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QUASI-MONTE CARLO INTEGRATION FOR AFFINE-PARAMETRIC, ELLIPTIC PDES: LOCAL SUPPORTS AND PRODUCT WEIGHTS*

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Abstract. We analyze convergence rates of first order quasi-Monte Carlo (QMC) integration with randomly shifted lattice rules and for higher order, interlaced polynomial lattice rules, for a class of countably parametric integrands that result from linear functionals of solutions of linear, elliptic diffusion equations with affine-parametric, uncertain coefficient function $a(x,\boldsymbol{y}) = \bar{a}(x) + \sum_{j\geq 1} y_j\psi_j(x)$ in a bounded domain $D \subset \mathbb{R}^d$. Extending the result in [F. Y. Kuo, Ch. Schwab, and I. H. Sloan, SIAM J. Numer. Anal., 50 (2012), pp. 3351-3374], where ψ_j was assumed to have global support in the domain D, we assume in the present paper that $\mathrm{supp}(\psi_j)$ is localized in D and that we have control on the overlaps of these supports. Under these conditions we prove *dimension independent convergence rates* in $[1/2, 1)$ of randomly shifted lattice rules with *product weights* and corresponding higher order convergence rates by higher order, interlaced polynomial lattice rules with product weights. The product structure of the QMC weights facilitates work bounds for the fast, component-by-component constructions of [D. Nuyens and R. Cools, Math. Comp., 75 (2006), pp. 903-920] which scale linearly with respect to the parameter dimension s. The dimension independent convergence rates are only limited by the degree of digit interlacing used in the construction of the higher order QMC quadrature rule and, for locally supported coefficient functions, by the summability of the locally supported coefficient sequence in the affine-parametric coefficient.

Key words. Quasi-Monte Carlo methods, uncertainty quantification, error estimates, high-dimensional quadrature, elliptic partial differential equations with random input

AMS subject classifications. 65D30, 65N30

1. Introduction. In this paper we analyze a particular type of quasi-Monte Carlo (QMC for short) integration for output functionals of solutions of a class of affine parametric, linear elliptic partial differential equations in divergence form,

$$-\nabla\cdot(a(x,\boldsymbol{y})\,\nabla u(x,\boldsymbol{y})) \,=\, f(x) \text{ in } D\subseteq\mathbb{R}^d\,,\quad u(x,\boldsymbol{y})\,=\,0 \text{ on } \partial D\,. \tag{1}$$

Here, $D\subset\mathbb{R}^d$ is a bounded domain with Lipschitz boundary. This appears in numerous applications, in particular in computational uncertainty quantification (UQ for short). The objective is to numerically compute the mean field, i.e., averages over all parameters, of functionals of (Galerkin approximations of) the parametric solution of (1) with QMC quadrature.

In (1), the gradients are understood with respect to $x\in D$ and the parameter vector $\boldsymbol{y}=(y_j)_{j\geq1}$ consists of a countable number of parameters $y_j\in[-1/2,1/2]$ so that $\boldsymbol{y}$ takes values in the parameter domain U, where

$$\boldsymbol{y}=(y_j)_{j\geq1}\in U:=\left[-\frac{1}{2},\frac{1}{2}\right]^{\mathbb{N}}\,. \tag{2}$$

The elements $(y_j)_{j\geq1}$ of the parameter vector are chosen to be independent and identically uniformly distributed, i.e., the distribution of $\boldsymbol{y}$ is given by the product measure

$$\mu(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j\geq1}\mathrm{d}y_j\,.$$

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The triple $(U, \bigotimes_{j\geq 1}\mathcal{B}([-1/2,1/2]),\mu)$ is a probability space and for a strongly μ-measurable, integrable mapping $F: U \to B$, where B is some Banach space over the reals, the mathematical expectation with respect to the product probability measure μ will be denoted by the Bochner integral

$$\mathbb{E}(F) = \int_U F(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}) \; . \tag{3}$$

The uncertain diffusion coefficient in (1) is assumed to be affine-parametric, i.e.,

$$a(x,\boldsymbol{y}) \;=\; \bar{a}(x) + \sum_{j\geq 1} y_j\psi_j(x), \qquad \text{a.e. } x \in D, \boldsymbol{y} \in U \; , \tag{4}$$

where the mean field of a, i.e., $\bar{a}$, and the functions $(\psi_j)_{j\geq 1}$ are bounded and measurable. Note the implicit scale invariance in the sum in (4).

Specifically, in the present paper we give a convergence rate analysis for the efficient computation by QMC integration of expected values of continuous linear functionals of the parametric solution of (1), or of the Finite Element (FE for short) approximation of the solution of (1). Suppose we are given a continuous, linear functional $G(\cdot): H_0^1(D) \to \mathbb{R}$, which is in UQ applications commonly referred to as *quantity of interest*. Then, we wish to compute (3) with

$$F(\boldsymbol{y}) \;:=\; G(u(\cdot,\boldsymbol{y})), \quad \text{or} \quad F_h(\boldsymbol{y}) \;:=\; G(u_h(\cdot,\boldsymbol{y})), \qquad \boldsymbol{y} \in U \; ,$$

where $\boldsymbol{y} \mapsto u_h(\cdot,\boldsymbol{y}) \in H_0^1(D)$ denotes a FE approximation of the parametric solution $\boldsymbol{y} \mapsto u(\cdot,\boldsymbol{y}) \in H_0^1(D)$.

The expected value (3) of the parametric integrand function F is an iterated integral of the functional $G(\cdot)$ of the parametric solution $U \ni \boldsymbol{y} \mapsto u(\cdot,\boldsymbol{y})$, i.e.,

$$\int_U F(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \;=\; \int_U G(u(\cdot,\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y} \; . \tag{5}$$

We note that this involves integration of (a functional of) the parametric solution over an infinite dimensional domain of integration, which for computational purposes has to be truncated or approximated by a sequence of (close) problems each depending on a finite number of parameters. In applications in computational uncertainty quantification for partial differential equations, parametric integrand evaluations at any point $\boldsymbol{y} \in U$ (as is required for quadrature and collocation) require the solution of a partial differential equation (PDE) for $u(x,\boldsymbol{y})$. This introduces, through numerical solution of the PDE, a discretization error which we bound with dimension-explicit error bounds. By this we mean that the bounds and convergence rates are explicit with respect to the dimension s of the parameters which are active in the approximation.

The parametric PDE (1) with (4) has recently attracted considerable attention, cp. [13, 3, 4] and also the reviews [12, 8] and the references there. The QMC error analysis in those references built on summability conditions of global bounds of the functions $(\psi_j)_{j\geq 1}$. Specifically, assumptions on the decay and p-summability of the sequence $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$, for some $0 < p \leq 1$, were made. There it was assumed for (4) that for some $p \in (0,1]$, it holds

$$\bar{a} \in L^\infty(D) \; , \qquad \sum_{j\geq 1} \|\psi_j\|^p_{L^\infty(D)} \;<\; \infty \; , \tag{6}$$

and also that the mean coefficient $\bar{a}$ and the infinite sum in (6) are such that for some positive numbers $a_{\min}$ and $a_{\max}$ we have

$$0 < a_{\min} \leq a(x, \boldsymbol{y}) \leq a_{\max} , \qquad \text{a.e. } x \in D , \quad \boldsymbol{y} \in U .$$

In particular, the problem in (1) was also considered in [13], where the theory in [13] was developed under the assumption of global supports of the functions $(\psi_j)_{j\geq 1}$. The main goal of this paper is to extend the QMC error analysis framework of [13] in order to be able to account for locality of the supports of the functions $(\psi_j)_{j\geq 1}$. As we shall show, and analogous to what has recently been pointed out for N-term approximation rates in [1], this results in some cases in significant improvement of the QMC convergence rate in cases where randomly shifted lattice rules or higher order interlaced polynomial lattice rules are applicable; equivalently, as we show here, in these cases the same rates as implied by the theory in [13, 5], which assumes global supports, can be achieved, under weaker summability conditions. Our QMC error analysis (which draws on techniques for N-term approximation estimates which were recently developed in [1]) reveals that the structure of the QMC weights changes in case of local support: rather than the product and order dependent (POD for short) weights which were found indispensable in [13] when the ψ_j, $j \geq 1$, in (4) have global support, we show in the present paper that *locally supported* ψ_j*,* $j \geq 1$, allow the use of product weights in the construction of the QMC integration rules. This is well known to imply *linear scaling of the computational work with respect to the parameter dimension* s*.* Hence, the cost of generating N QMC points in s dimensions is essentially comparable to the respective cost using Monte Carlo sampling, i.e., $\mathcal{O}(sN \log N)$ vs. $\mathcal{O}(sN)$. As a result, models with one parameter or dimension of integration per spatial degree of freedom are as feasible as in the case of Monte Carlo, but with higher convergence rates, which is illustrated in this manuscript by a numerical experiment for a model problem. Rather than bounding Legendre coefficients of polynomial chaos expansions as in [1], we use techniques from [1] to bound partial derivatives of integrands as required by QMC convergence theory. Our results are general, in that beyond the local support assumption they do not assume a specific representation system. Examples of systems of locally supported functions which are admissible in our QMC error analysis are indicator functions of a partition of D, B-splines or wavelets. However, for globally supported ψ_j, $j \geq 1$, or if no information about the support is available, the known QMC convergence theory from [13, 5] with POD weights achieves a certain convergence rate under weaker assumptions, where the construction of QMC points with POD weights has a higher cost, which is quadratic with respect to the dimension, cp. [13, 5].

To ensure uniform ellipticity of the parametric problem (1) with respect to the parameter sequence $\boldsymbol{y}$ in the domain U in (2), and also to preserve potential locality of supports, assuming that $\operatorname{ess\,inf}_{x\in D}\{\bar{a}(x)\} > 0$, motivates the condition that for some $\bar{\kappa} \in (0, 1)$

$$\textbf{(A1)} \qquad \operatorname*{ess\,inf}_{x\in D}\{\bar{a}(x)\} > 0 \quad \text{and} \quad \left\| \frac{\sum_{j\geq 1} |\psi_j|}{2\bar{a}} \right\|_{L^\infty(D)} \leq \bar{\kappa} < 1 ,$$

which readily implies that

$$0 < (1 - \bar{\kappa}) \operatorname*{ess\,inf}_{x\in D}\{\bar{a}(x)\} \leq a(x, \boldsymbol{y}), \quad \text{a.e. } x \in D, \boldsymbol{y} \in U . \tag{7}$$

For the sake of a concise notation, we assume that there exist $0 < \bar{a}_{\min} \leq \bar{a}_{\max}$ such that

$$0 < \bar{a}_{\min} \leq \bar{a}(x) \leq \bar{a}_{\max}, \quad \text{a.e. } x \in D \, ,$$

which is equivalent to $\operatorname{ess\,inf}_{x\in D}\{\bar{a}(x)\} > 0$ and $\|\bar{a}\|_{L^\infty(D)} < \infty$. Furthermore, we wish to exploit the decay of the sequence $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$, which is characterized in terms of a real-valued sequence $(b_j)_{j\geq 1}$ such that $0 < b_j \leq 1$ for every $j \in \mathbb{N}$. The condition (**A1**) is therefore generalized such that for some constant $\kappa \in (0,1)$

$$\text{(\textbf{A2})} \qquad \operatorname*{ess\,inf}_{x\in D}\{\bar{a}(x)\} > 0 \quad \text{and} \quad \left\| \frac{\sum_{j\geq 1} |\psi_j|/b_j}{2\bar{a}} \right\|_{L^\infty(D)} \leq \kappa < 1 \, ,$$

where the assumption in (**A1**) is trivially included for $b_j = 1$ for every $j \in \mathbb{N}$.

The "ensemble average" $\mathbb{E}(u)$ will be approximated with a QMC quadrature rule. Specifically, the sequence $(b_j)_{j\geq 1}$ will be used to obtain the weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$, i.e., set $\gamma_\emptyset := 1$ and for every $\emptyset \neq \mathfrak{u} \subset \mathbb{N}$ such that $|\mathfrak{u}| < \infty$ we define for $\alpha \in \mathbb{N}$

$$(8) \qquad \gamma_{\mathfrak{u}} := \begin{cases} \prod_{j\in\mathfrak{u}} (\rho_1 b_j)^2 & \text{if } \alpha = 1 \\ \prod_{j\in\mathfrak{u}} \sum_{\nu=1}^{\alpha} \left((\rho_1 b_j)^\nu \rho_2(\nu,\alpha)\nu!\right) & \text{else} \end{cases}$$

for some constants $\rho_1 > 0$ and $\rho_2 > 0$ that depends only on ν and α, which are of product type. The weights $\boldsymbol{\gamma}$ will be used to define certain Sobolev spaces of admissible integrands and $\boldsymbol{\gamma}$ will be an input for constructing the QMC points. In particular, the product in (8) is finite, since $|\mathfrak{u}| < \infty$. The membership $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ will result in the algebraic rate of convergence $\mathcal{O}(N^{-1/p})$ of a randomly shifted lattice QMC quadrature rule (formally the case of $\alpha = 1$ in (8)) for $p \in (1,2]$ and of a higher order interlaced polynomial lattice rule (the case of $\alpha > 1$ in (8)) for $p \in (0,1]$, where N is the number of sample points, and where all constants implied in $\mathcal{O}(\cdot)$ are bounded independent of s and N, cp. [19, 5]. These convergence rates coincide with the N-term rates for polynomial chaos expansions that are shown in [1, Theorem 1.2 and Equation (1.11)] in this setting. By equilibrating the QMC, dimension truncation, and the FE error contribution, we obtain in the case of wavelets *error vs. work bounds* that are independent of the dimension of integration.

