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Abstract

In this article we develop a framework for studying parabolic semilinear stochastic evolution equations (SEEs) with singularities in the initial condition and singularities at the initial time of the time-dependent coefficients of the considered SEE. We use this framework to establish existence, uniqueness, and regularity results for mild solutions of parabolic semilinear SEEs with singularities at the initial time. We also provide several counterexample SEEs that illustrate the optimality of our results.

Contents

1 Introduction

There are a number of existence, uniqueness, and regularity results for mild solutions of semilinear stochastic evolution equations (SEEs) in the literature; see, e.g., [10, 11, 4, 23, 15, 17, 19, 22] and the references mentioned therein. In this work we extend the above cited results by adding singularities in the initial condition and by introducing singularities at the initial time of the time-dependent coefficients of the considered SEE; cf., e.g., Chen & Dalang [7, 8]. To illustrate the results of this article, we assume the following setting throughout this introductory section. Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ be nontrivial separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $p \in [2,\infty)$, $\alpha \in [0,1)$, $\hat{\alpha} \in (-\infty,1)$, $\beta \in [0,1/2)$, $\hat{\beta} \in (-\infty,1/2)$, $L_0, \hat{L}_0, L_1, \hat{L}_1 \in [0,\infty)$ satisfy $\mathbb{1}_{(0,\infty)}(L_1)\cdot[\alpha+\hat{\alpha}] < 3/2$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $A\colon D(A)\subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with spectrum$(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$ (cf., e.g., [14, Definition 3.5.25]), let $\mathbf{F} \in \mathcal{M}\big(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})\otimes\mathcal{B}(H), \mathcal{B}(H_{-\alpha})\big)$, $\mathbf{B} \in \mathcal{M}\big(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})\otimes\mathcal{B}(H), \mathcal{B}(HS(U,H_{-\beta}))\big)$ satisfy for all $t\in(0,T]$, $X,Y \in \mathcal{L}^p(\mathbb{P};H)$ that

$$\|\mathbf{F}(t,X)-\mathbf{F}(t,Y)\|_{L^p(\mathbb{P};H_{-\alpha})} \leq L_0\,\|X-Y\|_{L^p(\mathbb{P};H)},\ \|\mathbf{F}(t,0)\|_{L^p(\mathbb{P};H_{-\alpha})} \leq \hat{L}_0\,t^{-\hat{\alpha}}, \tag{1}$$

$$\|\mathbf{B}(t,X)-\mathbf{B}(t,Y)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))} \leq L_1\|X-Y\|_{L^p(\mathbb{P};H)},\ \|\mathbf{B}(t,0)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))} \leq \hat{L}_1\,t^{-\hat{\beta}}, \tag{2}$$

for every $a,b\in(-\infty,1)$ let $\mathrm{E}_{a,b}\colon[0,\infty)\to[0,\infty)$ be the function with the property that for all $x\in[0,\infty)$ it holds that $\mathrm{E}_{a,b}[x] = 1+\sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1}\int_0^1 t^{-b}\,(1-t)^{k(1-b)-a}\,dt$ (cf., e.g., [13, Chapter 7]), for every $r\in[0,1]$ let $\chi_r\in(0,\infty)$ be the real number given by $\chi_r = \sup_{t\in(0,T]} t^r\,\|(\eta-A)^r e^{tA}\|_{L(H)}$, let $\kappa\in\{0,1\}$ be the real number given by $\kappa = \mathbb{1}_{(0,\infty)}(L_1)$, and for every $\lambda\in(-\infty,\frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)])$ let $\Theta_\lambda\in[0,\infty)$ be the real number given by

$$\Theta_\lambda = 2^{\kappa/2}\left|\mathrm{E}_{(1+\kappa)\lambda,\max\{\alpha,2\beta\kappa\}}\left[\left|\tfrac{\chi_\alpha\,L_0\,2^{\kappa/2}\,T^{(1-\alpha)}}{(1-\alpha)^{\kappa/2}} + \chi_\beta\,L_1\sqrt{p\,(p-1)\,T^{(1-2\beta)}}\right|^{(1+\kappa)}\right]\right|^{(2-\kappa)/2}. \tag{3}$$

In displays (4)–(12) below we illustrate the above framework through several examples and applications.

Our first result is a suitable *perturbation estimate* for predictable stochastic processes. More formally, in Proposition 2.5 below we prove for all $\delta\in\mathbb{R}$, $\lambda\in(-\infty,\frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)])$ and all $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $Y^1,Y^2\colon[0,T]\times\Omega\to H_\delta$ with $\cup_{k\in\{1,2\}}Y^k((0,T]\times\Omega)\subseteq H$ and $\limsup_{r\nearrow\frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]}\max_{k\in\{1,2\}}\sup_{t\in(0,T]} t^r\,\|Y^k_t\|_{L^p(\mathbb{P};H)}<\infty$ that $\forall\,t\in[0,T]\colon \mathbb{P}\big(\sum_{k=1}^2\int_0^t \|e^{(t-s)A}\mathbf{F}(s,Y^k_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s,Y^k_s)\|^2_{HS(U,H)}\,\mathrm{d}s<\infty\big)=1$ and

$$\begin{aligned}\sup_{t\in(0,T]}\big[t^\lambda\,\|Y^1_t-Y^2_t\|_{L^p(\mathbb{P};H)}\big] \leq \sup_{t\in(0,T]}\Bigg[t^\lambda\,\Bigg\|Y^1_t-\int_0^t e^{(t-s)A}\mathbf{F}(s,Y^1_s)\,\mathrm{d}s-\int_0^t e^{(t-s)A}\mathbf{B}(s,Y^1_s)\,\mathrm{d}W_s\\ +\int_0^t e^{(t-s)A}\mathbf{F}(s,Y^2_s)\,\mathrm{d}s+\int_0^t e^{(t-s)A}\mathbf{B}(s,Y^2_s)\,\mathrm{d}W_s-Y^2_t\Bigg\|_{L^p(\mathbb{P};H)}\Bigg]\,\Theta_\lambda.\end{aligned} \tag{4}$$

We note that the right hand side of (4) might be infinite. Moreover, we would like to emphasize that Y^1 and Y^2 in (4) are arbitrary $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes with $\cup_{k\in\{1,2\}}Y^k((0,T]\times\Omega)\subseteq H$ and $\limsup_{r\nearrow\frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]}\max_{k\in\{1,2\}}\sup_{t\in(0,T]} t^r\,\|Y^k_t\|_{L^p(\mathbb{P};H)}<\infty$ and, in particular, we emphasize that Y^1 and Y^2 do not need to be solution processes of some

SEEs. Estimate (4) follows from an appropriate application of a generalized Gronwall-type inequality (see the proof of Proposition 2.5 below for details).

We use inequality (4) to establish an existence, uniqueness, and regularity result for SEEs with singularities at the initial time. More precisely, in Theorem 2.7 below we prove that for all $\delta \in \big(-\infty, \frac{1}{2}[1 + \mathbb{1}_{\{0\}}(L_1)]\big)$, $\lambda \in \big[\max\{\delta, \alpha + \hat{\alpha} - 1, \beta + \hat{\beta} - 1/2\}, \frac{1}{2}[1 + \mathbb{1}_{\{0\}}(L_1)]\big)$, $\xi \in \mathcal{L}^p(\mathbb{P}|_{\mathcal{F}_0}; H_{-\delta^+})$ with $\sup_{t\in(0,T]} t^\delta \, \|e^{tA}\xi\|_{L^p(\mathbb{P};H)} < \infty$ it holds (i) that there exists an up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process $X\colon [0,T] \times \Omega \to H_{-\delta^+}$ which satisfies for all $t \in [0,T]$ that $X((0,T] \times \Omega) \subseteq H$, that $\sup_{s\in(0,T]} s^\lambda \, \|X_s\|_{L^p(\mathbb{P};H)} < \infty$, that $\mathbb{P}\big(\int_0^t \|e^{(t-s)A}\mathbf{F}(s, X_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s, X_s)\|^2_{HS(U,H)} \, \mathrm{d}s < \infty\big) = 1$, and $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}\mathbf{F}(s, X_s) \, \mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s, X_s) \, \mathrm{d}W_s \tag{5}$$

and (ii) that

$$\begin{split} &\sup_{t\in(0,T]} \Big[t^\lambda \, \|X_t\|_{L^p(\mathbb{P};H)}\Big] \leq T^\lambda \, \Theta_\lambda \\ &\cdot \left[\frac{\sup_{t\in(0,T]}(t^\delta \|e^{tA}\xi\|_{L^p(\mathbb{P};H)})}{T^\delta} + \frac{\chi_\alpha \, \hat{L}_0 \, \mathbb{B}(1-\alpha, 1-\hat{\alpha})}{T^{(\alpha+\hat{\alpha}-1)}} + \frac{\chi_\beta \, \hat{L}_1 \big|p\,(p-1)\, \mathbb{B}(1-2\beta, 1-2\hat{\beta})\big|^{1/2}}{\sqrt{2}\, T^{(\beta+\hat{\beta}-1/2)}} \right] < \infty. \end{split} \tag{6}$$

Inequality (6) follows from the perturbation estimate (4) (with $Y^1 = X$ and $Y^2 = 0$ in the notation of (4)). We now illustrate Theorem 2.7 and (5)–(6), respectively, by some examples. In particular, in Corollary 2.8 below we prove by an application of Theorem 2.7 that for all $F \in \mathrm{Lip}(H, H_{-\alpha})$, $B \in \mathrm{Lip}(H, HS(U, H_{-\beta}))$, $\hat{\delta} = \frac{1}{2}\big[1+\mathbb{1}_{\{0\}}(|B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))})\big]$ it holds (i) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^x\colon [0,T] \times \Omega \to H_{-\delta}$, $x \in H_{-\delta}$, $\delta \in [0, \hat{\delta})$, which fulfill for all $q \in [2, \infty)$, $\delta \in [0, \hat{\delta})$, $x \in H_{-\delta}$, $t \in [0,T]$ that $X^x((0,T] \times \Omega) \subseteq H$, that $\sup_{s\in(0,T]} s^\delta \, \|X^x_s\|_{L^q(\mathbb{P};H)} < \infty$, and $\mathbb{P}$-a.s. that

$$X^x_t = e^{tA}x + \int_0^t e^{(t-s)A}F(X^x_s) \, \mathrm{d}s + \int_0^t e^{(t-s)A}B(X^x_s) \, \mathrm{d}W_s \tag{7}$$

and (ii) that

$$\forall\, \delta \in [0, \hat{\delta}),\, q \in [2, \infty)\colon \sup_{\substack{x,y\in H_{-\delta},\\ x\neq y}} \sup_{t\in(0,T]} \max\left\{ \frac{t^\delta \, \|X^x_t\|_{L^q(\mathbb{P};H)}}{\max\{1, \|x\|_{H_{-\delta}}\}}, \frac{t^\delta \, \|X^x_t - X^y_t\|_{L^q(\mathbb{P};H)}}{\|x-y\|_{H_{-\delta}}} \right\} < \infty. \tag{8}$$

Here and below we denote for $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ by $\mathrm{Lip}(V, W)$ the set of all Lipschitz continuous functions from V to W and we denote for nontrivial $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ and a function $f \in \mathrm{Lip}(V, W)$ by $|f|_{\mathrm{Lip}(V,W)} \in [0, \infty)$ the real number given by $|f|_{\mathrm{Lip}(V,W)} = \sup_{v,w\in V,\, v\neq w} \frac{\|f(v)-f(w)\|_W}{\|v-w\|_V}$. The finitness of the second element in the set in the maximum in (8) follows from the perturbation estimate (4) (with $Y^1 = X^x$ and $Y^2 = X^y$ for $x, y \in H_{-\delta}$, $\delta \in [0, \hat{\delta})$ in the notation of (4)) and the finiteness of the first element in the set in the maximum in (8) is a consequence from (6), which, in turn, also follows from the perturbation estimate (4) (see above and the proof of Corollary 2.8 for details). Roughly speaking, Corollary 2.8 establishes the existence of mild solutions of the SEE (7) and also establishes the Lipschitz continuity of the solutions with respect to the initial conditions for any initial condition in $H_{-\delta}$ and any $\delta < \hat{\delta} = \frac{1}{2}\big[1 + \mathbb{1}_{\{0\}}(|B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))})\big]$ (see (8)). In Corollary 3.1, Proposition 3.2, Proposition 3.4, and Proposition 3.5 below we demonstrate that the *regularity barrier*

$$\hat{\delta} = \tfrac{1}{2}\big[1 + \mathbb{1}_{\{0\}}(|B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))})\big] = \begin{cases} 1/2 & : B \text{ is not a constant function} \\ 1 & : B \text{ is a constant function} \end{cases} \tag{9}$$

for the regularity of the initial conditions revealed in Corollary 2.8 (and Proposition 2.5 and Theorem 2.7, respectively) can, in general, not essentially be improved. In particular, Corollary 3.1 and Proposition 3.2 below prove in the case where $H = U = L^2((0,1);\mathbb{R})$, where $\beta \in (1/4, 1/2)$, where $A\colon D(A) \subseteq H \to H$ is the Laplacian with periodic boundary conditions on H, and where $B \in L(H, HS(H, H_{-\beta}))$ satisfies $\forall\, u, v \in H\colon B(v)u = v \cdot u$ (B is not a constant function) that it holds (i) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^x\colon [0,T] \times \Omega \to H_{-\delta}$, $x \in H_{-\delta}$, $\delta \in [0, 1/2)$, which fulfill for all $q \in [2,\infty)$, $\delta \in [0, 1/2)$, $x \in H_{-\delta}$, $t \in [0,T]$ that $X^x((0,T] \times \Omega) \subseteq H$, that $\sup_{s\in(0,T]} s^{\delta}\, \|X^x_s\|_{L^q(\mathbb{P};H)} < \infty$, and $\mathbb{P}$-a.s. that

$$X^x_t = e^{tA}x + \int_0^t e^{(t-s)A} B(X^x_s)\, \mathrm{d}W_s, \tag{10}$$

(ii) that

$$\forall\, \delta \in [0, 1/2),\ q \in [2,\infty),\ t \in (0,T]\colon \sup_{\substack{x,y\in H,\\ x\neq y}} \left[\frac{\|X^x_t - X^y_t\|_{L^q(\mathbb{P};H)}}{\|x-y\|_{H_{-\delta}}}\right] < \infty, \tag{11}$$

and (iii) that

$$\forall\, \delta \in (1/2, \infty),\ q \in [2,\infty),\ t \in (0,T]\colon \sup_{\substack{x,y\in H,\\ x\neq y}} \left[\frac{\|X^x_t - X^y_t\|_{L^q(\mathbb{P};H)}}{\|x-y\|_{H_{-\delta}}}\right] = \infty. \tag{12}$$

The SEE (10) is sometimes referred to as a continuous version of the *parabolic Anderson model* in the literature (see, e.g., Carmona & Molchanov [6]). In addition, Proposition 3.2 below *disproves* the existence of square integrable solutions of the SEE (10) with initial conditions in $(\cup_{\delta\in\mathbb{R}} H_\delta)\backslash H_{-1/2}$. The noise in the counterexample SEE (10) is spatially very rough and one might question whether the regularity barrier (9) can be overcome in the case of more regular spatially smooth noise. In Proposition 3.4 below we answer this question to the negative by presenting another counterexample SEE with a non-constant diffusion coefficient but a spatially smooth noise for which we disprove the existence of square integrable solutions with initial conditions in $(\cup_{\delta\in\mathbb{R}} H_\delta)\backslash H_{-1/2}$ (cf., however, also Proposition 3.3 below). Proposition 3.5 below also provides a further counterexample SEE which illustrates the sharpness of the regularity barrier (9) in the case where B is a constant function.

Proposition 2.5, Theorem 2.7, and Corollary 2.8 outlined above (see (4)–(8)) are of particular importance for establishing regularity properties for Kolmogorov backward equations associated to parabolic semilinear SEEs and, thereby, for establishing essentially sharp probabilistically *weak convergence rates* for numerical approximations of parabolic semilinear SEEs (cf., e.g., Lemmas 4.4–4.6 in Debussche [12], Lemma 3.3 in Wang & Gan [25], (4.2)–(4.3) in Andersson & Larsson [1], Propositions 5.1–5.2 and Lemma 5.4 in Bréhier [2], Lemma A.4 in Bréhier & Kopec [3], Lemma 3.3 in Wang [24], (79) in Conus et al. [9], Proposition 7.1, Lemma 10.5, and Lemma 10.10 in Kopec [18], and (183)–(184) in Jentzen & Kurniawan [16]). The analytically weak norm for the initial condition in (8) translates in an analytically weak norm for the approximation errors in the probabilistically weak error analysis which, in turn, results in essentially sharp probabilistically weak convergence rates (cf., e.g., Theorem 2.2 in [12], Theorem 2.1 in Wang & Gan [25], Theorem 1.1 in Andersson & Larsson [1], Theorem 1.1 in Bréhier [2], Theorem 5.1 in Bréhier & Kopec [3], Corollary 3.7 in Wang [24], Corollary 5.2 in Conus et al. [9], Theorem 6.1 in Kopec [18], and Corollary 8.2 in Jentzen & Kurniawan [16]). The perturbation inequality in Proposition 2.5 (see (4) above) is also useful to establish essentially sharp probabilistically *strong convergence rates* for numerical approximations and perturbations of SEEs (cf., e.g., Corollary 8.2.26 in [14], Proposition 4.1 in Conus et al. [9], and Proposition 4.3 in [16]).

