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SPARSE ADAPTIVE TENSOR GALERKIN APPROXIMATIONS OF STOCHASTIC PDE-CONSTRAINED CONTROL PROBLEMS*

ANGELA KUNOTH† AND CHRISTOPH SCHWAB‡

Abstract. For control problems constrained by linear elliptic or parabolic PDEs (partial differential equations) depending on countably many parameters, i.e., on σ_j with $j \in \mathbb{N}$, we proved in [KS] analytic parameter dependence of the state, the co-state and the control. Moreover, we established that these functions allow expansions in terms of sparse tensorized generalized polynomial chaos (gpc) bases. Their sparsity was quantified in terms of $\mathfrak{p}$-summability of the coefficient sequences for some $0 < \mathfrak{p} \leq 1$. Resulting a-priori estimates established the existence of an index set Λ, allowing for concurrent approximations of state, co-state and control for which the gpc approximations attain rates of best N-term approximation.

The regularity and N-term approximation results of [KS] serve as the analytical foundation for the development of adaptive Galerkin approximation methods in the present paper. Following the ideas in [Gi2, SG] and the realizations in [EGSZ, EGSZ, GAS] for a single PDE, we construct deterministic adaptive Galerkin approximations of state, co-state and control on the entire, possibly infinite-dimensional, parameter space. The starting point for these constructions are control problems formulated as abstract symmetric saddle point problems as in [KS]. Specifying this to adaptive wavelet based schemes in space and time, we prove convergence as well as optimal complexity estimates, when compared to best N-term approximations.

Key words. Linear-quadratic optimal control problems, stochastic or parametric coefficients, linear elliptic or parabolic PDE, analyticity, polynomial chaos approximation, symmetric saddle point problems, tensor Galerkin discretization, adaptivity, wavelets, convergence, optimal complexity.

AMS subject classifications. 41A, 65K10, 65N99, 49N10, 65C30.

1. Introduction.

The numerical solution of PDEs (partial differential equations) with random inputs, and the simulation of systems modelled by parametric or stochastic PDEs, has received increasing attention in recent years. On the one hand, in optimization PDEs on high–dimensional design spaces are approximated by sparse polynomial interpolation, or random field solutions of stochastic PDEs driven by noise are approached by means of Wiener chaos [GS, KX, Sch, W].

Solving for a parametric, deterministic representation of the law of the random solution of such problems leads to parametric PDEs depending on a large or even countably infinite number of parameters σ_j for $j \in \mathbb{N}$.

Already for a single scalar elliptic PDE, uncertainty quantification for such countably parametric PDEs pose enormous challenges for numerical simulations. Over the past years, the efficient numerical simulation of such problems has become a very active field of research, see, e.g., [BNT, BTZ, CDS1, GWZ, SG, ST] and the references therein. For optimal control problems governed by such PDEs, an additional challenge arises: even in the completely deterministic situation, during the optimization process arises the need to solve a system of coupled PDEs. At the core of efficient numerical simulation of PDE-control problems depending on countably many parameters is the

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question of *approximation*: state, co-state and control need to be approximated *as parametric functions on the entire parameter space*. Previous studies have focussed either on control problems involving none, or at most only finitely many parameters, see e.g., [CQR, GLL, HLM] and the references therein.

A new approximation paradigm for parametric PDE problems with infinitely many parameters is based on *sparsity* in the solutions' parameter dependence. It has been introduced in [CDS1, CDS2] for a single linear elliptic PDE with PDE operator depending affinely on the parameters $(\sigma_j)_{j\geq 1}$. Affine parameter dependence arises, e.g., for diffusion problems with diffusion coefficients expanded in terms of Karhunen–Loève approximations, see, e.g., [ST] for such expansions and their numerical analysis for elliptic PDEs with stochastic coefficients. It has since been generalized to large classes of countably parametric equations of state with both, affine and general nonlinear holomorphic parameter dependence in [CCS2].

The main line of argument in the new paradigm is as follows. First, it is shown that the solution of the PDE depends *analytically* on $(\sigma_j)_{j\geq 1}$. Even stronger, the solution is shown to be *regular* $\mathfrak{p}$*-analytic* for some $0 < \mathfrak{p} \leq 1$ if the PDE operator is; this notion is defined precisely below. This then allows an expansion of the solution into tensor products of Legendre polynomials with a sparse, $\mathfrak{p}$-summable coefficient sequence. In turn, this enables convergence rates of best N-term polynomial approximations of the parametric solution of the parametric PDE. Based on these theoretical results, corresponding greedy-type algorithms based on tensorized polynomials in the parameter domain and finite elements in the space domain were proposed in [Gi2, SG] together with convergence results and many practical considerations; for implementational aspects for elliptic PDEs with random input data, we refer to [EGSZ, EGSZ, GAS].

In our previous paper [KS], the first step of this paradigm is shown for a large class of PDE-constrained control problems involving linear elliptic and parabolic PDEs, depending on a possibly infinite sequence of parameters $(\sigma_j)_{j\geq 1}$. The main idea is to formulate the first order necessary conditions for optimality as a saddle point problem in the state and the costate variables and prove $\mathfrak{p}$-analyticity of the saddle point operator under the assumption that the underlying PDE operator for the constraining PDE is $\mathfrak{p}$-analytic for some $0 < \mathfrak{p} \leq 1$. Quite surprisingly, this result in turn allows for *concurrent* N-term truncated *gpc (generalized polynomial chaos)* tensorized Legendre expansions for state, adjoint state and control on the *entire* parameter domain at a rate $N^{-(1/\mathfrak{p}-1/2)}$ (in mean square with respect to a probability measure $\boldsymbol{\pi}$ on the space Γ of parameters). This means that *all* three parametric variables occurring in the control problem can be approximated at this rate with N degrees of freedom each in the parameter domain *with the same, finite set* Λ_N *(of cardinality at most* N*) of active polynomials of the parameter*. In addition, as we show in the present paper, the results of [KS] can be sharpened in the sense that the sets Λ_N can be chosen with *monotone structure* which renders them computationally accessible.

