Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

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Research Report No. 2016-02
January 2016

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
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Funding SNF: 200021_159940/1
Funding ERC: 247277
Funding: Knut and Alice Wallenberg foundation, the Swedish Research Council under Reg. No. 621-2014-3995

NUMERICAL ANALYSIS OF LOGNORMAL DIFFUSIONS ON $\mathbb{S}^2$

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Abstract. Numerical solutions of stationary diffusion equations on $\mathbb{S}^2$ with isotropic lognormal diffusion coefficients are considered. Hölder regularity in L^p sense for isotropic Gaussian random fields is obtained and related to the regularity of the driving lognormal coefficients. This yields regularity in L^p sense of the solution to the diffusion problem in Sobolev spaces. Convergence rate estimates of multilevel Monte Carlo Finite and Spectral Element discretizations of these problems on $\mathbb{S}^2$ are then deduced. Specifically, a convergence analysis is provided with convergence rate estimates in terms of the number of Monte Carlo samples of the solution to the considered diffusion equation and in terms of the total number of degrees of freedom of the spatial discretization, and with bounds for the total work required by the algorithm in the case of Finite Element discretizations. The obtained convergence rates are solely in terms of the decay of the angular power spectrum of the (logarithm) of the diffusion coefficient.

Contents

Date: January 11, 2016.

1991 *Mathematics Subject Classification.* 60G60, 60G15, 60G17, 33C55, 41A25, 60H15, 60H35, 65C30, 65N30.

Key words and phrases. Isotropic Gaussian random fields, lognormal random fields, Karhunen–Loève expansion, spherical harmonic functions, stochastic partial differential equations, random partial differential equations, regularity of random fields, Finite Element Methods, Spectral Galerkin Methods.

Acknowledgment. The work was supported in part by the European Research Council under ERC AdG 247277, the Swiss National Science Foundation under SNF 200021_159940/1, the Knut and Alice Wallenberg foundation, and the Swedish Research Council under Reg. No. 621-2014-3995.

1. Introduction

In the present paper, we are concerned with the existence, regularity, and approximation of solutions of elliptic partial differential equations (PDEs for short) with stochastic coefficients on the unit sphere $\mathbb{S}^2$. In particular, we are interested in PDEs with *isotropic lognormal random field* coefficients a, i.e., $T = \log a$ is an isotropic Gaussian random field (iGRF for short) on $\mathbb{S}^2$. Let, therefore, $(\Omega, \mathcal{A}, \mathbb{P})$ denote a probability space and write $\mathbb{S}^2$ for the unit sphere in $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3, \|x\| = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

For a given smooth, deterministic source term f, and for a positive random field a taking values in $C^0(\mathbb{S}^2)$, we consider the stochastic elliptic problem

$$-\nabla_{\mathbb{S}^2} \cdot (a \nabla_{\mathbb{S}^2} u) = f \quad \text{on } \mathbb{S}^2. \tag{1}$$

Since $\partial\mathbb{S}^2 = \emptyset$, no boundary conditions are required for the well-posedness of (1). Furthermore, the diffusion coefficient a in (1) is a random field, which yields that the solution u of (1) is strongly measurable from $(\Omega, \mathcal{A})$ to $(H^1(\mathbb{S}^2)/\mathbb{R}, \mathcal{B}(H^1(\mathbb{S}^2)/\mathbb{R}))$. The error and convergence rate analysis of Finite Element and Spectral Galerkin discretizations on $\mathbb{S}^2$ of the PDE (1) combined with multilevel Monte Carlo (MLMC for short) sampling is the purpose of the present paper.

While the combined Finite Element MLMC discretization of PDEs with random input data has received considerable attention in recent years (see, for example, [6, 15] and the survey [13] originating from Heinrich [19]), the invariance properties of the particular geometry $\mathbb{S}^2$ entail several specific consequences in the numerical analysis which allow more precise convergence results. Specifically, as we showed in [26, 20], the geometric setting of $\mathbb{S}^2$ allows for an essentially sharp characterization of Hölder regularity exponents of realizations of a in terms of the angular power spectrum of the Karhunen–Loève expansion of the Gaussian random field $T = \log a$. Furthermore, $\partial\mathbb{S}^2 = \emptyset$ implies the absence of corner singularities. We are therefore able to obtain elliptic regularity estimates in Sobolev scales, cp. [17], as well as Schauder estimates of classical elliptic regularity theory as presented for example in [12] and elaborated in detail for the presently considered PDE (1) in [20]. Based on these we derive explicit convergence rate bounds of discretizations of (1). Particularly, we obtain convergence rates with respect to the mesh width of Finite Element discretizations and to the spectral degree of Spectral Galerkin discretizations on $\mathbb{S}^2$ solely in terms of the decay of the angular power spectrum of the Gaussian random field $T = \log a$. These convergence rates are, in the Finite Element case, bounded by the polynomial degree of the basis functions. We confine our error analysis to sufficiently smooth source terms f in (1), which yields that the lack of smoothness of solutions is caused by the roughness of the lognormal random coefficients a.

Throughout the paper, we employ standard notation. We denote in particular by $H^s(\mathbb{S}^2)$ Sobolev spaces of square integrable functions of (not necessarily integer) order s on $\mathbb{S}^2$. By $\nabla_{\mathbb{S}^2}$, $\nabla_{\mathbb{S}^2}\cdot$, and by $\Delta_{\mathbb{S}^2} = \nabla_{\mathbb{S}^2} \cdot \nabla_{\mathbb{S}^2}$ we denote the spherical gradient, the spherical divergence, and the Laplace–Beltrami operator on $\mathbb{S}^2$, respectively.

The outline of the paper is as follows: In Section 2 we recapitulate basic properties of iGRFs from [28, 5]. We introduce standard notation and classical results from the differential geometry of surfaces as required in the ensuing developments. We also review results on the Hölder regularity of realizations of the random field from our earlier work [26], and relate the Hölder exponent to the angular power spectrum. We develop Hölder regularity here in the L^p sense. In Section 3 we review and establish basic results on existence, uniqueness, integrability, and regularity of solutions to the stochastic partial differential equation (SPDE for short) (1). In Section 4 we present isoparametric Finite Element (FE for short) discretizations of the SPDE (1) on $\mathbb{S}^2$ and establish apriori estimates on their convergence. Particular attention is given to the dependence of the convergence rate in terms of the Hölder regularity of the random field a. In Section 4, we prove convergence rate estimates for two families of discretizations of (1). Section 4.1 is devoted to the analysis of Finite Element discretizations, while Section 4.2 to the convergence analysis of Spectral Galerkin discretizations. In Section 5 we address the convergence of multilevel Monte Carlo methods for either variant of the Galerkin discretizations.

2. Isotropic Gaussian random fields on the sphere

In this section we introduce isotropic Gaussian random fields and their properties. We focus in particular on Karhunen–Loève expansions of these random fields. In doing so we follow closely the introduction of Gaussian random fields in [28, Chapter 5] and also present results from [20, 26].

2.1. Definitions and basic properties.

Let $(\mathbb{S}^2, d)$ be the compact metric space with the geodesic metric given by

$$d(x, x') := \arccos\langle x, x'\rangle_{\mathbb{R}^3}$$

for every $x, x' \in \mathbb{S}^2$. Let $\{(U_i, \eta_i), i \in \mathcal{I}\}$ be a finite C^∞ atlas of $\mathbb{S}^2$, where $\{U_i, i \in \mathcal{I}\}$ is a finite open cover of $\mathbb{S}^2$ and $\{\eta_i : U_i \to \eta_i(U_i) \subset \mathbb{R}^2, i \in \mathcal{I}\}$ are the respective coordinate charts, which are sometimes also simply called *coordinates*. Here and throughout, we do not separate indices for doubly sub- or superscripted functions and coefficients by a comma, with the understanding that the reader will recognize double indices as such. With this in mind, let g be the metric tensor which is expressed for any $x_0 \in \mathbb{S}^2$ locally in the coordinates $\{\eta_i, i \in \mathcal{I}\}$ as

$$g_{k\ell}(x_0) := \Big\langle \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}, \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^\ell} \Big\rangle_{\mathbb{R}^3}$$

for $k, \ell = 1, 2$, where $\hat{x}_0 = \eta_i(x_0)$ and $i \in \mathcal{I}$ is such that $x_0 \in U_i$. The matrix $g(x_0)$ induces an inner product on the *tangent space* $T_{x_0}\mathbb{S}^2$ *at* x_0 in the basis $\frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}$, $k = 1, 2$, i.e., for $v = \sum_{k=1}^2 v^k \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}$, $w = \sum_{k=1}^2 w^k \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k} \in T_{x_0}\mathbb{S}^2$, it holds that $\langle v, w\rangle_{\mathbb{R}^3} = \sum_{k,\ell=1}^2 g_{k\ell}(x_0) v^k w^\ell$. We denote the components of the inverse of g at any arbitrarily chosen $x_0 \in \mathbb{S}^2$ by $g^{k\ell}(x_0) := (g^{-1}(x_0))_{k\ell}$ for $k, \ell = 1, 2$ and further introduce $|g|(x_0) := \det(g(x_0))$. The *spherical gradient* $\nabla_{\mathbb{S}^2}$ and the *spherical divergence* $\nabla_{\mathbb{S}^2}\cdot$ are locally expressed with g, i.e., for any $x_0 \in \mathbb{S}^2$, $i \in \mathcal{I}$ such that $x_0 \in U_i$ and $\hat{x}_0 = \eta_i(x_0)$,

$$\nabla_{\mathbb{S}^2} f(x_0) := \sum_{k,\ell=1}^2 g^{k\ell}(x_0) \frac{\partial (f \circ \eta_i^{-1})(\hat{x}_0)}{\partial \hat{x}^k} \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^\ell}$$

and

$$\nabla_{\mathbb{S}^2} \cdot Z(x_0) := \frac{1}{\sqrt{|g|(x_0)}} \sum_{\ell=1}^2 \frac{\partial}{\partial \hat{x}^\ell}((\sqrt{|g|} Z^\ell) \circ \eta_i^{-1})(\hat{x}_0),$$

where $f : \mathbb{S}^2 \to \mathbb{R}$ is a function and $Z = \sum_{\ell=1}^2 Z^\ell \frac{\partial \eta_i^{-1}}{\partial \hat{x}^\ell}$ a vector field, cp. [23, (3.1.17), (3.1.19)]. We define the *spherical Laplacian*, also called *Laplace–Beltrami operator*, by

$$\Delta_{\mathbb{S}^2} := \nabla_{\mathbb{S}^2} \cdot \nabla_{\mathbb{S}^2}.$$

Furthermore, we denote by σ the *Lebesgue measure on the sphere* which admits for every $i \in \mathcal{I}$ the local representation

$$\mathrm{d}\sigma(x) = \sqrt{|g|(x)}\mathrm{d}\hat{x}^1 \mathrm{d}\hat{x}^2$$

on U_i by [23, (3.3.8)], where $x \in U_i$ and $\hat{x} = \eta_i(x)$. Note that for any $x_0 \in \mathbb{S}^2$, the inner product that is induced in $T_{x_0}\mathbb{S}^2$ by $g(x_0)$ does not depend on the choice of the coordinates $\{\eta_i, i \in \mathcal{I}\}$, cp. [23, (1.4.4), (1.4.5)]. For further details, the reader is referred to [23, Sections 1.4 and 3.1].

Definition 2.1. A mapping $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called a random field (RF for short) if T is measurable from $(\Omega \times \mathbb{S}^2, \mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{S}^2)$ and $\mathcal{B}(\mathbb{R})$ are the Borel σ-algebras of the respective sets.

A random field T is called *strongly isotropic* if for every $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$ and for $g \in \mathsf{SO}(3)$, the multivariate random variables $(T(x_1), \ldots, T(x_k))$ and $(T(gx_1), \ldots, T(gx_k))$ have the same law, where $\mathsf{SO}(3)$ denotes the group of rotations on $\mathbb{S}^2$.

It is called *n-weakly isotropic* for integer $n \geq 2$ if $\mathbb{E}(|T(x)|^n) < +\infty$ for every $x \in \mathbb{S}^2$ and if for $1 \leq k \leq n$, $x_1, \ldots, x_k \in \mathbb{S}^2$, and for every $g \in \mathsf{SO}(3)$,

$$\mathbb{E}(T(x_1) \cdots T(x_k)) = \mathbb{E}(T(gx_1) \cdots T(gx_k)).$$

Furthermore, the random field is called *Gaussian* if for every $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$, the multivariate random variable $(T(x_1), \ldots, T(x_k))$ is multivariate Gaussian distributed, i.e., $\sum_{i=1}^k a_i T(x_i)$ is a normally distributed random variable for every $a_i \in \mathbb{R}$, $i = 1, \ldots, k$.

In what follows, we focus on real-valued random fields. Similarly to a Gaussian random field (GRF for short) on $\mathbb{R}^d$, $d \in \mathbb{N}$, a GRF on $\mathbb{S}^2$ has the following property proven, e.g., in [28, Proposition 5.10(3)]:

Proposition 2.2. *Let T be a GRF on $\mathbb{S}^2$. Then, T is strongly isotropic if and only if T is 2-weakly isotropic.*

Therefore, 2-weakly and strongly isotropic Gaussian random fields will in the following be referred to as isotropic Gaussian random fields (iGRFs as introduced in Section 1).

2.2. **Spherical harmonics.** To introduce Karhunen–Loève expansions of iGRFs on $\mathbb{S}^2$ and also for the convergence analysis of Spectral Galerkin discretizations of (1) on $\mathbb{S}^2$, we introduce the surface spherical harmonics $Y_{\ell m}$ on $\mathbb{S}^2$. Being Karhunen–Loève eigenfunctions of iGRFs (see, e.g., [28]), in the ensuing analysis they will take a crucial role.

To define them, we recall the *classical Legendre polynomials* $(P_\ell, \ell \in \mathbb{N}_0)$ which are defined, for example, by Rodrigues' formula (see, e.g., [34])

$$P_\ell(\mu) := 2^{-\ell} \frac{1}{\ell!} \frac{\mathrm{d}^\ell}{\mathrm{d}\mu^\ell} (\mu^2 - 1)^\ell$$

for every $\ell \in \mathbb{N}_0$ and for every $\mu \in [-1, 1]$. The Legendre polynomials define the *associated Legendre functions* $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ by

$$P_{\ell m}(\mu) := (-1)^m (1 - \mu^2)^{m/2} \frac{\mathrm{d}^\ell}{\mathrm{d}\mu^\ell} P_\ell(\mu)$$

for $\ell \in \mathbb{N}_0$, $m = 0, \dots, \ell$, and $\mu \in [-1, 1]$. We further introduce the *surface spherical harmonic functions* $\mathcal{Y} := (Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ as mappings $Y_{\ell m} : [0, \pi] \times [0, 2\pi) \to \mathbb{C}$, which are given by

$$Y_{\ell m}(\vartheta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta) e^{im\varphi}$$

for $\ell \in \mathbb{N}_0$, $m = 0, \dots, \ell$, and $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$, and by $Y_{\ell m} := (-1)^m \overline{Y_{\ell -m}}$, for $\ell \in \mathbb{N}$ and $m = -\ell, \dots, -1$. In what follows, we set for $y \in \mathbb{S}^2$ $Y_{\ell m}(y) := Y_{\ell m}(\vartheta, \varphi)$, where $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$, i.e., we identify (with a slight abuse of notation) Cartesian and spherical coordinates of the point $y \in \mathbb{S}^2$.

For every $p \in [1, +\infty)$ and a measure space $(\mathcal{X}, \Sigma, \mu)$, we denote by $L^p(\mathcal{X}, \mu)$ all measurable functions $f : \mathcal{X} \to \mathbb{R}$ such that $|f|^p$ is μ-integrable, which we abbreviate with *L^p-integrable*. We equip $L^p(\mathcal{X}, \mu)$ with its usual norm, which makes it a Banach space. In the case that the measure is clear from the context of the measure space, we will omit it. Complex-valued L^p-integrable mappings are denoted by $L^p(\mathcal{X}, \mu; \mathbb{C})$. The usual inner product on the Hilbert spaces $L^2(\mathbb{S}^2)$ and $L^2(\mathbb{S}^2; \mathbb{C})$ is denoted by $(\cdot, \cdot)$. It is well-known that the functions of $\mathcal{Y}$ are orthonormal with respect to this inner product. For later use, we also record the following properties of the spherical harmonics which are for example proven in [30, Theorem 2.4.5].

Proposition 2.3. *Let $x, x' \in \mathbb{S}^2$. Then, for every $\ell \in \mathbb{N}_0$, there holds*

$$\sum_{|m| \le \ell} Y_{\ell m}(x) \overline{Y_{\ell m}(x')} = \frac{2\ell + 1}{4\pi} P_\ell(\langle x, x' \rangle_{\mathbb{R}^3})$$

as well as the identity

$$\sum_{|m| \le \ell} |Y_{\ell m}(x)|^2 = \frac{2\ell + 1}{4\pi} \tag{2}$$

for every $x \in \mathbb{S}^2$.

