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Convergence in Hölder norms with applications to Monte Carlo methods in infinite dimensions

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Abstract

We show that if a sequence of piecewise affine linear processes converges in the strong sense with a positive rate to a stochastic process which is strongly Hölder continuous in time, then this sequence converges in the strong sense even with respect to much stronger Hölder norms and the convergence rate is essentially reduced by the Hölder exponent. Our first application hereof establishes pathwise convergence rates of spectral Galerkin approximations of stochastic partial differential equations. Our second application derives strong convergence rates of multilevel Monte Carlo approximations of expectations of Banach space valued stochastic processes.

Contents

1 Introduction

In this article we study convergence rates for general stochastic processes in Hölder norms. In particular, in the main results of this work (see Corollary 2.8 and Corollary 2.9 in Section 2.2 below) we reveal estimates for uniform Hölder errors of general stochastic processes. In this introductory section we now sketch these results and thereafter outline several applications of the general estimates, which can be found in subsequent sections of this article (see Corollary 2.11 in Section 2.2, Corollary 3.5 in Section 3.3, and Corollary 4.15 in Section 4.3 below). To illustrate the key results of this work, we consider the following framework throughout this section. Let $T \in (0,\infty)$ be a real number, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \left\|\cdot\right\|_E)$ be an $\mathbb{R}$-Banach space, and for every function $f\colon [0,T] \to E$ and every natural number $N \in \mathbb{N}$ let $[f]_N\colon [0,T] \to E$ be the function which satisfies for all $n \in \{0,1,\ldots,N-1\}$, $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right]$ that

$$[f]_N(t) = \left(n + 1 - \tfrac{tN}{T}\right) \cdot f\left(\tfrac{nT}{N}\right) + \left(\tfrac{tN}{T} - n\right) \cdot f\left(\tfrac{(n+1)T}{N}\right) \tag{1.1}$$

(the piecewise affine linear interpolation of $f|_{\{0,T/N,2T/N,\ldots,(N-1)T/N,T\}}$, cf. (1.19) below).

Theorem 1.1. *Assume the above setting. Then for all $p \in (1,\infty)$, $\varepsilon \in (1/p, 1]$, $\alpha \in [0, \varepsilon - 1/p)$ there exists $C \in [0,\infty)$ such that for all $\beta \in [\varepsilon, 1]$, $N \in \mathbb{N}$ and all $(\mathscr{F}, \left\|\cdot\right\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths it holds that*

$$\begin{aligned} &\|X - [Y]_N\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \\ &\leq C N^{\varepsilon} \left(\sup_{n \in \{0,1,\ldots,N\}} \left\|X_{\frac{nT}{N}} - Y_{\frac{nT}{N}}\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} + N^{-\beta} \|X\|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \right). \end{aligned} \tag{1.2}$$

The Hölder and $\mathscr{L}^p$-norms in (1.2) are to be understood in the usual sense (see Section 1.1 below for details). Theorem 1.1 is a direct consequence of the more general result in Corollary 2.9 in Section 2.2 below, which establishes an estimate similar to (1.2) also for the case of non-equidistant time grids. Moreover, Corollary 2.8 in Section 2.2 provides an estimate similar to (1.2) but instead of $\|X - [Y]_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}$ for $\|X - Y\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}$ with an appropriate Hölder norm of Y occurring on the right hand side. Theorem 1.1 has a number of applications in the numerical approximation of stochastic processes, as the next corollary, Corollary 1.2, clarifies. Corollary 1.2 follows immediately from Theorem 1.1.

Corollary 1.2. *Assume the above setting, let $\beta \in (0,1]$, let $X\colon [0,T] \times \Omega \to E$ and $Y^N\colon [0,T] \times \Omega \to E$, $N \in \mathbb{N}$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths which satisfy for all $p \in (1,\infty)$ that $\forall\, N \in \mathbb{N}\colon Y^N = [Y^N]_N$ and*

$$\|X\|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \left[N^\beta \sup_{n\in\{0,1,\ldots,N\}} \big\| X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}} \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right] < \infty. \tag{1.3}$$

Then for all $p, \varepsilon \in (0,\infty)$ it holds that

$$\sup_{N\in\mathbb{N}} \left[N^{\beta-\varepsilon} \left(\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t - Y^N_t\|_E^p \right] \right)^{1/p} \right] < \infty. \tag{1.4}$$

It is assumed in (1.3) that a sequence of affine linearly interpolated $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $(Y^N)_{N\in\mathbb{N}}$ converges for every $p \in (1,\infty)$ in $\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$ to a $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic process X with a positive rate uniformly on all grid points and that this process X admits corresponding temporal Hölder regularity. Corollary 1.2 then shows that these assumptions are sufficient to obtain convergence for every $p \in (1,\infty)$ in the uniform $\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm with essentially the same rate. Corollary 2.11 in Section 2.2 below implies this result as a special case and includes the case of non-equidistant time grids. Moreover, Corollary 2.11 proves an analogous conclusion for convergence in uniform Hölder norms, where the obtained convergence rate is reduced by the considered Hölder exponent. Corollary 2.12 below demonstrates how this principle can be applied to Euler-Maryuama approximations for stochastic differential equations (SDEs) with globally Lipschitz coefficients. Arguments related to Corollary 2.11 can be found in Lemma A1 in Bally, Millet & Sanz-Solé [3] and in the second display on page 325 in [5].

Corollary 1.2 is particularly useful for the study of stochastic partial differential equations (SPDEs). In general, a solution of an SPDE fails to be a semimartingale. As a consequence, Doob's maximal inequality cannot be applied to obtain estimates with respect to the $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. However, convergence rates with respect to the $C([0,T], \|\cdot\|_{\mathscr{L}^2(\mathbb{P};\|\cdot\|_E)})$-norm are often feasible and Corollary 1.2 can then be applied to obtain convergence rates with respect to the $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. Estimates with respect to the $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm are useful for using standard localisation arguments in order to extend results for SPDEs with globally Lipschitz continuous nonlinearities to results for SPDEs with nonlinearities that are only Lipschitz continuous on bounded sets. We demonstrate this in Corollary 3.5 in Section 3.3 below in the case of pathwise convergence rates of Galerkin approximations. To be more specific, Corollary 3.5 proves essentially sharp pathwise convergence rates for spatial Galerkin and noise approximations for a large class of SPDEs with non-globally Lipschitz continuous nonlinearities. For example, Corollary 3.5 applies to stochastic Burgers', stochastic Ginzburg-Landau, stochastic Kuramoto-Sivashinsky, and Cahn-Hilliard-Cook equations.

Another prominent application of Corollary 1.2 are multilevel Monte Carlo methods in Banach spaces. For a random variable $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ convergence in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_E)$ of Monte Carlo approximations of the expectation $\mathbb{E}[X] \in E$ has only been established if E has so-called (Rademacher) type p for some $p \in (1,2]$ and in this case the convergence rate is given by $1 - 1/p$ (see, e.g., Heinrich [11] or Corollary 4.12 in Section 4.2 below). However, the space $C([0,T], E)$ fails to have type p for any $p \in (1,2]$. If X has more sample path regularity, this problem can nevertheless be bypassed. More precisely, if for some $\alpha \in (0,1]$, $p \in (1/\alpha, \infty)$ it holds that $X \in \mathscr{L}^2(\mathbb{P}; \|\cdot\|_{W^{\alpha,p}([0,T],E)})$, then Monte Carlo approximations of $\mathbb{E}[X] \in W^{\alpha,p}([0,T],E)$ have been shown to converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{W^{\alpha,p}([0,T],E)})$ with rate $1 - 1/\min\{2,p\}$ and, by the Sobolev embedding theorem, also converge in $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$ with the same rate. Informally speaking, in order to gain control over the variances appearing in multilevel Monte Carlo approximations it is therefore sufficient for the approximations to converge with respect to the $\mathscr{L}^2(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})$-norm for some $\alpha \in (0, 1]$. For more details, we refer the reader to Section 4 and, in particular, to Corollary 4.15, which formalizes this approach for the case of multilevel Monte Carlo approximations of expectations of Banach space valued stochastic processes.

1.1 Notation

The following notation is used throughout this article. For two sets A and B we denote by $\mathbb{M}(A, B)$ the set of all mappings from A to B. For measurable spaces $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ we denote by $\mathscr{M}(\mathscr{F}_1, \mathscr{F}_2)$ the set of all $\mathscr{F}_1/\mathscr{F}_2$-measurable mappings from Ω_1 to Ω_2. For topological spaces $(E, \mathscr{E})$ and $(F, \mathscr{F})$ we denote by $\mathscr{B}(E)$ the Borel σ-algebra on $(E, \mathscr{E})$ and we denote by $C(E, F)$ the set of all continuous functions from E to F. We denote by $|\cdot| : \mathbb{R} \to [0, \infty)$ the absolute value function on $\mathbb{R}$. We denote by $\Gamma\colon (0, \infty) \to (0, \infty)$ the Gamma function, that is, we denote by $\Gamma\colon (0, \infty) \to (0, \infty)$ the function which satisfies for all $x \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{(x-1)}\, e^{-t}\, \mathrm{d}t$. We denote by $\mathscr{E}_r\colon [0, \infty) \to [0, \infty)$, $r \in (0, \infty)$, the mappings which satisfy for all $r \in (0, \infty)$, $x \in [0, \infty)$ that

$$\mathscr{E}_r[x] = \left(\sum_{n=0}^{\infty} \frac{x^{2n}\Gamma(r)^n}{\Gamma(nr + 1)} \right)^{1/2} \tag{1.5}$$

(cf. Chapter 7 in Henry [13] and, e.g., Definition 1.3.1 in [20]). For a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a $\mathbb{K}$-vector space V, and a mapping $\|\cdot\| : V \to [0, \infty]$ which satisfies for all $v, w \in \{u \in V \colon \|u\| < \infty\}$, $\lambda \in \mathbb{K}\backslash\{0\}$ that $(\|v\| = 0 \Leftrightarrow v = 0)$, $\|\lambda v\| = \sqrt{[\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2}\, \|v\|$, and $\|v + w\| \leq \|v\| + \|w\|$ we call $\|\cdot\|$ an extended norm on V and we call $(V, \|\cdot\|)$ an extendedly normed vector space. For a metric space (M, d), an extendedly normed vector space $(E, \|\cdot\|)$, a real number $r \in [0, 1]$, and a set $A \subseteq (0, \infty)$ we denote by $|\cdot|_{C^{r,A}(M,\|\cdot\|)}, |\cdot|_{C^r(M,\|\cdot\|)}, \|\cdot\|_{C(M,\|\cdot\|)}, \|\cdot\|_{C^r(M,\|\cdot\|)} : \mathbb{M}(M, E) \to [0, \infty]$ the mappings which satisfy for all $f \in \mathbb{M}(M, E)$ that

$$|f|_{C^{r,A}(M,\|\cdot\|)} = \sup\Big(\Big\{ \tfrac{\|f(e_1)-f(e_2)\|}{|d(e_1,e_2)|^r} \colon e_1, e_2 \in M, d(e_1, e_2) \in A \Big\} \cup \{0\}\Big) \in [0, \infty], \tag{1.6}$$

$$|f|_{C^r(M,\|\cdot\|)} = |f|_{C^{r,(0,\infty)}(M,\|\cdot\|)} \in [0, \infty], \tag{1.7}$$

$$\|f\|_{C(M,\|\cdot\|)} = \sup(\{\|f(e)\| : e \in M\} \cup \{0\}) \in [0, \infty], \tag{1.8}$$

$$\|f\|_{C^r(M,\|\cdot\|)} = \|f\|_{C(M,\|\cdot\|)} + |f|_{C^r(M,\|\cdot\|)} \in [0, \infty] \tag{1.9}$$

and we denote by $C^r(M, \|\cdot\|)$ the set given by

$$C^r(M, \|\cdot\|) = \{f \in C(M, E) \colon \|f\|_{C^r(M,\|\cdot\|)} < \infty\}. \tag{1.10}$$

Note that for every $r \in [0, 1]$, every metric space (M, d), and every extendedly normed vector space $(E, \|\cdot\|)$ it holds that $\big(\mathbb{M}(M, E), \|\cdot\|_{C^r(M,\|\cdot\|)}\big)$, $\big(C(M, E), \|\cdot\|_{C^r(M,\|\cdot\|)}\big|_{C(M,E)}\big)$, and $\big(C^r(M, \|\cdot\|), \|\cdot\|_{C^r(M,\|\cdot\|)}\big|_{C^r(M,\|\cdot\|)}\big)$ are extendedly normed vector spaces. For Hilbert spaces $(H_i, \langle\cdot,\cdot\rangle_{H_i}, \|\cdot\|_{H_i})$, $i \in \{1, 2\}$, we denote by $(HS(H_1, H_2), \langle\cdot,\cdot\rangle_{HS(H_1,H_2)}, \|\cdot\|_{HS(H_1,H_2)})$ the Hilbert space of Hilbert-Schmidt operators from H_1 to H_2. For a measure space

$(\Omega, \mathscr{F}, \mu)$, a measurable space $(S, \mathscr{S})$, a set $R \subseteq S$, and a function $f\colon \Omega \to R$ we denote by $[f]_{\mu,\mathscr{S}}$ the set given by

$$[f]_{\mu,\mathscr{S}} = \big\{ g \in \mathscr{M}(\mathscr{F}, \mathscr{S})\colon (\exists A \in \mathscr{F}\colon \mu(A) = 0 \text{ and } \{\omega \in \Omega\colon f(\omega) \neq g(\omega)\} \subseteq A) \big\}. \tag{1.11}$$

For a measure space $(\Omega, \mathscr{F}, \mu)$, an extendedly normed vector space $(V, \|\cdot\|)$, and real numbers $p \in [0, \infty)$, $q \in (0, \infty)$ we denote by $\mathscr{L}^0(\mu; \|\cdot\|)$ the set given by

$$\mathscr{L}^0(\mu; \|\cdot\|) = \big\{ f \in \mathbb{M}(\Omega, V)\colon f \text{ is } (\mathscr{F}, \|\cdot\|)\text{-strongly measurable} \big\}, \tag{1.12}$$

we denote by $\|\cdot\|_{\mathscr{L}^q(\mu;\|\cdot\|)}\colon \mathscr{L}^0(\mu; \|\cdot\|) \to [0, \infty]$ the mapping which satisfies for all $f \in \mathscr{L}^0(\mu; \|\cdot\|)$ that

$$\|f\|_{\mathscr{L}^q(\mu;\|\cdot\|)} = \left[\int_\Omega \|f(\omega)\|^q \, \mu(\mathrm{d}\omega) \right]^{1/q} \in [0, \infty], \tag{1.13}$$

we denote by $\mathscr{L}^q(\mu; \|\cdot\|)$ the set given by

$$\mathscr{L}^q(\mu; \|\cdot\|) = \big\{ f \in \mathscr{L}^0(\mu; \|\cdot\|)\colon \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|)} < \infty \big\}, \tag{1.14}$$

we denote by $L^p(\mu; \|\cdot\|)$ the set given by

$$L^p(\mu; \|\cdot\|) = \big\{ \{g \in \mathscr{L}^0(\mu; \|\cdot\|)\colon \mu(f \neq g) = 0\} \subseteq \mathscr{L}^0(\mu; \|\cdot\|)\colon f \in \mathscr{L}^p(\mu; \|\cdot\|) \big\}, \tag{1.15}$$

and we denote by $\|\cdot\|_{L^q(\mu;\|\cdot\|)}\colon L^0(\mu; \|\cdot\|) \to [0, \infty]$ the function which satisfies for all $f \in \mathscr{L}^0(\mu; \|\cdot\|)$ that

$$\big\| \{g \in \mathscr{L}^0(\mu; \|\cdot\|)\colon \mu(f \neq g) = 0\} \big\|_{L^q(\mu;\|\cdot\|)} = \|f\|_{\mathscr{L}^q(\mu;\|\cdot\|)} \in [0, \infty]. \tag{1.16}$$

For a real number $T \in (0, \infty)$, a measurable space $(S, \mathscr{S})$, a normed vector space $(V, \|\cdot\|_V)$, and a mapping $X\colon [0, T] \times S \to V$ which satisfies for all $t \in [0, T]$ that $X_t\colon S \to V$ is an $(\mathscr{S}, \|\cdot\|_V)$-strongly measurable mapping we call X an $(\mathscr{S}, \|\cdot\|_V)$-strongly measurable stochastic process. For a real number $T \in (0, \infty)$ we denote by $\mathscr{P}_T$ the set given by

$$\mathscr{P}_T = \big\{ \theta \subseteq [0, T]\colon \{0, T\} \subseteq \theta \text{ and } \#(\theta) < \infty \big\}. \tag{1.17}$$

We denote by $d_{\max}, d_{\min}\colon \cup_{T\in(0,\infty)} \mathscr{P}_T \to \mathbb{R}$ the functions which satisfy for all $\theta = \{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\} \in \cup_{T\in(0,\infty)} \mathscr{P}_T$ with $\theta_0 < \theta_1 < \ldots < \theta_{\#(\theta)-1}$ that

$$d_{\max}(\theta) = \max_{j\in\{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}| \qquad \text{and} \qquad d_{\min}(\theta) = \min_{j\in\{1,2,\ldots,\#(\theta)-1\}} |\theta_j - \theta_{j-1}|. \tag{1.18}$$

For a normed vector space $(E, \|\cdot\|_E)$, an element $\theta = \{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\} \in \cup_{T\in(0,\infty)} \mathscr{P}_T$ with $\theta_0 < \theta_1 < \ldots < \theta_{\#(\theta)-1}$, and a function $f\colon [0, \theta_{\#(\theta)-1}] \to E$ we denote by $[f]_\theta\colon [0, \theta_{\#(\theta)-1}] \to E$ the piecewise affine linear interpolation of $f|_{\{\theta_0, \theta_1, \ldots, \theta_{\#(\theta)-1}\}}$, that is, we denote by $[f]_\theta\colon [0, \theta_{\#(\theta)-1}] \to E$ the function which satisfies for all $j \in \{1, 2, \ldots, \theta_{\#(\theta)-1}\}$, $s \in [\theta_{j-1}, \theta_j]$ that

$$[f]_\theta(s) = \frac{(\theta_j - s) f(\theta_{j-1})}{(\theta_j - \theta_{j-1})} + \frac{(s - \theta_{j-1}) f(\theta_j)}{(\theta_j - \theta_{j-1})}. \tag{1.19}$$

2 Convergence of stochastic processes in Hölder norms for vector-valued functions

2.1 Error bounds for the Hölder norm

Lemma 2.1 (An interpolation type inequality)**.** *Let $(E, \|\cdot\|_E)$ be a normed vector space, let (M, d) be a metric space, let $f\colon M \to E$ be a function, and let $c \in (0,\infty)$, $\alpha, \beta, \gamma \in [0,1]$ satisfy $\alpha \leq \beta \leq \gamma$. Then*

$$|f|_{C^\beta(M,\|\cdot\|_E)} \leq \max\Big\{ c^{\alpha-\beta}\, |f|_{C^{\alpha,(c,\infty)}(M,\|\cdot\|_E)}\, , c^{\gamma-\beta}\, |f|_{C^{\gamma,(0,c]}(M,\|\cdot\|_E)} \Big\} \tag{2.1}$$

and

$$|f|_{C^\beta(M,\|\cdot\|_E)} \leq \max\Big\{ c^{\alpha-\beta}\, |f|_{C^{\alpha,[c,\infty)}(M,\|\cdot\|_E)}\, , c^{\gamma-\beta}\, |f|_{C^{\gamma,(0,c)}(M,\|\cdot\|_E)} \Big\}. \tag{2.2}$$

Proof of Lemma 2.1. First of all, note that for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (c, \infty)$ it holds that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1,e_2)|^\beta} \leq |d(e_1,e_2)|^{\alpha-\beta}\, |f|_{C^{\alpha,(c,\infty)}(M,\|\cdot\|_E)} \leq c^{\alpha-\beta}\, |f|_{C^{\alpha,(c,\infty)}(M,\|\cdot\|_E)}\,. \tag{2.3}$$