Besides, the purpose of the present paper is to provide a practical construction of this index set and the corresponding approximation of the solution triple state, costate and control with respect to the parameters σ and with respect to time t and space x. First, building on the results for a single parameter-dependent PDE from [EGSZ, EGSZ, Gi2, GAS, SG], we show how this approximation in parameter space can be realized by employing greedy-type algorithms. Second, we tie this discretization to an appropriate one with respect to the space and time variables. For this purpose, we will employ adaptive tensor Galerkin approximations based on wavelets. Together,

this allows for sparse adaptive tensor discretizations of control problems constrained by linear elliptic and parabolic PDEs, along the lines as they were developed for the non-parametric case in [DK, GK, K].

The method of attack to derive these results for control problems is to formulate the main results first for a class of abstract saddle point problems as in [KS]. In order to do so, we first specify in Section 2 the classes of PDE-constrained control problems considered here and specify their first order optimality conditions as symmetric saddle point problems. We reformulate the main theoretical results for abstract parametric saddle point problems from [KS] in Section 3. Section 4 specifies the type of wavelet bases for the discretization with respect to space and time so that we can derive in Section 5 a representation of the saddle point problem as a bi-infinite linear system, in terms of parameter σ, space and time. Section 6 recalls the main ingredients of adaptive approximations for parametric elliptic and parabolic problems from [EGSZ, Gi2, GAS, SG] and extends them to the abstract saddle point problem. Section 7 highlights the corresponding tensor product structure for the biinfinite saddle point operator, followed by adaptive Galerkin discretizations in terms of a Riesz basis built by tensorizing a Legendre polynomial chaos in stochastic space and a multiresolution analysis in physical space in Section 8 and an approximation of the deterministic operator in Section 9. Finally, in Section 10, we discuss sparse, adaptive tensor discretizations of the parametric KKT operators.

2. PDE-constrained control problems.

2.1. First order necessary conditions. We begin by recalling some general statements about constrained optimization problems from, e.g., [NW], see also [BS]. Let Y, U be Hilbert spaces over $\mathbb{R}$ which shall host the *state* y of a system and a *control* by which the state can be influenced. Let $J : Y \times U \to \mathbb{R}$ be a functional which is twice differentiable with respect to y and u, and $K : Y \times U \to Y'$ be a (in y, u Fréchet-) differentiable function where Y' denotes the topological dual of Y. We shall be concerned with the *constrained minimization problem*

$$\inf_{(y,u)\in Y\times U} \quad J(y,u) \tag{2.1}$$

$$\text{subject to} \quad K(y,u) = 0. \tag{2.2}$$

For the constraints (2.2), we assume that there exists a unique solution $y \in Y$ for the case that $u \in U$ is given. A typical way to solve (2.1) subject to (2.2) is to compute the zeroes of the first order Fréchet derivatives of the corresponding *Lagrangian functional*. It is built by introducing a new variable p, the *costate* or *adjoint state* in terms of which the constraints (2.2) are appended to the functional (2.1), i.e.,

$$L(y,u,p) := J(y,u) + \langle K(y,u), p\rangle_{Y'\times Y} \tag{2.3}$$

with $L : Y \times U \times Y \to \mathbb{R}$. We always indicate spaces for duality pairings $\langle\cdot,\cdot\rangle$ by subscripts.

Denote by $L_z(y,u,p) := \frac{\partial}{\partial z}L(y,u,p)$ and $L_{zz}(y,u,p) := \frac{\partial^2}{\partial z^2}L(y,u,p)$ the first and second variation, respectively, of L with respect to $z = y, u, p$. Then the necessary conditions for optimality read

$$\delta L(y,u,p) := \begin{pmatrix} L_y(y,u,p) \\ L_u(y,u,p) \\ L_p(y,u,p) \end{pmatrix} = 0 \tag{2.4}$$

which is for (2.1), (2.2)

$$\begin{pmatrix} J_y(y,u) + \langle K_y(y,u), p\rangle_{Y'\times Y} \\ J_u(y,u) + \langle K_u(y,u), p\rangle_{Y'\times Y} \\ K(y,u) \end{pmatrix} = 0. \tag{2.5}$$

If J is now quadratic in both y, u, and K linear in y, u, system (2.5) can be specified to the linear system of equations

$$\begin{pmatrix} L_{yy} & L_{yu} & K_y^* \\ L_{uy} & L_{uu} & K_u^* \\ K_y & K_u & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = g \quad \Longleftrightarrow: \quad \begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} (y,u)^\top \\ p \end{pmatrix} = g$$
$$\Longleftrightarrow: \; G\,q = g \tag{2.6}$$

with some right hand side g. Here C^* denotes a linear operator which is the dual of C, i.e., $\langle C^* q, r\rangle := \langle q, Cr\rangle$. The Hessian of L or the *Karush-Kuhn-Tucker (KKT) operator* G has for such *linear-quadratic problems* constant entries. For linear-quadratic optimization problems, the necessary conditions are then also sufficient for the infimum. Moreover, if J or K do not contain products yu, one has $L_{yu} = L_{uy} = 0$ so that $\mathcal{A}$ is a block diagonal operator.

Typically, the quadratic functional (2.1) contains inner products so that the resulting Riesz operators L_{yy}, L_{uu} are *symmetric* which implies that $\mathcal{A}$ is *symmetric*. This is the situation we will always consider from now on.

Regarding the unique solvability of (2.6), one has the following result from, e.g., [BBF, Theorem 4.2.1].

