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IDENTIFICATION OF AN INCLUSION IN MULTIFREQUENCY ELECTRIC IMPEDANCE TOMOGRAPHY

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Abstract. The multifrequency electrical impedance tomography is considered in order to image a conductivity inclusion inside a homogeneous background medium by injecting one current. An original spectral decomposition of the solution of the forward conductivity problem is used to retrieve the Cauchy data corresponding to the extreme case of perfect conductor. Using results based on the unique continuation we then prove the uniqueness of multifrequency electrical impedance tomography and obtain rigorous stability estimates. Our results in this paper are quite surprising in inverse conductivity problem since in general infinitely many input currents are needed in order to obtain the uniqueness in the determination of the conductivity.

1. The Mathematical Model and main results

In this section we introduce the mathematical model of the multifrequency electrical impedance tomography (mfEIT). Let Ω be the open bounded smooth domain in $\mathbb{R}^d, d = 2, 3$, occupied by the sample under investigation and denote by $\partial\Omega$ its boundary. The mfEIT forward problem is to determine the potential $u(\cdot, \omega) \in H^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)\}$, solution to

$$\left\{ \begin{array}{lll} -\nabla \cdot (\sigma(x,\omega)\nabla u(x,\omega)) = 0 & \text{in} & \Omega, \\ \sigma(x,\omega)\partial_{\nu_\Omega} u(x,\omega)(x) = f(x) & \text{on} & \partial\Omega, \\ \int_{\partial\Omega} u(x,\omega) ds = 0, & & \end{array} \right. \tag{1}$$

where ω denotes the frequency, $\nu_\Omega(x)$ is the outward normal vector to $\partial\Omega$, $\sigma(x, \omega)$ is the conductivity distribution, and $f \in H^{\frac{1}{2}}_{\diamond}(\partial\Omega) := \{g \in H^{\frac{1}{2}}(\partial\Omega) : \int_{\partial\Omega} g \, ds = 0\}$ is the input current.

In this paper we are interested in the case where the frequency dependent conductivity distribution takes the form

$$\sigma(x, \omega) = k_0 + (k(\omega) - k_0)\chi_D(x) \tag{2}$$

with $\chi_D(x)$ being the characteristic function of a smooth inclusion D in Ω ($\overline{D} \subset \Omega$), $k(\omega) : \mathbb{R}_+ \to \mathbb{C} \setminus \overline{\mathbb{R}_-}$, being a continuous complex-valued function, and k_0 being a fixed positive constant (the conductivity of the background medium).

The mfEIT inverse problem is to recover the shape and the position of the inclusion D from measurements of the boundary voltages $u(x, \omega)$ on $\partial\Omega$ for $\omega \in (\underline{\omega}, \overline{\omega})$, $0 \leq \underline{\omega} < \overline{\omega}$. It has many important applications in biomedical imaging. Experimental research has found that the conductivity of many biological tissues varies strongly with respect to the frequency within certain frequency ranges [GPG]. In [AGGJS], using homogenization techniques, the authors analytically exhibit the fundamental mechanisms underlying the fact that effective biological tissue electrical properties and their frequency dependence reflect the tissue composition and physiology. There have been also several numerical studies on frequency-difference imaging. It was numerically shown that the approach can accommodate geometrical errors, including imperfectly known boundary [AAJS, JS, MSHA].

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1.1. **Main results.** Now, we introduce the class of inclusions on which we will study the uniqueness and stability of the mfEIT inverse problem. Without loss of generality we further assume that Ω contains the origin.

Let $b_1 = \mathrm{dist}(0, \partial\Omega)$ and let $b_0 < b_1$. For $\delta > 0$ small enough, and $m > 0$ large enough, define the set of inclusions:

$$\mathfrak{D} := \left\{ D := \{x \in \mathbb{R}^d : |x| < \Upsilon(\widehat{x}), \widehat{x} = \frac{x}{|x|}\};\ b_0 < \Upsilon(\widehat{x}) < b_1 - \delta;\ \|\Upsilon\|_{C^{2,\varsigma}} \leq m,\ \varsigma > 0 \right\}.$$

Then, the mfEIT inverse problem has a unique solution within the class $\mathfrak{D}$, and we have the following stability estimates.

Theorem 1.1. *Let D and $\widetilde{D}$ be two inclusions in $\mathcal{D}$. Denote by u (resp. $\widetilde{u}$) the solution of* (1) *with the inclusion D (resp. $\widetilde{D}$). Let*

$$\varepsilon = \sup_{x \in \partial\Omega, \omega \in (\underline{\omega}, \overline{\omega})} |u - \widetilde{u}|.$$

Then, there exist constants $C > 0$ and $\tau \in (0,1)$, such that the following estimate holds:

$$\left| D \Delta \widetilde{D} \right| \leq C \left(\frac{1}{\ln(\varepsilon^{-1})} \right)^{\tau}, \tag{3}$$

Here, Δ denotes the symmetric difference and the constants C and τ depend only on $f, \Omega, \mathfrak{D}$, and $\Sigma := \{k(\omega); \omega \in (\underline{\omega}, \overline{\omega})\}$.

Theorem 1.2. *Assume that $d = 2$, and let D and $\widetilde{D}$ be two analytic inclusions in $\mathfrak{D}$. Denote by u (resp. $\widetilde{u}$) the solution of* (1) *with the inclusion D (resp. $\widetilde{D}$). Let*

$$\varepsilon = \sup_{x \in \partial\Omega, \omega \in (\underline{\omega}, \overline{\omega})} |u - \widetilde{u}|.$$

Then, there exist constants $C > 0$ and $\tau' \in (0,1)$, such that the following estimate

$$\left| D \Delta \widetilde{D} \right| \leq C \varepsilon^{\tau'}, \tag{4}$$

holds. Here the constants C and τ' depend only on $f, \Omega, \mathfrak{D}$, and Σ.

These results show that the reconstruction of the inclusion from multi-frequency boundary voltage data is improving according to the regularity of the boundary of the inclusion. Precisely, the stability estimates vary from logarithmic to Hölder. They can also be extended to a larger class of inclusions as non-star shaped domains, and to measurements on only a small part of the boundary. In this paper for the sake of simplicity we do not handle such general cases.

The rest of the paper is organized as follows. In section 2, we introduce the variational Poincaré operator. We study in section 3 the regularity of the potential $u(x, \omega)$ as a function of the frequency function $k(\omega)$. Precisely, using a spectral decomposition based on the eigenfunctions of the variational Poincaré operator, we split the potential $u(x, \omega)$ into a frequency part $u_f(x, k(\omega))$ and a non-frequency part $k_0^{-1} u_0(x)$ (Theorem 3.1). Then, we recover the boundary Cauchy data for the non-frequency part from the boundary voltage data (Corollary 3.2). In section 4, we recover the shape and location of the inclusion from the knowledge of the boundary Cauchy data of the non-frequency part $k_0^{-1} u_0(x)$, and prove finally the main results of the paper.

2. The variational Poincaré operator

We first introduce an operator whose spectral decomposition will be later the corner stone of the identification of the inclusion D. Let $H^1_\diamond(\Omega)$ be the space of functions v in $H^1(\Omega)$ satisfying $\int_{\partial\Omega} v ds = 0$.

For $u \in H^1_\diamond(\Omega)$, we infer from the Riesz theorem that there exists a unique function $Tu \in H^1_\diamond(\Omega)$ such that for all $v \in H^1_\diamond(\Omega)$,

$$\int_\Omega \nabla Tu \cdot \nabla v dx = \int_D \nabla u \cdot \nabla v dx. \tag{5}$$

The variational Poincaré operator $T : H^1_\diamond(\Omega) \to H^1_\diamond(\Omega)$ is easily seen to be self-adjoint and bounded with norm $\|T\| \leq 1$.

