Weak convergence rates for spatial spectral Galerkin approximations of semilinear stochastic wave equations with multiplicative noise

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Abstract

Stochastic wave equations appear in several models for evolutionary processes subject to random forces, such as the motion of a strand of DNA in a liquid or heat flow around a ring. Semilinear stochastic wave equations can typically not be solved explicitly, but the literature contains a number of results which show that numerical approximation processes converge with suitable rates of convergence to solutions of such equations. In the case of approximation results for strong convergence rates, semilinear stochastic wave equations with both additive or multiplicative noise have been considered in the literature. In contrast, the existing approximation results for weak convergence rates assume that the diffusion coefficient of the considered semilinear stochastic wave equation is constant, that is, it is assumed that the considered wave equation is driven by additive noise, and no approximation results for multiplicative noise are known. The purpose of this work is to close this gap and to establish essentially sharp weak convergence rates for spatial spectral Galerkin approximations of semilinear stochastic wave equations with multiplicative noise. In particular, our weak convergence result establishes as a special case essentially sharp weak convergence rates for the hyperbolic Anderson model. Our method of proof makes use of the Kolmogorov equation, the Hölder-inequality for Schatten norms, and the mild Itô formula.

Contents

1 Introduction

In the field of numerical approximations for stochastic evolution equations one distinguishes between two conceptually fundamentally different error criteria, that is, strong convergence and weak convergence. In the case of finite dimensional stochastic ordinary differential equations, both strong and weak convergence are quite well understood nowadays; see, e.g., the standard monographs Kloeden & Platen [25] and Milstein [36]. In the case of infinite dimensional stochastic partial differential equations with regular nonlinearities strong convergence rates are essentially well understood, but weak convergence rates are still far away from being well comprehended (see, e.g., [1, 2, 3, 5, 6, 7, 10, 16, 17, 18, 19, 20, 21, 24, 26, 27, 28, 29, 31, 32, 35, 41, 44, 45, 46] for several weak convergence results in the literature). In this work we are interested in weak convergence rates for stochastic wave equations. Stochastic wave equations can be used for modelling several evolutionary processes subject to random forces. Examples include the motion of a DNA molecule floating in a fluid and the dilatation of shock waves throughout the sun (see, e.g., Section 1 in Dalang [14]), as well as heat conduction around a ring (see, e.g., Thomas [42]). Of course, these problems usually involve complicated nonlinearities and are inaccessible for current numerical analysis. Nonetheless, numerical examination of simpler model problems as the ones considered in the present work are a key first step. Even though a number of strong convergence rates for stochastic wave equations are available (see, e.g., [4, 8, 9, 29, 30, 39, 43, 44, 47]), the existing weak convergence results for stochastic wave equations in the literature (see, e.g., [21, 27, 28, 29, 44]) assume that the diffusion coefficient is constant, in other words, that the equation is driven by additive noise. The purpose of this work is to establish essentially sharp weak convergence rates for semilinear stochastic wave equations in the case of multiplicative noise.

To illustrate the main result of this article, we consider the following setting as a special case of our general framework (see Section 3.1 below). Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $(W_t)_{t \in [0,T]}$ be an id_U-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $\{e_n\}_{n \in \mathbb{N}} \subseteq H$ be an orthonormal basis of H, let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq (0, \infty)$ be an increasing sequence, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \{v \in H \colon \sum_{n \in \mathbb{N}} |\lambda_n \langle e_n, v \rangle_H|^2 < \infty\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{n \in \mathbb{N}} -\lambda_n \langle e_n, v \rangle_H e_n$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Definition 3.5.25 in [23]), let $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, be the family of $\mathbb{R}$-Hilbert spaces such that for all $r \in \mathbb{R}$ it holds that $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = (H_{r/2} \times H_{r/2 - 1/2}, \langle \cdot, \cdot \rangle_{H_{r/2} \times H_{r/2 - 1/2}}, \|\cdot\|_{H_{r/2} \times H_{r/2 - 1/2}})$, let $P_N \colon \bigcup_{r \in \mathbb{R}} H_r \to \bigcup_{r \in \mathbb{R}} H_r$, $N \in \mathbb{N} \cup \{\infty\}$, be the mappings such that for all $N \in \mathbb{N} \cup \{\infty\}$, $r \in \mathbb{R}$, $v \in H_r$ it holds that $P_N(v) = \sum_{n=1}^{N} \langle (\lambda_n)^{-r} e_n, v \rangle_{H_r} (\lambda_n)^{-r} e_n$, let $\mathbf{P}_N \colon \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \to \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $N \in \mathbb{N} \cup \{\infty\}$, be the mappings such that for all $N \in \mathbb{N} \cup \{\infty\}$, $r \in \mathbb{R}$, $(v, w) \in \mathbf{H}_r$ it holds that $\mathbf{P}_N(v, w) = (P_N(v), P_N(w))$, let $\mathbf{A} \colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ be the linear operator such that $D(\mathbf{A}) = \mathbf{H}_1$ and such that for all $(v, w) \in \mathbf{H}_1$ it holds that $\mathbf{A}(v, w) = (w, Av)$, and let $\gamma \in (0, \infty)$, $\beta \in (\gamma/2, \gamma]$, $\rho \in [0, 2(\gamma - \beta)]$, $C_{\mathbf{F}}, C_{\mathbf{B}} \in [0, \infty)$, $\xi \in L^2(\mathbb{P}|_{\mathcal{F}_0}; \mathbf{H}_{2(\gamma - \beta)})$, $\mathbf{F} \in \mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)$, $\mathbf{B} \in \mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$ satisfy that $(-A)^{-\beta} \in L_1(H_0)$, $\mathbf{F}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma - \beta)})$, $\mathbf{B}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho) \cap L(U, \mathbf{H}_\gamma))$, $\mathbf{F}|_{\bigcap_{r \in \mathbb{R}} \mathbf{H}_r} \in C_b^2(\bigcap_{r \in \mathbb{R}} \mathbf{H}_r, \mathbf{H}_0)$, $\mathbf{B}|_{\bigcap_{r \in \mathbb{R}} \mathbf{H}_r} \in C_b^2(\bigcap_{r \in \mathbb{R}} \mathbf{H}_r, L_2(U, \mathbf{H}_0))$, $C_{\mathbf{F}} = \sup_{x, v_1, v_2 \in \cap_{r \in \mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{F}''(x)(v_1, v_2)\|_{\mathbf{H}_0} < \infty$, and $C_{\mathbf{B}} = \sup_{x, v_1, v_2 \in \cap_{r \in \mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{B}''(x)(v_1, v_2)\|_{L_2(U, \mathbf{H}_0)} < \infty$.

Theorem 1.1. *Assume the above setting. Then*

(i) it holds that there exist up to modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic processes $\mathbf{X}^N = (X^N, \mathcal{X}^N) \colon [0, T] \times \Omega \to \mathbf{P}_N(\mathbf{H}_\rho)$, $N \in \mathbb{N} \cup \{\infty\}$, which satisfy for all $N \in \mathbb{N} \cup \{\infty\}$,

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$t \in [0, T]$ that $\sup_{s \in [0,T]} \|\mathbf{X}_s^N\|_{L^2(\mathbb{P};\mathbf{H}_\rho)} < \infty$ and $\mathbb{P}$-a.s. that

$$\mathbf{X}_t^N = e^{\mathbf{A}t}\mathbf{P}_N\xi + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{P}_N\mathbf{F}(\mathbf{X}_s^N)\,\mathrm{d}s + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{P}_N\mathbf{B}(\mathbf{X}_s^N)\,\mathrm{d}W_s \qquad (1.1)$$

(ii) and it holds that

$$\sup_{N \in \mathbb{N}} \sup_{\varphi \in C_b^2(\mathbf{H}_0,\mathbb{R}) \setminus \{0\}} \left(\frac{(\lambda_N)^{\gamma-\beta} \left| \mathbb{E}\big[\varphi(\mathbf{X}_T^\infty)\big] - \mathbb{E}\big[\varphi(\mathbf{X}_T^N)\big] \right|}{\|\varphi\|_{C_b^2(\mathbf{H}_0,\mathbb{R})}} \right)$$

$$\leq \left[\|\xi\|_{L^1(\mathbb{P};\mathbf{H}_{2(\gamma-\beta)})} + T \big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})} + 2T \|(-A)^{-\beta}\|_{L_1(H_0)} \big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,L(U,\mathbf{H}_\gamma))}^2 \right]$$

$$\cdot \left(1 \vee \left[T(C_{\mathbf{F}}^2 + 2C_{\mathbf{B}}^2) \right]^{1/2} \right) \exp\!\big(T \big[\tfrac{1}{2} + 3|\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + 4|\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}^2 \big] \big)$$

$$\cdot \left(1 \vee \|\xi\|_{L^2(\mathbb{P};\mathbf{H}_\rho)}^2 \right) \exp\!\left(T \Big[2\big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}^2 \Big] \right) < \infty. \qquad (1.2)$$

Theorem 1.1 is a consequence of the more general results in Remark 3.1 and Theorem 3.7 below (see Corollary 3.9). Let us add a few remarks regarding Theorem 1.1.

First, we briefly outline our proof of Theorem 1.1. As usual in the case of weak convergence analysis, the Kolmogorov equation (see (3.28) below) is used. Another key ingredient is Hölder inequality for Schatten norms (see (3.32) below). In addition, the proof of Theorem 1.1 employs the mild Itô formula (see Corollary 1 in Da Prato et al. [12]) to obtain suitable a priori estimates for solutions of (1.1) (see Lemma 3.2 in Section 3.2 below for details). The detailed proof of Theorem 1.1 and Theorem 3.7, respectively, can also be found in Section 3.2.

Second, we would like to emphasize that in the general setting of Theorem 1.1, the weak convergence rate established in Theorem 1.1 can *essentially not be improved*. More precisely, Theorem 1.1 together with Corollary 4.10 in Section 4.3 below proves that for every $\eta \in (0,\infty)$ there exist $\gamma, C_0 \in (0,\infty)$, $\rho \in [0,\gamma]$, $\{C_\varepsilon\}_{\varepsilon \in (0,\infty)} \subseteq [0,\infty)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, $A \colon D(A) \subseteq H \to H$, $\xi \in L^2(\mathbb{P}|_{\mathcal{F}_0}; \mathbf{H}_\gamma)$, $\varphi \in C_b^2(\mathbf{H}_0, \mathbb{R})$, $\mathbf{F} \in \mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)$, and $\mathbf{B} \in \mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$ such that $\mathbf{F}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\gamma)$, $\mathbf{B}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho) \cap L(U, \mathbf{H}_\gamma))$, $\mathbf{F}|_{\cap_{r \in \mathbb{R}} \mathbf{H}_r} \in C_b^2(\bigcap_{r \in \mathbb{R}} \mathbf{H}_r, \mathbf{H}_0)$, $\mathbf{B}|_{\cap_{r \in \mathbb{R}} \mathbf{H}_r} \in C_b^2(\bigcap_{r \in \mathbb{R}} \mathbf{H}_r, L_2(U, \mathbf{H}_0))$, $\sup_{x \in \cap_{r \in \mathbb{R}} \mathbf{H}_r} \sup_{v_1, v_2 \in \cap_{r \in \mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{F}''(x)(v_1,v_2)\|_{\mathbf{H}_0} < \infty$, $\sup_{x \in \cap_{r \in \mathbb{R}} \mathbf{H}_r} \sup_{v_1, v_2 \in \cap_{r \in \mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{B}''(x)(v_1,v_2)\|_{L_2(U,\mathbf{H}_0)} < \infty$, and such that for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N}$ it holds that

$$C_0 \cdot (\lambda_N)^{-\eta} \leq \left| \mathbb{E}\big[\varphi(\mathbf{X}_T^\infty)\big] - \mathbb{E}\big[\varphi(\mathbf{X}_T^N)\big] \right| \leq C_\varepsilon \cdot (\lambda_N)^{\varepsilon - \eta} \qquad (1.3)$$

(with $\gamma = 2\eta$, $\beta = \gamma/2 + \varepsilon$, $\rho = 0$, $\delta = (1-\gamma)/2$, and $p = 2/\gamma$ in the setting of Theorem 1.1 and Corollary 4.10).

Third, we illustrate Theorem 1.1 by a simple example (cf. Corollary 3.13). In the case where $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U) = \big(L^2(\lambda_{(0,1)};\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})} \big)$, $\xi = (\xi_0, \xi_1) \in H_0^1((0,1);\mathbb{R}) \times H$, $\mathbf{F} = 0$, where $A \colon D(A) \subseteq H \to H$ is the Laplacian with Dirichlet boundary conditions on H, and where $\mathbf{B} \colon H \times H_{-1/2} \to L_2(H, H \times H_{-1/2})$ is the mapping which satisfies for all $(v,w) \in H \times H_{-1/2}$, $u \in C([0,1],\mathbb{R})$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ that $\big(\mathbf{B}(v,w)u\big)(x) = \big(0, v(x) \cdot u(x)\big)$, the stochastic processes $X^N \colon [0,T] \times \Omega \to P_N(H)$, $N \in \mathbb{N} \cup \{\infty\}$, are mild solutions of the SPDEs

$$\ddot{X}_t(x) = \tfrac{\partial^2}{\partial x^2} X_t(x) + P_N X_t(x) \dot{W}_t(x) \qquad (1.4)$$

with $X_t(0) = X_t(1) = 0$, $X_0(x) = (P_N\xi_0)(x)$, $\dot{X}_0(x) = (P_N\xi_1)(x)$ for $t \in [0,T]$, $x \in (0,1)$, $N \in \mathbb{N} \cup \{\infty\}$. In the case $N = \infty$, (1.4) is known as the hyperbolic Anderson model in the literature (see, e.g., Conus et al. [11]). Theorem 1.1 applied to (1.4) ensures for all $\varphi \in C_b^2(H,\mathbb{R})$, $\varepsilon \in (0,\infty)$ that there exists a real number $C \in [0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left| \mathbb{E}\big[\varphi(X_T^\infty)\big] - \mathbb{E}\big[\varphi(X_T^N)\big] \right| \leq C \cdot N^{\varepsilon - 1} \qquad (1.5)$$

(see Corollary 3.13). We thus prove that the spectral Galerkin approximations converge with the weak rate 1- to the solution of the hyperbolic Anderson model. The weak rate 1- is exactly twice the well-known strong convergence rate of the hyperbolic Anderson model. To the best of our knowledge, Theorem 1.1 is the first result in the literature that establishes an essentially sharp weak convergence rate for the hyperbolic Anderson model. Theorem 1.1 also establishes essentially sharp weak convergence rates for more general semilinear stochastic wave equations (see Corollary 3.11 and Corollary 3.13 below).

The remainder of this article is organized as follows. In Sections 1.1 and 1.2 the general notation and framework are presented. Section 2.1 states mostly well-known existence, uniqueness, and regularity results, while Section 2.3 collects basic properties about the interpolation spaces and the semigroup associated to the deterministic wave equation. The main result of this article, Theorem 3.7 below, is stated and proven in Section 3.2. It establishes upper bounds for the convergence rates of weak errors. Section 3.3, in turn, shows how this abstract result can be applied to relevant problems, in particular, the hyperbolic Anderson model (see Corollary 3.11 and Corollary 3.13 below). Finally, Section 4 is dedicated to proving lower bounds for the convergence rates of weak errors to show that the rates from the previous section are essentially optimal (see Corollary 4.10).

1.1 Notation

Throughout this article the following notation is used. For a set A we denote by $\mathcal{P}(A)$ the power set of A and by $\mathcal{P}_0(A)$ the set of all finite subsets of A. Furthermore, for two sets A and B we denote by $A \triangle B$ be the set given by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and by $\mathbb{M}(A, B)$ the set of all mappings from A to B. Moreover, for two sets A and B we denote by $\mathbb{1}_B^A \colon A \to \mathbb{R}$ the indicator function of B on A, that is, it holds for all $a \in A \cap B$ that $\mathbb{1}_B^A(a) = 1$ and for all $a \in A \setminus B$ that $\mathbb{1}_B^A(a) = 0$. In addition, let $(\cdot) \wedge (\cdot), (\cdot) \vee (\cdot) \colon \mathbb{R}^2 \to \mathbb{R}$ be the mappings with the property that for all $x, y \in \mathbb{R}$ it holds that $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Moreover, let $\Gamma \colon (0, \infty) \to (0, \infty)$ be the Gamma function, that is, for all $x \in (0, \infty)$ it holds that $\Gamma(x) = \int_0^\infty t^{(x-1)} e^{-t} \, dt$, and let $\mathcal{E}_r \colon [0, \infty) \to [0, \infty)$, $r \in (0, \infty)$, be the mappings such that for all $r \in (0, \infty)$, $x \in [0, \infty)$ it holds that $\mathcal{E}_r[x] = \left[\sum_{n=0}^\infty \frac{x^{2n} \Gamma(r)^n}{\Gamma(nr+1)} \right]^{1/2}$ (cf. Chapter 7 in Henry [22] and, e.g., Definition 1.3.1 in [23]). Furthermore, for a metric space (E, d_E), a dense subset $A \subseteq E$, a complete metric space (F, d_F), a uniformly continuous mapping $f \colon A \to F$, and the unique mapping $\tilde{f} \in C(E, F)$ with the property that $\tilde{f}|_A = f$ (see, e.g., Proposition 2.5.19 in [23]), we often write, for simplicity of presentation, f instead of $\tilde{f}$ in the following. In addition, for two $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $V \neq \{0\}$, an open subset $U \subseteq V$, and a natural number $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let $|\cdot|_{C_b^k(U,W)}, \|\cdot\|_{C_b^k(U,W)} \colon C^k(U, W) \to [0, \infty]$ be the mappings with the property that for all $f \in C^k(U, W)$ it holds that

$$|f|_{C_b^k(U,W)} = \sup_{x \in U} \|f^{(k)}(x)\|_{L^{(k)}(V,W)} = \sup_{x \in U} \sup_{v_1, \ldots, v_k \in V \setminus \{0\}} \frac{\|f^{(k)}(x)(v_1, \ldots, v_k)\|_W}{\|v_1\|_V \cdot \ldots \cdot \|v_k\|_V}, \qquad (1.6)$$

$$\|f\|_{C_b^k(U,W)} = \|f(0)\|_W + \sum_{\ell=1}^k |f|_{C_b^\ell(U,W)} \qquad (1.7)$$

and let $C_b^k(U, W)$ be the set given by $C_b^k(U, W) = \{ f \in C^k(U, W) \colon \|f\|_{C_b^k(U,W)} < \infty \}$. Moreover, for two $\mathbb{R}$-Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ with $V \neq \{0\}$, an open subset $U \subseteq V$, and a number $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, let $|\cdot|_{\mathrm{Lip}^k(U,W)}, \|\cdot\|_{\mathrm{Lip}^k(U,W)} \colon C^k(U, W) \to [0, \infty]$ be the mappings

with the property that for all $f \in C^k(U, W)$ it holds that

$$|f|_{\mathrm{Lip}^k(U,W)} = \begin{cases} \sup\limits_{\substack{x,y \in U, \\ x \neq y}} \left(\frac{\|f(x)-f(y)\|_W}{\|x-y\|_V} \right) & : k = 0, \\ \sup\limits_{\substack{x,y \in U, \\ x \neq y}} \left(\frac{\|f^{(k)}(x)-f^{(k)}(y)\|_{L^{(k)}(V,W)}}{\|x-y\|_V} \right) & : k \in \mathbb{N}, \end{cases} \tag{1.8}$$

$$\|f\|_{\mathrm{Lip}^k(U,W)} = \|f(0)\|_W + \sum_{\ell=0}^{k} |f|_{\mathrm{Lip}^\ell(U,W)} \tag{1.9}$$

and let $\mathrm{Lip}^k(U, W)$ be the set given by $\mathrm{Lip}^k(U, W) = \{ f \in C^k(U, W) \colon \|f\|_{\mathrm{Lip}^k(U,W)} < \infty \}$. Additionally, for two normed $\mathbb{R}$-vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ let $\|\cdot\|_{\mathrm{LG}(V,W)} \colon \mathbb{M}(V, W) \to [0, \infty]$ be the mapping such that for all $f \in \mathbb{M}(V, W)$ it holds that $\|f\|_{\mathrm{LG}(V,W)} = \sup_{v \in V} \left(\frac{\|f(v)\|_W}{\max\{1, \|v\|_V\}} \right)$. For an $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ let $\mathcal{J}^H \colon L^{(2)}(H, \mathbb{R}) \to L(H)$ be the mapping with the property that for all $\beta \in L^{(2)}(H, \mathbb{R})$, $h_1, h_2 \in H$ it holds that $\beta(h_1, h_2) = \langle h_1, \mathcal{J}_\beta^H h_2 \rangle_H$. Furthermore, for two $\mathbb{R}$-inner product spaces $(V, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ and $(W, \langle \cdot, \cdot \rangle_W, \|\cdot\|_W)$ we denote by $(V \times W, \langle \cdot, \cdot \rangle_{V \times W}, \|\cdot\|_{V \times W})$ the $\mathbb{R}$-inner product space such that for all $x_1 = (v_1, w_1), x_2 = (v_2, w_2) \in V \times W$ it holds that $\langle x_1, x_2 \rangle_{V \times W} = \langle v_1, v_2 \rangle_V + \langle w_1, w_2 \rangle_W$. Moreover, for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$, an $\mathcal{F}/\mathcal{B}(H)$-measurable mapping $X \colon \Omega \to H$ with the property that for all $v \in H$ it holds that $\mathbb{E}\big[|\langle v, X \rangle_H|^2\big] < \infty$, we denote by $\mathrm{CovOp}(X) \in L(H)$ the covariance operator of the Hilbert space valued random variable X (see, e.g., Definition 5.3.13 and Definition 5.3.11 in [23]). Furthermore, for $\mathbb{R}$-Hilbert spaces $(H_i, \langle \cdot, \cdot \rangle_{H_i}, \|\cdot\|_{H_i})$, $i \in \{1, 2\}$, let $\|\cdot\|_{L_p(H_1, H_2)} \colon L(H_1, H_2) \to [0, \infty]$, $p \in [1, \infty)$, be the mappings with the property that for all $p \in [1, \infty)$, $A \in L(H_1, H_2)$ it holds that $\|A\|_{L_p(H_1, H_2)} = \big(\mathrm{trace}_{H_1}((A^\star A)^{p/2})\big)^{1/p}$, let $L_p(H_1, H_2)$ be the set given by $L_p(H_1, H_2) = \{ A \in L(H_1, H_2) \colon \|A\|_{L_p(H_1, H_2)} < \infty \}$, let $L_p(H_1)$ be the set given by $L_p(H_1) = L_p(H_1, H_1)$, and let $\|\cdot\|_{L_p(H_1)} \colon L(H_1) \to [0, \infty]$, $p \in [1, \infty)$, be the mappings which satisfy for all $p \in [1, \infty)$ that $\|\cdot\|_{L_p(H_1)} = \|\cdot\|_{L_p(H_1, H_1)}$. In addition, for an $\mathbb{R}$-Hilbert space $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$, an orthonormal basis $\mathbb{B} \subseteq H$ of H, a mapping $\lambda \colon \mathbb{B} \to \mathbb{R}$, a linear operator $A \colon D(A) \subseteq H \to H$ satisfying that $D(A) = \{ v \in H \colon \sum_{b \in \mathbb{B}} |\lambda_b \langle b, v \rangle_H|^2 < \infty \}$ and that for all $v \in D(A)$ it holds that $Av = \sum_{b \in \mathbb{B}} \lambda_b \langle b, v \rangle_H b$, and a mapping $\varphi \colon \mathbb{R} \to \mathbb{R}$, let $\varphi(A) \colon D(\varphi(A)) \subseteq H \to H$ be the linear operator satisfying that $D(\varphi(A)) = \{ v \in H \colon \sum_{b \in \mathbb{U}} |\varphi(\lambda_b) \langle b, v \rangle_H|^2 < \infty \}$ and that for all $v \in D(\varphi(A))$ it holds that $\varphi(A)v = \sum_{b \in \mathbb{B}} \varphi(\lambda_b) \langle b, v \rangle_H b$. Finally, for a Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ we denote by $\lambda_A \colon \mathcal{B}(A) \to [0, \infty]$ the Lebesgue-Borel measure on A.

1.2 Setting

Let $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, and let $(W_t)_{t \in [0,T]}$ be an id_U-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process.