THEOREM 1. *Let $\mathcal{V}$, $\mathcal{Q}$ be Hilbert spaces and let $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$, $\mathcal{B} : \mathcal{V} \to \mathcal{Q}'$ be linear continuous operators. Moreover, assume that* $\operatorname{Im}\mathcal{B} = \mathcal{Q}'$ *and that $\mathcal{A}$ is invertible on* $\operatorname{Ker}\mathcal{B}$. *Then the saddle point problem* (2.6) *has for $g \in \mathcal{V}' \times \mathcal{Q}'$ a unique solution $q \in \mathcal{V} \times \mathcal{Q}$.*

In view of the general argument to derive (2.6), we consider in the remainder of this work the design and analysis of adaptive stochastic Galerkin discretizations of parametric, *symmetric saddle point problems* (2.6) with a boundedly invertible linear mapping $G : \mathcal{X} \to \mathcal{X}'$ where $\mathcal{X} := Y \times U \times Y$, and where the operator G depends on possibly countably many parameters σ. Before detailing this, we present some standard examples from [KS] to which this scenario applies and specify the corresponding system (2.6).

2.2. Dirichlet problem with distributed control. Consider the standard weak formulation of a second order elliptic PDE with homogeneous boundary conditions. Choosing $Y := H_0^1(\Omega)$ and $U := Y'$, we consider for given $f \in Y'$ the linear operator equation

$$K(y,u) := Ay - f - u = 0 \tag{2.7}$$

and quadratic objective functional

$$J(y,u) := \frac{1}{2}\|y - y_*\|_Y^2 + \frac{\omega}{2}\|u\|_{Y'}^2, \tag{2.8}$$

for a given target state $y_* \in Y$ and any fixed weight parameter $\omega > 0$. We assume that $A : Y \to Y'$ is a linear boundedly invertible operator. However, we do *not* assume that A is self-adjoint. This is, in particular, essential for the treatment of

parabolic evolution equations, see Section 2.3. The norms in (2.8) can be norms on Hilbert spaces as long as they are induced by inner products and as long as the constrained optimization system (2.1) together with (2.2) is well-posed, leading to a unique solution pair (y, u) according to Theorem 1.

Denote by $R : Y \to Y'$ the *Riesz operator* defined by the inner product $(\cdot,\cdot)_Y$ inducing $\|\cdot\|_Y$,

$$\langle v, Rw\rangle_{Y\times Y'} := (v, w)_Y, \qquad v, w \in Y. \tag{2.9}$$

Since $(\cdot,\cdot)_Y$ is symmetric, R is also. The Lagrangian defined in (2.3) takes the form

$$L(y,u,p) = \frac{1}{2}\langle y-y_*, R(y-y_*)\rangle_{Y\times Y'} + \frac{\omega}{2}\langle u, R^{-1}u\rangle_{Y\times Y'} + \langle Ay-f-u, p\rangle_{Y'\times Y} \tag{2.10}$$

Thus, the system (2.6) becomes

$$\begin{pmatrix} R & 0 & A^* \\ 0 & \omega R^{-1} & -I \\ A & -I & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} Ry_* \\ 0 \\ f \end{pmatrix}, \tag{2.11}$$

i.e.,

$$\mathcal{A} = \operatorname{diag}(R, \omega R^{-1}) \quad \text{and} \quad \mathcal{B} = (A, -I)\,. \tag{2.12}$$

The system matrix G defined in (2.11) is symmetric since R is. Moreover, $\mathcal{A}$ is positive definite and $\mathcal{B}$ has full rank since, by assumption, the constraints (2.1) have a unique solution for given u. Thus, by Theorem 1, G is boundedly invertible.

Note that this form also applies to Dirichlet problems with Neumann boundary control by introducing an additional linear operator C applied to the control in (2.7), see [KS]. This results in a system (2.11) with $-I$ in the third row replaced by $-C$ and in the second row by $-C^*$, respectively.

The situation also applies to the Dirichlet problem with Dirichlet boundary control formulated as an (outer) saddle point problem of (inner) saddle point problems as in [KS].

2.3. Parabolic PDE with distributed control. The full space-time weak formulation of a linear parabolic evolution PDE from [SSt] in the variation from [ChSt] fits into this framework as well. The parabolic operator equation is formulated such that the resulting operator B is boundedly invertible from $\tilde{\mathcal{X}} := L^2(I)\otimes Y$ to $\tilde{\mathcal{X}} := (L^2(I)\otimes Y)\cap(H^1_T(I)\otimes Y')$ where $H^1_T(I)$ is the closure of the functions in $H^1(I)$ which vanish at end time T and $I := (0,T)$ denotes the time interval.

For the control problem, the constraints are of the form (2.7) with the parabolic evolution operator $B = \partial_t + A$ in full weak space-time form as in [ChSt] in place of A, see [GK] for details. Choosing the objective function then as in (2.8) with the obvious changes for the norms, i.e., using the norms for $\tilde{\mathcal{X}}$, $\tilde{\mathcal{Y}}$, we arrive at a system very similar to (2.11) with symmetric $\mathcal{A} = \operatorname{diag}(R_{\tilde{\mathcal{X}}}, \omega R_{\tilde{\mathcal{Y}}})$ with the respectively defined Riesz operators. The corresponding operator G is here a boundedly invertible mapping from $\mathcal{X} := \tilde{\mathcal{X}}\times\tilde{\mathcal{X}}\times\tilde{\mathcal{X}}$ onto $\mathcal{X}'$.

Similarly as for the elliptic control problem with Dirichlet boundary control, Dirichlet boundary controls can be handled for the case of the control problem subject to a parabolic PDE as well, by formulating the PDE in full space-time weak form and appending the Dirichlet boundary conditions by Lagrange multipliers. Thus, one has to deal again with an (outer) saddle point problem of (inner) saddle point problems involving a nonsymmetric operator $B = \partial_t + A$ in (2.7) in place of A. Again, the resulting saddle point operator G in (2.6) is symmetric and boundedly invertible.

3. Parametric saddle point problems. At the heart of our analysis are linear parametric saddle point operator equations where the amount of parameters may be countably infinite, such as they arise for elliptic PDEs with random coefficients expanded in Karhunen–Loève representations. We recall the main notions and results, first in an abstract setting, from [KS], specified to symmetric saddle point problems.