In addition, observe that for all $e_1, e_2 \in M$ with $d(e_1, e_2) \in (0, c]$ it holds that

$$\frac{\|f(e_1) - f(e_2)\|_E}{|d(e_1,e_2)|^\beta} \leq |d(e_1,e_2)|^{\gamma-\beta}\, |f|_{C^{\gamma,(0,c]}(M,\|\cdot\|_E)} \leq c^{\gamma-\beta}\, |f|_{C^{\gamma,(0,c]}(M,\|\cdot\|_E)}\,. \tag{2.4}$$

Combining (2.3) and (2.4) shows (2.1). The proof of (2.2) is analogous. This finishes the proof of Lemma 2.1. $\square$

Lemma 2.2 (Approximation error for affine linear interpolation)**.** *Let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, $\alpha \in [0,1]$, let $(E, \|\cdot\|_E)$ be a normed vector space, and let $f\colon [0,T] \to E$ be a function. Then*

$$\|f - [f]_\theta\|_{C([0,T],\|\cdot\|_E)} \leq \left|\tfrac{d_{\max}(\theta)}{2}\right|^\alpha |f|_{C^\alpha([0,T],\|\cdot\|_E)}\,. \tag{2.5}$$

Proof of Lemma 2.2. Throughout this proof let $N \in \mathbb{N}$ be a natural number, let $s \in [0,T]$, $\theta_0, \theta_1, \ldots, \theta_N \in [0,T]$ be real numbers, and let $j \in \{1, 2, \ldots, N\}$ be a natural number such that $0 = \theta_0 < \theta_1 < \ldots < \theta_N = T$, $\theta = \{\theta_0, \theta_1, \ldots, \theta_N\}$, and $s \in [\theta_{j-1}, \theta_j]$. Observe that for all $r \in [0,1]$ it holds that

$$\begin{aligned}
\|f(s) - [f]_\theta(s)\|_E &\leq \tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \|f(s) - f(\theta_{j-1})\|_E + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \|f(s) - f(\theta_j)\|_E \\
&\leq \tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \big(s - \theta_{j-1}\big)^r |f|_{C^r([0,T],\|\cdot\|_E)} + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \big(\theta_j - s\big)^r |f|_{C^r([0,T],\|\cdot\|_E)} \\
&= \Big(\tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} \Big(\tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})}\Big)^r + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \Big(\tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})}\Big)^r \Big) \big(\theta_j - \theta_{j-1}\big)^r |f|_{C^r([0,T],\|\cdot\|_E)}.
\end{aligned} \tag{2.6}$$

In particular, this implies that

$$\|f(s) - [f]_\theta(s)\|_E \leq \Big(\tfrac{(\theta_j - s)}{(\theta_j - \theta_{j-1})} + \tfrac{(s-\theta_{j-1})}{(\theta_j-\theta_{j-1})} \Big) |f|_{C^0([0,T],\|\cdot\|_E)} = |f|_{C^0([0,T],\|\cdot\|_E)}. \tag{2.7}$$

This proves (2.5) in the case $\alpha = 0$. It thus remains to prove (2.5) in the case $\alpha \in (0,1]$. For this let $g_r\colon [0,1] \to \mathbb{R}$, $r \in (0,1]$, be the functions which satisfy for all $r \in (0,1]$, $u \in [0,1]$ that $g_r(u) = (1-u)\, u^r + u\, (1-u)^r$. Next observe that for all $r \in (0,1]$, $u \in (0,1)$ it holds that $g_r(0) = 0 = g_r(1)$ and

$$\begin{aligned}
g_r'(u) &= r u^{r-1} - (r+1) u^r + (1-u)^r - r u (1-u)^{r-1} \\
&= -\big[(r+1)u - r\big]\, u^{r-1} + \big[1 - (r+1)u\big]\, (1-u)^{r-1},
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
g_r''(u) &= -(r+1)\,u^{r-1} - (r-1)\,\big[(r+1)u - r\big]u^{r-2} \\
&\quad - (r+1)\,(1-u)^{r-1} - (r-1)\,\big[1-(r+1)u\big]\,(1-u)^{r-2} \\
&= r\,\big[r-1-(r+1)u\big]\,u^{r-2} + r\,\big[(r+1)u-2\big]\,(1-u)^{r-2} \\
&= r\big\{(r+1)\,\big[u\,(1-u)^{r-2} + (1-u)\,u^{r-2}\big] - 2\,\big[(1-u)^{r-2}+u^{r-2}\big]\big\} \\
&\leq r\big\{2\,\big[u\,(1-u)^{r-2} + (1-u)\,u^{r-2}\big] - 2\,\big[(1-u)^{r-2}+u^{r-2}\big]\big\} < 0.
\end{aligned}
\tag{2.9}
$$

This implies for all $r \in (0,1]$ that $g_r'(\frac{1}{2}) = 0$ and $\sup_{u\in[0,1]} g_r(u) = g_r(\frac{1}{2}) = 2^{-r}$. Combining this with (2.6) ensures for all $r \in (0,1]$ that

$$
\|f(s) - [f]_\theta(s)\|_E \leq g_r\big(\tfrac{s-\theta_{j-1}}{\theta_j-\theta_{j-1}}\big)\,\big(\theta_j-\theta_{j-1}\big)^r\,|f|_{C^r([0,T],\|\cdot\|_E)} \leq \big(\tfrac{\theta_j-\theta_{j-1}}{2}\big)^r\,|f|_{C^r([0,T],\|\cdot\|_E)}\,. \tag{2.10}
$$

This completes the proof of Lemma 2.2. $\square$

The next result, Corollary 2.3, provides estimates for the Hölder norm differences of two functions by using the difference of the two functions on suitable grid points. Corollary 2.3 is a consequence of Lemma 2.1 and of Lemma 2.2.

Corollary 2.3. *Let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, $\beta \in [0,1]$, $\alpha \in [0,\beta]$, let $(E, \|\cdot\|_E)$ be a normed vector space, and let $f, g\colon [0,T] \to E$ be functions. Then*

$$
\begin{aligned}
&|f-g|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\leq \tfrac{2}{|d_{\max}(\theta)|^\alpha}\left[\sup_{t\in\theta}\big\|f(t)-g(t)\big\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta}\,\big(|f|_{C^\beta([0,T],\|\cdot\|_E)} + |g|_{C^\beta([0,T],\|\cdot\|_E)}\big)\right]
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
&\|f-g\|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\leq \left[\tfrac{2}{|d_{\max}(\theta)|^\alpha}+1\right]\left[\sup_{t\in\theta}\big\|f(t)-g(t)\big\|_E + \tfrac{|d_{\max}(\theta)|^\beta}{2^\beta}\,\big(|f|_{C^\beta([0,T],\|\cdot\|_E)} + |g|_{C^\beta([0,T],\|\cdot\|_E)}\big)\right].
\end{aligned}
\tag{2.12}
$$

Proof of Corollary 2.3. Lemma 2.1 and the triangle inequality ensure that

$$
\begin{aligned}
&|f-g|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\leq \max\Big\{|d_{\max}(\theta)|^{-\alpha}\,|f-g|_{C^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha}\,|f-g|_{C^\beta([0,T],\|\cdot\|_E)}\Big\} \\
&\leq \max\big\{2\,|d_{\max}(\theta)|^{-\alpha}\,\|f-g\|_{C([0,T],\|\cdot\|_E)}, |d_{\max}(\theta)|^{\beta-\alpha}\,\big(|f|_{C^\beta([0,T],\|\cdot\|_E)} + |g|_{C^\beta([0,T],\|\cdot\|_E)}\big)\big\}\,.
\end{aligned}
\tag{2.13}
$$

In addition, observe that Lemma 2.2 and the triangle inequality assure that

$$
\begin{aligned}
&\big\|f-g\big\|_{C([0,T],\|\cdot\|_E)} \leq \big\|f-[f]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[f]_\theta-[g]_\theta\big\|_{C([0,T],\|\cdot\|_E)} + \big\|[g]_\theta-g\big\|_{C([0,T],\|\cdot\|_E)} \\
&\leq \sup_{t\in\theta}\|f(t)-g(t)\|_E + \big|\tfrac{d_{\max}(\theta)}{2}\big|^\beta\,\big(|f|_{C^\beta([0,T],\|\cdot\|_E)} + |g|_{C^\beta([0,T],\|\cdot\|_E)}\big)\,.
\end{aligned}
\tag{2.14}
$$

Inserting (2.14) into (2.13) yields inequality (2.11). Moreover, adding inequality (2.11) and (2.14) results in inequality (2.12). This finishes the proof of Corollary 2.3. $\square$

Lemma 2.4. *Let $(E, \|\cdot\|_E)$ be a normed vector space, let $T, c \in (0,\infty)$, $\alpha \in [0,1]$, $\theta \in \mathscr{P}_T$, $N \in \mathbb{N}$, $\theta_0, \ldots, \theta_N \in [0,T]$ satisfy $0 = \theta_0 < \cdots < \theta_N = T$ and $\theta = \{\theta_0, \ldots, \theta_N\}$, and let $f\colon [0,T] \to E$ be a function. Then*

$$
|[f]_\theta|_{C^{\alpha,(0,c]}([0,T],\|\cdot\|_E)} \leq \tfrac{c^{1-\alpha}}{d_{\min}(\theta)}\big[\sup\nolimits_{j\in\{1,2,\ldots,N\}}\|f(\theta_j)-f(\theta_{j-1})\|_E\big]. \tag{2.15}
$$

Proof of Lemma 2.4. Observe that for all $s, t \in [0, T]$ with $t - s \in (0, c]$ it holds that

$$\begin{aligned}
\frac{\|[f]_\theta(t) - [f]_\theta(s)\|_E}{|t-s|^\alpha} &= \frac{\| \int_{(s,t)\setminus\theta} ([f]_\theta)'(u)\,\mathrm{d}u\|_E}{|t-s|^\alpha} \\
&\leq \frac{|t-s| \left[\sup_{u\in(s,t)\setminus\theta} \|([f]_\theta)'(u)\|_E\right]}{|t-s|^\alpha} \\
&\leq |t-s|^{1-\alpha} \left[\sup_{j\in\{1,2,\dots,N\}} \frac{\|f(\theta_j) - f(\theta_{j-1})\|_E}{|\theta_j - \theta_{j-1}|}\right] \\
&\leq \frac{c^{1-\alpha}}{d_{\min}(\theta)} \left[\sup_{j\in\{1,2,\dots,N\}} \|f(\theta_j) - f(\theta_{j-1})\|_E\right].
\end{aligned} \tag{2.16}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let $(E, \|\cdot\|_E)$ be a normed vector space, let $T \in (0,\infty)$, $\alpha \in [0,1]$, $\theta \in \mathscr{P}_T$, and let $f\colon [0,T] \to E$ be a function. Then $|[f]_\theta|_{C^\alpha([0,T],\|\cdot\|_E)} \leq |f|_{C^\alpha([0,T],\|\cdot\|_E)}$.*

Proof of Lemma 2.5. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \dots, \theta_N \in [0,T]$ be the real numbers which satisfy that $0 = \theta_0 < \theta_1 < \dots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \dots, \theta_N\}$ and let $n\colon [0,T] \to \mathbb{N}$ and $\rho\colon [0,T] \to [0,1]$ be the functions which satisfy for all $t \in [0,T]$ that

$$n(t) = \min\big\{k \in \{1,2,\dots,N\}\colon t \in [\theta_{k-1}, \theta_k]\big\} \qquad \text{and} \qquad \rho(t) = \frac{t - \theta_{n(t)-1}}{\theta_{n(t)} - \theta_{n(t)-1}}. \tag{2.17}$$

Note that for all $t \in [0,T]$ it holds that

$$[f]_\theta(t) = (1-\rho(t)) \cdot f(\theta_{n(t)-1}) + \rho(t) \cdot f(\theta_{n(t)}) = f(\theta_{n(t)-1}) + \rho(t) \cdot \big(f(\theta_{n(t)}) - f(\theta_{n(t)-1})\big). \tag{2.18}$$

Hence, we obtain for all $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $n(t_1) = n(t_2)$ that

$$\begin{aligned}
&\|[f]_\theta(t_1) - [f]_\theta(t_2)\|_E = \big\|\big[(1-\rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_1) \cdot f(\theta_{n(t_1)})\big] \\
&\quad - \big[(1-\rho(t_2)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_2) \cdot f(\theta_{n(t_1)})\big]\big\|_E \\
&= \big\|(\rho(t_2) - \rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + (\rho(t_1) - \rho(t_2)) \cdot f(\theta_{n(t_1)})\big\|_E \\
&= |\rho(t_1) - \rho(t_2)| \cdot \big\|f(\theta_{n(t_1)-1}) - f(\theta_{n(t_1)})\big\|_E \\
&\leq |\rho(t_1) - \rho(t_2)|\, |f|_{C^\alpha([0,T],\|\cdot\|_E)} \left|\theta_{n(t_1)-1} - \theta_{n(t_1)}\right|^\alpha \\
&= |\rho(t_1) - \rho(t_2)|^{1-\alpha}\, |f|_{C^\alpha([0,T],\|\cdot\|_E)} \left|(\rho(t_2) - \rho(t_1)) \cdot \theta_{n(t_1)-1} + (\rho(t_1) - \rho(t_2)) \cdot \theta_{n(t_1)}\right|^\alpha \\
&\leq |f|_{C^\alpha([0,T],\|\cdot\|_E)} \left|(\rho(t_2) - \rho(t_1)) \cdot \theta_{n(t_1)-1} + (\rho(t_1) - \rho(t_2)) \cdot \theta_{n(t_1)}\right|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\quad \cdot \left|\big[(1-\rho(t_1)) \cdot \theta_{n(t_1)-1} + \rho(t_1) \cdot \theta_{n(t_1)}\big] - \big[(1-\rho(t_2)) \cdot \theta_{n(t_2)-1} + \rho(t_2) \cdot \theta_{n(t_2)}\big]\right|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\quad \cdot \left|\big[\theta_{n(t_1)-1} + \rho(t_1) \cdot \big(\theta_{n(t_1)} - \theta_{n(t_1)-1}\big)\big] - \big[\theta_{n(t_2)-1} + \rho(t_2) \cdot \big(\theta_{n(t_2)} - \theta_{n(t_2)-1}\big)\big]\right|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)} \left|t_1 - t_2\right|^\alpha.
\end{aligned} \tag{2.19}$$

Moreover, (2.18) ensures for all $t_1, t_2 \in [0,T]$ with $n(t_1) < n(t_2)$ that

$$\begin{aligned}
&\|[f]_\theta(t_1) - [f]_\theta(t_2)\|_E = \big\|\big[(1-\rho(t_1)) \cdot f(\theta_{n(t_1)-1}) + \rho(t_1) \cdot f(\theta_{n(t_1)})\big] \\
&\quad - \big[(1-\rho(t_2)) \cdot f(\theta_{n(t_2)-1}) + \rho(t_2) \cdot f(\theta_{n(t_2)})\big]\big\|_E
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\rho(t_1))\,(1-\rho(t_2))\,\big\|f(\theta_{n(t_1)-1})-f(\theta_{n(t_2)-1})\big\|_E + \rho(t_1)\,\rho(t_2)\,\big\|f(\theta_{n(t_1)})-f(\theta_{n(t_2)})\big\|_E \\
&+ (1-\rho(t_1))\,\rho(t_2)\,\big\|f(\theta_{n(t_1)-1})-f(\theta_{n(t_2)})\big\|_E + \rho(t_1)\,(1-\rho(t_2))\,\big\|f(\theta_{n(t_1)})-f(\theta_{n(t_2)-1})\big\|_E \\
&\leq |f|_{C^\alpha([0,T],\|\cdot\|_E)}\,\big\{(1-\rho(t_1))\,(1-\rho(t_2))\,|\theta_{n(t_1)-1}-\theta_{n(t_2)-1}|^\alpha + \rho(t_1)\,\rho(t_2)\,|\theta_{n(t_1)}-\theta_{n(t_2)}|^\alpha \\
&\quad + (1-\rho(t_1))\,\rho(t_2)\,|\theta_{n(t_1)-1}-\theta_{n(t_2)}|^\alpha + \rho(t_1)\,(1-\rho(t_2))\,|\theta_{n(t_1)}-\theta_{n(t_2)-1}|^\alpha\big\}.
\end{aligned}
\tag{2.20}
$$

The concavity of the function $(-\infty,0] \ni x \mapsto |x|^\alpha \in \mathbb{R}$ hence proves for all $t_1, t_2 \in [0,T]$ with $n(t_1) < n(t_2)$ that

$$
\begin{aligned}
&\|[f]_\theta(t_1)-[f]_\theta(t_2)\|_E \\
&\leq |f|_{C^\alpha([0,T],\|\cdot\|_E)}\,\big|(1-\rho(t_1))\,(1-\rho(t_2))\,\big(\theta_{n(t_1)-1}-\theta_{n(t_2)-1}\big) + \rho(t_1)\,\rho(t_2)\,\big(\theta_{n(t_1)}-\theta_{n(t_2)}\big) \\
&\quad + (1-\rho(t_1))\,\rho(t_2)\,\big(\theta_{n(t_1)-1}-\theta_{n(t_2)}\big) + \rho(t_1)\,(1-\rho(t_2))\,\big(\theta_{n(t_1)}-\theta_{n(t_2)-1}\big)\big|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)}\,\big|(1-\rho(t_1))\,\theta_{n(t_1)-1} + \rho(t_1)\,\theta_{n(t_1)} - (1-\rho(t_2))\,\theta_{n(t_2)-1} - \rho(t_2)\,\theta_{n(t_2)}\big|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\quad \cdot \big|\big\{\theta_{n(t_1)-1} + \rho(t_1)\,\big[\theta_{n(t_1)}-\theta_{n(t_1)-1}\big]\big\} - \big\{\theta_{n(t_2)-1} + \rho(t_2)\,\big[\theta_{n(t_2)}-\theta_{n(t_2)-1}\big]\big\}\big|^\alpha \\
&= |f|_{C^\alpha([0,T],\|\cdot\|_E)}\,|t_1-t_2|^\alpha .
\end{aligned}
\tag{2.21}
$$

Combining this and (2.19) completes the proof of Lemma 2.5. □

Lemma 2.6 (Approximations by piecewise affine linear functions)**.** *Let $(E, \|\cdot\|_E)$ be a normed vector space, let $T \in (0,\infty)$, $\alpha \in [0,1]$, $\beta \in [\alpha,1]$, $\theta \in \mathscr{P}_T$, and let $f, g\colon [0,T] \to E$ be functions. Then*

$$
|f-[g]_\theta|_{C^\alpha([0,T],\|\cdot\|_E)} \leq \tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} \sup_{t\in\theta} \|f(t)-g(t)\|_E + 2\,|d_{\max}(\theta)|^{\beta-\alpha}\,|f|_{C^\beta([0,T],\|\cdot\|_E)}
\tag{2.22}
$$

and

$$
\begin{aligned}
&\|f-[g]_\theta\|_{C^\alpha([0,T],\|\cdot\|_E)} \\
&\leq \Big(\tfrac{2\,|d_{\max}(\theta)|^{1-\alpha}}{d_{\min}(\theta)} + 1\Big) \sup_{t\in\theta} \|f(t)-g(t)\|_E + \Big(\tfrac{2}{|d_{\max}(\theta)|^\alpha} + \tfrac{1}{2^\beta}\Big)\,|d_{\max}(\theta)|^\beta\,|f|_{C^\beta([0,T],\|\cdot\|_E)} .
\end{aligned}
\tag{2.23}
$$

Proof of Lemma 2.6. Throughout this proof let $N \in \mathbb{N}$, $\theta_0, \theta_1, \ldots, \theta_N \in [0,T]$ be the real numbers which satisfy that $0 = \theta_0 < \theta_1 < \cdots < \theta_N = T$ and $\theta = \{\theta_0, \theta_1, \ldots, \theta_N\}$. Note that Lemma 2.1 implies that

$$
\begin{aligned}
|f-[g]_\theta|_{C^\alpha([0,T],\|\cdot\|_E)} \leq \max\Big\{&|d_{\max}(\theta)|^{-\alpha}\,|f-[g]_\theta|_{C^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)}, \\
&|d_{\max}(\theta)|^{\beta-\alpha}\,|f-[g]_\theta|_{C^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)}\Big\}.
\end{aligned}
\tag{2.24}
$$