2 Preliminaries

2.1 Existence, uniqueness, and regularity results for stochastic evolution equations

Theorem 2.1 below is a direct consequence of Theorem 7.4 in Da Prato & Zabczyk [13].

Theorem 2.1. *Assume the setting in Section 1.2, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $S \colon [0, \infty) \to L(H)$ be a strongly continuous semigroup, and let $p \in [2, \infty)$, $F \in \mathrm{Lip}^0(H, H)$, $B \in \mathrm{Lip}^0(H, L_2(U, H))$, $\xi \in L^p(\mathbb{P}|_{\mathcal{F}_0}; H)$. Then there exists an up to modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic process $X \colon [0, T] \times \Omega \to H$ such that for all $t \in [0, T]$ it*

holds that $\sup_{s\in[0,T]}\|X_s\|_{L^p(\mathbb{P};H)} < \infty$ and $\mathbb{P}$-a.s. that

$$X_t = S_t\xi + \int_0^t S_{t-s}F(X_s)\,\mathrm{d}s + \int_0^t S_{t-s}B(X_s)\,\mathrm{d}W_s. \tag{2.1}$$

Remark 2.2. Assume the setting in Section 1.2, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $S\colon [0,\infty) \to L(H)$ be a strongly continuous semigroup, and let $F \in \mathrm{Lip}^0(H,H)$, $B \in \mathrm{Lip}^0(H, L_2(U,H))$. Then Theorem 2.1 shows that there exist up to modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^x\colon [0,T] \times \Omega \to H$, $x \in H$, such that for all $x \in H$, $t \in [0,T]$, $p \in [2,\infty)$ it holds that $\sup_{s\in[0,T]}\|X_s^x\|_{L^p(\mathbb{P};H)} < \infty$ and $\mathbb{P}$-a.s. that

$$X_t^x = S_t x + \int_0^t S_{t-s}F(X_s^x)\,\mathrm{d}s + \int_0^t S_{t-s}B(X_s^x)\,\mathrm{d}W_s. \tag{2.2}$$

Lemma 2.3. *Assume the setting in Section 1.2, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a finite-dimensional $\mathbb{R}$-vector space, let $A \in L(H)$, $F \in C_{\mathrm{b}}^2(H,H)$, $B \in C_{\mathrm{b}}^2(H, L_2(U,H))$, $\varphi \in C_{\mathrm{b}}^2(H,\mathbb{R})$, let $X^x\colon [0,T] \times \Omega \to H$, $x \in H$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes satisfying that for all $x \in H$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]}\|X_s^x\|_{L^2(\mathbb{P};H)} < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^x = \mathrm{e}^{At}x + \int_0^t \mathrm{e}^{A(t-s)}F(X_s^x)\,\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}B(X_s^x)\,\mathrm{d}W_s, \tag{2.3}$$

and let $u\colon [0,T] \times H \to \mathbb{R}$ be the mapping with the property that for all $t \in [0,T]$, $x \in H$ it holds that $u(t,x) = \mathbb{E}[\varphi(X_t^x)]$. Then

(i) *it holds that $u \in C^{1,2}([0,T] \times H, \mathbb{R})$,*

(ii) *it holds for all $(t,x) \in [0,T] \times H$ that*

$$\left(\tfrac{\partial}{\partial t}u\right)(t,x) = \left(\tfrac{\partial}{\partial x}u\right)(t,x)[Ax + F(x)] + \tfrac{1}{2}\sum_{u\in\mathbb{U}}\left(\tfrac{\partial^2}{\partial x^2}u\right)(t,x)(B(x)u, B(x)u), \tag{2.4}$$

(iii) *and it holds that*

$$\sup_{t\in[0,T]}|u(t,\cdot)|_{C_{\mathrm{b}}^1(H,\mathbb{R})} \le |\varphi|_{C_{\mathrm{b}}^1(H,\mathbb{R})}\left[\sup_{s\in[0,T]}\|\mathrm{e}^{As}\|_{L(H)}\right]$$
$$\cdot \exp\left(T\big[|F|_{C_{\mathrm{b}}^1(H,H)} + \tfrac{1}{2}|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\|\mathrm{e}^{As}\|_{L(H)}^2\right) < \infty, \tag{2.5}$$

$$\sup_{t\in[0,T]}|u(t,\cdot)|_{C_{\mathrm{b}}^2(H,\mathbb{R})}$$
$$\le \|\varphi\|_{C_{\mathrm{b}}^2(H,\mathbb{R})}\left(1 \vee \left[T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\right]^{1/2}\right)\left[\sup_{s\in[0,T]}\|\mathrm{e}^{As}\|_{L(H)}^3\right] \tag{2.6}$$
$$\cdot \exp\left(T\big[\tfrac{1}{2} + 3|F|_{C_{\mathrm{b}}^1(H,H)} + 4|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\|\mathrm{e}^{As}\|_{L(H)}^4\right) < \infty.$$

Proof of Lemma 2.3. It is well-known (cf., e.g., Sections 4 and 5 and Lemma 5.10 in Krylov [33]) that the assumption that H is finite-dimensional and the assumptions that $\varphi \in C_{\mathrm{b}}^2(H,\mathbb{R})$, $F \in C_{\mathrm{b}}^2(H,H)$, $B \in C_{\mathrm{b}}^2(H, L_2(U,H))$ imply that (i) and (ii) hold, that there exist up to modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^{x,v_1}, X^{x,v_1,v_2}\colon [0,T] \times \Omega \to H$, $x, v_1, v_2 \in H$, satisfying for all $x, v_1, v_2 \in H$, $t \in [0,T]$, $p \in [2,\infty)$ that $\sup_{s\in[0,T]}\big(\|X_s^{x,v_1}\|_{L^p(\mathbb{P};H)} + \|X_s^{x,v_1,v_2}\|_{L^p(\mathbb{P};H)}\big) <$

∞ and $\mathbb{P}$-a.s. that

$$X_t^{x,v_1} = \mathrm{e}^{At}v_1 + \int_0^t \mathrm{e}^{A(t-s)}F'(X_s^x)X_s^{x,v_1}\,\mathrm{d}s + \int_0^t \mathrm{e}^{A(t-s)}B'(X_s^x)X_s^{x,v_1}\,\mathrm{d}W_s, \tag{2.7}$$

$$X_t^{x,v_1,v_2} = \int_0^t \mathrm{e}^{A(t-s)}\big(F''(X_s^x)(X_s^{x,v_1},X_s^{x,v_2}) + F'(X_s^x)X_s^{x,v_1,v_2}\big)\,\mathrm{d}s$$
$$+ \int_0^t \mathrm{e}^{A(t-s)}\big(B''(X_s^x)(X_s^{x,v_1},X_s^{x,v_2}) + B'(X_s^x)X_s^{x,v_1,v_2}\big)\,\mathrm{d}W_s, \tag{2.8}$$

and that for all $(t,x) \in [0,T] \times H$, $v_1, v_2 \in H$ it holds that

$$\big(\tfrac{\partial}{\partial x}u\big)(t,x)v_1 = \mathbb{E}\big[\varphi'(X_t^x)X_t^{x,v_1}\big], \tag{2.9}$$
$$\big(\tfrac{\partial^2}{\partial x^2}u\big)(t,x)(v_1,v_2) = \mathbb{E}\big[\varphi''(X_t^x)(X_t^{x,v_1},X_t^{x,v_2}) + \varphi'(X_t^x)X_t^{x,v_1,v_2}\big]. \tag{2.10}$$

It thus remains to prove (iii). For this let $\psi_p\colon H \to \mathbb{R}$, $p \in [2,\infty)$, be the functions satisfying for all $p \in [2,\infty)$, $x \in H$ that $\psi_p(x) = \|x\|_H^p$. Then note for all $p \in [2,\infty)$, $x, v_1, v_2 \in H$ that $\psi_p \in C^2(H,\mathbb{R})$ and that

$$\psi_p'(x)v_1 = \begin{cases} 0 & : x = 0, \\ p\|x\|^{p-2}\langle x, v_1\rangle_H & : x \neq 0, \end{cases} \tag{2.11}$$

$$\psi_p''(x)(v_1,v_2) = \begin{cases} 2\langle v_1, v_2\rangle_H & : p = 2, \\ 0 & : (p \neq 2) \wedge (x = 0), \\ p\|x\|_H^{p-2}\langle v_1, v_2\rangle_H + p(p-2)\|x\|_H^{p-4}\langle x, v_1\rangle_H\langle x, v_2\rangle_H & : x \neq 0. \end{cases} \tag{2.12}$$

An application of the mild Itô formula in Corollary 1 in Da Prato et al. [12] on the test functions ψ_p, $p \in [2,\infty)$, and the Cauchy-Schwarz inequality hence yield for all $p \in [2,\infty)$, $x, v \in H$, $t \in [0,T]$ that

$$\mathbb{E}\big[\|X_t^{x,v}\|_H^p\big] = \mathbb{E}\big[\psi_p(X_t^{x,v})\big]$$
$$= \psi_p\big(\mathrm{e}^{At}v\big) + \int_0^t \mathbb{E}\big[\psi_p'\big(\mathrm{e}^{A(t-s)}X_s^{x,v}\big)\mathrm{e}^{A(t-s)}F'(X_s^x)X_s^{x,v}\big]\,\mathrm{d}s$$
$$+ \frac{1}{2}\sum_{u\in\mathbb{U}}\int_0^t \mathbb{E}\big[\psi_p''\big(\mathrm{e}^{A(t-s)}X_s^{x,v}\big)\big(\mathrm{e}^{A(t-s)}\big(B'(X_s^x)X_s^{x,v}\big)u, \mathrm{e}^{A(t-s)}\big(B'(X_s^x)X_s^{x,v}\big)u\big)\big]\,\mathrm{d}s$$
$$\leq \|v\|_H^p\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big] + p\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big]|F|_{C_{\mathrm{b}}^1(H,H)}\int_0^t \mathbb{E}\big[\|X_s^{x,v}\|_H^p\big]\,\mathrm{d}s$$
$$+ \tfrac{p}{2}\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big]|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\int_0^t \mathbb{E}\big[\|X_s^{x,v}\|_H^p\big]\,\mathrm{d}s \tag{2.13}$$
$$+ \tfrac{p(p-2)}{2}\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big]|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\int_0^t \mathbb{E}\big[\|X_s^{x,v}\|_H^p\big]\,\mathrm{d}s$$
$$= \|v\|_H^p\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big]$$
$$+ p\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big]\big(|F|_{C_{\mathrm{b}}^1(H,H)} + \tfrac{p-1}{2}|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big)\int_0^t \mathbb{E}\big[\|X_s^{x,v}\|_H^p\big]\,\mathrm{d}s.$$

Therefore, Gronwall's lemma shows for all $p \in [2,\infty)$, $x, v \in H$ that

$$\sup_{t\in[0,T]}\|X_t^{x,v}\|_{L^p(\mathbb{P};H)}$$
$$\leq \|v\|_H\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}\Big]\exp\Big(T\big[|F|_{C_{\mathrm{b}}^1(H,H)} + \tfrac{p-1}{2}|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^p\Big). \tag{2.14}$$

Furthermore, applying again Corollary 1 in Da Prato et al. [12] on the test function ψ_2, the Cauchy-Schwarz inequality, and the fact that $\forall a, b \in \mathbb{R}$ it holds that $ab \le \frac{a^2+b^2}{2}$ imply for all $x, v_1, v_2 \in H$, $t \in [0, T]$ that

$$
\begin{aligned}
&\mathbb{E}\big[\|X_t^{x,v_1,v_2}\|_H^2\big] \\
&= 2\int_0^t \mathbb{E}\big[\big\langle \mathrm{e}^{A(t-s)} X_s^{x,v_1,v_2}, \mathrm{e}^{A(t-s)}\big(F''(X_s^x)(X_s^{x,v_1}, X_s^{x,v_2}) + F'(X_s^x)X_s^{x,v_1,v_2}\big)\big\rangle_H\big] \,\mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\big[\big\|\mathrm{e}^{A(t-s)}\big(B''(X_s^x)(X_s^{x,v_1}, X_s^{x,v_2}) + B'(X_s^x)X_s^{x,v_1,v_2}\big)\big\|_{L_2(U,H)}^2\big]\,\mathrm{d}s \\
&\le \int_0^t \big\|\mathrm{e}^{A(t-s)}F''(X_s^x)(X_s^{x,v_1}, X_s^{x,v_2})\big\|_{L^2(\mathbb{P};H)}^2 + \big\|\mathrm{e}^{A(t-s)}X_s^{x,v_1,v_2}\big\|_{L^2(\mathbb{P};H)}^2 \,\mathrm{d}s \\
&\quad + 2\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\Big]|F|_{C_{\mathrm{b}}^1(H,H)}\int_0^t \mathbb{E}\big[\|X_s^{x,v_1,v_2}\|_H^2\big]\,\mathrm{d}s \\
&\quad + 2\int_0^t \mathbb{E}\big[\big\|\mathrm{e}^{A(t-s)}B''(X_s^x)(X_s^{x,v_1}, X_s^{x,v_2})\big\|_{L_2(U,H)}^2 + \big\|\mathrm{e}^{A(t-s)}B'(X_s^x)X_s^{x,v_1,v_2}\big\|_{L_2(U,H)}^2\big]\,\mathrm{d}s \\
&\le \Big[\sup_{s\in[0,T]}\|X_s^{x,v_1}\|_{L^4(\mathbb{P};H)}^2\|X_s^{x,v_2}\|_{L^4(\mathbb{P};H)}^2\Big]T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\Big] \\
&\quad + 2\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\Big]\big(\tfrac{1}{2} + |F|_{C_{\mathrm{b}}^1(H,H)} + |B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big)\int_0^t \mathbb{E}\big[\|X_s^{x,v_1,v_2}\|_H^2\big]\,\mathrm{d}s.
\end{aligned}
\tag{2.15}
$$

Gronwall's lemma and (2.14), therefore, imply for all $x, v_1, v_2 \in H$ that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|X_t^{x,v_1,v_2}\|_{L^2(\mathbb{P};H)} \\
&\le \Big[\sup_{s\in[0,T]}\|X_s^{x,v_1}\|_{L^4(\mathbb{P};H)}\|X_s^{x,v_2}\|_{L^4(\mathbb{P};H)}\Big]\Big[T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\Big]^{1/2} \\
&\quad \cdot \Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}\Big]\exp\Big(T\big[\tfrac{1}{2} + |F|_{C_{\mathrm{b}}^1(H,H)} + |B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\Big) \\
&\le \|v_1\|_H\|v_2\|_H\Big[T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\Big]^{1/2}\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^3\Big] \\
&\quad \cdot \exp\Big(T\big[\tfrac{1}{2} + 3|F|_{C_{\mathrm{b}}^1(H,H)} + 4|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^4\Big).
\end{aligned}
\tag{2.16}
$$

Next note that (2.9), (2.10), (2.14), and (2.16) ensure for all $(t, x) \in [0, T] \times H$, $v_1, v_2 \in H$ that

$$
\begin{aligned}
\big|\big(\tfrac{\partial}{\partial x}u\big)(t,x)v_1\big| = \big|\mathbb{E}\big[\varphi'(X_t^x)X_t^{x,v_1}\big]\big| &\le |\varphi|_{C_{\mathrm{b}}^1(H,\mathbb{R})}\mathbb{E}\big[\|X_t^{x,v_1}\|_H\big] \\
&\le \|v_1\|_H|\varphi|_{C_{\mathrm{b}}^1(H,\mathbb{R})}\Big[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}\Big] \\
&\quad \cdot \exp\Big(T\big[|F|_{C_{\mathrm{b}}^1(H,H)} + \tfrac{1}{2}|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\Big)
\end{aligned}
\tag{2.17}
$$

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and

$$\left|\left(\tfrac{\partial^2}{\partial x^2}u\right)(t,x)(v_1,v_2)\right| = \left|\mathbb{E}\big[\varphi''(X_t^x)(X_t^{x,v_1},X_t^{x,v_2}) + \varphi'(X_t^x)X_t^{x,v_1,v_2}\big]\right|$$

$$\leq |\varphi|_{C_{\mathrm{b}}^2(H,\mathbb{R})}\big\|X_t^{x,v_1}\big\|_{L^2(\mathbb{P};H)}\big\|X_t^{x,v_2}\big\|_{L^2(\mathbb{P};H)} + |\varphi|_{C_{\mathrm{b}}^1(H,\mathbb{R})}\,\mathbb{E}\big[\big\|X_t^{x,v_1,v_2}\big\|_H\big]$$

$$\leq \|v_1\|_H\|v_2\|_H|\varphi|_{C_{\mathrm{b}}^2(H,\mathbb{R})}\left[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\right]\exp\!\left(T\big[2|F|_{C_{\mathrm{b}}^1(H,H)} + |B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^2\right)$$

$$+ \|v_1\|_H\|v_2\|_H|\varphi|_{C_{\mathrm{b}}^1(H,\mathbb{R})}\Big[T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\Big]^{1/2}\left[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^3\right]$$

$$\cdot \exp\!\left(T\big[\tfrac{1}{2} + 3|F|_{C_{\mathrm{b}}^1(H,H)} + 4|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^4\right)$$

$$\leq \|v_1\|_H\|v_2\|_H\|\varphi\|_{C_{\mathrm{b}}^2(H,\mathbb{R})}\Big(1 \vee \Big[T\big(|F|_{C_{\mathrm{b}}^2(H,H)}^2 + 2|B|_{C_{\mathrm{b}}^2(H,L_2(U,H))}^2\big)\Big]^{1/2}\Big)\left[\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^3\right]$$

$$\cdot \exp\!\left(T\big[\tfrac{1}{2} + 3|F|_{C_{\mathrm{b}}^1(H,H)} + 4|B|_{C_{\mathrm{b}}^1(H,L_2(U,H))}^2\big]\sup_{s\in[0,T]}\big\|\mathrm{e}^{As}\big\|_{L(H)}^4\right).$$

$$(2.18)$$

This completes the proof of Lemma 2.3. □

2.2 Setting

Let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{H}\subseteq H$ be a non-empty orthonormal basis of H, let $\lambda\colon\mathbb{H}\to\mathbb{R}$ be a mapping such that $\sup_{h\in\mathbb{H}}\lambda_h < 0$, let $A\colon D(A)\subseteq H\to H$ be the linear operator such that $D(A) = \big\{v\in H\colon \sum_{h\in\mathbb{H}}|\lambda_h\langle h,v\rangle_H|^2 < \infty\big\}$ and such that for all $v\in D(A)$ it holds that $Av = \sum_{h\in\mathbb{H}}\lambda_h\langle h,v\rangle_H h$, let $(H_r,\langle\cdot,\cdot\rangle_{H_r},\|\cdot\|_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $(\mathbf{H}_r,\langle\cdot,\cdot\rangle_{\mathbf{H}_r},\|\cdot\|_{\mathbf{H}_r})$, $r\in\mathbb{R}$, be the family of $\mathbb{R}$-Hilbert spaces such that for all $r\in\mathbb{R}$ it holds that $(\mathbf{H}_r,\langle\cdot,\cdot\rangle_{\mathbf{H}_r},\|\cdot\|_{\mathbf{H}_r}) = \big(H_{r/2}\times H_{r/2-1/2},\langle\cdot,\cdot\rangle_{H_{r/2}\times H_{r/2-1/2}},\|\cdot\|_{H_{r/2}\times H_{r/2-1/2}}\big)$, and let $\mathbf{A}\colon D(\mathbf{A})\subseteq\mathbf{H}_0\to\mathbf{H}_0$ be the linear operator such that $D(\mathbf{A}) = \mathbf{H}_1$ and such that for all $(v,w)\in\mathbf{H}_1$ it holds that $\mathbf{A}(v,w) = (w,Av)$.

2.3 Basic properties of the deterministic wave equation

2.3.1 Basic properties of interpolation spaces associated to the deterministic wave equation

Lemma 2.4. *Assume the setting in Section 2.2 and let* $\mathbf{\Lambda}\colon D(\mathbf{\Lambda})\subseteq\mathbf{H}_0\to\mathbf{H}_0$ *be the linear operator such that* $D(\mathbf{\Lambda}) = \mathbf{H}_1$ *and such that for all* $(v,w)\in\mathbf{H}_1$ *it holds that*

$$\mathbf{\Lambda}(v,w) = \begin{pmatrix}\sum_{h\in\mathbb{H}}|\lambda_h|^{1/2}\langle h,v\rangle_{H_0}h \\ \sum_{h\in\mathbb{H}}|\lambda_h|^{1/2}\big\langle|\lambda_h|^{1/2}h,w\big\rangle_{H_{-1/2}}|\lambda_h|^{1/2}h\end{pmatrix}. \tag{2.19}$$

Then the $\mathbb{R}$-*Hilbert spaces* $(\mathbf{H}_r,\langle\cdot,\cdot\rangle_{\mathbf{H}_r},\|\cdot\|_{\mathbf{H}_r})$, $r\in\mathbb{R}$, *are a family of interpolation spaces associated to* $\mathbf{\Lambda}$.

Proof of Lemma 2.4. Observe that $\mathbf{\Lambda}\colon D(\mathbf{\Lambda})\subseteq\mathbf{H}_0\to\mathbf{H}_0$ is a symmetric diagonal linear operator

with $\inf(\sigma_{\mathrm{P}}(\mathbf{\Lambda})) > 0$ and that for all $r \in [0, \infty)$ it holds that

$$
\begin{aligned}
D(\mathbf{\Lambda}^r) &= \left\{ x \in \mathbf{H}_0 \colon \sum_{h \in \mathbb{H}} |\lambda_h|^r |\langle (h, 0), x \rangle_{\mathbf{H}_0}|^2 + |\lambda_h|^r |\langle (0, |\lambda_h|^{1/2} h), x \rangle_{\mathbf{H}_0}|^2 < \infty \right\} \\
&= \left\{ (v, w) \in \mathbf{H}_0 \colon \sum_{h \in \mathbb{H}} |\lambda_h|^r |\langle h, v \rangle_{H_0}|^2 + |\lambda_h|^r |\langle |\lambda_h|^{1/2} h, w \rangle_{H_{-1/2}}|^2 < \infty \right\} \\
&= \left\{ v \in H_0 \colon \sum_{h \in \mathbb{H}} |\lambda_h|^r |\langle h, v \rangle_{H_0}|^2 < \infty \right\} \times \left\{ w \in H_{-1/2} \colon \sum_{h \in \mathbb{H}} |\lambda_h|^{r-1} |\langle h, w \rangle_{H_0}|^2 < \infty \right\} \\
&= H_{r/2} \times H_{r/2 - 1/2} = \mathbf{H}_r.
\end{aligned}
\tag{2.20}
$$

Moreover, for all $r \in [0, \infty)$, $x_1 = (v_1, w_1), x_2 = (v_2, w_2) \in \mathbf{H}_r$ it holds that

$$
\begin{aligned}
\langle \mathbf{\Lambda}^r x_1, \mathbf{\Lambda}^r x_2 \rangle_{\mathbf{H}_0} &= \left\langle \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle h, v_1 \rangle_{H_0} h, \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle h, v_2 \rangle_{H_0} h \right\rangle_{H_0} \\
&\quad + \left\langle \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle |\lambda_h|^{1/2} h, w_1 \rangle_{H_{-1/2}} |\lambda_h|^{1/2} h, \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle |\lambda_h|^{1/2} h, w_2 \rangle_{H_{-1/2}} |\lambda_h|^{1/2} h \right\rangle_{H_{-1/2}} \\
&= \langle (-A)^{r/2} v_1, (-A)^{r/2} v_2 \rangle_{H_0} + \langle (-A)^{r/2} w_1, (-A)^{r/2} w_2 \rangle_{H_{-1/2}} \\
&= \langle v_1, v_2 \rangle_{H_{r/2}} + \langle w_1, w_2 \rangle_{H_{r/2 - 1/2}} = \langle x_1, x_2 \rangle_{\mathbf{H}_r}.
\end{aligned}
\tag{2.21}
$$

In addition, for all $r \in (-\infty, 0]$, $x = (v, w) \in \mathbf{H}_0$ it holds that

$$
\begin{aligned}
\|\mathbf{\Lambda}^r x\|_{\mathbf{H}_0}^2 &= \left\| \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle h, v \rangle_{H_0} h \right\|_{H_0}^2 + \left\| \sum_{h \in \mathbb{H}} |\lambda_h|^{r/2} \langle |\lambda_h|^{1/2} h, w \rangle_{H_{-1/2}} |\lambda_h|^{1/2} h \right\|_{H_{-1/2}}^2 \\
&= \|(-A)^{r/2} v\|_{H_0}^2 + \|(-A)^{r/2} w\|_{H_{-1/2}}^2 = \|v\|_{H_{r/2}}^2 + \|w\|_{H_{r/2 - 1/2}}^2 = \|x\|_{\mathbf{H}_r}^2.
\end{aligned}
\tag{2.22}
$$

This completes the proof of Lemma 2.4. $\qquad\square$

2.3.2 Basic properties of the deterministic linear wave equation

The next elementary and well-known lemma can be found in a slightly different form, e.g., in Section 5.3 in Lindgren [34].