It is well-known (see, e.g., [29, Theorem 2.13]) that the spherical harmonics $\mathcal{Y}$ are the eigenfunctions of $-\Delta_{\mathbb{S}^2}$ with eigenvalues $(\ell(\ell + 1), \ell \in \mathbb{N}_0)$, i.e.,

$$-\Delta_{\mathbb{S}^2} Y_{\ell m} = \ell(\ell + 1) Y_{\ell m}$$

on $\mathbb{S}^2$ for every $\ell \in \mathbb{N}_0$, $m = -\ell, \dots, \ell$. Furthermore, it is shown in [29, Theorem 2.42] that $L^2(\mathbb{S}^2; \mathbb{C})$ admits the direct sum decomposition

$$L^2(\mathbb{S}^2; \mathbb{C}) = \bigoplus_{\ell=0}^{+\infty} \mathcal{H}_\ell, \tag{3}$$

where the spherical harmonic spaces $(\mathcal{H}_\ell, \ell \in \mathbb{N}_0)$ are spanned by the spherical harmonic functions

$$\mathcal{H}_\ell := \operatorname{span}\{Y_{\ell m}, m = -\ell, \dots, \ell\},$$

i.e., $\mathcal{H}_\ell$ denotes the space of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ that correspond to the eigenvalue $\ell(\ell+1)$ for $\ell \in \mathbb{N}_0$. This direct sum decomposition, cp. (3), implies with the mentioned orthonormality that $\mathcal{Y}$ is an orthonormal basis of $L^2(\mathbb{S}^2; \mathbb{C})$. Every real-valued function f in $L^2(\mathbb{S}^2; \mathbb{C})$ admits the spherical harmonics series expansion

$$f = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m} \tag{4}$$

and the coefficients satisfy (cp., e.g., [28, Remark 3.37])

$$f_{\ell m} = (-1)^m \overline{f_{\ell -m}},$$

i.e., f can be represented in $L^2(\mathbb{S}^2)$ by the series expansion

$$f = \sum_{\ell=0}^{+\infty} \Big(f_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} (\operatorname{Re} f_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} f_{\ell m} \operatorname{Im} Y_{\ell m}) \Big). \tag{5}$$

We shall be partly concerned with spectral approximations on $\mathbb{S}^2$ which are obtained by truncation of the spherical harmonics expansion (4). To state results on convergence rates of such truncations, we introduce for any truncation levels $L_1 < L_2 \in \mathbb{N}_0$ the spaces

$$\mathcal{H}_{L_1:L_2} := \bigoplus_{\ell=L_1}^{L_2} \mathcal{H}_\ell \subset L^2(\mathbb{S}^2; \mathbb{C}) \tag{6}$$

and identify $\mathcal{H}_{L:L} := \mathcal{H}_L$ for any $L \in \mathbb{N}_0$. Evidently, $\mathcal{H}_{0:L}$ is a space of finite dimension that satisfies for $L \in \mathbb{N}$ that

$$L^2 \leq N_L := \dim(\mathcal{H}_{0:L}) = (L+1)^2 \leq 4L^2 \tag{7}$$

and thus, in particular, closed. For a function $f \in L^2(\mathbb{S}^2; \mathbb{C})$ and any $L \in \mathbb{N}_0$, we denote by $\Pi_L : L^2(\mathbb{S}^2; \mathbb{C}) \to \mathcal{H}_{0:L}$ the projector on $\mathcal{H}_{0:L}$ given by the truncated Karhunen–Loève series (4) of f, i.e.,

$$\Pi_L f := \sum_{\ell=0}^{L} \sum_{|m| \leq \ell} f_{\ell m} Y_{\ell m}. \tag{8}$$

To characterize the decay of the coefficients in the expansion (4) and, accordingly, also convergence rates of the projections Π_L in (8), we introduce for a smoothness index $s \in \mathbb{R}$ and $q \in (1, +\infty)$ the Sobolev spaces on $\mathbb{S}^2$ as $H^s_q(\mathbb{S}^2) := (\operatorname{Id} - \Delta_{\mathbb{S}^2})^{-s/2} L^q(\mathbb{S}^2)$. Then, for every $f \in H^s_q(\mathbb{S}^2)$,

$$\|f\|_{H^s_q(\mathbb{S}^2)} := \|(\operatorname{Id} - \Delta_{\mathbb{S}^2})^{s/2} f\|_{L^q(\mathbb{S}^2)}$$

defines a norm on $H^s_q(\mathbb{S}^2)$. For $s < 0$, the elements of $H^s_q(\mathbb{S}^2)$ have to be understood as distributions (cp. [33, Definition 4.1]). The positive definiteness is implied by [35, Theorem XI.2.5]. These spaces are discussed in [33, 35]. Recall the C^∞ atlas $\{(U_i, \eta_i), i \in \mathcal{I}\}$ from Section 2.1. Furthermore, let $\Psi = \{\Psi_i, i \in \mathcal{I}\}$ be a C^∞ partition of unity, which is subordinate to $\{U_i, i \in \mathcal{I}\}$, i.e., $\operatorname{supp}(\Psi_i) \subset U_i$ for every $i \in \mathcal{I}$. The support of a function is the closure of the points, where the function is non-zero. We infer from [37, Theorem 7.4.5] and [16, Theorem 3.9] that Sobolev spaces on $\mathbb{S}^2$ can be equivalently characterized via pullbacks with respect to general coordinates, i.e., $v \in H^s_q(\mathbb{S}^2)$ if and only if $(v\Psi_i) \circ \eta_i^{-1} \in H^s_q(\mathbb{R}^2)$ for every $i \in \mathcal{I}$, where $v\Psi_i$ has to be understood as pointwise multiplication, and

$$v \mapsto \Big(\sum_{i \in \mathcal{I}} \|(v\Psi_i) \circ \eta_i^{-1}\|^q_{H^s_q(\mathbb{R}^2)} \Big)^{1/q}$$

is an equivalent norm on $H^s_q(\mathbb{S}^2)$, where $H^s_q(\mathbb{R}^2)$ denote the usual Bessel potential spaces on $\mathbb{R}^2$, which are equal to the Sobolev–Slobodeckij spaces for $q = 2$ with equivalent norms, cp. [38, Definition 2.3.1(d), Theorem 2.3.2(d), Equation 4.4.1(8)]. More precisely, [37, Theorem 7.4.5] implies that $H^s_q(\mathbb{S}^2)$ can be equivalently characterized via pullbacks with respect to the geodesic normal coordinates. In [16, Theorem 3.9] it is shown that the characterization of Sobolev spaces on manifolds with bounded geometry, e.g., $\mathbb{S}^2$, via pullbacks with respect to arbitrary coordinates does not depend on the coordinates and different coordinates lead to equivalent norms. We remark that a function like $(v\Psi_i) \circ \eta_i^{-1}$ on $\eta_i(U_i)$ can be extended smoothly by zero to all of $\mathbb{R}^2$, since $\Psi_i \circ \eta_i^{-1}$ is smooth and compactly supported in $\eta_i(U_i)$. For details on the geodesic normal coordinates, we refer the reader to [16, Example 3]. They are sometimes also called (Riemannian) normal coordinates cp. [23, Definition 1.4.4]. For a detailed description of Bessel potential spaces, we refer the reader to [36, Chapter 2]. In the case $q = 2$ we omit q in our notation and simply write $H^s(\mathbb{S}^2)$. In this setting $H^0(\mathbb{S}^2)$ is identified with its dual space $H^0(\mathbb{S}^2)^*$ and $H^s(\mathbb{S}^2)^* = H^{-s}(\mathbb{S}^2)$ for every $s > 0$. Since the norm on $H^s_q(\mathbb{S}^2)$ is well-defined for every $s \in \mathbb{R}$ and every $q \in (1, +\infty)$, we obtain that

$$(\mathrm{Id} - \Delta_{\mathbb{S}^2})^{s/2} : H^t_q(\mathbb{S}^2) \to H^{t-s}_q(\mathbb{S}^2) \tag{9}$$

is bounded and surjective for every $t \in \mathbb{R}$. Since $\mathcal{Y}$ diagonalizes $-\Delta_{\mathbb{S}^2}$ and therefore

$$(\mathrm{Id} - \Delta_{\mathbb{S}^2})^{s/2} Y_{\ell m} = (1 + \ell(\ell+1))^{s/2} Y_{\ell m} \tag{10}$$

for every $Y_{\ell m} \in \mathcal{Y}$ by the spectral mapping theorem, cp. [31, Theorem 10.33(a)] applied to the bounded inverse of $(\mathrm{Id} - \Delta_{\mathbb{S}^2})$ on $L^2(\mathbb{S}^2)$, we obtain the following approximation result of the operator $(\mathrm{Id} - \Pi_L)$.

Proposition 2.4. *For every $-\infty < s \le t < +\infty$ and for every $f \in H^t(\mathbb{S}^2)$,*

$$\|f - \Pi_L f\|_{H^s(\mathbb{S}^2)} \le L^{-(t-s)} \|f\|_{H^t(\mathbb{S}^2)} \le 2^{t-s} N_L^{-(t-s)/2} \|f\|_{H^t(\mathbb{S}^2)}$$

for every $L \in \mathbb{N}_0$.

Proof. Let $-\infty < s \le t < +\infty$, and $f \in H^t(\mathbb{S}^2)$. Then, for $L \in \mathbb{N}$, it holds by (4) and (10) that

$$\begin{aligned}
\|f - \Pi_L f\|^2_{H^s(\mathbb{S}^2)} &= \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell+1))^s \\
&\le \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell+1))^s \Big(\frac{\ell(\ell+1)}{L(L+1)} \Big)^{t-s} \\
&\le L^{-2(t-s)} \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell+1))^t \le L^{-2(t-s)} \|f\|^2_{H^t(\mathbb{S}^2)}.
\end{aligned}$$

The relation between L and N_L in (7) implies the assertion. □

2.3. **Karhunen–Loève expansions.** The significance of the spherical harmonic functions lies in the fact that every 2-weakly isotropic random field admits a mean-square convergent Karhunen–Loève expansion. In this section we recall these facts from [28] similarly to [26] and start with an implication of the Peter–Weyl theorem applied to $\mathsf{SO}(3)$.

Theorem 2.5 ([28, Theorem 5.13])**.** *Let T be a 2-weakly isotropic random field on $\mathbb{S}^2$, then the following statements hold true:*

(1) T satisfies $\mathbb{P}$-almost surely

$$\int_{\mathbb{S}^2} T^2 \, d\sigma < +\infty.$$

(2) T admits a Karhunen–Loève expansion

$$T = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m} \tag{11}$$

with

$$a_{\ell m} = \int_{\mathbb{S}^2} T \overline{Y_{\ell m}} \, d\sigma$$

for $\ell \in \mathbb{N}_0$ and $m \in \{-\ell, \dots, \ell\}$.

(3) The series expansion (11) *converges in $L^2(\Omega \times \mathbb{S}^2)$, i.e.,*

$$\lim_{L \to +\infty} \mathbb{E}\Big(\int_{\mathbb{S}^2} (T - \Pi_L T)^2 \, d\sigma \Big) = 0.$$

(4) The series expansion (11) *converges in $L^2(\Omega)$, i.e., for every $x \in \mathbb{S}^2$, it holds that*

$$\lim_{L \to +\infty} \mathbb{E}\big((T(x) - \Pi_L T(x))^2 \big) = 0.$$

For the efficient computational simulation of iGRFs, special properties of the random coefficients $\mathbb{A} := (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ will be useful in statements on convergence of Karhunen–Loève expansions. Its properties are analogous to those of Fourier series representations of translation invariant GRFs on the torus (see, e.g., [25]). First of all, we recall [26, Lemma 2.4].

Lemma 2.6. *Let T be a strongly isotropic random field on $\mathbb{S}^2$ with Karhunen–Loève coefficients $\mathbb{A}$. The elements of the sequence $\mathbb{A}$ are, except for a_{00}, centered random variables, i.e., $\mathbb{E}(a_{\ell m}) = 0$ for every $\ell \in \mathbb{N}$ and $m = -\ell, \dots, \ell$. Furthermore, there exists a sequence $(A_\ell, \ell \in \mathbb{N}_0)$ of nonnegative real numbers such that*

$$\mathbb{E}(a_{\ell_1 m_1} \overline{a_{\ell_2 m_2}}) = A_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2}$$

for $\ell_1, \ell_2 \in \mathbb{N}$, and $m_i = -\ell_i, \dots, \ell_i$, $i = 1, 2$, where $\delta_{nm} = 1$ if $n = m$ and zero otherwise. For the first element a_{00}, it holds that

$$\mathbb{E}(a_{00} \overline{a_{\ell m}}) = (A_0 + \mathbb{E}(a_{00})^2) \delta_{0 \ell} \delta_{0 m}.$$

The sequence $(A_\ell, \ell \in \mathbb{N}_0)$ is called the angular power spectrum *of T.*

The random variables $a_{\ell m}$ and $a_{\ell -m}$ satisfy for $\ell \in \mathbb{N}$ and $m = 1, \dots, \ell$ that

$$a_{\ell m} = (-1)^m \, \overline{a_{\ell -m}}.$$

Using the properties of Gaussian random variables, we conclude this section with the following corollary, which can also be found in [26].

Corollary 2.7. *Let T be a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$. Then, T admits the Karhunen–Loève expansion*

$$T = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

where $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ is the sequence of spherical harmonic functions and the sequence $\mathbb{A} := (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ is a sequence of complex-valued, centered, Gaussian random variables with the following properties:

(1) $\mathbb{A}_+ := (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent, complex-valued Gaussian random variables.

(2) The elements of $\mathbb{A}_+$ *with* $m > 0$ *satisfy* $\operatorname{Re} a_{\ell m}$ *and* $\operatorname{Im} a_{\ell m}$ *are independent and* $\mathcal{N}(0, A_\ell/2)$ *distributed.*
(3) The elements of $\mathbb{A}_+$ *with* $m = 0$ *are real-valued and the elements* $a_{\ell 0}$ *are* $\mathcal{N}(0, A_\ell)$ *distributed for* $\ell \in \mathbb{N}$ *while* a_{00} *is* $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$ *distributed.*
(4) The elements of $\mathbb{A}$ *with* $m < 0$ *are deduced from those of* $\mathbb{A}_+$ *by the formulae*

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell -m}, \quad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell -m}.$$

2.4. **Hölder continuity, differentiability, and approximation.** So far, our analysis of iGRFs via the Karhunen–Loève expansion focused on mean square properties.

In this section, we consider Hölder regularity and differentiability of iGRFs in p-th moment, in relation to the decay of their angular power spectrum, and we investigate the resulting convergence rates of truncations of the Karhunen–Loève expansions with respect to the corresponding norms. To be more precise, let us start with the introduction of Hölder spaces on $\mathbb{S}^2$.

For $\iota \in \mathbb{N}_0$, we denote by $C^\iota(\mathbb{S}^2)$ the space of ι-times continuously differentiable functions taking values in $\mathbb{R}$ and, for $\gamma \in (0,1)$, by $C^{\iota,\gamma}(\mathbb{S}^2) \subset C^\iota(\mathbb{S}^2)$ the subspace of functions whose ι-th derivative is Hölder continuous with exponent γ. We identify $C^{\iota,0}(\mathbb{S}^2)$ with $C^\iota(\mathbb{S}^2)$. Let $\{(U_i, \eta_i), i \in \mathcal{I}\}$ be the finite C^∞ atlas of $\mathbb{S}^2$ with subordinate partition of unity $\{\Psi_i, i \in \mathcal{I}\}$ that was introduced in Section 2.1. We equip the Hölder spaces $C^{\iota,\gamma}(\mathbb{S}^2)$, $\iota \in \mathbb{N}_0$, $\gamma \in [0,1)$, with the norm $\|\cdot\|_{C^{\iota,\gamma}(\mathbb{S}^2)}$ given by

$$\|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)} := \max_{i \in \mathcal{I}} \|(v\Psi_i) \circ \eta_i^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)}$$

for every $v \in C^{\iota,\gamma}(\mathbb{S}^2)$. This norm is well-defined, since different choices of atlases and partitions of unity will lead to equivalent norms (cp. [20, Proposition 6.9]). The Hölder spaces satisfy the Sobolev embedding that $H^s_q(\mathbb{S}^2) \subset C^{\iota,\gamma}(\mathbb{S}^2)$ is continuously embedded for $s - 2/q \geq \iota + \gamma$, $\gamma \neq 0$, which is stated and proven in Theorem A.1 in the appendix.

Let $(B, \|\cdot\|_B)$ denote a Banach space. As a special case of L^p spaces, we introduce next *Bochner spaces* $L^p(\Omega; B)$, $p \in [1, +\infty)$, where the probability space is used as measure space and which consists of all strongly B-measurable functions $X : \Omega \to B$ such that $\|X\|_B$ is in $L^p(\Omega)$. For definiteness, we recall that the elements of $L^p(\Omega; B)$ are B-valued $\mathbb{P}$-equivalence classes under the equivalence relation $\sim$, where $X \sim Y$ if $\|X - Y\|_B = 0$, $\mathbb{P}$-a.s. The Bochner integral is well-defined even though the integrands may take values in a (possibly not separable) Banach space $(B, \|\cdot\|_B)$, e.g., Hölder spaces. The integrands will have a $\mathbb{P}$-almost separable range in $(B, \|\cdot\|_B)$. This is a consequence of the strong B-measurability (cp. Definition A.2) and of Theorem A.3. In particular, for the numerical analysis in the following sections, the use of the Bochner integral is very convenient, since numerical concepts can be applied directly on samples. For details on measurability, the reader is referred to Appendix A. Let us denote by $\|\cdot\|_{L^p(\Omega;B)}$ the corresponding norm, which is naturally given by

$$\|X\|_{L^p(\Omega;B)} := \Big(\mathbb{E}\big(\|X\|_B^p\big)\Big)^{1/p}$$

for $X \in L^p(\Omega; B)$, and which makes $(L^p(\Omega; B), \|\cdot\|_{L^p(\Omega;B)})$ a Banach space (cp. [9, Theorem III.6.6]).

Since $L^2(\Omega \times \mathbb{S}^2)$ and $L^2(\Omega; L^2(\mathbb{S}^2))$ are equal, Theorem 2.5 implies that every iGRF T is in $L^2(\Omega; L^2(\mathbb{S}^2))$ and satisfies by Proposition 2.3 and Corollary 2.7 that

$$\|T\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} = \sum_{\ell=0}^{+\infty} A_\ell \frac{2\ell+1}{4\pi},$$

i.e., $\sum_{\ell=0}^{+\infty} A_\ell \ell$ converges and is finite.

It was shown in [26, Theorem 4.6] that summability of the angular power spectrum with respect to higher orders of ℓ leads to regularity of the iGRF. More precisely, the finiteness of $\sum_{\ell=0}^{+\infty} A_\ell \ell^{1+\beta}$ implies the existence of a modification in $C^{\iota,\gamma}(\mathbb{S}^2)$ for $\iota + \gamma < \beta/2$.

In the present paper, we develop strong measurability and L^p-integrability of iGRFs, i.e., possible choices of ι and γ such that $T \in L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$, which will then imply [26, Theorem 4.6]. Furthermore, we will develop a convergence rate analysis of the projected fields in these spaces. Also it will be sufficient to obtain particular regularity for $\mathbb{P}$-a.e. sample. Additionally to Definition 2.1 we will only consider continuous iGRFs to ease the following presentation, which exists due to [26, Theorem 4.5] or [20, Theorem 5.4] whenever the angular power spectrum satisfies that

$$\sum_{\ell=0}^{+\infty} A_\ell \ell^{1+\beta} < +\infty \tag{12}$$

for some $\beta > 0$. These results are presented in the following theorem and can also be found as [20, Theorem 6.20].