1.1 Notation

Throughout this article the following notation is used. For two measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A}, \mathcal{B})$ the set of all $\mathcal{A}/\mathcal{B}$-measurable functions. For a set A we denote by $\mathcal{P}(A)$ the power set of A and we denote by $\#_A \colon \mathcal{P}(A) \to [0, \infty]$ the counting measure on A. For a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ we denote by $\mu_A \colon \mathcal{B}(A) \to [0, \infty]$ the Lebesgue-Borel measure on A. For a real number $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ (see, e.g., Definition 2.1.11 in [20]) we call the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ a stochastic basis. For a real number $T \in (0, \infty)$ and a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ we denote by $\operatorname{Pred}((\mathcal{F}_t)_{t\in[0,T]})$ the sigma-algebra given by $\operatorname{Pred}((\mathcal{F}_t)_{t\in[0,T]}) = \sigma_{[0,T]\times\Omega}\big(\{(s,t] \times A \colon s,t \in [0,T], s < t, A \in \mathcal{F}_s\} \cup \{\{0\} \times A \colon A \in \mathcal{F}_0\}\big)$ (the predictable sigma-algebra associated to $(\mathcal{F}_t)_{t\in[0,T]}$). We denote by $(\cdot)^+, (\cdot)^- \colon \mathbb{R} \to [0, \infty)$ the functions with the property that for all $x \in \mathbb{R}$ it holds that $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. We denote by $\lceil \cdot \rceil_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0, \infty)$, the functions with the property that for all $h \in (0, \infty)$, $t \in \mathbb{R}$ it holds that $\lceil t \rceil_h = \min([t, \infty) \cap \{0, h, -h, 2h, -2h, \dots\})$. For $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $\#_V(V) > 1$ we denote by $|\cdot|_{\operatorname{Lip}(V,W)} \colon \mathcal{C}(V, W) \to [0, \infty]$ and $\|\cdot\|_{\operatorname{Lip}(V,W)} \colon \mathcal{C}(V, W) \to [0, \infty]$ the functions with the property that for all $f \in \mathcal{C}(V, W)$ it holds that

$$
\begin{aligned}
|f|_{\operatorname{Lip}(V,W)} &= \sup_{x,y\in V, x\neq y} \left(\frac{\|f(x) - f(y)\|_W}{\|x - y\|_V} \right), \\
\|f\|_{\operatorname{Lip}(V,W)} &= \|f(0)\|_W + |f|_{\operatorname{Lip}(V,W)}
\end{aligned}
\tag{13}
$$

and we denote by $\operatorname{Lip}(V, W)$ the set given by $\operatorname{Lip}(V, W) = \{f \in \mathcal{C}(V, W) \colon |f|_{\operatorname{Lip}(V,W)} < \infty\}$. For a separable $\mathbb{R}$-Hilbert space $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, real numbers $T \in (0, \infty)$, $\eta \in \mathbb{R}$, $r \in$

$[0,1]$, and a generator of a strongly continuous analytic semigroup $A\colon D(A) \subseteq H \to H$ with spectrum$(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$ we denote by $\chi^{r,T}_{A,\eta} \in [0,\infty)$ the real number given by $\chi^{r,T}_{A,\eta} = \sup_{t\in(0,T]} t^r \, \|(\eta - A)^r e^{tA}\|_{L(H)}$ (see, e.g., Lemma 11.36 in Renardy & Rogers [21]). We denote by $\mathbb{B}\colon (0,\infty)^2 \to (0,\infty)$ the function with the property that for all $x, y \in (0,\infty)$ it holds that $\mathbb{B}(x,y) = \int_0^1 t^{(x-1)} \, (1-t)^{(y-1)} \, \mathrm{d}t$ (Beta function). We denote by $\mathrm{E}_{\alpha,\beta}\colon [0,\infty) \to [0,\infty)$, $\alpha, \beta \in (-\infty, 1)$, the functions with the property that for all $\alpha, \beta \in (-\infty,1)$, $x \in [0,\infty)$ it holds that $\mathrm{E}_{\alpha,\beta}[x] = 1 + \sum_{n=1}^{\infty} x^n \prod_{k=0}^{n-1} \mathbb{B}\big(1-\beta, k(1-\beta)+1-\alpha\big)$ (generalized exponential function; cf. Chapter 7 in Henry [13] and, e.g., Corollary 1.4.5 in [14] and Lemma 2.4 below). For real numbers $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $p \in [1,\infty)$, $a, \lambda \in (-\infty, 1)$, $b \in (-\infty, \frac{1}{2})$, a separable $\mathbb{R}$-Hilbert space $(H, \|\cdot\|_H, \langle \cdot, \cdot \rangle_H)$, and a generator $A\colon D(A) \subseteq H \to H$ of a strongly continuous analytic semigroup with spectrum$(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$ we denote by $\Theta^{a,b,\lambda}_{A,\eta,p,T}\colon [0,\infty)^2 \to [0,\infty]$ the function with the property that for all $L, \hat{L} \in [0,\infty)$ it holds that

$$\Theta^{a,b,\lambda}_{A,\eta,p,T}(L,\hat{L}) = \begin{cases} \sqrt{2}\left|E_{2\lambda,\max\{a,2b\}}\left[\left|\frac{\chi^{a,T}_{A,\eta}\, L\, \sqrt{2}\, T^{(1-a)}}{\sqrt{1-a}} + \chi^{b,T}_{A,\eta}\, \hat{L}\, \sqrt{p\,(p-1)\, T^{(1-2b)}}\right|^2\right]\right|^{1/2} & : (\lambda, \hat{L}) \in (-\infty, \frac{1}{2}) \times (0,\infty) \\ E_{\lambda,a}\left[\chi^{a,T}_{A,\eta}\, L\, T^{(1-a)}\right] & : \hat{L} = 0 \\ \infty & : \text{otherwise} \end{cases}. \tag{14}$$

For a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable space $(S, \mathcal{S})$, and an $\mathcal{F}/\mathcal{S}$-measurable function $f\colon \Omega \to S$ we denote by $[f]_{\mu,\mathcal{S}}$ the set given by

$$[f]_{\mu,\mathcal{S}} = \{g \in \mathcal{M}(\mathcal{F},\mathcal{S})\colon (\exists\, A \in \mathcal{F}\colon \mu(A) = 0 \text{ and } \{\omega \in \Omega\colon f(\omega) \neq g(\omega)\} \subseteq A)\}\,. \tag{15}$$

For a measure space $(\Omega, \mathcal{F}, \mu)$ and a measurable space $(S, \mathcal{S})$ we do as usual often not distinguish between an $\mathcal{F}/\mathcal{S}$-measurable function $f\colon \Omega \to S$ and its equivalence class $[f]_{\mu,\mathcal{S}}$.

2 Stochastic evolution equations (SEEs) with singularities at the initial time

2.1 Setting

Throughout this section the following setting is frequently used. Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ and $(U, \|\cdot\|_U, \langle\cdot,\cdot\rangle_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H(H) > 1$, let $T \in (0,\infty)$, $\eta \in \mathbb{R}$, $p \in [2,\infty)$, $\alpha \in [0,1)$, $\hat{\alpha} \in (-\infty,1)$, $\beta \in [0, 1/2)$, $\hat{\beta} \in (-\infty, 1/2)$, $L_0, \hat{L}_0, L_1, \hat{L}_1 \in [0,\infty)$ satisfy $\mathbb{1}_{(0,\infty)}(L_1) \cdot [\alpha + \hat{\alpha}] < 3/2$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $A\colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with spectrum$(A) \subseteq \{z \in \mathbb{C}\colon \mathrm{Re}(z) < \eta\}$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$ (cf., e.g., [14, Definition 3.5.25]), and let $\mathbf{F} \in \mathcal{M}\big(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}) \otimes \mathcal{B}(H), \mathcal{B}(H_{-\alpha})\big)$ and $\mathbf{B} \in$

$\mathcal{M}\big(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})\otimes\mathcal{B}(H),\mathcal{B}(HS(U,H_{-\beta}))\big)$ satisfy for all $t\in(0,T]$, $X,Y\in\mathcal{L}^p(\mathbb{P};H)$ that

$$\|\mathbf{F}(t,X)-\mathbf{F}(t,Y)\|_{L^p(\mathbb{P};H_{-\alpha})}\leq L_0\,\|X-Y\|_{L^p(\mathbb{P};H)},\ \|\mathbf{F}(t,0)\|_{L^p(\mathbb{P};H_{-\alpha})}\leq \hat{L}_0\,t^{-\hat{\alpha}},\tag{16}$$

$$\|\mathbf{B}(t,X)-\mathbf{B}(t,Y)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\leq L_1\|X-Y\|_{L^p(\mathbb{P};H)},\ \|\mathbf{B}(t,0)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\leq \hat{L}_1\,t^{-\hat{\beta}}.\tag{17}$$

2.2 Predictable stochastic processes with singularities at the initial time

The next result, Lemma 2.1, is an elementary lemma that slightly generalizes Proposition 3.6 (ii) in Da Prato & Zabczyk [10].

Lemma 2.1 (Existence of predictable modifications)**.** *Let $T\in[0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let (E,d_E) be a complete and separable metric space, and let $Y\colon[0,T]\times\Omega\to E$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process which satisfies for all $t\in(0,\infty)\cap(-\infty,T]$ that $\lim_{s\to t}\mathbb{E}\big[\min\{1,d_E(Y_s,Y_t)\}\big]=0$. Then there exists an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process $X\colon[0,T]\times\Omega\to E$ which satisfies for all $t\in[0,T]$ that $\mathbb{P}(X_t=Y_t)=1$.*

Proof. First, we observe that the assumption that $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space ensures that $\Omega\neq\emptyset$ and this implies that $[0,T]\times\Omega\neq\emptyset$. The assumption that $Y\colon[0,T]\times\Omega\to E$ is a mapping from $[0,T]\times\Omega$ to E therefore ensures that $E\neq\emptyset$. Hence, there exists an element $e_0\in E$. In the next step assume without loss of generality that $T>0$, let $Z^N\colon[0,T]\times\Omega\to E$, $N\in\mathbb{N}$, be the functions with the property that for all $N\in\mathbb{N}$, $t\in[0,T]$ it holds that $Z^N_t=Y_{\max\{\lceil t\rceil_{T/N}-T/N,0\}}$, and let $w\colon(0,T]\times\mathbb{N}\to[0,\infty)$ be the function with the property that for all $\varepsilon\in(0,T]$, $N\in\mathbb{N}$ it holds that

$$w(\varepsilon,N)=\sup_{\substack{t_1,t_2\in[\varepsilon,T],\\|t_1-t_2|\leq T/N}}\mathbb{E}\big[\min\big\{1,d_E(Y_{t_1},Y_{t_2})\big\}\big].\tag{18}$$

The assumption that $\forall\,t\in(0,T]\colon\ \lim_{s\to t}\mathbb{E}[\min\{1,d_E(Y_s,Y_t)\}]=0$ ensures that for all $\varepsilon\in(0,T]$ it holds that $\lim_{N\to\infty}w(\varepsilon,N)=0$. This implies that there exists a strictly increasing sequence $N_k\in\mathbb{N}$, $k\in\mathbb{N}$, with the property that for all $k\in\mathbb{N}$ it holds that

$$w(\tfrac{1}{k},N_k)<\tfrac{1}{2^k}.\tag{19}$$

Next let $X\colon[0,T]\times\Omega\to E$ be the mapping with the property that for all $(t,\omega)\in[0,T]\times\Omega$ it holds that

$$X_t(\omega)=\begin{cases}\lim_{k\to\infty}Z^{N_k}_t(\omega) & :(Z^{N_k}_t(\omega))_{k\in\mathbb{N}}\text{ is convergent}\\ e_0 & :\text{else}\end{cases}.\tag{20}$$

The fact that for all $N\in\mathbb{N}$ it holds that Z^N is $\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})/\mathcal{B}(E)$-measurable, the assumption that (E,d_E) is complete and separable, and, e.g., Lemma 5.3.19 in [14] imply that

$$\big\{(t,\omega)\in[0,T]\times\Omega\colon(Z^{N_k}_t(\omega))_{k\in\mathbb{N}}\text{ is convergent}\big\}\in\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}).\tag{21}$$

This together with the fact that for all $N\in\mathbb{N}$ it holds that Z^N is $\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})/\mathcal{B}(E)$-measurable, and, e.g., Theorem 2.4.7 in [14] ensure that X is $\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]})/\mathcal{B}(E)$-measurable.

It thus remains to prove that X is a modification of Y. For this we note that for all $N \in \mathbb{N}$, $t \in (\frac{T}{N}, T]$ it holds that

$$\mathbb{E}\big[\min\{1, d_E(Y_t, Z_t^N)\}\big] = \mathbb{E}\big[\min\{1, d_E(Y_t, Y_{\lceil t \rceil_{T/N} - T/N})\}\big] \leq w(t - \tfrac{T}{N}, N). \tag{22}$$

This together with (19), the fact that $\forall \varepsilon_1, \varepsilon_2 \in (0,T]$, $N \in \mathbb{N}$ with $\varepsilon_1 \leq \varepsilon_2$: $w(\varepsilon_1, N) \geq w(\varepsilon_2, N)$, and the fact that $\forall t \in (0,T]$, $k \in \mathbb{N} \cap (\frac{T+1}{t}, \infty)$: $\frac{1}{k} < t - \frac{T}{N_k}$ ensure that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{E}\Big[\min\Big\{1, d_E(Y_t, Z_t^{N_k})\Big\}\Big] = \sum_{k \in \mathbb{N}} \mathbb{E}\Big[\min\Big\{1, d_E(Y_t, Y_{\lceil t \rceil_{T/N_k} - T/N_k})\Big\}\Big] \\
&= \sum_{k \in \mathbb{N} \cap (0, (T+1)/t]} \mathbb{E}\Big[\min\Big\{1, d_E(Y_t, Y_{\lceil t \rceil_{T/N_k} - T/N_k})\Big\}\Big] \\
&\quad + \sum_{k \in \mathbb{N} \cap ((T+1)/t, \infty)} \mathbb{E}\Big[\min\Big\{1, d_E(Y_t, Y_{\lceil t \rceil_{T/N_k} - T/N_k})\Big\}\Big] \\
&\leq \frac{T+1}{t} + \sum_{k \in \mathbb{N} \cap ((T+1)/t, \infty)} w(t - \tfrac{T}{N_k}, N_k) \leq \frac{T+1}{t} + \sum_{k \in \mathbb{N} \cap ((T+1)/t, \infty)} w(\tfrac{1}{k}, N_k) \\
&\leq \frac{T+1}{t} + \sum_{k \in \mathbb{N} \cap ((T+1)/t, \infty)} \frac{1}{2^k} < \infty.
\end{aligned} \tag{23}$$

This implies that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that $\limsup_{k \to \infty} d_E(Z_t^{N_k}, Y_t) = 0$ (see, e.g., Lemma 10.2.1 in [14]). This and (20) ensure for all $t \in (0,T]$ that $\mathbb{P}(X_t = Y_t) = 1$. This and the fact that $\forall N \in \mathbb{N}$: $X_0 = Z_0^N = Y_0$ complete the proof of Lemma 2.1. ☐

Lemma 2.2 (Non-stochastic integral)**.** *Assume the setting in Section 2.1, let $\delta \in \mathbb{R}$, $\lambda \in (-\infty, 1)$, and let $Y\colon [0,T] \times \Omega \to H_\delta$ be an $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic process which satisfies $Y((0,T] \times \Omega) \subseteq H$ and $\sup_{t \in (0,T]} t^\lambda \, \|Y_t\|_{L^p(\mathbb{P}; H)} < \infty$. Then*

(i) for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A} \mathbf{F}(s, Y_s)\|_H \, ds < \infty$,

(ii) there exists an up-to-modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic process $\bar{Y}\colon [0,T] \times \Omega \to H$ such that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\bar{Y}_t = \int_0^t e^{(t-s)A} \mathbf{F}(s, Y_s) \, ds$,