We use standard notation. Throughout, V and H shall denote Hilbert spaces over the reals. By V^*, we denote the dual space of V. For Hilbert spaces H_1 and H_2, we let $L(H_1, H_2)$ denote the bounded linear operators from H_1 to H_2.

The paper is structured as follows. In Section 2, we present the variational formulation of the parametric problem (1). In Section 3 we recapitulate necessary facts about QMC integration rules such as randomly shifted lattice rules and interlaced polynomial lattice rules. The main parametric regularity estimates are derived in Section 4, which imply particular convergence rates that are discussed in Section 6. A combined error analysis taking into account QMC error, spatial discretization error, and truncation error of the series expansion in (4) is presented in Section 7, where the truncation error is estimated in Section 5. In Section 8, we present and analyze the concrete example of a spline wavelet representation of the parametric coefficient. Section 9 contains a numerical experiment for a model, parametric diffusion problem in one space dimension which allows for exact solutions of the parametric problem, thereby allowing to identify and monitor the QMC quadrature error. Section 10 indicates some conclusions from the present work, as well as possible generalizations.

2. Variational formulation. On the Hilbert space $V := H_0^1(D)$, we introduce the parametric bilinear form

$$\mathfrak{a}(\boldsymbol{y}; w, v) := \int_D a(x, \boldsymbol{y}) \nabla w(x) \cdot \nabla v(x) \, \mathrm{d}x \qquad \forall w, v \in V .$$

The weak (or variational) formulation of the parametric, elliptic PDE (1) for fixed $f \in V^*$ is standard: given $\boldsymbol{y} \in U$, find a parametric solution $U \ni \boldsymbol{y} \mapsto u(\cdot, \boldsymbol{y}) \in V$ such that

$$\mathfrak{a}(\boldsymbol{y}; u(\cdot, \boldsymbol{y}), v) = f(v) \qquad \forall v \in V . \tag{9}$$

The assumption in (**A1**) implies well-posedness of the variational formulation of (1). Specifically, (**A1**) implies that

$$0 < (1 - \bar{\kappa})\bar{a}_{\min} \leq \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x) = a(x, \boldsymbol{y}), \quad \text{a.e. } x \in D, \boldsymbol{y} \in U ,$$

and

$$a(x, \boldsymbol{y}) = \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x) \leq \bar{a}(x) + \bar{\kappa}\bar{a}(x) = (1 + \bar{\kappa})\bar{a}_{\max} \quad \text{a.e. } x \in D, \boldsymbol{y} \in U . \tag{10}$$

Hence, the parametric bilinear form $\mathfrak{a}(\boldsymbol{y}; \cdot, \cdot)$ is continuous and coercive on $V \times V$ uniformly with respect to $\boldsymbol{y} \in U$, i.e., for every $\boldsymbol{y} \in U$

$$\forall v, w \in V :$$
$$\mathfrak{a}(\boldsymbol{y}; v, v) \geq (1 - \bar{\kappa})\bar{a}_{\min} \|v\|_V^2 \quad \text{and} \quad |\mathfrak{a}(\boldsymbol{y}; v, w)| \leq (1 + \bar{\kappa})\bar{a}_{\max} \|v\|_V \|w\|_V .$$

The Lax–Milgram lemma implies that the parametric solution to (1) $u : U \to V$ exists, is unique, strongly μ-measurable (by the second Strang lemma), and that there holds

$$\|u(\cdot, \boldsymbol{y})\|_V \leq \frac{1}{(1 - \bar{\kappa})\bar{a}_{\min}} \|f\|_{V^*}, \quad \boldsymbol{y} \in U . \tag{11}$$

3. QMC integration. We recapitulate elements from randomly shifted lattice rules, interlaced polynomial lattice rules, and weighted function spaces on U which arise in the QMC convergence theory, cp. [13, Theorem 2.1] and [5, Theorem 3.10].

The purpose of QMC methods is the approximate evaluation of s-dimensional integrals

$$I_s(F) := \int_{[-\frac{1}{2}, \frac{1}{2}]^s} F(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} , \tag{12}$$

where $s \in \mathbb{N}$. These integrals arise from (3) by *anchored dimension truncation.* By this we mean that in $\boldsymbol{y}$ all components y_j with $j > s$ are set to zero. The parametric integrand function F will, in the presently considered case, consist of a bounded linear functional $G(\cdot)$ of the parametric solution $u(\cdot, \boldsymbol{y}_{\{1:s\}})$, where for $\boldsymbol{y} \in U$ we denote here and in the following $\boldsymbol{y}_{\{1:s\}} := (y_1, ..., y_s, 0, ...)$.

An N*-point QMC quadrature rule* for the s-dimensional integral (12) is an equal-weight integration rule of the form

$$Q_{s,N}(F) := \frac{1}{N} \sum_{i=0}^{N-1} F\left(\boldsymbol{y}^{(i)} - \boldsymbol{\frac{1}{2}}\right) , \tag{13}$$

with *judiciously* chosen points $\{\boldsymbol{y}^{(0)}, \dots, \boldsymbol{y}^{(N-1)}\} \subset [0,1]^s$ and $(\boldsymbol{\frac{1}{2}})_j = \frac{1}{2}$, $j = 1, \dots, s$; we refer to the surveys [12, 8] for more details and further references.

For the QMC error analysis, F in (12) will be assumed to belong to *weighted* and *unanchored* Sobolev spaces. Several choices of such spaces will be made, depending on the type of QMC which is employed. In the present paper, we consider two classes of QMC integration methods: first, we analyze a *randomly shifted lattice rule*, as considered in [13], there for globally supported functions $(\psi_j)_{j\geq 1}$. The error analysis of these rules involves the spaces of the type $\mathcal{W}_{s,\boldsymbol{\gamma}}$ which we now review. For a Hilbert space H and weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$, define the Hilbert space $\mathcal{W}_{s,\boldsymbol{\gamma}}(U;H)$ containing H-valued functions with square integrable mixed first derivatives. The norm in this Hilbert space is given, for arbitrary, finite dimension s, by the unanchored, mixed first derivative

$$\|F\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(U;H)} := \left(\sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac{1}{2},\frac{1}{2}]^{|\mathfrak{u}|}} \left\| \int_{[-\frac{1}{2},\frac{1}{2}]^{s-|\mathfrak{u}|}} \partial_{\boldsymbol{y}}^{\mathfrak{u}} F(\boldsymbol{y}_{\{1:s\}}) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right\|_H^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2}, \tag{14}$$

where the inner integral is understood as a Bochner integral (cp. [22, Chapter V.5]). In the case of $H = \mathbb{R}$, we omit H in the notation and write $\mathcal{W}_{s,\boldsymbol{\gamma}}(U)$. In (14) and throughout the following, $\{1 : s\}$ denotes the set of indices $\{1, 2, \dots, s\}$, and, for a finite, nonempty subset $\mathfrak{u}$ of $\mathbb{N}$, $\partial_{\boldsymbol{y}}^{\mathfrak{u}} F$ denotes the *mixed first derivative of F with respect to the variables y_j with $j \in \mathfrak{u}$*. This notation will be used in the context of randomly shifted lattice rules for historic reasons, cp. for example [13]. We also denote for a multi-index $\boldsymbol{\tau} \in \mathbb{N}_0^s$ by $\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} F$ the respective possibly higher order partial derivatives of F with respect to $\boldsymbol{y}$. Here and in what follows, the argument $\boldsymbol{y}_{\mathfrak{u}}$ signifies the $\mathfrak{u}$-projection of $\boldsymbol{y}$: $(\boldsymbol{y}_{\mathfrak{u}})_j = y_j$ if $j \in \mathfrak{u}$ and 0 otherwise for every $\mathfrak{u} \subset \mathbb{N}$. Similarly, for every $\mathfrak{u} \subset \mathbb{N}$ and $\boldsymbol{y} \in [-1/2, 1/2]^{|\mathfrak{u}|}$, $\boldsymbol{y}_{\mathfrak{u}} \in U$ denotes also the extension of $\boldsymbol{y}$ to the element in U such that $(\boldsymbol{y}_{\mathfrak{u}})_j = y_j$ for every $j \in \mathfrak{u}$ and 0 otherwise. We remark that the QMC quadrature rules $Q_{s,N}$ which are considered in the following depend implicitly also on the weight sequence $\boldsymbol{\gamma}$ in the definition of the norms (14) and in (15) ahead; we shall, however, not indicate this dependence explicitly in the notation for the formula $Q_{s,N}$. We recall the error estimate [13, Theorem 2.1] in the following theorem.

THEOREM 1. *Let $s, N \in \mathbb{N}$ be given and assume that $F \in \mathcal{W}_{s,\boldsymbol{\gamma}}(U)$ for a weight sequence $\boldsymbol{\gamma}$ with product weights. Then a randomly shifted lattice rule can be constructed using a fast component-by-component (CBC) algorithm from [17, 16] in $\mathcal{O}(sN \log N)$ operations such that the root-mean square error satisfies, for every $\lambda \in (\frac{1}{2}, 1]$,*

$$\sqrt{\mathbb{E}^{\Delta}(|I_s(F) - Q_{s,N}(F)|^2)} \le \left(\sum_{\emptyset\neq\mathfrak{u}\in\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^{\lambda}} \right)^{|\mathfrak{u}|} \right)^{1/(2\lambda)} (\varphi(N))^{-1/(2\lambda)} \|F\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(U)},$$

where $\varphi(\cdot)$ is Euler's totient function and where $\mathbb{E}^{\Delta}(\cdot)$ denotes expectation with respect to the random shift Δ.

We see that the (dimension-independent) rate of convergence of randomly shifted lattice rules is capped by one. The product structure of the QMC weights in Theorem 1 entails the cost estimate of $\mathcal{O}(sN\log N)$ operations for the computation of the QMC generating vector using the fast CBC construction of [17, 16].

A second class of QMC integration rules, the so-called *interlaced polynomial lattice rules*, has been proposed and analyzed in [5]. Their error analysis involves the weighted norms defined in [5, Definition 3.3] for scalar valued functions. Generally, for a Hilbert space H and weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$, we introduce the Banach space $\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,r}(U;H)$ of H-valued functions F that have finite $\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,r}(U;H)$-norm. This higher order, unanchored Sobolev norm of F is, for $1 \le q, r \le \infty$ and for arbitrary, finite dimension s,[1] given by

$$\begin{aligned}\|F\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,r}(U;H)} \\ := \Bigg(\sum_{\mathfrak{u}\subseteq\{1:s\}} \Bigg(\gamma_{\mathfrak{u}}^{-q} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}} \\ \int_{[-\frac12,\frac12]^{|\mathfrak{v}|}} \left\| \int_{[-\frac12,\frac12]^{s-|\mathfrak{v}|}} \partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\boldsymbol{0})} F(\boldsymbol{y}_{\{1:s\}})\,\mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{v}} \right\|_H^q \mathrm{d}\boldsymbol{y}_{\mathfrak{v}} \Bigg)^{r/q} \Bigg)^{1/r},\end{aligned} \tag{15}$$

with the obvious modifications if q or r is infinite. Also the inner integral in the definition (15) of the norm has to be interpreted as a Bochner integral, cp. [22, Chapter V.5]. Here, $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0}) \in \{0:\alpha\}^s$ denotes a multi-index such that $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = \alpha$ for $j \in \mathfrak{v}$, $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = \boldsymbol{\tau}_j$ for $j \in \mathfrak{u}\setminus\mathfrak{v}$, and $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}, \boldsymbol{0})_j = 0$ for $j \notin \mathfrak{u}$, for every $\mathfrak{u} \subseteq \{1:s\}, \mathfrak{v} \subseteq \mathfrak{u}, \boldsymbol{\tau} \in \{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}$. In the case of $H = \mathbb{R}$, we write $\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,r}(U)$. Finiteness of this norm for parametric integrand functions has been shown in [5] to imply dimension-independent convergence rates of higher order, interlaced polynomial lattice rules. We restate the result [5, Theorem 3.10] here for the readers' convenience.