3.1. Boundedly invertible operators. Let $\mathcal{X}$ be a reflexive Banach space over $\mathbb{R}$ with (topological) dual $\mathcal{X}'$. By $\mathcal{L}(\mathcal{X},\mathcal{X}')$, we denote the set of bounded linear operators mapping from $\mathcal{X}$ to $\mathcal{X}'$. Recall from, e.g., [Au], that Riesz' representation theorem associates with each $G\in\mathcal{L}(\mathcal{X},\mathcal{X}')$ a unique bilinear form $\mathcal{G}(\cdot,\cdot):\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ by means of

$$\mathcal{G}(v,w)=\langle w,Gv\rangle_{\mathcal{X}\times\mathcal{X}'}\quad\text{for all } v,w\in\mathcal{X}. \tag{3.1}$$

For solving linear operator equations $Gq=g$ with given data $g\in\mathcal{X}'$, we use the following existence and uniqueness result which is a straightforward consequence of the closed graph theorem, see, e.g., [BBF, NSV].

PROPOSITION 2. *A symmetric operator $G\in\mathcal{L}(\mathcal{X},\mathcal{X}')$ is* boundedly invertible *if and only if its associated bilinear form $\mathcal{G}(\cdot,\cdot)$ satisfies the* inf-sup conditions*: there exists a constant $\gamma>0$ such that*

$$\inf_{0\neq v\in\mathcal{X}}\sup_{0\neq w\in\mathcal{X}}\frac{\mathcal{G}(v,w)}{\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}}\geq\gamma\quad\text{and}\quad\inf_{0\neq w\in\mathcal{X}}\sup_{0\neq v\in\mathcal{X}}\frac{\mathcal{G}(v,w)}{\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}}}\geq\gamma \tag{3.2}$$

hold. In this case, for every $g\in\mathcal{X}'$ the operator equation

$$\text{find } q\in\mathcal{X}:\quad\mathcal{G}(q,v)=\langle g,v\rangle_{\mathcal{X}'\times\mathcal{X}}\quad\text{for all } v\in\mathcal{X} \tag{3.3}$$

has a unique solution $q\in\mathcal{X}$ with the a-priori estimate

$$\|q\|_{\mathcal{X}}\leq\frac{1}{\gamma}\,\|g\|_{\mathcal{X}'} \tag{3.4}$$

3.2. Parametric operator families. We shall be interested in *parametric* families of operators $G(\sigma)\in\mathcal{L}(\mathcal{X},\mathcal{X}')$. Specifically, we shall treat the practically most important case of countably infinite sequences $\sigma:=(\sigma_j)_{j\in\mathbb{S}}$ of (independent) parameters which we assume to take values in $\Gamma\subset\mathbb{R}^{\mathbb{N}}$, i.e., each realization of σ is a sequence of real numbers. We equip the parameter domain $\Gamma:=[-1,1]^{\mathbb{N}}$ with the product probability measure

$$\boldsymbol{\pi}(\sigma)=\bigotimes_{m\geq1}\pi_m(\sigma_m)\,. \tag{3.5}$$

For our regularity and approximation results, we require $G(\sigma)$ to be (real) analytic with respect to σ, i.e., the parameter family $G(\sigma)$ is infinitely differentiable with respect to σ and coincides in an open, nonempty neighborhood of each point with its Taylor series at that point. The precise format of the parameter dependence shall be the following.

ASSUMPTION 1. *The parametric operator family*

$$\{G(\sigma)\in\mathcal{L}(\mathcal{X},\mathcal{X}'):\sigma\in\Gamma\}$$

is regular $\mathfrak{p}$-analytic *for some $0<\mathfrak{p}\leq1$, i.e.,*

(i) $G(\sigma) \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ *is boundedly invertible for every* $\sigma \in \Gamma$ *with uniformly bounded inverses* $G(\sigma)^{-1} \in \mathcal{L}(\mathcal{X}', \mathcal{X})$*, i.e., there exists* $C_0 > 0$ *such that*

$$\sup_{\sigma\in\Gamma} \|G(\sigma)^{-1}\|_{\mathcal{L}(\mathcal{X}',\mathcal{X})} \le C_0; \tag{3.6}$$

(ii) for any fixed $\sigma \in [-1,1]^{\mathbb{N}}$*, the operators* $G(\sigma)$ *are analytic with respect to each* σ_j *and there exists a nonnegative sequence* $b = (b_j)_{j\ge1} \in \ell^{\mathfrak{p}}(\mathbb{N})$ *such that*

$$\text{for all } \nu \in \mathfrak{F}\backslash\{0\}: \quad \sup_{\sigma\in\Gamma} \left\|(G(0))^{-1}(\partial_\sigma^\nu G(\sigma))\right\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \le C_0 b^\nu \ . \tag{3.7}$$

Here, the index set

$$\mathfrak{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\operatorname{supp}\nu < \infty\} \tag{3.8}$$

appearing in (ii) where $\operatorname{supp}\nu := \{m \in \mathbb{N} : \nu_m \neq 0\}$ is countable; it contains all finitely supported sequences of nonnegative integers, i.e., sequences which only have a finite number of nonzero entries. For $\nu \in \mathfrak{F}$, b^ν denotes the (finite) product $b_1^{\nu_1} b_2^{\nu_2} \dots$ with $0^0 := 1$, and $\partial_\sigma^\nu G(\sigma) := \frac{\partial^{\nu_1}}{\partial\sigma_1}\frac{\partial^{\nu_2}}{\partial\sigma_2}\cdots G(\sigma)$.

We point out that the estimates in (3.7) are taken for all *finite order* derivatives from the countable index set $\mathfrak{F}$. This will be the key for dealing with infinitely many parameters σ. The regularity parameter $\mathfrak{p} \in (0,1]$ which controls the derivatives $\partial_\sigma^\nu (G(\sigma))^{-1}$ according to (3.7) will later determine the best N-term approximation rate in (3.31) below.