(iii) and it holds that

$$\begin{aligned}
&\sup_{t \in (0,T]} t^{(\max\{\lambda, \hat{\alpha}\} + \alpha - 1)} \, \|\bar{Y}_t\|_{L^p(\mathbb{P}; H)} \leq \sup_{t \in (0,T]} \max\big\{1, t^\lambda \, \|Y_t\|_{L^p(\mathbb{P}; H)}\big\} \\
&\cdot \big(L_0 + \hat{L}_0\big) \, |T \vee 1|^{|\lambda - \hat{\alpha}|} \, \mathbb{B}\big(1 - \alpha, 1 - \max\{\lambda, \hat{\alpha}\}\big) \, \chi_{A,\eta}^{\alpha,T} < \infty.
\end{aligned} \tag{24}$$

Proof. Throughout this proof let $K \in [0, \infty)$ be the real number given by

$$K = \sup_{t \in (0,T]} \max\big\{1, t^\lambda \, \|Y_t\|_{L^p(\mathbb{P}; H)}\big\} \tag{25}$$

and let $\kappa_r \in (0,\infty)$, $r \in [0,1]$, be the real numbers with the property that for all $r \in [0,1]$ it holds that

$$\kappa_r = \sup_{t\in(0,T]} \max\left\{ t^r \, \|(\eta - A)^r e^{tA}\|_{L(H)}, \frac{\|(\eta - A)^{-r}(e^{tA} - \mathrm{Id}_H)\|_{L(H)}}{t^r} \right\}. \tag{26}$$

We observe that (16) implies that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\int_0^t \left\| e^{(t-s)A} \mathbf{F}(s, Y_s) \right\|_{L^p(\mathbb{P};H)} \mathrm{d}s \\
&\leq \chi_{A,\eta}^{\alpha,T} \int_0^t (t-s)^{-\alpha} \left(\|\mathbf{F}(s,Y_s) - \mathbf{F}(s,0)\|_{L^p(\mathbb{P};H_{-\alpha})} + \|\mathbf{F}(s,0)\|_{L^p(\mathbb{P};H_{-\alpha})} \right) \mathrm{d}s \\
&\leq \chi_{A,\eta}^{\alpha,T} \int_0^t (t-s)^{-\alpha} \left(L_0 \, \|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_0 \, s^{-\hat{\alpha}} \right) \mathrm{d}s \\
&\leq K \big(L_0 + \hat{L}_0 \big) \, \chi_{A,\eta}^{\alpha,T} \int_0^t (t-s)^{-\alpha} \max\{ s^{-\lambda}, s^{-\hat{\alpha}} \} \, \mathrm{d}s \\
&\leq K \big(L_0 + \hat{L}_0 \big) \, \chi_{A,\eta}^{\alpha,T} \, |T \vee 1|^{|\lambda - \hat{\alpha}|} \int_0^t (t-s)^{-\alpha} \, s^{-\max\{\lambda,\hat{\alpha}\}} \, \mathrm{d}s \\
&\leq K \big(L_0 + \hat{L}_0 \big) \, \chi_{A,\eta}^{\alpha,T} \, |T \vee 1|^{|\lambda - \hat{\alpha}|} \, \mathbb{B}\big(1-\alpha, 1 - \max\{\lambda, \hat{\alpha}\} \big) \, t^{(1-\alpha-\max\{\lambda,\hat{\alpha}\})}.
\end{aligned} \tag{27}$$

In particular, this ensures that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A}\mathbf{F}(s,Y_s)\|_H \, ds < \infty$. Moreover, we note that for all $\varrho \in (0, 1-\alpha)$, $t_1, t_2 \in (0,T]$ with $t_1 < t_2$ it holds that

$$\begin{aligned}
&\left\| \int_0^{t_2} e^{(t_2-s)A} \mathbf{F}(s,Y_s) \, \mathrm{d}s - \int_0^{t_1} e^{(t_1-s)A} \mathbf{F}(s,Y_s) \, \mathrm{d}s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \int_0^{t_1} \left\| \left(e^{(t_2-s)A} - e^{(t_1-s)A} \right) \mathbf{F}(s,Y_s) \right\|_{L^p(\mathbb{P};H)} \mathrm{d}s + \int_{t_1}^{t_2} \left\| e^{(t_2-s)A} \mathbf{F}(s,Y_s) \right\|_{L^p(\mathbb{P};H)} \mathrm{d}s \\
&\leq \left\| \left(\mathrm{Id}_H - e^{(t_2-t_1)A} \right) \right\|_{L(H_\varrho, H)} \int_0^{t_1} \left\| e^{(t_1-s)A} \right\|_{L(H_{-\alpha}, H_\varrho)} \|\mathbf{F}(s,Y_s)\|_{L^p(\mathbb{P};H_{-\alpha})} \, \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} \left\| e^{(t_2-s)A} \right\|_{L(H_{-\alpha}, H)} \|\mathbf{F}(s,Y_s)\|_{L^p(\mathbb{P};H_{-\alpha})} \, \mathrm{d}s.
\end{aligned} \tag{28}$$

Assumption (16) hence implies that for all $\varrho \in (0, 1-\alpha)$, $t_1, t_2 \in (0,T]$ with $t_1 < t_2$ it holds that

$$\begin{aligned}
&\left\| \int_0^{t_2} e^{(t_2-s)A} \mathbf{F}(s,Y_s) \, \mathrm{d}s - \int_0^{t_1} e^{(t_1-s)A} \mathbf{F}(s,Y_s) \, \mathrm{d}s \right\|_{L^p(\mathbb{P};H)} \qquad (29) \\
&\leq \kappa_\varrho \, \kappa_{\varrho+\alpha} \, |t_2 - t_1|^\varrho \int_0^{t_1} (t_1 - s)^{-(\alpha+\varrho)} \left(L_0 \, \|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_0 \, s^{-\hat{\alpha}} \right) \mathrm{d}s \\
&\quad + \kappa_\alpha \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} \left(L_0 \, \|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_0 \, s^{-\hat{\alpha}} \right) \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\leq K\big(L_0+\hat{L}_0\big)\,\kappa_\varrho\,\kappa_{\varrho+\alpha}\,|t_2-t_1|^\varrho\int_0^{t_1}(t_1-s)^{-(\alpha+\varrho)}\max\big\{s^{-\lambda},s^{-\hat{\alpha}}\big\}\,\mathrm{d}s\\
&\quad+K\big(L_0+\hat{L}_0\big)\,\kappa_\alpha\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\max\big\{s^{-\lambda},s^{-\hat{\alpha}}\big\}\,\mathrm{d}s\\
&\leq K\big(L_0+\hat{L}_0\big)|T\vee 1|^{|\lambda-\hat{\alpha}|}\bigg[\kappa_\alpha\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\,s^{-\max\{\lambda,\hat{\alpha}\}}\,\mathrm{d}s\\
&\quad+\kappa_\varrho\,\kappa_{\varrho+\alpha}\,|t_2-t_1|^\varrho\int_0^{t_1}(t_1-s)^{-(\alpha+\varrho)}\,s^{-\max\{\lambda,\hat{\alpha}\}}\,\mathrm{d}s\bigg]\\
&\leq\bigg[\tfrac{\kappa_\alpha\,|t_2-t_1|^{(1-\alpha)}}{(1-\alpha)\min\{|t_1|^{\max\{\lambda,\hat{\alpha}\}},|t_2|^{\max\{\lambda,\hat{\alpha}\}}\}}+\tfrac{\kappa_\varrho\,\kappa_{\varrho+\alpha}\,|t_2-t_1|^\varrho\,\mathbb{B}(1-\alpha-\varrho,1-\max\{\lambda,\hat{\alpha}\})}{|t_1|^{(\varrho+\alpha+\max\{\lambda,\hat{\alpha}\}-1)}}\bigg]\\
&\quad\cdot K\big(L_0+\hat{L}_0\big)|T\vee 1|^{|\lambda-\hat{\alpha}|}.
\end{aligned}
$$

Combining (27), (29), and Lemma 2.1 completes the proof of Lemma 2.2. □

Lemma 2.3 (Stochastic integral)**.** *Assume the setting in Section 2.1, let $\delta,\lambda\in\mathbb{R}$ satisfy $\lambda\,\mathbb{1}_{(0,\infty)}(L_1)<1/2$, and let $Y\colon[0,T]\times\Omega\to H_\delta$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process which satisfies $Y((0,T]\times\Omega)\subseteq H$ and $\sup_{t\in(0,T]}t^\lambda\,\|Y_t\|_{L^p(\mathbb{P};H)}<\infty$. Then*

(i) for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t\|e^{(t-s)A}\mathbf{B}(s,Y_s)\|^2_{HS(U,H)}\,ds<\infty$,

(ii) there exists an up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process $\bar{Y}\colon[0,T]\times\Omega\to H$ such that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that $\bar{Y}_t=\int_0^t e^{(t-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s$,

(iii) and it holds that

$$
\begin{aligned}
&\sup_{t\in(0,T]}t^{(\max\{\lambda-(\lambda-\hat{\beta})\mathbb{1}_{\{0\}}(L_1),\hat{\beta}\}+\beta-1/2)}\,\|\bar{Y}_t\|_{L^p(\mathbb{P};H)}\\
&\leq\sqrt{\tfrac{p\,(p-1)}{2}\,\mathbb{B}\big(1-2\beta,1-2\max\{\lambda-(\lambda-\hat{\beta})\mathbb{1}_{\{0\}}(L_1),\hat{\beta}\}\big)}\\
&\quad\cdot(L_1+\hat{L}_1)\,|T\vee 1|^{|\lambda-\hat{\beta}|}\,\chi^{\beta,T}_{A,\eta}\sup_{t\in(0,T]}\max\big\{1,t^\lambda\|Y_t\|_{L^p(\mathbb{P};H)}\big\}<\infty.
\end{aligned}
\tag{30}
$$

Proof. Throughout this proof let $K\in[0,\infty)$ be the real number given by

$$
K=\sup_{t\in(0,T]}\max\big\{1,t^\lambda\|Y_t\|_{L^p(\mathbb{P};H)}\big\}\tag{31}
$$

and let $\kappa_r\in(0,\infty)$, $r\in[0,1]$, be the real numbers with the property that for all $r\in[0,1]$ it holds that

$$
\kappa_r=\sup_{t\in(0,T]}\max\bigg\{t^r\,\|(\eta-A)^re^{tA}\|_{L(H)},\frac{\|(\eta-A)^{-r}(e^{tA}-\mathrm{Id}_H)\|_{L(H)}}{t^r}\bigg\}.\tag{32}
$$

We observe that (17) implies for all $t \in (0,T]$ that

$$
\begin{aligned}
&\int_0^t \left\| e^{(t-s)A} \mathbf{B}(s, Y_s) \right\|^2_{L^p(\mathbb{P}; HS(U,H))} \, \mathrm{d}s \\
&\leq |\chi^{\beta,T}_{A,\eta}|^2 \int_0^t (t-s)^{-2\beta} \left(L_1 \, \|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_1 \, s^{-\hat{\beta}} \right)^2 \mathrm{d}s \\
&\leq K^2 \, |\chi^{\beta,T}_{A,\eta}|^2 \, (L_1 + \hat{L}_1)^2 \int_0^t (t-s)^{-2\beta} \max\bigl\{ s^{-2(\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1))}, s^{-2\hat{\beta}} \bigr\} \, \mathrm{d}s \\
&\leq K^2 \, |\chi^{\beta,T}_{A,\eta}|^2 \, (L_1 + \hat{L}_1)^2 \, |T \vee 1|^{2|\lambda - \hat{\beta}| \mathbb{1}_{(0,\infty)}(L_1)} \\
&\quad \cdot \int_0^t (t-s)^{-2\beta} \, s^{-2\max\{\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1), \hat{\beta}\}} \, \mathrm{d}s \\
&\leq K^2 \, |\chi^{\beta,T}_{A,\eta}|^2 \, (L_1 + \hat{L}_1)^2 \, |T \vee 1|^{2|\lambda - \hat{\beta}| \mathbb{1}_{(0,\infty)}(L_1)} \\
&\quad \cdot \mathbb{B}\bigl(1 - 2\beta, 1 - 2\max\{\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1), \hat{\beta}\}\bigr) \, t^{(1 - 2\beta - 2\max\{\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1), \hat{\beta}\})}.
\end{aligned}
\tag{33}
$$

This implies, in particular, that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A} \mathbf{B}(s, Y_s)\|^2_{HS(U,H)} \, ds < \infty$. In addition, (33) and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] ensure that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\left\| \int_0^t e^{(t-s)A} \mathbf{B}(s, Y_s) \, dW_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \left[\frac{p\,(p-1)}{2} \int_0^t \left\| e^{(t-s)A} \mathbf{B}(s, Y_s) \right\|^2_{L^p(\mathbb{P}; HS(U,H))} \mathrm{d}s \right]^{1/2} \\
&\leq K \sqrt{\tfrac{p\,(p-1)}{2}} \, \chi^{\beta,T}_{A,\eta} \, (L_1 + \hat{L}_1) \, |T \vee 1|^{|\lambda - \hat{\beta}| \mathbb{1}_{(0,\infty)}(L_1)} \\
&\quad \cdot \sqrt{\mathbb{B}\bigl(1 - 2\beta, 1 - 2\max\{\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1), \hat{\beta}\}\bigr)} \, t^{(1/2 - \beta - \max\{\lambda - (\lambda - \hat{\beta}) \mathbb{1}_{\{0\}}(L_1), \hat{\beta}\})}.
\end{aligned}
\tag{34}
$$

Furthermore, we observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] proves that for all $\varrho \in (0, 1/2 - \beta)$, $t_1, t_2 \in (0,T]$ with $t_1 < t_2$ it holds that

$$\begin{aligned}
&\left\| \int_0^{t_2} e^{(t_2-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s - \int_0^{t_1} e^{(t_1-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \left[\tfrac{p\,(p-1)}{2}\int_0^{t_1} \left\|\left(\mathrm{Id}_H - e^{(t_2-t_1)A}\right) e^{(t_1-s)A}\mathbf{B}(s,Y_s)\right\|^2_{L^p(\mathbb{P};HS(U,H))}\mathrm{d}s\right]^{1/2} \\
&\quad + \left[\tfrac{p\,(p-1)}{2}\int_{t_1}^{t_2} \left\|e^{(t_2-s)A}\mathbf{B}(s,Y_s)\right\|^2_{L^p(\mathbb{P};HS(U,H))}\mathrm{d}s\right]^{1/2} \\
&\leq \left[\tfrac{p\,(p-1)}{2}\int_{t_1}^{t_2} \left\|e^{(t_2-s)A}\right\|^2_{L(H_{-\beta},H)} \left\|\mathbf{B}(s,Y_s)\right\|^2_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\mathrm{d}s\right]^{1/2} \\
&\quad + \left[\tfrac{p\,(p-1)}{2}\int_0^{t_1} \left\|e^{(t_1-s)A}\right\|^2_{L(H_{-\beta},H_\varrho)} \left\|\mathbf{B}(s,Y_s)\right\|^2_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\mathrm{d}s\right]^{1/2} \\
&\quad \cdot \left\|\left(\mathrm{Id}_H - e^{(t_2-t_1)A}\right)\right\|_{L(H_\varrho,H)}.
\end{aligned} \tag{35}$$

Assumption (17) hence ensures that for all $\varrho \in (0, 1/2 - \beta)$, $t_1, t_2 \in (0,T]$ with $t_1 < t_2$ it holds that

$$\begin{aligned}
&\left\| \int_0^{t_2} e^{(t_2-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s - \int_0^{t_1} e^{(t_1-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \kappa_\varrho\,\kappa_{\varrho+\beta}\,|t_2-t_1|^\varrho \left[\tfrac{p\,(p-1)}{2}\int_0^{t_1} (t_1-s)^{-(2\beta+2\varrho)}\left(L_1\,\|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_1\,s^{-\hat{\beta}}\right)^2 \mathrm{d}s\right]^{1/2} \\
&\quad + \kappa_\beta \left[\tfrac{p\,(p-1)}{2}\int_{t_1}^{t_2} (t_2-s)^{-2\beta}\left(L_1\,\|Y_s\|_{L^p(\mathbb{P};H)} + \hat{L}_1\,s^{-\hat{\beta}}\right)^2 \mathrm{d}s\right]^{1/2} \\
&\leq K\sqrt{\tfrac{p\,(p-1)}{2}}\,\kappa_\varrho\,\kappa_{\varrho+\beta}(L_1+\hat{L}_1)\,|T\vee 1|^{|\lambda-\hat{\beta}|}\,|t_2-t_1|^\varrho \\
&\quad \cdot \left[\int_0^{t_1} (t_1-s)^{-(2\beta+2\varrho)} s^{-2\max\{\lambda-(\lambda-\hat{\beta})\mathbb{1}_{\{0\}}(L_1),\hat{\beta}\}}\,\mathrm{d}s\right]^{1/2} \\
&\quad + \frac{K\,\kappa_\beta\,|T\vee 1|^{|\lambda-\hat{\beta}|}\left(L_1+\hat{L}_1\right)\sqrt{\frac{p\,(p-1)}{2}}\,|t_2-t_1|^{(1-2\beta)}}{\min\{|t_1|^{\max\{\lambda,\hat{\beta}\}},|t_2|^{\max\{\lambda,\hat{\beta}\}}\}\sqrt{1-2\beta}} \\
&\leq \left[\frac{\kappa_\varrho\,\kappa_{\varrho+\beta}\,|t_2-t_1|^\varrho\sqrt{\mathbb{B}\left(1-2(\beta+\varrho),1-2\max\{\lambda-(\lambda-\hat{\beta})\mathbb{1}_{\{0\}}(L_1),\hat{\beta}\}\right)}}{|t_1|^{(\max\{\lambda-(\lambda-\hat{\beta})\mathbb{1}_{\{0\}}(L_1),\hat{\beta}\}+\varrho+\beta-1/2)}} + \frac{\kappa_\beta\,|t_2-t_1|^{(1/2-\beta)}}{\min\{|t_1|^{\max\{\lambda,\hat{\beta}\}},|t_2|^{\max\{\lambda,\hat{\beta}\}}\}\sqrt{1-2\beta}}\right] \\
&\quad \cdot \sqrt{\tfrac{p\,(p-1)}{2}}\,K\,|T\vee 1|^{|\lambda-\hat{\beta}|}\left(L_1+\hat{L}_1\right).
\end{aligned} \tag{36}$$

Combining (34), (36), and Lemma 2.1 completes the proof of Lemma 2.3. □

2.3 A perturbation estimate for stochastic processes

Lemma 2.4 is a generalized Gronwall inequality from Chapter 7 in Henry [13].