THEOREM 2. *Let $\alpha, s \in \mathbb{N}$ with $\alpha > 1$, $1 \le q \le \infty$, and let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ denote a collection of product weights. Let b be a prime number and let $m \in \mathbb{N}$ be arbitrary. Then, an interlaced polynomial lattice rule of order α with $N = b^m$ points $\{\boldsymbol{y}_0, \dots, \boldsymbol{y}_{N-1}\} \subset [0,1)^s$ can be constructed using a CBC algorithm, in $\mathcal{O}(sN\log N)$ operations such that for every $F \in \mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,\infty}(U)$*

$$|I_s(F) - Q_{N,s}(F)| \le \left(\frac{2}{N-1} \sum_{\emptyset\ne\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \rho_{\alpha,b}(\lambda)^{|\mathfrak{u}|} \right)^{1/\lambda} \|F\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,\infty}(U)},$$

for every $1/\alpha < \lambda \le 1$, where

$$\rho_{\alpha,b}(\lambda) = \left(C_{\alpha,b}\, b^{\alpha(\alpha-1)/2} \right)^{\lambda} \left(\left(1 + \frac{b-1}{b^{\alpha\lambda} - b} \right)^{\alpha} - 1 \right), \tag{16}$$

with $C_{\alpha,b}$ a positive constant that only depends on α and b.

The explicit value of the Walsh constant $C_{\alpha,b}$, $\alpha, b \in \mathbb{N}$, in the above theorem is stated in [5, Equation (3.11)]; we also refer to [21] for a mathematical justification of an improved value of the Walsh constant. The cost estimate $\mathcal{O}(s\,N\log N)$ for the CBC construction in the case of product weights is shown in [5, Section 3.4].

[1] The formula in [5, Equation (3.7)] is incorrectly stated. Expression (15) for $H = \mathbb{R}$ is the correct formula for the analysis in [5].

4. Parametric regularity. From the definitions (14), (15) of the weighted norms it is clear that higher order derivatives $\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} u(\cdot,\boldsymbol{y})$ of the parametric solution $u(\cdot,\boldsymbol{y})$ of (9) play a crucial role in QMC error bounds.

Let the assumption in (**A2**) hold for some $\kappa \in [\bar{\kappa},1)$ and for a sequence $(b_j)_{j\geq 1} \in (0,1]^{\mathbb{N}}$. In view of the ensuing QMC error analysis, we establish in this section derivative bounds with respect to the parameter vector $\boldsymbol{y}$. The idea is to extend the parameter domain U and introduce a dilated coordinate on the extended domain. Let us introduce the auxiliary parameter domain

$$\widetilde{U} := [-1,1]^{\mathbb{N}} ,$$

with parameter vectors $\boldsymbol{z} \in \widetilde{U}$. Consider $\eta \in (\kappa,1)$ being a scaling factor such that $\kappa/\eta < 1$, which represents by how much the parameter domain can potentially be extended. For every $\boldsymbol{y} \in U$ we define the affine mapping $T_{\boldsymbol{y}} : \widetilde{U} \to T_{\boldsymbol{y}}(\widetilde{U}) \subset \mathbb{R}^{\mathbb{N}}$ by

$$(T_{\boldsymbol{y}}(\boldsymbol{z}))_j := y_j + \frac{\eta^{-1} - 2|y_j|}{2b_j} z_j, \quad j \in \mathbb{N}, \boldsymbol{z} \in \widetilde{U}.$$

For fixed $\boldsymbol{y} \in U$ and interpreting $T_{\boldsymbol{y}}(\boldsymbol{z})$ as a parameter vector we denote by $\widetilde{u}_{\boldsymbol{y}}$ the solution to

$$-\nabla \cdot (\widetilde{a}_{\boldsymbol{y}}(x,\boldsymbol{z})\,\nabla \widetilde{u}_{\boldsymbol{y}}(x,\boldsymbol{z})) \;=\; f(x) \text{ in } D \subseteq \mathbb{R}^d , \quad \widetilde{u}_{\boldsymbol{y}}(x,\boldsymbol{z}) \;=\; 0 \text{ on } \partial D, \tag{17}$$

where for a.e. $x \in D$ and every $\boldsymbol{z} \in \widetilde{U}$, the affine-parametric coefficient in (17) reads

$$\widetilde{a}_{\boldsymbol{y}}(x,\boldsymbol{z}) := \bar{a}_{\boldsymbol{y}}(x) + \sum_{j\geq 1} z_j \psi_{\boldsymbol{y},j}(x), \quad \bar{a}_{\boldsymbol{y}}(x) := \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x),$$

and

$$\psi_{\boldsymbol{y},j}(x) := \frac{\eta^{-1} - 2|y_j|}{2b_j} \psi_j(x), \quad \text{for every } j \in \mathbb{N}.$$

We seek to verify an ellipticity condition for the diffusion coefficients $\{\widetilde{a}_{\boldsymbol{y}}(\cdot,\boldsymbol{z}) : \boldsymbol{y} \in U, \boldsymbol{z} \in \widetilde{U}\}$, which is uniform in $\boldsymbol{y} \in U$ and in $\boldsymbol{z} \in \widetilde{U}$. We recall that for every $\boldsymbol{y} \in U$, $\widetilde{a}_{\boldsymbol{y}}$ is parametrized over $\widetilde{U} = [-1,1]^{\mathbb{N}}$. By (**A1**) it holds that

$$\operatorname*{ess\,inf}_{x\in D} \{\bar{a}_{\boldsymbol{y}}(x)\} = \operatorname*{ess\,inf}_{x\in D} \left\{ \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x) \right\} \geq (1-\bar{\kappa})\bar{a}_{\min}$$

and for a.e. $x \in D$

$$\frac{\sum_{j\geq 1} |\psi_{\boldsymbol{y},j}(x)|}{\bar{a}_{\boldsymbol{y}}(x)} \leq \frac{\sum_{j\geq 1} |\psi_j(x)|/(2\eta b_j) - \sum_{j\geq 1}(|y_j|/b_j)\psi_j}{\bar{a}(x) - \sum_{j\geq 1}|y_j||\psi_j(x)|} \leq \frac{\sum_{j\geq 1}|\psi_j(x)|/b_j}{2\eta\bar{a}(x)} ,$$

where we used that $b_j \leq 1$ for every $j \in \mathbb{N}$. We conclude with (**A2**) an ellipticity condition that holds uniformly with respect to $\boldsymbol{y} \in U$, i.e., for every $\boldsymbol{y} \in U$

$$\operatorname*{ess\,inf}_{x\in D} \{\bar{a}_{\boldsymbol{y}}(x)\} \;\geq\; (1-\bar{\kappa})\bar{a}_{\min} > 0 \quad \text{and} \quad \left\| \frac{\sum_{j\geq 1}|\psi_{\boldsymbol{y},j}|}{\bar{a}_{\boldsymbol{y}}} \right\|_{L^\infty(D)} \leq \frac{\kappa}{\eta} < 1 . \tag{18}$$

This implies well-posedness of (17) analogous to the previously (cp. Section 2) established well-posedness of (1), which follows by the Lax–Milgram lemma. Specifically, the coercivity constant of the corresponding bilinear form of the diffusion coefficient $\widetilde{a}_{\boldsymbol{y}}$, when parametrized with $\boldsymbol{z} \in \widetilde{U}$, can be uniformly lower bounded in $\boldsymbol{y} \in U$: for every $\boldsymbol{y} \in U$

$$\widetilde{a}_{\boldsymbol{y}}(x, \boldsymbol{z}) \geq \left(1 - \frac{\kappa}{\eta}\right)(1 - \bar{\kappa})\bar{a}_{\min}, \quad \text{a.e. } x \in D, \boldsymbol{z} \in \widetilde{U} \ . \tag{19}$$

Therefore, for every $\boldsymbol{y} \in U$ there exists a unique $\widetilde{u}_{\boldsymbol{y}}$ and there holds the a-priori estimate

$$\|\widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z})\|_V \leq \frac{\eta}{(\eta - \kappa)(1 - \bar{\kappa})\bar{a}_{\min}} \|f\|_{V^*}, \quad \text{for every } \ \boldsymbol{z} \in \widetilde{U} \ ,$$

which follows from (18) in a similar way as (11) was shown in Section 2. By the well-posedness of (17) we arrive at the relation

$$\widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) = u(\cdot, T_{\boldsymbol{y}}(\boldsymbol{z})), \ \text{ in } \ V \ , \quad \forall \boldsymbol{y} \in U, \boldsymbol{z} \in \widetilde{U} \ . \tag{20}$$

The chain rule of differentiation and $T_{\boldsymbol{y}}$ being affine imply for every $\boldsymbol{\tau} \in \mathcal{F} = \{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\tau}| < \infty\}$

$$\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} = \left(\prod_{j \in \mathbb{N}} \left(\frac{\eta^{-1} - 2|y_j|}{2b_j}\right)^{\boldsymbol{\tau}_j}\right) \partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} u(\cdot, \boldsymbol{y}) \ . \tag{21}$$

The term on the left hand side of (21) is a Taylor coefficient of mixed derivatives. The idea to introduce an auxiliary parameter vector $\boldsymbol{z}$ and a dilation operator $T_{\boldsymbol{y}}$ to obtain a formula for the Taylor coefficients as (21) w.r.t. to a parameter dependent function system $(\psi_{\boldsymbol{y},j})_{j \geq 1}$ and nominal coefficient $\bar{a}_{\boldsymbol{y}}$ has been introduced to prove p-summability of Legendre coefficients in generalized polynomial chaos expansions of the parametric solution; see the proof of [1, Theorem 3.1]. Here, we exploit this observation to bound integrand derivatives; in the slightly different definition of the dilation operator, the differently scaled parameter set is taken into account. Summability of Taylor coefficients for problems of the type (1) has been studied in [4, 2, 1]. The square summability is proven in the following lemma, using techniques introduced in [1].

LEMMA 3. *Let the condition in* (**A1**) *be satisfied for* $\bar{\kappa} \in (0, 1)$*, and assume further that the condition in* (**A2**) *holds for some* $\kappa \in [\bar{\kappa}, 1)$*, and let* $\eta \in (\kappa, 1)$ *be arbitrary, fixed. Then, for every* $\boldsymbol{y} \in U$*, it holds that*

$$\sum_{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}}, |\boldsymbol{\tau}| < \infty} \frac{1}{(\boldsymbol{\tau}!)^2} \left\| \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_V^2 \leq \frac{\eta(1 + \bar{\kappa})}{(\eta - \kappa)(1 - \bar{\kappa})^3} \frac{\bar{a}_{\max}}{\bar{a}_{\min}^3} \|f\|_{V^*}^2 < \infty \ .$$

Proof. We recall the notation $\mathcal{F} = \{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\tau}| < \infty\}$. For the presentation in this proof, we introduce the parametric energy norm $\| \cdot \|_{\bar{a}_{\boldsymbol{y}}}$,

$$\|v\|_{\bar{a}_{\boldsymbol{y}}} := \left(\int_D \bar{a}_{\boldsymbol{y}} |\nabla v|^2 \mathrm{d}x\right)^{1/2}, \quad \text{for every } v \in V, \boldsymbol{y} \in U \ .$$

In the following, we will utilize the Taylor coefficients

$$t_{\boldsymbol{y},\boldsymbol{\tau}} := \frac{1}{\boldsymbol{\tau}!} \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \quad \boldsymbol{\tau} \in \mathcal{F}, \boldsymbol{y} \in U \ .$$

For every fixed $\boldsymbol{y} \in U$, the nominal part $\bar{a}_{\boldsymbol{y}}$ of $\widetilde{a}_{\boldsymbol{y}}(\cdot, \boldsymbol{z})$ and the fluctuations due to the functions $(\psi_{\boldsymbol{y},j})_{j\geq 1}$ satisfy the ellipticity condition in (18), which is uniform in $\boldsymbol{y}$ and implies a uniform coercivity constant, cp. (19). Therefore, the well-known recurrence relation of the Taylor coefficients $\{t_{\boldsymbol{y},\boldsymbol{\tau}} : \boldsymbol{\tau} \in \mathcal{F}\}$, cp. [2, Equation (3.1)] or [3, Equation (4.10)], holds for every fixed $\boldsymbol{y} \in U$. Specifically, for every $\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathcal{F}$ and every $\boldsymbol{y} \in U$

$$\int_D \bar{a}_{\boldsymbol{y}} \nabla t_{\boldsymbol{y},\boldsymbol{\tau}} \cdot \nabla v \mathrm{d}x = - \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D \psi_{\boldsymbol{y},j} \nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j} \cdot \nabla v \mathrm{d}x, \quad \forall v \in V \ ,$$

where $\mathrm{supp}(\boldsymbol{\tau}) := \{n \in \mathbb{N} : \boldsymbol{\tau}_j \neq 0\}$. As in [2] or as in the proof of [1, Lemma 2.1], for arbitrary $\boldsymbol{0} \neq \boldsymbol{\tau} \in \mathcal{F}$ and $v = t_{\boldsymbol{\tau}}$ we obtain with Young's inequality and (18) that for every $\boldsymbol{y} \in U$

$$\begin{aligned}
\int_D \bar{a}_{\boldsymbol{y}} |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \mathrm{d}x &\leq \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D |\psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j} \cdot \nabla t_{\boldsymbol{y},\boldsymbol{\tau}}| \mathrm{d}x \\
&\leq \frac{1}{2} \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D |\psi_{\boldsymbol{y},j}| (|\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j}|^2 + |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2) \mathrm{d}x \\
&\leq \frac{1}{2} \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D |\psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j}|^2 \mathrm{d}x + \frac{\kappa}{2\eta} \int_D \bar{a}_{\boldsymbol{y}} |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \mathrm{d}x,
\end{aligned}$$

which implies that

$$\left(1 - \frac{\kappa}{2\eta}\right) \|t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} \leq \frac{1}{2} \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D |\psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j}|^2 \mathrm{d}x. \tag{22}$$