Theorem 2.8. *Let T be a continuous iGRF that satisfies* (12) *for some $\beta > 0$. Then, for every $p \in [1, +\infty)$, $\iota \in \mathbb{N}_0$, and $\gamma \in (0, 1)$ with $\iota + \gamma < \beta/2$, it holds that $T \in L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$. Furthermore, there exists a constant $C_{p,\iota,\gamma}$, which is independent of $(A_\ell, \ell \in \mathbb{N}_0)$ such that for every $L \in \mathbb{N}_0$,*

$$\|T - \Pi_L T\|_{L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))} \le C_{p,\iota,\gamma} \Big(\sum_{\ell > L} A_\ell \ell^{1+\beta} \Big)^{1/2}.$$

Proof. It suffices to prove the theorem for p even, i.e., for $p = 2p'$ and $p' \in \mathbb{N}$. The result for all remaining $p \in [1, +\infty)$ follows then by Hölder's inequality. We set $T^L := \Pi_L T$ and show first that $(T^L, L \in \mathbb{N}_0)$ is a Cauchy sequence in $L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$. The smoothness of the spherical harmonics implies with Pettis' theorem (see Theorem A.3) that T^L is strongly measurable in every function space that contains $C^\infty(\mathbb{S}^2)$, $L \in \mathbb{N}_0$. In particular, T^L is strongly B-measurable, $B \in \{C^{\iota,\gamma}(\mathbb{S}^2), H_q^{\beta/2}(\mathbb{S}^2), q \in (1, +\infty)\}$ for every $L \in \mathbb{N}_0$. With the properties of spherical harmonics and the Karhunen–Loève expansion from Corollary 2.7, we observe that $\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$ is $\mathcal{N}(0, (2\ell+1)/(4\pi)\, A_\ell)$-distributed for every $\ell \in \mathbb{N}$ and $x \in \mathbb{S}^2$ as well as that $(\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \ell \in \mathbb{N}_0)$ is a sequence of independent random variables for every fixed $x \in \mathbb{S}^2$. Hence, for $L_1 > L_2 \in \mathbb{N}_0$, we obtain with Fubini's theorem and the fact that for centered Gaussian random variables, $\mathbb{E}(X^{2p'}) = (2p')!/(2^{p'} p'!)\, \mathbb{E}(X^2)^{p'}$ that

$$\begin{aligned} \|T^{L_1} - T^{L_2}\|^{2p'}_{L^{2p'}(\Omega;H^{\beta/2}_{2p'}(\mathbb{S}^2))} &= \int_{\mathbb{S}^2} \mathbb{E}\Big(\Big(\sum_{\ell=L_2+1}^{L_1} \sum_{m=-\ell}^{\ell} a_{\ell m} (1 + \ell(\ell+1))^{\beta/4} Y_{\ell m} \Big)^{2p'} \Big) \mathrm{d}\sigma \\ &= \frac{(2p')!}{2^{p'} p'!} |\mathbb{S}^2| \Big(\sum_{\ell=L_2+1}^{L_1} A_\ell \frac{2\ell+1}{4\pi} (1 + \ell(\ell+1))^{\beta/2} \Big)^{p'} < +\infty, \end{aligned}$$

where finiteness follows since (12) holds. This implies especially with the Sobolev embedding (cp. Theorem A.1) that there exists a constant C such that

$$\|T^{L_1} - T^{L_2}\|_{L^{2p'}(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))} \leq C\Big(4\pi\frac{(2p')!}{2^{p'}p'!}\Big)^{1/(2p')}\Big(\sum_{\ell=L_2+1}^{L_1} A_\ell \frac{2\ell+1}{4\pi}(1+\ell(\ell+1))^{\beta/2}\Big)^{1/2}$$

for $\beta/2-1/p' \geq \iota+\gamma$ and therefore that $(T^L, L\in\mathbb{N}_0)$ is a Cauchy sequence in $L^{2p'}(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))$ that converges due to completeness. Furthermore, the result extends by Hölder's inequality to $L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))$ for every $p\leq 2p'$. Since L^p limits are $\mathbb{P}$-almost surely unique and we know from Theorem 2.5 that $(T^L, L\in\mathbb{N}_0)$ converges to T in $L^2(\Omega;L^2(\mathbb{S}^2))$, $T\in L^{2p'}(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))$ holds also due to the assumed continuity.

For given $p\geq 1$, we choose $p'\in\mathbb{N}$ such that $p\leq 2p'$ and $\beta/2-1/p'\geq\iota+\gamma$ for fixed ι and γ. This implies that there exists a constant $C_{p'}$, i.e., $C_{p,\iota,\gamma}$, such that

$$\|T^{L_1} - T^{L_2}\|_{L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))} \leq C_{p,\iota,\gamma}\Big(\sum_{\ell=L_2+1}^{L_1} A_\ell \ell^{1+\beta}\Big)^{1/2}. \tag{13}$$

We obtain the claimed convergence rate by taking the limit $L_1\to+\infty$ in (13). □

Remark 2.9. Since $T\in L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))$ in the framework of Theorem 2.8, this theorem implies that $T\in C^{\iota,\gamma}(\mathbb{S}^2)$, $\mathbb{P}$-a.s. By setting $T(\omega):=0$ for every ω in the remaining set of measure zero, we obtain a modification that is in $C^{\iota,\gamma}(\mathbb{S}^2)$ and therefore we recover [26, Theorem 4.6] with a different approach under the continuity assumption on the random field, which can be established by the construction and existence of a continuous modification in [26, Theorem 4.5] or [20, Theorem 5.4] for all cases covered by [26, Theorem 4.6].

Remark 2.10. If T,ι,γ satisfy the conditions of Theorem 2.8, it implies that the iGRF T is strongly $C^{\iota,\gamma}(\mathbb{S}^2)$-measurable and hence by Theorem A.3 T is $\mathbb{P}$-almost separably-valued in $C^{\iota,\gamma}(\mathbb{S}^2)$, i.e., there exists a measurable set Ω^* satisfying $\mathbb{P}(\Omega^*)=1$ such that the range $\{T(\omega),\omega\in\Omega^*\}$ is separable in $C^{\iota,\gamma}(\mathbb{S}^2)$. This illustrates the applicability of the Bochner integral although $C^{\iota,\gamma}(\mathbb{S}^2)$ is not separable.

2.5. **Isotropic lognormal random fields.** In this section we consider isotropic lognormal random fields on $\mathbb{S}^2$, which have been considered in [20, 26]. Specifically, if T is a continuous iGRF on $\mathbb{S}^2$ we are interested in $a:=\exp(T)$ given by $a(x):=\exp(T(x))$ for every $x\in\mathbb{S}^2$. In our setting in Section 3, isotropic lognormal random fields are of interest as diffusion coefficients of elliptic differential operators. For the approximation of these lognormal random fields, we set for every $L\in\mathbb{N}_0$

$$a^L := \exp(\Pi_L T).$$

Our goal in this section is to establish regularity and approximation results for a^L converging to a that correspond to Theorem 2.8 in the case of a continuous iGRF. Questions of this kind have been addressed in [20, Section 7.2], whereas sample regularity exclusively is addressed in [26, Section 6]. First, we show that a and a^L, $L\in\mathbb{N}_0$, are in $L^p(\Omega;C^0(\mathbb{S}^2))$ for $p\in[1,+\infty)$. The following proposition can be found as [20, Proposition 7.5]. We present the proof for the sake of completeness.

Proposition 2.11. *Let $p\in[1,+\infty)$, and T be a continuous iGRF such that* (12) *is satisfied for some $\beta>0$. Then, $\exp(T),\exp(\Pi_L T)\in L^p(\Omega;C^0(\mathbb{S}^2))$ for every $L\in\mathbb{N}_0$ and the $L^p(\Omega;C^0(\mathbb{S}^2))$-norm of $\exp(\Pi_L T)$ can be bounded independently of L.*

Proof. It will be sufficient to prove the case that T is centered, i.e., $\mathbb{E}(T) = 0 \in C^0(\mathbb{S}^2)$. By Definition 2.1 $\mathbb{E}(T)$ is a constant function on $\mathbb{S}^2$, which implies $\| \exp(T)\|_{L^p(\Omega;C^0(\mathbb{S}^2))} = \| \exp(T - \mathbb{E}(T))\|_{L^p(\Omega;C^0(\mathbb{S}^2))} \exp(\mathbb{E}(T))$. Hence, the general case can be reduced to the case of a centered, continuous iGRF. So in the following we can assume that T is centered.

The idea of the proof is to apply Fernique's theorem, cp. [7, Theorem 2.7], on the separable Banach space $C^0(\mathbb{S}^2)$. Therefore, we have to establish that the law of T is a centered (symmetric) Gaussian measure on $C^0(\mathbb{S}^2)$, i.e., for every $\mathcal{G} \in C^0(\mathbb{S}^2)^*$, the dual space of $C^0(\mathbb{S}^2)$, there exists $\sigma_{\mathcal{G}} \in [0, +\infty)$ such that $\mathcal{G}(T) \sim \mathcal{N}(0, \sigma_{\mathcal{G}}^2)$. This is the first requirement in order to apply [7, Theorem 2.7]. We remark that in [7] the term 'symmetric' Gaussian measure is used instead of centered meaning the same. For every $L \in \mathbb{N}_0$, $\Pi_L T$ has the finite real expansion according to (5)

$$\Pi_L T = \sum_{\ell=0}^{L} \Big(a_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} (\operatorname{Re} a_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} a_{\ell m} \operatorname{Im} Y_{\ell m}) \Big).$$

From Corollary 2.7 we deduce that $\{a_{\ell 0}, \operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, \ell\}$ are independent real-valued random variables. Additionally this corollary implies that $a_{\ell 0} \sim \mathcal{N}(0, A_\ell)$ and $\operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m} \sim \mathcal{N}(0, A_\ell/2)$ for $\ell \in \mathbb{N}_0$ and $m = 1, \ldots, \ell$. Let $L \in \mathbb{N}_0$ and $\mathcal{G} \in C^0(\mathbb{S}^2)^*$ be arbitrary. Hence,

$$\mathcal{G}(\Pi_L T) = \sum_{\ell=0}^{L} \Big(a_{\ell 0} \mathcal{G}(Y_{\ell 0}) + 2 \sum_{m=1}^{\ell} (\operatorname{Re} a_{\ell m} \mathcal{G}(\operatorname{Re} Y_{\ell m}) - \operatorname{Im} a_{\ell m} \mathcal{G}(\operatorname{Im} Y_{\ell m})) \Big) \sim \mathcal{N}(0, \sigma_{\mathcal{G},L}^2)$$

and therefore the characteristic function $\varphi_{\mathcal{G},L}$ of $\mathcal{G}(\Pi_L T)$ is given by

$$\lambda \mapsto \varphi_{\mathcal{G},L}(\lambda) := \exp\Big(-\frac{1}{2} \lambda^2 \sigma_{\mathcal{G},L}^2 \Big),$$

where

$$\sigma_{\mathcal{G},L}^2 = \sum_{\ell=0}^{L} A_\ell \Big(\mathcal{G}(Y_{\ell 0})^2 + 2 \sum_{m=1}^{\ell} (\mathcal{G}(\operatorname{Re} Y_{\ell m})^2 + \mathcal{G}(\operatorname{Im} Y_{\ell m})^2) \Big).$$

Thus, $\Pi_L T$ is a centered Gaussian measure on $C^0(\mathbb{S}^2)$ for every $L \in \mathbb{N}_0$. The next step is to show that the sequence $(\sigma_{\mathcal{G},L}^2, L \in \mathbb{N}_0)$ is uniformly bounded. The Riesz representation theorem for $C^0(\mathbb{S}^2)$ (cp. [3, Theorem 7.10.4]) and [3, Theorem 3.1.1, Remark 3.1.5] imply that there exist a finite, positive measure ν on $(\mathbb{S}^2, \mathcal{B}(\mathbb{S}^2))$ and a measurable function g satisfying $|g(x)| = 1$ for every $x \in \mathbb{S}^2$ such that $\mathcal{G}(v) = \int_{\mathbb{S}^2} v g \mathrm{d}\nu$ for every $v \in C^0(\mathbb{S}^2)$, which implies with the Cauchy–Schwarz inequality that for every $v \in C^0(\mathbb{S}^2)$,

$$\mathcal{G}(v)^2 = \Big(\int_{\mathbb{S}^2} v g \mathrm{d}\nu \Big)^2 \leq \|v\|_{L^2(\mathbb{S}^2,\nu)}^2 \|g\|_{L^2(\mathbb{S}^2,\nu)}^2 = \|v\|_{L^2(\mathbb{S}^2,\nu)}^2 \nu(\mathbb{S}^2).$$

This implies with (2) from Proposition 2.3 that

$$\begin{aligned} &\mathcal{G}(Y_{\ell 0})^2 + 2 \sum_{m=1}^{\ell} (\mathcal{G}(\operatorname{Re} Y_{\ell m})^2 + \mathcal{G}(\operatorname{Im} Y_{\ell m})^2) \\ &\qquad \leq \Big(\int_{\mathbb{S}^2} Y_{\ell 0}^2 + 2 \sum_{m=1}^{\ell} ((\operatorname{Re} Y_{\ell m})^2 + (\operatorname{Im} Y_{\ell m})^2) \mathrm{d}\nu \Big) \nu(\mathbb{S}^2) \end{aligned}$$

$$= \Big(\int_{\mathbb{S}^2} \sum_{m=-\ell}^{\ell} |Y_{\ell m}|^2 \mathrm{d}\nu\Big)\nu(\mathbb{S}^2) = \nu(\mathbb{S}^2)^2 \frac{2\ell+1}{4\pi}.$$

Summing the previous inequality over ℓ implies with the finiteness of $\sum_{\ell\geq 0} A_\ell \frac{2\ell+1}{4\pi}$ that $(\sigma^2_{\mathcal{G},L}, L \in \mathbb{N}_0)$ is uniformly bounded in L. Hence, there exists a unique $\sigma_{\mathcal{G}} \in [0,+\infty)$ such that $\sigma^2_{\mathcal{G},L} \to \sigma^2_{\mathcal{G}}$ as $L \to +\infty$. Thus, $\lim_{L\to+\infty} \varphi_{\mathcal{G},L}(\lambda) = \exp(-1/2\ \lambda^2\sigma^2_{\mathcal{G}}) =: \varphi_{\mathcal{G}}(\lambda)$ for every $\lambda \in \mathbb{R}$. The $L^2(\Omega; C^0(\mathbb{S}^2))$-convergence of $\Pi_L T \to T$, which is implied by Theorem 2.8, yields that $\mathcal{G}(\Pi_L T) \to \mathcal{G}(T)$ in $L^2(\Omega)$ and thus in distribution. Lévy's continuity theorem, cp. [27, Theorem IV.13.2.B], implies that $\mathcal{G}(T) \sim \mathcal{N}(0, \sigma^2_{\mathcal{G}})$ and we conclude that the law of T is a centered (symmetric) Gaussian measure on $C^0(\mathbb{S}^2)$.

We infer from Theorem 2.8 that there exists an upper bound K of the $L^2(\Omega; C^0(\mathbb{S}^2))$-norm of T and of $\Pi_L T$, $L \in \mathbb{N}_0$, which is uniform in L. Let in the following $X \in \{T, \Pi_L T, L \in \mathbb{N}_0\}$. We choose $x_0 \in [1/(1+\exp(-2)), 1)$, which implies that $\log((1-x_0)/x_0) \leq -2$, and set $r_0 := K/\sqrt{1-x_0}$. We use the Chebychev inequality to obtain that

$$1 - \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0) = \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} > r_0) \leq \frac{\mathbb{E}(\|X\|^2_{C^0(\mathbb{S}^2)})}{r_0^2} \leq \frac{K^2}{r_0^2} = 1 - x_0,$$

which implies that $\mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0) \geq x_0$. We choose $\lambda > 0$ such that $\lambda \leq (1-x_0)/(32K^2)$, which implies that $32\lambda r_0^2 \leq 1$, and arrive with the monotonicity of the logarithm at the inequality

$$\log\Big(\frac{1 - \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0)}{\mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0)}\Big) + 32\lambda r_0^2 \leq \log\Big(\frac{1-x_0}{x_0}\Big) + 32\lambda r_0^2 \leq -1.$$

This is the second requirement for [7, Theorem 2.7]. Since X is a centered Gaussian measure on $C^0(\mathbb{S}^2)$, [7, Theorem 2.7] implies that

$$\mathbb{E}(\exp(\lambda\|X\|^2_{C^0(\mathbb{S}^2)})) \leq \exp(16\lambda r_0^2) + \frac{\exp(2)}{\exp(2)-1},$$

which is a bound that is independent of L, because the choices of r_0 and λ do not depend on L due to the uniformity of the bound K. Since $0 \leq (\sqrt{\lambda}x - p/(2\sqrt{\lambda}))^2$ implies that $px \leq \lambda x^2 + p^2/(4\lambda)$ for every $x \in \mathbb{R}$, we conclude that

$$\mathbb{E}(\|\exp(X)\|^p_{C^0(\mathbb{S}^2)}) \leq \mathbb{E}(\exp(p\|X\|_{C^0(\mathbb{S}^2)})) \leq \mathbb{E}(\exp(\lambda\|X\|^2_{C^0(\mathbb{S}^2)}) \exp\Big(\frac{p^2}{4\lambda}\Big),$$

which finishes the proof of the proposition. □

We remark that the last paragraph of the proof is similar to the proof of [5, Propositions 3.10], but here it is extended and in a slightly different context. Furthermore, Fernique's theorem was originally proven by Fernique in [11].

In the discussion of (1), several random quantities are of importance which are introduced in what follows. For a continuous iGRF T, we set already $a := \exp(T)$ and further define

$$\hat{a} := \max_{x\in\mathbb{S}^2} a(x) \quad \text{and} \qquad \check{a} := \min_{x\in\mathbb{S}^2} a(x),$$

and for every $L \in \mathbb{N}_0$ we set previously $a^L := \exp(\Pi_L T)$ and further define now

$$\hat{a}^L := \max_{x\in\mathbb{S}^2} a^L(x) \quad \text{and} \quad \check{a}^L := \min_{x\in\mathbb{S}^2} a^L(x),$$

which are elements of $L^p(\Omega)$, $p \in [1,+\infty)$, by the made assumptions.

Corollary 2.12. *Let T be a continuous iGRF, then $\hat{a}$, $\check{a}^{-1}$, $\hat{a}^L$, and $(\check{a}^L)^{-1}$ are in $L^p(\Omega)$ for every $p \in [1,+\infty)$ and every $L \in \mathbb{N}_0$, where the $L^p(\Omega)$-norm of $\hat{a}^L$ and $(\check{a}^L)^{-1}$ can be bounded independently of L.*

Proof. Since

$$\|\check{a}^{-1}\|_{L^p(\Omega)} = \|(\min_{x\in\mathbb{S}^2} a(x))^{-1}\|_{L^p(\Omega)} = \|\max_{x\in\mathbb{S}^2} \exp(-T(x))\|_{L^p(\Omega)} = \|\exp(-T)\|_{L^p(\Omega;C^0(\mathbb{S}^2))}$$

and

$$\|\hat{a}\|_{L^p(\Omega)} = \|\exp(T)\|_{L^p(\Omega;C^0(\mathbb{S}^2))},$$

the claim follows with Proposition 2.11 applied to $-T$ and T. The assertion for $\hat{a}^L$ and $\check{a}^L$ follows with the same arguments. □

Lemma 2.13. *Let $\iota \in \mathbb{N}_0$ and $\gamma \in (0,1)$ then there exists a constant $C_{\iota,\gamma}$ such that for every $v \in C^{\iota,\gamma}(\mathbb{S}^2)$,*

$$\|\exp(v)\|_{C^{\iota,\gamma}(\mathbb{S}^2)} \leq C_{\iota,\gamma}\|\exp(v)\|_{C^0(\mathbb{S}^2)}\big(1 + \|v\|^{\iota+1}_{C^{\iota,\gamma}(\mathbb{S}^2)}\big). \tag{14}$$

Proof. Generally, this proof is inspired by [22, Theorem A.8], but it achieves a specific result that is not covered by [22, Theorem A.8].

Let $D \subset \mathbb{R}^2$ be a bounded, convex open domain. The first step is to prove the estimate for Hölder spaces over Euclidean domains, i.e., $C^{\iota,\gamma}(\overline{D})$. We set $g := \exp$ and recall that the derivative g' is again equal to g.

