where the s^2 term is a consequence of the cost of the CBC construction with POD weights. For an error threshold $0<\delta=\mathcal{O}(M^{-(\tau+\tau')/d})$ and assuming $d/\nu\in(1,2)$,

$$\mathrm{error}=\mathcal{O}(\delta)\quad\text{is ensured with}\quad \mathrm{work}_{\mathrm{POD,\,KL}}=\mathcal{O}\left((\delta^{-1})^{d(\frac{1}{\nu}+1/\min\{\frac{\nu}{2},\tau+\tau'\})}\right).$$

As an illustration we now consider the two borderline cases $d/\nu\in\{1,2\}$ and compare the asymptotic complexity of the two algorithms. For $d/\nu=1$, we obtain that

$$\mathrm{work}_{\mathrm{PROD,\,wav}}=\mathcal{O}\left((\delta^{-1})^{\frac{4}{3}+\max\{\frac{2}{3},\frac{d}{\tau+\tau'}\}}\right)\quad\text{vs.}\quad \mathrm{work}_{\mathrm{POD,\,KL}}=\mathcal{O}\left((\delta^{-1})^{1+\max\{2,\frac{d}{\tau+\tau'}\}}\right).$$

Here, $\mathrm{work}_{\mathrm{PROD,\,wav}}$ is asymptotically smaller than $\mathrm{work}_{\mathrm{POD,\,KL}}$ in the regime that $d/(\tau+\tau')\le 5/3$. For $d=2$, this implies that QMC-FE with product weights and multiresolution representation is superior if $\tau+\tau'\ge5/6$. For the limiting case $d/\nu=2$, we *formally* obtain

$$\mathrm{work}_{\mathrm{PROD,\,wav}}=\mathcal{O}\left((\delta^{-1})^{2+\max\{2,\frac{d}{\tau+\tau'}\}}\right)\quad\text{vs.}\quad \mathrm{work}_{\mathrm{POD,\,KL}}=\mathcal{O}\left((\delta^{-1})^{2+\max\{4,\frac{d}{\tau+\tau'}\}}\right).$$

Then, the complexity for QMC-FE with product weights and multiresolution representation of the Gaussian random field is always asymptotically smaller. Notice that the appearance of 4 in the POD weight case, where in the product weight case there is 2, is a consequence of the construction cost of the QMC points which is, for POD weights, $\mathcal{O}(sN\log(N) + s^2N)$ and in the case of product weights $\mathcal{O}(sN\log(N))$.

Remark 22 *In [21, Equation (4.3)] the authors stipulated an error estimate for the dimension truncation with rate $2\nu/d$. As a consequence, for QMC-FE with POD weights and Karhunen–Loève expansion the truncation dimension would be chosen as $s = M^{(\tau+\tau')/(2\nu)}$. Then, for example in the case $d/\nu = 2/3$ and $d = 2$, QMC-FE with product weights and multiresolution representation would have a strictly smaller complexity than QMC-FE with POD weights and Karhunen–Loève expansion for $\tau + \tau' > 3$. For $\tau + \tau' \leq 3$, the complexity of both would be asymptotically equal (again ignoring* log*-factors and considering the formal case $\varepsilon = 0$). For more details on estimates of the truncation error with globally supported ψ_j, $j \geq 1$, the reader is referred to [4].*

11 Numerical experiments

To illustrate our error bounds, we consider a univariate model problem on the interval $D = (0, 1)$. The Gaussian random field Z will be a Wiener process conditioned on $Z(0) = 0 = Z(1)$ (a.k.a. "Brownian bridge") which is well-known to admit Karhunen–Loève expansions (i.e., the functions ψ_j are globally supported in D) and Lévy–Ciesielski expansions where the ψ_j have localized supports in D. Specifically,

$$Z = \sum_{\ell \geq 0} \sum_{k=0}^{2^\ell - 1} y_{\ell,k} \psi_{\mathrm{wav},\ell,k}, \quad \psi_{\mathrm{wav},\ell,k}(x) := 2^{-\widehat{\alpha}\ell} \psi(2^\ell x - k), \quad \ell \geq 0, k = 0, \ldots, 2^\ell - 1, \tag{42}$$