The condition in (18) implies with (22) that for every $k \in \mathbb{N}$

$$\begin{aligned}
\left(1 - \frac{\kappa}{2\eta}\right) \sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k} \|t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}} &\leq \frac{1}{2} \sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k} \sum_{j \in \mathrm{supp}(\boldsymbol{\tau})} \int_D |\psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}-\boldsymbol{e}_j}|^2 \mathrm{d}x \\
&= \frac{1}{2} \sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k-1} \sum_{j \geq 1} \int_D |\psi_{\boldsymbol{y},j}| |\nabla t_{\boldsymbol{y},\boldsymbol{\tau}}|^2 \mathrm{d}x \\
&\leq \frac{\kappa}{2\eta} \sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k-1} \|t_{\boldsymbol{y},\boldsymbol{\tau}}\|^2_{\bar{a}_{\boldsymbol{y}}}.
\end{aligned}$$

Thus,

$$\sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k} \frac{1}{(\boldsymbol{\tau}!)^2} \left\| \partial^{\boldsymbol{\tau}}_{\boldsymbol{z}} \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) \Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|^2_{\bar{a}_{\boldsymbol{y}}} \leq \frac{\kappa}{2\eta - \kappa} \sum_{\boldsymbol{\tau} \in \mathcal{F}, |\boldsymbol{\tau}| = k-1} \frac{1}{(\boldsymbol{\tau}!)^2} \left\| \partial^{\boldsymbol{\tau}}_{\boldsymbol{z}} \widetilde{u}_{\boldsymbol{y}}(\cdot, \boldsymbol{z}) \Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|^2_{\bar{a}_{\boldsymbol{y}}} . \tag{23}$$

By a geometric series argument, using (23), (10), and (11) we obtain

$$\begin{aligned}
&\sum_{\boldsymbol{\tau}\in\{0,\dots,\alpha\}^{\mathbb{N}},|\boldsymbol{\tau}|<\infty} \frac{1}{(\boldsymbol{\tau}!)^2} \left\| \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_V^2 \\
&\quad\leq \sum_{k\geq 0} \sum_{\boldsymbol{\tau}\in\mathcal{F},|\boldsymbol{\tau}|=k} \frac{1}{(\boldsymbol{\tau}!)^2} \frac{1}{\operatorname{ess\,inf}_{x\in D}\{\bar{a}_{\boldsymbol{y}}(x)\}} \left\| \partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_{\bar{a}_{\boldsymbol{y}}}^2 \\
&\quad\leq \sum_{k\geq 0} \left(\frac{\kappa}{2\eta-\kappa}\right)^k \frac{\|\bar{a}_{\boldsymbol{y}}\|_{L^\infty(D)}}{\operatorname{ess\,inf}_{x\in D}\{\bar{a}_{\boldsymbol{y}}(x)\}} \|u(\cdot,\boldsymbol{y})\|_V^2 \\
&\quad\leq \frac{\eta-\kappa/2}{\eta-\kappa} \frac{(1+\bar{\kappa})\bar{a}_{\max}}{((1-\bar{\kappa})\bar{a}_{\min})^3} \|f\|_{V^*}^2 \\
&\quad\leq \frac{\eta(1+\bar{\kappa})}{(\eta-\kappa)(1-\bar{\kappa})^3} \frac{\bar{a}_{\max}}{\bar{a}_{\min}^3} \|f\|_{V^*}^2 .
\end{aligned}$$

We applied in the above computation the correspondence between $\widetilde{u}_{\boldsymbol{y}}$ and u in (20). ∎

PROPOSITION 4. *Under the assumptions of Lemma 3, for every $s \in \mathbb{N}$ and every choice of weights $\boldsymbol{\gamma}$,*

$$\|u\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(U;V)} \leq \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}} \sqrt{\frac{\bar{a}_{\max}}{\bar{a}_{\min}^3}} \|f\|_{V^*} \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1/2} \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{1-\eta}\right).$$

Proof. We apply Jensen's inequality and observe (formally) $\mathfrak{u}! = 1$ to conclude with (21) the following bound of the $\mathcal{W}_{s,\boldsymbol{\gamma}}$-norm of u,

$$\begin{aligned}
\|u\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(U;V)}^2 &= \sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[-\frac12,\frac12]^{|\mathfrak{u}|}} \left\| \int_{[-\frac12,\frac12]^{s-|\mathfrak{u}|}} \partial_{\boldsymbol{y}}^{\mathfrak{u}} u(\cdot,\boldsymbol{y}_{\{1:s\}}) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right\|_V^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \\
&\leq \int_{[-\frac12,\frac12]^s} \sum_{\mathfrak{u}\subseteq\{1:s\}} \left\| \partial_{\boldsymbol{z}}^{\mathfrak{u}} \widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_V^2 \gamma_{\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{\eta^{-1}-2|y_j|}\right)^2 \mathrm{d}\boldsymbol{y} \\
&\leq \int_{[-\frac12,\frac12]^s} \sum_{\mathfrak{u}\subseteq\{1:s\}} \left\| \partial_{\boldsymbol{z}}^{\mathfrak{u}} \widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_V^2 \mathrm{d}\boldsymbol{y} \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{1-\eta}\right)^2 .
\end{aligned}$$

The assertion follows with Lemma 3 upon taking square roots. ∎

COROLLARY 5. *Under the assumptions of Proposition 4, for every $G(\cdot) \in V^*$ holds for $F(\boldsymbol{y}) := G(u(\cdot,\boldsymbol{y}))$, $\boldsymbol{y} \in U$,*

$$\begin{aligned}
\|F\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(U)} \leq{}& \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}} \sqrt{\frac{\bar{a}_{\max}}{\bar{a}_{\min}^3}} \|f\|_{V^*} \|G(\cdot)\|_{V^*} \\
&\times \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1/2} \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{1-\eta}\right) .
\end{aligned}$$

We extend the foregoing estimates to higher order norms.

PROPOSITION 6. *Under the assumptions of Lemma 3, for every $s \in \mathbb{N}$, $\alpha \in \mathbb{N}$*

and for every choice of weights $\boldsymbol{\gamma}$,

$$\|u\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U;V)} \leq \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}}\sqrt{\frac{\bar{a}_{\max}}{\bar{a}_{\min}^3}}\|f\|_{V^*} \times \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \sup_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \left(\left(\frac{2b_j}{1-\eta}\right)^{(\boldsymbol{\tau}_{\mathfrak{u}})_j} \sqrt{2}^{\delta((\boldsymbol{\tau}_{\mathfrak{u}})_j,\alpha)} (\boldsymbol{\tau}_{\mathfrak{u}})_j!\right),$$

where $\delta((\boldsymbol{\tau}_{\mathfrak{u}})_j,\alpha)=1$ *if* $(\boldsymbol{\tau}_{\mathfrak{u}})_j=\alpha$ *and* 0 *otherwise.*

Note that the $\|\cdot\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U;V)}$-norm corresponds to the norm defined in (15) with the choices $q=r=2$. Values of $q,r\in(2,\infty]$ are also possible (see, e.g., [12]).

Proof. We apply Jensen's inequality and account for multi-indices $(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\mathbf{0})$ appearing multiple times in the sum with the factor $2^{|\{j|(\boldsymbol{\tau}_{\mathfrak{u}})_j=\alpha\}|}$ to obtain that

$$\begin{aligned}
&\|u\|^2_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U;V)} \\
&\quad= \sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-2} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}} \\
&\qquad \int_{[-\frac12,\frac12]^{|\mathfrak{v}|}} \left\| \int_{[-\frac12,\frac12]^{s-|\mathfrak{v}|}} \partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\mathbf{0})} u(\cdot,\boldsymbol{y}_{\{1:s\}}) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{v}} \right\|_V^2 \mathrm{d}\boldsymbol{y}_{\mathfrak{v}} \qquad (24)\\
&\quad\leq \sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-2} \sum_{\mathfrak{v}\subseteq\mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}}\in\{1:\alpha\}^{|\mathfrak{u}\setminus\mathfrak{v}|}} \int_{[-\frac12,\frac12]^s} \left\|\partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}},\boldsymbol{\tau}_{\mathfrak{u}\setminus\mathfrak{v}},\mathbf{0})} u(\cdot,\boldsymbol{y})\right\|_V^2 \mathrm{d}\boldsymbol{y} \\
&\quad= \int_{[-\frac12,\frac12]^s} \sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-2} \sum_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} 2^{|\{j|(\boldsymbol{\tau}_{\mathfrak{u}})_j=\alpha\}|} \left\|\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}_{\mathfrak{u}}} u(\cdot,\boldsymbol{y})\right\|_V^2 \mathrm{d}\boldsymbol{y}\,.
\end{aligned}$$

In the second step of the proof, the following further modifications of (24)

$$\begin{aligned}
\|u\|^2_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U;V)} &\leq \int_{[-\frac12,\frac12]^s} \sum_{\mathfrak{u}\subseteq\{1:s\}} \sum_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \frac{1}{(\boldsymbol{\tau}_{\mathfrak{u}}!)^2} \left\|\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}_{\mathfrak{u}}} \widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\mathbf{0}}\right\|_V^2 \mathrm{d}\boldsymbol{y} \\
&\quad \times \gamma_{\mathfrak{u}}^{-2} 2^{|\{j|(\boldsymbol{\tau}_{\mathfrak{u}})_j=\alpha\}|} (\boldsymbol{\tau}_{\mathfrak{u}}!)^2 \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{1-\eta}\right)^{2(\boldsymbol{\tau}_{\mathfrak{u}})_j} \\
&\leq \int_{[-\frac12,\frac12]^s} \sum_{\boldsymbol{\tau}\in\{0,1,\dots,\alpha\}^s} \frac{1}{(\boldsymbol{\tau}!)^2} \left\|\partial_{\boldsymbol{z}}^{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{y}_{\{1:s\}}}(\cdot,\boldsymbol{z})\Big|_{\boldsymbol{z}=\mathbf{0}}\right\|_V^2 \mathrm{d}\boldsymbol{y} \\
&\quad \times \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-2} \sup_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \left(\left(\frac{2b_j}{1-\eta}\right)^{2(\boldsymbol{\tau}_{\mathfrak{u}})_j} 2^{\delta((\boldsymbol{\tau}_{\mathfrak{u}})_j,\alpha)} ((\boldsymbol{\tau}_{\mathfrak{u}})_j!)^2\right)
\end{aligned}$$

imply with (21) the asserted bound of the $\|\cdot\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U;V)}$-norm of u. The assertion follows with Lemma 3 upon taking square roots. ∎

COROLLARY 7. *Under the assumptions of Proposition 6, for every* $G(\cdot)\in V^*$

holds for $F(\boldsymbol{y}) := G(u(\cdot,\boldsymbol{y}))$, $\boldsymbol{y} \in U$,

$$\|F\|_{\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}(U)} \le \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}}\sqrt{\frac{\bar{a}_{\max}}{\bar{a}_{\min}^3}}\|f\|_{V^*}\|G(\cdot)\|_{V^*}$$
$$\times \sup_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \sup_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \prod_{j\in\mathfrak{u}} \left(\left(\frac{2b_j}{1-\eta}\right)^{(\boldsymbol{\tau}_{\mathfrak{u}})_j} \sqrt{2^{\delta((\boldsymbol{\tau}_{\mathfrak{u}})_j,\alpha)}(\boldsymbol{\tau}_{\mathfrak{u}})_j!}\right).$$

5. Dimension truncation. Regarding the impact of truncating the integration dimension we extend [13, Theorem 5.1] to the present setting. Specifically, we work under the assumptions in (**A1**) and in (**A2**). For every $s \in \mathbb{N}$, let us define the parametric solution of (1) for s-term truncated parameter vectors by

$$u^s(\cdot,\boldsymbol{y}) := u(\cdot,\boldsymbol{y}_{\{1:s\}}) \ \text{ in } \ V\ , \quad \boldsymbol{y} \in U\ .$$