For convenience, we will omit the set $\overline{D}$ if the context is clear. For $\iota = 0$, it is easily seen that for every $\tilde{v} \in C^{0,\gamma}(\overline{D})$, it holds that $\|g(\tilde{v})\|_{C^{0,\gamma}} \leq \|g(\tilde{v})\|_{C^0}(1 + \|\tilde{v}\|_{C^{0,\gamma}})$, cp. the proof of [22, Theorem A.8], which is the base case of an induction argument to the following induction hypothesis:

Let the estimate in (14) be satisfied for Hölder spaces over the Euclidean set $\overline{D}$, i.e., for functions $\tilde{v} \in C^{n,\gamma}(\overline{D})$ for every $n \in \{0,\dots,\iota-1\}$. We directly perform the induction step from $n = \iota - 1$ to $n+1 = \iota$. For $\tilde{v}_1, \tilde{v}_2 \in C^{\iota,\gamma}(\overline{D})$, the product estimate $\|\tilde{v}_1\tilde{v}_2\|_{C^{\iota,\gamma}} \leq \hat{C}_{\iota,\gamma}\|\tilde{v}_1\|_{C^{\iota,\gamma}}\|\tilde{v}_2\|_{C^{\iota,\gamma}}$ holds by [22, Theorem A.7], which implies with the chain rule from calculus and the induction hypothesis that

$$\begin{aligned}
\|g(\tilde{v})\|_{C^{\iota,\gamma}} &= \|g(\tilde{v})\|_{C^0} + \sum_{j=1,2}\|\partial_{x^j}(g\circ\tilde{v})\|_{C^{\iota-1,\gamma}} \\
&\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma}\sum_{j=1,2}\|g'(\tilde{v})\|_{C^{\iota-1,\gamma}}\|\partial_{x^j}\tilde{v}\|_{C^{\iota-1,\gamma}} \\
&\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma}\|g(\tilde{v})\|_{C^{\iota-1,\gamma}}\|\tilde{v}\|_{C^{\iota,\gamma}} \\
&\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma}\widetilde{C}_{\iota-1,\gamma}\|g(\tilde{v})\|_{C^0}(1 + \|\tilde{v}\|^{\iota}_{C^{\iota-1,\gamma}})\|\tilde{v}\|_{C^{\iota,\gamma}} \\
&\leq \widetilde{C}_{\iota,\gamma}\|g(\tilde{v})\|_{C^0}(1 + \|\tilde{v}\|^{\iota+1}_{C^{\iota,\gamma}}),
\end{aligned} \tag{15}$$

and finishes the induction step. Note that for convenience we used $\partial_{x^j} := \partial/\partial x^j$, $j = 1,2$.

Next let $\{(U_i,\eta_i), i \in \mathcal{I}\}$ be a finite C^∞ atlas and $\{\Psi_i, i \in \mathcal{I}\}$ a C^∞ partition of unity subordinate to $\{U_i, i \in \mathcal{I}\}$. We fix $j \in \mathcal{I}$ and choose another C^∞ partition of unity $\{\hat{\Psi}_i, i \in \mathcal{I}\}$ subordinate to $\{U_i, i \in \mathcal{I}\}$ such that $\hat{\Psi}_j = 1$ on $\operatorname{supp}(\Psi_j)$. We can assume that $\overline{D} :=$

$\operatorname{supp}(\Psi_j \circ \eta_j^{-1})$ and $\operatorname{supp}(\hat{\Psi}_j \circ \eta_j^{-1})$ are convex and observe with (15) that

$$\begin{aligned}\|(\exp(v)\Psi_j) \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)} &= \|(\exp(v)\Psi_j) \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\overline{D})} \\ &\le \hat{C}_{\iota,\gamma} \|\exp(v) \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\overline{D})} \|\Psi_j \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\overline{D})} \\ &\le C_{\iota,\gamma} \|\exp(v) \circ \eta_j^{-1}\|_{C^0(\overline{D})} (1 + \|v \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\overline{D})}^{\iota+1}) \\ &\le C_{\iota,\gamma} \|(\exp(v)\hat{\Psi}_j) \circ \eta_j^{-1}\|_{C^0(\mathbb{R}^2)} \big(1 + \|(v\hat{\Psi}_j) \circ \eta_j^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)}^{\iota+1}\big).\end{aligned} \tag{16}$$

We apply that different C^∞ partitions of unity result in equivalent norms on $C^{\iota,\gamma}(\mathbb{S}^2)$ and conclude the estimate of the lemma by taking the maximum over j on both sides of (16). □

A version of the following theorem (with a different proof) can be found in [20, Theorem 7.7].

Theorem 2.14. *Let $a = \exp(T)$ be an isotropic lognormal RF such that T is a continuous iGRF satisfying (12) for some $\beta > 0$. Then, for every $p \in [1, +\infty)$, $\iota \in \mathbb{N}_0$, and $\gamma \in (0,1)$ satisfying $\iota + \gamma < \beta/2$, and every $L \in \mathbb{N}_0$, it holds that $a, a^L \in L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$, where the $L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$-norm of a^L can be bounded independently of L. Furthermore, for every $\varepsilon \in (0,\beta)$, there exists a constant $C_{p,\varepsilon}$ such that for every $L \in \mathbb{N}_0$, it holds that*

$$\|a - a^L\|_{L^p(\Omega;C^0(\mathbb{S}^2))} \le C_{p,\varepsilon} \Big(\sum_{\ell > L} A_\ell \ell^{1+\varepsilon}\Big)^{\frac{1}{2}}.$$

Proof. Since the composition with the exponential function is a continuous mapping from $C^{\iota,\gamma}(\mathbb{S}^2)$ into itself and T is strongly $C^{\iota,\gamma}(\mathbb{S}^2)$-measurable by Theorem 2.8, Lemma A.4 implies strong $C^{\iota,\gamma}(\mathbb{S}^2)$-measurability of $a = \exp(T)$ and of $a^L = \exp(\Pi_L T)$ for every $L \in \mathbb{N}_0$.

Since T as $C^{\iota,\gamma}(\mathbb{S}^2)$-valued $\mathbb{P}$-equivalence class takes values in $C^{\iota,\gamma}(\mathbb{S}^2)$, Lemma 2.13 is applicable and implies with the Cauchy–Schwarz inequality that there exists a constant C that does not depend on T such that

$$\|a\|_{L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))} = \|\exp(T)\|_{L^p(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))} \le C \|\exp(T)\|_{L^{2p}(\Omega;C^0(\mathbb{S}^2))} (1 + \|T\|_{L^{2p(\iota+1)}(\Omega;C^{\iota,\gamma}(\mathbb{S}^2))}^{\iota+1}).$$

The first assertion is then obtained with Theorem 2.8 and Proposition 2.11. The second assertion about a^L is proven completely analogously and the $L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$-norm of a^L can be bounded independently of L due to Theorem 2.8 and Proposition 2.11.

For the proof of the third claim, note that the fundamental theorem of calculus implies for arbitrary $t, s \in \mathbb{R}$ that $|\exp(t) - \exp(s)| \le (\exp(t) + \exp(s))|t - s|$, which yields with the Cauchy–Schwarz inequality that

$$\|a - a^L\|_{L^p(\Omega;C^0(\mathbb{S}^2))} \le \big(\|a\|_{L^{2p}(\Omega;C^0(\mathbb{S}^2))} + \|a^L\|_{L^{2p}(\Omega;C^0(\mathbb{S}^2))}\big) \|T - \Pi_L T\|_{L^{2p}(\Omega;C^0(\mathbb{S}^2))}.$$

Therefore, the third assertion follows with Theorem 2.8, since the $L^{2p}(\Omega; C^0(\mathbb{S}^2))$-norms of a and a^L are bounded uniformly in L due to Proposition 2.11. □

3. Existence, uniqueness, and regularity of solutions

3.1. Existence and uniqueness.

We now turn to the basic existence and uniqueness result for the elliptic SPDE (1)

$$-\nabla_{\mathbb{S}^2} \cdot (a \nabla_{\mathbb{S}^2} u) = f, \tag{1}$$

where $a = \exp(T)$ is an isotropic lognormal RF resulting from a continuous iGRF T that satisfied (12) for some $\beta > 0$. The right hand side f of (1) is assumed to be deterministic.

Since $\mathbb{S}^2$ is a closed, compact submanifold of $\mathbb{R}^3$ without boundary, solutions of PDEs may exhibit nonuniqueness due to the random differential operator $-\nabla_{\mathbb{S}^2} \cdot (a\nabla_{\mathbb{S}^2})$ having a nontrivial kernel, i.e., $u \equiv$ const. is a solution of the homogeneous equation. Therefore, we shall work in factor spaces of function spaces which are orthogonal (in $L^2(\mathbb{S}^2)$) to constants. The closed subspace of $H^1(\mathbb{S}^2)$ that consists of all $v \in H^1(\mathbb{S}^2)$ whose inner product with 1 satisfies $(v, 1) = 0$ is denoted by $H^1(\mathbb{S}^2)/\mathbb{R}$. For every $v \in H^1(\mathbb{S}^2)/\mathbb{R}$,

$$\|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}} := \|\nabla_{\mathbb{S}^2} v\|_{L^2(\mathbb{S}^2)}$$

defines a norm on $H^1(\mathbb{S}^2)/\mathbb{R}$, since the second Poincaré inequality implies that

$$\|v\|_{L^2(\mathbb{S}^2)} \leq \frac{1}{\sqrt{2}} \|\nabla_{\mathbb{S}^2} v\|_{L^2(\mathbb{S}^2)}.$$

Remark 3.1. The closed subspace of $H^1(\mathbb{S}^2)$ of functions that are orthogonal to constants in $L^2(\mathbb{S}^2)$ and the factor space of $H^1(\mathbb{S}^2)$ with respect to the equivalence relation $\sim$, where $v_1, v_2 \in H^1(\mathbb{S}^2)$ satisfy that $v_1 \sim v_2$ if and only if there exists a constant c such that $v_1 = v_2 + c$ are isometrically isomorphic. That justifies to denote both by the symbol $H^1(\mathbb{S}^2)/\mathbb{R}$. The isometry between them will preserve strong measurability due to Lemma A.4. In the following we will use the first interpretation of $H^1(\mathbb{S}^2)/\mathbb{R}$ as a closed subspace of $H^1(\mathbb{S}^2)$.

Since $H^1(\mathbb{S}^2)/\mathbb{R}$ is a closed linear subspace of $H^1(\mathbb{S}^2)$ and the norm $\|\cdot\|_{H^1(\mathbb{S}^2)/\mathbb{R}}$ is induced by the inner product $(\nabla_{\mathbb{S}^2}\cdot, \nabla_{\mathbb{S}^2}\cdot)$, $H^1(\mathbb{S}^2)/\mathbb{R}$ is a Hilbert space.

We consider the variational formulation of the SPDE in (1) in $H^1(\mathbb{S}^2)/\mathbb{R}$ with right hand side $f \in H^{-1}(\mathbb{S}^2)$ such that $f(1) = 0$: find a strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable mapping u such that

$$(a\nabla_{\mathbb{S}^2} u, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}. \tag{17}$$

Moreover, we want to show that this mapping $u : \Omega \to H^1(\mathbb{S}^2)/\mathbb{R}$ is L^p-integrable. To this end, let us fix $f \in H^{-1}(\mathbb{S}^2)$ such that $f(1) = 0$. In the flavour of the Bochner spaces $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$, $p \in [1, +\infty)$, and $s \geq 0$, we will to this end consider the solution u as $H^1(\mathbb{S}^2)/\mathbb{R}$-valued $\mathbb{P}$-equivalence class.

Remark 3.2. Since all continuous modifications of a are in the same $C^0(\mathbb{S}^2)$-valued $\mathbb{P}$-equivalence class, this notion of solution is independent of a particularly chosen continuous representative a.

We recall certain elements of the deterministic existence and uniqueness theory in order to expose how it extends to the SPDE (17). In the following we frequently require the space

$$C^0_+(\mathbb{S}^2) := \{\tilde{a} \in C^0(\mathbb{S}^2), \min_{x \in \mathbb{S}^2} \tilde{a}(x) > 0\}.$$

For every $\tilde{a} \in C^0_+(\mathbb{S}^2)$, the bilinear form $(\tilde{a}\nabla_{\mathbb{S}^2}\cdot, \nabla_{\mathbb{S}^2}\cdot)$ is continuous and coercive on the space $H^1(\mathbb{S}^2)/\mathbb{R} \times H^1(\mathbb{S}^2)/\mathbb{R}$, i.e.,

$$(\tilde{a}\nabla_{\mathbb{S}^2} v, \nabla_{\mathbb{S}^2} w) \leq \|\tilde{a}\|_{C^0(\mathbb{S}^2)} \|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \|w\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \quad \forall v, w \in H^1(\mathbb{S}^2)/\mathbb{R} \tag{18}$$

and

$$\|v\|^2_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\tilde{a}\nabla_{\mathbb{S}^2} v, \nabla_{\mathbb{S}^2} v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}. \tag{19}$$

Existence and uniqueness of a solution $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$ to the respective variational PDE is then implied by the Lax–Milgram lemma, i.e.,

$$(\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}, \tag{20}$$

as well as the estimate

$$\|\tilde{u}\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \le \frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}\sqrt{\frac{3}{2}}\|f\|_{H^{-1}(\mathbb{S}^2)}, \tag{21}$$

where we used $\sup_{0\neq v\in H^1(\mathbb{S}^2)/\mathbb{R}} |f(v)|/\|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \le \sqrt{3/2}\|f\|_{H^{-1}(\mathbb{S}^2)}$.

The difference of two solutions with respect to different coefficients and the same right hand side can be estimated with a version of Strang's second lemma. This is stated as the next lemma, where the PDE in (20) will also be considered with respect to subspaces of $H^1(\mathbb{S}^2)/\mathbb{R}$ as well, to be applicable in Section 4.

Lemma 3.3. *Let $V \subset H^1(\mathbb{S}^2)/\mathbb{R}$ be a not necessarily strict subspace of $H^1(\mathbb{S}^2)/\mathbb{R}$ endowed with the $H^1(\mathbb{S}^2)/\mathbb{R}$-norm. For $\tilde{a}_1, \tilde{a}_2 \in C^0_+(\mathbb{S}^2)$, let $\tilde{u}_1, \tilde{u}_2 \in V$ satisfy*

$$(\tilde{a}_i\nabla_{\mathbb{S}^2}\tilde{u}_i, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in V$$

and (21) *for $i = 1, 2$. Then,*

$$\|\tilde{u}_1 - \tilde{u}_2\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \le \sqrt{\frac{3}{2}}\frac{\|f\|_{H^{-1}(\mathbb{S}^2)}}{(\min_{x\in\mathbb{S}^2}\tilde{a}_1(x))(\min_{x\in\mathbb{S}^2}\tilde{a}_2(x))}\|\tilde{a}_1 - \tilde{a}_2\|_{C^0(\mathbb{S}^2)}.$$

Proof. For $V = H^1(\mathbb{S}^2)/\mathbb{R}$, this is [20, Proposition 8.6] (with a different norm on f). The proof of [20, Proposition 8.6] also applies verbatim in the case that $V \subset H^1(\mathbb{S}^2)/\mathbb{R}$ is a strict subspace. □

We denote the solution map that maps the coefficient $\tilde{a} \in C^0_+(\mathbb{S}^2)$ to the respective unique solution $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$ by

$$\Phi_f : C^0_+(\mathbb{S}^2) \to H^1(\mathbb{S}^2)/\mathbb{R}. \tag{22}$$

The following proposition enables us to apply the deterministic theory to our stochastic framework. It is a direct consequence of Lemma 3.3.

Proposition 3.4. $\Phi_f : C^0_+(\mathbb{S}^2) \to H^1(\mathbb{S}^2)/\mathbb{R}$ *is continuous.*

Proof. For every sequence $(\tilde{a}_j, j \in \mathbb{N}_0)$ in $C^0_+(\mathbb{S}^2)$ satisfying $\|\tilde{a}_0 - \tilde{a}_j\|_{C^0(\mathbb{S}^2)} \to 0$ as $j \to +\infty$, Lemma 3.3 implies that $\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \to 0$ as $j \to +\infty$. Hence, Φ_f is continuous. □

We are now able to state the basic existence and uniqueness result for the weak formulation of the SPDE (17).

Theorem 3.5. *Let $a = \exp(T)$ be an isotropic lognormal RF such that T is a continuous iGRF satisfying* (12) *for some $\beta > 0$. Then, there exists a unique u such that $u \in L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ for every $p \in [1, +\infty)$ and u is the unique solution of the SPDE* (17).

Proof. Since by the continuity assumption of this theorem a takes values in $C^0_+(\mathbb{S}^2)$, we can set $u := \Phi_f(a)$, which solves (17) uniquely. The solution u is strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable by the continuity of Φ_f, cp. Proposition 3.4 and Lemma A.4. L^p-integrability follows with (21) and Corollary 2.12. □

Since the computation of the random coefficient $a = \exp(T)$ does not seem to be feasible in general due to the infinite Karhunen–Loève expansion of T, we consider solutions with respect to the coefficients $(a^L, L \in \mathbb{N}_0)$ in what follows and analyze the convergence of the resulting sequence of solutions in $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$, $p \in [1, +\infty)$. For every $L \in \mathbb{N}_0$, we consider the problem to find a strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable mapping u^L such that

$$(a^L \nabla_{\mathbb{S}^2} u^L, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}, \tag{23}$$

which is also L^p-integrable for every $p \in [1, +\infty)$.

Corollary 3.6. *Let the assumptions of Theorem 3.5 be satisfied. For every $L \in \mathbb{N}_0$, there exists a unique u^L such that u^L solves* (23) *uniquely and its $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$-norm is finite for every $p \in [1, +\infty)$ and can be bounded independently of L.*

Proof. Since a^L also takes values in $C^0_+(\mathbb{S}^2)$ we can set $u^L := \Phi_f(a^L)$, which is strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable due to Proposition 3.4 and Lemma A.4. The $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$-norm of u^L can be bounded uniformly in L due to (21) and Corollary 2.12. □

We show next that the sequence $(u^L, L \in \mathbb{N}_0)$ that results from the previous corollary converges to the unique solution of (17). The following result can be found in [20, Proposition 8.17]. We include its short proof for the convenience of the reader.

Proposition 3.7. *Let the assumptions of Theorem 3.5 be satisfied. Let u be the unique solution of* (17) *and $(u^L, L \in \mathbb{N}_0)$ be the sequence of unique solutions of* (23)*. For every $p \in [1, +\infty)$ and $\varepsilon \in (0, \beta)$, there exists a constant $C_{p,\varepsilon}$ such that for every $L \in \mathbb{N}_0$, it holds that*

$$\|u - u^L\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \le C_{p,\varepsilon} \Big(\sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \Big)^{1/2}.$$

Proof. For every $L \in \mathbb{N}_0$, Lemma 3.3 implies an estimate for $\|u - u^L\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})}$. A twofold application of Hölder's inequality implies the claim with Corollary 2.12, Theorem 2.14, and (21), i.e., there exists a constant $C_{p,\varepsilon}$ such that for every $L \in \mathbb{N}_0$, it holds that

$$\begin{aligned} \|u - u^L\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} &\le \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)} \|1/\check{a}\|_{L^{3p}(\Omega)} \|1/\check{a}^L\|_{L^{3p}(\Omega)} \|a - a^L\|_{L^{3p}(\Omega; C^0(\mathbb{S}^2))} \\ &\le C_{p,\varepsilon} \Big(\sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \Big)^{1/2}. \end{aligned}$$

□

3.2. **Regularity of solutions.** In the previous subsection we showed existence and uniqueness of the solution $u \in L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ to (17). To be able to perform numerical analysis with Finite Element Methods on samples we are interested in proving higher regularity of the stochastic solution u to (17). Precisely we will show that u takes values in $H^{1+s}(\mathbb{S}^2)$ for $s > 0$ such that the range of possible values for s depends only on the decay of the angular power spectrum of the underlying iGRF T satisfying $a = \exp(T)$ and the regularity of the right hand side f.