The spectral problem for T reads as: Find $(\lambda, w) \in \mathbb{R} \times H^1_\diamond(\Omega)$, $w \neq 0$ such that $\forall v \in H^1_\diamond(\Omega)$,

$$\lambda \int_\Omega \nabla w \cdot \nabla v dx = \int_D \nabla w \cdot \nabla v dx.$$

Integrating by parts, one immediately obtains that any eigenfunction w is harmonic in D and in $D' = \Omega \backslash \overline{D}$, and satisfies the transmission and boundary conditions

$$w|^+_{\partial D} = w|^-_{\partial D}, \qquad \partial_{\nu_D} w|^+_{\partial D} = (1 - \frac{1}{\lambda})\partial_{\nu_D} w|^-_{\partial D}, \qquad \partial_{\nu_\Omega} w = 0,$$

where $w|^\pm_{\partial D}(x) = \lim_{t\to 0} w(x \pm t\nu_D(x))$ for $x \in \partial D$. In other words, w is a solution to (1) for $k = k_0(1 - \frac{1}{\lambda})$ and $f = 0$.

Let $\mathfrak{H}_\diamond$ the space of harmonic functions in D and D', with zero mean $\int_{\partial\Omega} u ds(x) = 0$, and zero normal derivative $\partial_{\nu_\Omega} u = 0$ on $\partial\Omega$, and with finite energy semi-norm

$$\|u\|_{\mathfrak{H}_\diamond} = \int_\Omega |\nabla u|^2 dx.$$

Since the functions in $\mathfrak{H}_\diamond$ are harmonic in D', the $\mathfrak{H}_\diamond$ is a closed subspace of $H^1(\Omega)$. Later on, we will give a new characterization of the space $\mathfrak{H}_\diamond$ in terms of the single layer potential on ∂D associated with the Neumann function of Ω.

We remark that $Tu = 0$ for all u in $H^1_0(D')$, and $Tu = u$ for all u in $H^1_0(D)$ (the set of functions in $H^1(D)$ with trace zero).

We also remark that $T\mathfrak{H}_\diamond \subset \mathfrak{H}_\diamond$ and hence the restriction of T to $\mathfrak{H}_\diamond$ defines a linear bounded operator. Since we are interested in harmonic functions in D and D' we only consider the action of T on the closed space $\mathfrak{H}_\diamond$. We further keep the notation T for the restriction of T to $\mathfrak{H}_\diamond$. We will prove later that T has only isolated eigenvalues with an accumulation point $1/2$. We denote by $(\lambda^-_n)_{n\geq 1}$ the eigenvalues of T repeated according to their multiplicity, and ordered as follows

$$0 < \lambda^-_1 \leq \lambda^-_2 \leq \cdots < \frac{1}{2},$$

in $(0, 1/2]$ and, similarly,

$$1 > \lambda^+_1 \geq \lambda^+_2 \geq \cdots > \frac{1}{2}.$$

the eigenvalues in $[1/2, 1)$. The eigenvalue $1/2$ is the unique accumulation point of the spectrum.

Remark 2.1. *In contrast with the Dirichlet Poincaré variational spectral problem,* 0 *is not an eigenvalue of* T*. In fact if* w *is an eigenfunction associated to zero, then it satisfies*

$$\begin{cases} \Delta w(x) = 0 & \text{in} \quad D', \\ \nabla w(x) = 0 & \text{in} \quad D, \\ \partial_{\nu_\Omega} w(x) = 0 & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} w(x) ds(x) = 0. \end{cases}$$

Since this system of equations has only the trivial solution, zero is not in the point spectrum of T*.*

Next, we will characterize the spectrum of T via the mini-max principle.

Proposition 2.1. *The variational Poincaré operator has the following decomposition*

$$T = \frac{1}{2}I + K, \tag{6}$$

where K is a compact self-adjoint operator. Let $w_n^\pm$, $n \geq 1$ be the eigenfunctions associated to the eigenvalues $(\lambda_n^-)_{n\geq 0}$. Then

$$\begin{aligned}
\lambda_1^- &= \min_{0\neq w\in\mathfrak{H}_\diamond} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx},\\
\lambda_n^- &= \min_{0\neq w\in\mathfrak{H}_\diamond, w\perp w_1,\cdots,w_{n-1}} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx},\\
&= \min_{F_n\subset\mathfrak{H}_\diamond,\ dim(F_n)=n}\ \max_{w\in F_n} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx},
\end{aligned}$$

and similarly

$$\begin{aligned}
\lambda_1^+ &= \max_{0\neq w\in\mathfrak{H}_\diamond} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx},\\
\lambda_n^+ &= \min_{0\neq w\in\mathfrak{H}_\diamond, w\perp w_1,\cdots,w_{n-1}} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx},\\
&= \max_{F_n\subset\mathfrak{H}_\diamond,\ dim(F_n)=n}\ \min_{w\in F_n} \frac{\int_D |\nabla w(x)|^2dx}{\int_\Omega |\nabla w(x)|^2dx}.
\end{aligned}$$

Proof. We follow the approach of [BT] for the spectrum of the Poincaré operator in the whole space.

Define the operator $K : \mathfrak{H}_\diamond \to \mathfrak{H}_\diamond$ by

$$2\int_\Omega \nabla Ku\cdot\nabla v dx = \int_D \nabla u\cdot\nabla v dx - \int_{D'} \nabla u\cdot\nabla v dx. \tag{7}$$

Then K is a bounded self-adjoint operator with norm $\|K\| \leq 1$. The first step of the proof is to show that K is indeed a compact operator.

Let $\mathcal{N}(x,z)$ be the Neumann function for the Laplacian in Ω, that is, the solution to

$$\begin{cases} \Delta\mathcal{N}(x,z) = \delta_z & \text{in} \quad \Omega,\\ \partial_{\nu_\Omega}\mathcal{N}(x,z) = \frac{1}{|\partial\Omega|} & \text{on} \quad \partial\Omega,\\ \int_{\partial\Omega}\mathcal{N}(x,z)ds(x) = 0, \end{cases} \tag{8}$$

where δ_z is the Dirac mass at z.

Define the single layer potential $S_D : H^{-\frac{1}{2}}(\partial D) \to \mathfrak{H}_\diamond$ by

$$S_D[\varphi](x) = \int_{\partial D} \mathcal{N}(x,z)\varphi(z)ds(z).$$

Since the Neumann function and the Laplace Green function in the whole space have equivalent weak singularities as $x \to z$, (see for instance Lemma 2.14 in [AK]) the operator S_D satisfies the same jump relations through the boundary of D as the single layer of the Laplace Green function, that is,

$$\partial_{\nu_D}S_D[\varphi](x)|^\pm = \pm\frac{1}{2}\varphi(x) + K_D^*[\varphi](x),$$

for $x \in \partial D$, where $K_D^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$, defined by

$$K_D^*[\varphi](x) = \int_{\partial D} \partial_{\nu_D(x)} \mathcal{N}(x,z)\varphi(z)ds(z),$$

is a compact operator. Here, $H^s(\partial D)$ are the usual Sobolev spaces on ∂D.

It can also be shown that $S_D : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is invertible (this result is not true in general for the single layer the Laplace Green function in dimension two. Nevertheless, the operator S_D can be slightly modified to ensure invertibility [AnK]).