where $\psi(x) := \max\{1 - |2x - 1|, 0\}$. The Lévy–Ciesielski or Brownian bridge representation of the Gaussian random field is realized for $\widehat{\alpha} = 1/2$. The function system $\psi_{\mathrm{wav},\ell,k}$, $\ell \geq 0, k = 0, \ldots, 2^\ell - 1$, satisfies (**A1**) with $b_{\ell,k} := c2^{-\widehat{\beta}\ell}$, $\ell \geq 0, k = 0, \ldots, 2^\ell - 1$, for every $\widehat{\beta} \in (0, \widehat{\alpha})$ and some $c > 0$. The Karhunen–Loève expansion is for $\widehat{\alpha} = 1/2$ obtained with $\sqrt{\lambda_j}\psi_{\mathrm{KL},j}(x) := \sqrt{2}\sin(\pi jx)/(\pi j)$, $j \geq 1$. We consider the lognormal coefficient

$$a = \exp(\sigma Z)$$

with a scaling parameter $\sigma > 0$ and Z given by the expansion in (42). The functional $G(\cdot)$ is chosen to be point evaluation at $\bar{x} = 0.7$, i.e., $G(v) := v(0.7)$ for every $v \in H_0^1(D)$, which is continuous if D is an interval due to the continuous embedding $H^r(D) \subset C^t(\overline{D})$ for $0 < t < r - 1/2$.

We present numerical experiments with QMC by randomly shifted lattice rules from [28] with Gaussian weight functions, where the generating vector is computed by the fast CBC algorithm from [29, 30], and we use product weights according to (25). Note that the use of product weights simplifies the fast CBC algorithm which is for POD weights discussed in [28, Section 5.2]. We use first order FE to discretize the spatial variable and keep the mesh width h and truncation dimension s fixed to be able to examine the QMC error. The FE mesh and the number of terms in the truncated, piecewise linear expansion of Z with $\psi_{\mathrm{wav},\ell,k}$ is aligned such that the stiffness matrix can be computed exactly. This can be achived with $s = 2^{L+1} - 1$ and $h = 2^{-(L+1)}$, $L \geq 1$. We show numerical experiments for $L = 12$, which results in $s = 8191$ dimensions. Also we choose the borderline values $\widehat{\alpha} = \widehat{\beta}$ and $\varepsilon = 0$ in (25). We consider QMC points $N = 3^m$,

$m = 1, \ldots, 11$. The mean square error is approximated by the empirical variance of R samples Q_j corresponding to i.i.d. random shifts, with the unbiased estimator

$$\sqrt{\frac{1}{R-1}\sum_{j=1}^{R}(Q_j - \bar{Q})^2} \approx \sqrt{\mathbb{E}^{\mathbf{\Delta}}(|G(u^{s,h}) - Q^{\mathbf{\Delta}}_{s,N}(G(u^{s,h}))|^2)}.$$

The reference value $\bar{Q}$ is the average over R i.i.d. random shifts of $Q^{\mathbf{\Delta}}_{s,N}(G(u^{s,h}))$ with $N = 3^{12}$ QMC points. The FE solver and QMC integration routine is executed using Python. The empirical rates in Figures 1 and 2 are computed by a least squares approximation taking into account the 7 data points that correspond to $N = 3^m$, $m = 5, \ldots, 11$.