Next note that Lemma 2.2 ensures that

$$
\begin{aligned}
&|f-[g]_\theta|_{C^{0,(d_{\max}(\theta),\infty)}([0,T],\|\cdot\|_E)} \leq 2\,\|f-[g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\
&\leq 2\,\|f-[f]_\theta\|_{C([0,T],\|\cdot\|_E)} + 2\,\|[f]_\theta-[g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\
&\leq 2\,\big|\tfrac{d_{\max}(\theta)}{2}\big|^\beta\,|f|_{C^\beta([0,T],\|\cdot\|_E)} + 2\sup_{t\in\theta}\|f(t)-g(t)\|_E \\
&\leq 2\,|d_{\max}(\theta)|^\beta\,|f|_{C^\beta([0,T],\|\cdot\|_E)} + 2\cdot\frac{d_{\max}(\theta)}{d_{\min}(\theta)}\cdot\sup_{t\in\theta}\|f(t)-g(t)\|_E .
\end{aligned}
\tag{2.25}
$$

Moreover, observe that Lemma 2.4 and Lemma 2.5 imply that

$$
\begin{aligned}
&|f-[g]_\theta|_{C^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \le |f-[f]_\theta|_{C^\beta([0,T],\|\cdot\|_E)} + |[f-g]_\theta|_{C^{\beta,(0,d_{\max}(\theta)]}([0,T],\|\cdot\|_E)} \\
&\le |f|_{C^\beta([0,T],\|\cdot\|_E)} + |[f]_\theta|_{C^\beta([0,T],\|\cdot\|_E)} \\
&\quad + \tfrac{|d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} \Big[\sup_{j\in\{1,2,\dots,N\}} \big\| \big[f(\theta_j)-g(\theta_j)\big] - \big[f(\theta_{j-1})-g(\theta_{j-1})\big] \big\|_E \Big] \\
&\le 2\,|f|_{C^\beta([0,T],\|\cdot\|_E)} + \frac{2}{|d_{\max}(\theta)|^\beta} \cdot \frac{d_{\max}(\theta)}{d_{\min}(\theta)} \cdot \sup_{t\in\theta} \|f(t)-g(t)\|_E.
\end{aligned}
\tag{2.26}
$$

Substituting (2.26) and (2.25) into (2.24) proves (2.22). It thus remains to prove estimate (2.23). For this note that Lemma 2.2 yields that

$$
\begin{aligned}
\|f-[g]_\theta\|_{C([0,T],\|\cdot\|_E)} &\le \|f-[f]_\theta\|_{C([0,T],\|\cdot\|_E)} + \|[f]_\theta-[g]_\theta\|_{C([0,T],\|\cdot\|_E)} \\
&\le \left|\tfrac{d_{\max}(\theta)}{2}\right|^\beta |f|_{C^\beta([0,T],\|\cdot\|_E)} + \sup_{t\in\theta}\|f(t)-g(t)\|_E.
\end{aligned}
\tag{2.27}
$$

Combining (2.22) and (2.27) shows (2.23). The proof of Lemma 2.6 is thus completed. □

2.2 Upper error bounds for stochastic processes with Hölder continuous sample paths

We now turn to the result announced in the introduction which provides convergence of stochastic processes in Hölder norms given convergence on the grid-points. For this we first recall Kolmogorov's continuity criterion, cf., e.g., Theorem I.2.1 in Revuz & Yor [26].

Theorem 2.7. *There exists a function $\Xi = (\Xi_{T,p,\alpha,\beta})_{T,p,\alpha,\beta\in\mathbb{R}} \colon \mathbb{R}^4 \to \mathbb{R}$ such that for every $T\in[0,\infty)$, $p\in(1,\infty)$, $\beta\in(1/p,1]$, every Banach space $(E,\|\cdot\|_E)$, every probability space $(\Omega,\mathscr{F},\mathbb{P})$, and every $X\in C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})$ there exists an $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic process $Y\colon[0,T]\times\Omega\to E$ such that for every $\alpha\in[0,\beta-1/p)$ it holds that*

$$
\begin{aligned}
&\|Y\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \le \Xi_{T,p,\alpha,\beta}\,\|X\|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty \quad \textit{and} \\
&\forall\, t\in[0,T]\colon \mathbb{P}(X_t=Y_t)=1.
\end{aligned}
\tag{2.28}
$$

The next result, Corollary 2.8, follows directly from Corollary 2.3 and Kolmogorov's continuity criterion in Theorem 2.7 above.

Corollary 2.8 (Grid point approximations)**.** *Let $T\in(0,\infty)$, $\theta\in\mathscr{P}_T$, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, and let $(E,\|\cdot\|_E)$ be a Banach space. Then*

(i) it holds for all $p\in[1,\infty)$, $\beta\in[0,1]$, $\gamma\in[\beta,1]$ and all $(\mathscr{F},\|\cdot\|_E)$-strongly measurable stochastic processes $X,Y\colon[0,T]\times\Omega\to E$ that

$$
\begin{aligned}
\|X-Y\|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} &\le \Big(2\,|d_{\max}(\theta)|^{-\beta}+1\Big)\Big[\sup\nolimits_{t\in\theta}\|X_t-Y_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + |d_{\max}(\theta)|^\gamma \big(\,|X|_{C^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y|_{C^\gamma([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big)\Big]
\end{aligned}
\tag{2.29}
$$

(ii) *and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that*

$$\begin{aligned} &\|X - Y\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T], \|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,\beta} \|X - Y\|_{C^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \\ &\leq \Xi_{T,p,\alpha,\beta} \Big(2 \, |d_{\max}(\theta)|^{-\beta} + 1 \Big) \Big[\sup\nolimits_{t \in \theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \\ &\quad + |d_{\max}(\theta)|^\gamma \, \big(|X|_{C^\gamma([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} + |Y|_{C^\gamma([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \big) \Big]. \end{aligned} \tag{2.30}$$

The next result, Corollary 2.9, follows directly from Lemma 2.6 and Kolmogorov's continuity criterion in Theorem 2.7 above.

Corollary 2.9 (Piecewise affine linear stochastic processes)**.** *Let $T \in (0,\infty)$, $\theta \in \mathscr{P}_T$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $(E, \|\cdot\|_E)$ be a Banach space. Then*

(i) *it holds for all $p \in [1,\infty)$, $\beta \in [0,1]$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ that*

$$\begin{aligned} &\|X - [Y]_\theta\|_{C^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \leq \Big[\tfrac{2 \, |d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1 \Big] \sup_{t \in \theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \\ &\quad + \big[2 \, |d_{\max}(\theta)|^{-\beta} + 2^{-\gamma} \big] \, |d_{\max}(\theta)|^\gamma \, |X|_{C^\gamma([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \end{aligned} \tag{2.31}$$

(ii) *and it holds for all $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $\alpha \in [0, \beta - 1/p)$, $\gamma \in [\beta, 1]$ and all $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes $X, Y\colon [0,T] \times \Omega \to E$ with continuous sample paths that*

$$\begin{aligned} &\|X - [Y]_\theta\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T], \|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,\beta} \bigg(\Big[\tfrac{2 \, |d_{\max}(\theta)|^{1-\beta}}{d_{\min}(\theta)} + 1 \Big] \sup_{t \in \theta} \|X_t - Y_t\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \\ &\quad + \big[2 \, |d_{\max}(\theta)|^{-\beta} + 2^{-\gamma} \big] \, |d_{\max}(\theta)|^\gamma \, |X|_{C^\gamma([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \bigg). \end{aligned} \tag{2.32}$$

In (2.32) in Corollary 2.9 we assume beside other assumptions that α is strictly smaller than γ. In general, this assumption can not be omitted. To give an example, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $W\colon [0,1] \times \Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion. Then it clearly holds for all $p \in (0,\infty)$ that $\|W\|_{C^{1/2}([0,1], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)})} < \infty$. However, the law of the iterated logarithm (see, e.g., Corollary 3.1 in Arcones [2]) ensures that the sample paths of the Brownian motion are $\mathbb{P}$-a.s. not $1/2$-Hölder continuous, so that for all $p \in (0,\infty)$ it holds that $\|W - [W]_\theta\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^{1/2}([0,1], |\cdot|)})} = \infty$. The following corollary is related to Lemma A1 in Bally, Millet & Sanz-Solé [3].

Corollary 2.10 ($\mathscr{L}^p$–convergence in Hölder norms for a fixed $p \in [1,\infty)$)**.** *Let $T \in (0,\infty)$, $p \in [1,\infty)$, $\beta \in [0,1]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a Banach space, and let $Y^N\colon [0,T] \times \Omega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths which satisfy $\limsup_{N\to\infty} |Y^N|_{C^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} < \infty$ and $\forall\, t \in [0,T]\colon \limsup_{N\to\infty} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = 0$. Then*

(i) *it holds that $|Y^0|_{C^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} \leq \limsup_{N\to\infty} |Y^N|_{C^\beta([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} < \infty$,*

(ii) *it holds for all $\alpha \in [0,1] \cap (-\infty, \beta)$ that $\limsup_{N\to\infty} \|Y^0 - Y^N\|_{C^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)})} = 0$,* *(iii) and it holds for all $\alpha \in [0,1] \cap (-\infty, \beta - 1/p)$ that* $\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} = 0.$

Proof of Corollary 2.10. Throughout this proof let $\theta^n \in \mathscr{P}_T$, $n \in \mathbb{N}$, be the sequence which satisfies for all $n \in \mathbb{N}$ that $\theta^n = \{0, \frac{T}{n}, \frac{2T}{n}, \ldots, \frac{(n-1)T}{n}, T\} \in \mathscr{P}_T$. Observe that the assumption that $\forall\, t \in [0,T] \colon \limsup_{N\to\infty} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ and the assumption that $\limsup_{N\to\infty} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$ ensure that

$$
\begin{aligned}
&|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\tfrac{\|Y_t^0 - Y_s^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \\
&= \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \left[\tfrac{\limsup_{N\to\infty} \|(Y_t^N - Y_s^N) + (Y_t^0 - Y_t^N) + (Y_s^N - Y_s^0)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \\
&\leq \sup_{\substack{s,t\in[0,T],\\ s\neq t}} \limsup_{N\to\infty} \left[\tfrac{\|Y_t^N - Y_s^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{|t-s|^\beta}\right] \leq \limsup_{N\to\infty} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.
\end{aligned}
\tag{2.33}
$$

This establishes Item (i). In the next step we prove Item (ii). We apply Item (i) in Corollary 2.8 to obtain for all $\alpha \in [0,\beta]$, $n, N \in \mathbb{N}$ that

$$
\begin{aligned}
&\|Y^0 - Y^N\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \leq \big(2\,|d_{\max}(\theta^n)|^{-\alpha} + 1\big) \Big[\sup\nolimits_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + |d_{\max}(\theta^n)|^\beta \left(|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right)\Big] \\
&\leq \left(\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right) \sup\nolimits_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \\
&\quad + \left(\tfrac{2\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{T^\beta}{n^\beta}\right) \left(|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\right).
\end{aligned}
\tag{2.34}
$$

Item (i) and the assumption that $\forall\, t \in [0,T] \colon \limsup_{N\to\infty} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} = 0$ hence imply for all $\alpha \in [0,\beta]$, $n \in \mathbb{N}$ that

$$
\begin{aligned}
&\limsup_{N\to\infty} \|Y^0 - Y^N\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \leq \left[\tfrac{2\,T^{-\alpha}}{n^{-\alpha}} + 1\right] \left[\limsup_{N\to\infty} \sup_{t\in\theta^n} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\right] \\
&\quad + \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\
&= \left[\tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty.
\end{aligned}
\tag{2.35}
$$

Hence, we obtain for all $\alpha \in [0,\infty) \cap (-\infty,\beta)$ that

$$
\begin{aligned}
&\limsup_{N\to\infty} \|Y^0 - Y^N\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = \limsup_{n\to\infty} \limsup_{N\to\infty} \|Y^0 - Y^N\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \\
&\leq \left[\limsup_{n\to\infty} \tfrac{4\,T^{\beta-\alpha}}{n^{\beta-\alpha}} + \limsup_{n\to\infty} \tfrac{2\,T^\beta}{n^\beta}\right] \limsup_{N\to\infty} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0.
\end{aligned}
\tag{2.36}
$$

This shows Item (ii). It thus remains to establish Item (iii) to complete the proof of Corollary 2.10. For this we apply Item (ii) in Corollary 2.8 to obtain for all $r \in (1/p, \infty) \cap (-\infty, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$
\|Y^0 - Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,r} \, \|Y^0 - Y^N\|_{C^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}.
\tag{2.37}
$$

This and Item (ii) imply for all $r \in (1/p, \infty) \cap (-\infty, \beta)$, $\alpha \in [0, r - 1/p)$ that

$$
\limsup_{N\to\infty} \|Y^0 - Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,r} \limsup_{N\to\infty} \|Y^0 - Y^N\|_{C^r([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} = 0.
\tag{2.38}
$$

This establishes Item (iii). The proof of Corollary 2.10 is thus completed. □

The next result, Corollary 2.11 below, is a consequence from Corollary 2.8 and Corollary 2.9.

Corollary 2.11 (Convergence rates w.r.t. Hölder norms)**.** *Let $T \in (0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1]$, $(\theta^N)_{N\in\mathbb{N}} \subseteq \mathscr{P}_T$ satisfy $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a Banach space, let $Y^N \colon [0,T] \times \Omega \to E$, $N \in \mathbb{N}_0$, be $(\mathscr{F}, \|\cdot\|_E)$-strongly measurable stochastic processes with continuous sample paths satisfying $Y_0^0 \in \mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)$ and*

$$|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-\beta} \sup\nolimits_{t\in\theta^N} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right] < \infty, \tag{2.39}$$

and assume $\left(\left[\sup_{N\in\mathbb{N}} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty \right] \text{ or } \left[\sup_{N\in\mathbb{N}} {d_{\max}(\theta^N)}/{d_{\min}(\theta^N)} < \infty \text{ and } \forall\, N \in \mathbb{N} \colon Y^N = [Y^N]_{\theta^N} \right] \right)$. Then for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0,\infty)$ it holds that

$$\sup_{N\in\mathbb{N}} \left[\|Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} + \left| d_{\max}(\theta^N) \right|^{-(\beta-\alpha-1/p-\varepsilon)} \|Y^0 - Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \right] < \infty. \tag{2.40}$$

Proof of Corollary 2.11. Throughout this proof let $c_0 \in [0,\infty)$, $c_1, c_2 \in [0,\infty]$ be the extended real numbers given by

$$\begin{aligned} c_0 &= |Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-\beta} \sup\nolimits_{t\in\theta^N} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right], \\ c_1 &= \sup_{N\in\mathbb{N}} \left[\frac{d_{\max}(\theta^N)}{d_{\min}(\theta^N)} \right], \qquad \text{and} \qquad c_2 = \sup_{N\in\mathbb{N}} |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}. \end{aligned} \tag{2.41}$$

Next we observe that Item (ii) in Corollary 2.8 ensures for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned} &\left\|Y^0 - Y^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,r} \left(2 \left| d_{\max}(\theta^N) \right|^{-r} + 1 \right) \left[\sup_{t\in\theta^N} \|Y_t^0 - Y_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right. \\ &\quad \left. + \left| d_{\max}(\theta^N) \right|^\beta \left(|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} + |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \right) \right] \\ &\leq \Xi_{T,p,\alpha,r} \left(2 \left| d_{\max}(\theta^N) \right|^{(\beta-r)} + \left| d_{\max}(\theta^N) \right|^\beta \right) \left[c_0 + |Y^N|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} \right] \\ &\leq \Xi_{T,p,\alpha,r} \left(2 \left| d_{\max}(\theta^N) \right|^{(\beta-r)} + \left| d_{\max}(\theta^N) \right|^\beta \right) [c_0 + c_2] \\ &= \Xi_{T,p,\alpha,r} \left(2 + \left| d_{\max}(\theta^N) \right|^r \right) \left| d_{\max}(\theta^N) \right|^{(\beta-r)} [c_0 + c_2]. \end{aligned} \tag{2.42}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-(\beta-r)} \left\|Y^0 - Y^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \right] \leq \Xi_{T,p,\alpha,r} \left(2 + T^r \right) [c_0 + c_2]. \tag{2.43}$$

Hence, we obtain for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])} \left\|Y^0 - Y^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \right] \\ &\leq \Xi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]} \left(3 + T \right) (c_0 + c_2). \end{aligned} \tag{2.44}$$

This shows for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\left| d_{\max}(\theta^N) \right|^{-(\beta-\alpha-1/p-\varepsilon)} \left\|Y^0 - Y^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \right] \\ &\leq \Xi_{T,p,\alpha,\alpha+1/p+\varepsilon} \left(3 + T \right) (c_0 + c_2). \end{aligned} \tag{2.45}$$

In the next step we note that Item (ii) in Corollary 2.9 proves for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\big\|Y^0 - [Y^N]_{\theta^N}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \leq \Xi_{T,p,\alpha,r}\bigg(\Big[\tfrac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)} + 1\Big] \sup_{t\in\theta^N} \|Y^0_t - Y^N_t\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}$$
$$+ \big[2\,|d_{\max}(\theta^N)|^{-r} + 2^{-\beta}\big]\,\big|d_{\max}(\theta^N)\big|^{\beta}\,\big|Y^0\big|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\bigg). \tag{2.46}$$

This implies for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$, $N \in \mathbb{N}$ that

$$\begin{aligned} &\big\|Y^0 - [Y^N]_{\theta^N}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \\ &\leq c_0\,\big|d_{\max}(\theta^N)\big|^{\beta}\,\Xi_{T,p,\alpha,r}\Big(\tfrac{2\,|d_{\max}(\theta^N)|^{1-r}}{d_{\min}(\theta^N)} + 1 + 2\,|d_{\max}(\theta^N)|^{-r} + 2^{-\beta}\Big) \\ &\leq 2\,c_0\,\big|d_{\max}(\theta^N)\big|^{\beta}\,\Xi_{T,p,\alpha,r}\,\Big([c_1+1]\,\big|d_{\max}(\theta^N)\big|^{-r} + 1\Big). \end{aligned} \tag{2.47}$$

Hence, we obtain for all $r \in (1/p, \beta]$, $\alpha \in [0, r - 1/p)$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}}\Big[\big|d_{\max}(\theta^N)\big|^{-(\beta-r)}\,\big\|Y^0 - [Y^N]_{\theta^N}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big] \\ &\leq 2\,c_0\,\Xi_{T,p,\alpha,r}\,(c_1 + 1 + T^r) \leq 2\,c_0\,\Xi_{T,p,\alpha,r}\,(2 + T + c_1). \end{aligned} \tag{2.48}$$

This shows for all $\alpha \in [0, \beta - 1/p)$, $r \in (\alpha + 1/p, \beta]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}}\Big[\big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-[r-\alpha-1/p])}\,\big\|Y^0 - [Y^N]_{\theta^N}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big] \\ &\leq 2\,c_0\,\Xi_{T,p,\alpha,\alpha+1/p+[r-\alpha-1/p]}\,(2 + T + c_1). \end{aligned} \tag{2.49}$$

This establishes for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \beta - \alpha - 1/p]$ that

$$\begin{aligned} &\sup_{N\in\mathbb{N}}\Big[\big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-\varepsilon)}\,\big\|Y^0 - [Y^N]_{\theta^N}\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big] \\ &\leq 2\,c_0\,\Xi_{T,p,\alpha,\alpha+1/p+\varepsilon}\,(2 + T + c_1). \end{aligned} \tag{2.50}$$

Combining (2.45) and (2.50) assures for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0, \infty)$ that

$$\sup_{N\in\mathbb{N}}\Big[\big|d_{\max}(\theta^N)\big|^{-(\beta-\alpha-1/p-\varepsilon)}\,\|Y^0 - Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big] < \infty. \tag{2.51}$$

In addition, note that the assumption that $Y^0_0 \in \mathscr{L}^p(\mathbb{P};\|\cdot\|_E)$, the assumption that $|Y^0|_{C^\beta([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})} < \infty$, the assumption that Y^0 has continuous sample paths, and Theorem 2.7 ensure that for all $\alpha \in [0, \beta - 1/p)$ it holds that $\|Y^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} < \infty$. This and (2.51) complete the proof of Corollary 2.11. □

The next result, Corollary 2.12 below, illustrates Corollary 2.11 through a simple example. For this note that standard results for the Euler-Maruyama method show for every $p \in [2, \infty)$, $\beta \in [0, 1/2]$ that assumption (2.39) in Corollary 2.11 with uniform time steps is satisfied (cf., e.g., Section 10.6 in Kloeden & Platen [22]). Corollary 2.12 is related to Theorem 1.2 in [4] and Theorem 1.1 in [5].