3.3. Affine Parametric Operator Families. The dependence of $G(\sigma)$ on σ formulated in Assumption 1 allows for very general situations. One of the most frequently and particular relevant appearing case is the one where the operator family $G(\sigma)$ depends on σ in an *affine* fashion, sometimes also denotes as the *(stochastically) linear* case. This appears, for example, for diffusion problems with stochastic coefficients when the diffusion coefficients are represented in terms of a Karhunen–Loève expansion using a spectral approach, see, e.g., [ST]. Then there exists a family $\{G_j\}_{j\ge1} \subset \mathcal{L}(\mathcal{X}, \mathcal{X}')$ such that the parametric operator $G(\sigma)$ can be represented as

$$\forall \sigma \in \Gamma: \quad G(\sigma) = G_0 + \sum_{j\ge1} \sigma_j G_j \ , \tag{3.9}$$

with $G_0 = G(0)$ the *nominal* or *mean-field operator* and *fluctuation operators* G_j, $j \ge 1$. As in (3.1), we associate with each operator G_j a bilinear form $\mathcal{G}_j(\cdot,\cdot) : \mathcal{X}\times\mathcal{X} \to \mathbb{R}$. We impose the following assumptions on $\{G_j\}_{j\ge1}$.

Assumption 2. *The family of operators* $\{G_j\}_{j\ge1}$ *in* (3.9) *satisfies:*

(i) $G_0 \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ *is boundedly invertible with constant* $\gamma_0 > 0$ *in* (3.2)*;*

(ii) The $\{G_j\}_{j\ge1}$ *are small with respect to* G_0*: there exists a constant* $0 < \kappa < 1$ *such that*

$$\sum_{j\ge1} \|G_j\|_{\mathcal{X}\to\mathcal{X}'} \le \kappa\gamma_0 \ . \tag{3.10}$$

Assumption 2 on $\{G_j\}_{j\ge1}$ implies the following statements [KS, Thm. 2].

Theorem 3. *Let Assumption 2 on* $\{G_j\}_{j\ge1}$ *hold. Then one has:*

(i) The sum in (3.9) *converges absolutely and uniformly on* $[-1,1]^{\mathbb{N}}$*.*

(ii) The parametric operator $G(\sigma)$ *is boundedly invertible uniformly in* σ *with constant* $\gamma = (1-\kappa)\gamma_0 > 0$*.*

(iii) For every $g \in \mathcal{X}'$ and for every $\sigma \in \Gamma$, the parametric operator equation

$$\text{find } q(\sigma) \in \mathcal{X} \text{ satisfying } G(\sigma)\, q(\sigma) = g \tag{3.11}$$

has a unique solution $q(\sigma)$ which satisfies the a-priori estimate

$$\sup_{\sigma\in\Gamma} \|q(\sigma)\|_{\mathcal{X}} \le \frac{\|g\|_{\mathcal{X}'}}{(1-\kappa)\gamma_0}\,. \tag{3.12}$$

(iv) The affine parametric operator family $\{G_j\}_{j\ge1}$ satisfies Assumption 1 with $\mathfrak{p} = 1$ and

$$C_0 = \frac{1}{(1-\kappa)\gamma_0} \quad \text{and} \quad b_j := \frac{\|G_j\|_{\mathcal{X}\to\mathcal{X}'}}{(1-\kappa)\gamma_0} \quad \text{for all } j \ge 1\,.$$

Moreover, if

$$\sum_{j\ge1} \|G_j\|^{\mathfrak{p}}_{\mathcal{X}\to\mathcal{X}'} < \infty \tag{3.13}$$

holds for some $0 < \mathfrak{p} \le 1$, then the family $\{G_j\}_{j\ge1}$ satisfies Assumption 1 with the same $\mathfrak{p}$.

We remark that stronger assumptions than (3.10) on the sequence $\{G_j\}_{j\ge1}$ in the affine family (3.9) allow the verification of $\mathfrak{p}$-summability (ii) in Assumption 1 with condition (3.13) for a smaller exponent $\mathfrak{p}' < \mathfrak{p}$. As the convergence rates of adaptive stochastic Galerkin approximations are only based on Assumption 1, and yield approximation rates N^{-r}, $r := \frac{1}{\mathfrak{p}} - \frac{1}{2}$, with N gpc degrees of freedom, they will also apply in these cases. Optimality will be shown in Theorem 5 and Corollary 6.

3.4. Analytic parameter dependence of solutions. For the remainder of this Section, we return to the case of general parameter dependence from Subsection 3.2. Next we recall analytic dependence of the solution $q(\sigma)$ on σ, with precise bounds on the growth of the partial derivatives. This later allows for a-priori estimates for finite dimensional approximations of $q(\sigma)$ relative to σ, see (3.23) and Theorem 5 ahead.

THEOREM 4. *Let the parametric operator family $\{G(\sigma) \in \mathcal{L}(\mathcal{X},\mathcal{X}') : \sigma \in \Gamma\}$ satisfy Assumption 1 for some $0 < \mathfrak{p} \le 1$ and sequence $b = (b_j)_{j\ge1} \in \ell^{\mathfrak{p}}(\mathbb{N})$. Then,*

(i) for every $g \in \mathcal{X}'$ and every $\sigma \in \Gamma$, there exists a unique solution $q(\sigma) \in \mathcal{X}$ of the parametric operator equation (3.11)*,*

$$G(\sigma)\, q(\sigma) = g \qquad \text{in } \mathcal{X}';$$

(ii) the parametric solution family $\{q(\sigma) : \sigma \in \Gamma\}$ depends analytically on σ;

(iii) the partial derivatives of $q(\sigma)$ satisfy the bounds

$$\sup_{\sigma\in\Gamma} \|(\partial^\nu_\sigma q)(\sigma)\|_{\mathcal{X}} \le \frac{C_0}{\ln 2}\, \|g\|_{\mathcal{X}'}\, |\nu|!\, b^\nu \quad \text{for all } \nu \in \mathfrak{F}\,. \tag{3.14}$$