PROPOSITION 8. *Assume that* (**A1**) *is satisfied for* $\bar{\kappa} \in (0,1)$ *and that* (**A2**) *is satisfied for* $\kappa \in [\bar{\kappa},1)$ *and a sequence* $(b_j)_{j\ge1}$ *such that* $b_j \in (0,1]$. *Then, for every* $s \in \mathbb{N}$ *and every* $\boldsymbol{y} \in U$

$$\|u(\cdot,\boldsymbol{y}) - u^s(\cdot,\boldsymbol{y})\|_V \ \le\ \frac{\|f\|_{V^*}\bar{a}_{\max}}{(1-\bar{\kappa})^2(\bar{a}_{\min})^2}\max_{j\ge s+1}\{b_j\}\ . \tag{25}$$

Moreover, if there holds for κ *as in* (**A2**)

$$\frac{\bar{a}_{\max}}{(1-\bar{\kappa})\bar{a}_{\min}}\,\kappa\,\max_{j\ge s+1}\{b_j\} < 1\ , \tag{26}$$

then for every $G(\cdot) \in V^*$ *holds*

$$\begin{aligned} &|\mathbb{E}(G(u)) - I_s(G(u^s))| \\ &\le \frac{\|G(\cdot)\|_{V^*}\|f\|_{V^*}}{(1-\bar{\kappa})\bar{a}_{\min} - \bar{a}_{\max}\kappa\max_{j\ge s+1}\{b_j\}}\frac{\bar{a}_{\max}^2}{(1-\bar{\kappa})^2\bar{a}_{\min}^2}\left(\kappa\max_{j\ge s+1}\{b_j\}\right)^2. \end{aligned} \tag{27}$$

Proof. We readily obtain with the second Strang lemma that for every $\boldsymbol{y} \in U$

$$\begin{aligned}\|u(\cdot,\boldsymbol{y}) - u^s(\cdot,\boldsymbol{y})\|_V &\le \frac{\|f\|_{V^*}}{(1-\bar{\kappa})^2\bar{a}_{\min}^2}\Big\|\sum_{j\ge s+1}|y_j||\psi_j|\Big\|_{L^\infty(D)} \\ &\le \frac{\|f\|_{V^*}\bar{a}_{\max}}{(1-\bar{\kappa})^2\bar{a}_{\min}^2}\left\|\frac{\sum_{j\ge s+1}|\psi_j|/b_j}{2\bar{a}}\right\|_{L^\infty(D)}\max_{j\ge s+1}\{b_j\} \\ &\le \frac{\|f\|_{V^*}\bar{a}_{\max}}{(1-\bar{\kappa})^2\bar{a}_{\min}^2}\max_{j\ge s+1}\{b_j\}\ .\end{aligned}$$

The second part of the proof is a modification of the argument used in the proof of [13, Theorem 5.1]. We will therefore only present the necessary adaptations to exploit our conditions (**A1**) and (**A2**). Define $A(\boldsymbol{y}) := -\nabla\cdot(a(\cdot,\boldsymbol{y})\nabla)$ and $A_s(\boldsymbol{y}) := -\nabla\cdot(a(\cdot,\boldsymbol{y}_{\{1:s\}})\nabla)$. We will not indicate the dependence of A_s on the parameter sequence $\boldsymbol{y}$ for simplicity. Similarly, we obtain for κ as in (**A2**)

$$\begin{aligned}
\|A_s^{-1}(A-A_s)v\|_V &\leq \frac{\|(A-A_s)v\|_{V^*}}{(1-\bar{\kappa})\bar{a}_{\min}} \\
&\leq \frac{\|v\|_V}{(1-\bar{\kappa})\bar{a}_{\min}} \Big\| \sum_{j\geq s+1} |y_j||\psi_j| \Big\|_{L^\infty(D)} \\
&\leq \frac{\|v\|_V \bar{a}_{\max}}{(1-\bar{\kappa})\bar{a}_{\min}} \left\| \frac{\sum_{j\geq s+1}|\psi_j|/b_j}{2\bar{a}} \right\|_{L^\infty(D)} \max_{j\geq s+1}\{b_j\} \\
&\leq \frac{\|v\|_V \bar{a}_{\max}\kappa}{(1-\bar{\kappa})\bar{a}_{\min}} \max_{j\geq s+1}\{b_j\}\ .
\end{aligned} \tag{28}$$

By assumption (26), the bound (28) implies $\|A_s^{-1}(A-A_s)\|_{L(V)} < 1$. This implies in turn that the Neumann series

$$A^{-1} = (\mathcal{I} + A_s^{-1}(A-A_s))^{-1}A_s^{-1} = \sum_{k\geq 0}(-A_s^{-1}(A-A_s))^k A_s^{-1}$$

can be majorized by a convergent geometric series, which results for every $s\in\mathbb{N}$ in the representation

$$u - u^s = \sum_{k\geq 0}(-A_s^{-1}(A-A_s))^k A_s^{-1}f - u^s = \sum_{k\geq 1}(-A_s^{-1}(A-A_s))^k u^s\ . \tag{29}$$

We claim that the term with $k=1$ in this sum vanishes, i.e. that

$$\mathbb{E}[G(A_s^{-1}(A-A_s)u^s)] = 0\ .$$

To show this, we introduce the product σ-algebra

$$\mathcal{A}_s := \mathcal{B}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^s\right) \otimes \left\{\emptyset, \left[-\frac{1}{2},\frac{1}{2}\right]^{\mathbb{N}\setminus\{1:s\}}\right\} \subset \bigotimes_{j\geq 1}\mathcal{B}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right).$$

Since $A_s : U \to L(V,V^*)$, $A_s^{-1} : U \to L(V^*,V)$, and since $u^s : U \to V$ are measurable with respect to $\mathcal{A}_s$, it holds

$$\mathbb{E}(A_s^{-1}(A-A_s)u^s) = \mathbb{E}(\mathbb{E}(A_s^{-1}(A-A_s)u^s|\mathcal{A}_s)) = \mathbb{E}(A_s^{-1}\mathbb{E}(A-A_s|\mathcal{A}_s)u^s) = 0\ ,$$

where we have used that $\mathbb{E}(A|\mathcal{A}_s) = A_s$. Therefore, by continuity and linearity of $G(\cdot)$ it follows with (29) that

$$\mathbb{E}(G(u)) - I_s(G(u^s)) = \mathbb{E}(G(u-u^s)) = G(\mathbb{E}(u-u^s)) = \sum_{k\geq 2} G(\mathbb{E}((-A_s^{-1}(A-A_s))^k u^s))\ .$$

We conclude with (28) that

$$\begin{aligned}
&|\mathbb{E}(G(u)) - I_s(G(u^s))| \\
&\quad\leq \|G(\cdot)\|_{V^*} \sup_{\boldsymbol{y}\in U}\sum_{k\geq 2}\|A_s^{-1}(A-A_s)\|_{L(V)}^k \|u^s\|_V \\
&\quad\leq \frac{\|G(\cdot)\|_{V^*}\|u^s\|_V}{1-\frac{\bar{a}_{\max}\kappa\max_{j\geq s+1}\{b_j\}}{(1-\bar{\kappa})\bar{a}_{\min}}} \left(\frac{\bar{a}_{\max}\kappa\max_{j\geq s+1}\{b_j\}}{(1-\bar{\kappa})\bar{a}_{\min}}\right)^2 \\
&\quad= \frac{\|G(\cdot)\|_{V^*}\|f\|_{V^*}}{(1-\bar{\kappa})\bar{a}_{\min} - \bar{a}_{\max}\kappa\max_{j\geq s+1}\{b_j\}} \frac{\bar{a}_{\max}^2}{(1-\bar{\kappa})^2\bar{a}_{\min}^2}\left(\kappa\max_{j\geq s+1}\{b_j\}\right)^2 .
\end{aligned}$$

□

Remark 9. In the case that the sequence $(b_j)_{j\geq 1}$ is non-increasing and satisfies the assumptions in Proposition 8, the term $\max_{j\geq s+1}\{b_j\}$ in (25) and in (27) can be replaced with b_{s+1}. If $(b_j)_{j\geq 1}$ is majorized by a non-increasing $(\widehat{b}_j)_{j\geq 1} \in (0,1]^{\mathbb{N}}$, then $\max_{j\geq s+1}\{b_j\}$ in (25) and in (27) can be replaced by $\widehat{b}_{s+1}$.

For globally supported ψ_j, a sharper version of [13, Theorem 5.1] in the case of symmetric probability measure on the y_j has recently been obtained in [9].

6. QMC convergence rates for the exact solution. Based on the parametric regularity estimates obtained in Section 4, we now collect results on dimension independent convergence of first- and higher order QMC quadratures for functionals of the parametric solution. At this stage, we formulate these results under the assumption that the parametric problems can be solved exactly, for any realization of the parameter. Ahead, in Section 7, we shall address the impact of a Galerkin discretization, and also of dimension truncation, based on Proposition 8. Our first result pertains to first order, randomly shifted lattice rules.

THEOREM 10. *[Convergence rates of randomly shifted lattice rules]*
*Let the condition in (**A1**) be satisfied for $\bar{\kappa} \in (0,1)$, let the condition in (**A2**) be satisfied for $\kappa \in [\bar{\kappa}, 1)$, and let $\eta \in (\kappa, 1)$. Let $s \in \mathbb{N}$, $G(\cdot) \in V^*$ be given and let product weights $\boldsymbol{\gamma}$ be defined by*

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \left(\frac{2b_j}{1-\eta}\right)^2, \quad \mathfrak{u} \subset \mathbb{N},\ |\mathfrak{u}| < \infty\ . \tag{30}$$

For some $p \in (1,2]$ assume that $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$. Then for every $N \in \mathbb{N}$ a randomly shifted lattice rule can be constructed in $\mathcal{O}(sN\log N)$ operations using the fast CBC algorithm of [17, 16] such that the root-mean square error can be estimated independently of s and N, i.e.,

$$\sqrt{\mathbb{E}^{\Delta}(|I_s(G(u)) - Q_{s,N}(G(u))|^2)} \leq \mathcal{C}_p(\varphi(N))^{-1/p}, \tag{31}$$

where the finite constant $\mathcal{C}_p$ is independent of N and of s, and given explicitly as

$$\mathcal{C}_p = \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}}\sqrt{\frac{\bar{a}_{\max}}{\bar{a}^3_{\min}}}\|\mathcal{G}\|_{V^*}\|f\|_{V^*}\left(\sum_{\mathfrak{u}\subseteq\mathbb{N},|\mathfrak{u}|<\infty}\gamma_{\mathfrak{u}}^{p/2}\left(\frac{2\zeta(p)}{(2\pi^2)^{p/2}}\right)^{|\mathfrak{u}|}\right)^{1/p}. \tag{32}$$

Proof. The error estimate and the expression of the constant $\mathcal{C}_p$ in this theorem follow readily by combining Theorem 1 with $\lambda = p/2$ and the chosen weights $\boldsymbol{\gamma}$, and Corollary 5. The choice (30) of the weight sequence $\boldsymbol{\gamma}$ in the statement of the theorem also implies that

$$\begin{aligned}\sum_{\mathfrak{u}\subseteq\mathbb{N},|\mathfrak{u}|<\infty}\gamma_{\mathfrak{u}}^{p/2}\left(\frac{2\zeta(p)}{(2\pi^2)^{p/2}}\right)^{|\mathfrak{u}|} &= \sum_{\mathfrak{u}\subseteq\mathbb{N},|\mathfrak{u}|<\infty}\prod_{j\in\mathfrak{u}}\left(\left(\frac{2b_j}{1-\eta}\right)^p\frac{2\zeta(p)}{(2\pi^2)^{p/2}}\right)\\ &\leq \exp\left(\left(\frac{2}{1-\eta}\right)^p\frac{2\zeta(p)}{(2\pi^2)^{p/2}}\sum_{j\geq 1}b_j^p\right) < \infty,\end{aligned}$$

where we have applied [13, Lemma 6.3] in the last step. Thus, the constant $\mathcal{C}_p$ is finite and its value is independent of s or N. The linear work bound with respect to s for the CBC construction of the QMC generating vector was shown in [16], using that the weights (30) are product weights. ∎

COROLLARY 11. *Under the assumption of Theorem 10, for some $\varepsilon \in (0,1/2)$ set $q = 1-\varepsilon p$ and with the product weights $\boldsymbol{\gamma}$ defined by*

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} b_j^{2q}, \quad \mathfrak{u}\subset\mathbb{N}, |\mathfrak{u}|<\infty,$$

the convergence estimate (31) holds with convergence rate $1/p-\varepsilon$ independent of s and N, with constant

$$\mathcal{C} = \mathcal{C}_{(p/q)} \times \prod_{j\in\mathcal{I}} \left(\frac{2b_j^{1-q}}{1-\eta}\right) < \infty, \quad \mathcal{I} := \left\{ j\in\mathbb{N} : \frac{2b_j^{1-q}}{1-\eta} > 1 \right\}, \quad |\mathcal{I}|<\infty,$$

where $\mathcal{C}_{(p/q)}$ is given by (32) for chosen weights $\boldsymbol{\gamma}$.