Integrating by parts over D and D' in (7), using the jump conditions and the fact that u lies in $\mathfrak{H}_\diamond$, we obtain

$$\int_\Omega \nabla K u \cdot \nabla v dx = \int_{\partial D} K_D^* \left[S_D^{-1}[u|_{\partial D}]\right] v ds(x).$$

Since K_D^* is compact, the operator K is also compact.

A direct calculation shows that the operator T has the following decomposition

$$T = \frac{1}{2} I + K.$$

Then T is Fredholm operator of index zero and enjoy the same spectral decomposition as well as the min-max principle than the self-adjoint and compact operator K.

□

Remark 2.2. *We first remark that this result does not hold true if D is merely Lipschitz. Finally, the space $\mathfrak{H}_\diamond$ can be defined as follows*

$$\mathfrak{H}_\diamond := \left\{ S_D[\varphi]; \;\; \varphi \in H^{-\frac{1}{2}}(\partial D) \right\}.$$

Considering this characterization, it is clear that $\mathfrak{H}_\diamond$ is a closed subspace in $H^1_\diamond(\Omega)$.

We further normalize the eigenfunctions $w_n^\pm$, $n \geq 1$ in $\mathfrak{H}_\diamond$. A direct consequence of the previous result is the following spectral decomposition of functions in $\mathfrak{H}_\diamond$.

Corollary 2.1. *Let u be in $\mathfrak{H}_\diamond$. Then u has the following spectral decomposition in $\mathfrak{H}_\diamond$:*

$$u(x) = \sum_{n=1}^{\infty} u_n^\pm w_n^\pm(x),$$

where

$$u_n^\pm = \int_\Omega \nabla u(x) \cdot \nabla w_n^\pm(x) dx.$$

A similar spectral decomposition also holds for the Neumann function.

Corollary 2.2. *Let $\mathcal{N}(x,z)$ be the Neumann function defined in (8). Then*

$$\mathcal{N}(x,z) = -\sum_{n=1}^{\infty} w_n^\pm(x) w_n^\pm(z),$$

for all $x, z \in \Omega$ such that $x \neq z$.

3. Frequency dependence of the boundary data

We will first study the regularity of the solution $u(x,\omega)|_{\partial\Omega}$ as a function of the frequency function $k(\omega)$. We show that it is indeed meromorphic with poles of finite order. Then, we use the unique continuation property of meromorphic complex functions to determine the position of the poles and their corresponding singular parts.
It turns out that the poles are related to the plasmonic resonances of the inclusion [ADM, AnK, AKL]. We finally retrieve the non-frequency part of the potential from the plasmonic spectral information.

3.1. Spectral decomposition of the solution $u(x,\omega)$.

We have the following decomposition of $u(x,\omega)$ in the basis of the eigenfunctions of the variational Poincaré operator T.

Theorem 3.1. *Let $u(x,\omega)$ be the unique solution to the system* (1).

Then the following decomposition holds:

$$u(x,\omega) = k_0^{-1}u_0(x) + \sum_{n=1}^{\infty} \frac{\int_{\partial\Omega} f(z)w_n^{\pm}(z)ds(z)}{k_0 + \lambda_n^{\pm}(k(\omega)-k_0)} w_n^{\pm}(x), \quad x \in \Omega, \tag{9}$$

where $u_0(x) \in H^1_\diamond(\Omega)$ depends only on f and D, and is the unique solution to

$$\begin{cases} \Delta v = 0 & in \quad D', \\ \nabla v = 0 & in \quad D, \\ \partial_{\nu_\Omega} v = f & on \quad \partial\Omega. \end{cases} \tag{10}$$

Proof. Let $\mathfrak{f}$ be the unique solution in $H^1_\diamond(\Omega)$ to

$$\begin{cases} \Delta \mathfrak{f} = 0 & \text{in} \quad \Omega, \\ \partial_{\nu_\Omega} \mathfrak{f} = f & \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

The function $\mathfrak{f}$ can be written in terms of the Neumann function as follows:

$$\mathfrak{f}(x) = \int_{\partial\Omega} \mathcal{N}(x,z)f(z)ds(z).$$

Denote $\mathfrak{u} := u - k_0^{-1}\mathfrak{f}$. Then $\mathfrak{u}$ lies in $\mathfrak{H}_\diamond$, and has the following spectral decomposition:

$$\mathfrak{u}(x) = \sum_{n=1}^{\infty} \mathfrak{u}_n^{\pm} w_n^{\pm}(x),$$

where

$$\mathfrak{u}_n^{\pm} = \int_\Omega \nabla\mathfrak{u}(x)\cdot\nabla w_n^{\pm}(x)dx.$$

On the other hand, $\mathfrak{u}(x,\omega)$ is the unique solution to

$$\begin{cases} -\nabla\cdot(\sigma(x,\omega)\nabla\mathfrak{u}(x,\omega)) = k_0^{-1}\nabla\cdot(\sigma(x,\omega)\nabla\mathfrak{f}) & \text{in} \quad \Omega, \\ \sigma(x,\omega)\partial_{\nu_\Omega}\mathfrak{u}(x,\omega) = 0 & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega}\mathfrak{u}(x,\omega)ds = 0. \end{cases} \tag{12}$$

Multiplying the first equation in (12) by $w_n^{\pm}(x)$, and integrating by parts over Ω, we get

$$\mathfrak{u}_n^{\pm} = \frac{k_0^{-1}\int_\Omega \nabla\cdot(\sigma(x,\omega)\nabla\mathfrak{f})\, w_n^{\pm}dx}{k_0 + \lambda_n(k(w)-k_0)}.$$

Since $\nabla\cdot(\sigma(x,\omega)\nabla\mathfrak{f})$ lies in $H^{-1}(\Omega)$, the integral in the fraction above can be understood as a dual product between $H^{-1}(\Omega)$ and $H^1(\Omega)$, and can be simplified through integration by parts into

$$\int_\Omega \nabla\cdot(\sigma(x,\omega)\nabla\mathfrak{f})\, w_n^{\pm}dx = -(k_0 + \lambda_n(k(\omega)-k_0))\frac{1}{\lambda_n}\int_D \nabla\mathfrak{f}\cdot\nabla w_n^{\pm}dx + k_0\int_{\partial\Omega} f w_n^{\pm}ds(x).$$

Consequently, it follows that

$$\mathfrak{u}_n^{\pm} = -\frac{\int_D \nabla \mathfrak{f} \cdot \nabla w_n^{\pm} dx}{k_0 \lambda_n^{\pm}} + \frac{\int_{\partial\Omega} f w_n^{\pm} ds(x)}{k_0 + \lambda_n^{\pm}(k(\omega) - k_0)}.$$

Now we derive the orthogonal projection of $\mathfrak{f}$ onto $\mathfrak{H}_\diamond$. Let $\widetilde{f}(x) \in \mathfrak{H}_\diamond$ be the function that coincides with $\mathfrak{f}$ on D up to a constant, and solves the system of equations

$$\begin{cases} \Delta \widetilde{f} = 0 & \text{in} \quad D', \\ \\ \nabla \widetilde{f} = \nabla \mathfrak{f} & \text{in} \quad D, \\ \partial_{\nu_\Omega} \widetilde{f} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $w_n^{\pm}$ is an eigenfunction of T and $\widetilde{f}$ belongs to $\mathfrak{H}_\diamond$, we have

$$\int_D \nabla \mathfrak{f} \cdot \nabla w_n^{\pm} dx = \lambda_n^{\pm} \int_\Omega \nabla \widetilde{f} \cdot \nabla w_n^{\pm} dx,$$

which gives

$$\mathfrak{u}_n^{\pm} = -k_0^{-1} \int_\Omega \nabla \widetilde{f} \cdot \nabla w_n^{\pm} dx + \frac{\int_{\partial\Omega} f w_n^{\pm} ds(x)}{k_0 + \lambda_n(k(\omega) - k_0)}.$$

Finally, we obtain the desired decomposition for $u(x, \omega)$.