3.5. Stochastic Galerkin formulations. The bounds (3.14) are the basis for quantifying approximability of $q(\sigma)$ in the space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ This *Lebesgue-Bochner* space is the space of all functions v with finite norm

$$\|v\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} := \left(\int_\Gamma \|v(\sigma)\|^2_{\mathcal{X}}\, d\boldsymbol{\pi}(\sigma)\right)^{1/2}. \tag{3.15}$$

It is isometrically isomorphic to the Hilbert tensor product $L^2_{\boldsymbol{\pi}}(\Gamma)\otimes\mathcal{X}$ see, e.g., [Au], where $L^2_{\boldsymbol{\pi}}(\Gamma)$ is the Hilbert space of all functions v for which the norm $\|v\|_{L^2_{\boldsymbol{\pi}}(\Gamma)}$ defined by $\|v\|^2_{L^2_{\boldsymbol{\pi}}(\Gamma)} := \int_\Gamma |v(\sigma)|^2\, d\boldsymbol{\pi}(\sigma)$ is finite.

We shall be concerned with Galerkin-type approximations of (3.11). These formulations are based on the Bochner space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$. The weak stochastic formulation of the problem reads as follows: given $g\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X}')$, find $q\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ such that

$$\mathsf{E}(\langle G(\cdot)q(\cdot),v\rangle) = \mathsf{E}(\langle g,v\rangle) \quad \text{for all } v\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X}) \tag{3.16}$$

holds. Here, the mathematical expectation $\mathsf{E}(\cdot)$ is defined for $v\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ as Bochner integral

$$\mathsf{E}(v) := \int_\Gamma v(\sigma)\, d\boldsymbol{\pi}(\sigma) \in \mathcal{X}\ . \tag{3.17}$$

Note that the right hand side remains unchanged since g does not depend on σ and $\mathsf{E}(g) = g\int_\Gamma d\boldsymbol{\pi}(\sigma) = g$ on account of (3.5).

If the right hand side $g\in\mathcal{X}'$ is deterministic (as we have assumed up to now), problem (3.16) is well-posed for the solution space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ and test space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$. The resulting *parametric Galerkin formulation* reads: given $\sigma\in\Gamma$,

$$\text{find } q(\sigma)\in\mathcal{X}: \qquad \langle G(\sigma)q(\sigma),v\rangle_{\mathcal{X}'\times\mathcal{X}} = \langle g,v\rangle_{\mathcal{X}'\times\mathcal{X}} \qquad \text{for any } v\in\mathcal{X}\ . \tag{3.18}$$

The pointwise in σ variational formulation (3.18) is the foundation for the so-called "stochastic collocation" approaches, where (3.18) is "sampled" on judiciously chosen subsets $\Sigma_N\subset\Gamma$ of cardinality not exceeding N, either from sampling methods (eg. [SG, BNT, BTZ]) or selected adaptively (eg. [CCDS, CCS1]. At present, no rigorous results on adaptivity of stochastic collocation approaches seem to be available. Here, we are interested in the situation that we also discretize adaptively in the parameter domain Γ: the *stochastic Galerkin formulation* reads: given $g\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X}')$, find $q\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ such that for any $v\in L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ it holds

$$\int_\Gamma \langle G(\sigma)q(\sigma),v(\sigma)\rangle_{\mathcal{X}'\times\mathcal{X}}\, d\boldsymbol{\pi}(\sigma) = \int_\Gamma \langle g(\sigma),v(\sigma)\rangle_{\mathcal{X}'\times\mathcal{X}}\, d\boldsymbol{\pi}(\sigma)\ . \tag{3.19}$$

The stochastic Galerkin formulation (3.19) is the basis for adaptive stochastic Galerkin discretizations to be considered ahead. As it has been shown in [Gi1, Gi2, GAS] for elliptic and parabolic problems, these adaptive discretizations produce sequences of finitely truncated gpc approximations of which realize optimal convergence rates afforded by sparsity of the solutions, using techniques from adaptive wavelet approximations in [CDD1, CDD2]. The control problems for parametric operator equation as formulated in Section 2 always lead to symmetric operators $G(\sigma)$. Therefore, we work in the following with the stochastic Galerkin formulation (3.19).

Adaptive stochastic Galerkin algorithms produce approximations $q_\Lambda\in Q_\Lambda$ to the parametric solution $q(\sigma)$ of (3.19) which are supported on a finite index set $\Lambda\subset\mathfrak{F}$ of cardinality N. Here, the approximation space is defined as

$$Q_\Lambda := \left\{ v_\Lambda(\sigma) = \sum_{\lambda\in\Lambda} v_\mu P_\mu(\sigma), \quad v_\mu\in\mathcal{X} \right\} \subset L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})\ . \tag{3.20}$$

Here $\{P_\mu\}_{\mu\in\mathfrak{F}}$ is some basis for $L^2(\Gamma,\boldsymbol{\pi})$. which as well as the specific form of the index set Λ will be specified below in Section 3.6. The *Galerkin approximation* $q_\Lambda\in Q_\Lambda$ to

q is then defined as the unique solution of the problem to find $q_\Lambda \in Q_\Lambda$ such that

$$\int_\Gamma \langle G(\sigma)q(\sigma), v(\sigma)\rangle_{\mathcal{X}'\times\mathcal{X}}\, d\boldsymbol{\pi}(\sigma) = \int_{[-1,1]^{\mathbb{N}}} \langle g(\sigma), v(\sigma)\rangle_{\mathcal{X}'\times\mathcal{X}}\, d\boldsymbol{\pi}(\sigma) \text{ for all } v_\Lambda \in Q_\Lambda \tag{3.21}$$

holds. Note that (3.21) may be considered as a semi-discretization with respect to σ: with respect to space and time, the expansion coefficients in (3.20) are still taken from the infinite-dimensional space $\mathcal{X}$.