Lemma 2.5. *Assume the setting in Section 2.2 and let $\mathbf{S} \colon [0, \infty) \to L(\mathbf{H}_0)$ be the mapping such that for all $t \in [0, \infty)$, $(v, w) \in \mathbf{H}_0$ it holds that*

$$
\mathbf{S}_t(v, w) = \begin{pmatrix} \cos\!\big((-A)^{1/2} t\big) v + (-A)^{-1/2} \sin\!\big((-A)^{1/2} t\big) w \\ -(-A)^{1/2} \sin\!\big((-A)^{1/2} t\big) v + \cos\!\big((-A)^{1/2} t\big) w \end{pmatrix}.
\tag{2.23}
$$

Then $\mathbf{S} \colon [0, \infty) \to L(\mathbf{H}_0)$ is a strongly continuous semigroup of bounded linear operators on $\mathbf{H}_0$ and $\mathbf{A} \colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ is the generator of $\mathbf{S}$.

Lemma 2.6. *Assume the setting in Section 2.2. Then for all $t \in [0, \infty)$, $x \in \mathbf{H}_0$ it holds that $\|e^{\mathbf{A}t} x\|_{\mathbf{H}_0} = \|x\|_{\mathbf{H}_0}$ and $\sup_{s \in [0, \infty)} \|e^{\mathbf{A}s}\|_{L(\mathbf{H}_0)} = 1$.*

Proof of Lemma 2.6. Lemma 2.5 implies for all $t \in [0, \infty)$, $x = (v, w) \in \mathbf{H}_1$ that

$$
\begin{aligned}
\|\mathrm{e}^{\mathbf{A}t} x\|_{\mathbf{H}_0}^2 &= \left\|\cos\!\big((-A)^{1/2}t\big) v + (-A)^{-1/2} \sin\!\big((-A)^{1/2}t\big) w\right\|_{H_0}^2 \\
&\quad + \left\|-(-A)^{1/2} \sin\!\big((-A)^{1/2}t\big) v + \cos\!\big((-A)^{1/2}t\big) w\right\|_{H_{-1/2}}^2 \\
&= \left\|\cos\!\big((-A)^{1/2}t\big) v\right\|_{H_0}^2 + \left\|(-A)^{-1/2} \sin\!\big((-A)^{1/2}t\big) w\right\|_{H_0}^2 \\
&\quad + \left\|(-A)^{1/2} \sin\!\big((-A)^{1/2}t\big) v\right\|_{H_{-1/2}}^2 + \left\|\cos\!\big((-A)^{1/2}t\big) w\right\|_{H_{-1/2}}^2 \\
&\quad + 2\langle \cos\!\big((-A)^{1/2}t\big) v, (-A)^{-1/2} \sin\!\big((-A)^{1/2}t\big) w \rangle_{H_0} \\
&\quad - 2\langle \sin\!\big((-A)^{1/2}t\big) v, (-A)^{1/2} \cos\!\big((-A)^{1/2}t\big) w \rangle_{H_{-1/2}} \\
&= \left\|\cos\!\big((-A)^{1/2}t\big) v\right\|_{H_0}^2 + \left\|\sin\!\big((-A)^{1/2}t\big) v\right\|_{H_0}^2 \\
&\quad + \left\|\sin\!\big((-A)^{1/2}t\big) w\right\|_{H_{-1/2}}^2 + \left\|\cos\!\big((-A)^{1/2}t\big) w\right\|_{H_{-1/2}}^2 \\
&= \|v\|_{H_0}^2 + \|w\|_{H_{-1/2}}^2 = \|x\|_{\mathbf{H}_0}^2.
\end{aligned}
\tag{2.24}
$$

This completes the proof of Lemma 2.6. $\qquad\square$

Lemma 2.7. *Assume the setting in Section 2.2, let* $P_I \colon \bigcup_{r \in \mathbb{R}} H_r \to \bigcup_{r \in \mathbb{R}} H_r$, $I \in \mathcal{P}(\mathbb{H})$, *be the mappings such that for all* $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $v \in H_r$ *it holds that* $P_I(v) = \sum_{h \in I} \langle |\lambda_h|^{-r} h, v \rangle_{H_r} |\lambda_h|^{-r} h$, *and let* $\mathbf{P}_I \colon \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \to \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $I \in \mathcal{P}(\mathbb{H})$, *be the mappings such that for all* $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $(v, w) \in \mathbf{H}_r$ *it holds that* $\mathbf{P}_I(v, w) = \big(P_I(v), P_I(w)\big)$. *Then for all* $I \in \mathcal{P}(\mathbb{H})$, $x \in \mathbf{H}_1$ *it holds that* $\mathbf{A} \mathbf{P}_I(x) = \mathbf{P}_I \mathbf{A}(x)$ *and for all* $I \in \mathcal{P}(\mathbb{H})$, $t \in [0, \infty)$, $x \in \mathbf{H}_0$ *it holds that* $\mathrm{e}^{\mathbf{A}t} \mathbf{P}_I(x) = \mathbf{P}_I \mathrm{e}^{\mathbf{A}t}(x)$.

Proof of Lemma 2.7. For all $I \in \mathcal{P}(\mathbb{H})$, $x = (v, w) \in \mathbf{H}_1$ it holds that

$$
\mathbf{P}_I \mathbf{A}(x) = \mathbf{P}_I(w, Av) = (P_I(w), P_I Av) = (P_I(w), A P_I(v)) = \mathbf{A} \mathbf{P}_I(x).
\tag{2.25}
$$

In addition, Lemma 2.5 shows for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0, \infty)$, $x = (v, w) \in \mathbf{H}_0$ that

$$
\begin{aligned}
\mathrm{e}^{\mathbf{A}t} \mathbf{P}_I(x) &= \begin{pmatrix} \cos\!\big((-A)^{1/2}t\big) P_I(v) + (-A)^{-1/2} \sin\!\big((-A)^{1/2}t\big) P_I(w) \\ -(-A)^{1/2} \sin\!\big((-A)^{1/2}t\big) P_I(v) + \cos\!\big((-A)^{1/2}t\big) P_I(w) \end{pmatrix} \\
&= \begin{pmatrix} P_I\big(\cos\!\big((-A)^{1/2}t\big) v + (-A)^{-1/2} \sin\!\big((-A)^{1/2}t\big) w\big) \\ P_I\big(-(-A)^{1/2} \sin\!\big((-A)^{1/2}t\big) v + \cos\!\big((-A)^{1/2}t\big) w\big) \end{pmatrix} \\
&= \mathbf{P}_I \mathrm{e}^{\mathbf{A}t}(x).
\end{aligned}
\tag{2.26}
$$

The proof of Lemma 2.7 is thus completed. $\qquad\square$

3 Upper bounds for weak errors

3.1 Setting

Assume the setting in Section 1.2, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a mapping such that $\sup_{h \in \mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \big\{ v \in H \colon \sum_{h \in \mathbb{H}} |\lambda_h \langle h, v \rangle_H|^2 < \infty \big\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, be the family of $\mathbb{R}$-Hilbert spaces such that for all $r \in \mathbb{R}$ it holds that $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = \big(H_{r/2} \times H_{r/2-1/2}, \langle \cdot, \cdot \rangle_{H_{r/2} \times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2} \times H_{r/2-1/2}}\big)$, let $P_I \colon \bigcup_{r \in \mathbb{R}} H_r \to \bigcup_{r \in \mathbb{R}} H_r$, $I \in \mathcal{P}(\mathbb{H})$, be the mappings such that for all $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $v \in H_r$ it holds that $P_I(v) = \sum_{h \in I} \langle |\lambda_h|^{-r} h, v \rangle_{H_r} |\lambda_h|^{-r} h$, let $\mathbf{P}_I \colon \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \to \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $I \in \mathcal{P}(\mathbb{H})$, be the mappings such that for all $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$,

$(v,w) \in \mathbf{H}_r$ it holds that $\mathbf{P}_I(v,w) = \big(P_I(v), P_I(w)\big)$, let $\mathbf{A}\colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ be the linear operator such that $D(\mathbf{A}) = \mathbf{H}_1$ and such that for all $(v,w) \in \mathbf{H}_1$ it holds that $\mathbf{A}(v,w) = (w, Av)$, let $\mathbf{\Lambda}\colon D(\mathbf{\Lambda}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ be the linear operator such that $D(\mathbf{\Lambda}) = \mathbf{H}_1$ and such that for all $(v,w) \in \mathbf{H}_1$ it holds that $\mathbf{\Lambda}(v,w) = \big(\sum_{h \in \mathbb{H}} |\lambda_h|^{1/2} \langle h, v \rangle_{H_0} h, \sum_{h \in \mathbb{H}} |\lambda_h|^{1/2} \langle |\lambda_h|^{1/2} h, w \rangle_{H_{-1/2}} |\lambda_h|^{1/2} h \big)$, and let $\gamma \in (0,\infty)$, $\beta \in (\gamma/2, \gamma]$, $\rho \in [0, 2(\gamma - \beta)]$, $C_{\mathbf{F}}, C_{\mathbf{B}} \in [0,\infty)$, $\xi \in L^2(\mathbb{P}|_{\mathcal{F}_0}; \mathbf{H}_{2(\gamma-\beta)})$, $\mathbf{F} \in \mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)$, $\mathbf{B} \in \mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))$ satisfy that $\mathbf{\Lambda}^{-\beta} \in L_2(\mathbf{H}_0)$, $\mathbf{F}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma-\beta)})$, $\mathbf{B}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho) \cap L(U, \mathbf{H}_\gamma))$, $\mathbf{F}|_{\cap_{r\in\mathbb{R}} \mathbf{H}_r} \in C_b^2(\cap_{r\in\mathbb{R}} \mathbf{H}_r, \mathbf{H}_0)$, $\mathbf{B}|_{\cap_{r\in\mathbb{R}} \mathbf{H}_r} \in C_b^2(\cap_{r\in\mathbb{R}} \mathbf{H}_r, L_2(U, \mathbf{H}_0))$, $C_{\mathbf{F}} = \sup_{x \in \cap_{r\in\mathbb{R}} \mathbf{H}_r} \sup_{v_1, v_2 \in \cap_{r\in\mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{F}''(x)(v_1, v_2)\|_{\mathbf{H}_0} < \infty$, and $C_{\mathbf{B}} = \sup_{x \in \cap_{r\in\mathbb{R}} \mathbf{H}_r} \sup_{v_1, v_2 \in \cap_{r\in\mathbb{R}} \mathbf{H}_r, \|v_1\|_{\mathbf{H}_0} \vee \|v_2\|_{\mathbf{H}_0} \leq 1} \|\mathbf{B}''(x)(v_1, v_2)\|_{L_2(U, \mathbf{H}_0)} < \infty$.

3.2 Weak convergence rates for the Galerkin approximation

Remark 3.1. Assume the setting in Section 3.1. The assumptions that $\mathbf{F}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma-\beta)})$ and $\mathbf{B}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho) \cap L(U, \mathbf{H}_\gamma))$ then ensure that $\mathbf{F}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\rho)$ and $\mathbf{B}|_{\mathbf{H}_\rho} \in \mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))$ and Theorem 2.1 hence shows that there exist up to modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^I\colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, satisfying for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ that $\sup_{s\in[0,T]} \|X_s^I\|_{L^2(\mathbb{P}; \mathbf{H}_\rho)} < \infty$ and $\mathbb{P}$-a.s. that

$$X_t^I = \mathrm{e}^{\mathbf{A}t} \mathbf{P}_I \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{F}(X_s^I) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{B}(X_s^I) \, \mathrm{d}W_s. \tag{3.1}$$

The following lemma provides global L^2-bounds on the stochastic processes $X^I\colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, in Remark 3.1.

Lemma 3.2. *Assume the setting in Section 3.1 and let $X^I\colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]} \|X_s^I\|_{L^2(\mathbb{P}; \mathbf{H}_\rho)} < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^I = \mathrm{e}^{\mathbf{A}t} \mathbf{P}_I \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{F}(X_s^I) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{B}(X_s^I) \, \mathrm{d}W_s. \tag{3.2}$$

Then

$$\begin{aligned}
\sup_{I \in \mathcal{P}(\mathbb{H})} \sup_{t \in [0,T]} \big(1 \vee \big\|X_t^I\big\|_{L^2(\mathbb{P}; \mathbf{H}_\rho)}\big) &\leq \big(1 \vee \|\xi\|_{L^2(\mathbb{P}; \mathbf{H}_\rho)}\big) \\
&\cdot \exp\Big(T\Big[\big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\rho)} + \tfrac{1}{2}\big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))}^2\Big]\Big) < \infty.
\end{aligned} \tag{3.3}$$

Proof of Lemma 3.2. Corollary 1 in Da Prato et al. [12], Lemma 2.6, and the Cauchy-Schwarz inequality ensure for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ that

$$\begin{aligned}
\mathbb{E}\big[\big\|X_t^I\big\|_{\mathbf{H}_\rho}^2\big] &= \mathbb{E}\big[\big\|\mathrm{e}^{\mathbf{A}t} \mathbf{P}_I \xi\big\|_{\mathbf{H}_\rho}^2\big] + 2\int_0^t \mathbb{E}\big[\big\langle \mathrm{e}^{\mathbf{A}(t-s)} X_s^I, \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{F}(X_s^I)\big\rangle_{\mathbf{H}_\rho}\big] \, \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\big[\big\|\mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{B}(X_s^I)\big\|_{L_2(U, \mathbf{H}_\rho)}^2\big] \, \mathrm{d}s \\
&\leq \mathbb{E}\big[\|\mathbf{P}_I \xi\|_{\mathbf{H}_\rho}^2\big] + 2\int_0^t \Big(\|\mathbf{P}_I \mathbf{F}(0)\|_{\mathbf{H}_\rho} \mathbb{E}\big[\big\|X_s^I\big\|_{\mathbf{H}_\rho}\big] + \big|\mathbf{P}_I \mathbf{F}|_{\mathbf{H}_\rho}\big|_{\mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\rho)} \mathbb{E}\big[\big\|X_s^I\big\|_{\mathbf{H}_\rho}^2\big]\Big) \, \mathrm{d}s \\
&\quad + \int_0^t \Big(\|\mathbf{P}_I \mathbf{B}(0)\|_{L_2(U, \mathbf{H}_\rho)}^2 + 2\|\mathbf{P}_I \mathbf{B}(0)\|_{L_2(U, \mathbf{H}_\rho)} \big|\mathbf{P}_I \mathbf{B}|_{\mathbf{H}_\rho}\big|_{\mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))} \mathbb{E}\big[\big\|X_s^I\big\|_{\mathbf{H}_\rho}\big] \\
&\qquad + \big|\mathbf{P}_I \mathbf{B}|_{\mathbf{H}_\rho}\big|_{\mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))}^2 \mathbb{E}\big[\big\|X_s^I\big\|_{\mathbf{H}_\rho}^2\big]\Big) \, \mathrm{d}s \\
&\leq \mathbb{E}\big[\|\mathbf{P}_I \xi\|_{\mathbf{H}_\rho}^2\big] + \Big(2\big\|\mathbf{P}_I \mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_\rho)} + \big\|\mathbf{P}_I \mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, L_2(U, \mathbf{H}_\rho))}^2\Big) \int_0^t 1 \vee \mathbb{E}\big[\big\|X_s^I\big\|_{\mathbf{H}_\rho}^2\big] \, \mathrm{d}s.
\end{aligned} \tag{3.4}$$

Gronwall's lemma hence implies for all $I \in \mathcal{P}(\mathbb{H})$ that

$$
\sup_{t \in [0,T]} \left(1 \vee \|X_t^I\|_{L^2(\mathbb{P};\mathbf{H}_\rho)}\right) \leq \left(1 \vee \|\mathbf{P}_I \xi\|_{L^2(\mathbb{P};\mathbf{H}_\rho)}\right)
$$

$$
\cdot \exp\!\left(T\left[\big\|\mathbf{P}_I \mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \tfrac{1}{2}\big\|\mathbf{P}_I \mathbf{B}|_{\mathbf{H}_\rho}\big\|^2_{\mathrm{Lip}^0(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}\right]\right)
$$

$$
\leq \left(1 \vee \|\xi\|_{L^2(\mathbb{P};\mathbf{H}_\rho)}\right)
$$

$$
\cdot \exp\!\left(T\left[\big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \tfrac{1}{2}\big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|^2_{\mathrm{Lip}^0(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}\right]\right). \tag{3.5}
$$

The proof of Lemma 3.2 is thus completed. $\qquad\square$

Lemma 3.3. *Assume the setting in Section 3.1 and let $X^I\colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_0)$, $I \in \mathcal{P}(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]}\|X_s^I\|_{L^2(\mathbb{P};\mathbf{H}_0)} < \infty$ and $\mathbb{P}$-a.s. that*

$$
X_t^I = \mathrm{e}^{\mathbf{A}t}\mathbf{P}_I \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I \mathbf{F}(X_s^I)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I \mathbf{B}(X_s^I)\,\mathrm{d}W_s. \tag{3.6}
$$

Then it holds for all $I, J \in \mathcal{P}(\mathbb{H})$ that

$$
\sup_{t\in[0,T]} \big\|X_t^I - X_t^J\big\|_{L^2(\mathbb{P};\mathbf{H}_0)} \leq \sqrt{2}\,\mathcal{E}_1\!\left[\sqrt{2}\,T|\mathbf{P}_{I\cap J}\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{2T}|\mathbf{P}_{I\cap J}\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\right]
$$

$$
\cdot \sup_{t\in[0,T]} \big\|\mathbf{P}_{I\setminus J}X_t^I + \mathbf{P}_{J\setminus I}X_t^J\big\|_{L^2(\mathbb{P};\mathbf{H}_0)} < \infty. \tag{3.7}
$$

Proof of Lemma 3.3. Observe that Corollary 3.1 in Jentzen & Kurniawan [24] and Lemma 2.6 imply for all $I, J \in \mathcal{P}(\mathbb{H})$ that

$$
\sup_{t\in[0,T]} \big\|X_t^I - X_t^J\big\|_{L^2(\mathbb{P};\mathbf{H}_0)} \leq \sqrt{2}\,\mathcal{E}_1\!\left[\sqrt{2}\,T|\mathbf{P}_{I\cap J}\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{2T}|\mathbf{P}_{I\cap J}\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\right]
$$

$$
\cdot \sup_{t\in[0,T]} \Bigg\| X_t^I - \left[\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_{I\cap J}\mathbf{F}(X_s^I)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_{I\cap J}\mathbf{B}(X_s^I)\,\mathrm{d}W_s\right]
$$

$$
+ \left[\int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_{I\cap J}\mathbf{F}(X_s^J)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_{I\cap J}\mathbf{B}(X_s^J)\,\mathrm{d}W_s\right] - X_t^J \Bigg\|_{L^2(\mathbb{P};\mathbf{H}_0)}
$$

$$
= \sqrt{2}\,\mathcal{E}_1\!\left[\sqrt{2}\,T|\mathbf{P}_{I\cap J}\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{2T}|\mathbf{P}_{I\cap J}\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\right]
$$

$$
\cdot \sup_{t\in[0,T]} \Bigg\| X_t^I - \mathbf{P}_J\!\left(\mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I \mathbf{F}(X_s^I)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I \mathbf{B}(X_s^I)\,\mathrm{d}W_s\right)
$$

$$
+ \mathbf{P}_I\!\left(\mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J \mathbf{F}(X_s^J)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J \mathbf{B}(X_s^J)\,\mathrm{d}W_s\right) - X_t^J \Bigg\|_{L^2(\mathbb{P};\mathbf{H}_0)}
$$

$$
= \sqrt{2}\,\mathcal{E}_1\!\left[\sqrt{2}\,T|\mathbf{P}_{I\cap J}\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + \sqrt{2T}|\mathbf{P}_{I\cap J}\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\right]
$$

$$
\cdot \sup_{t\in[0,T]} \big\|\mathbf{P}_{I\setminus J}X_t^I - \mathbf{P}_{J\setminus I}X_t^J\big\|_{L^2(\mathbb{P};\mathbf{H}_0)}. \tag{3.8}
$$

This implies (3.7) and thus completes the proof of Lemma 3.3. $\qquad\square$

Remark 3.4. Assume the setting in Section 3.1. Then Remark 2.2 shows that there exist up to modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^{J,x}\colon [0,T] \times \Omega \to \mathbf{P}_J(\mathbf{H}_0)$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $J \in \mathcal{P}(\mathbb{H})$, satisfying that for all $J \in \mathcal{P}(\mathbb{H})$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]}\|X_s^{J,x}\|_{L^2(\mathbb{P};\mathbf{H}_0)} < \infty$ and $\mathbb{P}$-a.s. that

$$
X_t^{J,x} = \mathrm{e}^{\mathbf{A}t}x + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J \mathbf{F}(X_s^{J,x})\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J \mathbf{B}(X_s^{J,x})\,\mathrm{d}W_s. \tag{3.9}
$$

Lemma 3.5. *Assume the setting in Section 3.1, let $X^{J,x}\colon [0,T] \times \Omega \to \mathbf{P}_J(\mathbf{H}_0)$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $J \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $J \in \mathcal{P}_0(\mathbb{H})$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]}\|X_s^{J,x}\|_{L^2(\mathbb{P};\mathbf{P}_J(\mathbf{H}_0))} < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^{J,x} = \mathrm{e}^{\mathbf{A}t}x + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J\mathbf{F}(X_s^{J,x})\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J\mathbf{B}(X_s^{J,x})\,\mathrm{d}W_s, \tag{3.10}$$

let $\varphi \in C_{\mathrm{b}}^2(\mathbf{H}_0,\mathbb{R})$, and let $u^J\colon [0,T] \times \mathbf{P}_J(\mathbf{H}_0) \to \mathbb{R}$, $J \in \mathcal{P}_0(\mathbb{H})$, be the mappings which satisfy for all $J \in \mathcal{P}_0(\mathbb{H})$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ that $u^J(t,x) = \mathbb{E}\big[\varphi\big(X_t^{J,x}\big)\big]$. Then for all $J \in \mathcal{P}_0(\mathbb{H})$ it holds that $u^J \in C^{1,2}([0,T] \times \mathbf{P}_J(\mathbf{H}_0),\mathbb{R})$ and

$$\begin{aligned}
&\sup_{K\in\mathcal{P}_0(\mathbb{H})} \sup_{t\in[0,T]} |u^K(t,\cdot)|_{C_{\mathrm{b}}^1(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})} \\
&\leq |\varphi|_{C_{\mathrm{b}}^1(\mathbf{H}_0,\mathbb{R})} \exp\big(T\big[|\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + \tfrac{1}{2}|\mathbf{B}|^2_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\big]\big) < \infty,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\sup_{K\in\mathcal{P}_0(\mathbb{H})} \sup_{t\in[0,T]} |u^K(t,\cdot)|_{C_{\mathrm{b}}^2(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})} \leq \|\varphi\|_{C_{\mathrm{b}}^2(\mathbf{H}_0,\mathbb{R})} \Big(1 \vee \big[T\big(C_{\mathbf{F}}^2 + 2C_{\mathbf{B}}^2\big)\big]^{1/2}\Big) \\
&\cdot \exp\big(T\big[\tfrac{1}{2} + 3|\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + 4|\mathbf{B}|^2_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}\big]\big) < \infty.
\end{aligned} \tag{3.12}$$

Proof of Lemma 3.5. The assumptions that $\mathbf{F}|_{\bigcap_{r\in\mathbb{R}}\mathbf{H}_r} \in C_{\mathrm{b}}^2(\bigcap_{r\in\mathbb{R}}\mathbf{H}_r,\mathbf{H}_0)$ and $\mathbf{B}|_{\bigcap_{r\in\mathbb{R}}\mathbf{H}_r} \in C_{\mathrm{b}}^2(\bigcap_{r\in\mathbb{R}} \mathbf{H}_r, L_2(U,\mathbf{H}_0))$ and the fact that for all $J \in \mathcal{P}_0(\mathbb{H})$ it holds that $\mathbf{P}_J(\mathbf{H}_0)$ is a finite-dimensional $\mathbb{R}$-vector space ensure that $\mathbf{P}_J\mathbf{F}|_{\mathbf{P}_J(\mathbf{H}_0)} \in C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0),\mathbf{P}_J(\mathbf{H}_0))$ and $\mathbf{P}_J\mathbf{B}|_{\mathbf{P}_J(\mathbf{H}_0)} \in C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0), L_2(U,\mathbf{P}_J(\mathbf{H}_0)))$. Lemma 2.3 and Lemma 2.6 then prove for all $J \in \mathcal{P}_0(\mathbb{H})$ that

$$\begin{aligned}
&\sup_{t\in[0,T]} |u^J(t,\cdot)|_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})} \leq \big|\varphi|_{\mathbf{P}_J(\mathbf{H}_0)}\big|_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})} \\
&\cdot \exp\big(T\big[\big|\mathbf{P}_J\mathbf{F}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbf{P}_J(\mathbf{H}_0))} + \tfrac{1}{2}\big|\mathbf{P}_J\mathbf{B}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|^2_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),L_2(U,\mathbf{P}_J(\mathbf{H}_0)))}\big]\big)
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]} |u^J(t,\cdot)|_{C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})} \leq \big\|\varphi|_{\mathbf{P}_J(\mathbf{H}_0)}\big\|_{C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})} \\
&\cdot \Big(1 \vee \big[T\big(\big|\mathbf{P}_J\mathbf{F}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|^2_{C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0),\mathbf{P}_J(\mathbf{H}_0))} + 2\big|\mathbf{P}_J\mathbf{B}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|^2_{C_{\mathrm{b}}^2(\mathbf{P}_J(\mathbf{H}_0),L_2(U,\mathbf{P}_J(\mathbf{H}_0)))}\big)\big]^{1/2}\Big) \\
&\cdot \exp\big(T\big[\tfrac{1}{2} + 3\big|\mathbf{P}_J\mathbf{F}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbf{P}_J(\mathbf{H}_0))} + 4\big|\mathbf{P}_J\mathbf{B}|_{\mathbf{P}_J(\mathbf{H}_0)}\big|^2_{C_{\mathrm{b}}^1(\mathbf{P}_J(\mathbf{H}_0),L_2(U,\mathbf{P}_J(\mathbf{H}_0)))}\big]\big) < \infty.
\end{aligned} \tag{3.14}$$

This implies (3.11) and (3.12) and thus completes the proof of Lemma 3.5. $\qquad\square$

In the proof of the main result of this article, Theorem 3.7 below, we use the following elementary and well-known lemma.