Corollary 2.12 (Euler–Maruyama method)**.** *Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be an m-dimensional standard $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motion, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$*

be globally Lipschitz continuous functions, let $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ *be an* $(\mathscr{F}_t)_{t\in[0,T]}$*-adapted stochastic process with continuous sample paths which satisfies* $\forall\, p\in[1,\infty)\colon \mathbb{E}\big[\|X_0\|_{\mathbb{R}^d}^p\big]<\infty$ *and which satisfies for all* $t\in[0,T]$ *that*

$$[X_t]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)}=\left[X_0+\int_0^t\mu(X_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(\mathbb{R}^d)}+\int_0^t\sigma(X_s)\,\mathrm{d}W_s, \tag{2.52}$$

let $Y^N\colon[0,T]\times\Omega\to\mathbb{R}^d$*,* $N\in\mathbb{N}$*, be mappings satisfying for all* $N\in\mathbb{N}$*,* $n\in\{0,1,\dots,N-1\}$*,* $t\in[\frac{nT}{N},\frac{(n+1)T}{N}]$ *that* $Y_0^N=X_0$ *and*

$$Y_t^N=Y^N_{\frac{nT}{N}}+\big(t-\tfrac{nT}{N}\big)\cdot\mu\big(Y^N_{\frac{nT}{N}}\big)+\big(\tfrac{tN}{T}-n\big)\cdot\sigma\big(Y^N_{\frac{nT}{N}}\big)\big(W_{\frac{(n+1)T}{N}}-W_{\frac{nT}{N}}\big). \tag{2.53}$$

Then for all $\alpha\in[0,1/2)$*,* $\varepsilon\in(0,\infty)$*,* $p\in[1,\infty)$ *it holds that*

$$\sup_{N\in\mathbb{N}}\big(N^{1/2-\alpha-\varepsilon}\,\|X-Y^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_{\mathbb{R}^d})})}\big)<\infty. \tag{2.54}$$

2.3 Lower error bounds for stochastic processes with Hölder continuous sample paths

In this subsection we comment on the optimality of the convergence rate provided by Corollary 2.11 and Corollary 2.12, respectively. In particular, in the setting of Corollary 2.12, Theorem 3 in Müller-Gronbach [24] shows in the case $\alpha=0$ that there exists a class of SDEs in which the factors $N^{1/2-\varepsilon}$, $N\in\mathbb{N}$, on the left hand side of the estimate (2.54) can at best – up to a constant – be replaced by the factors $\frac{N^{1/2}}{\log(N)}$, $N\in\mathbb{N}$. In Proposition 2.14 below we show for every $\alpha\in[0,1/2)$ in the simple example $\mu=0$ and $\sigma=\big(\mathbb{R}\ni x\mapsto 1\in\mathbb{R}\big)$ in Corollary 2.12 that the factors $N^{1/2-\alpha-\varepsilon}$, $N\in\mathbb{N}$, on the left hand side of the estimate (2.54) can at best – up to a constant – be replaced by the factors $N^{1/2-\alpha}$, $N\in\mathbb{N}$. Our proof of Proposition 2.14 uses the following elementary lemma.

Lemma 2.13. *Let* $T\in(0,\infty)$*,* $p\in[1,\infty)$*,* $\alpha\in[0,1]$*, let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space, let* $(E,\|\cdot\|_E)$ *be a normed vector space, and let* $X\colon[0,T]\times\Omega\to E$ *be an* $(\mathscr{F},\|\cdot\|_E)$*-strongly measurable stochastic process. Then*

$$\max\big\{|X|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})},\,2^{(1/p-1)}\,\|X\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)})}\big\}\leq\|X\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\,. \tag{2.55}$$

The proof of Lemma 2.13 is clear. Instead we now present the promised proposition on the optimality of the convergence rate estimate in Corollary 2.12.

Proposition 2.14. *Let* $T\in(0,\infty)$*, let* $(\Omega,\mathscr{F},\mathbb{P})$ *be a probability space, let* $W\colon[0,T]\times\Omega\to\mathbb{R}$ *be a one-dimensional standard Brownian motion, and let* $W^N\colon[0,T]\times\Omega\to\mathbb{R}$*,* $N\in\mathbb{N}$*, be mappings which satisfy for all* $N\in\mathbb{N}$*,* $n\in\{0,1,\dots,N-1\}$*,* $t\in\big[\frac{nT}{N},\frac{(n+1)T}{N}\big]$ *that*

$$W_t^N=\big(n+1-\tfrac{tN}{T}\big)\cdot W_{\frac{nT}{N}}+\big(\tfrac{tN}{T}-n\big)\cdot W_{\frac{(n+1)T}{N}}\,. \tag{2.56}$$

Then it holds for all $\alpha\in[0,1/2]$*,* $p\in[1,\infty)$*,* $N\in\mathbb{N}$ *that*

$$\|W-W^N\|_{C([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}, \tag{2.57}$$

$$\tfrac{|W-W^N|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}=\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}}\in\left[\tfrac{1}{\sqrt{2}},1\right], \tag{2.58}$$

$$\frac{\|W-W^N\|_{C^\alpha([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} = \tfrac{T^\alpha}{2N^\alpha} + \tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \in \left[\tfrac{1}{\sqrt{2}}, \tfrac{2+T^\alpha}{2}\right], \tag{2.59}$$

$$\frac{\|W-W^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],|\cdot|)})}}{N^{(\alpha-1/2)}\,T^{-\alpha}\,\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}} \geq \tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}} \geq \tfrac{1}{\sqrt{2}}. \tag{2.60}$$

Proof of Proposition 2.14. Throughout this proof let $f\colon[0,1/2]\to(0,\infty)$ be the function which satisfies for all $x\in[0,1/2]$ that $f(x)=\frac{(1/2-x)^{(1/2-x)}}{2^x\,(1-x)^{(1-x)}}$ and let $g_\alpha\colon[0,1]^2\to\mathbb{R}$, $\alpha\in[0,1/2]$, be the functions which satisfy for all $x,y\in[0,1]$, $\alpha\in[0,1/2]$ with $x+y>0$ that

$$g_\alpha(x,y)=\frac{x\,(1-x)+y\,(1-y)}{(x+y)^{2\alpha}} \qquad\text{and}\qquad g_\alpha(0,0)=\begin{cases}0 & :\alpha<1/2\\ 1 & :\alpha=1/2\end{cases}. \tag{2.61}$$

We first prove (2.57). For this observe that for all $N\in\mathbb{N}$, $n\in\{0,1,\dots,N-1\}$, $t\in\left[\frac{nT}{N},\frac{(n+1)T}{N}\right]$ it holds that

$$\begin{aligned} W_t-W_t^N &= W_t-\left(n+1-\tfrac{tN}{T}\right)\cdot W_{\frac{nT}{N}}-\left(\tfrac{tN}{T}-n\right)\cdot W_{\frac{(n+1)T}{N}}\\ &=\left(n-\tfrac{tN}{T}\right)\cdot\left(W_{\frac{(n+1)T}{N}}-W_t\right)+\left(n+1-\tfrac{tN}{T}\right)\cdot\left(W_t-W_{\frac{nT}{N}}\right)\end{aligned} \tag{2.62}$$

This and the fact that $\forall\,N\in\mathbb{N}\colon\forall\,t\in[0,T]\colon\forall\,p\in[1,\infty)\colon\left\|{}^{W_t-W_t^N}\!/\!{}_{\|W_t-W_t^N\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}}\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}=\frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}}$ imply for all $N\in\mathbb{N}$, $p\in[1,\infty)$ that

$$\begin{aligned} &\left\|W-W^N\right\|_{C([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)})}=\sup\nolimits_{t\in[0,T]}\left\|W_t-W_t^N\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\ &=\sup\nolimits_{t\in[0,\frac{T}{N}]}\left\|W_t-W_t^N\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}}\left[\sup\nolimits_{t\in[0,\frac{T}{N}]}\left\|W_t-W_t^N\right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}\right]\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}}\left[\sup\nolimits_{t\in[0,\frac{T}{N}]}\left\|\tfrac{tN}{T}\cdot\left(W_t-W_{\frac{T}{N}}\right)+\left(1-\tfrac{tN}{T}\right)\cdot W_t\right\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}\right]\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}}\left[\sup\nolimits_{t\in[0,\frac{T}{N}]}\left[\left(\tfrac{tN}{T}\right)^2\cdot\left(\tfrac{T}{N}-t\right)+\left(1-\tfrac{tN}{T}\right)^2\cdot t\right]^{1/2}\right]\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}}\left[\sup_{t\in[0,1]}\sqrt{\left(t^2\cdot(1-t)+(1-t)^2\cdot t\right)}\right]\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}}\left[\sup_{t\in[0,1]}\sqrt{t\cdot(1-t)}\right]=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{2\sqrt{N}}.\end{aligned} \tag{2.63}$$

This establishes (2.57). In the next step we prove (2.58). For this observe that (2.62) shows for all $N\in\mathbb{N}$, $n\in\{1,2,\dots,N-1\}$, $t_1\in[0,\frac{T}{N}]$, $t_2\in[\frac{nT}{N},\frac{(n+1)T}{N}]$, $p\in[1,\infty)$ that

$$\begin{aligned} &\left\|\left(W_{t_2}-W_{t_2}^N\right)-\left(W_{t_1}-W_{t_1}^N\right)\right\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\ &=\bigg\|\left(n-\tfrac{t_2N}{T}\right)\cdot\left(W_{\frac{(n+1)T}{N}}-W_{t_2}\right)+\left(n+1-\tfrac{t_2N}{T}\right)\cdot\left(W_{t_2}-W_{\frac{nT}{N}}\right)\\ &\quad+\tfrac{t_1N}{T}\cdot\left(W_{\frac{T}{N}}-W_{t_1}\right)+\left(\tfrac{t_1N}{T}-1\right)\cdot W_{t_1}\bigg\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}}\bigg[\left(n-\tfrac{t_2N}{T}\right)^2\left(\tfrac{(n+1)T}{N}-t_2\right)\\ &\qquad+\left(n+1-\tfrac{t_2N}{T}\right)^2\left(t_2-\tfrac{nT}{N}\right)+\tfrac{|t_1|^2N^2}{T^2}\left(\tfrac{T}{N}-t_1\right)+\left(\tfrac{t_1N}{T}-1\right)^2t_1\bigg]^{\frac{1}{2}}\\ &=\tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{N}}\left[\left(\tfrac{t_2N}{T}-n\right)\left(n+1-\tfrac{t_2N}{T}\right)+\tfrac{t_1N}{T}\left(1-\tfrac{t_1N}{T}\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{2.64}$$

Moreover, (2.56) ensures for all $N \in \mathbb{N}$, $t_1, t_2 \in [0, \frac{T}{N}]$, $p \in [1, \infty)$ with $t_1 < t_2$ that

$$
\begin{aligned}
&\big\| \big(W_{t_2} - W^N_{t_2} \big) - \big(W_{t_1} - W^N_{t_1} \big) \big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \\
&= \Big\| \Big(W_{t_2} - \tfrac{t_2 N}{T} \cdot W_{\frac{T}{N}} \Big) - \Big(W_{t_1} - \tfrac{t_1 N}{T} \cdot W_{\frac{T}{N}} \Big) \Big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \Big\| W_{t_2} - W_{t_1} + \tfrac{(t_1 - t_2) N}{T} \cdot W_{\frac{T}{N}} \Big\|_{\mathscr{L}^2(\mathbb{P}; |\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \\
&\quad \cdot \Big\| \Big(1 + \tfrac{(t_1 - t_2) N}{T} \Big) \cdot (W_{t_2} - W_{t_1}) + \tfrac{(t_1 - t_2) N}{T} \cdot \Big(W_{\frac{T}{N}} - W_{t_2} \Big) + \tfrac{(t_1 - t_2) N}{T} \cdot W_{t_1} \Big\|_{\mathscr{L}^2(\mathbb{P}; |\cdot|)} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \left[\Big| 1 + \tfrac{(t_1 - t_2) N}{T} \Big|^2 \cdot (t_2 - t_1) + \tfrac{|t_1 - t_2|^2 N^2}{T^2} \cdot \big(\tfrac{T}{N} - t_2 \big) + \tfrac{|t_1 - t_2|^2 N^2}{T^2} \cdot t_1 \right]^{1/2} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \cdot \left[1 + \tfrac{2(t_1 - t_2) N}{T} + \tfrac{(t_1 - t_2)^2 N^2}{T^2} + \tfrac{|t_1 - t_2| \, N^2}{T^2} \cdot \big(\tfrac{T}{N} + t_1 - t_2 \big) \right]^{1/2} \cdot (t_2 - t_1)^{1/2} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \cdot \Big(1 + \tfrac{(t_1 - t_2) N}{T} \Big)^{1/2} \cdot (t_2 - t_1)^{1/2} .
\end{aligned}
\tag{2.65}
$$

Combining (2.64) and (2.65) proves for all $N \in \mathbb{N}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$
\begin{aligned}
&\big| W - W^N \big|_{C^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)})} = \sup_{t_1, t_2 \in [0,T], \, t_1 < t_2} \frac{\big\| \big(W_{t_2} - W^N_{t_2} \big) - \big(W_{t_1} - W^N_{t_1} \big) \big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{|t_1 - t_2|^\alpha} \\
&= \sup_{t_1 \in [0, \frac{T}{N}]} \sup_{t_2 \in (t_1, T]} \frac{\big\| \big(W_{t_2} - W^N_{t_2} \big) - \big(W_{t_1} - W^N_{t_1} \big) \big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{|t_1 - t_2|^\alpha} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \left| \max \left\{ \sup_{\substack{t_1, t_2 \in [0, \frac{T}{N}], \\ t_1 < t_2}} \frac{\big(1 + \frac{(t_1 - t_2) N}{T} \big)}{(t_2 - t_1)^{(2\alpha - 1)}}, \sup_{\substack{t_1 \in [0, \frac{T}{N}], \\ t_2 \in (\frac{T}{N}, \frac{2T}{N}]}} \frac{T \Big[\Big(\frac{t_2 N}{T} - 1 \Big) \Big(2 - \frac{t_2 N}{T} \Big) + \frac{t_1 N}{T} \Big(1 - \frac{t_1 N}{T} \Big) \Big]}{N \, (t_2 - t_1)^{2\alpha}} \right\} \right|^{\frac{1}{2}} .
\end{aligned}
\tag{2.66}
$$

This implies for all $N \in \mathbb{N}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$
\begin{aligned}
&\big| W - W^N \big|_{C^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)})} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \left| \big| \tfrac{T}{N} \big|^{(1 - 2\alpha)} \max \left\{ \sup_{x \in (0,1]} \frac{(1 - x)}{x^{(2\alpha - 1)}}, \sup_{\substack{x \in [0,1], \\ y \in (1,2]}} \frac{(y - 1)\,(2 - y) + x\,(1 - x)}{(y - x)^{2\alpha}} \right\} \right|^{\frac{1}{2}} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \, \big| \tfrac{T}{N} \big|^{\left(\frac{1}{2} - \alpha \right)} \left[\max \left\{ \sup_{x \in (0,1]} \frac{x\,(1 - x)}{x^{2\alpha}}, \sup_{\substack{x \in [0,1], \\ y \in (0,1]}} \frac{y\,(2 - (y + 1)) + x\,(1 - x)}{([y + 1] - [1 - x])^{2\alpha}} \right\} \right]^{\frac{1}{2}} .
\end{aligned}
\tag{2.67}
$$

Hence, we obtain for all $N \in \mathbb{N}$, $\alpha \in [0, 1/2]$, $p \in [1, \infty)$ that

$$
\begin{aligned}
&\big| W - W^N \big|_{C^\alpha([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)})} \\
&= \tfrac{\|W_T\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)}}{\sqrt{T}} \, \big| \tfrac{T}{N} \big|^{\left(\frac{1}{2} - \alpha \right)} \left[\max \left\{ \sup_{y \in (0,1]} \frac{y\,(1 - y)}{y^{2\alpha}}, \sup_{x \in [0,1]} \sup_{y \in (0,1]} \frac{x\,(1 - x) + y\,(1 - y)}{(x + y)^{2\alpha}} \right\} \right]^{\frac{1}{2}}
\end{aligned}
$$

$$= \frac{\|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)}}{\sqrt{T}} \left|\tfrac{T}{N}\right|^{(\frac{1}{2}-\alpha)} \left[\sup_{x\in[0,1]} \sup_{y\in(0,1]} \frac{x\,(1-x)+y\,(1-y)}{(x+y)^{2\alpha}} \right]^{1/2} \tag{2.68}$$
$$= N^{(\alpha-1/2)}\, T^{-\alpha}\, \|W_T\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \left[\sup_{x,y\in(0,1)} g_\alpha(x,y) \right]^{1/2}.$$

To complete the proof of (2.58), we study a few properties of the functions g_α, $\alpha \in [0, 1/2]$. In particular, note that

$$\sup_{x,y\in(0,1)} g_{\frac{1}{2}}(x,y) = \tfrac{x\,(1-x)}{(x+y)} + \tfrac{y\,(1-y)}{(x+y)} \leq \tfrac{x}{(x+y)} + \tfrac{y}{(x+y)} = 1 = \sup_{x\in(0,1)} \tfrac{2x\,(1-x)}{2x} = \sup_{x\in(0,1)} g_{\frac{1}{2}}(x,x). \tag{2.69}$$

Moreover, note that for all $\alpha \in [0, 1/2)$, $x, y \in [0,1]$ with $x + y > 0$ it holds that

$$\begin{aligned} |g_\alpha(x,y) - g_\alpha(0,0)| &= \left| \tfrac{x\,(1-x)}{(x+y)^{2\alpha}} + \tfrac{y\,(1-y)}{(x+y)^{2\alpha}} \right| \leq x^{(1-2\alpha)}\,(1-x) + y^{(1-2\alpha)}\,(1-y) \\ &\leq 2\,|\max\{x,y\}|^{(1-2\alpha)}. \end{aligned} \tag{2.70}$$

Hence, we obtain for all $\alpha \in [0, 1/2)$ that $g_\alpha\colon [0,1]^2 \to \mathbb{R}$ is continuous. Next note that the fact that for all $\alpha \in [0, 1/2)$, $x \in (0,1)$ it holds that

$$\tfrac{\partial}{\partial x}\big(x^{(1-2\alpha)}\,(1-x)\big) = (1-2\alpha)\,x^{-2\alpha}\,(1-x) - x^{(1-2\alpha)} = -2\,(1-\alpha)\,\big[x - \tfrac{1/2-\alpha}{1-\alpha}\big]\,x^{-2\alpha} \tag{2.71}$$

ensures that for all $\alpha \in [0, 1/2)$ it holds that

$$\begin{aligned} &\sup_{(x,y)\in\partial([0,1]^2)} g_\alpha(x,y) \leq \sup_{x\in(0,1]} \left[\tfrac{x\,(1-x)}{x^{2\alpha}}\right] = \sup_{x\in(0,1)} \left[x^{(1-2\alpha)}\,(1-x)\right] \\ &= \left[\tfrac{1/2-\alpha}{1-\alpha}\right]^{(1-2\alpha)} \left[1 - \tfrac{1/2-\alpha}{1-\alpha}\right] = \tfrac{1}{2}\left[\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{(1-\alpha)^{(1-\alpha)}}\right]^2 < 2^{(1-2\alpha)} \cdot \tfrac{1}{2} \cdot \left[\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{(1-\alpha)^{(1-\alpha)}}\right]^2 \\ &= 2^{(1-2\alpha)} \cdot \sup_{x\in(0,1)} \left[x^{(1-2\alpha)}\,(1-x)\right] = \sup_{x\in(0,1)} \left[\tfrac{2x\,(1-x)}{(2x)^{2\alpha}}\right] \\ &= \sup_{x\in(0,1)} g_\alpha(x,x) = 2^{-2\alpha} \cdot \left[\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{(1-\alpha)^{(1-\alpha)}}\right]^2 = \left[\tfrac{(1/2-\alpha)^{(1/2-\alpha)}}{2^\alpha\,(1-\alpha)^{(1-\alpha)}}\right]^2 = [f(\alpha)]^2. \end{aligned} \tag{2.72}$$