□

Corollary 3.1. *The function $u(x, \omega) = k_0^{-1} u_0(x) + u_f(x, k(\omega))$, where $k \to u_f(x, k)$ is meromorphic on $\mathbb{C}$. Furthermore, the poles of $u_f(x, k)$ are the complex values $(k_n^{\pm})_{n \geq 1}$ solutions to the dispersion equations*

$$k_0 + \lambda_n^{\pm}(k - k_0) = 0, \qquad n \geq 1$$

with $\lambda_n^{\pm}, n \geq 1$ being the eigenvalues of the variational Poincaré operator T.

3.2. **Retrieval of the frequency dependent part.** The idea here is to recover the frequency dependent part $u_f(x, k(\omega))$ from the knowledge of $u(x, \omega)$ for $\omega \in (\underline{\omega}, \overline{\omega})$.

The poles of $u_f(x, k)$ are given by $k_n^{\pm} := k_0(1 - \frac{1}{\lambda_n^{\pm}})$, and they can be ordered as follows:

$$-k_0 < \cdots \leq k_2^+ \leq k_1^+ < 0$$

in $(-k_0, 0)$ and, similarly,

$$k_1^- \leq k_2^- \leq \cdots < -k_0$$

in $(-\infty, -k_0)$. We remark that $-k_0$ is the only accumulation point of the sequence of poles, i.e., $k_n^{\pm}$ tends to $-k_0$ as $n \to \infty$.

The plasmonic resonances $(k_n^{\pm})_{n \geq 1}$ depend only on k_0, the shapes of the inclusion D and the background Ω. They can be experimentally measured and represents the plasmonic signature of the inclusion. One interesting inverse problem is to recover the inclusion from its plasmonic resonances [ACLZ]. The magnitude of k_1^- is related somehow to how flat is the domain D. More precisely, we have the following result.

Lemma 3.1. *There exists a constant $\widehat{\delta} > 0$ depending only on k_0 and $\mathfrak{D}$ such that*

$$k_n^{\pm} \geq -\widehat{\delta}^{-1}, \qquad \forall n \geq 1.$$

Proof. Let D be an inclusion in $\mathfrak{D}$. Then D is star-shaped and is given by

$$D := \{x \in \mathbb{R}^d : |x| < \Upsilon(\widehat{x}),\ \widehat{x} = \frac{x}{|x|}\},$$

where $\Upsilon : \mathbb{S}^d \to (b_0, b_1 - \delta)$, is $C^{2,\varsigma}$, $\varsigma > 0$.

A forward computation shows that the constant

$$r_D := \inf_{x\in\partial D} x\cdot\nu_D(x),$$

is strictly positive, and is lower and upper bounded by constants that depend only on $\mathfrak{D}$.

On the other hand, a simple modification of the proof of Theorem 2.2 in [AS] (see also Lemma 2.9 in [AK]), gives

$$-\infty < -1 - \left(\frac{r_D+2}{r_D}\right)^2 \leq k_1^-,$$

which completes the proof.

□

Since the function $u_f(x,k)$ have isolated poles, the complementary of the singular set is connected and the unique continuation of holomorphic functions implies the uniqueness in the identification of the poles $k_n^\pm, n\geq 1$, $k_0^{-1}u_0(x)$ and $u_f(x,k(\omega))$.

In order to derive stability estimates in the retrieval of the frequency independent part $k_0^{-1}u_0(x)$ of the solution, we need to obtain uniform bounds on the frequency dependent part u_f on the boundary $\partial\Omega$.

Theorem 3.2. *Let D be an inclusion in $\mathfrak{D}$. Then there exists a constant $C = C(\mathfrak{D},\Omega,k_0) > 0$ such that*

$$\|u(x,\omega) - k_0^{-1}u_0(x)\|_{C^0(\partial\Omega)} \leq \frac{C}{dist(k(\omega),[k_1^-,0])}\|f(x)\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \tag{13}$$

The constant C tends to $+\infty$ as $\widehat{\delta}$ tends to zero.

Proof. Recall that the function $u_f(x,k) = u(x,\omega) - k_0^{-1}u_0(x)$, defined in Lemma 3.1, lies in $\mathfrak{H}_\diamond$, and satisfies

$$k_0\partial_{\nu_D}u_f|_{\partial D}^+ - k\partial_{\nu_D}u_f|_{\partial D}^- = -\partial_{\nu_D}u_0|_{\partial D}^+.$$

Hence there exists a potential $\varphi_f\in H^{-\frac{1}{2}}(\partial D)$ satisfying

$$u_f(x,k) = S_D[\varphi_f](x),$$

for $x\in\Omega$. Note that the right-hand side term in the equality above is harmonic in both D and D' and continuous through the boundary ∂D. The transmission condition of $\partial_{\nu_D}u_f|_{\partial D}$ over ∂D implies

$$\left(\frac{k_0+k}{2(k_0-k)}I + K_D^*\right)[\varphi_f](x) \quad = \quad \frac{1}{k-k_0}\partial_{\nu_D}u_0(x)|_{\partial D}^+,$$

for $x\in\partial D$.

On the other hand, Calderon's identity holds for the operators K_D, K_D^* and S_D, and we have

$$S_DK_D^* \quad = \quad K_DS_D$$

Hence, K_D^* becomes a self-adjoint compact operator in the topology induced by the scalar product

$$\langle\cdot\,,\cdot\rangle_{-\frac{1}{2},S} = \langle -S_D\cdot\,,\cdot\rangle_{\frac{1}{2},-\frac{1}{2}}.$$

A direct calculation shows that $\|K_D^*\| = 1$ and the spectrum of K_D^* lies in $(-\frac{1}{2},\frac{1}{2}]$.

Moreover, the eigenvalues of K_D^* are given by

$$0,\ \frac{1}{2},\ \frac{k_0+k_n^\pm}{2(k_0-k_n^\pm)},\ n\geq 1.$$

Spectral decomposition of self-adjoint compact operator shows that

$$\|\varphi_f\|_{-\frac{1}{2},S} \leq \frac{1}{\operatorname{dist}(k,[k_1^-,0])}\frac{2}{k_0}\|\partial_{\nu_D}u_0|_+\|_{-\frac{1}{2},S}. \tag{14}$$

In order to derive precise estimates with constants that depend only on $\widehat{\delta}$ and Ω, we introduce the more conventional $H^{\frac{1}{2}}$-norm:

$$\|\psi\|_{\frac{1}{2}} = \|v_\psi\|_{H^1(D)},$$

where v_ψ is harmonic in D, that is, $\Delta v_\psi = 0$ on D, and satisfies $v_\psi|_{\partial D} = \psi$ on ∂D.