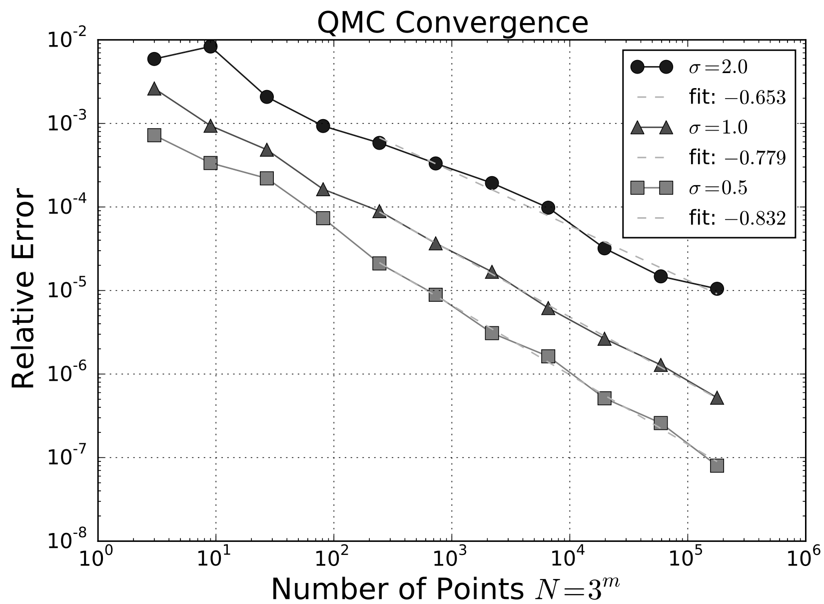

Figure 1: Parameter choices $\widehat{\alpha} = 1/2 = \widehat{\beta}$, $\sigma \in \{0.5, 1.0, 2.0\}$, $\alpha_{\mathrm{g}} = 1.05$, $c = 0.1$, $s = 8191$ dimensions, and $R = 20$ random shifts

In Figure 1, convergence tests with QMC with product weights and Gaussian weight functions are presented for the Brownian bridge ($\widehat{\alpha} = 1/2$). This is the borderline case of our theory on QMC with product weights and Gaussian weight functions. There, exponential weight functions are not applicable according to Theorem 13. In line with theory, a convergence rate of $\approx 1/2$ is expected, which is the same as for Monte Carlo sampling. However, the observed rate is higher and seems to depend on the scaling factor σ of the Gaussian random field Z. This dependence was also observed in the numerical experiments using the Karhunen–Loève expansion representation of Z and POD weights in [14, Tables 1 and 2]. In Figure 2, also QMC with product weights and Gaussian weight functions is used and the decay of the functions $\psi_{\mathrm{wav},\ell,k}$ is chosen to be stronger, i.e., $\widehat{\alpha} = 1$. Then, Theorem 11 implies a convergence rate of ≈ 0.75. Note that in this case with exponential weight functions, only a convergence rate of $\approx 1/2$ can be deduced from Theorem 13. Numerical experiments with QMC with product weights and exponential weight functions are reported by the authors in [19]. There, larger values of $\widehat{\alpha}$ were required to achieve the presently observed rates.

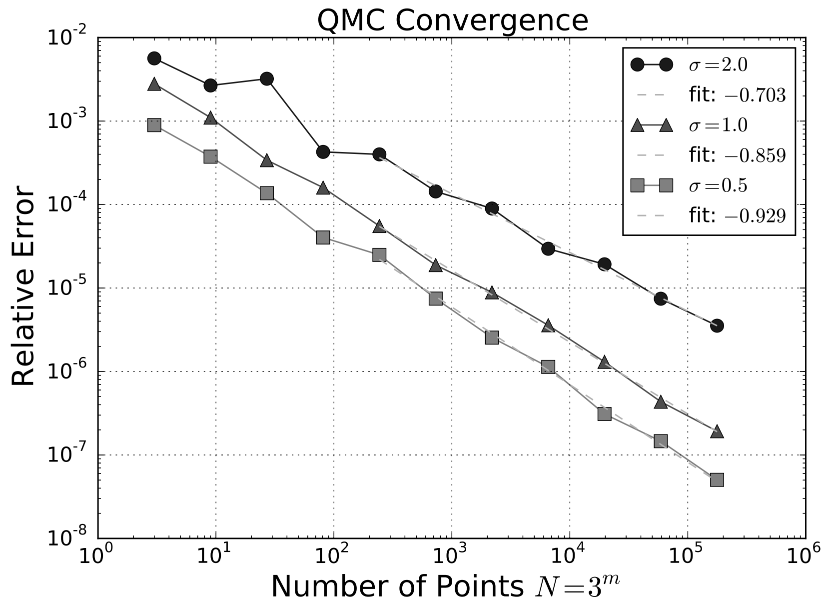

Figure 2: Parameter choices $\widehat{\alpha} = 1 = \widehat{\beta}$, $\sigma \in \{0.5, 1.0, 2.0\}$, $\alpha_{\mathrm{g}} = 2.05$, $c = 0.1$, $s = 8191$ dimensions, and $R = 20$ random shifts