For every $L \in \mathbb{N}_0$, Theorem 2.14 provides sufficient conditions for Hölder regularity of the isotropic lognormal RF a and L^p-bounds of the norms of a and a^L independently of L. We are going to treat the case of $H^{1+s}(\mathbb{S}^2)$-regularity and L^p-integrability for $s \in [0, 1)$ first and then recursively show higher order regularity with the known theory for the operator $\mathrm{Id} - \Delta_{\mathbb{S}^2}$ presented in Section 2.2. For $s \in (0, 1)$, the $H^{1+s}(\mathbb{S}^2)$-regularity will be concluded with a classical regularity estimate, which in the case of domains in Euclidean space is due to

Hackbusch, cp. [17, Theorem 9.1.8] (see also [12]). The explicit dependence on the coefficient of the elliptic operator in the estimate is of interest, which has been studied in [6] in the case of domains of Euclidean space and will be transferred to the case of $\mathbb{S}^2$ in the following proposition, which will be proven in Appendix B.

Proposition 3.8. *For some $0 \leq s < \gamma < 1$, let $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$, $f \in H^{-1+s}(\mathbb{S}^2)$, and $\tilde{a} \in C^{0,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$ satisfy (20), then, $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$ and there exists a constant C, which is independent of $\tilde{u}, f$, and $\tilde{a}$, such that*

$$\|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} \leq C \frac{\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}}{(\min_{x\in\mathbb{S}^2} \tilde{a}(x))^2} \|f\|_{H^{-1+s}(\mathbb{S}^2)}.$$

The following product estimate will be needed. Its proof is the translation of the estimate on domains of Euclidean space in [36, Theorem 3.3.2] to $\mathbb{S}^2$.

Proposition 3.9. *Let $q \in (1, +\infty)$ and let $\iota \in \mathbb{N}_0$, $\gamma \in (0,1)$, and $s \in \mathbb{R}$ be such that $|s| < \iota + \gamma$. If $v \in C^{\iota,\gamma}(\mathbb{S}^2)$ and $w \in H^s_q(\mathbb{S}^2)$, then $vw \in H^s_q(\mathbb{S}^2)$. Moreover the following product estimate holds: there exists a constant $C_{\iota,\gamma}$ such that for every $v \in C^{\iota,\gamma}(\mathbb{S}^2)$ and every $w \in H^s_q(\mathbb{S}^2)$,*

$$\|vw\|_{H^s_q(\mathbb{S}^2)} \leq C_{\iota,\gamma} \|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)} \|w\|_{H^s_q(\mathbb{S}^2)}.$$

Proof. Let $s \geq 0$. We recall the finite C^∞ atlas $\{(U_i, \eta_i), i \in \mathcal{I}\}$ of $\mathbb{S}^2$ with subordinate partition of unity $\{\Psi_i, i \in \mathcal{I}\}$ from Section 2. We fix $i \in \mathcal{I}$ and choose a subdomain $D \subset \mathbb{R}^2$ with smooth boundary satisfying $\operatorname{supp}(\Psi_i \circ \eta_i^{-1}) \subset\subset D \subset\subset \eta_i(U_i)$ and another partition of unity $\{\hat{\Psi}_j, j \in \mathcal{I}\}$ subordinate to our atlas such that $\hat{\Psi}_i = 1$ on $\eta_i^{-1}(\overline{D})$. Note that with the symbol '$\subset\subset$' we mean compact inclusion. Since, for non-integer order, the Hölder and Zygmund spaces agree (up to equivalent norms), cp. [38, Theorem 4.5.2.1(b), Remark 4.5.2.3], we conclude with [36, Theorem 3.3.2(ii)] that $(vw\Psi_i) \circ \eta_i^{-1} \in H^s_q(D)$ and that there exists a constant $\hat{C}_i$ such that

$$\begin{aligned} \|(vw\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(\mathbb{R}^2)} &= \|(vw\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(D)} \\ &\leq \hat{C}_i \|v \circ \eta_i^{-1}\|_{C^{\iota,\gamma}(\overline{D})} \|(w\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(D)} \\ &\leq \hat{C}_i \|(v\hat{\Psi}_i) \circ \eta_i^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)} \|(w\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(\mathbb{R}^2)} \\ &\leq \hat{C}_i \|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)} \|(w\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(\mathbb{R}^2)}. \end{aligned} \tag{24}$$

This argument can be repeated for the remaining indices in $\mathcal{I}$, which implies that $vw \in H^s_q(\mathbb{S}^2)$. Furthermore, we obtain the claimed estimate by summing the q^{th} power of (24) over i and then taking the q^{th} root. We obtain the assertion of the proposition with $C_{\iota,\gamma} := |\mathcal{I}| \max_{i\in\mathcal{I}} \hat{C}_i$. The other case that $s < 0$ follows now by duality, i.e., for every $\eta \in H^{|s|}(\mathbb{S}^2)$, it holds that

$$|(vw)(\eta)| = |w(v\eta)| \leq \|w\|_{H^s(\mathbb{S}^2)} \|v\eta\|_{H^{|s|}(\mathbb{S}^2)} \leq C_{\iota,\gamma} \|w\|_{H^s(\mathbb{S}^2)} \|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)} \|\eta\|_{H^{|s|}(\mathbb{S}^2)}. \qquad \square$$

In the following we argue with properties of the solution map Φ_f introduced in (22). We will analyse the domain and the respective range of Φ_f more precisely. We remark that the domain of Φ_f reflects the regularity of a coefficient $\tilde{a}$ in the elliptic operator in (20) and that the range of Φ_f reflects the regularity of the respective solution $\tilde{u} = \Phi_f(\tilde{a})$.

Proposition 3.10. *Let $\iota \in \mathbb{N}_0$, $\gamma \in (0,1)$, and $s \in [0,+\infty)$ satisfy $s < \iota + \gamma$. If $f \in H^{-1+s}(\mathbb{S}^2)$, then it holds that*

$$\Phi_f : C^{\iota,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2) \to H^{1+s}(\mathbb{S}^2)$$

is continuous.

Proof. The case $s \in [0,1)$ will serve as a base case for an induction argument. There the case $s = 0$ is already known from Proposition 3.4. So let $s \in (0,1)$ and assume that $\iota = 0$ and $\gamma \in (s,1)$. From Proposition 3.8 we infer that $\Phi_f(\tilde{a}) \in H^{1+s}(\mathbb{S}^2)$, which establishes the claimed domain and range of Φ_f. To prove the continuity of Φ_f let $(\tilde{a}_j, j \in \mathbb{N}_0)$ be a sequence in $C^{0,\gamma}(\mathbb{S}^2) \cap\ C^0_+(\mathbb{S}^2)$ such that $\|\tilde{a}_j - \tilde{a}_0\|_{C^{0,\gamma}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$. We observe that for every $j \in \mathbb{N}$, it holds that

$$(\tilde{a}_0 \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)), \nabla_{\mathbb{S}^2} v) = (-(\tilde{a}_0 - \tilde{a}_j)\nabla_{\mathbb{S}^2}\Phi_f(\tilde{a}_j), \nabla_{\mathbb{S}^2} v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}. \tag{25}$$

Since $\Phi_f(\tilde{a}_j) \in H^{1+s}(\mathbb{S}^2)$, $j \in \mathbb{N}$, we obtain with Proposition 3.9 that there exist constants C_1, C_2 such that

$$\begin{aligned}\|\nabla_{\mathbb{S}^2} \cdot ((\tilde{a}_0 - \tilde{a}_j)\nabla_{\mathbb{S}^2}\Phi_f(\tilde{a}_j))\|_{H^{-1+s}(\mathbb{S}^2)} &\le C_1 \|(\tilde{a}_0 - \tilde{a}_j)\nabla_{\mathbb{S}^2}\Phi_f(\tilde{a}_j)\|_{H^s(\mathbb{S}^2)} \\ &\le C_2 \|\tilde{a}_0 - \tilde{a}_j\|_{C^{0,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a}_j)\|_{H^{1+s}(\mathbb{S}^2)}.\end{aligned}$$

Hence, Proposition 3.8 applied to the setting in (25) implies that there exists a constant C, which is independent of $(\tilde{a}_j, j \in \mathbb{N}_0)$ and f, such that for every $j \in \mathbb{N}$, it holds that

$$\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+s}(\mathbb{S}^2)} \le C \frac{\|\tilde{a}_0\|_{C^{0,\gamma}(\mathbb{S}^2)}}{(\min_{x\in\mathbb{S}^2} \tilde{a}_0(x))^2} \frac{\|\tilde{a}_j\|_{C^{0,\gamma}(\mathbb{S}^2)}}{(\min_{x\in\mathbb{S}^2} \tilde{a}_j(x))^2} \|f\|_{H^{-1+s}(\mathbb{S}^2)} \|\tilde{a}_0 - \tilde{a}_j\|_{C^{0,\gamma}(\mathbb{S}^2)}.$$

We have $\|\tilde{a}_0 - \tilde{a}_j\|_{C^0(\mathbb{S}^2)} =: \epsilon_j \to 0$ as $j \to +\infty$, which implies that $\epsilon_j \le 1/2 \min_{x\in\mathbb{S}^2} \tilde{a}_0(x)$ for every j that are sufficiently large, i.e., $j > j_0$ for some $j_0 \in \mathbb{N}$. Since $\tilde{a}_j(x') \ge \min_{x\in\mathbb{S}^2} \tilde{a}_0(x) - \epsilon_j$ for every $x' \in \mathbb{S}^2$, we obtain that $1/\min_{x\in\mathbb{S}^2} \tilde{a}_j(x) \le 2/\min_{x\in\mathbb{S}^2} \tilde{a}_0(x)$ for every $j > j_0$. Since $\|\tilde{a}_j\|_{C^{0,\gamma}(\mathbb{S}^2)}$ and $(\min_{x\in\mathbb{S}^2} \tilde{a}_j(x))^{-2}$ can be bounded independently of j, it follows that $\|\Phi_f(\tilde{a}_j) - \Phi_f(\tilde{a}_0)\|_{H^{1+s}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$, i.e., $\Phi_f : C^{0,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2) \to H^{1+s}(\mathbb{S}^2)$ is continuous.

For $s \ge 1$, it must hold that $\iota \ge 1$. Since $\tilde{a} \in C^{\iota,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$ and $\tilde{u} := \Phi_f(\tilde{a})$, for any $w \in H^1(\mathbb{S}^2)$, we take $w/\tilde{a} - 1/|\mathbb{S}^2| \int_{\mathbb{S}^2} w/\tilde{a}\mathrm{d}\sigma \in H^1(\mathbb{S}^2)/\mathbb{R}$ as a test function and thus rewrite the PDE in (20) as

$$(\tilde{u}, w) + \Big(f, \frac{w}{\tilde{a}}\Big) = (\tilde{u}, w) + \Big(\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2}\Big(\frac{w}{\tilde{a}}\Big)\Big) = (\tilde{u}, w) + (\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2} w) - \Big(\frac{\nabla_{\mathbb{S}^2}\tilde{a} \cdot \nabla_{\mathbb{S}^2}\tilde{u}}{\tilde{a}}, w\Big),$$

where we applied that $(f, 1) = 0$. Hence, for every $w \in H^1(\mathbb{S}^2)$, it holds that

$$(\tilde{u}, w) + (\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2} w) = \Big(\frac{f + \nabla_{\mathbb{S}^2}\tilde{a} \cdot \nabla_{\mathbb{S}^2}\tilde{u}}{\tilde{a}}, w\Big) + (\tilde{u}, w),$$

which is stated with equality in $H^{-1}(\mathbb{S}^2)$ as

$$(\mathrm{Id} - \Delta_{\mathbb{S}^2})\tilde{u} = \frac{f + \nabla_{\mathbb{S}^2}\tilde{a} \cdot \nabla_{\mathbb{S}^2}\tilde{u}}{\tilde{a}} + \tilde{u} =: F. \tag{26}$$

We observe with (9) that $(\mathrm{Id} - \Delta_{\mathbb{S}^2})^{-1}$ is a linear and bounded operator from $H^r(\mathbb{S}^2)$ to $H^{r+2}(\mathbb{S}^2)$ for every $r \in \mathbb{R}$. The claim is now shown by induction. Let us write $s = \lfloor s \rfloor + \{s\}$, where $\{s\} \in [0,1)$ is the fractional part of s, and assume as induction hypothesis that $\Phi_f : C^{n,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2) \to H^{1+n+\{s\}}(\mathbb{S}^2)$ is continuous for every $n \in \{0, 1, \ldots, \lfloor s \rfloor - 1\}$, which we already showed for $n = 0$. Let $n \in \{0, 1, \ldots, \lfloor s \rfloor - 1\}$ and let $\tilde{a} \in C^{n+1,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$. Since by our induction hypothesis $\tilde{u} = \Phi_f(\tilde{a}) \in H^{1+n+\{s\}}(\mathbb{S}^2)$, we conclude with Proposition 3.9 that the right hand side F in (26) is in $H^{1+(n-1)+\{s\}}$. The fact that $(\mathrm{Id} - \Delta_{\mathbb{S}^2})^{-1}$ is a linear and bounded operator from $H^{1+(n-1)+\{s\}}(\mathbb{S}^2)$ to $H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$ implies that $\tilde{u} = \Phi_f(\tilde{a}) \in H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$. Moreover it implies with Proposition 3.9 a regularity estimate for $\tilde{u} = \Phi_f(\tilde{a})$, i.e., there exist constants C, C' that are independent of $\tilde{a}$ and f such that

$$\begin{aligned} &\|\Phi_f(\tilde{a})\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ &\quad \leq C\|F\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \\ &\quad \leq C'\Big(\|1/\tilde{a}\|_{C^{n,\gamma}(\mathbb{S}^2)}\big(\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{n+1,\gamma}(\mathbb{S}^2)}\|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}\big) \\ &\qquad + \|\Phi_f(\tilde{a})\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)}\Big). \end{aligned} \tag{27}$$

This implies the claimed domain and range of Φ_f. To prove continuity of Φ_f let $(\tilde{a}_j, j \in \mathbb{N}_0)$ be a sequence in $C^{n+1,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$ such that $\|\tilde{a}_j - \tilde{a}_0\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$ and let $(\tilde{u}_j = \Phi_f(\tilde{a}_j), j \in \mathbb{N}_0)$ be the sequence of respective solutions. The same manipulations that showed (26) imply with (25) that

$$(\mathrm{Id} - \Delta_{\mathbb{S}^2})(\tilde{u}_0 - \tilde{u}_j) = \frac{\nabla_{\mathbb{S}^2} \cdot ((\tilde{a}_0 - \tilde{a}_j)\nabla_{\mathbb{S}^2}\tilde{u}_j) + \nabla_{\mathbb{S}^2}\tilde{a}_0 \cdot \nabla_{\mathbb{S}^2}(\tilde{u}_0 - \tilde{u}_j)}{\tilde{a}_0} + (\tilde{u}_0 - \tilde{u}_j).$$

Similar estimates as in (27) imply that

$$\begin{aligned} &\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ &\quad \leq C'\Big(\|1/\tilde{a}_0\|_{C^{n,\gamma}(\mathbb{S}^2)}\big(\|\tilde{a}_0 - \tilde{a}_j\|_{C^{n+1,\gamma}(\mathbb{S}^2)}\|\Phi_f(\tilde{a}_0)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)} \\ &\qquad + \|\tilde{a}_0\|_{C^{n+1,\gamma}(\mathbb{S}^2)}\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}\big) \\ &\qquad + \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)}\Big). \end{aligned}$$

Since by our induction hypothesis $\Phi_f : C^{n,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2) \to H^{1+n+\{s\}}(\mathbb{S}^2)$ is continuous, $\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{n+\{s\}}(\mathbb{S}^2)} \leq \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$, which implies with $\|\tilde{a}_0 - \tilde{a}_j\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$ that $\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \to 0$ as $j \to +\infty$, i.e., $\Phi_f : C^{n+1,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2) \to H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$ is continuous. This finishes the induction and the proof of the proposition. ☐

The continuity of the solution map Φ_f also with respect to domains and ranges that reflect higher regularity will enable applicability of classical regularity theory to the stochastic solution u to (17), since strong measurability of u can be shown.

Theorem 3.11. *Let $a = \exp(T)$ be an isotropic lognormal RF such that T is a continuous iGRF satisfying* (12) *for some $\beta > 0$. Furthermore, let u be the unique solution of* (17) *and $(u^L, L \in \mathbb{N}_0)$ be the sequence of unique solutions of* (23). *Then, for every $s \in [0, \beta/2)$ and $L \in \mathbb{N}_0$, it holds that $u, u^L \in L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ for every $p \in [1, +\infty)$, if $f \in H^{-1+s}(\mathbb{S}^2)$. Moreover, for every $L \in \mathbb{N}_0$, the $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$-norm of u^L can be bounded independently of L.*

Proof. Let us write $s = \lfloor s \rfloor + \{s\}$, where $\{s\} \in [0, 1)$ is the fractional part of s, and then set $\iota := \lfloor s \rfloor \in \mathbb{N}_0$ and choose $\gamma \in (\{s\}, \min\{\beta/2 - \iota, 1\})$, which implies that $s < \iota + \gamma$. We deduce that $a, a^L \in L^{p'}(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$ for every $L \in \mathbb{N}_0$, $p' \in [1, +\infty)$, from Theorem 2.14. In particular, these RFs are strongly $C^{\iota,\gamma}(\mathbb{S}^2)$-measurable and positive. Hence, by the properties of the solution map Φ_f, cp. Proposition 3.10 and Lemma A.4, the mappings $u = \Phi_f(a)$ and $u^L = \Phi_f(a^L)$ are strongly H^{1+s}-measurable for every $L \in \mathbb{N}_0$.