Following [McL], we define the associated $H^{-\frac{1}{2}}$-norm by

$$\|\varphi\|_{-\frac{1}{2}} = \max_{0\neq\psi\in H^{\frac{1}{2}}(\partial D)} \frac{\left|\int_{\partial D}\varphi\psi ds\right|}{\|\psi\|_{\frac{1}{2}}}.$$

Now, we shall estimate $\|\varphi_f\|_{-\frac{1}{2}}$ in terms of the quantity $\|\varphi_f\|_{-\frac{1}{2},S}$. We have

$$\begin{aligned}\|\varphi_f\|_{-\frac{1}{2}} &= \max_{0\neq\psi\in H^{\frac{1}{2}}(\partial D)} \frac{\left|\int_{\partial D}\varphi_f\psi ds\right|}{\|\psi\|_{\frac{1}{2}}}\\ &\leq \max_{0\neq\psi\in H^{\frac{1}{2}}(\partial D)} \frac{\left|\int_{\Omega}\nabla S_D[\varphi_f]\cdot\nabla\widetilde{v}_\psi dx\right|}{\|\widetilde{v}_\psi\|_{H^1(D)}},\end{aligned}$$

where $\widetilde{v}_\psi \in H^1_\diamond(\Omega)$ is the unique solution to

$$\begin{cases}\Delta v = 0 & \text{in} \quad D',\\ \Delta v = 0 & \text{in} \quad D,\\ v = \psi & \text{on} \quad \partial D,\\ \partial_{\nu_\Omega} v = f & \text{on} \quad \partial\Omega.\end{cases}$$

Hence,

$$\begin{aligned}\|\varphi_f\|_{-\frac{1}{2}} \leq \left(\int_\Omega |\nabla S_D[\varphi_f]|^2\,dx\right)^{\frac{1}{2}} &\left(\max_{0\neq\psi\in H^{\frac{1}{2}}(\partial D)} \frac{\int_\Omega|\nabla\widetilde{v}_\psi|^2dx}{\int_D|\nabla\widetilde{v}_\psi|^2dx}\right)^{\frac{1}{2}}\\ &\leq \left(\lambda_1^-\right)^{-\frac{1}{2}}\|\varphi_f\|_{-\frac{1}{2},S}\\ &\leq \left(1-\frac{k_1^-}{k_0}\right)^{\frac{1}{2}}\|\varphi_f\|_{-\frac{1}{2},S}.\end{aligned}$$

Using the inequality satisfied by $k_n^\pm$ in Lemma 3.1, we obtain

$$\|\varphi_f\|_{-\frac{1}{2}} \leq C\left(1+\frac{1}{\widehat{\delta}k_0}\right)^{\frac{1}{2}}\|\varphi_f\|_{-\frac{1}{2},S},$$

where C depends only on Ω. Combining the last inequality and (14), we get

$$\|\varphi_f\|_{-\frac{1}{2}} \leq C\left(1+\frac{1}{\widehat{\delta}k_0}\right)^{\frac{1}{2}}\frac{1}{\operatorname{dist}(k,[k_1^-,0])}\frac{1}{k_0}\|\partial_{\nu_D}u_0|_+\|_{-\frac{1}{2},S}. \tag{15}$$

Next, we estimate $\|\partial_{\nu_D}u_0|_+\|_{-\frac{1}{2},S}$ in terms of $\|f\|_{H^{-\frac{1}{2}}(\partial\Omega)}$.

A direct calculation shows that

$$u_0 - S_D[\partial_{\nu_D}u_0|_+] = \mathfrak{f},$$

over Ω.

Therefore,

$$\|\partial_{\nu_D} u_0|_+\|_{-\frac{1}{2},S} = \int_\Omega |\nabla S_D[\partial_{\nu_D} u_0|_+]|^2\,dx \leq \int_\Omega |\nabla u_0|^2\,dx + \int_\Omega |\nabla \mathfrak{f}|^2\,dx. \tag{16}$$

On the other hand, we have

$$\int_\Omega |\nabla u_0|^2\,dx = -\int_{\partial\Omega} f u_0 ds \leq C_1 \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)} \left(\int_\Omega |\nabla u_0|^2\,dx\right)^{\frac{1}{2}},$$

where $C_1 > 0$ is the constant that appears in the trace theorem on $\partial\Omega$ and depends only on Ω and the dimension of the space. Hence,

$$\int_\Omega |\nabla u_0|^2\,dx \leq C_1^2 \|f\|^2_{-\frac{1}{2}}. \tag{17}$$

Since $\mathfrak{f}$ is the unique solution to the system (11), classical elliptic regularity implies

$$\int_\Omega |\nabla \mathfrak{f}|^2\,dx \leq C_2 \|f\|^2_{-\frac{1}{2}}, \tag{18}$$

where $C_2 > 0$ is a constant which depends only on Ω and the dimension of the space.
Combining inequalities (16), (17), and (18), we obtain

$$\|\partial_{\nu_D} u_0|_+\|_{-\frac{1}{2},S} \leq C\|f\|_{-\frac{1}{2}}. \tag{19}$$

Now we turn to inequality (15). Using the estimate above, we get

$$\|\varphi_f\|_{-\frac{1}{2}} \leq C\left(1 + \frac{1}{\widehat{\delta} k_0}\right)^{\frac{1}{2}} \frac{1}{\mathrm{dist}(k, [k_1^-, 0])} \frac{1}{k_0} \|f\|_{H^{-\frac{1}{2}}(\partial\Omega)}, \tag{20}$$

where C depends only on Ω.

Now, we are ready to prove the results of the theorem. Using the fact that $\mathrm{dist}\,(D, \partial\Omega) > \delta$, and

$$u_f(x) = \int_{\partial D} \mathcal{N}(x,z)\varphi_f(z) ds(z),$$

for all $x \in \partial D$, we deduce that

$$\|u_f(x)\|_{C^0(\partial\Omega)} \leq \left(\max_{x\in\partial\Omega} \|\mathcal{N}(x,.)\|_{H^1(\Omega_\delta)}\right) \|\varphi_f\|_{-\frac{1}{2}},$$

where $\Omega_\delta = \{x \in \mathbb{R}^d : \mathrm{dist}(x, \partial\Omega) > \delta\}$.

Finally, by using estimate (20) we achieve the proof of the theorem.

$\square$

Remark 3.1. *The results of Theorem 3.2 give a precise estimate on how the solution u to* (1) *blows up on the boundary when $k(\omega)$ approaches the plasmonic resonances. The estimates are uniform for inclusions within the set $\mathfrak{D}$, and are somehow a generalization of the results in* [KKL] *which are only valid in a sector of the complex plane.*

Theorem 3.3. *Let D be an inclusion in $\mathfrak{D}$. Then there exists a constant $C = C(\mathfrak{D}, k_0, \Omega) > 0$ such that*

$$\|u_0(x)\|_{C^0(\partial\Omega)} \leq C\left(1 + \frac{1}{dist(\Sigma, [k_1^-, 0])}\right) \|f(x)\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{21}$$

The constant C tends to $+\infty$ as $\widehat{\delta}$ tends to zero.

Proof. Recall that

$$u_0 - S_D[\partial_{\nu_D} u_0|_+] = \mathfrak{f}.$$

Then, similarly to the proof of Theorem 3.2, based on elliptic regularity, and the uniform bound (19) of $\partial_{\nu_D} u_0|_+$, we deduce the desired result.

□

Next, we show that the stability of the reconstruction of u_f depends in fact on the distance of the poles $k_n^{\pm}$ to the set $\Sigma = \{k(\omega); \omega \in (\underline{\omega}, \overline{\omega})\}$ in the complex plane.

Theorem 3.4. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Denote by u (resp. $\widetilde{u}$), the solution of* (1) *with inclusion D (resp. $\widetilde{D}$). Let*

$$\varepsilon = \sup_{x\in\partial\Omega, \omega\in(\underline{\omega},\overline{\omega})} |u - \widetilde{u}|.$$

Then, there exists a constant $\kappa > 0$, that depends only on $\Omega, \mathfrak{D}, k_0$, and Σ, such that

$$\sup_{x\in\partial\Omega, \omega\in(\underline{\omega},\overline{\omega})} |u_f - \widetilde{u}_f| \le C\varepsilon^{\kappa}, \tag{22}$$

where the constant $C > 0$ depends only on $f, \Omega, \mathfrak{D}$, and Σ.