12 Conclusions and generalizations

We extended and refined the QMC error analysis for the parametric, deterministic solutions of the linear elliptic partial differential equation (1) with log-Gaussian coefficient a as given in (2), (3). In particular, we considered locally supported functions $(\psi_j)_{j\geq 1}$ in (2). The assumed local support of the ψ_j and p-summability implied dimension-independent convergence rates of randomly shifted lattice rule quadratures of the parametric solution of (1) - (3) *with product weight sequences*. The present results constitute an extension of the convergence rate bounds in [14], wherein the global supports of the $(\psi_j)_{j\geq 1}$ implied POD weight sequences for the QMC quadratures. Moreover in Section 10 the presently developed QMC-FE theory is compared to the one from [14] for Gaussian random fields with Matérn covariance in terms of computational cost versus accuracy. For certain values of the Matérn smoothness parameter ν the computational cost of the resulting QMC-FEM is significantly smaller using the newly introduced QMC-FE with product weights and multiresolution representation of the Gaussian random field Z. In the case of Gaussian weight functions in the norm (9), in Theorem 11 the summability $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for $2/3 < p < 2$ was admissible in the present QMC convergence analysis for representations of Z in terms of locally supported ψ_j. This constitutes a refinement over the error vs. computation cost bounds resulting from [14]. Another insight of this paper is that for locally supported ψ_j, QMC-FE with Gaussian weight functions seems to satisfy error bounds that are superior to those obtained with exponential weight functions. As a byproduct of the present analysis, we also obtained dimension-independent convergence rate estimates for globally supported $(\psi_j)_{j\geq 1}$ as in [14], and with exponential weight functions. The use of product weights, however, entails stronger summability conditions on $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$ than those in [14] in order to achieve a certain convergence rate (Corollary 14, item 2.). This drawback may be offset by the linear with respect to dimension s scaling construction cost for the QMC quadrature rules provided stronger summability conditions are satisfied, for example for sufficiently large ν in the case of Gaussian random fields with Matérn covariance.

Numerical experiments in Section 11 confirm the theoretical error bounds and illustrate the practicality of QMC with product weights and Gaussian weight functions. The experiments considered piecewise linear multiscale approximations of the Gaussian random field. Another class of admissible models are piecewise constant approximations of Gaussian random fields as for example by Haar wavelets (considered in [12, Section 9] in the case of "affine-coefficients") or by indicator functions of s disjoint subsets of D multiplied by a suitable decreasing sequence. Then, the QMC convergence rate also does not depend on the dimension s.

The present work addressed only the *single-level* QMC Finite-Element algorithm, where the same FE space is employed for PDE discretization in all QMC points. The principal results of the present paper, Theorems 11 and 13, allow for *multi-level* extensions of the presently proposed algorithms, which can be designed and analyzed along the lines of [21]. These extensions, as well as additional numerical experiments, are developed by the authors in [18, 19].

References

[1] Markus Bachmayr, Albert Cohen, Ronald DeVore, and Giovanni Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. part II: lognormal coefficients. *ESAIM Math. Model. Numer. Anal*, 51(1):341–363, 2017.

[2] Markus Bachmayr, Albert Cohen, and Giovanni Migliorati. Representations of Gaussian random fields and approximation of elliptic PDEs with lognormal coefficients. *J. Fourier Anal. Appl.*, 2017.

[3] Vladimir I. Bogachev. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[4] Julia Charrier. Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.*, 50(1):216–246, 2012.

[5] Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4.

[6] Gyula Csató, Bernard Dacorogna, and Olivier Kneuss. *The Pullback Equation for Differential Forms*. Progress in Nonlinear Differential Equations and their Applications, 83. Birkhäuser/Springer, New York, 2012.

[7] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and Its Applications, Vol. 152. Cambridge: Cambridge University Press, second edition, 2014.

[8] Josef Dick, Frances Y. Kuo, and Ian H. Sloan. High-dimensional integration: the quasi-Monte Carlo way. *Acta Numer.*, 22:133–288, 2013.

[9] Jürgen Dölz, Helmut Harbrecht, and Christoph Schwab. Covariance regularity and H-matrix approximation for rough random fields. *Numer. Math.*, 135(4):1045–1071, 2017.

[10] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London, 1958.