The boundedness of the $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$-norm will be proved inductively. As a base case we apply the estimate of the $H^{1+\{s\}}(\mathbb{S}^2)$-norm of u from Proposition 3.8 and use the Cauchy–Schwarz inequality to obtain that there exists a constant $C > 0$ (independent of u, a, and f) such that

$$\|u\|_{L^p(\Omega;H^{1+\{s\}}(\mathbb{S}^2))} \leq C\|f\|_{H^{-1+\{s\}}(\mathbb{S}^2)}\|a\|_{L^{2p}(\Omega;C^{0,\gamma}(\mathbb{S}^2))}\|\check{a}^{-2}\|_{L^{2p}(\Omega)}.$$

We infer from Theorem 2.14 and Corollary 2.12 that the right hand side of the previous inequality is finite. Let us assume as induction hypothesis that the $L^p(\Omega; H^{1+n+\{s\}}(\mathbb{S}^2))$-norm of u is finite for every $n \in \{0, 1, \ldots, \lfloor s \rfloor - 1\}$, which we just established for $n = 0$. Let $n \in \{0, 1, \ldots, \lfloor s \rfloor - 1\}$ and let us apply the estimate of the $H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$-norm in (27) to u and apply the Cauchy–Schwarz inequality twice to obtain that there exists a constant C that is independent of u, a, and f such that

$$\begin{aligned}
&\|u\|_{L^p(\Omega;H^{1+(n+1)+\{s\}}(\mathbb{S}^2))} \\
&\quad \leq C\Big(\|1/a\|_{L^{3p}(\Omega;C^{n,\gamma}(\mathbb{S}^2))}\big(\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \\
&\qquad\qquad + \|a\|_{L^{3p}(\Omega;C^{n+1,\gamma}(\mathbb{S}^2))}\|u\|_{L^{3p}(\Omega;H^{1+n+\{s\}}(\mathbb{S}^2))}\big) \\
&\qquad + \|u\|_{L^p(\Omega;H^{1+(n-1)+\{s\}}(\mathbb{S}^2))}\Big).
\end{aligned}$$

Since $1/a = \exp(-T)$, Theorem 2.14 is applicable to the continuous iGRF $-T$, which satisfies (12) in the same way that T does. Hence, the $L^{3p}(\Omega; C^{n,\gamma}(\mathbb{S}^2))$-norm of $1/a$ is finite. The induction hypothesis, Theorem 2.14, and Corollary 2.12 imply that the right hand side of the previous inequality is finite. This completes the induction. We conclude that the $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$-norm of u is finite. The proof for u^L, $L \in \mathbb{N}_0$, is analogous. The uniform boundedness of the $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$-norm of u^L in $L \in \mathbb{N}_0$ is implied by Theorem 2.14 and Corollary 2.12. □

We remark that the $H^{1+s}(\mathbb{S}^2)$-regularity for every $s < \beta/2$ of the solution can also be deduced from higher order Hölder regularity, which is implied by Schauder estimates, cp. [12, Chapters 6 and 8], applied to pullbacks of the solution to the chart domains. Specifically, the continuous embedding $C^{\iota,\gamma}(\mathbb{S}^2) \subset H^{s'}(\mathbb{S}^2)$ for $\iota \in \mathbb{N}_0, \gamma \in (0, 1)$, and $s' \geq 0$ such that $\iota + \gamma > s'$, which is an immediate consequence of Proposition 3.9, would imply $H^{1+s}(\mathbb{S}^2)$-regularity. Since the explicit dependence of the coefficients of the elliptic operator in these estimates is analysed in [20, Section 8.2] also L^p-integrability could be deduced.

4. Discretization

4.1. Finite Elements on $\mathbb{S}^2$.

In Proposition 3.7 we analyzed the error that occurs when we consider the solution $u^L = \Phi_f(a^L)$ to the SPDE (17) with respect to the approximating isotropic lognormal RF $a^L = \exp(\Pi_L T)$ for $L \in \mathbb{N}_0$, where $a^L = \exp(\Pi_L T)$ can be simulated via the truncated Karhunen–Loève expansion of the iGRF T for every $L \in \mathbb{N}_0$. In this section we aim at a spatial discretization to numerically simulate realizations of u^L, $L \in \mathbb{N}_0$, with a *Galerkin* Finite Element Method and analyze the error in the $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$-norm for $p \in [1, +\infty)$.

We recall parts of the deterministic theory of Finite Elements on $\mathbb{S}^2$. Finite Elements on surfaces to approximate solutions of elliptic PDEs appear to have been first introduced in [10]. There, first order convergence estimates were obtained using affine approximations of the surface. Higher order estimates were shown in [8], where also an FE Method is defined on the surface so as to avoid a surface approximation error. We refer to [8, Section 2.6] for details.

Given a regular, quasiuniform triangulation $\mathcal{T}$ of $\mathbb{S}^2$ into parametric, curvilinear triangles $K \in \mathcal{T}$ of mesh width $h > 0$ (which we indicate by tagging $\mathcal{T}$ with the subscript h, i.e., by writing $\mathcal{T}_h$) we define $S^k(\mathbb{S}^2, \mathcal{T}_h)$ to be the space of continuous, piecewise parametric polynomials of degree $k \geq 1$ on the triangulation $\mathcal{T}_h$ of $\mathbb{S}^2$ and equip it with the $H^1(\mathbb{S}^2)$-norm. To approximate functions in $H^1(\mathbb{S}^2)/\mathbb{R}$ we define the subspace of $S^k(\mathbb{S}^2, \mathcal{T}_h)$ of functions that have zero average, i.e.,

$$V^{h,k} := \{v^h \in S^k(\mathbb{S}^2, \mathcal{T}_h), (v^h, 1) = 0\}.$$

Then, $V^{h,k} \subset H^1(\mathbb{S}^2)/\mathbb{R}$ and we equip it with the $H^1(\mathbb{S}^2)/\mathbb{R}$-norm. The FE spaces $S^k(\mathbb{S}^2, \mathcal{T}_h)$ and $V^{h,k}$, $h > 0$, are of finite dimension such that $\dim(S^k(\mathbb{S}^2, \mathcal{T}_h)) = \dim(V^{h,k}) + 1$. Also it holds that the degrees of freedom $N_h := \dim(V^{h,k}) = \mathrm{O}(h^{-2})$ as $h \to 0$ for fixed polynomial degree $k \in \mathbb{N}$. We refer to [32, Chapter 4] for details and remark that we will only tag elements of $V^{h,k}$ respectively $S^k(\mathbb{S}^2, \mathcal{T}_h)$ with the mesh width h keeping in mind that they implicitly also depend on the polynomial degree k of the FE space, i.e., let $v^h \in V^{h,k}$.

For every $\tilde{a} \in C^0_+(\mathbb{S}^2)$, $h > 0$, and $k \in \mathbb{N}$, we consider the variational problem (20) over the space $V^{h,k}$ to find a unique *Galerkin FE solution* $\tilde{u}^h \in V^{h,k}$ such that

$$(\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}^h, \nabla_{\mathbb{S}^2} v^h) = f(v^h) \quad \forall v^h \in V^{h,k}. \tag{28}$$

The conformity of the FE Method, i.e., $V^{h,k} \subset H^1(\mathbb{S}^2)/\mathbb{R}$, implies with (18) and (19) that the bilinear form $(\tilde{a}\nabla_{\mathbb{S}^2}\cdot, \nabla_{\mathbb{S}^2}\cdot)$ on $V^{h,k} \times V^{h,k}$ is continuous and coercive with coercivity constant $(\min_{x\in\mathbb{S}^2} \tilde{a}(x))^{-1}$ which is independent of h and of k.

Hence, by the Lax–Milgram lemma, the Galerkin approximation $\tilde{u}^h \in V^{h,k}$ exists and is the unique solution of (28). Also $\tilde{u}^h$ satisfies the estimate in (21) uniformly in $h > 0$, i.e.,

$$\|\tilde{u}^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \frac{1}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)}. \tag{29}$$

As in the previous section we introduce a solution mapping $\Phi^{h,k}_f$ that maps the coefficient $\tilde{a} \in C^0_+(\mathbb{S}^2)$ to the respective unique Galerkin FE solution $\tilde{u}^h \in V^{h,k}$ by

$$\Phi^{h,k}_f : C^0_+(\mathbb{S}^2) \to V^{h,k}.$$

Proposition 4.1. *$\Phi^{h,k}_f : C^0_+(\mathbb{S}^2) \to V^{h,k}$ is continuous for every $h > 0$ and $k \in \mathbb{N}$.*

Proof. Similarly to the proof of Proposition 3.4, the continuity of $\Phi^{h,k}_f$ follows, since for every sequence $(\tilde{a}_j, j \in \mathbb{N}_0)$ in $C^0_+(\mathbb{S}^2)$ satisfying $\|\tilde{a}_0 - \tilde{a}_j\|_{C^0(\mathbb{S}^2)} \to 0$ as $j \to +\infty$, Lemma 3.3 implies that $\|\Phi^{h,k}_f(\tilde{a}_0) - \Phi^{h,k}_f(\tilde{a}_j)\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \to 0$ as $j \to +\infty$. □

Functions in $H^{1+s}(\mathbb{S}^2)$ and in particular solutions to (20) can be approximated in $S^k(\mathbb{S}^2, \mathcal{T}_h)$, $s, h > 0$, and $k \in \mathbb{N}$, cp. [8, Proposition 2.7]. We will phrase this in terms of the solution mappings Φ_f and $\Phi^{h,k}_f$, $h > 0$ and $k \in \mathbb{N}$, in the following proposition. In the proof this well-known approximation property of $S^k(\mathbb{S}^2, \mathcal{T}_h)$, $h > 0$ and $k \in \mathbb{N}$, in $H^1(\mathbb{S}^2)$ will be combined with Céa's lemma.

Proposition 4.2. *Let $k \in \mathbb{N}$ be the polynomial degree of the FE spaces $V^{h,k}$, $h > 0$, and let $\iota \in \mathbb{N}_0$ and $\gamma \in (0,1)$. For every $s \in (0, \iota + \gamma)$ such that $f \in H^{-1+s}(\mathbb{S}^2)$, there exists a constant C_s such that for every $h > 0$ and every $\tilde{a} \in C^{\iota,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$, it holds that*

$$\|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq C_s \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \|\Phi_f(\tilde{a})\|_{H^{1+s}(\mathbb{S}^2)}\, h^{\min\{s,k\}}.$$

Proof. We observe that Proposition 3.10 implies that $\Phi_f(\tilde{a}) \in H^{1+s}(\mathbb{S}^2)$. The approximation property for integer orders, cp. [8, Proposition 2.7], implies by interpolation that for every h and k, there exists an interpolation operator $I^{h,k}$ which is, for every $s > 0$, continuous from $H^{1+s}(\mathbb{S}^2) \to S^k(\mathbb{S}^2, \mathcal{T}_h)$ and a constant $C_s > 0$ such that for every $h > 0$ and for every function $v \in H^{1+s}(\mathbb{S}^2)$, it holds that

$$\|v - I^{k,h} v\|_{H^1(\mathbb{S}^2)} \leq C_s\, h^{\min\{s,k\}}\, \|v\|_{H^{1+s}(\mathbb{S}^2)}, \tag{30}$$

where $C_s > 0$ is independent of h but depends on s. The coercivity and Galerkin orthogonality imply in the usual fashion that for every $v^h \in V^{h,k}$, it holds that

$$\begin{aligned}
\|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|^2_{H^1(\mathbb{S}^2)/\mathbb{R}} &\leq \frac{1}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} (\tilde{a}\nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})), \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a}))) \\
&= \frac{1}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} (\tilde{a}\nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})), \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - v^h)) \\
&\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}}.
\end{aligned} \tag{31}$$

When we equip $H^1(\mathbb{S}^2)/\mathbb{R}$ with the $H^1(\mathbb{S}^2)$-norm, the orthogonal decomposition $H^1(\mathbb{S}^2) = H^1(\mathbb{S}^2)/\mathbb{R} \oplus \operatorname{span}\{1\}$ holds, which implies with (31) that

$$\begin{aligned}
\|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} &\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in V^{h,k}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \\
&\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in V^{h,k}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)} \\
&= \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x\in\mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in S^k(\mathbb{S}^2, \mathcal{T}_h)} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)},
\end{aligned}$$

where we also used that $\Phi_f(\tilde{a}) \in H^1(\mathbb{S}^2)/\mathbb{R}$. Now the claim follows with (30). □

Since the mappings $\Phi_f^{h,k}$, $h > 0, k \in \mathbb{N}$, are continuous due to Proposition 4.1, the introduced theory on Galerkin FE Methods is applicable to our stochastic framework. Indeed, for every $L \in \mathbb{N}_0$, $h > 0$, and $k \in \mathbb{N}$, the problem to find a strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable $u^{L,h}$ such that

$$(a^L \nabla_{\mathbb{S}^2} u^{L,h}, \nabla_{\mathbb{S}^2} v^h) = f(v^h) \quad \forall v^h \in V^{h,k} \tag{32}$$

admits a unique solution by setting $u^{L,h} := \Phi_f^{h,k}(a^L)$, where we omit k in our notation of the solution. The strong $H^1(\mathbb{S}^2)/\mathbb{R}$-measurability of $u^{L,h}$ follows due to the strong $C^0(\mathbb{S}^2)$-measurability of a^L and the continuity of $\Phi_f^{h,k}$ with Lemma A.4. Moreover Corollary 2.12 implies with (29) that for every $p \in [1, +\infty)$, there exists a constant C_p such that for every

$L \in \mathbb{N}_0$ and every $h > 0$, it holds that

$$\|u^{L,h}\|_{L^p(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \leq \|1/\check{a}^L\|_{L^p(\Omega)}\sqrt{\frac{3}{2}}\|f\|_{H^{-1}(\mathbb{S}^2)} \leq C_p. \tag{33}$$

Theorem 4.3. *Let the assumptions of Theorem 3.11 be satisfied. Let $u = \Phi_f(a)$ be the unique solution of (17) and for every $h > 0$, let $u^{L,h} = \Phi_f^{h,k}(a^L)$ be the unique Galerkin FE solution of (32) for $k \in \mathbb{N}$. For every $s \in (0, \beta/2)$ such that $f \in H^{-1+s}(\mathbb{S}^2)$ and every $p \in [1, +\infty)$, there exists a constant $C_{p,s}$ such that for every $h > 0$ and every $L \in \mathbb{N}_0$, it holds that*

$$\|u - u^{L,h}\|_{L^p(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_{p,s}(L^{-s} + h^{\min\{s,k\}}).$$

Proof. For every $L \in \mathbb{N}_0$, let us set $u^L := \Phi_f(a^L)$. A twofold application of Hölder's inequality implies with Proposition 4.2 that there exists a constant C_s such that for every $L \in \mathbb{N}_0$ and every $h > 0$, it holds that

$$\|u^L - u^{L,h}\|_{L^p(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s\|a^L\|_{L^{3p}(\Omega;C^0(\mathbb{S}^2))}\|1/\check{a}^L\|_{L^{3p}(\Omega)}\|u^L\|_{L^{3p}(\Omega;H^{1+s}(\mathbb{S}^2))}\ h^{\min\{s,k\}}.$$

Due to Proposition 2.11, Corollary 2.12, and Theorem 3.11 there exists a constant $\hat{C}_{p,s}$ such that for every $L \in \mathbb{N}_0$, it holds that

$$C_s\|a^L\|_{L^{3p}(\Omega;C^0(\mathbb{S}^2))}\|1/\check{a}^L\|_{L^{3p}(\Omega)}\|u^L\|_{L^{3p}(\Omega;H^{1+s}(\mathbb{S}^2))} \leq C_{p,s}.$$

Let $\varepsilon := \beta - 2s \in (0, \beta)$. We apply the triangle inequality and conclude with Proposition 3.7 that there exists a constant that we also denote by $\hat{C}_{p,s}$ such that for every $L \in \mathbb{N}_0$ and every $h > 0$, it holds that

$$\|u - u^{L,h}\|_{L^p(\Omega;H^1(\mathbb{S}^2))} \leq \hat{C}_{p,s}\Big(\sum_{\ell > L} A_\ell \ell^{1+\varepsilon}\Big)^{\frac{1}{2}} + \hat{C}_{p,s}\ h^{\min\{s,k\}}.$$

We further estimate

$$\sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \leq (L^{-1})^{\beta-\varepsilon}\sum_{\ell > L} A_\ell \ell^{1+\beta} \leq (L^{-1})^{2s}\sum_{\ell \geq 0} A_\ell \ell^{1+\beta}.$$

Since $\sum_{\ell \geq 0} A_\ell \ell^{1+\beta} < +\infty$ by assumption, we conclude the proof of the theorem. □

4.2. **Spectral Methods on $\mathbb{S}^2$.** In Theorem 4.3 we established a rate of convergence for Galerkin approximations of the stochastic solution from subspaces $V^{h,k}$ of continuous, piecewise polynomial functions on a quasiuniform triangulation $\mathcal{T}_h$ on $\mathbb{S}^2$. The obtained bound for the convergence rate in Theorem 4.3 indicated an asymptotic convergence order $N_h^{-\min\{s,k\}/2}$ as $N_h = \dim(V^{h,k}) \to +\infty$, i.e., the convergence rate is limited by the regularity of the solutions (as expressed in the Sobolev scale parameter $s \geq 0$) and by the polynomial degree $k \in \mathbb{N}$ of the Finite Elements used in the discretization. If, in particular, the Sobolev regularity of solutions is high, i.e., if $s > 0$ is large, the convergence of the Galerkin FE approximations $u^{L,h}$ defined in (32) is limited by the order k of the used Finite Elements. Spectral Elements do not have this drawback.

To introduce it, we recall the space $\mathcal{H}_{0:L^u} \subset H^1(\mathbb{S}^2)$ spanned by spherical harmonics of order at most L^u defined in (6). Since we are interested in a conforming method, we restrict ourselves to the functions that are orthogonal to constants as in the FE case, i.e., we consider $\mathcal{H}_{1:L^u}$ as Spectral Element spaces, $L^u \in \mathbb{N}$. In the following the index L^a refers to the degree of the approximation of a and L^u refers to the degree of the Spectral Element space. Its dimension is $N_{L^u} := \dim(\mathcal{H}_{1:L^u}) = \mathrm{O}((L^u)^2)$ as $L^u \to +\infty$, and is also referred to as degrees of freedom. Let $a = \exp(T)$ be an isotropic lognormal RF that results from a continuous iGRF T satisfying (12) for some $\beta > 0$. Similarly to (32), for every $L^a, L^u \in \mathbb{N}_0$, we define a Galerkin approximation as the solution of the problem to find a strongly $H^1(\mathbb{S}^2)/\mathbb{R}$-measurable u^{L^a,L^u} that takes values in $\mathcal{H}_{1:L^u}$ such that

$$(a^{L^a}\nabla_{\mathbb{S}^2} u^{L^a,L^u}, \nabla_{\mathbb{S}^2} v^{L^u}) = f(v^{L^u}) \quad \forall v^{L^u} \in \mathcal{H}_{1:L^u}.$$

The coercivity of the bilinear form $(a^{L^a}\nabla_{\mathbb{S}^2}\cdot, \nabla_{\mathbb{S}^2}\cdot)$ implies that u^{L^a,L^u} exists and is unique, since $\mathcal{H}_{1:L^u} \subset H^1(\mathbb{S}^2)/\mathbb{R}$ is a closed subspace. Strong $H^1(\mathbb{S}^2)/\mathbb{R}$-measurability of u^{L^a,L^u} follows in the same way as in Section 4.1.

Theorem 4.4. *Let the assumptions of Theorem 3.11 be satisfied. For every $s \in (0, \beta/2)$ such that $f \in H^{-1+s}(\mathbb{S}^2)$ and for every $p \in [1, +\infty)$, there exists a constant $C_{p,s}$ such that for every $L^a, L^u \in \mathbb{N}_0$, it holds that*

$$\|u - u^{L^a,L^u}\|_{L^p(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \le C_{p,s}\Big((L^a)^{-s} + (L^u)^{-s}\Big).$$

Proof. The proof is similar to that of Theorem 4.3. We use the approximation result Proposition 3.7, the quasioptimality, the regularity result Theorem 3.11, and the approximation property of $\mathcal{H}_{0:L^u}$ in Proposition 2.4 to conclude the assertion. □

5. MLMC convergence analysis

In this section we aim at approximating the expectation $\mathbb{E}(u)$. So far we established for FE approximations in Section 4.1 that the constructed double indexed sequence $(u^{L,h}, L \in \mathbb{N}_0, h > 0)$ converges to u in $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ for every $p \in [1, +\infty)$ with a particular convergence rate, cp. Theorem 4.3. The remaining part of the numerical analysis is to approximate $\mathbb{E}(u^{L,h})$ for $L \in \mathbb{N}_0$ and $h > 0$. To this end, we apply an MLMC estimator in order to reduce the computational cost that a conventional Monte Carlo simulation would incur.