This ensures that for all $\alpha \in [0, 1/2)$ there exist $(x_0, y_0) \in (0,1)^2$ such that

$$g_\alpha(x_0, y_0) = \sup\nolimits_{x,y\in[0,1]} g_\alpha(x,y). \tag{2.73}$$

Moreover, note that for all $\alpha \in [0, 1/2)$, $k \in \{1,2\}$, $x_1, x_2 \in (0,1)$ it holds that

$$\begin{aligned} \big(\tfrac{\partial}{\partial x_k} g_\alpha\big)(x_1,x_2) &= \frac{\partial}{\partial x_k}\left(\frac{x_1\,(1-x_1) + x_2\,(1-x_2)}{(x_1+x_2)^{2\alpha}}\right) \\ &= \frac{(1-2x_k)}{(x_1+x_2)^{2\alpha}} - \frac{2\alpha\,[x_1\,(1-x_1) + x_2\,(1-x_2)]}{(x_1+x_2)^{2\alpha+1}}. \end{aligned} \tag{2.74}$$

This implies that for all $\alpha \in [0, 1/2)$, $x, y \in (0,1)$ with $\frac{\partial}{\partial x} g_\alpha(x,y) = \frac{\partial}{\partial y} g_\alpha(x,y) = 0$ it holds that $x = y$. This and (2.73) ensure for all $\alpha \in [0, 1/2)$ that

$$\sup_{x,y\in[0,1]^2} g_\alpha(x,y) = \sup_{x\in(0,1)} g_\alpha(x,x). \tag{2.75}$$

This and (2.72) show for all $\alpha \in [0, 1/2)$ that

$$\sup_{x,y\in[0,1]^2} g_\alpha(x,y) = \sup_{x\in(0,1)} g_\alpha(x,x) = \big[f(\alpha)\big]^2. \tag{2.76}$$

This and (2.69) prove that for all $\alpha \in [0, 1/2]$ it holds that

$$\sup_{x,y\in[0,1]^2} g_\alpha(x,y) = \sup_{x\in(0,1)} g_\alpha(x,x) = [f(\alpha)]^2 . \tag{2.77}$$

Next note that for all $\alpha \in (0, 1/2)$ it holds that

$$f(\alpha) = \exp\big(\big(\tfrac{1}{2} - \alpha\big) \cdot \ln\big(\tfrac{1}{2} - \alpha\big) + (\alpha - 1) \cdot \ln(1 - \alpha) - \alpha \cdot \ln(2)\big) . \tag{2.78}$$

Moreover, observe that for all $\alpha \in (0, 1/2)$ it holds that

$$\begin{aligned} &\tfrac{\partial}{\partial\alpha} \big(\big(\tfrac{1}{2} - \alpha\big) \cdot \ln\big(\tfrac{1}{2} - \alpha\big) + (\alpha - 1) \cdot \ln(1 - \alpha) - \alpha \cdot \ln(2)\big) \\ &= -\ln\big(\tfrac{1}{2} - \alpha\big) + 1 + \ln(1 - \alpha) - 1 - \ln(2) = \ln(1 - \alpha) - \ln\big(\tfrac{1}{2} - \alpha\big) - \ln(2) \\ &= \ln\big(\tfrac{1-\alpha}{1-2\alpha}\big) > 0. \end{aligned} \tag{2.79}$$

This and (2.78) ensure that f is strictly increasing. Equation (2.77) hence proves that

$$\sup_{x,y\in[0,1]^2} g_\alpha(x,y) = \sup_{x\in(0,1)} g_\alpha(x,x) = [f(\alpha)]^2 \in \Big[|f(0)|^2, \big|f(\tfrac{1}{2})\big|^2\Big] = \big[\tfrac{1}{2}, 1\big] . \tag{2.80}$$

Putting this into (2.68) establishes (2.58). Combining (2.57) with (2.58) proves (2.59). Moreover, (2.58) and Lemma 2.13 imply (2.60). The proof of Proposition 2.14 is thus completed. □

3 Convergence of Galerkin approximations in Hölder norms

3.1 Setting

Throughout this section the following setting is frequently used. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $T, \iota \in (0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathscr{F}_t)_{t\in[0,T]}$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathscr{F}_t)_{t\in[0,T]}$-Wiener process, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function with $\sup_{h\in\mathbb{H}} \lambda_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator such that

$$D(A) = \Big\{ v \in H \colon \sum_{h\in\mathbb{H}} |\lambda_h \langle h, v\rangle_H|^2 < \infty \Big\} \tag{3.1}$$

and such that for all $v \in D(A)$ it holds that

$$Av = \sum_{h\in\mathbb{H}} \lambda_h \langle h, v\rangle_H h, \tag{3.2}$$

let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Definition 3.5.25 in [20]), let $\gamma \in \mathbb{R}$, $\alpha \in [0,1)$, $\beta \in [0, 1/2)$, $\chi \in [\beta, 1/2)$, $F \in C(H_\gamma, H_{\gamma-\alpha})$, $B \in C(H_\gamma, HS(U, H_{\gamma-\beta}))$ satisfy that for all bounded sets $E \subseteq H_\gamma$ it holds that

$$|F|_E|_{C^1(E,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_E|_{C^1(E,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} < \infty, \tag{3.3}$$

let $\mathbb{H}_N \subseteq \mathbb{H}$, $N \in \mathbb{N}_0$, be sets satisfying $\mathbb{H}_0 = \mathbb{H}$ and $\sup_{N\in\mathbb{N}} N^\iota \sup\big(\{1/|\lambda_h| \colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big) < \infty$, let $P_N \in L(H_{\min\{0,\gamma-1\}})$, $N \in \mathbb{N}_0$, and $\mathscr{P}_N \in L(U)$, $N \in \mathbb{N}_0$, be linear operators satisfying for all $N \in \mathbb{N}_0$, $v \in H$ that

$$P_N(v) = \sum_{h\in\mathbb{H}_N} \langle h, v\rangle_H \, h, \tag{3.4}$$

and let $X^N\colon [0,T]\times\Omega\to H_\gamma$, $N\in\mathbb{N}_0$, be $(\mathscr{F}_t)_{t\in[0,T]}$-adapted stochastic processes with continuous sample paths satisfying for all $N\in\mathbb{N}_0$, $t\in[0,T]$ that

$$\big[X_t^N\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} = \left[e^{tA}P_N X_0^0 + \int_0^t e^{(t-s)A}P_N F(X_s^N)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A}P_N B(X_s^N)\mathscr{P}_N\,\mathrm{d}W_s. \tag{3.5}$$

3.2 Strong convergence of Galerkin approximations in Hölder norms for SEEs with globally Lipschitz continuous nonlinearities

The next lemma, Lemma 3.1 below, follows directly from, e.g., Proposition 7.1.16 in [20] and, e.g., Proposition 7.1.7 in [20].

Lemma 3.1. *Assume the setting in Subsection 3.1, let $p\in[2,\infty)$, $\eta\in\big[\max\{\alpha,2\beta\},1\big)$, $N\in\mathbb{N}_0$, and assume that*

$$\mathbb{E}\big[\|X_0^0\|_{H_\gamma}^p\big] + |F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} < \infty. \tag{3.6}$$

Then

$$\begin{aligned}\sup_{t\in[0,T]}\big\|\max\{1,\|X_t^N\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} &\leq \sqrt{2}\,\big\|\max\{1,\|X_0^0\|_{H_\gamma}\}\big\|_{\mathscr{L}^p(\mathbb{P};|\cdot|)} \\ &\cdot \mathscr{E}_{(1-\eta)}\left[\tfrac{T^{1-\eta}\sqrt{2}}{\sqrt{1-\eta}}\left(\sup_{v\in H_\gamma}\tfrac{\|F(v)\|_{H_{\gamma-\eta}}}{\max\{1,\|v\|_{H_\gamma}\}}\right) + \sqrt{T^{1-\eta}p(p-1)}\left(\sup_{v\in H_\gamma}\tfrac{\|B(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\eta/2})}}{\max\{1,\|v\|_{H_\gamma}\}}\right)\right].\end{aligned} \tag{3.7}$$

Lemma 3.2. *Assume the setting in Subsection 3.1, let $p\in[2,\infty)$, $\eta\in\big[\max\{\alpha,2\beta\},1\big)$, $N\in\mathbb{N}_0$, and assume that*

$$\mathbb{E}\big[\|X_0^0\|_{H_\gamma}^p\big] + |F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} < \infty. \tag{3.8}$$

Then

$$\begin{aligned}\sup_{t\in[0,T]}\big\|X_t^0 - X_t^N\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} &\leq \Bigg[\sqrt{2}\sup_{t\in[0,T]}\big\|(P_0-P_N)X_t^0\big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\ &+ \tfrac{T^{1/2-\chi}\sqrt{p\,(p-1)}}{\sqrt{1-2\chi}}\left(1+\sup_{t\in[0,T]}\|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right)\left(\sup_{v\in H_\gamma}\frac{\|B(v)(\mathscr{P}_0-\mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}}\right)\Bigg] \\ &\cdot \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right] < \infty.\end{aligned} \tag{3.9}$$

Proof of Lemma 3.2. First of all, observe that Lemma 3.1 ensures that

$$\sup_{t\in[0,T]}\max\big\{\|X_t^0\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})},\|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big\} < \infty. \tag{3.10}$$

We can hence apply Proposition 7.1.4 in [20] to obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|P_N F(\cdot)|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|P_N B(\cdot)\mathscr{P}_0|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\eta/2})})}\right] \\
&\quad \cdot \sqrt{2} \sup_{t\in[0,T]} \Bigg\| \left[X_t^0 - \int_0^t e^{(t-s)A} P_N F(X_s^0)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} - \int_0^t e^{(t-s)A} P_N B(X_s^0)\mathscr{P}_0\,\mathrm{d}W_s \\
&\quad + \left[\int_0^t e^{(t-s)A} P_N F(X_s^N)\,\mathrm{d}s - X_t^N\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N)\mathscr{P}_0\,\mathrm{d}W_s \Bigg\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}.
\end{aligned} \tag{3.11}
$$

This shows

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right] \\
&\quad \cdot \sqrt{2} \sup_{t\in[0,T]} \left\| \left[(P_0 - P_N)X_t^0\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B(X_s^N)(\mathscr{P}_0 - \mathscr{P}_N)\,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};\|\cdot\|_{H_\gamma})}.
\end{aligned} \tag{3.12}
$$

The Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [6] hence implies

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\| X_t^0 - X_t^N \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \\
&\leq \mathscr{E}_{(1-\eta)}\left[\frac{T^{1-\eta}\sqrt{2}\,|F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\eta}})}}{\sqrt{1-\eta}} + \sqrt{T^{1-\eta}p(p-1)}\,|B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\eta/2})})}\,\|\mathscr{P}_0\|_{L(U)}\right] \\
&\quad \cdot \sqrt{2}\left[\sup_{t\in[0,T]} \left\|(P_0 - P_N)X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right. \\
&\quad \left. + \sup_{s\in[0,T]} \left\|B(X_s^N)[\mathscr{P}_0 - \mathscr{P}_N]\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\chi})})} \sqrt{\tfrac{p\,(p-1)}{2} \sup_{t\in[0,T]} \int_0^t (t-s)^{-2\chi}\,\mathrm{d}s}\,\right].
\end{aligned} \tag{3.13}
$$

This and (3.10) complete the proof of Proposition 3.2. □

Corollary 3.3. *Assume the setting in Subsection 3.1, let* $\vartheta \in \big[0, \min\{1-\alpha, 1/2 - \beta\}\big)$, $p \in [2,\infty)$, *and assume that* $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ *and*

$$
\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] + |F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} < \infty, \tag{3.14}
$$

$$
\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{N^{\iota\vartheta}\,\|B(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}}\right] < \infty. \tag{3.15}
$$

Then it holds that

$$
\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(\|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})} + \|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\chi})})}\big) < \infty \tag{3.16}
$$

and

$$
\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(N^{\iota\vartheta}\,\|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big) < \infty. \tag{3.17}
$$

Proof of Corollary 3.3. Combining the assumptions that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ and $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$ with, e.g., Proposition 7.1.16 in [20] and, e.g., Proposition 7.1.7 in [20] ensures that $\forall\, t \in [0,T]\colon \mathbb{P}\big(X_t^0 \in H_{\gamma+\vartheta}\big) = 1$ and $\sup_{t\in[0,T]} \mathbb{E}\big[\|\mathbb{1}_{\{X_t^0 \in H_{\gamma+\vartheta}\}} X_t^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$. This and the assumption that $\sup_{N\in\mathbb{N}} N^\iota \sup\!\big(\{1/|\lambda_h|\colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big) < \infty$ imply that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left[N^{\iota\vartheta} \left\|(P_0 - P_N) X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right] \\
&\leq \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left[N^{\iota\vartheta} \left\|(-A)^{-\vartheta}(P_0 - P_N)\right\|_{L(H_\gamma)} \left\|\mathbb{1}_{\{X_t^0 \in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} \right] \\
&\leq \left[\sup_{N\in\mathbb{N}} N^{\iota\vartheta} \left\|(-A)^{-1}(\mathrm{Id}_{H_\gamma} - P_N)\right\|_{L(H_\gamma)}^{\vartheta} \right] \left[\sup_{t\in[0,T]} \left\|\mathbb{1}_{\{X_t^0 \in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} \right] \\
&= \left[\sup_{N\in\mathbb{N}} N^{\iota\vartheta} \left[\sup\!\big(\{1/|\lambda_h|\colon h \in \mathbb{H}\backslash\mathbb{H}_N\} \cup \{0\}\big)\right]^{\vartheta} \right] \left[\sup_{t\in[0,T]} \left\|\mathbb{1}_{\{X_t^0 \in H_{\gamma+\vartheta}\}} X_t^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} \right] \\
&< \infty.
\end{aligned} \tag{3.18}$$

In addition, observe that (3.15) and Lemma 3.1 ensure that

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} < \infty. \tag{3.19}$$

The triangle inequality and (3.15) hence prove that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|B(X_t^N)\mathscr{P}_N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\chi})})} \\
&\leq \left(1 + \sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \left\|X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right) \left(\sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \tfrac{\|B(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}} \right) \\
&\leq \left(1 + \sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \left\|X_t^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\right) \\
&\quad \cdot \left(\sup_{v\in H_\gamma} \tfrac{\|B(v)\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}} + \sup_{N\in\mathbb{N}_0} \sup_{v\in H_\gamma} \tfrac{\|B(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}} \right) < \infty.
\end{aligned} \tag{3.20}$$

In the next step we combine (3.19), (3.18), and (3.15) with Lemma 3.2 to obtain that

$$\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \big(N^{\iota\vartheta} \, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}\big) < \infty. \tag{3.21}$$

Furthermore, observe that (3.19) assures that $\sup_{N\in\mathbb{N}_0} \sup_{t\in[0,T]} \|F(X_t^N)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\alpha}})} < \infty$. This, (3.20), and (3.21) complete the proof of Corollary 3.3. □

Corollary 3.4. *Assume the setting in Subsection 3.1, let $\vartheta \in \big(0, \min\{1-\alpha, 1/2 - \beta\}\big)$, $p \in (1/\vartheta, \infty)$, and assume that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$, $\mathbb{E}\big[\|X_0^0\|_{H_{\gamma+\vartheta}}^p\big] < \infty$, $|F|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} < \infty$, $|B|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} < \infty$, and*

$$\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{\|B(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \, \|B(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}} \right] < \infty. \tag{3.22}$$

Then for all $\delta \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$ it holds that

$$\sup_{N\in\mathbb{N}} \big(\|X^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})})} + N^{\iota\,(\vartheta-\delta-1/p-\varepsilon)} \, \|X^0 - X^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})})}\big) < \infty. \tag{3.23}$$

Proof of Corollary 3.4. Throughout this proof let $\eta \in \mathbb{R}$ be the real number given by $\eta = \max\{\alpha, 2\beta\}$ and let $\theta^N \in \mathscr{P}_T$, $N \in \mathbb{N}$, be a sequence of sets such that

$$\sup_{N\in\mathbb{N}} \left[\frac{d_{\max}(\theta^N)}{N^{-\iota}} + \frac{N^{-\iota}}{d_{\min}(\theta^N)} \right] < \infty. \tag{3.24}$$

In particular, this ensures that $\limsup_{N\to\infty} d_{\max}(\theta^N) = 0$. In addition, Corollary 3.3 proves that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \left[\left|d_{\max}(\theta^N)\right|^{-\vartheta} \sup_{t\in\theta^N} \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right] \\ &\leq \left[\sup_{N\in\mathbb{N}} \tfrac{\left|d_{\max}(\theta^N)\right|^{-\vartheta}}{N^{\iota\vartheta}} \right] \left(\sup_{N\in\mathbb{N}} \sup_{t\in\theta^N} N^{\iota\vartheta} \, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right) \\ &\leq \left[\sup_{N\in\mathbb{N}} \tfrac{N^{-\iota}}{d_{\min}(\theta^N)} \right]^{\vartheta} \left(\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} N^{\iota\vartheta} \, \|X_t^0 - X_t^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})} \right) < \infty. \end{aligned} \tag{3.25}$$

In the next step note that, e.g., Corollary 11.3.2 in [20] shows that for all $N \in \mathbb{N}_0$, $\varepsilon \in \big(0, \min\{1+\gamma-\eta, 1/2+\gamma-\beta\} - \gamma\big)$ it holds that

$$\sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{|\min\{t_1,t_2\}|^{\max\{\gamma+\varepsilon-(\gamma+\vartheta),0\}} \left\|X_{t_1}^N - X_{t_2}^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{|t_1-t_2|^{\varepsilon}} \right) \tag{3.26}$$

$$\begin{aligned} &\leq \left\|X_0^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\min\{\gamma+\vartheta,\gamma+\varepsilon\}}})} + \left[\sup_{s\in[0,T]} \left\|P_N F(X_s^N)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1+\gamma-\eta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-\eta-\varepsilon)} \\ &\quad + \left[\sup_{s\in[0,T]} \left\|P_N B(X_s^N)\mathscr{P}_N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\beta})})} \right] \frac{\sqrt{2}\,T^{(1/2+\gamma-\beta-\min\{\gamma+\vartheta,\gamma+\varepsilon\})}}{(1-2\beta-2\varepsilon)^{1/2}} < \infty. \end{aligned}$$

This and the fact that $\min\{1+\gamma-\eta, 1/2+\gamma-\beta\} - \gamma = \min\{1-\max\{\alpha,2\beta\}, 1/2-\beta\} = \min\{1-\alpha, 1/2-\beta\} > \vartheta > 0$ imply that

$$\begin{aligned} &\sup_{N\in\mathbb{N}_0} \sup_{\substack{t_1,t_2\in[0,T],\\ t_1\neq t_2}} \left(\frac{\left\|X_{t_1}^N - X_{t_2}^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})}}{|t_1-t_2|^{\vartheta}} \right) \\ &\leq \sup_{N\in\mathbb{N}_0} \left\|X_0^N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \left\|F(X_s^N)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \frac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)} \\ &\quad + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \left\|B(X_s^N)\mathscr{P}_N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\beta})})} \right] \frac{\sqrt{2}\,T^{(1/2-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{1/2}}. \end{aligned} \tag{3.27}$$

Corollary 3.3 and estimate (3.22) hence prove that

$$\begin{aligned} &\sup_{N\in\mathbb{N}_0} \left|X^N\right|_{C^{\vartheta}([0,T],\|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_\gamma})})} \\ &\leq \left\|X_0^0\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma+\vartheta}})} + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \left\|F(X_s^N)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{H_{\gamma-\eta}})} \right] \tfrac{2\,T^{(1-\eta-\vartheta)}}{(1-\eta-\vartheta)} \\ &\quad + \left[\sup_{N\in\mathbb{N}_0} \sup_{s\in[0,T]} \left\|B(X_s^N)\mathscr{P}_N\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{HS(U,H_{\gamma-\beta})})} \right] \tfrac{\sqrt{2}\,T^{(1/2-\beta-\vartheta)}}{(1-2\beta-2\vartheta)^{1/2}} < \infty. \end{aligned} \tag{3.28}$$

This, (3.25), and the fact that $\vartheta \in (1/p, 1]$ allow us to apply Corollary 2.11 to obtain for all $\delta \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$ that

$$\sup_{N\in\mathbb{N}} \Big[\|X^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})})} + \big|d_{\max}(\theta^N)\big|^{-(\vartheta-\delta-1/p-\varepsilon)} \|X^0 - X^N\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})})} \Big] < \infty. \tag{3.29}$$

Combining this with the fact that $\sup_{N\in\mathbb{N}} \big[\frac{d_{\max}(\theta^N)}{N^{-\iota}}\big] < \infty$ completes the proof of Corollary 3.4. ☐

3.3 Almost sure convergence of Galerkin approximations in Hölder norms for SEEs with semi-globally Lipschitz continuous nonlinearities

The proof of the following corollary employs a standard localisation argument; see, e.g., [9, 25].