[11] William Feller. *An Introduction to Probability Theory and Its Applications. Vol. I.* Third edition. John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[12] Robert N. Gantner, Lukas Herrmann, and Christoph Schwab. Quasi–Monte Carlo integration for affine-parametric, elliptic PDEs: local supports and product weights. *SIAM J. Numer. Anal.*, 56(1):111–135, 2018.

[13] Robert N. Gantner and Christoph Schwab. Computational higher order quasi-Monte Carlo integration. In *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC, Leuven, Belgium, April 2014*, volume 163, pages 271–288, 2016.

[14] Ivan G. Graham, Frances Y. Kuo, James A. Nichols, Robert Scheichl, Christoph Schwab, and Ian H. Sloan. Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients. *Numer. Math.*, 131(2):329–368, 2015.

[15] Pierre Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[16] Lukas Herrmann. Strong convergence analysis of iterative solvers for random operator equations. Technical Report 2017-35, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2017.

[17] Lukas Herrmann, Annika Lang, and Christoph Schwab. Numerical analysis of lognormal diffusions on the sphere. *Stoch. PDE: Anal. Comp.*, 2017.

[18] Lukas Herrmann and Christoph Schwab. Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients. Technical Report 2017-19, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2017.

[19] Lukas Herrmann and Christoph Schwab. QMC algorithms with product weights for lognormal-parametric, elliptic PDEs. Technical Report 2017-04, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2017.

[20] Viet Ha Hoang and Christoph Schwab. N-term Wiener chaos approximation rate for elliptic PDEs with lognormal Gaussian random inputs. *Math. Models Methods Appl. Sci.*, 24(4):797–826, 2014.

[21] Frances Y. Kuo, Robert Scheichl, Christoph Schwab, Ian H. Sloan, and Elisabeth Ullmann. Multilevel Quasi-Monte Carlo methods for lognormal diffusion problems. *Math. Comp.*, 86(308):2827–2860, 2017.

[22] Frances Y. Kuo, Christoph Schwab, and Ian H. Sloan. Quasi-Monte Carlo methods for high-dimensional integration: the standard (weighted Hilbert space) setting and beyond. *ANZIAM J.*, 53(1):1–37, 2011.

[23] Frances Y. Kuo, Christoph Schwab, and Ian H. Sloan. Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.*, 50(6):3351–3374, 2012.

[24] Frances Y. Kuo, Ian H. Sloan, Grzegorz W. Wasilkowski, and Benjamin J. Waterhouse. Randomly shifted lattice rules with the optimal rate of convergence for unbounded integrands. *J. Complexity*, 26(2):135–160, 2010.

[25] Annika Lang and Christoph Schwab. Isotropic Gaussian random fields on the sphere: regularity, fast simulation and stochastic partial differential equations. *Ann. Appl. Probab.*, 25(6):3047–3094, 2015.

[26] Michel Loève. *Probability Theory I*. Springer-Verlag, New York, fourth edition, 1977. Graduate Texts in Mathematics, Vol. 45.

[27] Bertil Matérn. *Spatial variation*, volume 36 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, second edition, 1986. With a Swedish summary.

[28] James A. Nichols and Frances Y. Kuo. Fast CBC construction of randomly shifted lattice rules achieving $\mathcal{O}(n^{-1+\delta})$ convergence for unbounded integrands over $\mathbb{R}^s$ in weighted spaces with POD weights. *J. Complexity*, 30(4):444–468, 2014.

[29] Dirk Nuyens and Ronald Cools. Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. *Math. Comp.*, 75(254):903–920 (electronic), 2006.

[30] Dirk Nuyens and Ronald Cools. Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points. *J. Complexity*, 22(1):4–28, 2006.

[31] Aretha L. Teckentrup, Robert Scheichl, Mike B. Giles, and Elisabeth Ullmann. Further analysis of multilevel Monte Carlo methods for elliptic PDEs with random coefficients. *Numer. Math.*, 125(3):569–600, 2013.

[32] Sundaram Thangavelu. *Lectures on Hermite and Laguerre expansions*, volume 42 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1993. With a preface by Robert S. Strichartz.

[33] Hans Triebel. *Theory of Function Spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.

[34] Hans Triebel. *Function Spaces and Wavelets on Domains*, volume 7 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2008.

[35] Kôsaku Yosida. *Functional Analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, sixth edition, 1980.

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