The error analysis of MLMC discretizations is standard, by now, and our development is analogous to those carried out in [14, 2, 1]. In particular, in [14] the error from truncating a Karhunen–Loève expansion of the Gaussian random field was considered. In contrast to the situation there, we will benefit in our analysis from the knowledge of the properties of iGRFs and of the behavior of their Karhunen–Loève expansions that we develop in Sections 2.3 – 2.5. This relieves us from additional assumptions on the Karhunen–Loève eigenfunctions, on the behavior of the truncated Karhunen–Loève expansion, and on the iGRF itself, apart from summability assumptions on the angular power spectrum.

We introduce the usual Monte Carlo (MC) estimator and the MLMC estimator in a general setting. Let $(V, \|\cdot\|_V)$ be a separable Hilbert space. For every $v \in L^2(\Omega; V)$, let $(\hat{v}_i, i \in \mathbb{N})$ be a sequence of independent, identically distributed random variables in $L^2(\Omega; V)$ such that they are independent from v and have the same law. For every $M \in \mathbb{N}$, the MC estimator E_M of v is then defined by

$$E_M(v) := \frac{1}{M}\sum_{i=1}^{M} \hat{v}_i.$$

It is well-known that for every $v \in L^2(\Omega; V)$ and every $M \in \mathbb{N}$, it holds that

$$\|\mathbb{E}(v) - E_M(v)\|^2_{L^2(\Omega;V)} = \frac{1}{M}\|v - \mathbb{E}(v)\|^2_{L^2(\Omega;V)} = \frac{1}{M}(\|v\|^2_{L^2(\Omega;V)} - \|\mathbb{E}(v)\|^2_V). \tag{34}$$

For every $L^2(\Omega;V)$-valued sequence $(v^j, j \in \mathbb{N}_0)$, we consider a finite telescoping sum expansion with the convention that $v^{-1} = 0$, i.e., for every $J' \in \mathbb{N}_0$, it holds that

$$v^{J'} = \sum_{j=0}^{J'} v^j - v^{j-1},$$

and define for every $\mathbb{N}$-valued sequences $(M_j, j = 0, \dots, J)$, $J \in \mathbb{N}_0$, the MLMC estimator E^J of v^J by

$$E^J(v^J) := \sum_{j=0}^{J} E_{M_j}(v^j - v^{j-1}) \tag{35}$$

such that the MC estimators $(E_{M_j}(v^j - v^{j-1}), j = 0, \dots, J)$ are independent.

Lemma 5.1. *For every $L^2(\Omega;V)$-valued sequence $(v^j, j \in \mathbb{N}_0)$ and every integer-valued $(M_j, j = 0, \dots, J)$ sequences with finite $J \in \mathbb{N}_0$, the MLMC estimator $E^J(v^J)$ satisfies that*

$$\| \mathbb{E}(v^J) - E^J(v^J)\|^2_{L^2(\Omega;V)} = \sum_{j=0}^{J} \frac{1}{M_j}\Big(\|v^j - v^{j-1}\|^2_{L^2(\Omega;V)} - \| \mathbb{E}(v^j - v^{j-1})\|^2_V\Big).$$

Proof. The independence of the MC estimators in (35) on the different levels $(M_j, j = 0, \dots, J)$ and (34) imply that

$$\begin{aligned}\| \mathbb{E}(v^J) - E^J(v^J)\|^2_{L^2(\Omega;V)} &= \| \sum_{j=0}^{J} \mathbb{E}(v^j - v^{j-1}) - E_{M_j}(v^j - v^{j-1})\|^2_{L^2(\Omega;V)} \\ &= \sum_{j=0}^{J} \| \mathbb{E}(v^j - v^{j-1}) - E_{M_j}(v^j - v^{j-1})\|^2_{L^2(\Omega;V)} \\ &= \sum_{j=0}^{J} \frac{1}{M_j}(\|v^j - v^{j-1}\|^2_{L^2(\Omega;V)} - \| \mathbb{E}(v^j - v^{j-1})\|^2_V).\end{aligned}$$

□

Theorem 5.2. *Let the assumptions of Theorem 4.3 be satisfied. Further, let u be the unique solution to (17), $(L_j, j \in \mathbb{N}_0)$ be an increasing $\mathbb{N}$-valued and $(h_j, j \in \mathbb{N}_0)$ a decreasing $(0,+\infty)$-valued sequence, and let $(u^{L_j,h_j}, j \in \mathbb{N}_0)$ be the corresponding sequence of FE solutions to (32), i.e., let $k \in \mathbb{N}$ be such that for every $j \in \mathbb{N}_0$, it holds that*

$$(a^{L_j}\nabla_{\mathbb{S}^2} u^{L_j,h_j}, \nabla_{\mathbb{S}^2} v^{h_j}) = f(v^{h_j}) \qquad \forall v^{h_j} \in V^{h_j,k}.$$

Then, for every $s \in (0, \beta/2)$, there exists a constant C_s such that for every $\mathbb{N}$-valued sequences $(M_j, j = 0.\dots, J)$, $J \in \mathbb{N}_0$, it holds that

$$\| \mathbb{E}(u) - E^J(u^{L_J,h_J})\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \le C_s\Big(\frac{1}{M_0} + \sum_{j=1}^{J} \frac{L_{j-1}^{-2s} + h_{j-1}^{2\min\{s,k\}}}{M_j} + L_J^{-2s} + h_J^{2\min\{s,k\}}\Big)^{1/2}.$$

Proof. Theorem 4.3 implies that there exists a constant $\hat{C}_s$ independent of $(L_j, j \in \mathbb{N}_0)$, $(h_j, j \in \mathbb{N}_0)$, and of J such that for every $j = 1, \dots, J$, it holds that

$$\begin{aligned}&\|u^{L_j,h_j} - u^{L_{j-1},h_{j-1}}\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \\ &\quad\le \|u - u^{L_j,h_j}\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} + \|u - u^{L_{j-1},h_{j-1}}\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})}\end{aligned}$$

$$\leq \hat{C}_s(L_j^{-s} + h_j^{\min\{s,k\}} + L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}}) \leq 2\hat{C}_s(L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}}),$$

where we applied that $(L_j, j \in \mathbb{N}_0)$ is increasing and $(h_j, j \in \mathbb{N}_0)$ is decreasing and recall that the elements of $(u^{L_j,h_j},\ j \in \mathbb{N}_0)$ depend on the polynomial degree k of $V^{h,k}$. Another implication of Theorem 4.3 is that for the same constant $\hat{C}_s$, it holds that

$$\| \mathbb{E}(u) - \mathbb{E}(u^{L_J,h_J})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \|u - u^{L_J,h_J}\|_{L^1(\Omega;H^1(\mathbb{S}^2))} \leq \hat{C}_s(L_J^{-s} + h_J^{\min\{s,k\}}),$$

and due to (33) there exists a constant $\hat{C}$ that is independent of L_0 and h_0 such that

$$\|u^{L_0,h_0}\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \leq \hat{C}.$$

Hence, we conclude the claim of this theorem with the triangle inequality, Lemma 5.1, and with the elementary inequality that $(r_1 + r_2)^2 \leq 2(r_1^2 + r_2^2)$ for every $r_1, r_2 \in \mathbb{R}$. Specifically, for $C_s := 4\max\{\hat{C}_s, \hat{C}\}$, it holds that

$$\begin{aligned}
&\| \mathbb{E}(u) - E^J(u^{L_J,h_J})\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \\
&\quad\leq \| \mathbb{E}(u) - \mathbb{E}(u^{L_J,h_J})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} + \| \mathbb{E}(u^{L_J,h_J}) - E^J(u^{L_J,h_J})\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \\
&\quad\leq \sqrt{2}\Big(\hat{C}_s^2(L_J^{-s} + h_J^{\min\{s,k\}})^2 + \frac{\hat{C}^2}{M_0} + 4\hat{C}_s^2 \sum_{j=1}^{J} \frac{(L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}})^2}{M_j}\Big)^{1/2} \\
&\quad\leq C_s\Big(\frac{1}{M_0} + \sum_{j=1}^{J} \frac{(L_{j-1}^{-2s} + h_{j-1}^{2\min\{s,k\}})}{M_j} + L_J^{-2s} + h_J^{2\min\{s,k\}}\Big)^{1/2}.
\end{aligned}$$

□

Remark 5.3. The convergence analysis of the MC estimator is also covered by Theorem 5.2 for the choice $J = 0$.

It is natural to set

$$h_j := 2^{-j}h_0 \quad \text{for every } j \in \mathbb{N}_0,$$

for some initial mesh width $h_0 > 0$. Generally, one attempts to equilibrate the error contributions. From Theorem 4.3 or Theorem 5.2 we see that to equilibrate the error contributions from the truncation of the Karhunen–Loève expansion of the continuous iGRF and the error contribution from the Galerkin FE approximation we need to choose the increasing sequence $(L_j, j \in \mathbb{N}_0)$ *comparably* to $(h_j^{-1}, j \in \mathbb{N}_0)$, i.e., there exists a constant C with $C^{-1}h_j \leq (L_j)^{-1} \leq Ch_j$ for every $j \in \mathbb{N}_0$, which implies that $(L_j, j \in \mathbb{N}_0)$ is comparable to $(2^jL_0, j \in \mathbb{N}_0)$. Hence, without loss of generality we can consider

$$L_j = 2^jL_0 \quad \text{for every } j \in \mathbb{N}_0$$

for some initial truncation level $L_0 \in \mathbb{N}$. Under our only assumption that the angular power spectrum of the continuous iGRF satisfies (12) for some $\beta > 0$ we obtained with Theorem 3.11 that the unique solution u to (17) is in $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ for every $s \in [0, \beta/2)$ and every $p \in [1, +\infty)$. To determine the sample sizes for a given $\beta > 0$, we fix $s \in (0, \beta/2)$ such that $s \leq k$, where k denotes the polynomial degree of the FE space. A possible choice of the sample numbers $(M_j, j = 0, \ldots, J)$, $J \in \mathbb{N}_0$, in the MLMC estimator is to equilibrate the error contributions of the MLMC estimator across the discretization levels according to Theorem 5.2. This leads to the following choice: for a given maximal discretization level $J \in \mathbb{N}_0$, we set

$$M_0 = \lceil h_J^{-2s}\kappa\rceil = \lceil 2^{2sJ}h_0^{-2s}\kappa\rceil \quad \text{and} \quad M_j = \Big\lceil \Big(\frac{h_{j-1}}{h_J}\Big)^{2s} j^{1+\varepsilon}\kappa\Big\rceil = \lceil 2^{2s(J-j+1)}j^{1+\varepsilon}\kappa\rceil \tag{36}$$

for $j = 1, \dots, J$, a scaling factor $\kappa \geq 2^{-2s}$ (allow $\kappa > 0$ if $J = 0$), and a positive constant $\varepsilon > 0$. If $s > k$ we make the same choices as in (36) with s replaced by k.

Corollary 5.4. *Let $J \in \mathbb{N}_0$, let the conditions of Theorem 5.2 be satisfied for some $\beta > 0$ and with the made choices for $(L_j, h_j, M_j, j = 0, \dots, J)$, and let ζ denote the Riemann zeta function. Then, for every $s \in (0, \beta/2)$, and for $\varepsilon > 0$ as in (36), there exists $C_s > 0$ such that*

$$\| \mathbb{E}(u) - E^J(u^{L_J,h_J})\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s\Big(\zeta(1+\varepsilon)\frac{1}{\kappa} + 1\Big)^{1/2} h_J^{\min\{s,k\}}.$$

If for $\eta_1 > 0$ and $\eta_2 \geq 0$, the work to compute one sample of u^{L_j,h_j} is comparable to $h_j^{-2\eta_1} \log^{\eta_2}(h_j^{-2})$, $j = 0, \dots, J$, then the total work to compute $E^J(u^{L_J,h_J})$ satisfies

$$\mathcal{W}_J = \begin{cases} \mathrm{O}(h_J^{-2\min\{s,k\}}\kappa) = \mathrm{O}(2^{2\min\{s,k\}J}\kappa) & \min\{s,k\} > \eta_1 \\ \mathrm{O}(h_J^{-2\eta_1}\max\{J,1\}^{\eta_2+2+\varepsilon}\kappa) = \mathrm{O}(2^{2J\eta_1}\max\{J,1\}^{\eta_2+2+\varepsilon}\kappa) & \min\{s,k\} \leq \eta_1 \end{cases},$$

where the contributions of $h_0^{-1}, L_0, \eta_1, \eta_2$ are absorbed into the Landau symbols.

Proof. Let $s_0 := \min\{s,k\}$. The error estimate follows by the choices of the values for $(L_j, h_j, M_j, j = 0, \dots, J)$ due to Theorem 5.2, i.e., we conclude that

$$\begin{aligned} &\| \mathbb{E}(u) - E^J(u^{L_J,h_J})\|_{L^2(\Omega;H^1(\mathbb{S}^2)/\mathbb{R})} \\ &\leq \hat{C}_s\Big(\frac{1}{M_0} + \sum_{j=1}^{J} \frac{L_{j-1}^{-2s} + h_{j-1}^{2s_0}}{M_j} + L_J^{-2s} + h_J^{2s_0}\Big)^{1/2} \\ &\leq \hat{C}_s\Big(\frac{1}{\kappa} + \frac{1}{\kappa}\Big(\frac{1}{(L_0h_0)^{2s_0}} + 1\Big)\zeta(1+\varepsilon) + \frac{1}{(L_0h_0)^{2s_0}} + 1\Big)^{1/2} h_J^{s_0}, \end{aligned}$$

where $\hat{C}_s$ is the constant from Theorem 5.2. Since $\zeta(1+\varepsilon) > 1$ for every $\varepsilon > 0$, we obtain the claimed estimate with $C_s := \hat{C}_s\sqrt{(L_0h_0)^{-2s_0} + 2}$. To prove the bound on the computational work, we insert the values for M_j and h_j and obtain with $C_1 > 0$ and a constant $C_2 > 0$ depending on L_0, h_0, η_1, η_2 that

$$\begin{aligned} \mathcal{W}_J &\leq C_1\Big(M_0 h_0^{-2\eta_1}\log^{\eta_2}(h_0^{-2}) + \sum_{j=1}^{J} M_j(h_j^{-2\eta_1}\log^{\eta_2}(h_j^{-2}) + h_{j-1}^{-2\eta_1}\log^{\eta_2}(h_{j-1}^{-2}))\Big) \\ &\leq C_2\kappa\Big(2^{2s_0J} + \sum_{j=1}^{J} 2^{2s_0(J-j+1)+2j\eta_1} j^{\eta_2+1+\varepsilon}\Big). \end{aligned}$$

If $s_0 \leq \eta_1$, then $\mathcal{W}_J = \mathrm{O}(2^{2J\eta_1}J^{\eta_2+2+\varepsilon}\kappa) = \mathrm{O}(h_J^{-2\eta_1}J^{\eta_2+2+\varepsilon}\kappa)$. In the other case that $s_0 > \eta_1$, it follows with the fact $\sum_{j\geq 1}\rho^j j^{\eta_2+1+\varepsilon} < +\infty$ for every $\rho \in (0,1)$ that

$$\mathcal{W}_J \leq C_2 2^{2s_0J}\kappa(1 + \sum_{j=1}^{J} 2^{-2j(s_0-\eta_1)} j^{\eta_2+1+\varepsilon}) = \mathrm{O}(2^{2s_0J}\kappa) = \mathrm{O}(h_J^{-2s_0}\kappa). \qquad \square$$

Note that the choices $(M_j, j = 0, \dots, J)$, $J \in \mathbb{N}_0$ in (36) depend on the regularity of the solution u to (17). However, the closer s is to $\beta/2$ the harder it should be to observe the convergence behavior that is theoretically guaranteed by Theorem 5.2, because constants may become arbitrarily large. We conclude the paper with several remarks on the convergence bounds.

Remark 5.5. For smooth source terms $f \in C^\infty(\mathbb{S}^2) = \bigcap_{s'>0} H^{-1+s'}(\mathbb{S}^2)$, the convergence rate for Spectral Methods is given by s without further restrictions, cp. Theorem 4.4 and Remark 5.7. The decay of the angular power spectrum of the logarithm of the stochastic diffusion coefficient, i.e., $T = \log(a)$ satisfies (12) for some $\beta > 0$, is the only constraint of the convergence rate s with $s < \beta/2$. For Finite Element Methods, the convergence rate is additionally bounded by the polynomial degree k of the Finite Element space, cp. Theorem 5.2.

We conclude that we have essentially determined the achievable convergence rates of MLMC FE and Spectral Methods solely with the decay of the angular power spectrum of the logarithm of the isotropic lognormal diffusion coefficient, which in the FE case are bounded by the polynomial degree of the basis functions.

Remark 5.6. There exists an algorithm to compute samples of an iGRF that has a complexity behaving as $\mathrm{O}(N \log^2(N))$, cp. [18], where N is the number of sample points. In the FE case, iterative solvers such as multigrid, cp. [4], suggest to have a complexity that is linear in the degrees of freedom, where here the resulting linear systems do not render the classical theory, since condition numbers of system matrices may be close to degenerate due to the lognormal diffusion coefficient. In the setting of Corollary 5.4, this would allow for $\eta_1 = 1$ and $\eta_2 = 2$.

Remark 5.7. The proof of Theorem 5.2 is not restricted to the considered FE Methods above. If the conditions of Theorem 4.4 are satisfied with $\beta > 0$, an analogous argument implies the respective statement in the case of Spectral Methods, i.e., for every $s \in (0, \beta/2)$, there exists a constant $C_s > 0$ such that for $J \in \mathbb{N}_0$,

$$\| \mathbb{E}(u) - E^J(u^{L^a_J, L^u_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \Big(\frac{1}{M_0} + \sum_{j=1}^{J} \frac{1}{M_j} L_{j-1}^{-2s} + L_J^{-2s} \Big)^{1/2},$$

where the degrees of $a^{L^a_j}$ and of $\mathcal{H}_{1:L^u_j}$, $j \in \mathbb{N}_0$, are chosen as increasing sequences that define $L_j := \min\{L^a_j, L^u_j\}$, $j \in \mathbb{N}_0$. As in the FE case the number of samples to equilibrate the MC errors on the levels can be chosen $M_0 := \lceil L_J^{2s} \kappa \rceil$ and $M_j := \lceil (L_J/L_{j-1})^{2s} j^{1+\varepsilon} \kappa \rceil$, $j = 1, \ldots, J$, for a positive constant $\varepsilon > 0$ and $\kappa \geq (L_J/L_{J-1})^{-2s}$ (allow $\kappa > 0$ if $J = 0$). Hence, there exists $C_s > 0$ such that

$$\| \mathbb{E}(u) - E^J(u^{L^a_J, L^u_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \Big(\zeta(1+\varepsilon) \frac{1}{\kappa} + 1 \Big)^{1/2} L_J^{-s}.$$

Remark 5.8. Since the degrees of freedom of either of the considered spatial discretizations relate to the discretization parameter with $N_{h_J} = \dim(V^{h,k}) = \mathrm{O}(h_J^{-2})$ and $N_{L^u} = \dim(\mathcal{H}_{1:L^u}) = \mathrm{O}((L^u)^2)$, cp. Section 4, respective convergence estimates and work bounds from Corollary 5.4 and Remark 5.7 in the degrees of freedom are implied.