3.6. Gpc (generalized polynomial chaos) approximations and N-term approximation rates. Our goal is to compute the Galerkin approximation $q_N := q_\Lambda$ to q having at most $\#\Lambda = N$ degrees of freedom such that

$$\|q - q_N\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \lesssim N^{-r} \tag{3.22}$$

with some largest possible rate $r > 0$. Here the symbol $\lesssim$ denotes inequality up to constants that are specifically independent of N. If we are able to establish such an estimate with a rate r exceeding the Monte-Carlo benchmark rate $1/2$, this approximation would converge faster than Monte-Carlo methods, assuming equal, uniform cost for the solution of each system (3.11). In particular, this holds in the case of only finitely many parameters σ. After having established the type of smoothness for q with respect to σ in Section 3.4, such an error rate with possibly high r depends, in addition, on the choice of the approximation space Q_Λ. A second fundamental aspect of the spectral approximation methods proposed in [KS] which we recall next is that they allow for reduced resolution in x and t in large parts of the parameter space. This stands in contrast to Monte-Carlo sampling which mandates the same level of resolution in all sampling points.

Our error analysis begins by applying a generalization of Céa's lemma to solutions of saddle point problems, called quasi-best approximation property in [NSV, Theorem 3.2]. Applying the same reasoning used there with respect to the discretization for σ yields

$$\|q - q_N\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \le C_\gamma \inf_{v_N\in Q_\Lambda} \|q - v_N\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \tag{3.23}$$

with constant $C_\gamma := \gamma^{-1}(\sup_{\sigma\in\Gamma}\|G(\sigma)\|_{\mathcal{L}(\mathcal{X},\mathcal{X}')})$. Note that in this case the LBB-constant with respect to the discretization for σ does not depend on Λ since γ is a uniform lower bound for all σ, see Theorem 3, (ii).

Thus, the approximation problem (3.22) is reduced to establishing an *a-priori error estimate* for the *best N-term approximation* v_N of q, independent of the concrete Galerkin problem (3.19). This means that v_N possesses at most N degrees of freedom and minimizes the error to q with respect to the norm of the Bochner space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$. Naturally, this approximation is tied to the choice of the closed subspace Q_Λ of $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ and its Riesz basis.

In the context of random parameters, the *spectral approach* introduced in [GS], see also [KX, Sch], is based on representation of the parametric state and control as so-called *Wiener* or *generalized polynomial chaos (gpc) expansions*. These expansions are performed in terms of tensor products of *orthogonal polynomials* with respect to countable product probability measure in $L^2(-1,1)$. The choice of this interval stems from choosing the parameter space Γ which then determines the choice of orthogonal polynomials. The expression "polynomial chaos" goes back to N. Wiener [W] for the spectral representation of Gaussian processes in terms of Hermite polynomials

of i.i.d. Gaussian random variables, and 'generalized' expresses that one works with orthogonal polynomials with respect to the inner product for $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ weighted by the uniform probability measure $d\boldsymbol{\pi}(\sigma)$ in (3.5).

Such spectral approaches offer at least three distinct advantages over other methods: (I) It is a (deterministic) spectral expansion in terms of orthogonal polynomials which entails that one can achieve an *exponential convergence rate* in the polynomial degree N. Such an optimal result was first established in [BTZ] for a finite number $\mathcal{K}$ of random coefficients σ for elliptic PDEs with random inputs. However, the constant in the error estimate depends strongly on $\mathcal{K}$ which means that, as $\mathcal{K}\to\infty$ in the convergence analysis, this result requires an assumption of truncation with respect to $\mathcal{K}$. Error estimates for the general case of infinitely many parameters were for the first time established in [TS] for the same problem class under the assumption that the Karhunen–Loève expansion of the stochastic diffusion coefficient decays exponentially to zero in the L^∞ norm. This was relaxed in [CDS1] to the more realistic assumption of algebraic decay, as in (3.14). (II) The spectral approach allows for an immediate computation of the mean field $\mathsf{E}(q_N)$, as detailed below in (3.32). (III) All the inner products in relation to σ for different polynomial degrees vanish which is a significant computational advantage that is not tied to the particular choice of probability measure $\boldsymbol{\pi}$.

We base our approximations on univariate orthogonal polynomials which provide an orthogonal polynomial basis for $L^2_{\pi_m}(-1,1)$ with respect to the measure π_m (we refer to [EMSU] for an analysis of such polynomials in the representation of random fields). To this end, let P^m_n denote the orthogonal (with respect to the measure π_m) polynomial of degree $n\geq 0$ defined via the recursion formula

$$a^m_{n+1}P^m_{n+1}(\sigma_m) := a^m_n\, s\, P^m_n(\sigma_m) + a^m_{n-1}P^m_{n-1}(\sigma_m), \quad \sigma_m\in(-1,1),\ m,n\geq 1, \tag{3.24}$$

where, in the case of the uniform measure, ie. $d\pi_m(\sigma_m) = d\sigma_m/2$, $a^m_{n+1} = n+1$, $a^m_n := 2n+1$ and $a^m_{n-1} := -n$, and with the initialization $P^m_0(s) := 1$ and $P^m_1(s) := s$. In anticipation of their use for the parameter space and the measure (3.5), we normalize them such that

$$\int_{-1}^{1} |P^m_n(\sigma_m)|^2\, d\pi_m(\sigma_m) = 1, \quad m,n\in\mathbb{N} \tag{3.25}$$

so that for all $m = 1,2,...$, the collection $\{P^m_n\}_{n\geq 0}$ is an orthonormal basis of $L^2_{\pi_m}(-1,1)$. For $\nu\in\mathfrak{F}$ with multi-index $\mathfrak{F}$ from (3.8), we define the *tensorized orthogonal polynomials*

$$P_\nu(\sigma) := \prod_{m\geq 1} P^m_{\nu_m}(\sigma_m), \quad \sigma\in\Gamma\,. \tag{3.26}$$