Lemma 3.6. *Let $p \in [0,\infty)$, let $\mathcal{J}_n$, $n \in \mathbb{N}_0$, be sets such that $\bigcup_{n\in\mathbb{N}} \mathcal{J}_n = \mathcal{J}_0$ and such that for all $n \in \mathbb{N}$ it holds that $\mathcal{J}_n \subseteq \mathcal{J}_{n+1}$, and let $g\colon \mathcal{J}_0 \to (0,\infty)$ be a mapping with the property that $\sum_{h\in\mathcal{J}_0}(g_h)^p < \infty$. Then*

$$\lim_{n\to\infty} \sup\big(\{g_h\colon h \in \mathcal{J}_0 \setminus \mathcal{J}_n\} \cup \{0\}\big) = 0. \tag{3.15}$$

Proof of Lemma 3.6. Without loss of generality we assume that $p \in (0,\infty)$ (otherwise (3.15) is clear). Then observe that for all $n \in \mathbb{N}$ it holds that

$$\sum_{h\in\mathcal{J}_n}(g_h)^p + \big[\sup\big(\{g_h\colon h \in \mathcal{J}_0 \setminus \mathcal{J}_n\} \cup \{0\}\big)\big]^p \leq \sum_{h\in\mathcal{J}_n}(g_h)^p + \sum_{h\in\mathcal{J}_0\setminus\mathcal{J}_n}(g_h)^p = \sum_{h\in\mathcal{J}_0}(g_h)^p < \infty. \tag{3.16}$$

Moreover, note that Lebesgue's theorem of dominated convergence proves that

$$\lim_{n\to\infty} \sum_{h\in\mathcal{J}_n}(g_h)^p = \sum_{h\in\mathcal{J}_0}(g_h)^p. \tag{3.17}$$

Letting $n \to \infty$ in (3.16) and combining this with (3.17) complete the proof of Lemma 3.6. $\quad\square$

Theorem 3.7. *Assume the setting in Section 3.1, let $X^I\colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, and $X^{J,x}\colon [0,T] \times \Omega \to \mathbf{P}_J(\mathbf{H}_0)$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $J \in \mathcal{P}_0(\mathbb{H})$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{H})$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $t \in [0,T]$ it holds that $\sup_{s\in[0,T]}\big(\|X_s^I\|_{L^2(\mathbb{P};\mathbf{H}_\rho)} + \|X_s^{J,x}\|_{L^2(\mathbb{P};\mathbf{H}_0)}\big) < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^I = \mathrm{e}^{\mathbf{A}t}\mathbf{P}_I\xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I\mathbf{F}(X_s^I)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I\mathbf{B}(X_s^I)\,\mathrm{d}W_s, \tag{3.18}$$

$$X_t^{J,x} = \mathrm{e}^{\mathbf{A}t}x + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J\mathbf{F}(X_s^{J,x})\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_J\mathbf{B}(X_s^{J,x})\,\mathrm{d}W_s, \tag{3.19}$$

let $\varphi \in C_\mathrm{b}^2(\mathbf{H}_0, \mathbb{R})$, and let $u^J\colon [0,T] \times \mathbf{P}_J(\mathbf{H}_0) \to \mathbb{R}$, $J \in \mathcal{P}_0(\mathbb{H})$, be the mappings which satisfy for all $J \in \mathcal{P}_0(\mathbb{H})$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ that $u^J(t,x) = \mathbb{E}\big[\varphi(X_t^{J,x})\big]$. Then it holds for all $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ that

$$\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big|$$
$$\leq \left(\left[\sup_{K\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\big|u^K(t,\cdot)\big|_{C_\mathrm{b}^1(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})}\right]\left[\|\xi\|_{L^1(\mathbb{P};\mathbf{H}_{2(\gamma-\beta)})} + \sup_{K\in\mathcal{P}_0(\mathbb{H})}\int_0^T \mathbb{E}\big[\big\|\mathbf{F}(X_s^K)\big\|_{\mathbf{H}_{2(\gamma-\beta)}}\big]\,\mathrm{d}s\right]\right.$$
$$\left. + \|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^2\left[\sup_{K\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\big|u^K(t,\cdot)\big|_{C_\mathrm{b}^2(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})}\right]\sup_{K\in\mathcal{P}_0(\mathbb{H})}\int_0^T \mathbb{E}\big[\big\|\mathbf{B}(X_s^K)\big\|_{L(U,\mathbf{H}_\gamma)}^2\big]\,\mathrm{d}s\right)$$
$$\cdot \left[\inf_{h\in\mathbb{H}\setminus I}|\lambda_h|\right]^{\beta-\gamma} < \infty. \tag{3.20}$$

Proof of Theorem 3.7. Throughout this proof let $v^J, v_{1,0}^J\colon [0,T] \times \mathbf{P}_J(\mathbf{H}_0) \to \mathbb{R}$, $J \in \mathcal{P}_0(\mathbb{H})$, and $v_{0,\ell}^J\colon [0,T] \times \mathbf{P}_J(\mathbf{H}_0) \to L^{(\ell)}(\mathbf{P}_J(\mathbf{H}_0), \mathbb{R})$, $\ell \in \{1,2\}$, $J \in \mathcal{P}_0(\mathbb{H})$, be the mappings such that for all $J \in \mathcal{P}_0(\mathbb{H})$, $(k,\ell) \in \{(1,0),(0,1),(0,2)\}$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ it holds that $v^J(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^{J,x})\big]$ and $v_{k,\ell}^J(t,x) = \big(\frac{\partial^{k+\ell}}{\partial t^k \partial x^\ell}v^J\big)(t,x)$. Note that for all $J \in \mathcal{P}_0(\mathbb{H})$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ it holds that $v^J(t,x) = u^J(T-t,x)$. Next observe for all $J \in \mathcal{P}_0(\mathbb{H})$, $x \in \mathbf{P}_J(\mathbf{H}_0)$ that

$$\varphi(x) = \mathbb{E}[\varphi(x)] = u^J(0,x) = v^J(T,x). \tag{3.21}$$

Moreover, note for all $J \in \mathcal{P}_0(\mathbb{H})$ that

$$\mathbb{E}\big[\varphi(X_T^J)\big] = \mathbb{E}\big[u^J(T, X_0^J)\big] = \mathbb{E}\big[v^J(0, X_0^J)\big]. \tag{3.22}$$

Combining (3.21) and (3.22) shows for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$\big|\mathbb{E}\big[\varphi(X_T^J)\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big| = \big|\mathbb{E}\big[\varphi(X_T^I)\big] - \mathbb{E}\big[\varphi(X_T^J)\big]\big|$$
$$= \big|\mathbb{E}\big[v^J(T, X_T^I)\big] - \mathbb{E}\big[v^J(0, X_0^J)\big]\big| \tag{3.23}$$
$$\leq \big|\mathbb{E}\big[v^J(T, X_T^I)\big] - \mathbb{E}\big[v^J(0, X_0^I)\big]\big| + \big|\mathbb{E}\big[v^J(0, X_0^I)\big] - \mathbb{E}\big[v^J(0, X_0^J)\big]\big|.$$

In a first step we establish an estimate for the second summand on the right hand side of (3.23). For this observe for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$\big|\mathbb{E}\big[v^J(0, X_0^I)\big] - \mathbb{E}\big[v^J(0, X_0^J)\big]\big| = \left|\mathbb{E}\left[\int_0^1 v_{0,1}^J\big(0, X_0^I + \tau(X_0^J - X_0^I)\big)(X_0^J - X_0^I)\,\mathrm{d}\tau\right]\right|$$
$$\leq \big|u^J(T,\cdot)\big|_{C_\mathrm{b}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})}\mathbb{E}\big[\|X_0^J - X_0^I\|_{\mathbf{H}_0}\big] \tag{3.24}$$
$$= \big|u^J(T,\cdot)\big|_{C_\mathrm{b}^1(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})}\big\|\mathbf{P}_I(X_0^J) - \mathbf{P}_J(X_0^J)\big\|_{L^1(\mathbb{P};\mathbf{H}_0)}.$$

In addition, it holds for all $x \in \mathbf{H}_{2(\gamma-\beta)}$, $I, J \in \mathcal{P}(\mathbb{H})$ with $I \neq J$ that

$$
\begin{aligned}
\|\mathbf{P}_I(x) - \mathbf{P}_J(x)\|_{\mathbf{H}_0} &\leq \|\mathbf{\Lambda}^{2(\beta-\gamma)}\mathbf{P}_{I\triangle J}\|_{L(\mathbf{H}_0)}\|\mathbf{P}_{I\triangle J}(x)\|_{\mathbf{H}_{2(\gamma-\beta)}} \\
&= \left[\inf_{h\in I\triangle J}|\lambda_h|\right]^{\beta-\gamma}\|\mathbf{P}_{I\triangle J}(x)\|_{\mathbf{H}_{2(\gamma-\beta)}} \leq \left[\inf_{h\in I\triangle J}|\lambda_h|\right]^{\beta-\gamma}\|x\|_{\mathbf{H}_{2(\gamma-\beta)}}
\end{aligned}
\tag{3.25}
$$

(cf., e.g., Proposition 8.1.4 in [23]). Putting (3.24) and (3.25) together proves for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{\mathbb{H}\}$ that

$$
\begin{aligned}
\left|\mathbb{E}\big[v^J\big(0, X_0^I\big)\big] - \mathbb{E}\big[v^J\big(0, X_0^J\big)\big]\right| &\leq \left[\sup_{K\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\big|u^K(t,\cdot)\big|_{C_b^1(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})}\right]\|\xi\|_{L^1(\mathbb{P};\mathbf{H}_{2(\gamma-\beta)})} \\
&\quad \cdot \left[\inf_{h\in\mathbb{H}\setminus I}|\lambda_h|\right]^{\beta-\gamma} < \infty.
\end{aligned}
\tag{3.26}
$$

Inequality (3.26) provides an estimate for the second summand on the right hand side of (3.23). In a second step we establish an estimate for the fist summand on the right hand side of (3.23). The chain rule and Lemma 2.3 show that for all $J \in \mathcal{P}_0(\mathbb{H})$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ it holds that

$$
v_{1,0}^J(t,x) = -v_{0,1}^J(t,x)\big[\mathbf{A}x + \mathbf{P}_J\mathbf{F}(x)\big] - \tfrac{1}{2}\sum_{u\in\mathbb{U}}v_{0,2}^J(t,x)(\mathbf{P}_J\mathbf{B}(x)u, \mathbf{P}_J\mathbf{B}(x)u).
\tag{3.27}
$$

The standard Itô formula and (3.27) prove for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$
\begin{aligned}
\mathbb{E}\big[v^J\big(T, X_T^I\big)\big] - \mathbb{E}\big[v^J\big(0, X_0^I\big)\big] &= \int_0^T \mathbb{E}\big[v_{1,0}^J\big(s, X_s^I\big)\big]\,\mathrm{d}s + \int_0^T \mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\mathbf{A}X_s^I\big]\,\mathrm{d}s \\
&\quad + \int_0^T \mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\mathbf{P}_I\mathbf{F}(X_s^I)\big]\,\mathrm{d}s + \tfrac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T \mathbb{E}\big[v_{0,2}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{B}(X_s^I)u, \mathbf{P}_I\mathbf{B}(X_s^I)u\big)\big]\,\mathrm{d}s \\
&= \int_0^T \mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\mathbf{P}_I\mathbf{F}(X_s^I)\big]\,\mathrm{d}s - \int_0^T \mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\mathbf{P}_J\mathbf{F}(X_s^I)\big]\,\mathrm{d}s \\
&\quad + \tfrac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T \Big(\mathbb{E}\big[v_{0,2}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{B}(X_s^I)u, \mathbf{P}_I\mathbf{B}(X_s^I)u\big)\big] \\
&\quad\quad - \mathbb{E}\big[v_{0,2}^J\big(s, X_s^I\big)\big(\mathbf{P}_J\mathbf{B}(X_s^I)u, \mathbf{P}_J\mathbf{B}(X_s^I)u\big)\big]\Big)\,\mathrm{d}s.
\end{aligned}
\tag{3.28}
$$

This shows for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$
\begin{aligned}
\left|\mathbb{E}\big[v^J\big(T, X_T^I\big)\big] - \mathbb{E}\big[v^J\big(0, X_0^I\big)\big]\right| &\leq \int_0^T \left|\mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{F}(X_s^I) - \mathbf{P}_J\mathbf{F}(X_s^I)\big)\big]\right|\,\mathrm{d}s \\
&\quad + \left|\tfrac{1}{2}\sum_{u\in\mathbb{U}}\int_0^T \mathbb{E}\big[v_{0,2}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{B}(X_s^I)u + \mathbf{P}_J\mathbf{B}(X_s^I)u, \mathbf{P}_I\mathbf{B}(X_s^I)u - \mathbf{P}_J\mathbf{B}(X_s^I)u\big)\big]\,\mathrm{d}s\right|.
\end{aligned}
\tag{3.29}
$$

Inequality (3.25), Lemma 3.5, and Lemma 3.2 thus prove for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{J\}$ that

$$
\begin{aligned}
&\int_0^T \left|\mathbb{E}\big[v_{0,1}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{F}(X_s^I) - \mathbf{P}_J\mathbf{F}(X_s^I)\big)\big]\right|\,\mathrm{d}s \\
&\leq \int_0^T \mathbb{E}\big[\big|v_{0,1}^J\big(s, X_s^I\big)\big(\mathbf{P}_I\mathbf{F}(X_s^I) - \mathbf{P}_J\mathbf{F}(X_s^I)\big)\big|\big]\,\mathrm{d}s \\
&\leq \sup_{t\in[0,T]}\big|u^J(t,\cdot)\big|_{C_b^1(\mathbf{P}_J(\mathbf{H}_0),\mathbb{R})}\int_0^T \mathbb{E}\big[\big\|\mathbf{P}_I\mathbf{F}(X_s^I) - \mathbf{P}_J\mathbf{F}(X_s^I)\big\|_{\mathbf{H}_0}\big]\,\mathrm{d}s \\
&\leq \left[\sup_{K\in\mathcal{P}_0(\mathbb{H})}\sup_{t\in[0,T]}\big|u^K(t,\cdot)\big|_{C_b^1(\mathbf{P}_K(\mathbf{H}_0),\mathbb{R})}\right]\sup_{K\in\mathcal{P}_0(\mathbb{H})}\int_0^T \mathbb{E}\big[\big\|\mathbf{F}(X_s^K)\big\|_{\mathbf{H}_{2(\gamma-\beta)}}\big]\,\mathrm{d}s \\
&\quad \cdot \left[\inf_{h\in J\setminus I}|\lambda_h|\right]^{\beta-\gamma} < \infty.
\end{aligned}
\tag{3.30}
$$

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This estimates the first summand on the right hand side of (3.29). Next we consider the second summand on the right hand side of (3.29). Observe for all $J \in \mathcal{P}_0(\mathbb{H})$, $s \in [0,T]$, $I \in \mathcal{P}(J)$, $\omega \in \Omega$ that $v_{0,2}^J(s, X_s^I(\omega)) \in L^{(2)}(\mathbf{P}_J(\mathbf{H}_0), \mathbb{R})$. Let $R_{I,J,s} \colon \Omega \to L(\mathbf{P}_J(\mathbf{H}_0))$, $I \in \mathcal{P}(J)$, $J \in \mathcal{P}_0(\mathbb{H})$, $s \in [0,T]$, be the mappings with the property that for all $s \in [0,T]$, $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$, $\omega \in \Omega$ it holds that

$$R_{I,J,s}(\omega) = \mathcal{J}_{v_{0,2}^J(s,X_s^I(\omega))}^{\mathbf{P}_J(\mathbf{H}_0)}. \tag{3.31}$$

Then note that for all $s \in [0,T]$, $J \in \mathcal{P}_0(\mathbb{H})$, $y_1, y_2 \in \mathbf{P}_J(\mathbf{H}_0)$, $I \in \mathcal{P}(J)$ it holds that $v_{0,2}^J(s, X_s^I)(y_1, y_2) = \langle y_1, R_{I,J,s} y_2 \rangle_{\mathbf{H}_0}$. The Hölder inequality for Schatten norms therefore implies for all $s \in [0,T]$, $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$
\begin{aligned}
&\left| \sum_{u \in \mathbb{U}} \mathbb{E}\left[v_{0,2}^J(s, X_s^I)\left((\mathbf{P}_I + \mathbf{P}_J)\mathbf{B}(X_s^I)u, (\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I)u \right) \right] \right| \\
&= \left| \mathbb{E}\left[\sum_{u \in \mathbb{U}} \langle (\mathbf{P}_I + \mathbf{P}_J)\mathbf{B}(X_s^I)u, R_{I,J,s}(\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I)u \rangle_{\mathbf{H}_0} \right] \right| \\
&= \left| \mathbb{E}\left[\text{trace}_U(\mathbf{B}(X_s^I)^\star (\mathbf{P}_I + \mathbf{P}_J)^\star R_{I,J,s}(\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I)) \right] \right| \\
&\leq \mathbb{E}\left[\left\| \mathbf{B}(X_s^I)^\star (\mathbf{P}_I + \mathbf{P}_J)^\star R_{I,J,s}(\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I) \right\|_{L_1(U)} \right] \\
&\leq \mathbb{E}\left[\left\| \mathbf{B}(X_s^I)^\star (\mathbf{P}_I + \mathbf{P}_J)^\star \right\|_{L_{(2\beta)/\gamma}(\mathbf{H}_0, U)} \|R_{I,J,s}\|_{L(\mathbf{P}_J(\mathbf{H}_0))} \left\| (\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I) \right\|_{L_{(2\beta)/(2\beta-\gamma)}(U, \mathbf{H}_0)} \right].
\end{aligned}
\tag{3.32}
$$

Moreover, observe for all $s \in [0,T]$, $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{J\}$ that

$$
\begin{aligned}
\left\| \mathbf{B}(X_s^I)^\star (\mathbf{P}_I + \mathbf{P}_J)^\star \right\|_{L_{(2\beta)/\gamma}(\mathbf{H}_0, U)} &= \left\| \mathbf{B}(X_s^I)^\star \mathbf{\Lambda}^\gamma \mathbf{\Lambda}^{-\gamma}(\mathbf{P}_I + \mathbf{P}_J)^\star \right\|_{L_{(2\beta)/\gamma}(\mathbf{H}_0, U)} \\
&\leq \left\| \mathbf{B}(X_s^I)^\star \mathbf{\Lambda}^\gamma \right\|_{L(\mathbf{H}_0, U)} \|\mathbf{\Lambda}^{-\gamma}\|_{L_{(2\beta)/\gamma}(\mathbf{H}_0)} \|(\mathbf{P}_I + \mathbf{P}_J)^\star\|_{L(\mathbf{H}_0)} \\
&= \left\| \mathbf{B}(X_s^I) \right\|_{L(U, \mathbf{H}_\gamma)} \|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^{\gamma/\beta} \|\mathbf{P}_I + \mathbf{P}_J\|_{L(\mathbf{H}_0)} \leq 2 \left\| \mathbf{B}(X_s^I) \right\|_{L(U, \mathbf{H}_\gamma)} \|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^{\gamma/\beta} < \infty
\end{aligned}
\tag{3.33}
$$

and

$$
\begin{aligned}
\left\| (\mathbf{P}_I - \mathbf{P}_J)\mathbf{B}(X_s^I) \right\|_{L_{(2\beta)/(2\beta-\gamma)}(U, \mathbf{H}_0)} &\leq \|\mathbf{P}_I - \mathbf{P}_J\|_{L_{(2\beta)/(2\beta-\gamma)}(\mathbf{H}_\gamma, \mathbf{H}_0)} \left\| \mathbf{B}(X_s^I) \right\|_{L(U, \mathbf{H}_\gamma)} \\
&\leq \|(\mathbf{P}_I - \mathbf{P}_J)\mathbf{\Lambda}^{2(\beta-\gamma)}\|_{L(\mathbf{H}_0)} \|\mathbf{\Lambda}^{2(\gamma-\beta)}\|_{L_{(2\beta)/(2\beta-\gamma)}(\mathbf{H}_\gamma, \mathbf{H}_0)} \left\| \mathbf{B}(X_s^I) \right\|_{L(U, \mathbf{H}_\gamma)} \\
&= \left[\inf_{h \in J \setminus I} |\lambda_h| \right]^{\beta-\gamma} \|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^{(2\beta-\gamma)/\beta} \left\| \mathbf{B}(X_s^I) \right\|_{L(U, \mathbf{H}_\gamma)}.
\end{aligned}
\tag{3.34}
$$

In addition, Lemma 3.5 implies for all $s \in [0,T]$, $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J)$ that

$$\|R_{I,J,s}\|_{L(\mathbf{P}_J(\mathbf{H}_0))} \leq \sup_{t \in [0,T]} \left| u^J(t, \cdot) \right|_{C_b^2(\mathbf{P}_J(\mathbf{H}_0), \mathbb{R})} < \infty. \tag{3.35}$$

Inequalities (3.32)–(3.35), Lemma 3.2, and Lemma 3.5 prove for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{J\}$ that

$$
\begin{aligned}
&\left| \frac{1}{2} \sum_{u \in \mathbb{U}} \int_0^T \mathbb{E}\left[v_{0,2}^J(s, X_s^I)\left(\mathbf{P}_I \mathbf{B}(X_s^I)u + \mathbf{P}_J \mathbf{B}(X_s^I)u, \mathbf{P}_I \mathbf{B}(X_s^I)u - \mathbf{P}_J \mathbf{B}(X_s^I)u \right) \right] ds \right| \\
&\leq \|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^2 \left[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \left| u^K(t, \cdot) \right|_{C_b^2(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \right] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\left[\left\| \mathbf{B}(X_s^K) \right\|_{L(U, \mathbf{H}_\gamma)}^2 \right] ds \\
&\quad \cdot \left[\inf_{h \in J \setminus I} |\lambda_h| \right]^{\beta-\gamma} < \infty.
\end{aligned}
\tag{3.36}
$$