Corollary 3.5. *Assume the setting in Subsection 3.1, let $\vartheta \in \big(0, \min\{1-\alpha, 1/2 - \beta\}\big)$, assume that $\mathbb{P}\big(X_0^0 \in H_{\gamma+\vartheta}\big) = 1$, and assume for all bounded sets $E \subseteq H_\gamma$ that*

$$\sup_{N\in\mathbb{N}} \sup_{v\in E} \left[\frac{\|B(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \|B(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1 + \|v\|_{H_\gamma}} \right] < \infty. \tag{3.30}$$

Then for all $\delta \in [0,\vartheta)$, $\varepsilon \in (0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$\sup_{N\in\mathbb{N}} \big[N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0 - X^N\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \big] < \infty. \tag{3.31}$$

Proof of Corollary 3.5. Throughout this proof we assume w.l.o.g. that $X_0^0(\Omega) \subseteq H_{\gamma+\vartheta}$ and throughout this proof let $\delta \in [0,\vartheta)$, let $\phi_{r,M} \colon H_r \to H_r$, $r \in \mathbb{R}$, $M \in (0,\infty)$, be the mappings which satisfy for all $r \in \mathbb{R}$, $M \in (0,\infty)$, $v \in H_r$ that

$$\phi_{r,M}(v) = v \cdot \min\!\left\{1, \frac{M+1}{1+\|v\|_{H_r}}\right\}, \tag{3.32}$$

let $\xi_M \colon \Omega \to H_\gamma$, $M \in \mathbb{N}$, be the mappings which satisfy for all $M \in \mathbb{N}$ that $\xi_M = \phi_{\gamma+\vartheta,M}(X_0^0)$, let $F_M \colon H_\gamma \to H_{\gamma-\alpha}$, $M \in \mathbb{N}$, and $B_M \colon H_\gamma \to HS(U, H_{\gamma-\beta})$, $M \in \mathbb{N}$, be the mappings which satisfy for all $M \in \mathbb{N}$ that $F_M = F \circ \phi_{\gamma,M}$ and $B_M = B \circ \phi_{\gamma,M}$, and let $S_M \subseteq H_\gamma$, $M \in \mathbb{N}$, be the sets which satisfy for all $M \in \mathbb{N}$ that $S_M = \{v \in H_\gamma \colon \|v\|_{H_\gamma} \leq M+1\}$. Observe that for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\|_{H_\gamma} \\
&= \left\| \frac{v\,(1+\|w\|_{H_\gamma})\min\{1+\|v\|_{H_\gamma}, M+1\} - w\,(1+\|v\|_{H_\gamma})\min\{1+\|w\|_{H_\gamma}, M+1\}}{(1+\|v\|_{H_\gamma})\,(1+\|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v-w\|_{H_\gamma} \\
&\quad + \left\| \frac{w\,\big[(1+\|w\|_{H_\gamma})\min\{1+\|v\|_{H_\gamma}, M+1\} - (1+\|v\|_{H_\gamma})\min\{1+\|w\|_{H_\gamma}, M+1\}\big]}{(1+\|v\|_{H_\gamma})\,(1+\|w\|_{H_\gamma})} \right\|_{H_\gamma} \\
&\leq \|v-w\|_{H_\gamma} \\
&\quad + \frac{\big|(1+\|w\|_{H_\gamma})\min\{1+\|v\|_{H_\gamma}, M+1\} - (1+\|v\|_{H_\gamma})\min\{1+\|w\|_{H_\gamma}, M+1\}\big|}{(1+\|v\|_{H_\gamma})}.
\end{aligned} \tag{3.33}$$

This ensures that for all $v, w \in H_\gamma$, $M \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\|\phi_{\gamma,M}(v) - \phi_{\gamma,M}(w)\|_{H_\gamma} \\
&\leq \|v-w\|_{H_\gamma} + \frac{\big|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\big| \min\{1+\|v\|_{H_\gamma}, M+1\}}{(1+\|v\|_{H_\gamma})} \\
&\quad + \frac{(1+\|v\|_{H_\gamma})\,\big|\min\{1+\|v\|_{H_\gamma}, M+1\} - \min\{1+\|w\|_{H_\gamma}, M+1\}\big|}{(1+\|v\|_{H_\gamma})} \\
&\leq \|v-w\|_{H_\gamma} + \big|\|w\|_{H_\gamma} - \|v\|_{H_\gamma}\big| \\
&\quad + \big|\min\{1+\|v\|_{H_\gamma}, M+1\} - \min\{1+\|w\|_{H_\gamma}, M+1\}\big| \\
&\leq 3\,\|v-w\|_{H_\gamma}\,.
\end{aligned} \tag{3.34}$$

Hence, we obtain for all $M \in \mathbb{N}$ that $|\phi_{\gamma,M}|_{C^1(H_\gamma,\|\cdot\|_{H_\gamma})} \leq 3$. This, the fact that $\forall\, M \in \mathbb{N}\colon |F|_{S_M}|_{C^1(S_M,\|\cdot\|_{H_{\gamma-\alpha}})} + |B|_{S_M}|_{C^1(S_M,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} + |\phi_{\gamma,M}|_{C^1(H_\gamma,\|\cdot\|_{H_\gamma})} < \infty$, and the fact that $\forall\, M \in \mathbb{N}\colon \phi_{\gamma,M}(H_\gamma) \subseteq S_M$ ensure that for all $M \in \mathbb{N}$, $p \in [1,\infty)$ it holds that

$$|F_M|_{C^1(H_\gamma,\|\cdot\|_{H_{\gamma-\alpha}})} + |B_M|_{C^1(H_\gamma,\|\cdot\|_{HS(U,H_{\gamma-\beta})})} + \mathbb{E}\big[\|\xi_M\|^p_{H_{\gamma+\vartheta}}\big] < \infty. \tag{3.35}$$

Therefore, there exist $(\mathscr{F}_t)_{t\in[0,T]}$-adapted stochastic processes $\mathscr{X}^{N,M}\colon [0,T]\times\Omega \to H_\gamma$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, with continuous sample paths such that for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0,T]$ it holds that

$$\begin{aligned}
\big[\mathscr{X}^{N,M}_t\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} &= \left[e^{tA} P_N \xi_M + \int_0^t e^{(t-s)A} P_N F_M(\mathscr{X}^{N,M}_s)\,\mathrm{d}s\right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\
&\quad + \int_0^t e^{(t-s)A} P_N B_M(\mathscr{X}^{N,M}_s) \mathscr{P}_N \,\mathrm{d}W_s
\end{aligned} \tag{3.36}$$

(cf., e.g., Proposition 7.1.16 in [20] or Theorem 6.1 in van Neerven, Veraar & Weis [27]). We now introduce a bit more notation. Let $\tau_{N,M}\colon \Omega \to [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$, be the mappings which satisfy for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\tau_{N,M} = \min\Big\{T\,\mathbb{1}_{\{\|X^0_0\|_{H_{\gamma+\vartheta}}\leq M\}}, \inf\Big(\big\{t \in [0,T]\colon \|\mathscr{X}^{N,M}_t\|_{H_\gamma} \geq M\big\} \cup \{T\}\Big)\Big\}, \tag{3.37}$$

let $\varUpsilon \in \mathscr{F}$ be the set given by

$$\begin{aligned}
&\varUpsilon = \\
&\big[\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\dots\}} \{\tau_{N,m} = T\}\big] \cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big\{\|\mathscr{X}^{N,M}\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty\Big\}\Big] \\
&\cap \Big[\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big(\big\{\|X^0_0\|_{H_{\gamma+\vartheta}} > M\big\} \cup \big\{\forall\, t \in [0,T]\colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\big\}\Big)\Big] \\
&\cap \Big[\cap_{M,n\in\mathbb{N}} \Big\{\sup_{N\in\mathbb{N}} \big(N^{\iota(\vartheta-\delta-1/n)}\,\|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})}\big) < \infty\Big\}\Big]\,,
\end{aligned} \tag{3.38}$$

let $\mathscr{M}\colon \varUpsilon \to \mathbb{N}$ be the mapping which satisfies for all $\omega \in \varUpsilon$ that

$$\mathscr{M}(\omega) = \min\big\{M \in \mathbb{N}\cap\big(\|X^0_0(\omega)\|_{H_{\gamma+\vartheta}}, \infty\big)\colon \forall\, m \in \{M, M+1, \dots\}\colon \tau_{0,m}(\omega) = T\big\}, \tag{3.39}$$

and let $\mathscr{N}\colon \varUpsilon \to \mathbb{N}$ be the mapping which satisfies for all

$$\omega \in \varUpsilon \subseteq \Big\{\omega \in \Omega\colon \Big[\forall\, M \in \mathbb{N}\colon \limsup_{N\to\infty} \|\mathscr{X}^{0,M}(\omega) - \mathscr{X}^{N,M}(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} = 0\Big]\Big\} \tag{3.40}$$

that

$$\mathscr{N}(\omega) = \min\Big\{N \in \mathbb{N} \colon \sup_{n \in \{N,N+1,\dots\}} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{n,2\mathscr{M}(\omega)}(\omega)\|_{C([0,T],\|\cdot\|_{H_\gamma})} < 1\Big\}. \tag{3.41}$$

Observe that (3.38) ensures that for all $\omega \in \varUpsilon$, $N \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0, \tau_{N,M}(\omega)]$ with $M \geq \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}$ it holds that

$$\mathscr{X}_t^{N,M}(\omega) = X_t^N(\omega). \tag{3.42}$$

This, the fact that $\forall\, \omega \in \varUpsilon, N \in \mathbb{N}_0 \colon \exists\, M \in \mathbb{N} \colon \forall\, m \in \{M, M+1, \dots\} \colon \tau_{N,m}(\omega) = T$, and the fact that $\forall\, \omega \in \varUpsilon, N \in \mathbb{N}_0, m \in \mathbb{N} \colon \|\mathscr{X}^{N,m}(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty$ prove that for all $\omega \in \varUpsilon$, $N \in \mathbb{N}_0$ it holds that

$$\|X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty. \tag{3.43}$$

Next note that (3.39) ensures that for all $\omega \in \varUpsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \dots\}$ it holds that

$$\tau_{0,M}(\omega) = T \qquad \text{and} \qquad M \geq \mathscr{M}(\omega) > \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}}. \tag{3.44}$$

This and (3.42) show that for all $\omega \in \varUpsilon$, $M \in \{\mathscr{M}(\omega), \mathscr{M}(\omega)+1, \dots\}$, $t \in [0,T]$ it holds that

$$\mathscr{X}_t^{0,M}(\omega) = X_t^0(\omega) = \mathscr{X}_t^{0,\mathscr{M}(\omega)}(\omega). \tag{3.45}$$

This, (3.44), and (3.37) prove that for all $\omega \in \varUpsilon$ it holds that

$$\sup_{t \in [0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}\|_{H_\gamma} = \sup_{t \in [0,T]} \|\mathscr{X}_t^{0,\mathscr{M}(\omega)}\|_{H_\gamma} \leq \mathscr{M}(\omega). \tag{3.46}$$

The triangle inequality and (3.41) hence assure that for all $\omega \in \varUpsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \dots\}$ it holds that

$$\begin{aligned} &\sup_{t \in [0,T]} \|\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &\leq \sup_{t \in [0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + \sup_{t \in [0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} \\ &< \sup_{t \in [0,T]} \|\mathscr{X}_t^{0,2\mathscr{M}(\omega)}(\omega)\|_{H_\gamma} + 1 \leq \mathscr{M}(\omega) + 1 \leq 2\mathscr{M}(\omega). \end{aligned} \tag{3.47}$$

This and the fact that $\forall\, \omega \in \varUpsilon \colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ prove for all $\omega \in \varUpsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \dots\}$ that $\tau_{N,2\mathscr{M}(\omega)} = T$. Again the fact that $\forall\, \omega \in \varUpsilon \colon \|X_0^0(\omega)\|_{H_{\gamma+\vartheta}} < \mathscr{M}(\omega) \leq 2\mathscr{M}(\omega)$ and (3.42) hence show for all $\omega \in \varUpsilon$, $N \in \{\mathscr{N}(\omega), \mathscr{N}(\omega)+1, \dots\}$, $t \in [0,T]$ that $\mathscr{X}_t^{N,2\mathscr{M}(\omega)}(\omega) = X_t^N(\omega)$. This and (3.45) prove for all $\omega \in \varUpsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned} &\sup_{N \in \mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq \sup_{N \in \{1,2,\dots,\mathscr{N}(\omega)\}} N^{\iota\vartheta} \|X^0(\omega) - X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\quad + \sup_{N \in \{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\dots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq [\mathscr{N}(\omega)]^{\iota\vartheta} \Big[\|X^0(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} + \sup_{N \in \{1,2,\dots,\mathscr{N}(\omega)\}} \|X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})}\Big] \\ &\quad + \sup_{N \in \{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\dots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})}. \end{aligned} \tag{3.48}$$

Combining this with (3.43) and (3.38) ensures for all $\omega \in \varUpsilon$, $\varepsilon \in (0,\infty)$ that

$$\begin{aligned} &\sup_{N \in \mathbb{N}} N^{\iota(\vartheta-\delta-\varepsilon)} \|X^0(\omega) - X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\leq [\mathscr{N}(\omega)]^{\iota\vartheta} \textstyle\sum_{N=0}^{\mathscr{N}(\omega)} \|X^N(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \\ &\quad + \sup_{N \in \{\mathscr{N}(\omega),\mathscr{N}(\omega)+1,\dots\}} N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,2\mathscr{M}(\omega)}(\omega) - \mathscr{X}^{N,2\mathscr{M}(\omega)}(\omega)\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} < \infty. \end{aligned} \tag{3.49}$$

It thus remains to prove that $\mathbb{P}(\varUpsilon) = 1$ to complete the proof of Corollary 3.5. For this observe that the assumption (3.30) shows that for all $M \in \mathbb{N}$ it holds that

$$\begin{aligned} &\sup_{N\in\mathbb{N}} \sup_{v\in H_\gamma} \left[\frac{\|B_M(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \|B_M(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}} \right] \\ &\leq \sup_{N\in\mathbb{N}} \sup_{v\in S_M} \left[\frac{\|B(v)\mathscr{P}_N\|_{HS(U,H_{\gamma-\beta})} + N^{\iota\vartheta} \|B(v)(\mathrm{Id}_U - \mathscr{P}_N)\|_{HS(U,H_{\gamma-\chi})}}{1+\|v\|_{H_\gamma}} \right] < \infty. \end{aligned} \tag{3.50}$$

We can hence apply Corollary 3.4 to obtain that for all $p \in (1/\vartheta, \infty)$, $r \in [0, \vartheta - 1/p)$, $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ it holds that

$$\sup_{N\in\mathbb{N}} \left(\|\mathscr{X}^{N,M}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^r([0,T],\|\cdot\|_{H_\gamma})})} + N^{\iota\,(\vartheta-r-\varepsilon)} \, \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^r([0,T],\|\cdot\|_{H_\gamma})})} \right) < \infty. \tag{3.51}$$

A standard Borel-Cantelli-type argument (see, e.g., Lemma 2.1 in Kloeden & Neuenkirch [21]) hence shows that for all $\varepsilon \in (0,\infty)$, $M \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$\sup\nolimits_{N\in\mathbb{N}} \left(N^{\iota(\vartheta-\delta-\varepsilon)} \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \right) < \infty. \tag{3.52}$$

Hence, we obtain

$$\mathbb{P}\Big(\forall\, M, n \in \mathbb{N} \colon \sup\nolimits_{N\in\mathbb{N}} \left[N^{\iota(\vartheta-\delta-1/n)} \, \|\mathscr{X}^{0,M} - \mathscr{X}^{N,M}\|_{C^\delta([0,T],\|\cdot\|_{H_\gamma})} \right] < \infty \Big) = 1. \tag{3.53}$$

In addition, observe that (3.51) proves that for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ it holds that $\mathbb{P}\big(\mathscr{X}^{N,M} \in C^\delta([0,T], \|\cdot\|_{H_\gamma})\big) = 1$. This, in turn, ensures that

$$\mathbb{P}\Big(\forall\, M \in \mathbb{N}, N \in \mathbb{N}_0 \colon \mathscr{X}^{N,M} \in C^\delta([0,T], \|\cdot\|_{H_\gamma}) \Big) = 1. \tag{3.54}$$

Next observe that for all $t \in [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ it holds that

$$\begin{aligned} &\big[\mathscr{X}_t^{N,M} - e^{tA} P_N \mathscr{X}_0^{0,M}\big]_{\mathbb{P},\mathscr{B}(H_\gamma)} \mathbb{1}_{\{t\leq\tau_{N,M}\}} \\ &= \left(\left[\int_0^t e^{(t-s)A} P_N F_M(\mathscr{X}_s^{N,M}) \,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} + \int_0^t e^{(t-s)A} P_N B_M(\mathscr{X}_s^{N,M}) \mathscr{P}_N \,\mathrm{d}W_s \right) \mathbb{1}_{\{t\leq\tau_{N,M}\}} \\ &= \Bigg(\left[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N F_M(\mathscr{X}_s^{N,M}) \,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &\qquad + \int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} e^{(t-s)A} P_N B_M(\mathscr{X}_s^{N,M}) \mathscr{P}_N \,\mathrm{d}W_s \Bigg) \mathbb{1}_{\{t\leq\tau_{N,M}\}} \\ &= \Bigg(\left[\int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N F(\mathscr{X}_s^{N,M}) \,\mathrm{d}s \right]_{\mathbb{P},\mathscr{B}(H_\gamma)} \\ &\qquad + \int_0^t \mathbb{1}_{\{s<\tau_{N,M}\}} \, e^{(t-s)A} P_N B(\mathscr{X}_s^{N,M}) \mathscr{P}_N \,\mathrm{d}W_s \Bigg) \mathbb{1}_{\{t\leq\tau_{N,M}\}}. \end{aligned} \tag{3.55}$$

Therefore, we obtain that for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ it holds that

$$\mathbb{P}\Big(\forall\, t \in [0,T] \colon \mathbb{1}_{\{\mathscr{X}_0^{N,M} = X_0^N\}} \mathscr{X}_{\min\{t,\tau_{N,M}\}}^{N,M} = \mathbb{1}_{\{\mathscr{X}_0^{N,M} = X_0^N\}} X_{\min\{t,\tau_{N,M}\}}^N \Big) = 1 \tag{3.56}$$

(cf., e.g., Proposition 7.1.7 in [20] or Lemma 8.2 in van Neerven, Veraar & Weis [27]). This implies that for all $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ it holds that

$$\mathbb{P}\big(\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\} \cup \big\{ \forall\, t \in [0,T] \colon \mathscr{X}_{\min\{t,\tau_{N,M}\}}^{N,M} = X_{\min\{t,\tau_{N,M}\}}^N \big\} \big) = 1. \tag{3.57}$$

Hence, we obtain

$$\mathbb{P}\Big(\cap_{M\in\mathbb{N},N\in\mathbb{N}_0} \Big[\{\|X_0^0\|_{H_{\gamma+\vartheta}} > M\} \cup \big\{\forall\, t \in [0,T] \colon \mathscr{X}^{N,M}_{\min\{t,\tau_{N,M}\}} = X^N_{\min\{t,\tau_{N,M}\}}\big\}\Big]\Big) = 1. \tag{3.58}$$

In the next step we combine this with (3.37) to obtain that for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ it holds $\mathbb{P}$-a.s. that

$$\tau_{N,M} = \min\Big\{T\, \mathbb{1}_{\{\|X_0^0\|_{H_{\gamma+\vartheta}}\leq M\}}, \inf\big(\big\{t \in [0,T] \colon \|X_t^N\|_{H_\gamma} \geq M\big\} \cup \{T\}\big)\Big\}. \tag{3.59}$$

This shows that for all $N \in \mathbb{N}_0$, $M_1, M_2 \in \mathbb{N}$ with $M_1 \leq M_2$ it holds that $\mathbb{P}\big(\tau_{N,M_1} \leq \tau_{N,M_2}\big) = 1$. This, (3.59), and the fact that $\forall\, \omega \in \Omega, N \in \mathbb{N}_0 \colon \sup_{t\in[0,T]} \|X_t^N(\omega)\|_{H_\gamma} < \infty$ imply that for all $N \in \mathbb{N}_0$ it holds that $\mathbb{P}\big(\cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\dots\}} \{\tau_{N,m} = T\}\big) = 1$. This, in turn, proves that

$$\mathbb{P}\big(\cap_{N\in\mathbb{N}_0} \cup_{M\in\mathbb{N}} \cap_{m\in\{M,M+1,\dots\}} \{\tau_{N,m} = T\}\big) = 1. \tag{3.60}$$

Combining (3.60), (3.54), (3.58), and (3.53) proves that $\mathbb{P}(\Upsilon) = 1$. The proof of Corollary 3.5 is thus completed. ☐

4 Cubature methods in Banach spaces

We first discuss in Section 4.1 a number of preliminary definitions related to the Monte Carlo method in Banach spaces. In Section 4.2 we present an elementary error estimate for the Monte Carlo method in Corollary 4.12. In Section 4.3 we then illustrate how expectations of Banach space valued functions of stochastic processes can be approximated.