Lemma 2.4. *Let $\alpha, \beta \in (-\infty, 1)$, $a, b \in [0, \infty)$, $T \in (0, \infty)$, $e \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}([0,\infty])\big)$ satisfy for all $t \in (0,T]$ that $\int_0^T e(s)\,\mathrm{d}s < \infty$ and $e(t) \leq \frac{a}{t^\alpha} + \int_0^t \frac{b\,e(s)}{(t-s)^\beta}\,\mathrm{d}s$. Then for all $t \in (0,T]$ it holds that $e(t) \leq \frac{a}{t^\alpha} \mathrm{E}_{\alpha,\beta}\big[b\,t^{(1-\beta)}\big]$.*

Lemma 2.4 is, e.g., proved as Corollary 1.4.5 in [14]. We next prove a strong perturbation result that will be used several times throughout the paper.

Proposition 2.5 (Perturbation estimate)**.** *Assume the setting in Section 2.1, let $\delta \in \mathbb{R}$, and let $Y^1, Y^2 \colon [0,T] \times \Omega \to H_\delta$ be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy $\cup_{k\in\{1,2\}} Y^k((0,T]\times\Omega) \subseteq H$ and $\limsup_{\lambda \nearrow \frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]} \max_{k\in\{1,2\}} \sup_{t\in(0,T]} t^\lambda \, \|Y^k_t\|_{L^p(\mathbb{P};H)} < \infty$. Then it holds that $\inf_{t\in[0,T]} \mathbb{P}\big(\sum_{k=1}^2 \int_0^t \|e^{(t-s)A}\mathbf{F}(s,Y^k_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s,Y^k_s)\|^2_{HS(U,H)}\,\mathrm{d}s < \infty\big) = 1$ and it holds for all $\lambda \in \big(-\infty, \frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]\big)$ that*

$$
\begin{aligned}
&\sup_{t\in(0,T]} \big[t^\lambda \, \|Y^1_t - Y^2_t\|_{L^p(\mathbb{P};H)}\big] \leq \Theta^{\alpha,\beta,\lambda}_{A,\eta,p,T}(L_0, L_1) \\
&\cdot \sup_{t\in(0,T]} \bigg[t^\lambda \, \bigg\| Y^1_t - \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^1_s)\,\mathrm{d}s - \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^1_s)\,\mathrm{d}W_s \\
&+ \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^2_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^2_s)\,\mathrm{d}W_s - Y^2_t \bigg\|_{L^p(\mathbb{P};H)}\bigg].
\end{aligned}
\tag{37}
$$

Proof. Throughout this proof let $r \in (-\infty, \frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)])$. We observe that Lemma 2.2.i and Lemma 2.3.i prove that $\inf_{t\in[0,T]} \mathbb{P}\big(\sum_{k=1}^2 \int_0^t \|e^{(t-s)A}\mathbf{F}(s,Y^k_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s,Y^k_s)\|^2_{HS(U,H)}\,\mathrm{d}s < \infty\big) = 1$. It thus remains to prove (37). For this we assume without loss of generality that

$$
\begin{aligned}
&\sup_{t\in(0,T]} \bigg[t^r \, \bigg\| Y^1_t - \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^1_s)\,\mathrm{d}s - \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^1_s)\,\mathrm{d}W_s \\
&+ \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^2_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^2_s)\,\mathrm{d}W_s - Y^2_t \bigg\|_{L^p(\mathbb{P};H)}\bigg] < \infty.
\end{aligned}
\tag{38}
$$

Next we note that the triangle inequality shows that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
\big\|Y^1_t - Y^2_t\big\|_{L^p(\mathbb{P};H)} &\leq \bigg\| Y^1_t - \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^1_s)\,\mathrm{d}s - \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^1_s)\,\mathrm{d}W_s \\
&\quad + \int_0^t e^{(t-s)A}\mathbf{F}(s,Y^2_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,Y^2_s)\,\mathrm{d}W_s - Y^2_t \bigg\|_{L^p(\mathbb{P};H)} \\
&\quad + \bigg\| \int_0^t e^{(t-s)A}\big(\mathbf{F}(s,Y^1_s) - \mathbf{F}(s,Y^2_s)\big)\,\mathrm{d}s \bigg\|_{L^p(\mathbb{P};H)} \\
&\quad + \bigg\| \int_0^t e^{(t-s)A}\big(\mathbf{B}(s,Y^1_s) - \mathbf{B}(s,Y^2_s)\big)\,\mathrm{d}W_s \bigg\|_{L^p(\mathbb{P};H)}.
\end{aligned}
\tag{39}
$$

This and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] imply that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\left\|Y_t^1 - Y_t^2\right\|_{L^p(\mathbb{P};H)} \\
&\leq t^{-r}\left[\sup_{u\in(0,T]} u^r \left\|Y_u^1 - \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^1)\,\mathrm{d}s - \int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^1)\,\mathrm{d}W_s\right.\right. \\
&\left.\left. + \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^2)\,\mathrm{d}s + \int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^2)\,\mathrm{d}W_s - Y_u^2\right\|_{L^p(\mathbb{P};H)}\right] \\
&+ \chi_{A,\eta}^{\alpha,T}\, L_0 \int_0^t (t-s)^{-\alpha}\,\|Y_s^1 - Y_s^2\|_{L^p(\mathbb{P};H)}\,\mathrm{d}s \\
&+ \chi_{A,\eta}^{\beta,T}\, L_1 \left[\tfrac{p\,(p-1)}{2}\int_0^t (t-s)^{-2\beta}\,\|Y_s^1 - Y_s^2\|_{L^p(\mathbb{P};H)}^2\,\mathrm{d}s\right]^{1/2}.
\end{aligned}
\tag{40}
$$

Combining this with Lemma 2.4 proves (37) in the case $L_1 = 0$. It thus remains to prove (37) in the case $L_1 > 0$. For this we observe that (40) together with Hölder's inequality ensures that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\left\|Y_t^1 - Y_t^2\right\|_{L^p(\mathbb{P};H)} \\
&\leq t^{-r}\left[\sup_{u\in(0,T]} u^r \left\|Y_u^1 - \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^1)\,\mathrm{d}s - \int_0^u e^{(t-s)A}\mathbf{B}(s,Y_s^1)\,\mathrm{d}W_s\right.\right. \\
&\left.\left. + \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^2)\,\mathrm{d}s + \int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^2)\,\mathrm{d}W_s - Y_u^2\right\|_{L^p(\mathbb{P};H)}\right] \\
&+ \chi_{A,\eta}^{\alpha,T}\, L_0 \left[T^{\max\{2\beta-\alpha,0\}}\int_0^t (t-s)^{-\alpha}\,\mathrm{d}s \int_0^t (t-s)^{-\max\{\alpha,2\beta\}}\,\|Y_s^1 - Y_s^2\|_{L^p(\mathbb{P};H)}^2\,\mathrm{d}s\right]^{1/2} \\
&+ \chi_{A,\eta}^{\beta,T}\, L_1 \left[\tfrac{p\,(p-1)}{2}\,T^{\max\{\alpha-2\beta,0\}}\int_0^t (t-s)^{-\max\{\alpha,2\beta\}}\,\|Y_s^1 - Y_s^2\|_{L^p(\mathbb{P};H)}^2\,\mathrm{d}s\right]^{1/2}.
\end{aligned}
\tag{41}
$$

The fact that $\forall\, a,b \in \mathbb{R}\colon (a+b)^2 \leq 2a^2 + 2b^2$ hence yields that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\|Y_t^1 - Y_t^2\|_{L^p(\mathbb{P};H)}^2 \leq \int_0^t (t-s)^{-\max\{\alpha,2\beta\}}\,\|Y_s^1 - Y_s^2\|_{L^p(\mathbb{P};H)}^2\,\mathrm{d}s \\
&\cdot\left[\chi_{A,\eta}^{\alpha,T}\, L_0\, \tfrac{\sqrt{2}\,T^{1/2-\alpha+\max\{\beta,\alpha/2\}}}{\sqrt{1-\alpha}} + \chi_{A,\eta}^{\beta,T}\, L_1 \sqrt{p(p-1)}T^{\max\{\alpha/2,\beta\}-\beta}\right]^2 \\
&+ \frac{2}{t^{2r}}\left[\sup_{u\in(0,T]} u^r \left\|Y_u^1 - \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^1)\,\mathrm{d}s - \int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^1)\,\mathrm{d}W_s\right.\right. \\
&\left.\left. + \int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^2)\,\mathrm{d}s + \int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^2)\,\mathrm{d}W_s - Y_u^2\right\|_{L^p(\mathbb{P};H)}\right]^2.
\end{aligned}
\tag{42}
$$

Combining this with Lemma 2.4 and the fact that

$$E_{2r,\max\{\alpha,2\beta\}}\left[T^{(1-\max\{\alpha,2\beta\})}\left|\chi_{A,\eta}^{\alpha,T}\,L_0\,\tfrac{\sqrt{2}\,T^{1/2-\alpha+\max\{\beta,\alpha/2\}}}{\sqrt{1-\alpha}}+\chi_{A,\eta}^{\beta,T}\,L_1\sqrt{p\,(p-1)}\,T^{\max\{\alpha/2,\beta\}-\beta}\right|^2\right]$$
$$=E_{2r,\max\{\alpha,2\beta\}}\left[\left|\chi_{A,\eta}^{\alpha,T}\,L_0\,\tfrac{\sqrt{2}\,T^{(1-\alpha)}}{\sqrt{1-\alpha}}+\chi_{A,\eta}^{\beta,T}\,L_1\sqrt{p\,(p-1)\,T^{(1-2\beta)}}\right|^2\right] \tag{43}$$

ensures that for all $t \in (0,T]$ it holds that

$$\begin{aligned}\|Y_t^1-Y_t^2\|^2_{L^p(\mathbb{P};H)} &\leq E_{2r,\max\{\alpha,2\beta\}}\left[\left|\chi_{A,\eta}^{\alpha,T}\,L_0\,\tfrac{\sqrt{2}\,T^{(1-\alpha)}}{\sqrt{1-\alpha}}+\chi_{A,\eta}^{\beta,T}\,L_1\sqrt{p\,(p-1)\,T^{(1-2\beta)}}\right|^2\right]\\ &\cdot\frac{2}{t^{2r}}\left[\sup_{u\in(0,T]}u^r\left\|Y_u^1-\int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^1)\,\mathrm{d}s-\int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^1)\,\mathrm{d}W_s\right.\right.\\ &\left.\left.+\int_0^u e^{(u-s)A}\mathbf{F}(s,Y_s^2)\,\mathrm{d}s+\int_0^u e^{(u-s)A}\mathbf{B}(s,Y_s^2)\,\mathrm{d}W_s-Y_u^2\right\|_{L^p(\mathbb{P};H)}\right]^2.\end{aligned} \tag{44}$$

Hence, we obtain that

$$\begin{aligned}&\sup_{t\in(0,T]}\left[t^r\,\|Y_t^1-Y_t^2\|_{L^p(\mathbb{P};H)}\right]\\ &\leq\sqrt{2}\left|E_{2r,\max\{\alpha,2\beta\}}\left[\left|\chi_{A,\eta}^{\alpha,T}\,L_0\,\tfrac{\sqrt{2}\,T^{(1-\alpha)}}{\sqrt{1-\alpha}}+\chi_{A,\eta}^{\beta,T}\,L_1\sqrt{p\,(p-1)\,T^{(1-2\beta)}}\right|^2\right]\right|^{1/2}\\ &\cdot\sup_{t\in(0,T]}\left[t^r\left\|Y_t^1-\int_0^t e^{(t-s)A}\mathbf{F}(s,Y_s^1)\,\mathrm{d}s-\int_0^t e^{(t-s)A}\mathbf{B}(s,Y_s^1)\,\mathrm{d}W_s\right.\right.\\ &\left.\left.+\int_0^t e^{(t-s)A}\mathbf{F}(s,Y_s^2)\,\mathrm{d}s+\int_0^t e^{(t-s)A}\mathbf{B}(s,Y_s^2)\,\mathrm{d}W_s-Y_t^2\right\|_{L^p(\mathbb{P};H)}\right].\end{aligned} \tag{45}$$

This finishes the proof of Proposition 2.5. □

Corollary 2.6 (Initial conditions)**.** *Assume the setting in Section 2.1, let* $\delta \in \left[0, \tfrac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]\right)$*, and let* $X^1, X^2\colon [0,T]\times\Omega \to H_{-\delta}$ *be* $(\mathcal{F}_t)_{t\in[0,T]}$*-predictable stochastic processes which fulfill for all* $k \in \{1,2\}$*,* $t \in [0,T]$ *that* $X^k((0,T]\times\Omega) \subseteq H$*, that* $\limsup_{\lambda\nearrow\frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]}\sup_{s\in(0,T]} s^\lambda\,\|X_s^k\|_{L^p(\mathbb{P};H)} < \infty$*, and* $\mathbb{P}$*-a.s. that*

$$X_t^k = e^{tA}X_0^k+\int_0^t e^{(t-s)A}\mathbf{F}(s,X_s^k)\,\mathrm{d}s+\int_0^t e^{(t-s)A}\mathbf{B}(s,X_s^k)\,\mathrm{d}W_s. \tag{46}$$

Then it holds for all $\lambda \in \left[\delta, \tfrac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]\right)$ *that*

$$\sup_{t\in(0,T]}\left[t^\lambda\,\|X_t^1-X_t^2\|_{L^p(\mathbb{P};H)}\right] \leq \chi_{A,\eta}^{\delta,T}\,T^{(\lambda-\delta)}\|X_0^1-X_0^2\|_{L^p(\mathbb{P};H_{-\delta})}\,\Theta_{A,\eta,p,T}^{\alpha,\beta,\lambda}(L_0,L_1). \tag{47}$$

2.4 Existence, uniqueness, and regularity for SEEs with singularities at the initial time

Theorem 2.7. *Assume the setting in Section 2.1 and let* $\delta \in \big(-\infty, \frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]\big)$, $\lambda \in \big[\max\{\delta, \alpha+\hat{\alpha}-1, \beta+\hat{\beta}-1/2\}, \frac{1}{2}[1+\mathbb{1}_{\{0\}}(L_1)]\big)$, $\xi \in \mathcal{L}^p(\mathbb{P}|_{\mathcal{F}_0}; H_{-\delta+})$ *satisfy* $\sup_{t\in(0,T]} t^\delta \, \|e^{tA}\xi\|_{L^p(\mathbb{P};H)} < \infty$*. Then*

(i) there exists an up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$*-predictable stochastic process* $X\colon [0,T]\times\Omega \to H_{-\delta+}$ *which satisfies for all* $t \in [0,T]$ *that* $X((0,T]\times\Omega) \subseteq H$*, that* $\sup_{s\in(0,T]} s^\lambda \, \|X_s\|_{L^p(\mathbb{P};H)} < \infty$*, that* $\mathbb{P}\big(\int_0^t \|e^{(t-s)A}\mathbf{F}(s,X_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s,X_s)\|^2_{HS(U,H)}\,\mathrm{d}s < \infty\big) = 1$*, and* $\mathbb{P}$*-a.s. that*

$$X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}\mathbf{F}(s,X_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,X_s)\,\mathrm{d}W_s \tag{48}$$