Combining this with (3.29) and (3.30) ensures for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{\mathbb{H}\}$ that

$$
\begin{aligned}
& \left| \mathbb{E}\big[v^J\big(T, X_T^I\big) \big] - \mathbb{E}\big[v^J\big(0, X_0^I\big) \big] \right| \\
& \leq \Bigg(\bigg[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^1(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \bigg] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{F}(X_s^K) \big\|_{\mathbf{H}_{2(\gamma - \beta)}} \big] \, \mathrm{d}s \\
& \quad + \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)}^2 \bigg[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^2(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \bigg] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{B}(X_s^K) \big\|_{L(U, \mathbf{H}_\gamma)}^2 \big] \, \mathrm{d}s \Bigg) \\
& \quad \cdot \bigg[\inf_{h \in \mathbb{H} \setminus I} |\lambda_h| \bigg]^{\beta - \gamma} < \infty.
\end{aligned}
\tag{3.37}
$$

This constitutes an estimate for the first summand on the right hand side of (3.23). Inequalities (3.37), (3.23), and (3.26) show for all $J \in \mathcal{P}_0(\mathbb{H})$, $I \in \mathcal{P}(J) \setminus \{\mathbb{H}\}$ that

$$
\begin{aligned}
& \left| \mathbb{E}\big[\varphi\big(X_T^J \big) \big] - \mathbb{E}\big[\varphi\big(X_T^I \big) \big] \right| \\
& \leq \Bigg(\bigg[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^1(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \bigg] \bigg[\| \xi \|_{L^1(\mathbb{P}; \mathbf{H}_{2(\gamma - \beta)})} + \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{F}(X_s^K) \big\|_{\mathbf{H}_{2(\gamma - \beta)}} \big] \, \mathrm{d}s \bigg] \\
& \quad + \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)}^2 \bigg[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^2(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \bigg] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{B}(X_s^K) \big\|_{L(U, \mathbf{H}_\gamma)}^2 \big] \, \mathrm{d}s \Bigg) \\
& \quad \cdot \bigg[\inf_{h \in \mathbb{H} \setminus I} |\lambda_h| \bigg]^{\beta - \gamma} < \infty.
\end{aligned}
\tag{3.38}
$$

In a third step Lemma 3.3, Lemma 2.6, Minkowski's integral inequality, the Burkholder-Davis-Gundy inequality (see, e.g., Lemma 7.7 in Da Prato & Zabczyk [13]), and the Cauchy-Schwarz inequality imply for all $n \in \mathbb{N}$, $\{J_k\}_{k \in \mathbb{N}_0} \subseteq \mathcal{P}(\mathbb{H})$ with $\bigcup_{k \in \mathbb{N}} J_k = J_0$ and $\forall k \in \mathbb{N}$: $J_k \subseteq J_{k+1} \in \mathcal{P}_0(\mathbb{H})$ that

$$
\begin{aligned}
& \sup_{t \in [0,T]} \big\| X_t^{J_0} - X_t^{J_n} \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)} \\
& \leq \sqrt{2} \mathcal{E}_1 \big[\sqrt{2} T | \mathbf{P}_{J_n} \mathbf{F} |_{\mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)} + \sqrt{2T} | \mathbf{P}_{J_n} \mathbf{B} |_{\mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))} \big] \sup_{t \in [0,T]} \big\| \mathbf{P}_{J_0 \setminus J_n} X_t^{J_0} \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)} \\
& \leq \sqrt{2} \mathcal{E}_1 \big[\sqrt{2} T | \mathbf{F} |_{\mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)} + \sqrt{2T} | \mathbf{B} |_{\mathrm{Lip}^0(\mathbf{H}_0, L_2(U, \mathbf{H}_0))} \big] \\
& \quad \cdot \Bigg(\| \mathbf{P}_{J_0 \setminus J_n} \xi \|_{L^2(\mathbb{P}; \mathbf{H}_0)} + \bigg[T \int_0^T \big\| \mathbf{P}_{J_0 \setminus J_n} \mathbf{F}(X_s^{J_0}) \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)}^2 \, \mathrm{d}s \bigg]^{1/2} \\
& \quad + \bigg[\int_0^T \big\| \mathbf{P}_{J_0 \setminus J_n} \mathbf{B}(X_s^{J_0}) \big\|_{L^2(\mathbb{P}; L_2(U, \mathbf{H}_0))}^2 \, \mathrm{d}s \bigg]^{1/2} \Bigg).
\end{aligned}
\tag{3.39}
$$

Therefore, Lebesgue's theorem of dominated convergence proves for all $\{J_k\}_{k \in \mathbb{N}_0} \subseteq \mathcal{P}(\mathbb{H})$ with $\bigcup_{k \in \mathbb{N}} J_k = J_0$ and $\forall k \in \mathbb{N}$: $J_k \subseteq J_{k+1} \in \mathcal{P}_0(\mathbb{H})$ that

$$
\lim_{n \to \infty} \sup_{t \in [0,T]} \big\| X_t^{J_0} - X_t^{J_n} \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)} = 0.
\tag{3.40}
$$

Moreover, observe that (3.38) ensures for all $n \in \mathbb{N}$, $I \in \mathcal{P}_0(\mathbb{H}) \setminus \{\mathbb{H}\}$, $\{J_k\}_{k \in \mathbb{N}} \subseteq \mathcal{P}_0(\mathbb{H})$ with

$\bigcup_{k \in \mathbb{N}} J_k = \mathbb{H}$ and $\forall k \in \mathbb{N}: I \subseteq J_k \subseteq J_{k+1}$ that

$$
\begin{aligned}
\left| \mathbb{E}\big[\varphi\big(X_T^{\mathbb{H}} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^I \big) \big] \right| &\leq \left| \mathbb{E}\big[\varphi\big(X_T^{\mathbb{H}} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^{J_n} \big) \big] \right| + \left| \mathbb{E}\big[\varphi\big(X_T^{J_n} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^I \big) \big] \right| \\
&\leq |\varphi|_{\mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)} \big\| X_T^{\mathbb{H}} - X_T^{J_n} \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)} \\
&\quad + \Bigg(\Big[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^1(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \Big] \Big[\| \xi \|_{L^1(\mathbb{P}; \mathbf{H}_{2(\gamma - \beta)})} + \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{F}(X_s^K) \big\|_{\mathbf{H}_{2(\gamma - \beta)}} \big] \, \mathrm{d}s \Big] \\
&\quad + \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)}^2 \Big[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^2(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \Big] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{B}(X_s^K) \big\|_{L(U, \mathbf{H}_\gamma)}^2 \big] \, \mathrm{d}s \Bigg) \\
&\quad \cdot \Big[\inf_{h \in \mathbb{H} \setminus I} |\lambda_h| \Big]^{\beta - \gamma}.
\end{aligned}
\tag{3.41}
$$

Letting $n \to \infty$ in (3.41) and (3.40) complete the proof of Theorem 3.7 in the case that $I \in \mathcal{P}_0(\mathbb{H}) \setminus \{\mathbb{H}\}$. In a last step we prove the remaining cases. Estimate (3.41) ensures for all $n \in \mathbb{N}$, $I_0 \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$, $\{I_k\}_{k \in \mathbb{N}} \subseteq \mathcal{P}_0(I_0)$ with $\bigcup_{k \in \mathbb{N}} I_k = I_0$ and $\forall k \in \mathbb{N}: I_k \subseteq I_{k+1}$ that

$$
\begin{aligned}
\left| \mathbb{E}\big[\varphi\big(X_T^{\mathbb{H}} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^{I_0} \big) \big] \right| &\leq \left| \mathbb{E}\big[\varphi\big(X_T^{\mathbb{H}} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^{I_n} \big) \big] \right| + \left| \mathbb{E}\big[\varphi\big(X_T^{I_0} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^{I_n} \big) \big] \right| \\
&\leq \Bigg(\Big[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^1(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \Big] \Big[\| \xi \|_{L^1(\mathbb{P}; \mathbf{H}_{2(\gamma - \beta)})} + \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{F}(X_s^K) \big\|_{\mathbf{H}_{2(\gamma - \beta)}} \big] \, \mathrm{d}s \Big] \\
&\quad + \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)}^2 \Big[\sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^2(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \Big] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \int_0^T \mathbb{E}\big[\big\| \mathbf{B}(X_s^K) \big\|_{L(U, \mathbf{H}_\gamma)}^2 \big] \, \mathrm{d}s \Bigg) \\
&\quad \cdot \Big[\inf_{h \in \mathbb{H} \setminus I} |\lambda_h| \Big]^{\beta - \gamma} + |\varphi|_{\mathrm{Lip}^0(\mathbf{H}_0, \mathbf{H}_0)} \big\| X_T^{I_0} - X_T^{I_n} \big\|_{L^2(\mathbb{P}; \mathbf{H}_0)}.
\end{aligned}
\tag{3.42}
$$

Equation (3.40) and Lemma 3.6 thus complete the proof of Theorem 3.7. $\qquad \square$

The next corollary is a direct consequence of Theorem 3.7 and Lemma 3.2.

Corollary 3.8. *Assume the setting in Section 3.1, let* $X^I \colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, *and* $X^{J,x} \colon [0,T] \times \Omega \to \mathbf{P}_J(\mathbf{H}_0)$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $J \in \mathcal{P}_0(\mathbb{H})$, *be* $(\mathcal{F}_t)_{t \in [0,T]}$-*predictable stochastic processes such that for all* $I \in \mathcal{P}(\mathbb{H})$, $J \in \mathcal{P}_0(\mathbb{H})$, $x \in \mathbf{P}_J(\mathbf{H}_0)$, $t \in [0,T]$ *it holds that* $\sup_{s \in [0,T]} \big(\| X_s^I \|_{L^2(\mathbb{P}; \mathbf{H}_\rho)} + \| X_s^{J,x} \|_{L^2(\mathbb{P}; \mathbf{H}_0)} \big) < \infty$ *and* $\mathbb{P}$-*a.s. that*

$$
X_t^I = \mathrm{e}^{\mathbf{A}t} \mathbf{P}_I \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{F}(X_s^I) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_I \mathbf{B}(X_s^I) \, \mathrm{d}W_s,
\tag{3.43}
$$

$$
X_t^{J,x} = \mathrm{e}^{\mathbf{A}t} x + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_J \mathbf{F}(X_s^{J,x}) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_J \mathbf{B}(X_s^{J,x}) \, \mathrm{d}W_s,
\tag{3.44}
$$

let $\varphi \in C_{\mathrm{b}}^2(\mathbf{H}_0, \mathbb{R})$, *and let* $u^J \colon [0,T] \times \mathbf{P}_J(\mathbf{H}_0) \to \mathbb{R}$, $J \in \mathcal{P}_0(\mathbb{H})$, *be the mappings which satisfy for all* $J \in \mathcal{P}_0(\mathbb{H})$, $(t,x) \in [0,T] \times \mathbf{P}_J(\mathbf{H}_0)$ *that* $u^J(t,x) = \mathbb{E}\big[\varphi\big(X_t^{J,x} \big) \big]$. *Then it holds for all* $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ *that*

$$
\begin{aligned}
\big| \mathbb{E}\big[& \varphi\big(X_T^{\mathbb{H}} \big) \big] - \mathbb{E}\big[\varphi\big(X_T^I \big) \big] \big| \\
&\leq \Big[\max_{i \in \{1,2\}} \sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big| u^K(t, \cdot) \big|_{C_{\mathrm{b}}^i(\mathbf{P}_K(\mathbf{H}_0), \mathbb{R})} \Big] \sup_{K \in \mathcal{P}_0(\mathbb{H})} \sup_{t \in [0,T]} \big(1 \vee \mathbb{E}\big[\| X_t^K \|_{\mathbf{H}_\rho}^2 \big] \big) \\
&\quad \cdot \Big(\| \xi \|_{L^1(\mathbb{P}; \mathbf{H}_{2(\gamma - \beta)})} + T \big\| \mathbf{F}|_{\mathbf{H}_\rho} \big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, \mathbf{H}_{2(\gamma - \beta)})} + T \| \mathbf{\Lambda}^{-\beta} \|_{L_2(\mathbf{H}_0)}^2 \big\| \mathbf{B}|_{\mathbf{H}_\rho} \big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho, L(U, \mathbf{H}_\gamma))}^2 \Big) \\
&\quad \cdot \Big[\inf_{h \in \mathbb{H} \setminus I} |\lambda_h| \Big]^{\beta - \gamma} < \infty.
\end{aligned}
\tag{3.45}
$$

The last result in this section, Corollary 3.9 below, follows immediately from Corollary 3.8, Lemma 3.5, and Lemma 3.2.

Corollary 3.9. *Assume the setting in Section 3.1 and let $X^I \colon [0,T] \times \Omega \to \mathbf{P}_I(\mathbf{H}_\rho)$, $I \in \mathcal{P}(\mathbb{H})$, be $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic processes such that for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ it holds that $\sup_{s \in [0,T]} \|X_s^I\|_{L^2(\mathbb{P};\mathbf{H}_\rho)} < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^I = \mathrm{e}^{\mathbf{A}t}\mathbf{P}_I\xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I\mathbf{F}(X_s^I)\,\mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{P}_I\mathbf{B}(X_s^I)\,\mathrm{d}W_s. \tag{3.46}$$

Then it holds for all $\varphi \in C_{\mathrm{b}}^2(\mathbf{H}_0,\mathbb{R})$, $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ that

$$\begin{aligned}
\big|\mathbb{E}\big[\varphi\big(X_T^{\mathbb{H}}\big)\big] - \mathbb{E}\big[\varphi\big(X_T^I\big)\big]\big| &\leq \|\varphi\|_{C_{\mathrm{b}}^2(\mathbf{H}_0,\mathbb{R})}\big(1 \vee \|\xi\|_{L^2(\mathbb{P};\mathbf{H}_\rho)}^2\big) \\
&\quad \cdot \Big(\|\xi\|_{L^1(\mathbb{P};\mathbf{H}_{2(\gamma-\beta)})} + T\big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_{2(\gamma-\beta)})} + T\|\mathbf{\Lambda}^{-\beta}\|_{L_2(\mathbf{H}_0)}^2\big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,L(U,\mathbf{H}_\gamma))}^2\Big) \\
&\quad \cdot \Big(1 \vee \big[T\big(C_{\mathbf{F}}^2 + 2C_{\mathbf{B}}^2\big)\big]^{1/2}\Big)\exp\big(T\big[\tfrac{1}{2} + 3|\mathbf{F}|_{\mathrm{Lip}^0(\mathbf{H}_0,\mathbf{H}_0)} + 4|\mathbf{B}|_{\mathrm{Lip}^0(\mathbf{H}_0,L_2(U,\mathbf{H}_0))}^2\big]\big) \\
&\quad \cdot \exp\Big(T\Big[2\big\|\mathbf{F}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,\mathbf{H}_\rho)} + \big\|\mathbf{B}|_{\mathbf{H}_\rho}\big\|_{\mathrm{Lip}^0(\mathbf{H}_\rho,L_2(U,\mathbf{H}_\rho))}^2\Big]\Big)\Big[\inf_{h \in \mathbb{H}\setminus I}|\lambda_h|\Big]^{\beta-\gamma} < \infty.
\end{aligned} \tag{3.47}$$

3.3 Semilinear stochastic wave equations and the hyperbolic Anderson model

The following elementary lemma is well-known (cf., e.g., Example 37.1 in Sell & You [40]).

Lemma 3.10. *Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ be a $\mathbb{K}$-Hilbert space, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $A\colon D(A) \subseteq H \to H$ be a symmetric diagonal linear operator with $\inf(\sigma_{\mathrm{P}}(A)) > 0$, and let $(H_r,\langle\cdot,\cdot\rangle_{H_r},\|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to A. Then*

(i) for all $v \in \bigcup_{s \in \mathbb{R}} H_s$, $r \in \mathbb{R}$ it holds that $v \in H_r$ if and only if

$$\sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\setminus\{0\}} \frac{|\langle w,v\rangle_{H_0}|}{\|w\|_{H_{-r}}} < \infty, \tag{3.48}$$

(ii) for all $s \in \mathbb{R}$, $v \in H_{-s}$, $r \in [-s,\infty)$ it holds that $v \in H_r$ if and only if

$$\sup_{w \in H_s\setminus\{0\}} \frac{|\langle w,v\rangle_{H_0}|}{\|w\|_{H_{-r}}} < \infty, \tag{3.49}$$

(iii) and for all $r \in \mathbb{R}$, $v \in H_r$, $s \in [-r,\infty)$ it holds that

$$\|v\|_{H_r} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\setminus\{0\}} \frac{|\langle w,v\rangle_{H_0}|}{\|w\|_{H_{-r}}} = \sup_{w \in H_s\setminus\{0\}} \frac{|\langle w,v\rangle_{H_0}|}{\|w\|_{H_{-r}}}. \tag{3.50}$$

Proof of Lemma 3.10. Note that for all $r \in \mathbb{R}$, $v \in H_r$, $w \in H_{-r}$ it holds that

$$|\langle w,v\rangle_{H_0}| = |\langle A^{-r}w, A^r v\rangle_{H_0}| \leq \|w\|_{H_{-r}}\|v\|_{H_r}. \tag{3.51}$$

This proves the "$\Rightarrow$" direction in the statement of (i). Next we consider the "$\Leftarrow$" direction in the statement of (i). Let $s, r \in \mathbb{R}$ and $v \in H_s$ satisfy (3.48). Without loss of generality we assume that $s \leq r$ (otherwise it clearly holds that $v \in H_r$). It then follows that

$$\sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\setminus\{0\}} \frac{|\langle v,w\rangle_{H_0}|}{\|w\|_{H_{-r}}} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\setminus\{0\}} \frac{|\langle A^r v, A^r w\rangle_{H_0}|}{\|A^{2r}w\|_{H_{-r}}} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\setminus\{0\}} \frac{|\langle v,w\rangle_{H_r}|}{\|w\|_{H_r}} < \infty. \tag{3.52}$$

Proposition 2.5.19 in [23] and the Riesz representation theorem hence prove the existence of a unique element $\tilde{v} \in H_r$ which satisfies that

$$\|\tilde{v}\|_{H_r} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H}) \setminus \{0\}} \frac{|\langle v, w \rangle_{H_r}|}{\|w\|_{H_r}} \tag{3.53}$$

and which satisfies for all $w \in \mathrm{span}_{H_0}(\mathbb{H})$ that $\langle \tilde{v}, w \rangle_{H_r} = \langle v, w \rangle_{H_r}$. This implies for all $w \in \mathrm{span}_{H_0}(\mathbb{H})$ that $\langle \tilde{v}, w \rangle_{H_s} = \langle v, w \rangle_{H_s}$. In addition, let $\{w_n\}_{n \in \mathbb{N}} \subseteq \mathrm{span}_{H_0}(\mathbb{H})$ satisfy that $\lim_{n \to \infty} \|\tilde{v} - v - w_n\|_{H_s} = 0$. It then follows that

$$\|\tilde{v} - v\|_{H_s}^2 = \lim_{n \to \infty} \langle \tilde{v} - v, w_n \rangle_{H_s} = 0, \tag{3.54}$$

which shows that $\tilde{v} = v \in H_r$. This proves (i) and together with (3.52) and (3.53) the first equality in (3.50). Next note that the "$\Rightarrow$" direction in the statement of (ii) follows directly from (3.51), while the "$\Leftarrow$" direction in the statement of (ii) is a consequence of (i). Finally, (3.51) also shows for all $r \in \mathbb{R}$, $v \in H_r$, $s \in [-r, \infty)$ that

$$\|v\|_{H_r} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H}) \setminus \{0\}} \frac{|\langle w, v \rangle_{H_0}|}{\|w\|_{H_{-r}}} \leq \sup_{w \in H_s \setminus \{0\}} \frac{|\langle w, v \rangle_{H_0}|}{\|w\|_{H_{-r}}} \leq \sup_{w \in H_{-r} \setminus \{0\}} \frac{|\langle w, v \rangle_{H_0}|}{\|w\|_{H_{-r}}} \leq \|v\|_{H_r}. \tag{3.55}$$

This completes the proof of Lemma 3.10. $\qquad\square$

In the next result, Corollary 3.11, we illustrate Corollary 3.8 above by a simple example. Corollary 3.11 is an elementary consequence from Corollary 3.8.

Corollary 3.11. *Let* $T, \vartheta \in (0, \infty)$, $\gamma \in (1/4, 1/2)$, $\rho \in [0, 1/2]$, $r \in [1/6, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a normal filtration* $(\mathcal{F}_t)_{t \in [0,T]}$, *let* $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ *be the* $\mathbb{R}$-*Hilbert space given by* $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)}; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda_{(0,1)}; \mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)}; \mathbb{R})})$, *let* $(W_t)_{t \in [0,T]}$ *be an* id_H-*cylindrical* $(\mathcal{F}_t)_{t \in [0,T]}$-*Wiener process, let* $\{e_n\}_{n \in \mathbb{N}} \subseteq H$ *satisfy for all* $n \in \mathbb{N}$ *and* $\lambda_{(0,1)}$-*a.e.* $x \in (0, 1)$ *that* $e_n(x) = \sqrt{2} \sin(n \pi x)$, *let* $A \colon D(A) \subseteq H \to H$ *be the Laplacian with Dirichlet boundary conditions on* H *multiplied by* ϑ, *let* $(H_s, \langle \cdot, \cdot \rangle_{H_s}, \|\cdot\|_{H_s})$, $s \in \mathbb{R}$, *be a family of interpolation spaces associated to* $-A$, *let* $\mathbf{P}_N \colon H_0 \times H_{-1/2} \to H_0 \times H_{-1/2}$, $N \in \mathbb{N} \cup \{\infty\}$, *be the mappings which satisfy for all* $N \in \mathbb{N} \cup \{\infty\}$, $(v, w) \in H_0 \times H_{-1/2}$ *that* $\mathbf{P}_N(v, w) = \sum_{n=1}^{N} (\langle e_n, v \rangle_H e_n, \langle \sqrt{\vartheta} \pi n e_n, w \rangle_{H_{-1/2}} \sqrt{\vartheta} \pi n e_n)$, *let* $\mathbf{A} \colon D(\mathbf{A}) \subseteq H_0 \times H_{-1/2} \to H_0 \times H_{-1/2}$ *be the linear operator such that* $D(\mathbf{A}) = H_{1/2} \times H_0$ *and such that for all* $(v, w) \in H_{1/2} \times H_0$ *it holds that* $\mathbf{A}(v, w) = (w, Av)$, *let* $\xi \in L^2(\mathbb{P}|_{\mathcal{F}_0}; H_{1/2} \times H_0)$, $\varphi \in C_{\mathrm{b}}^2(H_0 \times H_{-1/2}, \mathbb{R})$, $f \in \mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})$, $B \in \mathrm{Lip}^0(H_0, L_2(H_0, H_{-1/2}))$ *satisfy that* $B|_{H_\rho} \in \mathrm{Lip}^0(H_\rho, L_2(H_0, H_{\rho-1/2}) \cap L(H_0, H_{\gamma-1/2}))$, $B|_{H_r} \in C_{\mathrm{b}}^2(H_r, L_2(H_0, H_{-1/2}))$, *and* $\sup_{x \in H_r} \sup_{v_1, v_2 \in H_r, \|v_1\|_{H_0} \vee \|v_2\|_{H_0} \leq 1} \|B''(x)(v_1, v_2)\|_{L_2(H_0, H_{-1/2})} < \infty$, *and let* $\mathbf{F} \colon H_0 \times H_{-1/2} \to H_{1/2} \times H_0$ *and* $\mathbf{B} \colon H_0 \times H_{-1/2} \to L_2(H_0, H_0 \times H_{-1/2})$ *be the mappings which satisfy for all* $(v, w) \in H_0 \times H_{-1/2}$ *and* $\lambda_{(0,1)}$-*a.e.* $x \in (0, 1)$ *that* $(\mathbf{F}(v, w))(x) = (0, f(x, v(x)))$ *and* $\mathbf{B}(v, w) = (0, B(v))$. *Then*