4.1 Preliminaries

As mentioned in the introduction, the rate of convergence of Monte Carlo approximations in a Banach space depends on the so-called *type* of the Banach space; cf., e.g., Section 9.2 in Ledoux & Talagrand [23]. In order to define the type of a Banach space, we first reconsider a few concepts from the literature.

Definition 4.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let J be a set, and let $r_j \colon \Omega \to \{-1, 1\}$, $j \in J$, be a family of independent random variables with $\forall\, j \in J \colon \mathbb{P}\big(r_j = 1\big) = \mathbb{P}\big(r_j = -1\big)$. Then we say that $(r_j)_{j\in J}$ is a $\mathbb{P}$-Rademacher family.*

Definition 4.2. *Let $p \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\mathscr{T}_p(E) \in [0, \infty]$ the extended real number given by*

$$\mathscr{T}_p(E) = \sup\left(\left\{ r \in [0,\infty) \colon \begin{array}{c} \exists\, \text{probability space } (\Omega, \mathscr{F}, \mathbb{P}) \colon \\ \exists\, \mathbb{P}\text{-Rademacher family } (r_j)_{j\in\mathbb{N}} \colon \\ \exists\, k \in \mathbb{N} \colon \exists\, x_1, x_2, \dots, x_k \in E\backslash\{0\} \colon \\ r = \frac{\left(\mathbb{E}\left[\|\sum_{j=1}^k r_j x_j\|_E^p\right]\right)^{1/p}}{\left(\sum_{j=1}^k \|x_j\|_E^p\right)^{1/p}} \end{array} \right\} \cup \{0\}\right) \tag{4.1}$$

and we call $\mathscr{T}_p(E)$ the type p-constant of E.

Definition 4.3. *Let $p \in (0, \infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space which satisfies that $\mathscr{T}_p(E) < \infty$. Then we say that E has type p.*

Note that for all $p \in (0,\infty)$, all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with type p, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j\in\mathbb{N}}$, and all $k \in \mathbb{N}$, $x_1, x_2, \dots, x_k \in E$ it holds that

$$\Big\| \sum_{j=1}^{k} r_j x_j \Big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \leq \mathscr{T}_p(E) \bigg(\sum_{j=1}^{k} \|x_j\|_E^p \bigg)^{1/p}. \tag{4.2}$$

In addition, observe that for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$, all probability spaces $(\Omega, \mathscr{F}, \mathbb{P})$, all $\mathbb{P}$-Rademacher families $(r_j)_{j\in\mathbb{N}}$, and all $p \in [2,\infty)$, $k \in \mathbb{N}$, $x \in E\backslash\{0\}$ it holds that

$$\mathscr{T}_p(E) \geq \frac{\big\| \sum_{j=1}^{k} r_j x \big\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}}{\Big[\sum_{j=1}^{k} \|x\|_E^p \Big]^{1/p}} \geq \frac{\|x\|_E \, \big\| \sum_{j=1}^{k} r_j \big\|_{\mathscr{L}^2(\mathbb{P};|\cdot|)}}{k^{1/p}\,\|x\|_E} = \frac{k^{1/2}\,\|x\|_E}{k^{1/p}\,\|x\|_E} = k^{(1/2-1/p)}. \tag{4.3}$$

In particular, for all $p \in (2,\infty)$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ it holds that $\mathscr{T}_p(E) = \infty$. Furthermore, observe that Jensen's inequality together with the fact that for all normed $\mathbb{R}$-vector spaces $(E, \|\cdot\|_E)$ and all $p \in (0,\infty)$, $q \in [p,\infty)$, $k \in \mathbb{N}$, $x_1, \dots, x_k \in E$ it holds that

$$\Big(\textstyle\sum_{j=1}^{k} \|x_j\|_E^q \Big)^{1/q} \leq \Big(\textstyle\sum_{j=1}^{k} \|x_j\|_E^p \Big)^{1/p} \tag{4.4}$$

assures that for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ and all $p, q \in (0,\infty)$ with $p \leq q$ it holds that $\mathscr{T}_p(E) \leq \mathscr{T}_q(E)$. Hence, it holds for every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ that the function $(0,\infty) \ni p \mapsto \mathscr{T}_p(E) \in [0,\infty]$ is non-decreasing. This and the triangle inequality ensure that for all $p \in (0,1]$ and all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ with $E \neq \{0\}$ it holds that $\mathscr{T}_p(E) = 1$. In particular, note that for all $\mathbb{R}$-Banach spaces $(E, \|\cdot\|_E)$ it holds that $\sup_{p\in(0,1]} \mathscr{T}_p(E) \leq 1 < \infty$. Additionally, observe that for all $p \in (0,2]$ and all $\mathbb{R}$-Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ with $H \neq \{0\}$ it holds that $\mathscr{T}_p(H) = 1$. Furthermore, we note that for every probability space $(\Omega, \mathscr{F}, \mathbb{P})$, every $p, q \in [1,\infty)$, and every $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ with type q it holds that $L^p(\mathbb{P}; \|\cdot\|_E)$ has type $\min\{p,q\}$; cf., e.g., Proposition 7.4 in Hytönen et al. [19], Section 9.2 in Ledoux & Talagrand [23], or Theorem 6.2.14 in Albiac & Kalton [1]. In particular, it holds for every $p \in [1,\infty)$ and every probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that $L^p(\mathbb{P}; |\cdot|)$ has type $\min\{p, 2\}$.

Definition 4.4. *Let $p, q \in (0,\infty)$. Then we denote by $\mathscr{K}_{p,q} \in [0,\infty]$ the extended real number given by*

$$\mathscr{K}_{p,q} = \sup \left\{ r \in [0,\infty) \colon \begin{array}{c} \exists\, \mathbb{R}\text{-Banach space } (E, \|\cdot\|_E)\colon \\ \exists\, \text{probability space } (\Omega, \mathscr{F}, \mathbb{P})\colon \\ \exists\, \mathbb{P}\text{-Rademacher family } (r_j)_{j\in\mathbb{N}}\colon \exists\, k \in \mathbb{N}\colon \\ \exists\, x_1, x_2, \dots, x_k \in E\backslash\{0\}\colon r = \frac{(\mathbb{E}[\| \sum_{j=1}^{k} r_j x_j \|_E^p])^{1/p}}{(\mathbb{E}[\| \sum_{j=1}^{k} r_j x_j \|_E^q])^{1/q}} \end{array} \right\} \tag{4.5}$$

and we call $\mathscr{K}_{p,q}$ the (p,q)-Kahane-Khintchine constant.

The celebrated *Kahane-Khintchine inequality* asserts that for all $p, q \in (0,\infty)$ it holds that $\mathscr{K}_{p,q} < \infty$; see, e.g., Theorem 6.2.5 in Albiac & Kalton [1]. Observe that Jensen's inequality ensures for all $p, q \in (0,\infty)$ with $p \leq q$ that $\mathscr{K}_{p,q} = 1$. The nontrivial assertion of the Kahane-Khintchine inequality is the fact that for all $p, q \in (0,\infty)$ with $p > q$ it holds that $\mathscr{K}_{p,q} < \infty$. In our analysis below we also use the following two abbreviations.

Definition 4.5. *Let $p, q \in (0,\infty)$ and let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space. Then we denote by $\Theta_{p,q}(E) \in [0,\infty]$ the extended real number given by $\Theta_{p,q}(E) = 2\mathscr{T}_q(E)\mathscr{K}_{p,q}$.*

Definition 4.6. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $p \in (0,\infty)$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, and let $X \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$. Then we denote by $\sigma_{p,E}(X) \in [0,\infty]$ the extended real number given by $\sigma_{p,E}(X) = \big(\mathbb{E}\big[\|X - \mathbb{E}[X]\|_E^p\big]\big)^{1/p}$.*

4.2 Monte Carlo methods in Banach spaces

In this section we collect a few elementary results on sums of random variables with values in Banach spaces. The next result, Lemma 4.7 below, can be found, e.g., in Section 2.2 of Ledoux & Talagrand [23].

Lemma 4.7 (Symmetrization lemma). *Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_E)$ be independent mappings which satisfy that $\mathbb{E}\big[\|\tilde{\xi}\|_E\big] < \infty$ and $\mathbb{E}[\tilde{\xi}] = 0$, and let $\varphi\colon [0,\infty) \to [0,\infty)$ be a convex function. Then*

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big]. \tag{4.6}$$

Proof of Lemma 4.7. Jensen's inequality assures that

$$\begin{aligned}
\mathbb{E}\big[\varphi(\|\xi\|_E)\big] = \mathbb{E}\big[\varphi(\|\xi - \mathbb{E}[\tilde{\xi}]\|_E)\big] &= \int_\Omega \varphi\bigg(\bigg\|\int_\Omega \xi(\omega) - \tilde{\xi}(\tilde{\omega})\,\mathbb{P}(\mathrm{d}\tilde{\omega})\bigg\|_E\bigg)\,\mathbb{P}(\mathrm{d}\omega) \\
&\le \int_\Omega \int_\Omega \varphi(\|\xi(\omega) - \tilde{\xi}(\tilde{\omega})\|_E)\,\mathbb{P}(\mathrm{d}\tilde{\omega})\mathbb{P}(\mathrm{d}\omega) \\
&= \int_E \int_E \varphi(\|x - y\|_E)\,\big(\tilde{\xi}(\mathbb{P})\big)(\mathrm{d}y)\big(\xi(\mathbb{P})\big)(\mathrm{d}x) \\
&= \int_{E\times E} \varphi(\|x - y\|_E)\,\big((\xi,\tilde{\xi})(\mathbb{P})\big)(\mathrm{d}x,\mathrm{d}y) = \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big].
\end{aligned} \tag{4.7}$$

This completes the proof of Lemma 4.7. □

Corollary 4.8 (Symmetrization corollary). *Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\xi, \tilde{\xi} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$ be independent and identically distributed mappings which satisfy that $\mathbb{E}[\xi] = 0$, and let $\varphi\colon [0,\infty) \to [0,\infty)$ be a convex function. Then*

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \mathbb{E}\big[\varphi(2\|\xi\|_E)\big]. \tag{4.8}$$

Proof of Corollary 4.8. Lemma 4.7 and the fact that φ is non-decreasing show that

$$\mathbb{E}\big[\varphi(\|\xi\|_E)\big] \le \mathbb{E}\big[\varphi(\|\xi - \tilde{\xi}\|_E)\big] \le \tfrac{1}{2}\mathbb{E}\big[\varphi(2\|\xi\|_E)\big] + \tfrac{1}{2}\mathbb{E}\big[\varphi(2\|\tilde{\xi}\|_E)\big] = \mathbb{E}\big[\varphi(2\|\xi\|_E)\big]. \tag{4.9}$$

The proof of Corollary 4.8 is thus completed. □

As a straightforward application we obtain the following randomisation result, cf., e.g., Lemma 6.3 in Ledoux & Talagrand [23].

Lemma 4.9 (Randomisation). *Let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $k \in \mathbb{N}$, let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1,\dots,k\}$, satisfy for all $j \in \{1,\dots,k\}$ that $\mathbb{E}[\xi_j] = 0$, and let $r_j\colon \Omega \to \{-1,1\}$, $j \in \{1,\dots,k\}$, be a $\mathbb{P}$-Rademacher family such that $\xi_1, \xi_2, \dots, \xi_k, r_1, r_2, \dots, r_k$ are independent. Then for all $p \in [1,\infty)$ it holds that*

$$\bigg\|\sum_{j=1}^k \xi_j\bigg\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \le 2\,\bigg\|\sum_{j=1}^k r_j\xi_j\bigg\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}. \tag{4.10}$$

Proof of Lemma 4.9. Throughout this proof let $\boldsymbol{\Omega} = \Omega \times \Omega$, let $\boldsymbol{\xi}_j \colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, $\tilde{\boldsymbol{\xi}}_j \colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\mathbf{r}_j \colon \boldsymbol{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings which satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \boldsymbol{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$, $\tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega}) = \xi_j(\tilde{\omega})$, and $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\omega)$, and let $\mathbf{P} = \mathbb{P} \otimes \mathbb{P}$. The fact that $\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j \colon \boldsymbol{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\mathbf{r}_j \colon \boldsymbol{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$ are independent and the symmetry of $\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j$, $j \in \{1, \ldots, k\}$, prove for all $p \in [1, \infty)$ that

$$
\begin{aligned}
&\int_{\boldsymbol{\Omega}} \Bigg\| \sum_{j=1}^{k} \mathbf{r}_j(\boldsymbol{\omega}) \big(\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega}) \big) \Bigg\|_E^p \, \mathbf{P}(\mathrm{d}\boldsymbol{\omega}) \\
&= \int_{(\{-1,1\} \times E)^k} \Bigg\| \sum_{j=1}^{k} z_j x_j \Bigg\|_E^p \big((\mathbf{r}_1, \boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \ldots, \mathbf{r}_k, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P}) \big)(\mathrm{d}z_1, \mathrm{d}x_1, \ldots, \mathrm{d}z_k, \mathrm{d}x_k) \\
&= \int_{\{-1,1\}} \int_E \cdots \int_{\{-1,1\}} \int_E \; \cdots \\
&\quad \Bigg\| \sum_{j=1}^{k} x_j \Bigg\|_E^p \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1)(\mathbf{P}) \big)(\mathrm{d}x_1) \big((\mathbf{r}_1)(\mathbf{P}) \big)(\mathrm{d}z_1) \; \ldots \; \big((\boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P}) \big)(\mathrm{d}x_k) \big((\mathbf{r}_k)(\mathbf{P}) \big)(\mathrm{d}z_k) \\
&= \int_{E^k} \Bigg\| \sum_{j=1}^{k} x_j \Bigg\|_E^p \big((\boldsymbol{\xi}_1 - \tilde{\boldsymbol{\xi}}_1, \ldots, \boldsymbol{\xi}_k - \tilde{\boldsymbol{\xi}}_k)(\mathbf{P}) \big)(x_1, \ldots, x_k) \\
&= \int_{\boldsymbol{\Omega}} \Bigg\| \sum_{j=1}^{k} \big(\boldsymbol{\xi}_j(\boldsymbol{\omega}) - \tilde{\boldsymbol{\xi}}_j(\boldsymbol{\omega}) \big) \Bigg\|_E^p \, \mathbf{P}(\mathrm{d}\boldsymbol{\omega}).
\end{aligned} \tag{4.11}
$$

Furthermore, the fact that $\sum_{j=1}^{k} \boldsymbol{\xi}_j \colon \boldsymbol{\Omega} \to E$ and $\sum_{j=1}^{k} \tilde{\boldsymbol{\xi}}_j \colon \boldsymbol{\Omega} \to E$ are independent and identically distributed, the facts that $\mathbb{E}\big[\|\sum_{j=1}^{k} \boldsymbol{\xi}_j\|_E\big] < \infty$ and $\mathbb{E}\big[\sum_{j=1}^{k} \boldsymbol{\xi}_j\big] = 0$, Lemma 4.7, and (4.11) imply for all $p \in [1, \infty)$ that

$$
\begin{aligned}
&\Bigg\| \sum_{j=1}^{k} \xi_j \Bigg\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \Bigg\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \Bigg\|_{\mathscr{L}^p(\mathbf{P}; \|\cdot\|_E)} \leq \Bigg\| \sum_{j=1}^{k} (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \Bigg\|_{\mathscr{L}^p(\mathbf{P}; \|\cdot\|_E)} \\
&= \Bigg\| \sum_{j=1}^{k} \mathbf{r}_j (\boldsymbol{\xi}_j - \tilde{\boldsymbol{\xi}}_j) \Bigg\|_{\mathscr{L}^p(\mathbf{P}; \|\cdot\|_E)} \leq \Bigg\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \Bigg\|_{\mathscr{L}^p(\mathbf{P}; \|\cdot\|_E)} + \Bigg\| \sum_{j=1}^{k} \mathbf{r}_j \tilde{\boldsymbol{\xi}}_j \Bigg\|_{\mathscr{L}^p(\mathbf{P}; \|\cdot\|_E)} \\
&= 2 \Bigg\| \sum_{j=1}^{k} r_j \xi_j \Bigg\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}.
\end{aligned} \tag{4.12}
$$

The proof of Lemma 4.9 is thus completed. □

The next result, Proposition 4.10 below, is the key to estimate the statistical error term in the Banach space valued Monte Carlo method in the next section. Proposition 4.10 is similar to, e.g., Proposition 9.11 in Ledoux & Talagrand [23].

Proposition 4.10 (Sums of independent, centered, Banach space valued random variables)**.** *Let $k \in \mathbb{N}$, $q \in [1, 2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, k\}$, be independent mappings which satisfy for all $j \in \{1, \ldots, k\}$ that $\mathbb{E}[\xi_j] = 0$. Then for all $p \in [q, \infty)$ it holds that*

$$
\Bigg\| \sum_{j=1}^{k} \xi_j \Bigg\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \leq \Theta_{p,q}(E) \Bigg(\sum_{j=1}^{k} \|\xi_j\|^q_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} \Bigg)^{1/q}. \tag{4.13}
$$

Proof of Proposition 4.10. Throughout this proof let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ be a probability space, let $r_j \colon \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be a $\tilde{\mathbb{P}}$-Rademacher family, and let $\boldsymbol{\xi}_j \colon \Omega \times \tilde{\Omega} \to E$, $j \in \{1, \ldots, k\}$, and $\mathbf{r}_j \colon \Omega \times \tilde{\Omega} \to \{-1, 1\}$, $j \in \{1, \ldots, k\}$, be the mappings which satisfy for all $\boldsymbol{\omega} = (\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$, $j \in \{1, \ldots, k\}$ that $\boldsymbol{\xi}_j(\boldsymbol{\omega}) = \xi_j(\omega)$ and $\mathbf{r}_j(\boldsymbol{\omega}) = r_j(\tilde{\omega})$. Lemma 4.9 and the triangle inequality show that

$$
\begin{aligned}
&\Big\| \sum_{j=1}^{k} \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \Big\| \sum_{j=1}^{k} \boldsymbol{\xi}_j \Big\|_{\mathscr{L}^p(\mathbb{P}\otimes\tilde{\mathbb{P}}; \|\cdot\|_E)} \leq 2 \Big\| \sum_{j=1}^{k} \mathbf{r}_j \boldsymbol{\xi}_j \Big\|_{\mathscr{L}^p(\mathbb{P}\otimes\tilde{\mathbb{P}}; \|\cdot\|_E)} \\
&= 2 \Big\| \sum_{j=1}^{k} r_j \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{\mathscr{L}^p(\tilde{\mathbb{P}}; \|\cdot\|_E)})} \leq 2\mathscr{K}_{p,q} \Big\| \sum_{j=1}^{k} r_j \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{\mathscr{L}^q(\tilde{\mathbb{P}}; \|\cdot\|_E)})} \\
&\leq 2\mathscr{K}_{p,q} \mathscr{T}_q(E) \Big\| \Big(\sum_{j=1}^{k} \|\xi_j\|_E^q \Big)^{1/q} \Big\|_{\mathscr{L}^p(\mathbb{P}; |\cdot|)} = 2\mathscr{K}_{p,q} \mathscr{T}_q(E) \Big\| \sum_{j=1}^{k} \|\xi_j\|_E^q \Big\|_{\mathscr{L}^{p/q}(\mathbb{P}; |\cdot|)}^{1/q} \\
&\leq 2\mathscr{K}_{p,q} \mathscr{T}_q(E) \Big(\sum_{j=1}^{k} \|\xi_j\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)}^q \Big)^{1/q}
\end{aligned}
\tag{4.14}
$$

(cf., e.g., Proposition 7.4 in Hytönen et al. [19]). This finishes the proof of Proposition 4.10. □

The result in Corollary 4.11 blow is a direct consequence of Proposition 4.10.