(ii) and it holds that

$$\begin{split}&\sup_{t\in(0,T]}\Big[t^\lambda \, \|X_t\|_{L^p(\mathbb{P};H)}\Big] \leq T^\lambda \, \Theta^{\alpha,\beta,\lambda}_{A,\eta,p,T}(L_0,L_1)\\ &\cdot\left[\tfrac{\sup_{t\in(0,T]}(t^\delta\|e^{tA}\xi\|_{L^p(\mathbb{P};H)})}{T^\delta} + \tfrac{\chi^{\alpha,T}_{A,\eta}\,\hat{L}_0\,\mathbb{B}(1-\alpha,1-\hat{\alpha})}{T^{(\alpha+\hat{\alpha}-1)}} + \tfrac{\chi^{\beta,T}_{A,\eta}\,\hat{L}_1\left|p\,(p-1)\,\mathbb{B}(1-2\beta,1-2\hat{\beta})\right|^{1/2}}{\sqrt{2}\,T^{(\beta+\hat{\beta}-1/2)}}\right] < \infty.\end{split} \tag{49}$$

Proof. Throughout this proof let $\mathcal{L}$ and $\mathbb{L}$ be the sets given by

$$\mathcal{L} = \left\{\substack{X\in\mathcal{M}(\mathrm{Pred}((\mathcal{F}_t)_{t\in[0,T]}),\mathcal{B}(H_{-\delta+}))\colon X((0,T]\times\Omega)\subseteq H,\\ \|X_0\|_{L^p(\mathbb{P};H_{-\delta+})}+\sup_{t\in(0,T]} t^\lambda\,\|X_t\|_{L^p(\mathbb{P};H)}<\infty}\right\} \tag{50}$$

and $\mathbb{L} = \big\{\{Y \in \mathcal{L}\colon \inf_{t\in[0,T]} \mathbb{P}(Y_t = X_t) = 1\} \subseteq \mathcal{L}\colon X \in \mathcal{L}\big\}$ and let $|\cdot|_{\mathbb{L},r} \colon \mathbb{L} \to [0,\infty)$, $r \in \mathbb{R}$, and $\|\cdot\|_{\mathbb{L},r} \colon \mathbb{L} \to [0,\infty)$, $r \in \mathbb{R}$, be the mappings with the property that for all $r \in \mathbb{R}$, $X \in \mathbb{L}$ it holds that $|X|_{\mathbb{L},r} = \sup_{t\in(0,T]}\big[e^{rt}\,t^\lambda\,\|X_t\|_{L^p(\mathbb{P};H)}\big]$ and $\|X\|_{\mathbb{L},r} = \|X_0\|_{L^p(\mathbb{P};H_{-\delta+})} + |X|_{\mathbb{L},r}$. Here and below we do not distinguish between an element $X \in \mathcal{L}$ and its equivalent class $\{Y \in \mathcal{L}\colon \inf_{t\in[0,T]}\mathbb{P}(Y_t = X_t) = 1\} \in \mathbb{L}$. We observe that for all $t \in (0,T]$ it holds that

$$t^\lambda\|e^{tA}\xi\|_{L^p(\mathbb{P};H)} \leq T^{(\lambda-\delta)} \sup_{s\in(0,T]} s^\delta\|e^{sA}\xi\|_{L^p(\mathbb{P};H)} < \infty. \tag{51}$$

This ensures that $\big([0,T]\times\Omega \ni (t,\omega) \mapsto e^{tA}\xi(\omega) \in H_{-\delta+}\big) \in \mathcal{L}$. Combining this with Lemma 2.2 and Lemma 2.3 shows that there exists a unique mapping $\Phi\colon \mathbb{L} \to \mathbb{L}$ with the property that for all $Y \in \mathbb{L}$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\Phi(Y)_t = e^{tA}\xi + \int_0^t e^{(t-s)A}\mathbf{F}(s,Y_s)\,\mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,Y_s)\,\mathrm{d}W_s. \tag{52}$$

Our next aim is to prove that there exists a real number $r \in \mathbb{R}$ such that Φ is a contraction on the $\mathbb{R}$-Banach space $(\mathbb{L}, \|\cdot\|_{\mathbb{L},r})$. Banach's fixed point theorem will then allow us to prove (i). Observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] proves that for all $Y, Z \in \mathbb{L}$, $r \in \mathbb{R}$, $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|\Phi(Y)_t - \Phi(Z)_t\|_{L^p(\mathbb{P};H)} \leq \left\| \int_0^t e^{(t-s)A} \left(\mathbf{F}(s,Y_s) - \mathbf{F}(s,Z_s) \right) \mathrm{d}s \right\|_{L^p(\mathbb{P};H)} \\
&\quad + \left\| \int_0^t e^{(t-s)A} \left(\mathbf{B}(s,Y_s) - \mathbf{B}(s,Z_s) \right) \mathrm{d}W_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \int_0^t \left\| e^{(t-s)A} \right\|_{L(H_{-\alpha},H)} \|\mathbf{F}(Y_s) - \mathbf{F}(Z_s)\|_{L^p(\mathbb{P};H_{-\alpha})} \, \mathrm{d}s \\
&\quad + \left[\tfrac{p\,(p-1)}{2} \int_0^t \left\| e^{(t-s)A} \right\|^2_{L(H_{-\beta},H)} \|\mathbf{B}(Y_s) - \mathbf{B}(Z_s)\|^2_{L^p(\mathbb{P};HS(U,H_{-\beta}))} \, \mathrm{d}s \right]^{1/2} \\
&\leq \chi^{\alpha,T}_{A,\eta} \int_0^t L_0 \, (t-s)^{-\alpha} \, \|Y_s - Z_s\|_{L^p(\mathbb{P};H)} \, \mathrm{d}s \\
&\quad + \chi^{\beta,T}_{A,\eta} \left[\tfrac{p\,(p-1)}{2} \int_0^t |L_1|^2 \, (t-s)^{-2\beta} \, \|Y_s - Z_s\|^2_{L^p(\mathbb{P};H)} \, \mathrm{d}s \right]^{\frac{1}{2}} \\
&\leq \chi^{\alpha,T}_{A,\eta} \, |Y - Z|_{\mathbb{L},r} \int_0^t L_0 \, (t-s)^{-\alpha} \, s^{-\lambda} \, e^{-rs} \, \mathrm{d}s \\
&\quad + \chi^{\beta,T}_{A,T} \, |Y - Z|_{\mathbb{L},r} \left[\tfrac{p\,(p-1)}{2} \int_0^t |L_1|^2 \, (t-s)^{-2\beta} \, s^{-2\lambda} \, e^{-2rs} \, \mathrm{d}s \right]^{\frac{1}{2}} \\
&\leq \left[\chi^{\alpha,T}_{A,\eta} \textstyle\int_0^t L_0 \, e^{-rs} \, (t-s)^{-\alpha} \, s^{-\lambda} \, \mathrm{d}s + \chi^{\beta,T}_{A,T} \left[\tfrac{p\,(p-1)}{2} \textstyle\int_0^t |L_1|^2 \, e^{-2rs} \, (t-s)^{-2\beta} \, s^{-2\lambda} \, \mathrm{d}s \right]^{\frac{1}{2}} \right] \\
&\quad \cdot |Y - Z|_{\mathbb{L},r} < \infty.
\end{aligned} \tag{53}$$

Hence, we obtain that for all $Y, Z \in \mathbb{L}$, $r \in (-\infty, 0]$ it holds that

$$\begin{aligned}
&\|\Phi(Y) - \Phi(Z)\|_{\mathbb{L},r} \\
&= \|\Phi(Y)_0 - \Phi(Z)_0\|_{L^p(\mathbb{P};H_{-\delta^+})} + \sup_{t \in (0,T]} \left[e^{rt} \, t^{\lambda} \, \|\Phi(Y)_t - \Phi(Z)_t\|_{L^p(\mathbb{P};H)} \right] \\
&\leq \sup_{t \in (0,T]} \left[\chi^{\alpha,T}_{A,\eta} \textstyle\int_0^t \tfrac{L_0 \, e^{r(t-s)} \, t^{\lambda}}{(t-s)^{\alpha} \, s^{\lambda}} \, \mathrm{d}s + \chi^{\beta,T}_{A,T} \left[\tfrac{p\,(p-1)}{2} \textstyle\int_0^t \tfrac{|L_1|^2 \, e^{2r(t-s)} \, t^{2\lambda}}{(t-s)^{2\beta} \, s^{2\lambda}} \, \mathrm{d}s \right]^{\frac{1}{2}} \right] |Y - Z|_{\mathbb{L},r} \\
&\leq \left(\chi^{\alpha,T}_{A,\eta} \sup\nolimits_{t \in (0,T]} \left[\textstyle\int_0^1 \tfrac{L_0 \, e^{rts} \, t^{(1-\alpha)}}{s^{\alpha} \, (1-s)^{\lambda}} \, \mathrm{d}s \right] + \chi^{\beta,T}_{A,T} \left[\tfrac{p\,(p-1)}{2} \sup\nolimits_{t \in (0,T]} \left[\textstyle\int_0^1 \tfrac{|L_1|^2 \, e^{2rts} \, t^{(1-2\beta)}}{s^{2\beta} \, (1-s)^{2\lambda}} \, \mathrm{d}s \right] \right]^{\frac{1}{2}} \right) \\
&\quad \cdot |Y - Z|_{\mathbb{L},r} \, .
\end{aligned} \tag{54}$$

Next note that Lebesgue's theorem of dominated convergence ensures that for all $r \in \mathbb{R}$ it holds that the functions

$$[0,T] \ni t \mapsto \int_0^1 \frac{L_0 \, e^{rts} \, t^{(1-\alpha)}}{s^{\alpha} \, (1-s)^{\lambda}} \, \mathrm{d}s = L_0 \, t^{(1-\alpha)} \int_0^1 \frac{e^{rts}}{s^{\alpha} \, (1-s)^{\lambda}} \, \mathrm{d}s \in [0, \infty) \tag{55}$$

and

$$[0,T] \ni t \mapsto \int_0^1 \frac{|L_1|^2 \, e^{2rts} \, t^{(1-2\beta)}}{s^{2\beta} \, (1-s)^{2\lambda}} \, \mathrm{d}s = t^{(1-2\beta)} \int_0^1 \frac{|L_1|^2 \, e^{2rts}}{s^{2\beta} \, (1-s)^{2\lambda}} \, \mathrm{d}s \in [0,\infty) \tag{56}$$

are continuous. This and the fact that for all $t \in [0,T]$ it holds that

$$\limsup_{r \to -\infty} \left[\int_0^1 \frac{L_0 \, e^{rts} \, t^{(1-\alpha)}}{s^{\alpha} \, (1-s)^{\lambda}} \, \mathrm{d}s \right] = \limsup_{r \to -\infty} \left[\int_0^1 \frac{|L_1|^2 \, e^{2rts} \, t^{(1-2\beta)}}{s^{2\beta} \, (1-s)^{2\lambda}} \, \mathrm{d}s \right] = 0 \tag{57}$$

allows us to apply Dini's theorem (see, e.g., [14, Proposition 10.3.1]) to obtain that

$$\limsup_{r \to -\infty} \left(\chi^{\alpha,T}_{A,\eta} \sup_{t \in (0,T]} \left[\int_0^1 \tfrac{L_0 \, e^{rts} \, t^{(1-\alpha)}}{s^{\alpha} \, (1-s)^{\lambda}} \, \mathrm{d}s \right] + \chi^{\beta,T}_{A,T} \left[\tfrac{p\,(p-1)}{2} \sup_{t \in (0,T]} \left[\int_0^1 \tfrac{|L_1|^2 \, e^{2rts} \, t^{(1-2\beta)}}{s^{2\beta} \, (1-s)^{2\lambda}} \, \mathrm{d}s \right] \right]^{\frac{1}{2}} \right) = 0. \tag{58}$$

The Banach fixed point theorem together with (54) hence establishes (i), that is, there exists an up-to-modifications unique $X \in \mathcal{L}$ which fulfills that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A}\mathbf{F}(s,X_s)\|_H + \|e^{(t-s)A}\mathbf{B}(s,X_s)\|^2_{HS(U,H)} \, \mathrm{d}s < \infty$ and $X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(s,X_s) \, \mathrm{d}s + \int_0^t e^{(t-s)A} B(s,X_s) \, \mathrm{d}W_s$. It thus remains to prove (ii). For this we apply Proposition 2.5 (with $Y^1 = X$, $Y^2 = 0$, and $r = \lambda$ in the notation of Proposition 2.5) to obtain that

$$\begin{aligned}
&\sup_{t \in (0,T]} \left[t^{\lambda} \, \|X_t\|_{L^p(\Omega;H)} \right] \leq \Theta^{\alpha,\beta,\lambda}_{A,\eta,p,T}(L_0,L_1) \\
&\cdot \sup_{t \in (0,T]} \Bigg[t^{\lambda} \, \Bigg\| X_t - \int_0^t e^{(t-s)A}\mathbf{F}(s,X_s) \, \mathrm{d}s - \int_0^t e^{(t-s)A}\mathbf{B}(s,X_s) \, \mathrm{d}W_s \\
&\qquad + \int_0^t e^{(t-s)A}\mathbf{F}(s,0) \, \mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,0) \, \mathrm{d}W_s \Bigg\|_{L^p(\mathbb{P};H)} \Bigg] \\
&= \Theta^{\alpha,\beta,\lambda}_{A,\eta,p,T}(L_0,L_1) \\
&\quad \cdot \sup_{t \in (0,T]} \left[t^{\lambda} \left\| e^{tA}\xi + \int_0^t e^{(t-s)A}\mathbf{F}(s,0) \, \mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,0) \, \mathrm{d}W_s \right\|_{L^p(\mathbb{P};H)} \right].
\end{aligned} \tag{59}$$

Next we note that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [10] implies that for all $t \in [0,T]$ it holds that

$$\begin{aligned}
&\left\| e^{tA}\xi + \int_0^t e^{(t-s)A}\mathbf{F}(s,0) \, \mathrm{d}s + \int_0^t e^{(t-s)A}\mathbf{B}(s,0) \, \mathrm{d}W_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \left\| e^{tA}\xi \right\|_{L^p(\mathbb{P};H)} + \int_0^t \left\| e^{(t-s)A}\mathbf{F}(s,0) \right\|_{L^p(\mathbb{P};H)} \mathrm{d}s + \left[\tfrac{p(p-1)}{2} \int_0^t \left\| e^{(t-s)A}\mathbf{B}(s,0) \right\|^2_{L^p(\mathbb{P};H)} \mathrm{d}s \right]^{1/2} \\
&\leq \left\| e^{tA}\xi \right\|_{L^p(\mathbb{P};H)} + \chi^{\alpha,T}_{A,\eta} \, \hat{L}_0 \int_0^t (t-s)^{-\alpha} \, s^{-\hat{\alpha}} \, \mathrm{d}s + \chi^{\beta,T}_{A,\eta} \, \hat{L}_1 \left[\tfrac{p\,(p-1)}{2} \int_0^t (t-s)^{-2\beta} \, s^{-2\hat{\beta}} \, \mathrm{d}s \right]^{1/2} \\
&\leq \left\| e^{tA}\xi \right\|_{L^p(\mathbb{P};H)} + \tfrac{\chi^{\alpha,T}_{A,\eta} \, \hat{L}_0 \, \mathbb{B}(1-\alpha,1-\hat{\alpha})}{t^{(\alpha+\hat{\alpha}-1)}} + \tfrac{\chi^{\beta,T}_{A,\eta} \, \hat{L}_1 \sqrt{p\,(p-1) \, \mathbb{B}(1-2\beta,1-2\hat{\beta})}}{\sqrt{2} \, t^{(\beta+\hat{\beta}-1/2)}}.
\end{aligned} \tag{60}$$

This shows that

$$\begin{aligned}&\sup_{t\in(0,T]}\left[t^{\lambda}\left\|e^{tA}\xi+\int_0^t e^{(t-s)A}\mathbf{F}(s,0)\,\mathrm{d}s+\int_0^t e^{(t-s)A}\mathbf{B}(s,0)\,\mathrm{d}W_s\right\|_{L^p(\mathbb{P};H)}\right]\\&\leq T^{(\lambda-\delta)}\sup_{t\in(0,T]}\left[t^{\delta}\|e^{tA}\xi\|_{L^p(\mathbb{P};H)}\right]+\chi_{A,\eta}^{\alpha,T}\,\hat{L}_0\,T^{(\lambda+1-\alpha-\hat{\alpha})}\,\mathbb{B}(1-\alpha,1-\hat{\alpha})\\&\quad+\tfrac{\chi_{A,\eta}^{\beta,T}\,\hat{L}_1\,T^{(\lambda+1/2-\beta-\hat{\beta})}\sqrt{p\,(p-1)\,\mathbb{B}(1-2\beta,1-2\hat{\beta})}}{\sqrt{2}}<\infty.\end{aligned}\tag{61}$$