(i) it holds that $\mathbf{F} \in \mathrm{Lip}^0(H_0 \times H_{-1/2}, H_{1/2} \times H_0)$, $\mathbf{F}|_{H_r \times H_{r-1/2}} \in \mathrm{Lip}^2(H_r \times H_{r-1/2}, H_{1/2} \times H_0)$, $\mathbf{B} \in \mathrm{Lip}^0(H_0 \times H_{-1/2}, L_2(H_0, H_0 \times H_{-1/2}))$, $\mathbf{B}|_{H_\rho \times H_{\rho-1/2}} \in \mathrm{Lip}^0(H_\rho \times H_{\rho-1/2}, L_2(H_0, H_\rho \times H_{\rho-1/2}) \cap L(H_0, H_\gamma \times H_{\gamma-1/2}))$, $\mathbf{B}|_{H_r \times H_{r-1/2}} \in C_{\mathrm{b}}^2(H_r \times H_{r-1/2}, L_2(H_0, H_0 \times H_{-1/2}))$, *and*

$$\forall \delta \in (-\infty, 1/4) \colon \sup_{\substack{x \in H_r \times H_{r-1/2}, \\ v_1, v_2 \in H_r \times H_{r-1/2} \setminus \{0\}}} \frac{\|\mathbf{F}''(x)(v_1, v_2)\|_{H_\delta \times H_{\delta-1/2}} + \|\mathbf{B}''(x)(v_1, v_2)\|_{L_2(H_0, H_0 \times H_{-1/2})}}{\|v_1\|_{H_0 \times H_{-1/2}} \|v_2\|_{H_0 \times H_{-1/2}}} < \infty, \tag{3.56}$$

(ii) there exist up to modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-*predictable stochastic processes* $X^N \colon [0, T] \times \Omega \to \mathbf{P}_N(H_\rho \times H_{\rho-1/2})$, $N \in \mathbb{N} \cup \{\infty\}$, *which satisfy for all* $N \in \mathbb{N} \cup \{\infty\}$, $t \in [0, T]$ *that* $\sup_{s \in [0,T]} \|X_s^N\|_{L^2(\mathbb{P}; H_\rho \times H_{\rho-1/2})} < \infty$ *and* $\mathbb{P}$-*a.s. that*

$$X_t^N = e^{\mathbf{A}t} \mathbf{P}_N \xi + \int_0^t e^{\mathbf{A}(t-s)} \mathbf{P}_N \mathbf{F}(X_s^N) \, \mathrm{d}s + \int_0^t e^{\mathbf{A}(t-s)} \mathbf{P}_N \mathbf{B}(X_s^N) \, \mathrm{d}W_s, \tag{3.57}$$

(iii) and for all $\varepsilon \in (4(\nicefrac{1}{2} - \gamma), \infty)$ there exists a real number $C \in [0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left| \mathbb{E}\big[\varphi\big(X_T^\infty\big)\big] - \mathbb{E}\big[\varphi\big(X_T^N\big)\big] \right| \leq C \cdot N^{\varepsilon - 1}. \tag{3.58}$$

Proof of Corollary 3.11. Throughout this proof let $f_{k,\ell} \colon (0,1) \times \mathbb{R} \to \mathbb{R}$, $k, \ell \in \{0, 1, 2\}$ with $k + \ell \leq 2$, be the mappings such that for all $k, \ell \in \{0, 1, 2\}$, $(x, y) \in (0, 1) \times \mathbb{R}$ with $k + \ell \leq 2$ it holds that $f_{k,\ell}(x, y) = \big(\frac{\partial^{k+\ell}}{\partial x^k \partial y^\ell} f\big)(x, y)$ and let $F \colon H_0 \to H_0$ be the mapping such that for all $v \in H_0$ and $\lambda_{(0,1)}$-a.e. $x \in (0, 1)$ it holds that $\big(F(v)\big)(x) = f(x, v(x))$. Then note for all $u, v \in H_0$, $w \in H_{-\nicefrac{1}{2}}$ that $\mathbf{F}(v, w) = \big(0, F(v)\big)$ and

$$\|F(u) - F(v)\|_{H_0} = \left(\int_0^1 |f(x, u(x)) - f(x, v(x))|^2 \, \mathrm{d}x \right)^{1/2} \leq |f|_{\mathrm{Lip}^0((0,1)\times\mathbb{R},\mathbb{R})} \|u - v\|_{H_0}. \tag{3.59}$$

This proves that $F \in \mathrm{Lip}^0(H_0, H_0)$. Hence, we obtain that $\mathbf{F} \in \mathrm{Lip}^0(H_0 \times H_{-\nicefrac{1}{2}}, H_{\nicefrac{1}{2}} \times H_0)$. Next observe that the Sobolev embedding theorem ensures for all $\delta \in [1, 6]$ that

$$\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^\delta(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_r}} < \infty. \tag{3.60}$$

Moreover, it holds for all $v, h \in H_0$ and $\lambda_{(0,1)}$-a.e. $x \in (0, 1)$ that

$$\begin{aligned}
&|f(x, v(x) + h(x)) - f(x, v(x)) - f_{0,1}(x, v(x))h(x)| \\
&= \left| \int_0^1 \big[f_{0,1}(x, v(x) + yh(x)) - f_{0,1}(x, v(x)) \big] h(x) \, \mathrm{d}y \right| \leq |f|_{\mathrm{Lip}^1((0,1)\times\mathbb{R},\mathbb{R})} |h(x)|^2.
\end{aligned} \tag{3.61}$$

This, Hölder's inequality, and (3.60) imply for all $v \in H_r$, $h \in H_r \setminus \{0\}$ that

$$\begin{aligned}
&\frac{1}{\|h\|_{H_r}} \left(\int_0^1 |f(x, v(x) + h(x)) - f(x, v(x)) - f_{0,1}(x, v(x))h(x)|^2 \, \mathrm{d}x \right)^{1/2} \\
&\leq |f|_{\mathrm{Lip}^1((0,1)\times\mathbb{R},\mathbb{R})} \frac{\|h\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^2}{\|h\|_{H_r}} \leq |f|_{\mathrm{Lip}^1((0,1)\times\mathbb{R},\mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^4(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_r}} \right)^2 \|h\|_{H_r} < \infty.
\end{aligned} \tag{3.62}$$

In addition, it holds for all $v, h \in H_r$ that

$$\begin{aligned}
\left(\int_0^1 |f_{0,1}(x, v(x))h(x)|^2 \, \mathrm{d}x \right)^{1/2} &\leq |f|_{C_{\mathrm{b}}^1((0,1)\times\mathbb{R},\mathbb{R})} \|h\|_{H_0} \leq |f|_{C_{\mathrm{b}}^1((0,1)\times\mathbb{R},\mathbb{R})} \|h\|_{H_r} \\
&= |f|_{\mathrm{Lip}^0((0,1)\times\mathbb{R},\mathbb{R})} \|h\|_{H_r} < \infty.
\end{aligned} \tag{3.63}$$

Inequalities (3.62) and (3.63) prove that $F|_{H_r} \colon H_r \to H_0$ is Fréchet differentiable, that for all $v, h \in H_r$ and $\lambda_{(0,1)}$-a.e. $x \in (0, 1)$ it holds that

$$\big(F'(v)h\big)(x) = f_{0,1}(x, v(x))h(x), \tag{3.64}$$

and that $\sup_{v \in H_r} \|F'(v)\|_{L(H_r, H_0)} \leq |f|_{C_{\mathrm{b}}^1((0,1)\times\mathbb{R},\mathbb{R})} < \infty$. Furthermore, Hölder's inequality and (3.60) show for all $u, v, h \in H_r$ that

$$\begin{aligned}
\|(F'(u) - F'(v))h\|_{H_0} &= \left(\int_0^1 \big| \big[f_{0,1}(x, u(x)) - f_{0,1}(x, v(x)) \big] h(x) \big|^2 \, \mathrm{d}x \right)^{1/2} \\
&\leq |f|_{\mathrm{Lip}^1((0,1)\times\mathbb{R},\mathbb{R})} \|u - v\|_{L^4(\lambda_{(0,1)};\mathbb{R})} \|h\|_{L^4(\lambda_{(0,1)};\mathbb{R})} \\
&\leq |f|_{\mathrm{Lip}^1((0,1)\times\mathbb{R},\mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^4(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_r}} \right)^2 \|u - v\|_{H_r} \|h\|_{H_r} < \infty.
\end{aligned} \tag{3.65}$$

This ensures that $F|_{H_r} \in \mathrm{Lip}^1(H_r, H_0)$. Similarly, observe for all $v, h, g \in H_0$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ that

$$
\begin{aligned}
&|f_{0,1}(x, v(x) + g(x))h(x) - f_{0,1}(x, v(x))h(x) - f_{0,2}(x, v(x))h(x)g(x)| \\
&= \left| \int_0^1 \left[f_{0,2}(x, v(x) + yg(x)) - f_{0,2}(x, v(x)) \right] h(x)g(x)\, \mathrm{d}y \right| \leq |f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})} |h(x)| |g(x)|^2.
\end{aligned} \tag{3.66}
$$

This, Hölder's inequality, and (3.60) establish for all $v, h \in H_r$, $g \in H_r \setminus \{0\}$ that

$$
\begin{aligned}
&\frac{1}{\|g\|_{H_r}} \left(\int_0^1 |f_{0,1}(x, v(x) + g(x))h(x) - f_{0,1}(x, v(x))h(x) - f_{0,2}(x, v(x))h(x)g(x)|^2\, \mathrm{d}x \right)^{1/2} \\
&\leq \frac{|f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})}}{\|g\|_{H_r}} \left(\int_0^1 |h(x)|^2 |g(x)|^4\, \mathrm{d}x \right)^{1/2} \\
&\leq |f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})} \frac{\|h\|_{L^6(\lambda_{(0,1)}; \mathbb{R})} \|g\|^2_{L^6(\lambda_{(0,1)}; \mathbb{R})}}{\|g\|_{H_r}} \\
&\leq |f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^6(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_r}} \right)^3 \|h\|_{H_r} \|g\|_{H_r} < \infty.
\end{aligned} \tag{3.67}
$$

Furthermore, Hölder's inequality and (3.60) also prove for all $v, h, g \in H_r$ that

$$
\begin{aligned}
\left(\int_0^1 |f_{0,2}(x, v(x))h(x)g(x)|^2\, \mathrm{d}x \right)^{1/2} &\leq |f|_{C_{\mathrm{b}}^2((0,1) \times \mathbb{R}, \mathbb{R})} \|h\|_{L^4(\lambda_{(0,1)}; \mathbb{R})} \|g\|_{L^4(\lambda_{(0,1)}; \mathbb{R})} \\
&\leq |f|_{C_{\mathrm{b}}^2((0,1) \times \mathbb{R}, \mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_r}} \right)^2 \|h\|_{H_r} \|g\|_{H_r} \\
&= |f|_{\mathrm{Lip}^1((0,1) \times \mathbb{R}, \mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_r}} \right)^2 \|h\|_{H_r} \|g\|_{H_r} < \infty.
\end{aligned} \tag{3.68}
$$

Combining (3.67) and (3.68) ensures that $F|_{H_r} \colon H_r \to H_0$ is twice Fréchet differentiable, that for all $v, h, g \in H_r$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ it holds that

$$
\big(F''(v)(h, g) \big)(x) = f_{0,2}(x, v(x))h(x)g(x), \tag{3.69}
$$

and that

$$
\sup_{v \in H_r} \|F''(v)\|_{L^{(2)}(H_r, H_0)} \leq |f|_{C_{\mathrm{b}}^2((0,1) \times \mathbb{R}, \mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^4(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_r}} \right)^2 < \infty. \tag{3.70}
$$

In addition, Hölder's inequality and (3.60) establish for all $u, v, h, g \in H_r$ that

$$
\begin{aligned}
\|(F''(u) - F''(v))(h, g)\|_{H_0} &= \left(\int_0^1 \left| \left[f_{0,2}(x, u(x)) - f_{0,1}(x, v(x)) \right] h(x)g(x) \right|^2\, \mathrm{d}x \right)^{1/2} \\
&\leq |f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})} \|u - v\|_{L^6(\lambda_{(0,1)}; \mathbb{R})} \|h\|_{L^6(\lambda_{(0,1)}; \mathbb{R})} \|g\|_{L^6(\lambda_{(0,1)}; \mathbb{R})} \\
&\leq |f|_{\mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})} \left(\sup_{w \in H_r \setminus \{0\}} \frac{\|w\|_{L^6(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_r}} \right)^3 \|u - v\|_{H_r} \|h\|_{H_r} \|g\|_{H_r} < \infty.
\end{aligned} \tag{3.71}
$$

This shows that $F|_{H_r} \in \mathrm{Lip}^2(H_r, H_0)$. Hence, we obtain that $\mathbf{F}|_{H_r \times H_{r-1/2}} \in \mathrm{Lip}^2(H_r \times H_{r-1/2}, H_{1/2} \times H_0)$. Next, note that the assumptions that $B \in \mathrm{Lip}^0(H_0, L_2(H_0, H_{-1/2}))$, $B|_{H_\rho} \in \mathrm{Lip}^0(H_\rho, L_2(H_0, H_{\rho-1/2}) \cap L(H_0, H_{\gamma-1/2}))$, and $B|_{H_r} \in C_{\mathrm{b}}^2(H_r, L_2(H_0, H_{-1/2}))$ ensure that $\mathbf{B} \in \mathrm{Lip}^0(H_0 \times H_{-1/2}, L_2(H_0, H_0 \times H_{-1/2}))$, $\mathbf{B}|_{H_\rho \times H_{\rho-1/2}} \in \mathrm{Lip}^0(H_\rho \times H_{\rho-1/2}, L_2(H_0, H_\rho \times H_{\rho-1/2}) \cap L(H_0, H_\gamma \times H_{\gamma-1/2}))$, and

$\mathbf{B}|_{H_r \times H_{r-1/2}} \in C_b^2(H_r \times H_{r-1/2}, L_2(H_0, H_0 \times H_{-1/2}))$. In addition, Lemma 3.10 proves for all $\delta \in (-\infty, 1/4)$ that

$$\|F''(v)(h,g)\|_{H_{\delta-1/2}} = \sup_{w \in H_{1/2-\delta}\backslash\{0\}} \frac{\langle w, F''(v)(h,g)\rangle_{H_0}}{\|w\|_{H_{1/2-\delta}}}$$

$$\leq |f|_{C_b^2((0,1)\times\mathbb{R},\mathbb{R})} \left(\sup_{w \in H_{1/2-\delta}\backslash\{0\}} \frac{\|w\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{1/2-\delta}}}\right) \|h\|_{H_0}\|g\|_{H_0} < \infty. \tag{3.72}$$

This and the assumption that $\sup_{x,v_1,v_2 \in H_r, \|v_1\|_{H_0}\vee\|v_2\|_{H_0}\leq 1}\|B''(x)(v_1,v_2)\|_{L_2(H_0,H_{-1/2})} < \infty$ show (3.56). The proof of (i) is thus completed. Furthermore, observe that (ii) follows directly from (i) and Remark 3.1. It thus remains to prove (iii). For this let $\varepsilon \in (4(1/2 - \gamma), 1]$, $\beta \in (1/2, 2\gamma]$ and $\lambda_n \in \mathbb{R}$, $n \in \mathbb{N}$, be real numbers which satisfy for all $n \in \mathbb{N}$ that $\beta = 1/2 + (\varepsilon-4(1/2-\gamma))/2$ and $\lambda_n = -\vartheta\pi^2 n^2$ and let $\mathbf{\Lambda}: D(\mathbf{\Lambda}) \subseteq H_0 \times H_{-1/2} \to H_0 \times H_{-1/2}$ be the linear operator such that $D(\mathbf{\Lambda}) = H_{1/2} \times H_0$ and such that for all $(v,w) \in H_{1/2} \times H_0$ it holds that

$$\mathbf{\Lambda}(v,w) = \left(\begin{array}{c} \sum_{n=1}^{\infty}|\lambda_n|^{1/2}\langle e_n, v\rangle_{H_0} e_n \\ \sum_{n=1}^{\infty}|\lambda_n|^{1/2}\langle|\lambda_n|^{1/2}e_n, w\rangle_{H_{-1/2}}|\lambda_n|^{1/2}e_n \end{array}\right). \tag{3.73}$$

Then note for all $v \in H_1$ that $Av = \sum_{n=1}^{\infty}\lambda_n\langle e_n, v\rangle_{H_0} e_n$ and $\|\mathbf{\Lambda}^{-\beta}\|_{L_2(H_0 \times H_{-1/2})} < \infty$. Furthermore, observe that (i) and the fact that $2\gamma - \beta = (1-\varepsilon)/2$ imply that $\mathbf{F} \in \mathrm{Lip}^0(H_0 \times H_{-1/2}, H_{2\gamma-\beta} \times H_{2\gamma-\beta-1/2})$. This and again (i) enable us to apply Corollary 3.8 to obtain that there exists a real number $C \in [0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left|\mathbb{E}[\varphi(X_T^\infty)] - \mathbb{E}[\varphi(X_T^N)]\right| \leq C|\lambda_{N+1}|^{\beta-2\gamma} \leq \vartheta^{(\varepsilon-1)/2}\cdot C\cdot N^{\varepsilon-1}. \tag{3.74}$$

The proof of Corollary 3.11 is thus completed. $\qquad\square$

In the proof of Corollary 3.13 below we use the following elementary and well-known result, Lemma 3.12.

Lemma 3.12. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a $\mathbb{K}$-Hilbert space, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $A: D(A) \subseteq H \to H$ be a symmetric diagonal linear operator with $\inf(\sigma_P(A)) > 0$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to A, and let $q \in \mathbb{R}$, $p \in [q, \infty)$, $s \in \mathbb{R}$, $r \in [s, \infty)$. Then*

(i) for all $B \in L(H_q, H_s)$ it holds that $B \in L(H_q, H_r)$ if and only if

$$\left(B\left(\mathrm{span}_{H_0}(\mathbb{H})\right) \subseteq H_r \quad and \quad \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\backslash\{0\}} \frac{\|Bw\|_{H_r}}{\|w\|_{H_q}} < \infty\right), \tag{3.75}$$

(ii) for all $B \in L(H_q, H_s)$ it holds that $B \in L(H_q, H_r)$ if and only if

$$\left(B(H_p) \subseteq H_r \quad and \quad \sup_{w \in H_p\backslash\{0\}} \frac{\|Bw\|_{H_r}}{\|w\|_{H_q}} < \infty\right), \tag{3.76}$$

(iii) and for all $B \in L(H_q, H_r)$ it holds that

$$\|B\|_{L(H_q,H_r)} = \sup_{w \in \mathrm{span}_{H_0}(\mathbb{H})\backslash\{0\}} \frac{\|Bw\|_{H_r}}{\|w\|_{H_q}} = \sup_{w \in H_p\backslash\{0\}} \frac{\|Bw\|_{H_r}}{\|w\|_{H_q}}. \tag{3.77}$$

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Corollary 3.13 (Hyperbolic Anderson model). *Let $T, \vartheta \in (0, \infty)$, $\alpha, \beta \in \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be the $\mathbb{R}$-Hilbert space given by $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) = \big(L^2(\lambda_{(0,1)}; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda_{(0,1)}; \mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)}; \mathbb{R})}\big)$, let $(W_t)_{t \in [0,T]}$ be an id_H-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $\{e_n\}_{n \in \mathbb{N}} \subseteq H$ satisfy for all $n \in \mathbb{N}$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ that $e_n(x) = \sqrt{2} \sin(n\pi x)$, let $A \colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H multiplied by ϑ, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $\mathbf{P}_N \colon H_0 \times H_{-1/2} \to H_0 \times H_{-1/2}$, $N \in \mathbb{N} \cup \{\infty\}$, be the mappings which satisfy for all $N \in \mathbb{N} \cup \{\infty\}$, $(v, w) \in H_0 \times H_{-1/2}$ that $\mathbf{P}_N(v, w) = \sum_{n=1}^{N}(\langle e_n, v \rangle_H e_n, \langle \sqrt{\vartheta}\pi n e_n, w \rangle_{H_{-1/2}} \sqrt{\vartheta}\pi n e_n)$, let $\mathbf{A} \colon D(\mathbf{A}) \subseteq H_0 \times H_{-1/2} \to H_0 \times H_{-1/2}$ be the linear operator such that $D(\mathbf{A}) = H_{1/2} \times H_0$ and such that for all $(v, w) \in H_{1/2} \times H_0$ it holds that $\mathbf{A}(v, w) = (w, Av)$, let $\xi \in L^2(\mathbb{P}|_{\mathcal{F}_0}; H_{1/2} \times H_0)$, $\varphi \in C_b^2(H_0 \times H_{-1/2}, \mathbb{R})$, $f \in \mathrm{Lip}^2((0,1) \times \mathbb{R}, \mathbb{R})$, and let $\mathbf{F} \colon H_0 \times H_{-1/2} \to H_{1/2} \times H_0$ and $\mathbf{B} \colon H_0 \times H_{-1/2} \to L_2(H_0, H_0 \times H_{-1/2})$ be the mappings which satisfy for all $(v, w) \in H_0 \times H_{-1/2}$, $u \in H_1$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ that $\big(\mathbf{F}(v, w)\big)(x) = \big(0, f(x, v(x))\big)$ and $\big(\mathbf{B}(v, w)u\big)(x) = \big(0, (\alpha + \beta v(x))u(x)\big)$. Then*

(i) *there exist up to modifications unique $(\mathcal{F}_t)_{t \in [0,T]}$-predictable stochastic processes $X^N \colon [0,T] \times \Omega \to \bigcap_{\rho \in [0, 1/4)} \mathbf{P}_N(H_\rho \times H_{\rho - 1/2})$, $N \in \mathbb{N} \cup \{\infty\}$, which satisfy for all $\rho \in [0, 1/4)$, $N \in \mathbb{N} \cup \{\infty\}$, $t \in [0,T]$ that $\sup_{s \in [0,T]} \|X_s^N\|_{L^2(\mathbb{P}; H_\rho \times H_{\rho - 1/2})} < \infty$ and $\mathbb{P}$-a.s. that*

$$X_t^N = \mathrm{e}^{\mathbf{A}t} \mathbf{P}_N \xi + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_N \mathbf{F}(X_s^N) \, \mathrm{d}s + \int_0^t \mathrm{e}^{\mathbf{A}(t-s)} \mathbf{P}_N \mathbf{B}(X_s^N) \, \mathrm{d}W_s \tag{3.78}$$

(ii) *and for all $\varepsilon \in (0, \infty)$ there exists a real number $C \in [0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\big|\mathbb{E}\big[\varphi(X_T^\infty)\big] - \mathbb{E}\big[\varphi(X_T^N)\big]\big| \leq C \cdot N^{\varepsilon - 1}. \tag{3.79}$$

Proof of Corollary 3.13. Throughout this proof let $B \colon H_0 \to L_2(H_0, H_{-1/2})$ be the mapping which satisfies for all $v \in H_0$, $u \in H_1$ and $\lambda_{(0,1)}$-a.e. $x \in (0,1)$ that $\big(B(v)u\big)(x) = (\alpha + \beta v(x))u(x)$. Section 7.2.1 in [23] then implies for all $\rho \in [0, 1/4)$ that $B \in \mathrm{Lip}^0(H_0, L_2(H_0, H_{\rho - 1/2}))$. Remark 3.1 and (i) in Corollary 3.11 thus prove (i). Next observe that the Sobolev embedding theorem proves for all $\rho \in (0, 1/4)$ that

$$\left[\sup_{w \in H_1 \setminus \{0\}} \frac{\|w\|_{L^{1/(2\rho)}(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_{1/4 - \rho}}}\right] \vee \left[\sup_{w \in H_\rho \setminus \{0\}} \frac{\|w\|_{L^{2/(1-4\rho)}(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_\rho}}\right] < \infty. \tag{3.80}$$

This and Hölder's inequality ensure for all $\rho \in (0, 1/4)$, $v \in H_\rho$, $u \in H_1$ that

$$
\begin{aligned}
&\sup_{w \in H_1 \setminus \{0\}} \frac{|\langle w, B(v)u \rangle_{H_0}|}{\|w\|_{H_{1/4 - \rho}}} \\
&\leq \left[\sup_{w \in H_1 \setminus \{0\}} \frac{\|w\|_{L^{1/(2\rho)}(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_{1/4 - \rho}}}\right] \|\alpha + \beta v\|_{L^{2/(1-4\rho)}(\lambda_{(0,1)}; \mathbb{R})} \|u\|_{L^2(\lambda_{(0,1)}; \mathbb{R})} \\
&\leq \left[\sup_{w \in H_1 \setminus \{0\}} \frac{\|w\|_{L^{1/(2\rho)}(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_{1/4 - \rho}}}\right] \left[\sup_{w \in H_\rho \setminus \{0\}} \frac{\|w\|_{L^{2/(1-4\rho)}(\lambda_{(0,1)}; \mathbb{R})}}{\|w\|_{H_\rho}}\right] \|\alpha + \beta v\|_{H_\rho} \|u\|_{H_0} < \infty.
\end{aligned} \tag{3.81}
$$

Lemma 3.10 hence shows for all $\rho \in (0, 1/4)$, $v \in H_\rho$, $u \in H_1$ that $B(v)u \in H_{\rho - 1/4}$. In addition, (3.81), Lemma 3.10, and Lemma 3.12 prove for all $\rho \in (0, 1/4)$, $v \in H_\rho$ that $B(v) \in L(H_0, H_{\rho - 1/4})$. Furthermore, Hölder's inequality shows for all $\varepsilon \in (0, 1]$, $\gamma \in (1/2 - \varepsilon/4, 1/2)$, $\rho \in [\gamma - 1/4, 1/4)$,

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$v, w \in H_\rho$, $u \in H_1$ that

$$
\begin{aligned}
\|(B(v) - B(w))u\|_{H_{\gamma-1/2}} &= \sup_{w \in H_1 \setminus \{0\}} \frac{|\langle w, (B(v) - B(w))u\rangle_{H_0}|}{\|w\|_{H_{1/4-\rho}}} \\
&\leq \left[\sup_{w \in H_1 \setminus \{0\}} \frac{\|w\|_{L^{1/(2\rho)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_{1/4-\rho}}} \right] \left[\sup_{w \in H_\rho \setminus \{0\}} \frac{\|w\|_{L^{2/(1-4\rho)}(\lambda_{(0,1)};\mathbb{R})}}{\|w\|_{H_\rho}} \right] |\beta| \|v - w\|_{H_\rho} \|u\|_{H_0} < \infty.
\end{aligned}
\tag{3.82}
$$

This, Lemma 3.10, and Lemma 3.12 establish for all $\varepsilon \in (0,1]$, $\gamma \in (1/2 - \varepsilon/4, 1/2)$, $\rho \in [\gamma - 1/4, 1/4)$ that $B|_{H_\rho} \in \mathrm{Lip}^0(H_\rho, L(H_0, H_{\gamma-1/2}))$. Corollary 3.11 thus completes the proof of Corollary 3.13. $\quad\square$

4 Lower bounds for weak errors

The weak convergence rates for the Galerkin approximation obtained in Section 3 provide upper bounds for weak errors. In this section we prove lower bounds for weak errors in the case of concrete examples of stochastic wave equations with additive noise and without drift nonlinearity which allow to show that the weak convergence rates established in Theorem 3.7 are essentially sharp. We argue similar to the reasoning in Section 6 in Conus et al. [10] and Section 9 in Jentzen & Kurniawan [24]. First results on lower bounds for strong errors for two examples of stochastic heat equations were achieved in Davie & Gaines [15]. Furthermore, lower bounds for strong errors for examples and whole classes of stochastic heat equations have been established in Müller-Gronbach et al. [38] and in Müller-Gronbach & Ritter [37], respectively. Results on lower bounds for weak errors in in the case of a few specific examples of stochastic heat equations can be found in Conus et al. [10] and in Jentzen & Kurniawan [24].