Proof. Since $(b_j)_{j\geq 1}\in\ell^p(\mathbb{N})$, there exists $J\in\mathbb{N}$ such that for every $j\geq J$, $2/(1-\eta)b_j^{1-q}\leq 1$. This implies that $|\mathcal{I}|\leq J<\infty$. Hence,

$$\sup_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1/2} \prod_{j\in\mathfrak{u}} \frac{2b_j}{1-\eta} = \sup_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{2}{1-\eta} b_j^{1-q} \leq \prod_{j\in\mathcal{I}} \frac{2}{1-\eta} b_j^{1-q} =: C.$$

The number C is in particular independent of s. The claimed convergence estimate holds with the constant C multiplied by $\mathcal{C}_{p'}$ in (32) for $p' = p/q = p/(1-\varepsilon p)$, which is bounded independently of s. This yields the dimension independent convergence rate $q/p = 1/p-\varepsilon$. ∎

THEOREM 12. *[Convergence of higher order, interlaced polynomial lattice rules] Let the condition in (**A1**) be satisfied for $\bar{\kappa}\in(0,1)$, let the condition in (**A2**) be satisfied for $\kappa\in[\bar{\kappa},1)$, and let $\eta\in(\kappa,1)$. Let $s\in\mathbb{N}$, b a prime number, and $G(\cdot)\in V^*$ be given and let product weights $\boldsymbol{\gamma}$ be defined by*

$$\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \left(\sum_{\nu=1}^{\alpha} \left(\frac{2b_j}{1-\eta}\right)^{\nu} \sqrt{2^{\delta(\nu,\alpha)}\nu!} \right), \quad \mathfrak{u}\subset\mathbb{N},\ |\mathfrak{u}|<\infty\,. \tag{33}$$

For some $p\in(0,1]$ assume that $(b_j)_{j\geq 1}\in\ell^p(\mathbb{N})$. Then for every $N=b^m$, $m\in\mathbb{N}$, an interlaced polynomial lattice rule of order $\alpha = \lfloor 1/p\rfloor + 1$ can be constructed using a fast CBC algorithm of [5], with cost $\mathcal{O}(\alpha\, s\, N\log N)$ operations, such that the absolute error can be bounded independently of s and of N, i.e.,

$$|I_s(G(u)) - Q_{s,N}(G(u))| \leq \mathcal{C}_p \left(\frac{2}{N-1}\right)^{1/p}, \tag{34}$$

where the constant $\mathcal{C}_p$ is independent of N and of s, and given explicitly as

$$\mathcal{C}_p = \frac{\sqrt{2}}{\sqrt{(\eta-\kappa)(1-\bar{\kappa})^3}} \sqrt{\frac{\bar{a}_{\max}}{\bar{a}_{\min}^3}} \|\mathcal{G}(\cdot)\|_{V^*} \|f\|_{V^*} \left(\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^p \rho_{\alpha,b}(p)^{|\mathfrak{u}|} \right)^{1/p}, \tag{35}$$

with $\rho_{\alpha,b}(p)$ defined in (16).

Proof. We note that $\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,2}\subset\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},q,\infty}$ with continuous embedding for every $q\in[1,\infty]$. In particular, it follows from the definition in (15) that $\|F\|_{s,\alpha,\boldsymbol{\gamma},q,\infty}\leq \|F\|_{s,\alpha,\boldsymbol{\gamma},q,2}$. Then, the error estimate and the expression of the constant $\mathcal{C}_p$ in this theorem follow by combining Theorem 2 with $\lambda = p$, $q = 2$ and the chosen weights $\boldsymbol{\gamma}$, and Corollary 7. The choice (33) of weights $\boldsymbol{\gamma}$ also implies that

$$\sum_{\mathfrak{u}\subseteq\mathbb{N},|\mathfrak{u}|<\infty} \gamma_{\mathfrak{u}}^p \rho_{\alpha,b}(p)^{|\mathfrak{u}|} = \sum_{\mathfrak{u}\subseteq\mathbb{N},|\mathfrak{u}|<\infty} \prod_{j\in\mathfrak{u}} \left(\left(\sum_{\nu=1}^{\alpha} \left(\frac{2b_j}{1-\eta}\right)^{\nu} \sqrt{2^{\delta(\nu,\alpha)}}\nu! \right)^p \rho_{\alpha,b}(p) \right)$$
$$\leq \exp\left(\sum_{\nu=1}^{\alpha} \left(\sum_{j\geq 1} b_j^{\nu p} \right) \left(\frac{2}{1-\eta}\right)^p \left(\sqrt{2^{\delta(\nu,\alpha)}}\nu!\right)^p \rho_{\alpha,b}(p) \right),$$

where we have applied [13, Lemma 6.3] in the last step and used that $p \leq 1$. Since $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$, $\mathcal{C}_p$ is bounded independently of s and of N. □

COROLLARY 13. *Under the assumption of Theorem 12, for some $\varepsilon \in (0,1)$ set $q = 1 - \varepsilon p$ and the product weights $\boldsymbol{\gamma}$ defined by*

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} \sum_{\nu=1}^{\alpha} \left((b_j^q)^{\nu} \sqrt{2^{\delta(\nu,\alpha)}}\nu! \right), \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty ,$$

the convergence estimate in (34) holds with convergence rate $1/p - \varepsilon$ independent of s and N, with constant

$$\mathcal{C} = \mathcal{C}_{(p/q)} \times \prod_{j\in\mathcal{I}} \left(\frac{2b_j^{1-q}}{1-\eta}\right)^{\alpha} < \infty, \quad \mathcal{I} := \left\{ j \in \mathbb{N} : \frac{2b_j^{1-q}}{1-\eta} > 1 \right\}, \quad |\mathcal{I}| < \infty,$$

where $\mathcal{C}_{(p/q)}$ is given by (35) for chosen product weights $\boldsymbol{\gamma}$.

This corollary follows from Theorem 12 along the lines of the proof of Corollary 11.

The presently developed QMC convergence analysis also implies dimension independent convergence rates of either of the QMC rules studied in this article with product weights in the case of globally supported functions $(\psi_j)_{j\geq 1}$ considered in [13, 14].

COROLLARY 14. *Under the* sparsity *assumption that (6) holds for $p \in (0, 2/3]$ and under the* smallness *assumption that for some $\bar{\kappa} \in (0,1)$, $\sum_{j\geq 1} \|\psi_j\|_{L^\infty(D)}/(2\bar{a}_{\min}) \leq \bar{\kappa}$, define the sequence $(b_j)_{j\geq 1}$ by*

$$b_j := \left(1 + \frac{\bar{a}_{\min}(1-\bar{\kappa})}{\sum_{j\geq 1} \|\psi_j\|_{L^\infty(D)}^p} \|\psi_j\|_{L^\infty(D)}^{p-1} \right)^{-1}, \quad j \in \mathbb{N} .$$

Then, it holds with implied constants independent of the truncation dimension that:

1. *Higher order QMC integration by interlaced polynomial lattice rules is applicable with product weights for $p \in (0, 1/2]$ with convergence rate $1/p - 1$.*
2. *First order QMC integration by randomly shifted lattice rules, with product weights for $p \in (1/2, 2/3]$ yields a (mean square w.r. to shift averages) convergence rate $1/p - 1$.*

This corollary follows by Theorems 10 and 12, since (**A1**) and (**A2**) are satisfied with $\bar{\kappa}$ and $\kappa = (\bar{\kappa} + 1)/2$ and $(b_j)_{j\geq 1} \in \ell^{p/(1-p)}(\mathbb{N})$ in the setting of this corollary. In the case of randomly shifted lattice rules, a convergence rate of essentially equal to 1 for $p \approx 1/2$ for product weights and globally supported ψ_j was already noted in [13, p. 3368]. In this case, however, it was shown in [13] that the use of POD weights reduces the summability necessary for QMC convergence rate essentially 1 to $p \approx 2/3$. The cost of CBC construction of generating vectors with POD weights scales, however, quadratically with respect to s.

Remark 15. In the setting of Theorem 12, QMC by interlaced polynomial lattice rules is also applicable for $p \in (1,2]$ with convergence rate $\mathcal{O}(N^{-1/p})$. This can be achieved by applying [5, Theorem 3.5] with product weights $\boldsymbol{\gamma}$ defined in (33) with respect to $(b_j)_{j\geq 1}$ and the $\mathcal{W}_{s,\alpha,\boldsymbol{\gamma},2,2}$-norm of $F = G(u)$. The respective *worst case error bound* $e_{s,\alpha,\boldsymbol{\gamma},2}(\mathcal{S})$ (in the notation of [5]) can be estimated by $\sqrt{e_{s,\alpha,\widetilde{\boldsymbol{\gamma}},1}(\mathcal{S})}$ (see also [12, Lemma 3.3]), where $\widetilde{\boldsymbol{\gamma}}$ are product weights of type (33) with respect to the sequence $(b_j^2)_{j\geq 1}$. Further, $e_{s,\alpha,\widetilde{\boldsymbol{\gamma}},1}(\mathcal{S})$ can be bounded by an expression of the type $\mathcal{O}(N^{-2/p}(\sum_{\mathfrak{u}} \widetilde{\gamma}_{\mathfrak{u}}^{p/2} c^{|\mathfrak{u}|})^{2/p})$, cp. [5, Theorem 3.9], where we remark that Theorem 2 (i.e. [5, Theorem 3.10]) is a combination of [5, Theorems 3.5 and 3.9]. Since $(b_j^2)_{j\geq 1} \in \ell^{p/2}(\mathbb{N})$, the convergence rate $\mathcal{O}(N^{-1/p})$ can be shown as in the proof of Theorem 12.

Remark 16. Due to Theorem 10, QMC by randomly shifted lattice rules is applicable if $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (1,2]$ with dimension independent convergence rate $\mathcal{O}(\varphi(N)^{-1/p})$. For $p = 2$, this convergence rate agrees with the rate of Monte Carlo sampling, where for randomly shifted lattice rules the convergence rate is in terms of $\varphi(N)$. Monte Carlo sampling, however, is applicable with a convergence rate of $1/2$ also for every $p \in (2, \infty]$.

7. Combined QMC Finite Element discretization. Up to this point, we considered the convergence of QMC quadratures for the countably parametric integrands $F(\boldsymbol{y}) = G(u(\cdot,\boldsymbol{y}))$, $\boldsymbol{y} \in U$. In practice, the numerical evaluation of $Q_{N,s}(F)$ in (13) requires approximate integrand evaluations $G(u(\cdot,\boldsymbol{y}))$, in points $\boldsymbol{y}^{(i)}$.

We consider here Galerkin approximations of the parametric variational formulation (9). In the error analysis of Galerkin FE methods, we impose to simplify the presentation also the hypothesis

$$(\mathbf{A3}) \qquad D \subset \mathbb{R}^d \text{ is a bounded polyhedron with plane faces .}$$

For a one-parametric family $\{\mathcal{T}_h\}_{h>0}$ of nested, shape regular, simplicial triangulations of the polygonal resp. polyhedral domain D and with maximal diameter h, we denote by V_h the corresponding family of continuous, piecewise polynomial functions of (total) degree $r \geq 1$ on $\mathcal{T}_h$ in D which vanish on ∂D. Then, $V_h \subset V$ is a subspace of finite dimension $M_h = \dim(V_h)$, with $M_h = \mathcal{O}(h^{-d})$ as $h \to 0$.

To obtain convergence rates of the FE solutions, we require spatial regularity of the parametric coefficient function $a(\cdot,\boldsymbol{y})$ to hold, uniformly with respect to $\boldsymbol{y}$: we assume there exists a constant $C > 0$ and $t_0 \in \mathbb{R}_{>0}\backslash\mathbb{N}$ such that
$(\mathbf{A4})$

$$\text{for some } t_0 > 0, \quad a(\cdot,\boldsymbol{y}) \in W^{t_0,\infty}(D) \quad \text{and} \quad \|a(\cdot,\boldsymbol{y})\|_{W^{t_0,\infty}(D)} \leq C, \quad \boldsymbol{y} \in U\ ,$$

where, for every $t > 0$ not an integer, $W^{t,\infty}(D)$ is identified with the Hölder space $C^t(\overline{D})$. (Bi)orthogonal Spline multiresolution analyses allow for stable expansions of parametric, smooth functions $a(x,\boldsymbol{y})$ in terms of locally supported functions. Analogous to what is classic for Fourier expansions in D, where coefficient decay encodes the spatial regularity in Sobolev scales, wavelets are well-known to encode Besov regularity in the coefficient decay, while affording locally supported expansion functions.