Combining this with (59) completes the proof of Theorem 2.7. □

Corollary 2.8. *Let $(H,\|\cdot\|_H,\langle\cdot,\cdot\rangle_H)$ and $(U,\|\cdot\|_U,\langle\cdot,\cdot\rangle_U)$ be separable $\mathbb{R}$-Hilbert spaces with $\#_H(H)>1$, let $T\in(0,\infty)$, $\eta\in\mathbb{R}$, $\alpha\in[0,1)$, $\beta\in[0,1/2)$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, let $A\colon D(A)\subseteq H\to H$ be a generator of a strongly continuous analytic semigroup with the property that* $\operatorname{spectrum}(A)\subseteq\{z\in\mathbb{C}\colon\operatorname{Re}(z)<\eta\}$, *let $(H_r,\|\cdot\|_{H_r},\langle\cdot,\cdot\rangle_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $\eta-A$, and let $F\in\operatorname{Lip}(H,H_{-\alpha})$, $B\in\operatorname{Lip}(H,HS(U,H_{-\beta}))$, $\hat{\delta}=\frac{1}{2}\big[1+\mathbb{1}_{\{0\}}(|B|_{\operatorname{Lip}(H,HS(U,H_{-\beta}))})\big]$. Then*

(i) there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^x\colon[0,T]\times\Omega\to H_{-\delta}$, $x\in H_{-\delta}$, $\delta\in[0,\hat{\delta})$, which fulfill for all $p\in[2,\infty)$, $\delta\in[0,\hat{\delta})$, $x\in H_{-\delta}$, $t\in[0,T]$ that $X^x((0,T]\times\Omega)\subseteq H$, that $\sup_{s\in(0,T]}s^{\delta}\,\|X_s^x\|_{L^p(\mathbb{P};H)}<\infty$, and $\mathbb{P}$-a.s. that

$$X_t^x=e^{tA}x+\int_0^t e^{(t-s)A}F(X_s^x)\,\mathrm{d}s+\int_0^t e^{(t-s)A}B(X_s^x)\,\mathrm{d}W_s,\tag{62}$$

(ii) for all $p\in[2,\infty)$, $\delta\in[0,\hat{\delta})$ it holds that

$$\begin{aligned}&\sup_{x\in H_{-\delta}}\sup_{t\in(0,T]}\left[\frac{t^{\delta}\,\|X_t^x\|_{L^p(\mathbb{P};H)}}{\max\{1,\|x\|_{H_{-\delta}}\}}\right]\leq\Theta_{A,\eta,p,T}^{\alpha,\beta,\delta}\big(|F|_{\operatorname{Lip}(H,H_{-\alpha})},|B|_{\operatorname{Lip}(H,HS(U,H_{-\beta}))}\big)\\&\cdot\left[\chi_{A,\eta}^{\delta,T}+\tfrac{\chi_{A,\eta}^{\alpha,T}\|F(0)\|_{H_{-\alpha}}\,T^{(\delta+1-\alpha)}}{(1-\alpha)}+\tfrac{\sqrt{p\,(p-1)}\,\chi_{A,\eta}^{\beta,T}\|B(0)\|_{HS(U,H_{-\beta})}\,T^{(\delta+1/2-\beta)}}{\sqrt{2-4\beta}}\right]<\infty,\end{aligned}\tag{63}$$

(iii) and for all $p\in[2,\infty)$, $\delta\in[0,\hat{\delta})$ it holds that

$$\begin{aligned}&\sup_{\substack{x,y\in H_{-\delta},\\ x\neq y}}\sup_{t\in(0,T]}\left[\frac{t^{\delta}\,\|X_t^x-X_t^y\|_{L^p(\mathbb{P};H)}}{\|x-y\|_{H_{-\delta}}}\right]\\&\leq\chi_{A,\eta}^{\delta,T}\,\Theta_{A,\eta,p,T}^{\alpha,\beta,\delta}\big(|F|_{\operatorname{Lip}(H,H_{-\alpha})},|B|_{\operatorname{Lip}(H,HS(U,H_{-\beta}))}\big)<\infty.\end{aligned}\tag{64}$$

Proof of Corollary 2.8. Throughout this proof let $L_0,L_1,\hat{L}_0,\hat{L}_1\in[0,\infty)$ be the real numbers given by $L_0=|F|_{\operatorname{Lip}(H,H_{-\alpha})}$, $L_1=|B|_{\operatorname{Lip}(H,HS(U,H_{-\beta}))}$, $\hat{L}_0=\|F(0)\|_{H_\alpha}$, and $\hat{L}_1=\|B(0)\|_{HS(U,H_\beta)}$. We note that for all $t\in(0,T]$, $X,Y\in\mathcal{L}^p(\mathbb{P};H)$ it holds that

$$\|F(X)-F(Y)\|_{L^p(\mathbb{P};H_{-\alpha})}\leq L_0\,\|X-Y\|_{L^p(\mathbb{P};H)},\quad\|F(0)\|_{L^p(\mathbb{P};H_{-\alpha})}\leq\hat{L}_0,\tag{65}$$

$$\|B(X)-B(Y)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\leq L_1\|X-Y\|_{L^p(\mathbb{P};H)},\quad\|B(0)\|_{L^p(\mathbb{P};HS(U,H_{-\beta}))}\leq\hat{L}_1.\tag{66}$$

We can hence apply Corollary 2.6 and Theorem 2.7. More specifically, an application of Theorem 2.7 (with $\lambda = \delta$, $\hat{\alpha} = \hat{\beta} = 0$, $L_0 = |F|_{\mathrm{Lip}(H,H_{-\alpha})}$, $\hat{L}_0 = \|F(0)\|_{H_{-\alpha}}$, $L_1 = |B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))}$, and $\hat{L}_1 = \|B(0)\|_{HS(U,H_{-\beta})}$ for $\delta \in [0,\hat{\delta})$ in the notation of Theorem 2.7) proves (i) and assures that for all $p \in [2,\infty)$, $\delta \in [0,\hat{\delta})$, $x \in H_{-\delta}$ it holds that

$$\begin{aligned}&\sup_{t\in(0,T]}\Big[t^{\delta}\,\|X_t^x\|_{L^p(\mathbb{P};H)}\Big] \leq \Theta^{\alpha,\beta,\delta}_{A,\eta,p,T}\big(|F|_{\mathrm{Lip}(H,H_{-\alpha})},|B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))}\big)\\ &\cdot\Bigg[\chi^{\delta,T}_{A,\eta}\,\|x\|_{H_{-\delta}} + \chi^{\alpha,T}_{A,\eta}\,\|F(0)\|_{H_{-\alpha}}\,T^{(\delta+1-\alpha)}\,\mathbb{B}(1-\alpha,1)\\ &\quad + \frac{\chi^{\beta,T}_{A,\eta}\,\|B(0)\|_{HS(U,H_{-\beta})}\,T^{(\delta+1/2-\beta)}|p\,(p-1)\,\mathbb{B}(1-2\beta,1)|^{1/2}}{\sqrt{2}}\Bigg] < \infty.\end{aligned} \tag{67}$$

This implies (ii). In addition, an application of Corollary 2.6 (with $X^1 = X^x$, $X^2 = X^y$, $\lambda = \delta$, and $r = \delta$ for $x, y \in H_{-\delta}$, $\delta \in [0,\hat{\delta})$ in the notation of Corollary 2.6) ensures that for all $p \in [2,\infty)$, $\delta \in [0,\hat{\delta})$, $x, y \in H_{-\delta}$ it holds that

$$\begin{aligned}&\sup_{t\in(0,T]}\big[t^{\delta}\,\|X_t^x - X_t^y\|_{L^p(\mathbb{P};H)}\big]\\ &\leq \chi^{\delta,T}_{A,\eta}\,\|x-y\|_{H_{-\delta}}\Theta^{\alpha,\beta,\delta}_{A,\eta,p,T}\big(|F|_{\mathrm{Lip}(H,H_{-\alpha})},|B|_{\mathrm{Lip}(H,HS(U,H_{-\beta}))}\big) < \infty.\end{aligned} \tag{68}$$

This establishes (iii). The proof of Corollary 2.8 is thus completed. □

3 Examples and counterexamples for SEEs with irregular initial values

3.1 Stochastic heat equations with linear multiplicative noise

In the next result, Corollary 3.1, we illustrate Corollary 2.8 in the case of a simple example.

Corollary 3.1. *Let* $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = \big(L^2(\mu_{(0,1)};\mathbb{R}), \|\cdot\|_{L^2(\mu_{(0,1)};\mathbb{R})}, \langle\cdot,\cdot\rangle_{L^2(\mu_{(0,1)};\mathbb{R})}\big)$*, let* $T, \eta \in (0,\infty)$*,* $\beta \in (1/4, 1/2)$*,* $f, b \in \mathrm{Lip}(\mathbb{R},\mathbb{R})$*, let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ *be a stochastic basis, let* $(W_t)_{t\in[0,T]}$ *be an* Id_H*-cylindrical* $(\mathcal{F}_t)_{t\in[0,T]}$*-Wiener process, let* $A\colon D(A) \subseteq H \to H$ *be the Laplacian with periodic boundary conditions on* H*, let* $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$*,* $r \in \mathbb{R}$*, be a family of interpolation spaces associated to* $\eta - A$*, and let* $F\colon H \to H$ *and* $B\colon H \to HS(H, H_{-\beta})$ *satisfy for all* $v \in \mathcal{L}^2(\mu_{(0,1)};\mathbb{R})$*,* $u \in \mathcal{C}([0,1],\mathbb{R})$ *that* $F([v]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}) = [\{f(v(x))\}_{x\in(0,1)}]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}$ *and* $B\big([v]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}\big)[u|_{(0,1)}]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})} = \big[\{b(v(x))\cdot u(x)\}_{x\in(0,1)}\big]_{\mu_{(0,1)},\mathcal{B}(\mathbb{R})}$*. Then*

(i) there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$*-predictable stochastic processes* $X^x\colon [0,T]\times\Omega \to H_{-\delta}$*,* $x \in H_{-\delta}$*,* $\delta \in [0, 1/2)$*, which fulfill for all* $p \in [2,\infty)$*,* $\delta \in [0,1/2)$*,* $x \in H_{-\delta}$*,* $t \in [0,T]$ *that* $X^x((0,T]\times\Omega) \subseteq H$*, that* $\sup_{s\in(0,T]} s^{\delta}\,\|X_s^x\|_{L^p(\mathbb{P};H)} < \infty$*, and* $\mathbb{P}$*-a.s. that*

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^x)\,\mathrm{d}s + \int_0^t e^{(t-s)A}B(X_s^x)\,\mathrm{d}W_s \tag{69}$$

(ii) *and for all* $p \in [2, \infty)$, $\delta \in [0, 1/2)$ *it holds that*

$$\sup_{\substack{x,y \in H_{-\delta},\\ x \neq y}} \sup_{t \in (0,T]} \left[\frac{t^\delta \, \|X_t^x\|_{L^p(\mathbb{P};H)}}{\max\{1, \|x\|_{H_{-\delta}}\}} + \frac{t^\delta \, \|X_t^x - X_t^y\|_{L^p(\mathbb{P};H)}}{\|x-y\|_{H_{-\delta}}} \right] < \infty. \tag{70}$$

Proposition 3.2. *Let* $(H, \|\cdot\|_H, \langle \cdot, \cdot \rangle_H) = \big(L^2(\mu_{(0,1)}; \mathbb{R}), \|\cdot\|_{L^2(\mu_{(0,1)};\mathbb{R})}, \langle \cdot, \cdot \rangle_{L^2(\mu_{(0,1)};\mathbb{R})}\big)$, *let* $T, \eta, \nu \in (0, \infty)$, $r \in [0, \infty)$, $\delta \in \mathbb{R}$, $\beta \in (\frac{1}{4}, \frac{1}{2})$, *let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ *be a stochastic basis, let* $(W_t)_{t \in [0,T]}$ *be an* Id_H*-cylindrical* $(\mathcal{F}_t)_{t \in [0,T]}$*-Wiener process, let* $(b_n)_{n \in \mathbb{Z}} \subseteq H$ *satisfy for all* $n \in \mathbb{N}$ *that* $b_0 = [\{1\}_{x \in (0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})}$, $b_n = [\{\sqrt{2} \cos(2n\pi x)\}_{x \in (0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})}$, *and* $b_{-n} = [\{\sqrt{2} \sin(2n\pi x)\}_{x \in (0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})}$, *let* $A \colon D(A) \subseteq H \to H$ *be the linear operator which satisfies* $D(A) = \big\{ v \in H \colon \sum_{n \in \mathbb{Z}} |n^2 \langle b_n, v \rangle_H|^2 < \infty \big\}$ *and which satisfies for all* $v \in D(A)$ *that* $Av = -\nu \sum_{n \in \mathbb{Z}} n^2 \langle b_n, v \rangle_H b_n$, *let* $(H_q, \|\cdot\|_{H_q}, \langle \cdot, \cdot \rangle_{H_q})$, $q \in \mathbb{R}$, *be a family of interpolation spaces associated to* $\eta - A$, *let* $\xi \in \mathcal{M}\big(\mathcal{F}_0, \mathcal{B}(H_\delta)\big)$, $B \in L(H, HS(H, H_{-\beta}))$ *satisfy for all* $v \in \mathcal{L}^2(\mu_{(0,1)}; \mathbb{R})$, $u \in \mathcal{C}\big([0,1], \mathbb{R}\big)$ *that* $B\big([v]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})}\big)[u|_{(0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})} = [\{v(x) \cdot u(x)\}_{x \in (0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})}$, *and let* $X \colon [0,T] \times \Omega \to H_\delta$ *be an* $(\mathcal{F}_t)_{t \in [0,T]}$*-predictable stochastic process which satisfies for all* $t \in (0,T]$ *that* $X((0,T] \times \Omega) \subseteq H$, *that* $\mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + \int_0^t \mathbb{E}\big[\|e^{(t-s)A} B(X_s)\|_{HS(H, H_{-r})}^2\big] \, ds < \infty$, *and* $\mathbb{P}$*-a.s. that* $X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} B(X_s) \, dW_s$. *Then for all* $t \in (0,T]$ *it holds that* $\mathbb{P}\big(\xi \in H_{-1/2}\big) = 1$ *and* $2^{-1/2} \, \eta^{-r} \, (1 - e^{-2\eta t})^{1/2} \, \|\xi\|_{L^2(\mathbb{P}; H_{-1/2})} \leq \|X_t\|_{L^2(\mathbb{P}; H_{-r})} < \infty$.