4.1 Setting

Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$, let $(W_t)_{t \in [0,T]}$ be an id_H-cylindrical $(\mathcal{F}_t)_{t \in [0,T]}$-Wiener process, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $\lambda \colon \mathbb{H} \to \mathbb{R}$ be a mapping such that $\sup_{h \in \mathbb{H}} \lambda_h < 0$, let $A \colon D(A) \subseteq H \to H$ be the linear operator such that $D(A) = \{v \in H \colon \sum_{h \in \mathbb{H}} |\lambda_h \langle h, v \rangle_H|^2 < \infty\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{h \in \mathbb{H}} \lambda_h \langle h, v \rangle_H h$, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$, let $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r})$, $r \in \mathbb{R}$, be the family of $\mathbb{R}$-Hilbert spaces such that for all $r \in \mathbb{R}$ it holds that $(\mathbf{H}_r, \langle \cdot, \cdot \rangle_{\mathbf{H}_r}, \|\cdot\|_{\mathbf{H}_r}) = (H_{r/2} \times H_{r/2-1/2}, \langle \cdot, \cdot \rangle_{H_{r/2} \times H_{r/2-1/2}}, \|\cdot\|_{H_{r/2} \times H_{r/2-1/2}})$, let $P_I \colon \bigcup_{r \in \mathbb{R}} H_r \to \bigcup_{r \in \mathbb{R}} H_r$, $I \in \mathcal{P}(\mathbb{H})$, be the mappings such that for all $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $v \in H_r$ it holds that $P_I(v) = \sum_{h \in I} \langle |\lambda_h|^{-r} h, v \rangle_{H_r} |\lambda_h|^{-r} h$, let $\mathbf{P}_I \colon \bigcup_{r \in \mathbb{R}} \mathbf{H}_r \to \bigcup_{r \in \mathbb{R}} \mathbf{H}_r$, $I \in \mathcal{P}(\mathbb{H})$, be the mappings such that for all $I \in \mathcal{P}(\mathbb{H})$, $r \in \mathbb{R}$, $(v, w) \in \mathbf{H}_r$ it holds that $\mathbf{P}_I(v, w) = (P_I(v), P_I(w))$, let $\mathbf{A} \colon D(\mathbf{A}) \subseteq \mathbf{H}_0 \to \mathbf{H}_0$ be the linear operator such that $D(\mathbf{A}) = \mathbf{H}_1$ and such that for all $(v, w) \in \mathbf{H}_1$ it holds that $\mathbf{A}(v, w) = (w, Av)$, let $\mu \colon \mathbb{H} \to \mathbb{R}$ be a mapping such that $\sum_{h \in \mathbb{H}} \frac{|\mu_h|^2}{|\lambda_h|} < \infty$, let $\mathbf{B} \in L_2(H, \mathbf{H}_0)$ be the linear operator such that for all $v \in H$ it holds that $\mathbf{B}v = (0, \sum_{h \in \mathbb{H}} \mu_h \langle h, v \rangle_H h)$, let $\mathbf{X}^I = (X^{I,1}, X^{I,2}) \colon \Omega \to \mathbf{P}_I(\mathbf{H}_0)$, $I \in \mathcal{P}(\mathbb{H})$, be random variables such that for all $I \in \mathcal{P}(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that $\mathbf{X}^I = \int_0^T \mathrm{e}^{\mathbf{A}(T-s)} \mathbf{P}_I \mathbf{B} \, \mathrm{d}W_s$.

4.2 Lower bounds for the squared norm

Lemma 4.1. *Assume the setting in Section 4.1. Then for all $I \in \mathcal{P}(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that*

$$
\mathbf{X}^I = \mathbf{P}_I \mathbf{X}^{\mathbb{H}} = \begin{pmatrix} X^{I,1} \\ X^{I,2} \end{pmatrix} = \begin{pmatrix} \sum_{h \in I} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin(|\lambda_h|^{1/2}(T-s)) \, \mathrm{d}\langle h, W_s \rangle_H \right) h \\ \sum_{h \in I} \left(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos(|\lambda_h|^{1/2}(T-s)) \, \mathrm{d}\langle h, W_s \rangle_H \right) |\lambda_h|^{1/2} h \end{pmatrix}.
\tag{4.1}
$$

Proof of Lemma 4.1. Lemma 2.5 proves that it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\mathbf{X}^{\mathbb{H}} &= \int_0^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\,\mathrm{d}W_s = \sum_{h\in\mathbb{H}} \int_0^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}h\,\mathrm{d}\langle h, W_s\rangle_H \\
&= \sum_{h\in\mathbb{H}} \begin{pmatrix} \mu_h \int_0^T (-A)^{-1/2}\sin\big((-A)^{1/2}(T-s)\big)h\,\mathrm{d}\langle h, W_s\rangle_H \\ \mu_h \int_0^T \cos\big((-A)^{1/2}(T-s)\big)h\,\mathrm{d}\langle h, W_s\rangle_H \end{pmatrix} \\
&= \sum_{h\in\mathbb{H}} \begin{pmatrix} \frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin\big(|\lambda_h|^{1/2}(T-s)\big)h\,\mathrm{d}\langle h, W_s\rangle_H \\ \mu_h \int_0^T \cos\big(|\lambda_h|^{1/2}(T-s)\big)h\,\mathrm{d}\langle h, W_s\rangle_H \end{pmatrix} \\
&= \begin{pmatrix} \sum_{h\in\mathbb{H}} \Big(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin\big(|\lambda_h|^{1/2}(T-s)\big)\,\mathrm{d}\langle h, W_s\rangle_H \Big) h \\ \sum_{h\in\mathbb{H}} \Big(\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos\big(|\lambda_h|^{1/2}(T-s)\big)\,\mathrm{d}\langle h, W_s\rangle_H \Big) |\lambda_h|^{1/2} h \end{pmatrix}.
\end{aligned}
\tag{4.2}
$$

Furthermore, Lemma 2.7 shows that for all $I \in \mathcal{P}(\mathbb{H})$ it holds $\mathbb{P}$-a.s. that

$$
\mathbf{P}_I\mathbf{X}^{\mathbb{H}} = \int_0^T \mathbf{P}_I\mathrm{e}^{\mathbf{A}(T-s)}\mathbf{B}\,\mathrm{d}W_s = \int_0^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{P}_I\mathbf{B}\,\mathrm{d}W_s = \mathbf{X}^I.
\tag{4.3}
$$

This and (4.2) complete the proof of Lemma 4.1. $\qquad\square$

Lemma 4.2. *Assume the setting in Section 4.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then*

(i) *it holds that $\langle h, X^{I,1}\rangle_{H_0}$, $h \in \mathbb{H}$, is a family of independent centred Gaussian random variables,*

(ii) *it holds that $\big\langle |\lambda_h|^{1/2}h, X^{I,2}\big\rangle_{H_{-1/2}}$, $h \in \mathbb{H}$, is a family of independent centred Gaussian random variables,*

(iii) *and for all $h \in \mathbb{H}$ it holds that*

$$
\mathrm{Var}\big(\langle h, X^{I,1}\rangle_{H_0}\big) = \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\frac{1}{2}\left(T - \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right),
\tag{4.4}
$$

$$
\mathrm{Var}\Big(\big\langle |\lambda_h|^{1/2}h, X^{I,2}\big\rangle_{H_{-1/2}}\Big) = \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\frac{1}{2}\left(T + \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right),
\tag{4.5}
$$

$$
\mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \big\langle |\lambda_h|^{1/2}h, X^{I,2}\big\rangle_{H_{-1/2}}\Big) = \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\left(\frac{1 - \cos\big(2|\lambda_h|^{1/2}T\big)}{4|\lambda_h|^{1/2}}\right).
\tag{4.6}
$$

Proof of Lemma 4.2. Observe that Lemma 4.1 implies (i) and (ii). It thus remains to prove (iii). Lemma 4.1 implies for all $h \in \mathbb{H}$ that it holds $\mathbb{P}$-a.s. that

$$
\langle h, X^{I,1}\rangle_{H_0} = \mathbb{1}_I^{\mathbb{H}}(h)\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \sin\big(|\lambda_h|^{1/2}(T-s)\big)\,\mathrm{d}\langle h, W_s\rangle_H,
\tag{4.7}
$$

$$
\big\langle |\lambda_h|^{1/2}h, X^{I,2}\big\rangle_{H_{-1/2}} = \mathbb{1}_I^{\mathbb{H}}(h)\frac{\mu_h}{|\lambda_h|^{1/2}} \int_0^T \cos\big(|\lambda_h|^{1/2}(T-s)\big)\,\mathrm{d}\langle h, W_s\rangle_H.
\tag{4.8}
$$

Itô's isometry hence shows for all $h \in \mathbb{H}$ that

$$
\begin{aligned}
\mathrm{Var}\big(\langle h, X^{I,1}\rangle_{H_0}\big) &= \mathbb{E}\big[|\langle h, X^{I,1}\rangle_{H_0}|^2\big] = \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\int_0^T \big|\sin\big(|\lambda_h|^{1/2}(T-s)\big)\big|^2 \, ds \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\frac{1}{2}\left(T - \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right),
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
\mathrm{Var}\big(\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\big) &= \mathbb{E}\big[\big|\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\big|^2\big] \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\int_0^T \big|\cos\big(|\lambda_h|^{1/2}(T-s)\big)\big|^2 \, ds \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\frac{1}{2}\left(T + \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right).
\end{aligned}
\tag{4.10}
$$

Furthermore, observe for all $h \in \mathbb{H}$ that

$$
\begin{aligned}
\mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big) &= \mathbb{E}\Big[\langle h, X^{I,1}\rangle_{H_0}\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big] \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\int_0^T \sin\big(|\lambda_h|^{1/2}(T-s)\big)\cos\big(|\lambda_h|^{1/2}(T-s)\big) \, ds \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\left(\frac{\big|\sin\big(|\lambda_h|^{1/2}T\big)\big|^2}{2|\lambda_h|^{1/2}}\right) \\
&= \mathbb{1}_I^{\mathbb{H}}(h)\frac{|\mu_h|^2}{|\lambda_h|}\left(\frac{1 - \cos\big(2|\lambda_h|^{1/2}T\big)}{4|\lambda_h|^{1/2}}\right).
\end{aligned}
\tag{4.11}
$$

The proof of Lemma 4.2 is thus completed. $\qquad\square$

Corollary 4.3. *Assume the setting in Section 4.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then for all $(v,w) \in \mathbf{H}_0$ it holds that*

$$
\begin{aligned}
\mathrm{CovOp}(\mathbf{X}^I)(v,w) = \frac{1}{2}\frac{|\mu_h|^2}{|\lambda_h|}\sum_{h \in I}\Bigg[&\left(T - \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right)\langle h, v\rangle_{H_0}\begin{pmatrix} h \\ 0 \end{pmatrix} \\
&+ \left(\frac{1 - \cos\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right)\langle |\lambda_h|^{1/2}h, w\rangle_{H_{-1/2}}\begin{pmatrix} h \\ 0 \end{pmatrix} \\
&+ \left(\frac{1 - \cos\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right)\langle h, v\rangle_{H_0}\begin{pmatrix} 0 \\ |\lambda_h|^{1/2}h \end{pmatrix} \\
&+ \left(T + \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{2|\lambda_h|^{1/2}}\right)\langle |\lambda_h|^{1/2}h, w\rangle_{H_{-1/2}}\begin{pmatrix} 0 \\ |\lambda_h|^{1/2}h \end{pmatrix}\Bigg].
\end{aligned}
\tag{4.12}
$$

Proof of Corollary 4.3. Lemma 4.2 and Lemma 4.1 prove for all $x_1 = (v_1, w_1)$, $x_2 = (v_2, w_2) \in \mathbf{H}_0$

that

$$\langle x_1, \mathrm{CovOp}(\mathbf{X}^I)x_2\rangle_{\mathbf{H}_0} = \mathrm{Cov}\big(\langle x_1, \mathbf{X}^I\rangle_{\mathbf{H}_0}, \langle x_2, \mathbf{X}^I\rangle_{\mathbf{H}_0}\big) = \mathbb{E}\big[\langle x_1, \mathbf{X}^I\rangle_{\mathbf{H}_0}\langle x_2, \mathbf{X}^I\rangle_{\mathbf{H}_0}\big]$$

$$= \mathbb{E}\big[\big(\langle v_1, X^{I,1}\rangle_{H_0} + \langle w_1, X^{I,2}\rangle_{H_{-1/2}}\big)\big(\langle v_2, X^{I,1}\rangle_{H_0} + \langle w_2, X^{I,2}\rangle_{H_{-1/2}}\big)\big]$$

$$= \sum_{h\in\mathbb{H}} \mathrm{Var}\big(\langle h, X^{I,1}\rangle_{H_0}\big)\langle h, v_1\rangle_{H_0}\langle h, v_2\rangle_{H_0}$$

$$+ \sum_{h\in\mathbb{H}} \mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle h, v_1\rangle_{H_0}\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}$$

$$+ \sum_{h\in\mathbb{H}} \mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle h, v_2\rangle_{H_0}\langle |\lambda_h|^{1/2}h, w_1\rangle_{H_{-1/2}}$$

$$+ \sum_{h\in\mathbb{H}} \mathrm{Var}\Big(\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle |\lambda_h|^{1/2}h, w_1\rangle_{H_{-1/2}}\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}$$

$$= \Big\langle v_1, \sum_{h\in\mathbb{H}}\Big[\mathrm{Var}\big(\langle h, X^{I,1}\rangle_{H_0}\big)\langle h, v_2\rangle_{H_0}$$

$$+ \mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}\Big]h\Big\rangle_{H_0} \qquad (4.13)$$

$$+ \Big\langle w_1, \sum_{h\in\mathbb{H}}\Big[\mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle h, v_2\rangle_{H_0}$$

$$+ \mathrm{Var}\Big(\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}\Big]|\lambda_h|^{1/2}h\Big\rangle_{H_{-1/2}}$$

$$= \Big\langle x_1, \sum_{h\in\mathbb{H}}\Big[\mathrm{Var}\big(\langle h, X^{I,1}\rangle_{H_0}\big)\langle h, v_2\rangle_{H_0}$$

$$+ \mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}\Big]\begin{pmatrix} h \\ 0 \end{pmatrix}\Big\rangle_{\mathbf{H}_0}$$

$$+ \Big\langle x_1, \sum_{h\in\mathbb{H}}\Big[\mathrm{Cov}\Big(\langle h, X^{I,1}\rangle_{H_0}, \langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle h, v_2\rangle_{H_0}$$

$$+ \mathrm{Var}\Big(\langle |\lambda_h|^{1/2}h, X^{I,2}\rangle_{H_{-1/2}}\Big)\langle |\lambda_h|^{1/2}h, w_2\rangle_{H_{-1/2}}\Big]\begin{pmatrix} 0 \\ |\lambda_h|^{1/2}h \end{pmatrix}\Big\rangle_{\mathbf{H}_0}.$$

This and again Lemma 4.2 complete the proof of Corollary 4.3. $\qquad\square$

Lemma 4.4. *Assume the setting in Section 4.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then for all $i \in \{1,2\}$ it holds that $\mathbf{X}^I \in L^2(\mathbb{P}; \mathbf{H}_0)$ and*

$$\mathbb{E}\big[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2\big] = T\sum_{h\in I}\frac{|\mu_h|^2}{|\lambda_h|} < \infty, \qquad (4.14)$$

$$\mathbb{E}\big[\|X^{I,i}\|_{H_{1/2-i/2}}^2\big] = \frac{1}{2}\sum_{h\in I}\frac{|\mu_h|^2}{|\lambda_h|}\left(T + \frac{\sin\big(2|\lambda_h|^{1/2}T\big)}{(-1)^i 2|\lambda_h|^{1/2}}\right) < \infty. \qquad (4.15)$$

Proof of Lemma 4.4. Itô's isometry and Lemma 2.6 imply that

$$\mathbb{E}\big[\|\mathbf{X}^I\|_{\mathbf{H}_0}^2\big] = \mathbb{E}\left[\left\|\int_0^T \mathrm{e}^{\mathbf{A}(T-s)}\mathbf{P}_I\mathbf{B}\,\mathrm{d}W_s\right\|_{\mathbf{H}_0}^2\right] = T\|\mathbf{P}_I\mathbf{B}\|_{L_2(H,\mathbf{H}_0)}^2 = T\sum_{h\in I}\frac{|\mu_h|^2}{|\lambda_h|} < \infty. \qquad (4.16)$$

In addition, Lemma 4.2 shows for all $i \in \{1, 2\}$ that

$$
\mathbb{E}\left[\|X^{I,i}\|^2_{H_{1/2-i/2}}\right] = \sum_{h \in \mathbb{H}} \mathbb{E}\left[\left|\left\langle|\lambda_h|^{i/2-1/2}h, X^{I,i}\right\rangle_{H_{1/2-i/2}}\right|^2\right]
$$
$$
= \frac{1}{2}\sum_{h \in I}\frac{|\mu_h|^2}{|\lambda_h|}\left(T + \frac{\sin\left(2|\lambda_h|^{1/2}T\right)}{(-1)^i 2|\lambda_h|^{1/2}}\right) < \infty. \tag{4.17}
$$

The proof of Lemma 4.4 is thus completed. □

Proposition 4.5. *Assume the setting in Section 4.1 and let* $\inf_{h \in \mathbb{H}}|\mu_h| > 0$. *Then for all* $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ *it holds that*

$$
\mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right] - \mathbb{E}\left[\|\mathbf{X}^I\|^2_{\mathbf{H}_0}\right] = \mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}\setminus I}\|^2_{\mathbf{H}_0}\right] \geq T\inf_{h \in \mathbb{H}}|\mu_h|^2\sum_{h \in \mathbb{H}\setminus I}\frac{1}{|\lambda_h|} > 0. \tag{4.18}
$$

Proof of Proposition 4.5. Orthogonality and Lemma 4.1 imply for all $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ that

$$
\mathbb{E}\left[\|\mathbf{X}^I\|^2_{\mathbf{H}_0}\right] + \mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}\setminus I}\|^2_{\mathbf{H}_0}\right] = \mathbb{E}\left[\|\mathbf{P}_I\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right] + \mathbb{E}\left[\|\mathbf{P}_{\mathbb{H}\setminus I}\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right]
$$
$$
= \mathbb{E}\left[\|(\mathbf{P}_I + \mathbf{P}_{\mathbb{H}\setminus I})\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right] = \mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right]. \tag{4.19}
$$

This and Lemma 4.4 show for all $I \in \mathcal{P}(\mathbb{H}) \setminus \{\mathbb{H}\}$ that

$$
\mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right] - \mathbb{E}\left[\|\mathbf{X}^I\|^2_{\mathbf{H}_0}\right] = \mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}\setminus I}\|^2_{\mathbf{H}_0}\right]
$$
$$
= T\sum_{h \in \mathbb{H}\setminus I}\frac{|\mu_h|^2}{|\lambda_h|} \geq T\inf_{h \in \mathbb{H}}|\mu_h|^2\sum_{h \in \mathbb{H}\setminus I}\frac{1}{|\lambda_h|} > 0. \tag{4.20}
$$

The proof of Proposition 4.5 is thus completed. □

In Corollary 4.7 and Corollary 4.8 below lower bounds on the weak approximation error with the squared norm as test function are presented. Our proofs of Corollary 4.7 and Corollary 4.8 use the following elementary and well-known lemma (cf., e.g., Proposition 6.4 in Conus et al. [10]).