Proof. The proof of the theorem is based on the unique continuation of holomorphic functions.

For $x \in \partial\Omega$ fixed, Lemma 3.1 implies that $\alpha(k) = k_0^{-1} u_0(x) + u_f(x, k)$, is a meromorphic function with poles $(k_n^{\pm})_{n\ge1}$.

Similarly for $x \in \partial\Omega$ fixed, we have $\widetilde{\alpha}(k) = k_0^{-1}\widetilde{u}_0(x) + \widetilde{u}_f(x, k)$, is a meromorphic function with poles $(\widetilde{k}_n^{\pm})_{n\ge1}$ (the plasmonic surface resonances of the inclusion $\widetilde{D}$).

We consider $\widetilde{u}(x, \omega)$ as a perturbation of the meromorphic function $u(x, \omega)$ on $\mathbb{C}$. We will use the concept of harmonic measure to estimate the difference $\alpha(k) - \widetilde{\alpha}(k)$ on a complex contour that encirles the poles of both functions $\alpha(k)$ and $\widetilde{\alpha}(k)$.

Let $\mathcal{C}_+$ be a Jordan complex contour with interior $\mathring{\mathcal{C}}_+$ that contains $[-\widehat{\delta}^{-1}, 0) \cup \overline{\Sigma}$. Let $\mathcal{C}_-$ be a Jordan complex contour in $\mathring{\mathcal{C}}_+$, with interior $\mathring{\mathcal{C}}_-$ that contains $[-\widehat{\delta}^{-1}, 0]$ and does not intersect Σ, that is, $\mathring{\mathcal{C}}_- \cap \Sigma = \emptyset$. Finally, let $\mathcal{C}$ be a Jordan complex contour in $\mathring{\mathcal{C}}_+ \setminus \mathring{\mathcal{C}}_-$ such that $[-\widehat{\delta}^{-1}, 0] \subset \mathring{\mathcal{C}}$, and $\mathring{\mathcal{C}} \cap \Sigma = \emptyset$.

Let ω be a fixed frequency in $(\underline{\omega}, \overline{\omega})$. Since the poles $(k_n^{\pm})_{n\ge1}, (\widetilde{k}_n^{\pm})_{n\ge1}$ are inside $\mathring{\mathcal{C}}$, and $k(\omega)$ lies in the exterior of $\mathcal{C}$, we deduce from Lemma 3.1 that

$$u_f(x, k(\omega)) - \widetilde{u}_f(x, k(\omega)) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{\alpha(k) - \widetilde{\alpha}(k)}{k - k(\omega)} dk.$$

Consequently,

$$|u_f(x, k(\omega)) - \widetilde{u}_f(x, k(\omega))| \le \frac{1}{\mathrm{dist}(\Sigma, \mathcal{C})} \|\alpha(k) - \widetilde{\alpha}(k)\|_{L^\infty(\mathcal{C})}. \tag{23}$$

Now, define $w(z)$ to be the harmonic measure of $\overline{\Sigma}$ in $\mathring{\mathcal{C}}_+ \setminus \overline{\mathring{\mathcal{C}}_-}$, which is holomorphic in $\mathring{\mathcal{C}}_+ \setminus \overline{\mathring{\mathcal{C}}_-}$ and statifies $w(z) = 1$ on $\overline{\Sigma}$, $w(z) = 0$ on $\mathcal{C}_- \cup \mathcal{C}_+$.

Then the two-constants theorem implies

$$|\alpha(k) - \widetilde{\alpha}(k)| \le M^{1-w(k)}\varepsilon^{w(k)},$$

for all z in $\overset{\circ}{\mathcal{C}}_+ \setminus \overset{\circ}{\mathcal{C}}_-$ where $M = \max_{k\in \overset{\circ}{\mathcal{C}}_+ \setminus \overset{\circ}{\mathcal{C}}_-} (|\alpha(k)| + |\widetilde{\alpha}(k)|)$.

We deduce from Theorems 3.2 and 3.3 that

$$M \leq \frac{C}{\mathrm{dist}([-\widehat{\delta}^{-1}, 0], \mathcal{C}_-)} \|f\|_{H^{\frac{1}{2}}(\partial\Omega)},$$

where $C > 0$ depends only on $\mathfrak{D}, \Omega, \Sigma$ and k_0.

Taking $\kappa = \min_{k\in\mathcal{C}} w(k)$, we obtain

$$\|\alpha(k) - \widetilde{\alpha}(k)\|_{L^\infty(\mathcal{C})} \leq C\varepsilon^\kappa \tag{24}$$

with $C > 0$ being a constant that depends only on the contours $\mathcal{C}, \mathcal{C}_\pm, M$ and Σ.

Combining the inequalities (23) and (24), we get the estimate (22) of the theorem.

□

Remark 3.2. *Since the position of Σ in the complex plane is known, the contours $\mathcal{C}_\pm$ and $\mathcal{C}$ can be explicitly given, and the harmonic measure $w(z)$ can be explicitly constructed using known conformal maps. Hence, the constant κ can be precisely estimated in terms of the distance between the sets Σ and $[-\widehat{\delta}^{-1}, 0]$.*

A direct consequence of the Theorem 3.4 is the estimation of the frequency independent part of the data from the complete collected data over Σ.

Corollary 3.2. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Denote u (resp. $\widetilde{u}$), the solution of* (1) *with inclusion D (resp. $\widetilde{D}$). Let*

$$\varepsilon = \sup_{x\in\partial\Omega, \omega\in(\underline{\omega}, \overline{\omega})} |u - \widetilde{u}|.$$

Then, there exists a constant $\kappa > 0$, that depends only on $\Omega, \mathfrak{D}$ and Σ, such that

$$\sup_{x\in\partial\Omega, \omega\in(\underline{\omega}, \overline{\omega})} |u_0 - \widetilde{u}_0| \leq C\varepsilon^\kappa, \tag{25}$$

where the constant $C > 0$ only depends on $f, \Omega, \mathfrak{D}$ and Σ.

4. Reconstruction of the inclusion from the Cauchy data of the perfectly conductor solution.

In this section we construct the inclusion D from the knowledge of the Cauchy data of the frequency independent part u_0 on the boundary $\partial\Omega$. Precisely, we derive the uniqueness and stability of the reconstruction within the set of inclusions $\mathfrak{D}$. The results are quite surprising in inverse conductivity problem since in general infinitely many input currents are needed in order to obtain the uniqueness in the determination of the conductivity. Here the fact that the solution is constant inside the inclusion is essential to derive such results.

Based on quantitative estimates of the unique continuation for Laplace operator we evaluate how the solution on the boundary of the perturbed inclusion is sensitive to errors made in the Cauchy data on $\partial\Omega$. Using the fact that the solution is constant inside the inclusion we then obtain the variation of the solution on the perturbed inclusion, and again using the interior unique continuation [GL, HL] we estimate the variation of the inclusion induced by the errors in Cauchy data. The methods developped here are similar to the ones used in determining parts of the boundaries under zero Dirichlet or Neumann conditions on unknown sub-boundaries [BV, BCY1, ABRV, Rd1, Rd2, Is1].

The conditional stability in our inverse problem depends heavily on the one in a Cauchy problem for the Laplace equation.