Proof. Throughout this proof let $\kappa_k \in [0, \infty]$, $k \in \mathbb{Z}$, be the extended real numbers which satisfy for all $k \in \mathbb{Z}$ that

$$\kappa_k = \sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|_{H_{-r-1}}^2}{\|x\|_{H_{-r}}^2}. \tag{71}$$

Observe that the product rule for differentiation and the fact that the mapping $\mathcal{C}\big([0,1], \mathbb{R}\big) \ni v \mapsto [v|_{(0,1)}]_{\mu_{(0,1)}, \mathcal{B}(\mathbb{R})} \in H_{1/2}$ is continuous ensures that for all $n \in \mathbb{N}$ it holds that $\forall\, u, v \in \cap_{s \in \mathbb{R}} H_s \colon u \cdot v \in \cap_{s \in \mathbb{R}} H_s$ and $\sup_{u,v \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|u \cdot v\|_{H_n}}{\|u\|_{H_n} \|v\|_{H_n}} < \infty$. This implies that for all $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
&\sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|_{H_{-n}}}{\|x\|_{H_{-n}}} = \sup_{x,u \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{|\langle u, B(x) b_k \rangle_H|}{\|x\|_{H_{-n}} \|u\|_{H_n}} \\
&= \sup_{x,u \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{|\langle u \cdot b_k, x \rangle_H|}{\|x\|_{H_{-n}} \|u\|_{H_n}} = \sup_{x,u \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{|\langle (\eta - A)^n (u \cdot b_k), (\eta - A)^{-n} x \rangle_H|}{\|x\|_{H_{-n}} \|u\|_{H_n}} \\
&\leq \sup_{u \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|u \cdot b_k\|_{H_n}}{\|u\|_{H_n}} < \infty.
\end{aligned} \tag{72}$$

Hence, we obtain that for all $k \in \mathbb{Z}$ it holds that

$$
\begin{aligned}
\kappa_k &= \sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|^2_{H_{-r-1}}}{\|x\|^2_{H_{-r}}} \\
&\leq \|(\eta - A)^{-r-1-\lceil -r-1 \rceil_1}\|^2_{L(H)} \left[\sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|_{H_{\lceil -r-1 \rceil_1}}}{\|x\|_{H_{-r}}} \right]^2 \\
&\leq \|(\eta - A)^{-r-1-\lceil -r-1 \rceil_1}\|^2_{L(H)} \left[\sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|x\|_{H_{\lceil -r-1 \rceil_1}}}{\|x\|_{H_{-r}}} \right]^2 \\
&\quad \cdot \left[\sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|_{H_{\lceil -r-1 \rceil_1}}}{\|x\|_{H_{\lceil -r-1 \rceil_1}}} \right]^2 \\
&= \|(\eta - A)^{-r-1+\lceil -r-1 \rceil_1}\|^2_{L(H)} \, \|(\eta - A)^{r+\lceil -r-1 \rceil_1}\|^2_{L(H)} \\
&\quad \cdot \left[\sup_{x \in (\cap_{s \in \mathbb{R}} H_s) \setminus \{0\}} \frac{\|B(x) b_k\|_{H_{\lceil -r-1 \rceil_1}}}{\|x\|_{H_{\lceil -r-1 \rceil_1}}} \right]^2 < \infty.
\end{aligned}
\tag{73}
$$

In the next step we observe that for all $t \in (0, T]$ it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\|X_t\|^2_{H_{-r}} &= \left\| e^{tA} \xi + \int_0^t e^{(t-s)A} B(X_s) \, dW_s \right\|^2_{H_{-r}} \\
&= \left\| e^{tA} \xi \right\|^2_{H_{-r}} + 2 \left\langle e^{tA} \xi, \int_0^t e^{(t-s)A} B(X_s) \, dW_s \right\rangle_{H_{-r}} + \left\| \int_0^t e^{(t-s)A} B(X_s) \, dW_s \right\|^2_{H_{-r}}.
\end{aligned}
\tag{74}
$$

Combining (74) with Itô's isometry and the assumption that $\forall \, t \in (0, T] \colon \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + \int_0^t \mathbb{E}\big[\|e^{(t-s)A} B(X_s)\|^2_{HS(H, H_{-r})}\big] \, ds < \infty$ proves that for all $t \in (0, T]$ it holds that

$$
\begin{aligned}
\mathbb{E}\Big[\|X_t\|^2_{H_{-r}}\Big] &= \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + 2 \, \mathbb{E}\left[\left\langle e^{tA} \xi, \int_0^t e^{(t-s)A} B(X_s) \, dW_s \right\rangle_{H_{-r}} \right] \\
&\quad + \mathbb{E}\left[\left\| \int_0^t e^{(t-s)A} B(X_s) \, dW_s \right\|^2_{H_{-r}} \right] \\
&= \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + 2 \, \mathbb{E}\left[\left\langle e^{tA} \xi, \mathbb{E}\left[\int_0^t e^{(t-s)A} B(X_s) \, dW_s \middle| \mathcal{F}_0 \right] \right\rangle_{H_{-r}} \right] \\
&\quad + \int_0^t \mathbb{E}\Big[\big\| e^{(t-s)A} B(X_s) \big\|^2_{HS(H, H_{-r})} \Big] \, ds \\
&= \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + \int_0^t \mathbb{E}\Big[\big\| e^{(t-s)A} B(X_s) \big\|^2_{HS(H, H_{-r})} \Big] \, ds \\
&= \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + \sum_{k \in \mathbb{Z}} \int_0^t \mathbb{E}\Big[\big\| e^{(t-s)A} B(X_s) b_k \big\|^2_{H_{-r}} \Big] \, ds < \infty.
\end{aligned}
\tag{75}
$$

Moreover, we note that for all $k \in \mathbb{Z}$, $t \in (0,T]$, $s \in (0,t)$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\big\|e^{(t-s)A}B(X_s)\,b_k\big\|_{H_{-r}}^2 = \bigg\|e^{(t-s)A}B\bigg(e^{sA}\xi + \int_0^s e^{(s-u)A}B(X_u)\,dW_u\bigg)b_k\bigg\|_{H_{-r}}^2 \\
&= \big\|e^{(t-s)A}B\big(e^{sA}\xi\big)b_k\big\|_{H_{-r}}^2 + \bigg\|e^{(t-s)A}B\bigg(\int_0^s e^{(s-u)A}B(X_u)\,dW_u\bigg)b_k\bigg\|_{H_{-r}}^2 \\
&\quad + 2\left\langle e^{(t-s)A}B\big(e^{sA}\xi\big)b_k, e^{(t-s)A}B\bigg(\int_0^s e^{(s-u)A}B(X_u)\,dW_u\bigg)b_k\right\rangle_{H_{-r}} \\
&\geq \big\|e^{(t-s)A}B\big(e^{sA}\xi\big)b_k\big\|_{H_{-r}}^2 \\
&\quad + 2\left\langle e^{(t-s)A}B\big(e^{sA}\xi\big)b_k, \int_0^s e^{(t-s)A}B\big(e^{(s-u)A}B(X_u)\,dW_u\big)b_k\right\rangle_{H_{-r}}.
\end{aligned} \tag{76}$$

This and the assumption that $\forall\, t \in (0,T] \colon \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + \int_0^t \mathbb{E}\big[\|e^{(t-s)A}B(X_s)\|_{HS(H,H_{-r})}^2\big]\,ds < \infty$ imply that for all $k \in \mathbb{Z}$, $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\mathbb{E}\bigg[\sum_{n\in\mathbb{Z}}\int_0^s \big\|e^{(t-s)A}B\big(e^{(s-u)A}B(X_u)\,b_n\big)b_k\big\|_{H_{-r}}^2\,du\bigg] \\
&\leq \big\|e^{(t-s)A}\big\|_{L(H_{-r-1},H_{-r})}^2\,\mathbb{E}\bigg[\sum_{n\in\mathbb{Z}}\int_0^s \big\|B\big(e^{(s-u)A}B(X_u)\,b_n\big)b_k\big\|_{H_{-r-1}}^2\,du\bigg] \\
&\leq \kappa_k\,\big\|e^{(t-s)A}\big\|_{L(H_{-r-1},H_{-r})}^2\,\mathbb{E}\bigg[\sum_{n\in\mathbb{Z}}\int_0^s \big\|e^{(s-u)A}B(X_u)\,b_n\big\|_{H_{-r}}^2\,du\bigg] \\
&= \kappa_k\,\big\|e^{(t-s)A}\big\|_{L(H_{-r-1},H_{-r})}^2\,\mathbb{E}\bigg[\int_0^s \big\|e^{(s-u)A}B(X_u)\big\|_{HS(H,H_{-r})}^2\,du\bigg] < \infty
\end{aligned} \tag{77}$$

and

$$\mathbb{E}\Big[\big\|e^{(t-s)A}B(e^{sA}\xi)b_k\big\|_{H_{-r}}^2\Big] \leq \kappa_k\,\big\|e^{(t-s)A}\big\|_{L(H_{-r-1},H_{-r})}^2\,\mathbb{E}\Big[\big\|e^{sA}\xi\big\|_{H_{-r}}^2\Big] < \infty. \tag{78}$$

Combining (76) with (77)–(78) proves that for all $k \in \mathbb{Z}$, $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big\|e^{(t-s)A}B(X_s)b_k\big\|_{H_{-r}}^2\Big] \geq \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)b_k\big\|_{H_{-r}}^2\Big] \\
&\quad + 2\,\mathbb{E}\bigg[\left\langle e^{(t-s)A}B\big(e^{sA}\xi\big)\,b_k, \int_0^s e^{(t-s)A}B\big(e^{(s-u)A}B(X_u)\,dW_u\big)\,b_k\right\rangle_{H_{-r}}\bigg] \\
&= \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\,b_k\big\|_{H_{-r}}^2\Big] \\
&\quad + 2\,\mathbb{E}\bigg[\left\langle e^{(t-s)A}B\big(e^{sA}\xi\big)\,b_k, \mathbb{E}\bigg[\int_0^s e^{(t-s)A}B\big(e^{(s-u)A}B(X_u)\,dW_u\big)\,b_k\,\bigg|\,\mathcal{F}_0\bigg]\right\rangle_{H_{-r}}\bigg] \\
&= \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\,b_k\big\|_{H_{-r}}^2\Big].
\end{aligned} \tag{79}$$

Combining this with (75) ensures that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
\infty &> \mathbb{E}\Big[\|X_t\|_{H_{-r}}^2\Big] \geq \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + \sum_{k\in\mathbb{Z}} \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\, b_k\big\|_{H_{-r}}^2\Big]\, ds \\
&\geq \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|_{HS(H,H_{-r})}^2\Big]\, ds = \int_0^t \mathbb{E}\Big[\big\|(\eta-A)^{-r}e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|_{HS(H)}^2\Big]\, ds \\
&= \int_0^t \mathbb{E}\Big[\big\|B\big(e^{sA}\xi\big)\, e^{(t-s)A}(\eta-A)^{-r}\big\|_{HS(H)}^2\Big]\, ds \\
&= \sum_{n\in\mathbb{Z}} \int_0^t \mathbb{E}\Big[\big\|B(e^{sA}\xi)\, e^{(t-s)A}\,(\eta-A)^{-r}b_n\big\|_H^2\Big]\, ds \\
&\geq \int_0^t \mathbb{E}\Big[\big\|B(e^{sA}\xi)\, e^{(t-s)A}\,(\eta-A)^{-r}b_0\big\|_H^2\Big]\, ds \\
&= \eta^{-2r}\int_0^t \mathbb{E}\Big[\big\|B(e^{sA}\xi)\, b_0\big\|_H^2\Big]\, ds = \eta^{-2r}\int_0^t \mathbb{E}\big[\|e^{sA}\xi\|_H^2\big]\, ds \\
&= \eta^{-2r}\sum_{n\in\mathbb{Z}}\int_0^t \mathbb{E}\Big[\big|\big\langle e^{sA}b_n,\xi\big\rangle_H\big|^2\Big]\, ds = \eta^{-2r}\sum_{n\in\mathbb{Z}}\int_0^t e^{-2(\nu n^2+\eta)s}\, e^{2\eta s}\,\mathbb{E}\big[|\langle b_n,\xi\rangle_H|^2\big]\, ds \\
&\geq \eta^{-2r}\sum_{n\in\mathbb{Z}}\int_0^t e^{-2(\nu n^2+\eta)s}\,\mathbb{E}\big[|\langle b_n,\xi\rangle_H|^2\big]\, ds = \eta^{-2r}\sum_{n\in\mathbb{Z}} \frac{(1-e^{-2(\nu n^2+\eta)t})\,\mathbb{E}\big[|\langle b_n,\xi\rangle_H|^2\big]}{2\,(\nu n^2+\eta)} \\
&= \frac{1}{2\eta^{2r}}\sum_{n\in\mathbb{Z}}\big(1-e^{-2(\nu n^2+\eta)t}\big)\,\mathbb{E}\Big[\big|\big\langle(\eta-A)^{-1/2}b_n,\xi\big\rangle_H\big|^2\Big] \\
&\geq \frac{\big(1-e^{-2\eta t}\big)}{2\eta^{2r}}\sum_{n\in\mathbb{Z}}\mathbb{E}\Big[\big|\big\langle(\eta-A)^{-1/2}b_n,\xi\big\rangle_H\big|^2\Big].
\end{aligned}
\tag{80}
$$

In particular, we obtain that $\mathbb{E}\big[\sum_{n\in\mathbb{Z}}\big|\big\langle(\eta-A)^{-1/2}b_n,\xi\big\rangle_H\big|^2\big] < \infty$. Therefore, it holds that $\mathbb{P}\big(\xi \in H_{-1/2}\big) = 1$. This and (80) complete the proof of Proposition 3.2. □

Proposition 3.3 (Positive example)**.** *Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be a separable $\mathbb{R}$-Hilbert space, let $k \in \mathbb{N}$, $T \in (0,\infty)$, $\eta,\delta \in \mathbb{R}$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W = (W^{(1)},W^{(2)},\ldots,W^{(k)})\colon [0,T]\times\Omega \to \mathbb{R}^k$ be a k-dimensional standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $A\colon D(A)\subseteq H \to H$ be a diagonal linear operator (see, e.g., [14, Definition 3.5.1]) with* spectrum$(A) \subseteq \{z\in\mathbb{C}\colon \mathrm{Re}(z) < \eta\}$*, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $\xi \in \mathcal{M}\big(\mathcal{F}_0,\mathcal{B}(H_\delta)\big)$, $(L_i)_{i\in\{1,2,\ldots,k\}} \subseteq L(H)$, $B \in L(H, HS(\mathbb{R}^k,H))$ satisfy for all $i,j \in \{1,2,\ldots,k\}$, $v \in H$, $u \in D(A)$, $\mathbf{y} = (y_1,y_2,\ldots,y_k) \in \mathbb{R}^k$ that $L_i(D(A)) \subseteq D(A)$, $L_iL_ju - L_jL_iu = L_iAu - AL_iu = 0$, and $B(v)\mathbf{y} = \sum_{l=1}^k y_l L_l v$, and let $X\colon [0,T]\times\Omega \to H_\delta$ satisfy for all $t \in [0,T]$ that*

$$
X_t = \exp\big(tA + \textstyle\sum_{i=1}^k\big[W_t^{(i)}L_i - \frac{1}{2}t(L_i)^2\big]\big)\,\xi. \tag{81}
$$

Then X has continuous sample paths and for all $r \in \mathbb{R}$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A}B(X_s)\|_{HS(\mathbb{R}^k,H_r)}^2\, ds < \infty$ and $X_t = e^{tA}\xi + \int_0^t e^{(t-s)A}B(X_s)\, dW_s$.

Proof. Throughout this proof let $r \in [0,\infty)$ and let $\varphi \in \mathcal{C}\big([0,T] \times \mathbb{R}^k \times H_r, H_r\big)$ be the mapping with the property that for all $t \in [0,T]$, $\mathbf{y} = (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$, $v \in H_r$ it holds that $\varphi(t,\mathbf{y},v) = \exp\big(\sum_{i=1}^k [y_i L_i - \frac{1}{2} t (L_i)^2]\big) v$. Note that the assumption that W has continuous sample paths ensures that X also has continuous sample paths. Next observe that $\varphi \in \mathcal{C}^2\big([0,T] \times \mathbb{R}^k \times H_r, H_r\big)$. Itô's formula (cf., e.g., Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [5]) therefore implies that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\int_0^t \big\| \big(\tfrac{\partial}{\partial \mathbf{y}} \varphi\big)(s, W_s, e^{tA}\xi) \big\|^2_{HS(\mathbb{R}^k, H_r)} \, \mathrm{d}s = \int_0^t \textstyle\sum_{i=1}^k \big\| \big(\tfrac{\partial}{\partial y_i} \varphi\big)(s, W_s, e^{tA}\xi) \big\|^2_{H_r} \, \mathrm{d}s \\
&= \int_0^t \textstyle\sum_{i=1}^k \big\| e^{(t-s)A} \big(\tfrac{\partial}{\partial y_i} \varphi\big)(s, W_s, e^{sA}\xi) \big\|^2_{H_r} \, \mathrm{d}s = \int_0^t \| e^{(t-s)A} B(X_s) \|^2_{HS(\mathbb{R}^k, H_r)} \, \mathrm{d}s < \infty
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
X_t = \varphi(t, W_t, e^{tA}\xi) &= \varphi(0,0,e^{tA}\xi) + \int_0^t \big(\tfrac{\partial}{\partial s} \varphi\big)(u, W_u, e^{tA}\xi) \, \mathrm{d}u \\
&\quad + \int_0^t \big(\tfrac{\partial}{\partial \mathbf{y}} \varphi\big)(s, W_s, e^{tA}\xi) \, \mathrm{d}W_s + \frac{1}{2} \sum_{i=1}^k \int_0^t \big(\tfrac{\partial^2}{\partial y_i^2} \varphi\big)(s, W_s, e^{tA}\xi) \, \mathrm{d}s \\
&= e^{tA}\xi - \int_0^t \frac{1}{2} \sum_{i=1}^k (L_i)^2 \varphi(s, W_s, e^{tA}\xi) \, \mathrm{d}s + \int_0^t e^{(t-s)A} B(X_s) \, \mathrm{d}W_s \\
&\quad + \frac{1}{2} \sum_{i=1}^k \int_0^t (L_i)^2 \varphi(s, W_s, e^{tA}\xi) \, \mathrm{d}s = e^{tA}\xi + \int_0^t e^{(t-s)A} B(X_s) \, \mathrm{d}W_s.
\end{aligned} \tag{83}$$

This, (82), and the fact that $X_0 = \xi$ complete the proof of Proposition 3.3. □