We recall that for QMC integration, the infinite sum in (4) is truncated to a finite number of terms, denoted by s. The Galerkin discretization of the dimensionally truncated, parametric variational problem (9) reads: for every $\boldsymbol{y} \in [-\frac{1}{2},\frac{1}{2}]^s$, find

$u^h(\cdot, \boldsymbol{y}_{\{1:s\}}) \in V_h$ such that

$$\int_D a(x, \boldsymbol{y}_{\{1:s\}}) \nabla u^h(x, \boldsymbol{y}_{\{1:s\}}) \cdot \nabla v(x) \, \mathrm{d}x = f(v) \qquad \forall v \in V_h . \tag{36}$$

By the coercivity (7), which remains valid also for $\boldsymbol{y}_{\{1:s\}} \in U$ uniformly with respect to s, for every $\boldsymbol{y} \in [-\frac{1}{2}, \frac{1}{2}]^s$ the parametric Galerkin solution $u^h(\cdot, \boldsymbol{y}_{\{1:s\}})$ exists, and is quasioptimal, uniformly with respect to the parameter $\boldsymbol{y}$, and the truncation dimension s.

To bound the discretization error $u(\cdot, \boldsymbol{y}) - u^h(\cdot, \boldsymbol{y})$ incurred by the Galerkin approximation (36), we assume that (**A4**) holds for some $t_0 \in \mathbb{N}$. For convergence rate bounds, we also assume the data f and $G(\cdot)$ to have extra regularity: for real parameters $t > 0$ and $t' \geq 0$, there holds

$$f \in H^{-1+t}(D) , \quad G(\cdot) \in H^{-1+t'}(D) . \tag{A5}$$

Here, the space $H^{-1}(D) = (H_0^1(D))^* = V^*$ and, for $r > -1$, the spaces $H^r(D)$ denote the usual Sobolev spaces over D.

Under the regularity assumptions (**A4**) and (**A5**), and due to the physical domain D being a polyhedron with plane sides by (**A3**), the parametric solutions are known to have regularity in Sobolev scales uniformly with respect to $\boldsymbol{y}$ in D. The functional $G(u^h(\cdot, \boldsymbol{y}))$ of the parametric FE solution $u^h(\cdot, \boldsymbol{y})$ converges with rate $\mathcal{O}(h^{t+t'})$: there exists a constant $C > 0$ such that, for every $\boldsymbol{y} \in U$, there holds the asymptotic error bound

$$\left| G(u(\cdot, \boldsymbol{y})) - G(u^h(\cdot, \boldsymbol{y})) \right| \leq C h^\tau \|f\|_{H^{-1+t}(D)} \|G(\cdot)\|_{H^{-1+t'}(D)} , \tag{37}$$

with the convergence rate $\tau = \min\{t, \bar{t}\} + \min\{t', \bar{t}\}$ and where $\bar{t}$ depends on the maximal convergence rate of the Finite Element approximation in D (which, in turn, depends on the regularity shift of the operator $(-\mathrm{div}(a(\cdot, \boldsymbol{y})\nabla \cdot))^{-1}$, that also depends on the value t_0 from (**A4**), and on the order of the Finite Element discretization). This error bound is obtained by combining regularity of the parametric solution in Sobolev scales with approximation error bounds on regular triangulations $\mathcal{T}_h$ (possibly with local refinements towards the singular support of the parametric solutions which do not depend on the parameter instances $\boldsymbol{y}$, cp. [15]), and an Aubin–Nitsche duality argument. The preceding discussion assumes quasiuniform, regular simplicial triangulations $\mathcal{T}_h$ of D of meshwidth h; in general, corners and edges of ∂D induce singularities in the parametric solutions $u(\cdot, \boldsymbol{y})$ of (1), which in turn limit the maximal regularity $\bar{t}$ of the solution $u(\cdot, \boldsymbol{y})$ in the Sobolev scales $H^{1+t}(D)$. Full regularity shifts hold in *weighted Sobolev scales* which, upon combination with *graded triangulations* $\mathcal{T}_h$ of D allow for FE convergence rates (37) where $\bar{t}$ is only limited by the approximation order of the elements and by the regularity of the parametric coefficient $a(x, \boldsymbol{y})$. We refer to [15] for details. The self-adjointness of the differential operator in (1) allows to refer to [15] also in the analysis of the dual problem. Combining the bounds on the QMC integration error in Theorem 12 in the case of interlaced polynomial lattice rules and in Theorem 10 in the case of randomly shifted lattice rules, the dimension truncation and the Galerkin error bound (37), we obtain the following combined error bounds.

THEOREM 17. *Let the regularity assumption* (**A4**) *and* (**A5**) *be satisfied for some* $t_0, t > 0$ *and* $t' \geq 0$ *such that* $t, t' \leq \bar{t} < t_0$. *Let* (**A1**) *and* (**A2**) *be satisfied for* $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ *for some* $p \in (0, 2]$. *For the error incurred in the approximation* *of the integral (3) of the parametric integrand function $F(\boldsymbol{y}) = G(u(\cdot,\boldsymbol{y}))$ with an N-point QMC quadrature applied to the s-variate, dimensionally truncated integral $I_s(F_h)$, with approximate integrand function $F_h(\boldsymbol{y}_{\{1:s\}})$ holds:*

1. *For $p \in (0,1]$, with an interlaced polynomial lattice rule of order $\alpha = \lfloor 1/p \rfloor + 1$ the error is bounded by*

$$|\mathbb{E}(F) - Q_{N,s}(F_h)| \le C \left(N^{-1/p} + h^{t+t'} + \left(\max_{j \ge s+1} \{b_j\} \right)^2 \right) . \tag{38}$$

2. *For $p \in (1,2]$, with a randomly shifted lattice rule the error is bounded by*

$$\sqrt{\mathbb{E}^{\Delta}(|\mathbb{E}(F) - Q_{N,s}(F_h)|^2)} \le C \left(\varphi(N)^{-1/p} + h^{t+t'} + \left(\max_{j \ge s+1} \{b_j\} \right)^2 \right) . \tag{39}$$

The constant C in the bounds (38) and (39) is in particular independent of N, h, and s.

Note that $\varphi(N)^{-1} \le N^{-1} \cdot (e^{\widehat{\gamma}} \log\log N + 3/\log\log N)$, for every $N \ge 3$, where $\widehat{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant.

The first part of this result follows from the dimension truncation error bound (27) in Proposition 8 and the Galerkin error bound (37), together with the QMC error bound Theorem 12. The second part follows analogously with Theorem 10. Also note that (26) is satisfied for sufficiently large s.

8. Multiresolution representation of $a(x,\boldsymbol{y})$. We now consider a particular case of the affine-parametric expansion (4), in a polyhedral domain (i.e., Assumption (**A3**) holds). In the domain D, consider a multiresolution analysis (MRA) $\boldsymbol{\Psi} = \{\psi_\lambda : \lambda \in \triangledown\}$ which is stable in $L^2(D)$ and whose members ψ_λ are indexed by $\lambda \in \triangledown$, and are obtained from one or from a finite number of generating elements ψ by translation and scaling, i.e.,

$$\psi_\lambda(x) = \psi(2^{|\lambda|}x - k) , \quad k \in \triangledown_{|\lambda|} , \tag{40}$$

where the index set $\triangledown_{|\lambda|}$ is of cardinality $\mathcal{O}(2^{d|\lambda|})$, and where $\operatorname{diam}\operatorname{supp}(\psi_\lambda) = \mathcal{O}(2^{-|\lambda|})$. We assume that there exists a suitable enumeration of elements of the index set $\triangledown$, i.e., a bijective mapping $j : \triangledown \to \mathbb{N}$, which we denote by $j(\lambda)$, $\lambda \in \triangledown$. The amount of overlap of the supports at refinement level $|\lambda|$ is assumed to be bounded by an absolute multiple K times $\mathcal{O}(2^{-d|\lambda|})$ such that

$$|\{\lambda \in \triangledown : |\lambda| = \ell, \psi_\lambda(x) \ne 0\}| \le K, \quad \text{for all} \quad x \in D, \ell \ge 0 .$$

Rather than normalizing the ψ_λ in $L^2(D)$, we scale here the functions ψ_λ to enforce the decay $\|\psi_{|\lambda|,k}\|_{L^\infty(\mathbb{R})} \le \sigma 2^{-\widehat{\alpha}|\lambda|}$ for parameters $\sigma > 0, \widehat{\alpha} > 0$ at our disposal. From the estimates

$$\begin{aligned} \left\| \frac{\sum_{\lambda\in\triangledown} |\psi_\lambda|}{2\bar{a}} \right\|_{L^\infty(D)} &= \left\| \frac{\sum_{\ell\ge 0}\sum_{k\in\triangledown_\ell} |\psi_{\ell,k}|}{2\bar{a}} \right\|_{L^\infty(D)} \\ &\le \frac{\sigma K}{2\bar{a}_{\min}} \sum_{\ell\ge 0} 2^{-\widehat{\alpha}\ell} = \frac{\sigma K}{2\bar{a}_{\min}} \frac{2^{\widehat{\alpha}}}{2^{\widehat{\alpha}}-1} \le \bar{\kappa} , \end{aligned}$$

we obtain the sufficient condition for ellipticity: (**A1**) is satisfied if

$$\sigma \leq \frac{2(2^{\widehat{\alpha}}-1)\bar{a}_{\min}\bar{\kappa}}{2^{\widehat{\alpha}}K} \,. \tag{41}$$

To choose σ, we assume equality in (41) and define for some $0<\widehat{\beta}<\widehat{\alpha}$ and $\delta>1/2$ at our disposal

$$b_{j(\lambda)} = b_\lambda := \left(1+\frac{\bar{a}_{\min}(1-\bar{\kappa})(1-2^{\widehat{\beta}-\widehat{\alpha}})}{\sigma\delta K}2^{\widehat{\beta}|\lambda|}\right)^{-1} , \quad \lambda\in\nabla \,. \tag{42}$$

We observe that (**A2**) is satisfied with $\kappa=\frac{(2\delta-1)\bar{\kappa}+1}{2\delta}$. Also, it follows by the choice in (42) that $b_j\sim j^{-\widehat{\beta}/d}$, $j\in\mathbb{N}$. We assume the generating elements ψ in (40) to be sufficiently regular in order for (**A4**) to hold.

Specifically, for sufficiently smooth wavelets, the decay property that for some constant $C>0$ and $t_0>0$ not an integer

$$\|\psi_\lambda\|_{L^\infty(D)} \leq C2^{-t_0|\lambda|}, \quad \lambda\in\nabla,$$

implies, if also $\bar{a}\in C^{t_0}(\overline{D})$, that for every $\boldsymbol{y}\in U$, $a(\cdot,\boldsymbol{y})\in C^{t_0}(\overline{D})$ with a $C^{t_0}(\overline{D})$-norm that is uniformly bounded in $\boldsymbol{y}$, e.g. for the case of orthogonal wavelets cp. [20, Theorem 4.23], where we note that for non-integer $t_0>0$, the Hölder space $C^{t_0}(\overline{D})$ agrees with the Besov space $B^{t_0}_{\infty,\infty}(D)$ with equivalent norms. Note the continuous embedding $C^{t_0}(\overline{D})\subset W^{\lfloor t_0\rfloor,\infty}(D)$ to imply differentiability of integer order.

Remark 18. Since $(b_j)_{j\geq 1}$ in (42) satisfies $b_j\sim j^{-\widehat{\beta}/d}$, $j\in\mathbb{N}$, QMC by Theorem 17 becomes applicable if $\widehat{\beta}>d/2$ with rate $\mathcal{O}(N^{-\widehat{\beta}/d+\varepsilon})$ for every sufficiently small $\varepsilon>0$ with implied constants independent of the number of QMC points N and the truncation dimension.