Remark 5.9. A decrease of the choices of samples $(M_j, j = 0, \ldots, J)$ in (36), i.e., if $\kappa < 1$, will increase the MC error contribution in Theorem 5.2 basically by the inverted square root of κ due to a larger MC error contribution. For instance, applying [24, Theorem 1] in our setting yields sample numbers scaled by a factor of $2^{-2\min\{s,k\}}$ for $j = 1, \ldots, J$ sacrificing an increase in the corresponding constant C_2 (in the notation of [24, Theorem 1]) of the error estimate that is scaled by a factor of $2^{2\min\{s,k\}}$ and appears under a square root, which is a natural consequence of the convergence rate of 1/2 of MC methods.

Appendix A. Measure theory and functional analysis

In this appendix we collect additional background material for completeness of the presentation and for the convenience of the reader.

Theorem A.1. *If $s \in (0, +\infty), q \in (1, +\infty), \iota \in \mathbb{N}_0$, and $\gamma \in (0, 1)$ satisfy $s - 2/q \geq \iota + \gamma$, then the embedding $H^s_q(\mathbb{S}^2) \subset C^{\iota,\gamma}(\mathbb{S}^2)$ is continuous.*

Proof. Let $\{(U_i, \eta_i), i \in \mathcal{I}\}$ be a finite C^∞ atlas of $\mathbb{S}^2$ with subordinate partition of unity $\{\Psi_i, i \in \mathcal{I}\}$. In [38, Theorem 4.6.1(e)] it is shown that the embedding $H^s_q(\mathbb{R}^2) \subset C^{\iota,\gamma}(\mathbb{R}^2)$ is continuous with $p = q$, $n = 2$, and $t = \iota + \gamma$ in the framework of the theorem. Hence, every $v \in H^s_q(\mathbb{S}^2)$ has a representative, also denoted by v, such that $(v\Psi_i) \circ \eta_i^{-1} \in C^{\iota,\gamma}(\mathbb{R}^2)$ for every $i \in \mathcal{I}$ and satisfies the estimate

$$\|(v\Psi_i) \circ \eta_i^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)} \leq C\|(v\Psi_i) \circ \eta_i^{-1}\|_{H^s_q(\mathbb{R}^2)} \leq C\|v\|_{H^s_q(\mathbb{S}^2)},$$

where C does not depend on v. We take the maximum of the left hand side and obtain the assertion. □

Let $(B, \|\cdot\|_B)$ be a Banach space with dual space B^*. We recall the definition of weak and strong measurability and mappings being $\mathbb{P}$-almost separably-valued in the following definition, which corresponds to [21, Definitions 3.5.3 and 3.5.4].

Definition A.2. Let $X : \Omega \to B$.

(1) X is called *weakly measurable* if for every $\mathcal{G} \in B^*$, the real-valued function $\mathcal{G}(X)$ is measurable.
(2) X is called *countably-valued* if X assumes at most a countable set of values in B on countably many, disjoint measurable subsets.
(3) X is called *strongly measurable* if there exists a sequence of countably-valued mappings $(X_n, n \in \mathbb{N})$, $X_n : \Omega \to B$, such that $\lim_{n\to+\infty} X_n(\omega) = X(\omega)$ in B for $\mathbb{P}$-a.e. $\omega \in \Omega$.
(4) X is called $\mathbb{P}$-*almost separably-valued* if there exists a measurable set N with $\mathbb{P}(N) = 0$ such that the set $\{X(\omega), \omega \in \Omega\backslash N\}$ is separable in B.

Weak and strong measurability are connected through Pettis' theorem, which can be found in [21, Theorem 3.5.3].

Theorem A.3 (Pettis' theorem)**.** *A B-valued mapping on Ω is strongly measurable if and only if it is weakly measurable and $\mathbb{P}$-almost separably-valued.*

In applications various Banach spaces occur, which is why we include them in our notation and write 'strongly B-measurable' as well as 'weakly B-measurable'.

Lemma A.4. *Let B_1, B_2 be Banach spaces and let $\varphi : B_1 \to B_2$ be continuous. If $f : \Omega \to B_1$ is strongly B_1-measurable, then $\varphi \circ f$ is strongly B_2-measurable.*

This follows directly from the definition of strong measurability. We remark that a mapping $X : \Omega \to B$ is Bochner integrable if and only if it is strongly B-measurable and the real-valued function $\|X\|_B$ is integrable, cp. [21, Theorem 3.7.4]. The strong B-measurability of X implies that the real-valued function $\|X\|_B$ is measurable.

Appendix B. Details to Section 3.2

In this part of the appendix, we provide the details that are missing in Section 3.2.

Lemma B.1. *For some* $0 \le s < \gamma < 1$, *let* $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$, $f \in H^{-1+s}(\mathbb{S}^2)$, *and* $\tilde{a} \in C^{0,\gamma}(\mathbb{S}^2) \cap C^0_+(\mathbb{S}^2)$ *satisfy that*

$$(\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2}v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}, \tag{37}$$

then $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$ *and there exists a constant* C, *which is independent of* $\tilde{u}, f$, *and* $\tilde{a}$, *such that*

$$\|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} \le C\Big(\frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}(\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)}\Big).$$

Proof. Let $\{(\eta_j, U_j), j \in \mathcal{I}\}$ be a C^∞-atlas and $\{\Psi_j, j \in \mathcal{I}\}$ be the subordinate partition of unity. Let us fix $i \in \mathcal{I}$. We observe with the product rule and (37) that $\tilde{u}\Psi_i$ satisfies for every $v \in H^1(\mathbb{S}^2)/\mathbb{R}$ that

$$\begin{aligned}(\tilde{a}\nabla_{\mathbb{S}^2}(\tilde{u}\Psi_i), \nabla_{\mathbb{S}^2}v) &= (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2}(v\Psi_i)) - (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}\cdot\nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u}\nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v)\\ &= f(v\Psi_i - \frac{1}{|\mathbb{S}^2|}\int v\Psi_i \mathrm{d}\sigma) - (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}\cdot\nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u}\nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v)\\ &= f(v\Psi_i) - \frac{1}{|\mathbb{S}^2|}f(1)(\Psi_i, v) - (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}\cdot\nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u}\nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v),\end{aligned}$$

where we remark that for every $v \in H^1(\mathbb{S}^2)/\mathbb{R}$, it holds that $v\Psi_i - 1/|\mathbb{S}^2|\int v\Psi_i \mathrm{d}\sigma \in H^1(\mathbb{S}^2)/\mathbb{R}$. Let $V_i := \eta_i(U_i)$ and let $D \subset \mathbb{R}^2$ with smooth boundary be such that $\mathrm{supp}(\Psi_i \circ \eta_i^{-1}) \subset\subset D \subset\subset V_i$. We recall that for two functions $w_1, w_2 : \mathbb{S}^2 \to \mathbb{R}$, the first fundamental form of their gradients satisfies with respect to the coordinate chart η_i that on V_i it holds that

$$(\nabla_{\mathbb{S}^2}w_1 \cdot \nabla_{\mathbb{S}^2}w_2)\circ\eta_i^{-1} = \sum_{k,\ell=1}^{2} g^{k\ell}\circ\eta_i^{-1}\frac{\partial(w_1\circ\eta_i^{-1})}{\partial x^k}\frac{\partial(w_2\circ\eta_i^{-1})}{\partial x^\ell}.$$

Furthermore, there exists a constant $\lambda_g > 0$ such that for every $y \in U_i$, $\sum_{k,\ell=1}^2 g^{k\ell}(y)\xi_k\xi_\ell \ge \lambda_g \sum_{k=1}^2 \xi_k^2$ for every $\xi \in T_y\mathbb{S}^2$. We also recall that with respect to the coordinate chart η_i it holds that $\mathrm{d}\sigma(y) = \sqrt{|g|(y)}\mathrm{d}x$, where $|g|(y) = \det(g(y))$ and $y = \eta_i^{-1}(x)$, and $|g|(y) > 0$ for every $x \in V_i$. We choose $\chi \in C^\infty(\mathbb{R}^2)$ such that $\chi = 1$ on $\mathrm{supp}(\Psi_i \circ \eta_i^{-1})$ and $\chi = 0$ on the complement of D. We define the matrix-valued function

$$A := \begin{cases} ((\sqrt{|g|}\tilde{a}g^{-1})\circ\eta_i^{-1})\,\chi + \min_{y\in U_i}\{\sqrt{|g|(y)}\tilde{a}(y)\}\lambda_g\,(1-\chi)\,\mathrm{Id}_{\mathbb{R}^2} & \text{on } V_i\\ \min_{y\in U_i}\{\sqrt{|g|}\tilde{a}\}\lambda_g\,(1-\chi)\,\mathrm{Id}_{\mathbb{R}^2} & \text{else}\end{cases}$$

and the functions

$$b := \begin{cases} ((\sqrt{|g|}\tilde{a})\circ\eta_i^{-1})\chi\, \sum_{k,l=1}^2 g^{k\ell}\circ\eta_i^{-1}\frac{\partial(\tilde{u}\circ\eta_i^{-1})}{\partial x^k}\frac{\partial(\Psi_i\circ\eta_i^{-1})}{\partial x^\ell} & \text{on } V_i\\ \chi & \text{else}\end{cases},$$

$$c := \begin{cases} ((\sqrt{|g|}\tilde{a}\tilde{u})\circ\eta_i^{-1})\,\chi\, \sum_{k=1}^2 (g^{k1}, g^{k2})^\top\circ\eta_i^{-1}\frac{\partial(\Psi_i\circ\eta_i^{-1})}{\partial x^k} & \text{on } V_i\\ (\chi,\chi)^\top & \text{else}\end{cases}.$$

We use these three functions to define the functional F for every $w \in H^1(\mathbb{R}^2)$ by

$$w \mapsto F(w) := f(((w\chi)\circ\eta_i)\Psi_i) - \frac{1}{|\mathbb{S}^2|}f(1)(\Psi_i, (w\chi)\circ\eta_i) - \int_{\mathbb{R}^2} bw\,\mathrm{d}x + \int_{\mathbb{R}^2} c\cdot\nabla w\,\mathrm{d}x. \tag{38}$$

We observe that for every $w \in H^1(\mathbb{R}^2)$, the function $((w\chi)\circ\eta_i)$ can be extended to a function $\tilde{w} \in H^1(\mathbb{S}^2)/\mathbb{R}$, which then satisfies that

$$F(w) = f(\tilde{w}\Psi_i) - \frac{1}{|\mathbb{S}^2|}f(1)(\Psi_i,\tilde{w}) - (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}\cdot\nabla_{\mathbb{S}^2}\Psi_i,\tilde{w}) + (\tilde{a}\tilde{u}\nabla_{\mathbb{S}^2}\Psi_i,\nabla_{\mathbb{S}^2}\tilde{w})$$

and

$$\int_{V_i} A\nabla((u\Psi_i)\circ\eta_i^{-1})\cdot\nabla w \,\mathrm{d}x = (\tilde{a}\nabla_{\mathbb{S}^2}(u\Psi_i),\nabla_{\mathbb{S}^2}\tilde{w}),$$

where we used that $\chi = 1$ on $\mathrm{supp}(\Psi_i\circ\eta_i^{-1})$. Since $\mathrm{supp}(\Psi_i\circ\eta_i^{-1}) \subset V_i$, we obtain that

$$\int_{\mathbb{R}^2} A\nabla((u\Psi_i)\circ\eta_i^{-1})\cdot\nabla w \,\mathrm{d}x = F(w) \quad \forall w \in H^1(\mathbb{R}^2). \tag{39}$$

We now aim to prove finiteness of the $H^{-1+s}(\mathbb{R}^2)$-norm of F and to find a suitable bound. Let $\{\hat{\Psi}_j, j\in\mathcal{I}\}$ be another partition of unity subordinate to the open cover $\{U_j, j\in\mathcal{I}\}$ such that $\hat{\Psi}_i\circ\eta_i^{-1} = 1$ on $\mathrm{supp}(\chi) \supset \mathrm{supp}(\Psi_i\circ\eta_i^{-1})$, which necessarily implies that $\hat{\Psi}_j = 0$ on $\mathrm{supp}(\Psi_i)$ for every $j \neq i$. Thus we obtain with the characterization of the $H^{1-s}(\mathbb{S}^2)$-norm on chart domains, the partition of unity property of $\{\hat{\Psi}_j, j\in\mathcal{I}\}$, Proposition 3.9, and [36, Theorem 3.3.2(ii)] that there are constants C_1, C_2, C_3 such that for every $w \in H^{1-s}(\mathbb{R}^2)$, it holds that

$$\begin{aligned}|f(((w\chi)\circ\eta_i)\Psi_i)| &\le C_1\|f\|_{H^{-1+s}(\mathbb{S}^2)}\|(((w\chi)\circ\eta_i)\hat{\Psi}_i)\circ\eta_i^{-1}\|_{H^{1-s}(V_i)}\|\Psi_i\|_{C^1(\mathbb{S}^2)}\\ &\le C_2\|f\|_{H^{-1+s}(\mathbb{S}^2)}\|w\chi\|_{H^{1-s}(V_i)}\|\hat{\Psi}_i\circ\eta_i^{-1}\|_{C^1(\overline{V_i})}\\ &\le C_3\|f\|_{H^{-1+s}(\mathbb{S}^2)}\|w\|_{H^{1-s}(\mathbb{R}^2)}\|\chi\|_{C^1(\overline{V_i})}.\end{aligned}$$

The forth summand in the definition of F in (38) can be written in a distributional sense as $w \mapsto -\int_{\mathbb{R}^2}(\nabla\cdot c)w\,\mathrm{d}x$, where we applied that c is compactly supported in V_i. Note that for $\ell = 1,2$ and $s\in\mathbb{R}$, the linear operators $\frac{\partial}{\partial x^\ell} : H^s(\mathbb{R}^2) \to H^{s-1}(\mathbb{R}^2)$ are bounded, Hence, we conclude as in the proof of Proposition 3.9 with [36, Theorem 3.3.2(ii)] and the property that $\chi = 1$ on $\mathrm{supp}(\hat{\Psi}_i\circ\eta_i^{-1})$ that there exist constants C_1, C_2, C_3, C_4 such that

$$\begin{aligned}\|\nabla\cdot c\|_{H^{-1+s}(\mathbb{R}^2)} \le C_1\|c\|_{H^s(\mathbb{R}^2)} &\le C_2\|(\tilde{a}\circ\eta_i^{-1})\chi\|_{C^{0,\gamma}(\overline{V_i})}\|\Psi_i\circ\eta_i^{-1}\|_{C^1(\overline{V_i})}\|(\tilde{u}\circ\eta_i^{-1})\chi\|_{H^s(V_i)}\\ &\le C_3\|(\tilde{a}\hat{\Psi}_i)\circ\eta_i^{-1}\|_{C^{0,\gamma}(\overline{V_i})}\|(\tilde{u}\hat{\Psi}_i)\circ\eta_i^{-1}\|_{H^s(V_i)}\|\chi\|^2_{C^{0,\gamma}(\overline{V_i})}\\ &\le C_4\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^s(\mathbb{S}^2)},\end{aligned}$$

where we applied that derivatives of smooth compactly supported functions, e.g., $\Psi_i\circ\eta_i^{-1}$ and χ, are bounded. Their norms have been included into the constants appearing in the above inequalities. The $H^{-1+s}(\mathbb{R}^2)$-norm of the third summand in (38) can be treated similarly, i.e., there exists a constant C such that $\|b\|_{H^{-1+s}(\mathbb{R}^2)} \le C\|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)}$. The second summand in (38) poses no difficulty. Hence, we conclude that $F \in H^{-1+s}(\mathbb{R}^2)$ and that there exists a constant C, which is independent of $\tilde{a}, \tilde{u}$, and f, such that

$$\|F\|_{H^{-1+s}(\mathbb{R}^2)} \le C(\|f\|_{H^{-1+s}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^s(\mathbb{S}^2)}). \tag{40}$$

We observe that for every $\xi\in\mathbb{R}^2$, it holds that $\xi^\top A\xi \ge \lambda_g \min_{x\in V_i}\sqrt{|g(x)|}\min_{x\in\mathbb{S}^2}\tilde{a}(x)\xi^\top\xi$ on $\mathbb{R}^2$. Since the matrix-valued function A is constant on the complement of V_i, we observe that there exists a constant C such that

$$\sup_{x,y\in\mathbb{R}^2, x\neq y}\frac{\|A(x)-A(y)\|_{\mathbb{R}^{2\times 2}}}{\|x-y\|^\gamma_{\mathbb{R}^2}} \le C\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}. \tag{41}$$

We are now in the situation to apply the regularity estimate in [6, Lemma 3.2] to the problem in (39), which implies that $(u\Psi_i)\circ\eta_i^{-1} \in H^{1+s}(\mathbb{R}^2)$. Also it implies together with the estimates in (40) and in (41) that there exist constants C_1, C_2, C_3 such that

$$\begin{aligned}
\|(\tilde{u}\Psi_i)\circ\eta_i^{-1}\|_{H^{1+s}(\mathbb{R}^2)} &\leq C_1 \frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}(\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|(\tilde{u}\Psi_i)\circ\eta_i^{-1}\|_{H^1(\mathbb{R}^2)} + \|F\|_{H^{-1+s}(\mathbb{R}^2)}) \\
&\quad + C_1\|(\tilde{u}\Psi_i)\circ\eta_i^{-1}\|_{H^1(\mathbb{R}^2)} \\
&\leq C_2 \frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}(\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)} \\
&\quad + \|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^s(\mathbb{S}^2)}) + C_1\|\tilde{u}\|_{H^1(\mathbb{S}^2)} \\
&\leq C_3\Big(\frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}(\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)}\Big),
\end{aligned}$$

where the first inequality is the estimate from [6, Lemma 3.2] applied to our setting.

This argument can be repeated for all remaining $i \in \mathcal{I}$, which implies that $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$, and therefore we can establish the previous estimate for every $i \in \mathcal{I}$. Hence, we sum this squared estimate over all $i \in \mathcal{I}$ and take the square root. We maximize the constants over the finite index set $\mathcal{I}$ which establishes the estimate claimed in the lemma. □

Proof of Proposition 3.8. From Lemma B.1 we readily conclude that $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$. Also this lemma implies with the $H^1(\mathbb{S}^2)/\mathbb{R}$-estimate in (21) that there exist constants C_1, C_2 such that

$$\begin{aligned}
\|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} &\leq C_1\Big(\frac{1}{\min_{x\in\mathbb{S}^2}\tilde{a}(x)}(\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)}\Big) \\
&\leq C_2 \frac{\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}}{(\min_{x\in\mathbb{S}^2}\tilde{a}(x))^2}\|f\|_{H^{-1+s}(\mathbb{S}^2)},
\end{aligned}$$

where we applied that $\|\tilde{a}\|_{C^0(\mathbb{S}^2)}/(\min_{x\in\mathbb{S}^2}\tilde{a}(x)) \geq 1$ and summarized the resulting terms. □

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