3.2 Stochastic heat equations with nonlinear multiplicative noise

Proposition 3.4. *Let $(H, \|\cdot\|_H, \langle \cdot, \cdot \rangle_H)$ be a separable $\mathbb{R}$-Hilbert space with $\#_H(H) > 1$, let $T \in (0,\infty)$, $\eta, \delta \in \mathbb{R}$, $r, \beta \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W \colon [0,T] \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $A \colon D(A) \subseteq H \to H$ be a diagonal linear operator with* $\operatorname{spectrum}(A) \subseteq \{z \in \mathbb{C} \colon \operatorname{Re}(z) < \eta\}$, *let $(H_q, \|\cdot\|_{H_q}, \langle \cdot, \cdot \rangle_{H_q})$, $q \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $w \in H_{-\beta} \setminus \{0\}$, $\xi \in \mathcal{M}\big(\mathcal{F}_0, \mathcal{B}(H_\delta)\big)$, $B \in \mathcal{C}(H, HS(\mathbb{R}, H_{-\beta}))$ satisfy for all $v \in H$, $u \in \mathbb{R}$ that $B(v)u = u \|v\|_H w$, and let $X \colon [0,T] \times \Omega \to H_\delta$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process which fulfills for all $t \in (0,T]$ that $X((0,T] \times \Omega) \subseteq H$, that $\mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + \int_0^t \mathbb{E}\big[\|e^{(t-s)A} B(X_s)\|^2_{HS(\mathbb{R}, H_{-r})}\big] \, ds < \infty$, and $\mathbb{P}$-a.s. that $X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} B(X_s) \, dW_s$. Then for all $t \in (0,T]$ it holds that $\mathbb{P}\big(\xi \in H_{-1/2}\big) = 1$ and*

$$2^{-1/2} \, e^{-|\eta| t} \left(1 - e^{-2[\eta - \sup(\sigma_p(A))]t}\right)^{1/2} \|e^{tA} w\|_{H_{-r}} \|\xi\|_{L^2(\mathbb{P}; H_{-1/2})} \leq \|X_t\|_{L^2(\mathbb{P}; H_{-r})} < \infty. \tag{84}$$

Proof. Throughout this proof let $\mathbb{B} \subseteq H$ be an orthonormal basis of H and let $\lambda \colon \mathbb{B} \to \mathbb{R}$ be a mapping which satisfies $\sup_{b\in\mathbb{B}} -\lambda_b < \eta$, which satisfies $D(A) = \{v \in H \colon \sum_{b\in\mathbb{B}} |\lambda_b \langle b, v \rangle_H|^2 <$

$\infty\}$, and which satisfies for all $v \in D(A)$ that $Av = -\sum_{b\in\mathbb{B}} \lambda_b \langle b, v\rangle_H \, b$. Note that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} \|X_t\|_{H_{-r}}^2 &= \big\|e^{tA}\xi\big\|_{H_{-r}}^2 + 2\left\langle e^{tA}\xi, \int_0^t e^{(t-s)A} B(X_s)\, dW_s \right\rangle_{H_{-r}} \\ &\quad + \left\| \int_0^t e^{(t-s)A} B(X_s)\, dW_s \right\|_{H_{-r}}^2 . \end{aligned} \tag{85}$$

Equation (85) together with Itô's isometry and the assumption that $\forall\, t \in [0,T] \colon \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + \int_0^t \mathbb{E}\big[\|e^{(t-s)A}B(X_s)\|_{HS(\mathbb{R},H_{-r})}^2\big]\, ds < \infty$ hence prove that for all $t \in (0,T]$ it holds that

$$\begin{aligned} \mathbb{E}\Big[\|X_t\|_{H_{-r}}^2\Big] &= \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + 2\,\mathbb{E}\left[\left\langle e^{tA}\xi, \int_0^t e^{(t-s)A} B(X_s)\, dW_s \right\rangle_{H_{-r}}\right] \\ &\quad + \mathbb{E}\left[\left\| \int_0^t e^{(t-s)A} B(X_s)\, dW_s \right\|_{H_{-r}}^2\right] \\ &= \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + 2\,\mathbb{E}\left[\left\langle e^{tA}\xi, \mathbb{E}\left[\int_0^t e^{(t-s)A} B(X_s)\, dW_s \,\middle|\, \mathcal{F}_0\right] \right\rangle_{H_{-r}}\right] \\ &\quad + \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B(X_s)\big\|_{HS(\mathbb{R},H_{-r})}^2\Big]\, ds \\ &= \mathbb{E}\big[\|e^{tA}\xi\|_{H_{-r}}^2\big] + \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B(X_s)\big\|_{HS(\mathbb{R},H_{-r})}^2\Big]\, ds < \infty. \end{aligned} \tag{86}$$

Next we note that for all $t \in (0,T]$, $s \in (0,t)$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\big\|e^{(t-s)A}B(X_s)\big\|_{HS(\mathbb{R},H_{-r})}^2 = \left\| e^{(t-s)A} B\left(e^{sA}\xi + \int_0^s e^{(s-u)A} B(X_u)\, dW_u \right) \right\|_{HS(\mathbb{R},H_{-r})}^2 \\ &= \left\| e^{sA}\xi + \int_0^s e^{(s-u)A} B(X_u)\, dW_u \right\|_H^2 \big\|e^{(t-s)A} w\big\|_{H_{-r}}^2 \\ &= \big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|_{HS(\mathbb{R},H_{-r})}^2 + \left\| e^{(t-s)A} B\left(\int_0^s e^{(s-u)A} B(X_u)\, dW_u \right) \right\|_{HS(\mathbb{R},H_{-r})}^2 \\ &\quad + 2\,\big\|e^{(t-s)A} w\big\|_{H_{-r}}^2 \left\langle e^{sA}\xi, \int_0^s e^{(s-u)A} B(X_u)\, dW_u \right\rangle_H \\ &\geq \big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|_{HS(\mathbb{R},H_{-r})}^2 + 2\,\big\|e^{(t-s)A} w\big\|_{H_{-r}}^2 \left\langle e^{sA}\xi, \int_0^s e^{(s-u)A} B(X_u)\, dW_u \right\rangle_H \\ &= \big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|_{HS(\mathbb{R},H_{-r})}^2 \\ &\quad + 2\,\big\|e^{(t-s)A} w\big\|_{H_{-r}}^2 \left\langle (\eta - A)^r e^{sA}\xi, \int_0^s (\eta - A)^{-r} e^{(s-u)A} B(X_u)\, dW_u \right\rangle_H . \end{aligned} \tag{87}$$

In addition, the assumption that $\forall\, t \in (0,T]\colon \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] < \infty$ implies that for all $t \in (0,T]$ it holds that

$$\mathbb{E}\big[\|e^{tA}\xi\|^2_{H_r}\big] \le \|e^{\frac{t}{2}A}\|^2_{L(H_{-r},H_r)}\, \mathbb{E}\big[\|e^{\frac{t}{2}A}\xi\|^2_{H_{-r}}\big] < \infty. \tag{88}$$

Itô's isometry and the assumption that $\forall\, t \in (0,T]\colon \int_0^t \mathbb{E}\big[\|e^{(t-s)A}B(X_s)\|^2_{HS(\mathbb{R},H_{-r})}\big]\, ds < \infty$ hence prove that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\big\|e^{(t-s)A}B(X_s)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big] \ge \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big] \\
&\quad + 2\,\big\|e^{(t-s)A}w\big\|^2_{H_{-r}}\, \mathbb{E}\Big[\Big\langle (\eta-A)^r e^{sA}\xi, \int_0^s (\eta-A)^{-r} e^{(s-u)A} B(X_u)\, dW_u \Big\rangle_H\Big] \\
&= \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big] \\
&\quad + 2\,\big\|e^{(t-s)A}w\big\|^2_{H_{-r}}\, \mathbb{E}\Big[\Big\langle (\eta-A)^r e^{sA}\xi, \mathbb{E}\Big[\int_0^s (\eta-A)^{-r} e^{(s-u)A} B(X_u)\, dW_u \Big| \mathcal{F}_0\Big]\Big\rangle_H\Big] \\
&= \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big].
\end{aligned} \tag{89}$$

Furthermore, we observe that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\|e^{tA}w\|_{H_{-r}} \le \|e^{sA}\|_{L(H)}\, \|e^{(t-s)A}w\|_{H_{-r}} \le e^{\eta s}\, \|e^{(t-s)A}w\|_{H_{-r}} \le e^{\eta^+ t}\, \|e^{(t-s)A}w\|_{H_{-r}}. \tag{90}$$

Combining (89) with (86) and (90) ensures that for all $t \in (0,T]$ it holds that

$$\begin{aligned}
&\infty > \mathbb{E}\Big[\|X_t\|^2_{H_{-r}}\Big] \ge \mathbb{E}\big[\|e^{tA}\xi\|^2_{H_{-r}}\big] + \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big]\, ds \\
&\ge \int_0^t \mathbb{E}\Big[\big\|e^{(t-s)A}B\big(e^{sA}\xi\big)\big\|^2_{HS(\mathbb{R},H_{-r})}\Big]\, ds = \int_0^t \big\|e^{(t-s)A}w\big\|^2_{H_{-r}}\, \mathbb{E}\big[\|e^{sA}\xi\|^2_H\big]\, ds \\
&\ge e^{-2\eta^+ t}\, \big\|e^{tA}w\big\|^2_{H_{-r}} \int_0^t \mathbb{E}\big[\|e^{sA}\xi\|^2_H\big]\, ds = e^{-2\eta^+ t}\, \big\|e^{tA}w\big\|^2_{H_{-r}} \sum_{b\in\mathbb{B}} \int_0^t \mathbb{E}\Big[\big|\big\langle e^{sA}b, \xi\big\rangle_H\big|^2\Big]\, ds \\
&= e^{-2\eta^+ t}\, \big\|e^{tA}w\big\|^2_{H_{-r}} \sum_{b\in\mathbb{B}} \int_0^t e^{-2(\lambda_b+\eta)s}\, e^{2\eta s}\, \mathbb{E}\big[|\langle b, \xi\rangle_H|^2\big]\, ds \\
&\ge e^{-2|\eta| t}\, \big\|e^{tA}w\big\|^2_{H_{-r}} \sum_{b\in\mathbb{B}} \int_0^t e^{-2(\lambda_b+\eta)s}\, \mathbb{E}\big[|\langle b, \xi\rangle_H|^2\big]\, ds \\
&= e^{-2|\eta| t}\, \big\|e^{tA}w\big\|^2_{H_{-r}} \sum_{b\in\mathbb{B}} \frac{(1-e^{-2(\lambda_b+\eta)t})\mathbb{E}\big[|\langle b, \xi\rangle_H|^2\big]}{2\,(\lambda_b+\eta)} \\
&= \frac{\|e^{tA}w\|^2_{H_{-r}}}{2e^{2|\eta| t}} \sum_{b\in\mathbb{B}} \big(1-e^{-2(\lambda_b+\eta)t}\big)\, \mathbb{E}\Big[\big|\big\langle (\eta-A)^{-1/2}b, \xi\big\rangle_H\big|^2\Big] \\
&\ge \frac{\big(1-e^{-2(\inf_{b\in\mathbb{B}}\lambda_b+\eta)t}\big)\|e^{tA}w\|^2_{H_{-r}}}{2e^{2|\eta| t}} \sum_{b\in\mathbb{B}} \mathbb{E}\Big[\big|\big\langle (\eta-A)^{-1/2}b, \xi\big\rangle_H\big|^2\Big].
\end{aligned} \tag{91}$$

This and the assumption that $w \neq 0$, in particular, assure that $\mathbb{E}\big[\sum_{b\in\mathbb{B}} \big|\big\langle (\eta - A)^{-1/2} b, \xi \big\rangle_H\big|^2\big] < \infty$. Hence, we obtain that $\mathbb{P}\big(\xi \in H_{-1/2}\big) = 1$. This and (91) complete the proof of Proposition 3.4. $\square$

3.3 Nonlinear heat equations

Proposition 3.5. *Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be a separable $\mathbb{R}$-Hilbert space with $\#_H(H) > 1$, let $\mathbb{B} \subseteq H$ be an orthonormal basis of H, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T \in (0,\infty)$, $\eta, \delta \in \mathbb{R}$, let $\lambda\colon \mathbb{B} \to \mathbb{R}$ be a function which satisfies $\sup_{b\in\mathbb{B}} -\lambda_b < \eta$, let $A\colon D(A) \subseteq H \to H$ be a linear operator which satisfies $D(A) = \big\{v \in H\colon \sum_{b\in\mathbb{B}} |\lambda_b \langle b, v\rangle_H|^2 < \infty\big\}$ and which satisfies for all $v \in D(A)$ that $Av = -\sum_{b\in\mathbb{B}} \lambda_b \langle b, v\rangle_H\, b$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $w \in H$, $b_0 \in \mathbb{B}$, $\xi \in \mathcal{M}\big(\mathcal{F}, \mathcal{B}(H_\delta)\big)$, $F \in \mathcal{C}(H, H)$ satisfy for all $v \in H$ that $\langle b_0, w\rangle_H > 0$, $w = \langle b_0, w\rangle_H\, b_0$, and $F(v) = \|v\|_H\, w$, and let $X \in \mathcal{M}\big(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B}(H_\delta)\big)$ satisfy for all $t \in (0,T]$ that $X((0,T] \times \Omega) \subseteq H$, $\mathbb{P}$-a.s. that $\int_0^t \|e^{(t-s)A} F(X_s)\|_{H_\delta}\, ds < \infty$, and $\mathbb{P}$-a.s. that $X_t = e^{tA}\xi + \int_0^t e^{(t-s)A} F(X_s)\, ds$. Then for all $t \in (0,T]$ it holds that $\mathbb{P}(\xi \in H_{-1}) = 1$ and $\mathbb{P}$-a.s. that*

$$\begin{aligned} &\langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + |\eta|)t} \big[1 - e^{-(\inf_{b\in\mathbb{B}} \lambda_b + \eta)t}\big] \, \|\xi - \langle b_0, \xi\rangle_H\, b_0\|_{H_{-1}} \\ &\leq \big\langle b_0, X_t - e^{tA}\xi \big\rangle_H \leq \big\|X_t - e^{tA}\xi\big\|_H < \infty. \end{aligned} \tag{92}$$

Proof. Throughout this proof let $P \in L(H_{-\delta^-})$ be the linear operator with the property that for all $v \in H$ it holds that $P(v) = v - \langle b_0, v\rangle_H\, b_0$. We observe that the assumption that $X((0,T] \times \Omega) \subseteq H$ implies that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\infty > \big\|X_t - e^{tA}\xi\big\|_H = \big\langle b_0, X_t - e^{tA}\xi\big\rangle_H = \int_0^t \big\langle b_0, e^{(t-s)A} F(X_s)\big\rangle_H\, ds \\ &= \int_0^t \big\langle b_0, e^{(t-s)A} w\big\rangle_H \|X_s\|_H\, ds = \int_0^t \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + \eta)(t-s)}\, e^{\eta(t-s)}\, \|X_s\|_H\, ds \\ &\geq \langle b_0, w\rangle_H\, e^{-\eta^- t} \int_0^t e^{-(\lambda_{b_0} + \eta)(t-s)}\, \|P X_s\|_H\, ds \\ &= \langle b_0, w\rangle_H\, e^{-\eta^- t} \int_0^t e^{-(\lambda_{b_0} + \eta)(t-s)}\, \|e^{sA} P\xi\|_H\, ds \geq \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + \eta^+)t} \int_0^t \|e^{sA} P\xi\|_H\, ds \\ &= \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + \eta^+)t} \int_0^t \Big[\textstyle\sum_{b\in\mathbb{B}\setminus\{b_0\}} \big|e^{-(\lambda_b + \eta)s}\, e^{\eta s}\, \langle b, \xi\rangle_H\big|^2\Big]^{1/2} ds. \end{aligned} \tag{93}$$

This and the Minkowski integral inequality imply that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} &\infty > \big\|X_t - e^{tA}\xi\big\|_H \geq \big\langle b_0, X_t - e^{tA}\xi\big\rangle_H \\ &\geq \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + |\eta|)t} \left[\textstyle\sum_{b\in\mathbb{B}\setminus\{b_0\}} \left|\int_0^t \big|e^{-(\lambda_b + \eta)s}\, \langle b, \xi\rangle_H\big|\, ds\right|^2\right]^{1/2} \\ &= \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + |\eta|)t} \left[\textstyle\sum_{b\in\mathbb{B}\setminus\{b_0\}} \frac{\left[1 - e^{-(\lambda_b + \eta)t}\right]^2 |\langle b, \xi\rangle_H|^2}{|\lambda_b + \eta|^2}\right]^{1/2} \\ &\geq \langle b_0, w\rangle_H\, e^{-(\lambda_{b_0} + |\eta|)t} \big[1 - e^{-(\inf_{b\in\mathbb{B}} \lambda_b + \eta)t}\big] \left[\textstyle\sum_{b\in\mathbb{B}\setminus\{b_0\}} |\langle (\eta - A)^{-1} b, \xi\rangle_H|^2\right]^{1/2}. \end{aligned} \tag{94}$$

The assumption that $\langle b_0, w \rangle_H > 0$ hence implies that it holds $\mathbb{P}$-a.s. that

$$\textstyle\sum_{b \in \mathbb{B}} \left| \langle (\eta - A)^{-1} b, \xi \rangle_H \right|^2 < \infty. \tag{95}$$

This ensures that $\mathbb{P}(\xi \in H_{-1}) = 1$. This together with (94) completes the proof of Proposition 3.5. □

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