Corollary 4.11 (Sums of independent Banach space valued random variables)**.** *Let $M \in \mathbb{N}$, $q \in [1, 2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$, be independent. Then for all $p \in [q, \infty)$ it holds that*

$$
\sigma_{p,E}\Big(\sum_{j=1}^{M} \xi_j \Big) \leq \Theta_{p,q}(E) \Big(\sum_{j=1}^{M} |\sigma_{p,E}(\xi_j)|^q \Big)^{1/q}. \tag{4.15}
$$

Corollary 4.12 (Monte Carlo methods in Banach spaces)**.** *Let $M \in \mathbb{N}$, $q \in [1, 2]$, let $(E, \|\cdot\|_E)$ be an $\mathbb{R}$-Banach space with type q, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $\xi_j \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_E)$, $j \in \{1, \ldots, M\}$, be independent and identically distributed. Then for all $p \in [q, \infty)$ it holds that*

$$
\Big\| \mathbb{E}[\xi_1] - \frac{1}{M} \sum_{j=1}^{M} \xi_j \Big\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_E)} = \frac{\sigma_{p,E}\big(\sum_{j=1}^{M} \xi_j \big)}{M} \leq \frac{\Theta_{p,q}(E)\, \sigma_{p,E}(\xi_1)}{M^{1-1/q}}. \tag{4.16}
$$

Results on lower and upper error bounds related to Corollary 4.12 can be found, e.g., in Theorem 1 in Daun & Heinrich [7] and in Corollary 2 in Heinrich & Hinrichs [12].

4.3 Multilevel Monte Carlo methods in Banach spaces

In many situations the work required to obtain a certain accuracy of an approximation using the Monte Carlo method can be improved by using a multilevel Monte Carlo method. Heinrich [10, 11] was first to observe this and established multilevel Monte Carlo methods concerning convergence in a Banach (function) space. However, these methods do not apply to SDEs. Then Giles [8] derived the complexity reduction of multilevel Monte Carlo methods for SDEs. The minor contribution of Proposition 4.13 below to the literature on multilevel Monte Carlo methods is to combine the approaches of Heinrich [10] and of Giles [8] into a single result on multilevel Monte Carlo methods in Banach spaces.

The following useful observation of Proposition 4.13 generalizes the discussion in Section 4 of Heinrich [11]. Corollary 4.12 does not imply convergence if the underlying Banach space $(E, \|\cdot\|_E)$ has only type 1, in the sense that for all $q \in (1,\infty)$ it holds that $\mathscr{T}_q(E) = \infty$. If, however, a point in a type 1 Banach space is approximated with random variables having values in a finite-dimensional subspace (which always has type 2) then it is essential to know how large the corresponding embedding constant is.

Proposition 4.13 (Abstract multilevel Monte Carlo methods in Banach spaces)**.** *Let $q \in [1,2]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(V_1, \|\cdot\|_{V_1})$ be an $\mathbb{R}$-Banach space with type q, let $(V_2, \|\cdot\|_{V_2})$ be an $\mathbb{R}$-Banach space with $V_1 \subseteq V_2$ continuously, let $v \in V_2$, $L \in \mathbb{N}$, $M_1, \ldots, M_L \in \mathbb{N}$, and for every $\ell \in \{1, \ldots, L\}$ let $D_{\ell,k} \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $k \in \{1, \ldots, M_\ell\}$, be independent and identically distributed. Then for all $p \in [q, \infty)$ it holds that*

$$\begin{aligned} &\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}\big[D_{\ell,1}\big] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{\mathscr{L}(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}. \end{aligned} \tag{4.17}$$

Proof of Proposition 4.13. The triangle inequality and Corollary 4.12 imply for all $p \in [q, \infty)$ that

$$\begin{aligned} &\left\| v - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \left\| \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] - \sum_{\ell=1}^{L} \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{\mathscr{L}(V_1,V_2)} \sum_{\ell=1}^{L} \left\| \mathbb{E}[D_{\ell,1}] - \frac{1}{M_\ell} \sum_{k=1}^{M_\ell} D_{\ell,k} \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_1})} \\ &\leq \left\| v - \sum_{\ell=1}^{L} \mathbb{E}[D_{\ell,1}] \right\|_{V_2} + \|\mathrm{Id}_{V_1}\|_{\mathscr{L}(V_1,V_2)}\, \Theta_{p,q}(V_1) \sum_{\ell=1}^{L} \frac{\sigma_{p,V_1}(D_{\ell,1})}{(M_\ell)^{1-1/q}}. \end{aligned} \tag{4.18}$$

This completes the proof of Proposition 4.13. □

Corollary 4.14 (Multilevel Monte Carlo methods in Banach spaces)**.** *Let $q \in [1,2]$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $\big(V_i, \|\cdot\|_{V_i}\big)$, $i \in \{1,2,3\}$, be $\mathbb{R}$-Banach spaces such that $\big(V_1, \|\cdot\|_{V_1}\big)$ has type q and such that $V_1 \subseteq V_2$ continuously, let $f\colon V_3 \to V_2$ be a $\mathscr{B}(V_3)/\mathscr{B}(V_2)$-measurable mapping, let $g\colon V_3 \to V_1$ be a $\mathscr{B}(V_3)/\mathscr{B}(V_1)$-measurable mapping, let $X \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_{V_3})$ satisfy that $\mathbb{E}\big[\|f(X)\|_{V_2}\big] < \infty$, for every $N \in \mathbb{N}$ let $Y^{N,\ell,k} \in \mathscr{L}^0(\mathbb{P}; \|\cdot\|_{V_3})$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$, be independent and identically distributed mappings which satisfy that $\mathbb{E}\big[\|g(Y^{N,0,1})\|_{V_1}\big] < \infty$, and let $L \in \mathbb{N}_0$, $M_0, M_1, \ldots, M_{L+1}, N_0, N_1, \ldots, N_L \in \mathbb{N}$. Then for all $p \in [q, \infty)$ it holds that*

$$\begin{aligned} &\left\| \mathbb{E}[f(X)] - \tfrac{1}{M_0} \sum_{k=1}^{M_0} g(Y^{N_0,0,k}) - \sum_{\ell=1}^{L} \tfrac{1}{M_\ell} \sum_{k=1}^{M_\ell} \big[g(Y^{N_\ell,\ell,k}) - g(Y^{N_{(\ell-1)},\ell,k})\big] \right\|_{\mathscr{L}^p(\mathbb{P}; \|\cdot\|_{V_2})} \\ &\leq \big\| \mathbb{E}[f(X)] - \mathbb{E}\big[g(Y^{N_L,0,1})\big] \big\|_{V_2} \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
&+ \|\mathrm{Id}_{V_1}\|_{\mathscr{L}(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\tfrac{\sigma_{p,V_1}\left(g(Y^{N_0,0,1})\right)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{\sigma_{p,V_1}\left(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right)}{(M_\ell)^{1-1/q}} \right) \\
&\leq \left\| \mathbb{E}[f(X)] - \mathbb{E}\big[g(Y^{N_L,0,1})\big] \right\|_{V_2} \\
&+ \|\mathrm{Id}_{V_1}\|_{\mathscr{L}(V_1,V_2)}\, \Theta_{p,q}(V_1) \left(\tfrac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \tfrac{4\left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}} \right).
\end{aligned}
$$

Proof of Corollary 4.14. Proposition 4.13 and the identity

$$
\mathbb{E}\big[g(Y^{N_L,0,1})\big] = \mathbb{E}\big[g(Y^{N_0,0,1})\big] + \sum_{\ell=1}^{L} \mathbb{E}\big[g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\big] \tag{4.20}
$$

imply the first inequality in (4.19). Next note that the triangle inequality implies for all $\xi \in \mathscr{L}^1(\mathbb{P}; \|\cdot\|_{V_1})$, $p \in [q, \infty)$ that $\sigma_{p,V_1}(\xi) \leq 2\|\xi\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}$. This and again the triangle inequality show for all $p \in [q, \infty)$ that

$$
\begin{aligned}
&\tfrac{\sigma_{p,V_1}\left(g(Y^{N_0,0,1})\right)}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{\sigma_{p,V_1}\left(g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right)}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\left\|g(Y^{N_0,0,1})\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=1}^{L} \tfrac{2\left\|g(Y^{N_\ell,0,1}) - g(Y^{N_{(\ell-1)},0,1})\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} + 2\left\|g(Y^{N_0,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} \\
&\quad + \sum_{\ell=1}^{L} \tfrac{2\left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} + 2\left\|g(Y^{N_{(\ell-1)},0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_\ell)^{1-1/q}} \\
&\leq \tfrac{2\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(M_0)^{1-1/q}} + \sum_{\ell=0}^{L} \tfrac{4\left\|g(Y^{N_\ell,0,1}) - g(X)\right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}}{(\min\{M_\ell, M_{\ell+1}\})^{1-1/q}}.
\end{aligned} \tag{4.21}
$$

This implies the second inequality in (4.19). The proof of Corollary 4.14 is thus completed. □

Corollary 4.15 (Convergence of multilevel Monte Carlo approximations)**.** *Let $T \in (0,\infty)$, $\beta \in (0,1)$, $\alpha \in [0,\beta)$, $c, r \in [0,\infty)$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space with type 2, let $X\colon [0,T] \times \Omega \to E$ be a stochastic process with continuous sample paths which satisfies for all $p \in [1,\infty)$, $\gamma \in [0,\beta)$ that $X \in C^\gamma\big([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\big)$, for every $N \in \mathbb{N}$ let $Y^{N,\ell,k}\colon [0,T] \times \Omega \to E$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$, be independent and identically distributed stochastic processes which satisfy for all $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \left[\frac{nT}{N}, \frac{(n+1)T}{N}\right]$, $p \in [1,\infty)$, $\rho \in [0,\beta)$ that*

$$
Y_t^{N,\ell,k} = \left(n + 1 - \tfrac{tN}{T}\right) \cdot Y^{N,\ell,k}_{\frac{nT}{N}} + \left(\tfrac{tN}{T} - n\right) \cdot Y^{N,\ell,k}_{\frac{(n+1)T}{N}}, \tag{4.22}
$$

$$
\sup_{M \in \mathbb{N}} \sup_{m \in \{0,1,\ldots,M\}} \left(M^\rho \left\| X_{\frac{mT}{M}} - Y^{M,0,1}_{\frac{mT}{M}} \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)} \right) < \infty, \tag{4.23}
$$

and let $f\colon C([0,T],E) \to C([0,T],E)$ be a $\mathscr{B}\big(C([0,T],E)\big)/\mathscr{B}\big(C([0,T],E)\big)$-measurable function which satisfies for all $v, w \in C^\alpha([0,T], \|\cdot\|_E)$ that

$$
\|f(v) - f(w)\|_{C^\alpha([0,T],\|\cdot\|_E)} \leq c\Big(1 + \|v\|^r_{C^\alpha([0,T],\|\cdot\|_E)} + \|w\|^r_{C^\alpha([0,T],\|\cdot\|_E)}\Big) \|v - w\|_{C^\alpha([0,T],\|\cdot\|_E)}. \tag{4.24}
$$

Then it holds that

$$\mathbb{E}\big[\|f(X)\|_{C^\alpha([0,T],\|\cdot\|_E)}\big] < \infty, \tag{4.25}$$

it holds for all $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that

$$\sup_{N\in\mathbb{N}} \big(N^\rho \|f(X) - f(Y^{N,0,1})\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\big) < \infty, \tag{4.26}$$

and it holds for all $p \in [1,\infty)$, $\gamma \in [0,\alpha)$, $\rho \in [0,\beta-\alpha) \setminus \{\frac{1}{2}\}$ that

$$\sup_{L\in\mathbb{N}} \left[2^{L(\rho\wedge\frac{1}{2})} \left\| \mathbb{E}[f(X)] - \sum_{k=1}^{2^L} \tfrac{f(Y^{1,0,k})}{2^L} - \sum_{\ell=1}^{L}\sum_{k=1}^{2^{L-\ell}} \tfrac{f(Y^{2^\ell,\ell,k}) - f(Y^{2^{(\ell-1)},\ell,k})}{2^{L-\ell}} \right\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\gamma([0,T],\|\cdot\|_E)})} \right] < \infty. \tag{4.27}$$

Proof of Corollary 4.15. Throughout this proof let $\gamma \in [0,\alpha)$, let $(V_i, \|\cdot\|_{V_i})$, $i \in \{1,2,3\}$, be the $\mathbb{R}$-Banach spaces which satisfy that

$$(V_1, \|\cdot\|_{V_1}) = \big(W^{\alpha,2/(\alpha-\gamma)}([0,T],E), \|\cdot\|_{W^{\alpha,2/(\alpha-\gamma)}([0,T],E)}\big), \tag{4.28}$$

$$(V_2, \|\cdot\|_{V_2}) = \Big(C^\gamma([0,T],\|\cdot\|_E), \|\cdot\|_{C^\gamma([0,T],\|\cdot\|_E)}\big|_{C^\gamma([0,T],\|\cdot\|_E)}\Big), \tag{4.29}$$

$$(V_3, \|\cdot\|_{V_3}) = \Big(C([0,T],E), \|\cdot\|_{C([0,T],\|\cdot\|_E)}\big|_{C([0,T],E)}\Big), \tag{4.30}$$

and let $\tilde{f}\colon V_3 \to V_2$ and $g\colon V_3 \to V_1$ be the functions which satisfy for all $v \in V_3$ that $\tilde{f}(v) = g(v) = \mathbb{1}_{C^\alpha([0,T],\|\cdot\|_E)}(v) f(v)$. Kolmogorov's continuity criterion (see Theorem 2.7) together with the assumptions that $X \in \cap_{p\in[1,\infty)} \cap_{\eta\in[0,\beta)} C^\eta\big([0,T], \|\cdot\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_E)}\big)$ and that X has continuous sample paths implies that $X \in \cap_{p\in[1,\infty)} \mathscr{L}^p(\mathbb{P}; \|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})$. Next observe that assumption (4.24), Hölder's inequality, and Corollary 2.11 show for all $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \Big(N^\rho\, \mathbb{E}\Big[\|\tilde{f}(X) - g(Y^{N,0,1})\|_{C^\alpha([0,T],\|\cdot\|_E)}\Big]\Big) \\
&\leq \sup_{N\in\mathbb{N}} \left(N^\rho \left(\mathbb{E}\Big[\|f(X) - f(Y^{N,0,1})\|^p_{C^\alpha([0,T],\|\cdot\|_E)}\Big]\right)^{1/p}\right) \\
&\leq \sup_{N\in\mathbb{N}} \left(N^\rho \bigg|\mathbb{E}\bigg[\Big(c\big(1 + \|X\|^r_{C^\alpha([0,T],\|\cdot\|_E)} + \|Y^{N,0,1}\|^r_{C^\alpha([0,T],\|\cdot\|_E)}\big)\right. \\
&\qquad\qquad \left.\cdot \|X - Y^{N,0,1}\|_{C^\alpha([0,T],\|\cdot\|_E)}\Big)^p\bigg]\bigg|^{1/p}\right) \\
&\leq c\left(1 + \|X\|^r_{\mathscr{L}^{2pr}(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} + \sup_{N\in\mathbb{N}} \|Y^{N,0,1}\|^r_{\mathscr{L}^{2pr}(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\right) \\
&\qquad \cdot \sup_{N\in\mathbb{N}} \Big(N^\rho \|X - Y^{N,0,1}\|_{\mathscr{L}^{2p}(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big) < \infty.
\end{aligned} \tag{4.31}$$

Again assumption (4.24) implies for all $p \in [1,\infty)$ that

$$\begin{aligned}
&\mathbb{E}\Big[\|f(X)\|_{C^\alpha([0,T],\|\cdot\|_E)}\Big] \leq \|f(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} \\
&\leq \|f(0)\|_{C^\alpha([0,T],\|\cdot\|_E)} + c\Big(\|X\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})} + \|X\|^{r+1}_{\mathscr{L}^{(r+1)p}(\mathbb{P};\|\cdot\|_{C^\alpha([0,T],\|\cdot\|_E)})}\Big) < \infty.
\end{aligned} \tag{4.32}$$

Next observe that $(V_1, \|\cdot\|_{V_1})$ has type 2 and note that the Sobolev embedding theorem proves that $V_1 \subseteq V_2$ continuously. Combining (4.32) with (4.31) and the fact that

$C^\alpha([0,T],\|\cdot\|_E) \subseteq V_1$ continuously hence implies for all $N \in \mathbb{N}$, $p \in [1,\infty)$, $\rho \in [0,\beta-\alpha)$ that $\mathbb{E}\big[\|\tilde{f}(X)\|_{V_2}\big] + \mathbb{E}\big[\|g(Y^{N,0,1})\|_{V_1}\big] < \infty$, $\|g(X)\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})} < \infty$, and

$$\sup_{N\in\mathbb{N}}\Big(N^\rho\,\mathbb{E}\Big[\big\|\tilde{f}(X)-g(Y^{N,0,1})\big\|_{V_2}\Big]\Big)+\sup_{N\in\mathbb{N}}\Big(N^\rho\|g(X)-g(Y^{N,0,1})\|_{\mathscr{L}^p(\mathbb{P};\|\cdot\|_{V_1})}\Big) < \infty. \tag{4.33}$$

In addition, observe for all $L \in \mathbb{N}$, $\rho \in [0,\beta-\alpha)\setminus\{\frac{1}{2}\}$ that

$$\sum_{\ell=1}^{L}(2^\ell)^{-\rho}\,2^{-\frac{1}{2}(L-\ell)} = 2^{-\frac{L}{2}}\sum_{\ell=1}^{L}2^{(\frac{1}{2}-\rho)\ell} = 2^{-\frac{L}{2}}\,\tfrac{1-2^{(\frac{1}{2}-\rho)L}}{2^{\rho-\frac{1}{2}}-1} = 2^{-(\rho\wedge\frac{1}{2})L}\,\tfrac{1-2^{-|\frac{1}{2}-\rho|L}}{|1-2^{\rho-\frac{1}{2}}|} \le \tfrac{2^{-(\rho\wedge\frac{1}{2})L}}{|1-2^{\rho-\frac{1}{2}}|}. \tag{4.34}$$

Combining Corollary 4.14 with (4.33) and (4.34) implies (4.27). This finishes the proof of Corollary 4.15. □

Corollary 4.15 can be applied to many SDEs. Under general conditions on the coefficient functions of the SDEs (see, e.g., Theorem 1.3 and Section 3.1 in [14]), suitable stopped-tamed Euler approximations (cf. (6) in [18] or (10) in [16]) converge in the strong sense with convergence rate $\frac{1}{2}$. We note that the classical Euler-Maruyama approximations do not satisfy condition (4.23) for most SDEs with superlinearly growing coefficients; see Theorem 2.1 in [15] and Theorem 2.1 in [17]. Moreover, under general conditions on the coefficients it holds that the solution process is strongly $\frac{1}{2}$-Hölder continuous in time. So Corollary 4.15 can be applied to many SDEs with $\beta = \frac{1}{2}$.

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