Lemma 4.1. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let u_0 (resp. $\widetilde{u}_0$) be the solution in $H^1_\diamond(\Omega)$ of (10) with inclusion D (resp. $\widetilde{D}$), and assume that*

$$0 < \varepsilon = \sup_{x \in \partial\Omega} |u_0 - \widetilde{u}_0| < 1.$$

Then, there exist constants $C > 0$ and $\mu > 0$, such that the following estimate holds:

$$\|u_0 - \widetilde{u}_0\|_{C^0(\partial(D \cup \widetilde{D}))} \leq C \left(\frac{1}{\ln(\varepsilon^{-1})} \right)^\mu . \tag{26}$$

Here, the constants C and μ depend only on f, Ω, and $\mathfrak{D}$.

If, in addition $d = 2$, and the inclusions D and $\widetilde{D}$ are analytic, then we have

$$\|u_0 - \widetilde{u}_0\|_{C^0(\partial(D \cup \widetilde{D}))} \leq C\varepsilon^{\mu'}, \tag{27}$$

where the constants C and μ' depend only on f, Ω, and $\mathfrak{D}$.

Proof. The stability estimate is well known for smooth boundaries ($\partial \left(D \cup \widetilde{D} \right)$ is Lipschitz). The proof of this Lemma can be found in [BCY1, BCY2, CHY, Is1, Is2, Is3, BV] for the Laplacian operator, and in [ABRV] for an elliptic operator in a divergence form. It is based on three facts. The first one is that $\Omega \setminus \overline{D_1 \cup D_2}$ satisfies a uniform cone property (see for instance [BCY1] for dimension two, the proof can be extended easily to higher dimensions). If D_1 and D_2 are not star-shaped, it is proved in [Rd1, ABRV] that $\Omega \setminus \overline{D_1 \cup D_2}$ satisfies the cone property if D_1 and D_2 are too close (which can be verified for general elliptic operators if ε is too small through a first rough stability estimate using the three sphere inequality [GL]). The second fact is to evaluate the value of $u_0 - \widetilde{u}_0$ on a point $p \in \partial \left(D \cup \widetilde{D} \right)$ by evaluating the unique continuation in a cone included in $\Omega \setminus \overline{D_1 \cup D_2}$ with vertex p using the explicit expression of the harmonic measure (Lemma 4.2 in [CHY], Lemma 3.5 in [Rd1], c) in proof of Lemma 3.6 in [Is3]). The third fact is a Hölder Cauchy stability estimate in any smooth domain lying in $\Omega \setminus \overline{D_1 \cup D_2}$, neighboring $\partial\Omega$, and at a finite distance from the boundary $\partial \left(D \cup \widetilde{D} \right)$ proved in [Pa] (extended in [Tr] for elliptic equations in a divergence form with Lipschitz coefficients). Finally, Hölder stability estimate type is obtained for analytic curves in [BCY1].

□

Recall that $\nabla u_0 = 0$ (resp. $\nabla \widetilde{u}_0 = 0$) in D (resp. $\widetilde{D}$). We further denote ϱ (resp. $\widetilde{\varrho}$), the constant value of $u_0|_D$ (resp. $\widetilde{u}_0|_{\widetilde{D}}$).

Lemma 4.2. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let u_0 (resp. $\widetilde{u}_0$) be the solution in $H^1_\diamond(\Omega)$ of (10) with inclusion D (resp. $\widetilde{D}$). Then, the following estimate holds:*

$$|\varrho - \widetilde{\varrho}| \leq \|u_0 - \widetilde{u}_0\|_{C^0(\partial(D \cup \widetilde{D}))} . \tag{28}$$

Proof. Since D and $\widetilde{D}$ are star-shaped and contain the point 0, $D \cap \widetilde{D}$ is not empty. Then we have two different cases.

Case (1): $\partial D \cap \partial\widetilde{D}$ is not empty. In this case the estimate is trivial.

Case (2): $\partial D \cap \partial\widetilde{D}$ is empty, and hence we have $D \subset \widetilde{D}$ or $\widetilde{D} \subset D$. Without any loss of generality, we will further assume that $D \subset \widetilde{D}$. By Green's formula inside the domain $\Omega \setminus \overline{D}$, we have $\int_{\partial D} \partial_{\nu_D} u_0(x)|^+ ds(x) = 0$, and hence $\partial_{\nu_D} u_0(x)$ can not have a constant sign on ∂D. Since u_0 is constant on ∂D, we deduce then from Hopf's Lemma that u_0 does not take its maximum or minimum on ∂D. The fact that u_0 is harmonic on $\widetilde{D} \setminus \overline{D}$ implies that u_0 reaches its minimum and maximum in $\overline{\widetilde{D}} \setminus D$ on $\partial\widetilde{D}$. Consequently u_0 takes the value ϱ on $\partial\widetilde{D} = \partial \left(D \cup \widetilde{D} \right)$, which concludes the proof of the estimate.

□

Recall that $u_0 - \varrho$ (resp. $\widetilde{u}_0 - \widetilde{\varrho}$) is harmonic in $\Omega \setminus \overline{D}$ (resp. $\Omega \setminus \overline{\widetilde{D}}$, and satisfies a zero Dirichlet boundary condition on ∂D (resp. $\partial \widetilde{D}$). Consequently,

$$\max_{x \in \widetilde{D} \setminus \overline{D}} |u_0(x) - \varrho| \leq \max_{x \in \partial \widetilde{D} \setminus \overline{D}} |u_0(x) - \widetilde{u}_0(x) + \widetilde{\varrho} - \varrho|,$$
$$\max_{x \in D \setminus \overline{\widetilde{D}}} |\widetilde{u}_0(x) - \varrho| \leq \max_{x \in \partial D \setminus \overline{\widetilde{D}}} |\widetilde{u}_0(x) - u_0(x) + \varrho - \widetilde{\varrho}|.$$

Using the estimate stated in Lemma 4.2, we have

$$\max_{x \in \widetilde{D} \setminus \overline{D}} |u_0(x) - \varrho| + \max_{x \in D \setminus \overline{\widetilde{D}}} |\widetilde{u}_0(x) - \widetilde{\varrho}| \leq 4 \left\| u_0 - \widetilde{u}_0 \right\|_{C^0\left(\partial\left(D \cup \widetilde{D}\right)\right)}.$$

Then we immediately obtain the following result.

Lemma 4.3. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let u_0 (resp. $\widetilde{u}_0$) be the solution in $H^1_\diamond(\Omega)$ of* (10) *with inclusion D (resp. $\widetilde{D}$).*

Then, the following estimate holds:

$$\int_{D \setminus \overline{\widetilde{D}}} |\widetilde{u}_0(x) - \widetilde{\varrho}|^2 dx + \int_{\widetilde{D} \setminus \overline{D}} |u_0(x) - \varrho|^2 dx \leq C \left\| u_0 - \widetilde{u}_0 \right\|^2_{C^0\left(\partial\left(D \cup \widetilde{D}\right)\right)}. \tag{29}$$

Here, $C > 0$ depends only on $\mathfrak{D}$.

Proposition 4.1. *Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let u_0 (resp. $\widetilde{u}_0$) be the solution in $H^1_\diamond(\Omega)$ of* (10) *with inclusion D (resp. $\widetilde{D}$), and $x_0 \in \partial D$ (resp. $x_0 \in \partial \widetilde{D}$).*

Then, for any $r > 0$ and $R \geq r$, we have

$$\int_{B_R(x_0) \cap \Omega} |u_0(x) - \varrho|^2 dx \leq C \left(\frac{R}{r}\right)^K \int_{B_r(x_0) \cap \Omega} |u_0(x) - \varrho|^2 dx,$$
$$\int_{B_R(x_0) \cap \Omega} |\widetilde{u}_0(x) - \widetilde{\varrho}|^2 dx \leq C \left(\frac{R}{r}\right)^K \int_{B_r(x_0) \cap \Omega} |\widetilde{u}_0(x) - \widetilde{\varrho}|^2 dx,$$

where $C > 1$ and $K > 0$ depend on f, Ω, Σ, and $\mathfrak{D}$.