Lemma 4.6. *Let* $p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$. *Then for all* $N \in \mathbb{N}$ *it holds that*

$$
\sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \geq \frac{N^{p(2\delta-1)+1}}{[p(1-2\delta)-1]2^{p(1-2\delta)-1}}. \tag{4.21}
$$

Proof of Lemma 4.6. Observe that the assumption that $\delta \in (-\infty, 1/2 - 1/(2p))$ ensures that $p(2\delta - 1) \in (-\infty, -1)$. This implies for all $N \in \mathbb{N}$ that

$$
\sum_{n=N+1}^{\infty} n^{p(2\delta-1)} = \sum_{n=N+1}^{\infty}\int_n^{n+1} n^{p(2\delta-1)}\,\mathrm{d}x \geq \sum_{n=N+1}^{\infty}\int_n^{n+1} x^{p(2\delta-1)}\,\mathrm{d}x
$$
$$
= \int_{N+1}^{\infty} x^{p(2\delta-1)}\,\mathrm{d}x = -\frac{(N+1)^{p(2\delta-1)+1}}{p(2\delta-1)+1} \tag{4.22}
$$
$$
\geq \frac{N^{p(2\delta-1)+1}}{[p(1-2\delta)-1]2^{p(1-2\delta)-1}}.
$$

This completes the proof of Lemma 4.6. □

Corollary 4.7. *Assume the setting in Section 4.1, let* $c \in (0, \infty), p \in (1, \infty)$, *let* $e\colon \mathbb{N} \to \mathbb{H}$ *be a bijection such that for all* $n \in \mathbb{N}$ *it holds that* $\lambda_{e_n} = -cn^p$, *and let* $I_N \in \mathcal{P}(\mathbb{H}), N \in \mathbb{N}$, *be the sets which satisfy for all* $N \in \mathbb{N}$ *that* $I_N = \{e_1, e_2, \ldots, e_N\} \subseteq \mathbb{H}$. *Then for all* $N \in \mathbb{N}$ *it holds that*

$$
\mathbb{E}\left[\|\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\right] - \mathbb{E}\left[\|\mathbf{X}^{I_N}\|^2_{\mathbf{H}_0}\right] \geq \frac{T\inf_{h \in \mathbb{H}}|\mu_h|^2 N^{1-p}}{c(p-1)2^{p-1}}. \tag{4.23}
$$

Proof of Corollary 4.7. Proposition 4.5 and Lemma 4.6 prove for all $N \in \mathbb{N}$ that

$$\mathbb{E}\big[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2\big] - \mathbb{E}\big[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2\big] \geq T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{h \in \mathbb{H} \setminus I_N} \frac{1}{|\lambda_h|} = c^{-1} T \inf_{h \in \mathbb{H}} |\mu_h|^2 \sum_{n=N+1}^{\infty} \frac{1}{n^p}$$

$$\geq \frac{T \inf_{h \in \mathbb{H}} |\mu_h|^2 N^{1-p}}{c(p-1)2^{p-1}}. \tag{4.24}$$

The proof of Corollary 4.7 is thus completed. $\qquad\square$

Corollary 4.8. *Assume the setting in Section 4.1, let $c, p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$, let $e \colon \mathbb{N} \to \mathbb{H}$ be a bijection such that for all $n \in \mathbb{N}$ it holds that $\lambda_{e_n} = -cn^p$, let $\mu \colon \mathbb{H} \to \mathbb{R}$ be the mapping which satisfies for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^\delta$, and let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$. Then for all $N \in \mathbb{N}$ it holds that*

$$\mathbb{E}\big[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2\big] - \mathbb{E}\big[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2\big] \geq \frac{T c^{2\delta-1} N^{p(2\delta-1)+1}}{[p(1-2\delta)-1]2^{p(1-2\delta)-1}}. \tag{4.25}$$

Proof of Corollary 4.8. Proposition 4.5, Lemma 4.4, and Lemma 4.6 show for all $N \in \mathbb{N}$ that

$$\mathbb{E}\big[\|\mathbf{X}^{\mathbb{H}}\|_{\mathbf{H}_0}^2\big] - \mathbb{E}\big[\|\mathbf{X}^{I_N}\|_{\mathbf{H}_0}^2\big] = T \sum_{h \in \mathbb{H} \setminus I_N} \frac{|\mu_h|^2}{|\lambda_h|} = T \sum_{h \in \mathbb{H} \setminus I_N} |\lambda_h|^{2\delta-1}$$

$$= T c^{2\delta-1} \sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \geq \frac{T c^{2\delta-1} N^{p(2\delta-1)+1}}{[p(1-2\delta)-1]2^{p(1-2\delta)-1}}. \tag{4.26}$$

This completes the proof of Corollary 4.8. $\qquad\square$

4.3 Lower bounds for the weak error of a particular regular test function

The next proposition, Proposition 4.9 below, follows directly from Lemma 4.2 above and Lemma 9.5 in Jentzen & Kurniawan [24].

Proposition 4.9. *Assume the setting in Section 4.1 and let $\varphi_i \colon \mathbf{H}_0 \to \mathbb{R}$, $i \in \{1, 2\}$, be the mappings which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp\big(-\|v_i\|_{H_{1/2-i/2}}^2\big)$. Then for all $i \in \{1, 2\}$, $I \in \mathcal{P}(\mathbb{H})$ it holds that $\varphi_i \in C_{\mathrm{b}}^2(\mathbf{H}_0, \mathbb{R})$ and*

$$\mathbb{E}[\varphi_i(\mathbf{X}^I)] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \geq \frac{\mathbb{E}\big[\|X^{\mathbb{H},i}\|_{H_{1/2-i/2}}^2\big] - \mathbb{E}\big[\|X^{I,i}\|_{H_{1/2-i/2}}^2\big]}{\exp\big(6\,\mathbb{E}\big[\|X^{\mathbb{H},i}\|_{H_{1/2-i/2}}^2\big]\big)}. \tag{4.27}$$

Corollary 4.10. *Assume the setting in Section 4.1, let $c, p \in (0, \infty)$, $\delta \in (-\infty, 1/2 - 1/(2p))$, let $e \colon \mathbb{N} \to \mathbb{H}$ be a bijection such that for all $n \in \mathbb{N}$ it holds that $\lambda_{e_n} = -cn^p$, let $\mu \colon \mathbb{H} \to \mathbb{R}$ be the mapping which satisfies for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^\delta$, let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$, and let $\varphi_i \colon \mathbf{H}_0 \to \mathbb{R}$, $i \in \{1, 2\}$, be the mappings which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp\big(-\|v_i\|_{H_{1/2-i/2}}^2\big)$. Then for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ it holds that $\varphi_i \in C_{\mathrm{b}}^2(\mathbf{H}_0, \mathbb{R})$ and*

$$\mathbb{E}[\varphi_i(\mathbf{X}^{I_N})] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \geq \Big[1 + \inf_{x \in [2c^{1/2}T, \infty)} \frac{\sin(x)}{(-1)^i x}\Big] \frac{T c^{2\delta-1} 2^{p(2\delta-1)} N^{p(2\delta-1)+1}}{[p(1-2\delta)-1] \exp\big(\frac{6p(2\delta-1)Tc^{2\delta-1}}{p(2\delta-1)+1}\big)} > 0. \tag{4.28}$$

Proof of Corollary 4.10. Lemma 4.4, Lemma 4.6, and the fact that $\forall x \in (0, \infty) \colon \left| \frac{\sin(x)}{x} \right| < 1$ prove for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ that

$$
\begin{aligned}
\mathbb{E}\Big[\|X^{\mathbb{H},i}\|^2_{H_{1/2-i/2}}\Big] - \mathbb{E}\Big[\|X^{I,i}\|^2_{H_{1/2-i/2}}\Big] &= \frac{1}{2} \sum_{h \in \mathbb{H} \setminus I_N} \frac{|\mu_h|^2}{|\lambda_h|} \left(T + \frac{\sin\big(2|\lambda_h|^{1/2} T\big)}{(-1)^i 2 |\lambda_h|^{1/2}} \right) \\
&\geq \left(1 + \inf_{h \in \mathbb{H}} \frac{\sin\big(2|\lambda_h|^{1/2} T\big)}{(-1)^i 2 |\lambda_h|^{1/2} T} \right) \frac{T}{2} \sum_{h \in \mathbb{H} \setminus I_N} |\lambda_h|^{2\delta-1} \\
&\geq \left(1 + \inf_{x \in [2c^{1/2} T, \infty)} \frac{\sin(x)}{(-1)^i x} \right) \frac{T c^{2\delta-1}}{2} \sum_{n=N+1}^{\infty} n^{p(2\delta-1)} \\
&\geq \left(1 + \inf_{x \in [2c^{1/2} T, \infty)} \frac{\sin(x)}{(-1)^i x} \right) \frac{T c^{2\delta-1} 2^{p(2\delta-1)} N^{p(2\delta-1)+1}}{[p(1-2\delta)-1]} > 0.
\end{aligned}
\tag{4.29}
$$

Furthermore, the assumption that $\delta \in (-\infty, 1/2 - 1/(2p))$ ensures that $p(2\delta - 1) \in (-\infty, -1)$. Hence, we obtain that

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{p(2\delta-1)} &\leq 1 + \sum_{n=1}^{\infty} \int_n^{n+1} x^{p(2\delta-1)} \, \mathrm{d}x = 1 + \int_1^{\infty} x^{p(2\delta-1)} \, \mathrm{d}x \\
&= 1 - \frac{1}{p(2\delta-1)+1} = \frac{p(2\delta-1)}{p(2\delta-1)+1}.
\end{aligned}
\tag{4.30}
$$

Lemma 4.4 and (4.30) imply for all $i \in \{1, 2\}$ that

$$
\begin{aligned}
\exp\Big(-6 \, \mathbb{E}\Big[\|X^{\mathbb{H},i}\|^2_{H_{1/2-i/2}}\Big]\Big) &\geq \exp\big(-6 \, \mathbb{E}\big[\|\mathbf{X}^{\mathbb{H}}\|^2_{\mathbf{H}_0}\big]\big) = \exp\left(-6 T c^{2\delta-1} \sum_{n=1}^{\infty} n^{p(2\delta-1)} \right) \\
&\geq \exp\left(-\frac{6 p(2\delta-1) T c^{2\delta-1}}{p(2\delta-1)+1} \right) > 0.
\end{aligned}
\tag{4.31}
$$

Combining this and (4.29) with Proposition 4.9 concludes the proof of Corollary 4.10. $\qquad\square$

Corollary 4.11 below specifies Corollary 4.10 to the case where the linear operator $A \colon D(A) \subseteq H \to H$ in the setting in Section 4.1 is the Laplacian with Dirichlet boundary conditions on H. It is an immediate consequence of Corollary 4.10.

Corollary 4.11. *Assume the setting in Section 4.1, let $\delta \in (-\infty, 1/4)$, let $e \colon \mathbb{N} \to \mathbb{H}$ be a bijection such that for all $n \in \mathbb{N}$ it holds that $\lambda_{e_n} = -\pi^2 n^2$, let $\mu \colon \mathbb{H} \to \mathbb{R}$ be the mapping which satisfies for all $h \in \mathbb{H}$ that $|\mu_h| = |\lambda_h|^{\delta}$, let $I_N \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, be the sets which satisfy for all $N \in \mathbb{N}$ that $I_N = \{e_1, e_2, \dots, e_N\} \subseteq \mathbb{H}$, and let $\varphi_i \colon \mathbf{H}_0 \to \mathbb{R}$, $i \in \{1, 2\}$, be the mappings which satisfy for all $i \in \{1, 2\}$, $(v_1, v_2) \in \mathbf{H}_0$ that $\varphi_i(v_1, v_2) = \exp\big(-\|v_i\|^2_{H_{1/2-i/2}}\big)$. Then for all $i \in \{1, 2\}$, $N \in \mathbb{N}$ it holds that $\varphi_i \in C_b^2(\mathbf{H}_0, \mathbb{R})$ and*

$$
\mathbb{E}[\varphi_i(\mathbf{X}^{I_N})] - \mathbb{E}[\varphi_i(\mathbf{X}^{\mathbb{H}})] \geq \left[1 + \inf_{x \in [2\pi T, \infty)} \frac{\sin(x)}{(-1)^i x} \right] \frac{T (4\pi^2)^{2\delta-1} N^{4\delta-1}}{[1-4\delta] \exp\big(\frac{12(2\delta-1) T \pi^{4\delta-2}}{4\delta-1} \big)} > 0.
\tag{4.32}
$$

References

[1] ANDERSSON, A., KOVÁCS, M., AND LARSSON, S. Weak error analysis for semilinear stochastic Volterra equations with additive noise. *ArXiv e-prints* (Nov. 2014). arXiv: 1411.6476 [math.NA].

[2] ANDERSSON, A., KRUSE, R., AND LARSSON, S. Duality in refined Sobolev-Malliavin spaces and weak approximations of SPDE. *ArXiv e-prints* (Dec. 2013). arXiv: 1312.5893 [math.PR].

[3] ANDERSSON, A., AND LARSSON, S. Weak convergence for a spatial approximation of the nonlinear stochastic heat equation. *ArXiv e-prints* (Dec. 2012). arXiv: 1212.5564 [math.NA].

[4] ANTON, R., COHEN, D., LARSSON, S., AND WANG, X. Full discretisation of semi-linear stochastic wave equations driven by multiplicative noise. *ArXiv e-prints* (Feb. 2015). arXiv: 1503.00073 [math.NA].

[5] BRÉHIER, C.-E., AND KOPEC, M. Approximation of the invariant law of SPDEs: error analysis using a Poisson equation for a full-discretization scheme. *ArXiv e-prints* (Nov. 2013). arXiv: 1311.7030 [math.NA].

[6] BRÉHIER, C.-E. Approximation of the invariant measure with an Euler scheme for stochastic PDEs driven by space-time white noise. *Potential Anal.* 40, 1 (2014), 1–40. ISSN: 0926-2601. DOI: 10.1007/s11118-013-9338-9. URL: http://dx.doi.org/10.1007/s11118-013-9338-9.

[7] BRÉHIER, C.-E. Strong and weak orders in averaging for SPDEs. *Stochastic Process. Appl.* 122, 7 (2012), 2553–2593. ISSN: 0304-4149. DOI: 10.1016/j.spa.2012.04.007. URL: http://dx.doi.org/10.1016/j.spa.2012.04.007.

[8] COHEN, D., LARSSON, S., AND SIGG, M. A trigonometric method for the linear stochastic wave equation. *SIAM J. Numer. Anal.* 51, 1 (2013), 204–222. ISSN: 0036-1429. DOI: 10.1137/12087030X. URL: http://dx.doi.org/10.1137/12087030X.

[9] COHEN, D., AND QUER-SARDANYONS, L. A fully discrete approximation of the one-dimensional stochastic wave equation. *IMA Journal of Numerical Analysis* (2015). DOI: 10.1093/imanum/drv006. eprint: http://imajna.oxfordjournals.org/content/early/2015/03/11/imanum.drv006.full.pdf+html. URL: http://imajna.oxfordjournals.org/content/early/2015/03/11/imanum.drv006.abstract.

[10] CONUS, D., JENTZEN, A., AND KURNIAWAN, R. Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. *ArXiv e-prints* (Aug. 2014). arXiv: 1408.1108 [math.PR].

[11] CONUS, D., JOSEPH, M., KHOSHNEVISAN, D., AND SHIU, S.-Y. Intermittency and chaos for a stochastic non-linear wave equation in dimension 1. *ArXiv e-prints* (Dec. 2011). arXiv: 1112.1909 [math.PR].

[12] DA PRATO, G., JENTZEN, A., AND ROECKNER, M. A mild Ito formula for SPDEs. *ArXiv e-prints* (Sept. 2010). arXiv: 1009.3526 [math.PR].

[13] DA PRATO, G., AND ZABCZYK, J. *Stochastic equations in infinite dimensions*. Vol. 44. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992, xviii+454. ISBN: 0-521-38529-6. DOI: 10.1017/CBO9780511666223. URL: http://dx.doi.org/10.1017/CBO9780511666223.

[14] DALANG, R. C. The Stochastic Wave Equation. In: *A Minicourse on Stochastic Partial Differential Equations*. Ed. by KHOSHNEVISAN, D., AND RASSOUL-AGHA, F. Vol. 1962. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2009, 39–71. ISBN: 978-3-540-85993-2. DOI: 10.1007/978-3-540-85994-9_2. URL: http://dx.doi.org/10.1007/978-3-540-85994-9_2.

[15] DAVIE, A. M., AND GAINES, J. G. Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Math. Comp.* 70, 233 (2001), 121–134. ISSN: 0025-5718. DOI: 10.1090/S0025-5718-00-01224-2. URL: http://dx.doi.org/10.1090/S0025-5718-00-01224-2.

[16] DE BOUARD, A., AND DEBUSSCHE, A. Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation. *Appl. Math. Optim.* 54, 3 (2006), 369–399. ISSN: 0095-4616. DOI: 10.1007/s00245-006-0875-0. URL: http://dx.doi.org/10.1007/s00245-006-0875-0.

[17] DEBUSSCHE, A. Weak approximation of stochastic partial differential equations: the nonlinear case. *Math. Comp.* 80, 273 (2011), 89–117. ISSN: 0025-5718. DOI: 10.1090/S0025-5718-2010-02395-6. URL: http://dx.doi.org/10.1090/S0025-5718-2010-02395-6.

[18] DEBUSSCHE, A., AND PRINTEMS, J. Weak order for the discretization of the stochastic heat equation. *Math. Comp.* 78, 266 (2009), 845–863. ISSN: 0025-5718. DOI: 10.1090/S0025-5718-08-02184-4. URL: http://dx.doi.org/10.1090/S0025-5718-08-02184-4.

[19] GEISSERT, M., KOVÁCS, M., AND LARSSON, S. Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise. *BIT* 49, 2 (2009), 343–356. ISSN: 0006-3835. DOI: 10.1007/s10543-009-0227-y. URL: http://dx.doi.org/10.1007/s10543-009-0227-y.

[20] HAUSENBLAS, E. Weak approximation for semilinear stochastic evolution equations. In: *Stochastic analysis and related topics VIII*. Vol. 53. Progr. Probab. Birkhäuser, Basel, 2003, 111–128.

[21] HAUSENBLAS, E. Weak approximation of the stochastic wave equation. *J. Comput. Appl. Math.* 235, 1 (2010), 33–58. ISSN: 0377-0427. DOI: 10.1016/j.cam.2010.03.026. URL: http://dx.doi.org/10.1016/j.cam.2010.03.026.

[22] HENRY, D. *Geometric theory of semilinear parabolic equations*. Vol. 840. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1981, iv+348. ISBN: 3-540-10557-3.

[23] JENTZEN, A. Stochastic Partial Differential Equations: Analysis and Numerical Approximations. Lecture Notes, ETH Zürich. 2015. URL: https://www2.math.ethz.ch/education/bachelor/lectures/fs2015/math/numsol/NASPDE_128.pdf.

[24] JENTZEN, A., AND KURNIAWAN, R. Weak convergence rates for Euler-type approximations of semilinear stochastic evolution equations with nonlinear diffusion coefficients. *ArXiv e-prints* (Jan. 2015). arXiv: 1501.03539 [math.PR].

[25] KLOEDEN, P. E., AND PLATEN, E. *Numerical solution of stochastic differential equations*. Vol. 23. Applications of Mathematics (New York). Springer-Verlag, Berlin, 1992, xxxvi+632. ISBN: 3-540-54062-8. DOI: 10.1007/978-3-662-12616-5. URL: http://dx.doi.org/10.1007/978-3-662-12616-5.

[26] KOPEC, M. Quelques contributions à l'analyse numérique d'équations stochastiques. PhD thesis. Ecole normale supérieure de Rennes - ENS Rennes, June 2014. URL: https://tel.archives-ouvertes.fr/tel-01064811.

[27] KOVÁCS, M., LINDNER, F., AND SCHILLING, R. L. Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise. *ArXiv e-prints* (Nov. 2014). arXiv: 1411.1051 [math.PR].

[28] KOVÁCS, M., LARSSON, S., AND LINDGREN, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise. *BIT* 52, 1 (2012), 85–108. ISSN: 0006-3835. DOI: 10.1007/s10543-011-0344-2. URL: http://dx.doi.org/10.1007/s10543-011-0344-2.

[29] KOVÁCS, M., LARSSON, S., AND LINDGREN, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II. Fully discrete schemes. *BIT* 53, 2 (2013), 497–525. ISSN: 0006-3835.

[30] KOVÁCS, M., LARSSON, S., AND SAEDPANAH, F. Finite element approximation of the linear stochastic wave equation with additive noise. *SIAM J. Numer. Anal.* 48, 2 (2010), 408–427. ISSN: 0036-1429. DOI: 10.1137/090772241. URL: http://dx.doi.org/10.1137/090772241.

[31] KOVÁCS, M., AND PRINTEMS, J. Weak convergence of a fully discrete approximation of a linear stochastic evolution equation with a positive-type memory term. *J. Math. Anal. Appl.* 413, 2 (2014), 939–952. ISSN: 0022-247X. DOI: 10.1016/j.jmaa.2013.12.034. URL: http://dx.doi.org/10.1016/j.jmaa.2013.12.034.

[32] KRUSE, R. *Strong and weak approximation of semilinear stochastic evolution equations.* Vol. 2093. Lecture Notes in Mathematics. Springer, Cham, 2014, xiv+177. ISBN: 978-3-319-02230-7; 978-3-319-02231-4. DOI: 10.1007/978-3-319-02231-4. URL: http://dx.doi.org/10.1007/978-3-319-02231-4.

[33] KRYLOV, N. V. On Kolmogorov's equations for finite-dimensional diffusions. In: *Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998).* Vol. 1715. Lecture Notes in Math. Springer, Berlin, 1999, 1–63. DOI: 10.1007/BFb0092417. URL: http://dx.doi.org/10.1007/BFb0092417.

[34] LINDGREN, F. On weak and strong convergence of numerical approximations of stochastic partial differential equations. PhD thesis. Chalmers University of Technology, 2012. ISBN: 978-91-7385-787-1.

[35] LINDNER, F., AND SCHILLING, R. L. Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *Potential Anal.* 38, 2 (2013), 345–379. ISSN: 0926-2601. DOI: 10.1007/s11118-012-9276-y. URL: http://dx.doi.org/10.1007/s11118-012-9276-y.

[36] MILSTEIN, G. N. *Numerical integration of stochastic differential equations.* Vol. 313. Mathematics and its Applications. Translated and revised from the 1988 Russian original. Kluwer Academic Publishers Group, Dordrecht, 1995, viii+169. ISBN: 0-7923-3213-X. DOI: 10.1007/978-94-015-8455-5. URL: http://dx.doi.org/10.1007/978-94-015-8455-5.

[37] MÜLLER-GRONBACH, T., AND RITTER, K. Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Comput. Math.* 7, 2 (2007), 135–181. ISSN: 1615-3375. DOI: 10.1007/s10208-005-0166-6. URL: http://dx.doi.org/10.1007/s10208-005-0166-6.

[38] MÜLLER-GRONBACH, T., RITTER, K., AND WAGNER, T. Optimal pointwise approximation of infinite-dimensional Ornstein-Uhlenbeck processes. *Stoch. Dyn.* 8, 3 (2008), 519–541. ISSN: 0219-4937. DOI: 10.1142/S0219493708002433. URL: http://dx.doi.org/10.1142/S0219493708002433.

[39] QUER-SARDANYONS, L., AND SANZ-SOLÉ, M. Space semi-discretisations for a stochastic wave equation. *Potential Anal.* 24, 4 (2006), 303–332. ISSN: 0926-2601. DOI: 10.1007/s11118-005-9002-0. URL: http://dx.doi.org/10.1007/s11118-005-9002-0.

[40] SELL, G. R., AND YOU, Y. *Dynamics of evolutionary equations.* Vol. 143. Applied Mathematical Sciences. Springer-Verlag, New York, 2002, xiv+670. ISBN: 0-387-98347-3. DOI: 10.1007/978-1-4757-5037-9. URL: http://dx.doi.org/10.1007/978-1-4757-5037-9.

[41] SHARDLOW, T. Weak convergence of a numerical method for a stochastic heat equation. *BIT* 43, 1 (2003), 179–193. ISSN: 0006-3835. DOI: 10.1023/A:1023661308243. URL: http://dx.doi.org/10.1023/A:1023661308243.

[42] THOMAS, L. E. Persistent energy flow for a stochastic wave equation model in nonequilibrium statistical mechanics. *J. Math. Phys.* 53, 9 (2012), 095208, 10. ISSN: 0022-2488. DOI: 10.1063/1.4728986. URL: http://dx.doi.org/10.1063/1.4728986.

[43] WALSH, J. B. On numerical solutions of the stochastic wave equation. *Illinois J. Math.* 50, 1-4 (2006), 991–1018 (electronic). ISSN: 0019-2082. URL: `http://projecteuclid.org/euclid.ijm/1258059497`.

[44] WANG, X. An exponential integrator scheme for time discretization of nonlinear stochastic wave equation. *J. Sci. Comput.* 64, 1 (2015), 234–263. ISSN: 0885-7474. DOI: `10.1007/s10915-014-9931-0`. URL: `http://dx.doi.org/10.1007/s10915-014-9931-0`.

[45] WANG, X. Weak error estimates of the exponential Euler scheme for semi-linear SPDEs without Malliavin calculus. *Discrete Contin. Dyn. Syst.* 36, 1 (2016), 481–497. ISSN: 1078-0947. DOI: `10.3934/dcds.2016.36.481`. URL: `http://dx.doi.org/10.3934/dcds.2016.36.481`.

[46] WANG, X., AND GAN, S. Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise. *J. Math. Anal. Appl.* 398, 1 (2013), 151–169. ISSN: 0022-247X. DOI: `10.1016/j.jmaa.2012.08.038`. URL: `http://dx.doi.org/10.1016/j.jmaa.2012.08.038`.

[47] WANG, X., GAN, S., AND TANG, J. Higher order strong approximations of semilinear stochastic wave equation with additive space-time white noise. *SIAM J. Sci. Comput.* 36, 6 (2014), A2611–A2632. ISSN: 1064-8275. DOI: `10.1137/130937524`. URL: `http://dx.doi.org/10.1137/130937524`.

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