Here the local support property of the MRA $(\psi_\lambda)_{\lambda\in\nabla}$ implies that the cost to assemble $a(x,\boldsymbol{y}_{\{1:s\}})$ is $\mathcal{O}(\log s)$ for every $x\in D$ under the assumption that the mother wavelets can be evaluated at unit cost. In the setting that the dimension s is coupled to the spatial discretization, i.e., $s\sim M_h=\dim(V_h)$, for a given choice of lattice point, one matrix-vector multiplication used, e.g., in preconditioned iterative linear system solvers, can be effected in $\mathcal{O}(M_h\log M_h)$ operations, where cost estimates of this type for similar settings are also obtained in [6] using FFT. The total cost of the quadrature is $\mathcal{O}(sN\log N+NM_h\log M_h)=\mathcal{O}(M_h^{1+p\tau/d}\log M_h)$, where we applied the cost estimate of the fast CBC construction of generating vectors with product weights and chose $N=\mathcal{O}(M_h^{p\tau/d})$ to equilibrate error contributions according to Theorem 17. The summability parameter p is assumed to satisfy $d/\widehat{\beta}<p\leq 2$. Taking $\varepsilon=\mathcal{O}(M_h^{-\tau/d})$ as error threshold implies by Theorem 17 assuming $\tau<2\widehat{\beta}$,

$$\text{error}=\mathcal{O}(\varepsilon) \quad \text{ensured with} \quad \text{work}=\mathcal{O}(\varepsilon^{-(d/\tau+p)}\log(\varepsilon^{-1})). \tag{43}$$

More generally, choosing $s\sim M_h^{\tau/(2\widehat{\beta})}$ implies with the same choice of N by Theorem 17

$$\text{error}=\mathcal{O}(\varepsilon) \quad \text{ensured with} \quad \text{work}=\mathcal{O}(\varepsilon^{-(d/\min\{\tau,2\widehat{\beta}\}+p)}\log(\varepsilon^{-1})). \tag{44}$$

The constants which are implied in the Landau symbols $\mathcal{O}(\cdot)$ in the error vs. work bounds (43) and (44) do not depend on the parameter dimension s. In their derivation, we assumed the availability of an iterative linear complexity PDE solver such as multigrid. This cost estimate can in some cases be reduced to essentially the cost of the solution of the corresponding deterministic elliptic PDE with multilevel QMC, cp. [10].

9. Numerical experiments. We illustrate the demonstrated convergence results with a model, affine-parametric diffusion problem (1) in the interval $D = (0, 1)$, in space dimension $d = 1$. To this end, we use a wavelet representation of the diffusion coefficient.

9.1. Description. We consider an affine-parametric diffusion coefficient as in (4), where we parametrize the piecewise constant fluctuations in a Haar wavelet system. Haar wavelets are piecewise constant functions, which are obtained as in (40) from ψ, given by

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Here, (40) reads, with ℓ denoting the level index $|\lambda|$,

$$\psi_{\ell,k}(x) = \sigma 2^{-\widehat{\alpha}\ell}\psi(2^\ell x - k), \quad \ell \in \mathbb{N}_0,\, k = 0, \dots, 2^\ell - 1 \,. \tag{45}$$

Every finite truncation of the series expansion (4) (with a suitable enumeration $j = j(\ell, k)$) in terms of $\psi_{\ell,k}(t)$) to comprise contributions of resolution $\ell = 0, ..., L$ then yields a simple function on a uniform partition of D of width $\mathcal{O}(2^{-L})$, whose values depend on the numerical values of the parameters y_j in a unique way. This, in turn, implies in (1) that *for* $f(x)$ *in* (1) being a polynomial of degree $r \geq 0$, for any instance of the parameter vector $\boldsymbol{y}$, the parametric solution $x \mapsto u(x, \boldsymbol{y})$ of (1) will be a piecewise polynomial function of degree $r + 2$ which belongs to $H_0^1(D)$.

The enumeration $(\ell, k) \mapsto j$ which we use in (45) is given by $j(\ell, k) = 2^\ell + k \in \mathbb{N}$, if we consider the MRA $(\psi_j)_{j\geq1}$ with $\psi_j = \psi_{j(\ell,k)} = \psi_{\ell,k}$ for $\ell \geq 0$, $k \in \{0, \dots, 2^\ell - 1\}$. Conversely, for given j, we obtain ℓ, k by $\ell = \lfloor \log_2(j) \rfloor$ and $k = j - 2^\ell$. For a parameter sequence $\boldsymbol{y} = (y_j)_{j\geq1} \in U = [-1/2, 1/2]^{\mathbb{N}}$, we now consider the parametric coefficient $a(x, \boldsymbol{y})$ with coefficient functions ψ_j from the Haar wavelet system $\psi_j(x) = \psi_{j(\ell,k)}(x)$, $j \in \mathbb{N}$:

$$a(x, \boldsymbol{y}) = \bar{a}(x) + \sum_{j\geq1} y_j \psi_j(x) \,.$$

We choose the nominal coefficient to be constant, $\bar{a} \equiv 1$. Fixing a maximal level L in the multiscale representation naturally yields the truncation dimension $s = 2^{L+1} - 1$, corresponding to the number of wavelet coefficients for the fluctuations.

The weights (33) can then be bounded by the product weights

$$\gamma_{\mathfrak{u}} \leq \prod_{j\in\mathfrak{u}} \sum_{\nu=1}^{\alpha} \left(\nu! 2^{\delta(\nu,\alpha)} \beta_j^\nu\right), \quad \beta_j = 2b_j/(1 - \eta) \,\,, \tag{46}$$

with $(b_j)_{j\geq1}$ as in (42). For (46) the fast CBC construction of interlaced polynomial lattice rules developed in [5] scales linearly with respect to the integration dimension s. Depending on the values for η, the magnitude of β_j may be large in the first few

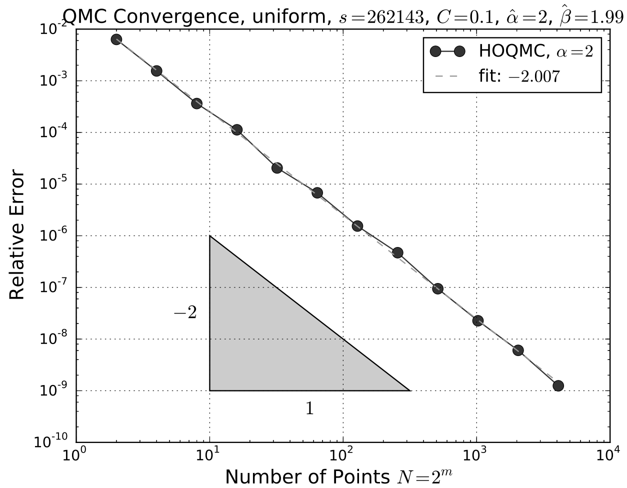

FIG. 1. *Convergence of QMC approximation using interlaced polynomial lattice rules with product weights in $s = 262143$ dimensions. By Theorem 12, the convergence rate is at least $\mathcal{O}(N^{-\widehat{\beta}+\varepsilon})$ for sufficiently small $\varepsilon > 0$. The empirical convergence rate measured by a linear least squares fit is consistent with the rate predicted by our theory. The rate is clearly visible already for numbers N of QMC points which are considerably smaller than the dimension $s = 262143$ of the parameter domain.*

coordinates, which was found in [11] to yield ill-suited generating vectors. We choose a numerical value for the Walsh constant in the CBC construction which was smaller than the (conservative) value suggested by theory: in the presented numerical results we use lattice rule which were computed with the value $C = 0.1$.

We solve (36) using the finite element method with piecewise quadratic basis functions on an equidistant mesh with meshwidth $h = 2^{-L-1}$, where $L \geq 0$ is the discretization level of the wavelet system. For the piecewise constant coefficient function represented in the Haar system, and for the choice $f(x) = 15$ for the right-hand side, we then obtain the exact solution, to within machine precision. As output quantity of interest, we consider point evaluation of the solution at the point $\bar{x} = 0.7$, i.e., $G(v) := v(\bar{x})$ for every $v \in V$.

9.2. QMC convergence. We consider in all computations the nominal value $\bar{a} = 1$, exponent $\widehat{\alpha}$ in (45) of coefficient decay $\widehat{\alpha} = 2$, decay $\widehat{\beta} = 1.99$ of the sequence $(\beta_j)_{j\geq 1}$, $\delta = 2$, $\bar{\kappa} = 0.1$, and perturbation size parameter $\sigma = 0.15$. For the interlaced polynomial lattice rules, we use digit interlacing factor $\alpha = 2$ and Walsh constant $C = 0.1$ (see [11] for details). The sequence $(\beta_j)_{j\geq 1}$ that was used in the CBC construction is of the form $\beta_{j(\ell,k)} = c_1\big(1 + c_2 2^{\widehat{\beta}\ell}\big)^{-1}$, $j \in \mathbb{N}$, where $c_1 = 1$ can be justified by Corollary 13 and $c_2 = \bar{a}(1-\bar{\kappa})(1-2^{\widehat{\beta}-\widehat{\alpha}})/(\sigma\delta K) \approx 0.021$ results from (42) with $K = 1$. Numerical results are presented in Figure 1.

9.3. Dimension truncation. In order to verify the bound (27) on the dimension truncation error when considering the integral over $\boldsymbol{y}$ of a goal functional G, we consider the same parametrization of $a(x,\boldsymbol{y})$ using the Haar wavelet basis as above. As model problem, we consider the PDE $-(a(x,\boldsymbol{y})u'(x,\boldsymbol{y}))' = 1$ in $D = (0,1)$ with mixed boundary conditions $u'(0,\boldsymbol{y}) = 0$ and $u(1,\boldsymbol{y}) = 0$. Truncating the wavelet expansion of a at wavelet level L, we have $s_L = 2^{L+1} - 1$ parameters and

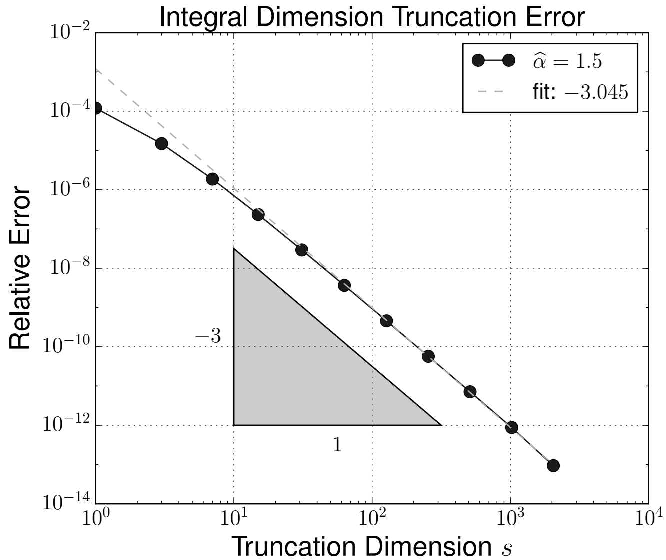

FIG. 2. *Convergence of the dimension truncation error of the integral $\int_U G(q_L(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}$ where the dimension on level L is $s_L = 2^{L+1} - 1$. The empirical convergence rate measured by a linear least squares fit on the last five data points is consistent with the rate $\mathcal{O}(s^{-2\widehat{\alpha}})$ predicted in* (27).

$M_h = 2^{L+1}$ elements. We denote by a_j^L the (constant) value that the truncated expansion $a(x, \boldsymbol{y})$ assumes on element $j = 1, \ldots, M_h$. Combined with the goal functional $G(u(\cdot, \boldsymbol{y})) = u(\bar{x}, \boldsymbol{y}) - u(0, \boldsymbol{y})$ with $\bar{x} = 0.5$ we obtain for $J = M_h/2 = 2^L$ the exact formula $G(u(\boldsymbol{y})) = -\sum_{j=1}^{J} \frac{(j-1/2)h^2}{a_j^L}$. Figure 2 shows measurements of the dimension truncation error $\left|\int_U G(u(\boldsymbol{y}) - u_L(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{y}\right|$ for $L = 0, \ldots, 10$, $\sigma = 0.1$. The results provide evidence that the bound from (27), which implies the rate $\mathcal{O}(s^{-2\widehat{\alpha}})$, is sharp. As reference value $G(u(\boldsymbol{y}))$, the computation on level $L = 11$ was used, where in all cases we applied interlaced polynomial lattice rules with generating vectors constructed with the CBC algorithm based on product weights (46) as above, and chose $N = 2^{20}$ QMC points. We note that this experiment also indicates that, for locally supported ψ_j in the affine-parametric representation (4), there is no extra order of convergence in the dimension truncation error, as was recently shown for globally supported, Karhúnen-Loève based global expansion systems in [9].

10. Conclusions and generalizations. In the present paper, we considered so-called *single level QMC Galerkin discretizations*: all QMC parameter samples of the parametric differential equation (1) are solved on one, common, discretization level. As is well known, *multilevel QMC Finite Element* discretizations can afford substantial gains in efficiency; for coefficient functions $(\psi_j)_{j\geq 1}$ with global supports, multilevel versions of the current QMC discretizations (with first and higher order QMC formulas) have been first proposed and analyzed in [14] (first order, multilevel QMC) and in [7] (higher order, multilevel QMC Galerkin). The results in [14, 7] were based on the corresponding single level results in [13, 5], both of which are extended in the present paper. *Based on the present single level results for QMC integration with product weights for ψ_j with local support, multilevel extensions can be obtained in analogy to [14, 7]:* on each discretization level in the physical domain, QMC quadratures with level-dependent sample numbers as derived in [14, 7] can be used. Here, however, the QMC quadratures admit product weights. Details and further numerical experiments are reported by us in [10]. We also considered here only the 'local', parametric diffusion equation. The present analysis can be generalized to more general, affine parametric linear operator equations considered in [18].

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