Furthermore, we have

$$\int_\Omega |u_0(x) - \varrho|^2 dx \geq C_0,$$

where $C_0 > 0$ depends on f, Ω, Σ, and $\mathfrak{D}$.

Proof. The doubling inequalities are obtained in [AE] for general elliptic operators in a divergence form (Theorem 1.1). In [ABRV], a more explicit evaluation of the constants C and K in terms of the problem a priori data is derived (Proposition 4.5). □

Now, we are ready to prove Theorems 1.1 and 1.2. We follow the ideas developed in the proof of Theorem 2.2 in [ABRV].

Proof. For $\widehat{x} \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere, we further denote by

$$\Upsilon_m(\widehat{x}) = \min(\Upsilon(\widehat{x}), \widetilde{\Upsilon}(\widehat{x})), \quad \Upsilon_M(\widehat{x}) = \max(\Upsilon(\widehat{x}), \widetilde{\Upsilon}(\widehat{x})).$$

A direct computation shows that Υ_m and Υ_M belong to $C^{0,1}(\mathbb{S}^{d-1})$.

We introduce the domains

$$D_m(\widehat{x}) = \begin{cases} D & \text{if} \quad \Upsilon_m(\widehat{x}) = \Upsilon(\widehat{x}), \\ \widetilde{D} & \text{if} \quad \Upsilon_m(\widehat{x}) = \widetilde{\Upsilon}(\widehat{x}), \end{cases}$$

and

$$D_M(\widehat{x}) = \begin{cases} D & \text{if} \quad \Upsilon_M(\widehat{x}) = \Upsilon(\widehat{x}), \\ \widetilde{D} & \text{if} \quad \Upsilon_M(\widehat{x}) = \widetilde{\Upsilon}(\widehat{x}). \end{cases}$$

Let $r_M : \mathbb{S}^{d-1} \to \mathbb{R}_+$ be defined by

$$r_M(\widehat{x}) = \max\{r : 0 \leq r \leq \Upsilon_M(\widehat{x}) - \Upsilon_m(\widehat{x}) \text{ and } B_r(\Upsilon_m(\widehat{x})\widehat{x}) \setminus \overline{D_m(\widehat{x})} \subset D_M(\widehat{x}) \setminus \overline{D_m(\widehat{x})}\}.$$

Then $r_M(\widehat{x})$ attains its maximum $r_0 > 0$ over $\mathbb{S}^{d-1}$ at $\widehat{x}_0$, that is

$$r_0 := r_M(\widehat{x}_0) = \max_{\widehat{x} \in \mathbb{S}^{d-1}} r_M(\widehat{x}).$$

Now, let $\widehat{x}_M \in \mathbb{S}^{d-1}$, be such that

$$d_0 := \Upsilon_M(\widehat{x}_M) - \Upsilon_m(\widehat{x}_M) = \max_{\widehat{x} \in \mathbb{S}^{d-1}} \left(\Upsilon_M(\widehat{x}) - \Upsilon_m(\widehat{x})\right).$$

Obviously, we have $r_0 \leq d_0 \leq 2m$.

Lemma 4.4. *There exist constants $c_1 > 0$ and $c_2 > 1$, such that the following estimates hold:*

$$|D \Delta \widetilde{D}| \leq c_1 m^{d-1} d_0,$$
$$\min\left(b_0 \sqrt{\frac{d_0}{m}}, d_0\right) \leq c_2 r_0.$$

The constants $c_i,\ i = 1, 2,$ only depend on the dimension of the space.

Proof. Without any loss of generality, we assume that $\Upsilon_M(\widehat{x}_M) = \widetilde{\Upsilon}(\widehat{x}_M)$, $\Upsilon_m(\widehat{x}_M) = \Upsilon(\widehat{x}_M)$, and denote by $x_M = \Upsilon(\widehat{x}_M)\widehat{x}_M \in \partial D$.

The first inequality immediately follows from the definition of d_0.

Now, from the C^2 regularity of the function $\widehat{x} \to \Upsilon(\widehat{x}) - \widetilde{\Upsilon}(\widehat{x})$, we deduce that for $t_0 = \frac{1}{2\sqrt{m}}$, we have $d_0/2 \leq \Upsilon_M(\widehat{x}) - \Upsilon_m(\widehat{x}) \leq d_0$ for all $\widehat{x} \in B_{t_0\sqrt{d_0}}(\widehat{x}_M) \cap \mathbb{S}^{d-1}$.

A forward calculation shows that for $s_0 = \frac{1}{4} \min\left(b_0 \sqrt{\frac{d_0}{m}}, d_0\right)$, we have

$$B_{s_0}(\Upsilon(\widehat{x}_M)\widehat{x}_M) \setminus \overline{D} \subset \widetilde{D} \setminus \overline{D}.$$

From the definition of r_0, we obtain

$$s_0 \leq r_0,$$

which finishes the proof of the lemma.

□

Without any loss of generality, we assume that $\Upsilon_M(\widehat{x}_0) = \widetilde{\Upsilon}(\widehat{x}_0)$, $\Upsilon_m(\widehat{x}_0) = \Upsilon(\widehat{x}_0)$, and denote by $x_0 = \Upsilon(\widehat{x}_0)\widehat{x}_0 \in \partial D$.

Now, let $\widetilde{r}_0 > 0$ such that $B_{\widetilde{r}_0}(x_0) \cap \Omega = \Omega$. Then, we immediately have $\widetilde{r}_0 > r_0$. Obviously, $\widetilde{r}_0$ only depends on $\mathfrak{D}$ and Ω.

Taking $R = \widetilde{r}_0$ and $r = r_0$ in Lemma 4.3, we obtain

$$\int_{B_{\widetilde{r}_0}(x_0)\cap\Omega} |u_0(x)-\varrho|^2dx \leq C\left(\frac{\widetilde{r}_0}{r_0}\right)^K \int_{B_{r_0}(x_0)\cap\Omega} |u_0(x)-\varrho|^2dx,$$

which implies that

$$r_0^K \int_\Omega |u_0(x)-\varrho|^2dx \leq C\widetilde{r}_0^K \int_{B_{r_0}(x_0)\cap\Omega} |u_0(x)-\varrho|^2dx.$$

Since $u_0(x)-\varrho$ vanishes inside D, and $B_{r_0}(x_0)\setminus\overline{D}\subset\widetilde{D}\setminus\overline{D}$, we have

$$\begin{aligned} r_0^K \int_\Omega |u_0(x)-\varrho|^2dx &\leq C\widetilde{r}_0^K \int_{B_{r_0}(x_0)\cap\Omega} |u_0(x)-\varrho|^2dx, \\ &\leq C\widetilde{r}_0^K \int_{\widetilde{D}\setminus\overline{D}} |u_0(x)-\varrho|^2dx. \end{aligned}$$

From Lemma 4.3, we deduce that

$$r_0\left(\int_\Omega |u_0(x)-\varrho|^2dx\right)^{\frac{1}{K}} \leq C\,\|u_0-\widetilde{u}_0\|^{\frac{2}{K}}_{C^0(\partial(D\cup\widetilde{D}))}.$$

Now, combining estimates of Lemmas 4.1, 4.4, and Proposition 4.1, we finally obtain the results of the main theorems.

□

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