On account of taking $\nu\in\mathfrak{F}$, there are only finitely many nontrivial factors in this product, and each $P_\nu(\sigma)$ depends only on finitely many of the σ_j. It is shown, e.g., in [EMSU, Gi1] that the countable collection

$$\boldsymbol{P} := \{P_\nu(\sigma) : \nu\in\mathfrak{F}\} \tag{3.27}$$

is a Riesz basis for the function space $L^2_{\boldsymbol{\pi}}(\Gamma)$, that is, a dense, orthonormal family in $L^2_{\boldsymbol{\pi}}(\Gamma)$. This means that each $v\in L^2_{\boldsymbol{\pi}}(\Gamma)$ admits an orthonormal expansion

$$v(\sigma) = \sum_{\nu\in\mathfrak{F}} v_\nu P_\nu(\sigma)\,, \qquad \text{where} \quad v_\nu := \int_\Gamma v(\sigma)P_\nu(\sigma)d\boldsymbol{\pi}(\sigma)\in\mathcal{X}\,. \tag{3.28}$$

The sequence $\{v_\nu : \nu \in \mathfrak{F}\}$ satisfies Parseval's equality, ie.

$$\|v\|^2_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} = \sum_{\nu\in\mathfrak{F}} \|v_\nu\|^2_{\mathcal{X}} \tag{3.29}$$

and the synthesis map $T_{\boldsymbol{P}}$ is an orthogonal isomorphism which is given by

$$T_{\boldsymbol{P}} \colon \ell^2(\mathfrak{F}) \to L^2_{\boldsymbol{\pi}}(\Gamma)\,, \quad (c_\mu)_{\mu\in\mathfrak{F}} \mapsto \sum_{\mu\in\mathfrak{F}} c_\mu P_\mu\,. \tag{3.30}$$

The polynomial chaos representation (3.28) is the basis for the analysis of best N-term approximation rates. Denote by $\Lambda \subset \mathfrak{F}$ a subset of cardinality $N := \#\Lambda < \infty$. Denote by q_ν the Legendre coefficients of the solution q of the parametric operator equation (3.11). From (3.23) and Parseval's equality (3.29), we bound the Galerkin error as

$$\big\|q - q_N\big\|^2_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \le C_\gamma \inf_{v_N\in Q_\Lambda} \|q - v_N\|^2_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} = C_\gamma \sum_{\nu\notin\Lambda} \|q_\nu\|^2_{\mathcal{X}}\,.$$

Best N-term approximation rates in $\|\cdot\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})}$ now follow from summability of the norms $\|q_\nu\|_{\mathcal{X}}$ of the Legendre coefficients by *Stechkin's Lemma* (see, eg. [CDS2]). We proved in [KS] that the sequence of the norms of the Legendre coefficients satisfies $(\|q_\nu\|_{\mathcal{X}})_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F})$ for the same value of the regularity parameter $\mathfrak{p} \in (0,1]$ for which $G(\sigma)$ satisfies Assumption 1. Since $\mathfrak{p} \le 1$, we refer to such sequences as *sparse* and to the corresponding expansions of the type (3.28) *sparse generalized polynomial chaos (gpc) expansions*. Finally, we arrive at the following result on convergence rates of best N-term polynomial approximations of the parametric solution $q(\sigma)$ of the parametric operator equation (3.11).

THEOREM 5. *[KS] Let the parametric operator family $\{G(\sigma) \in \mathcal{L}(\mathcal{X},\mathcal{X}') : \sigma \in \Gamma\}$ satisfy Assumption 1 for some $0 < \mathfrak{p} \le 1$. Assume moreover that the probability measures $\pi_m(\sigma_m)$ which charge the parameter σ_m are uniform, ie. that $\pi_m = \lambda^1/2$ where λ^1 denotes the Lebesgue measure in $\mathbb{R}^1$. Then there exists a nested sequence $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathfrak{F}$ of index sets of cardinality less or equal N such that*

$$\|q - q_N\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \lesssim N^{-r}\,, \quad r = \frac{1}{\mathfrak{p}} - \frac{1}{2}\,. \tag{3.31}$$

Here, $q_N := q_{\Lambda_N}$ where q_{Λ_N} denotes the sequence in $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ whose entries q_ν equal those of the sequence q if $\nu \in \Lambda_N \subset \mathfrak{F}$ and which equal zero otherwise.

From Theorem 5, we can immediately derive from q_N an approximation to the mean field or "ensemble average" $\overline{q} := \mathsf{E}(q)$ as

$$\overline{q_N} := \mathsf{E}(q_N) = \sum_{\nu\in\Lambda_N} e_\nu q_\nu\,. \tag{3.32}$$

Here q_ν are the expansion coefficients

$$q_\nu := \int_\Gamma q(\sigma)\, P_\nu(\sigma)\, d\boldsymbol{\pi}(\sigma) \in \mathcal{X} \tag{3.33}$$

and e_ν are the *νth moments* of the Legendre polynomials

$$e_\nu := \mathsf{E}(P_\nu(\sigma)) = \int_\Gamma P_\nu(\sigma) d\boldsymbol{\pi}(\sigma)\,.$$

In the case of a uniform probability measure (3.5), the orthogonality and normalization properties of the Legendre polynomials (3.25) yield $e_\nu = 0$ for all ν except $\nu = 0$ in which case we have $e_0 = 1$. Thus, the mathematical expectation $\mathsf{E}(\cdot)$ of the Galerkin approximation with respect to the probability measure $\boldsymbol{\pi}$ in (3.5) of q_N is just given by the 0th polynomial chaos expansion coefficient of q_N, i.e., by

$$\overline{q_N} = \mathsf{E}(q_N) = q_0 = \int_{[-1,1]^{\mathbb{N}}} q(\sigma) d\boldsymbol{\pi}(\sigma) \,. \tag{3.34}$$

Moreover, one has by the Jensen and the Cauchy-Schwarz inequality the estimate

$$\|\overline{q} - \overline{q_N}\|_{\mathcal{X}} \leq \int_\Gamma \|q(\sigma,\cdot) - q_N(\sigma,\cdot)\|_{\mathcal{X}} d\boldsymbol{\pi}(\sigma) \leq \|q - q_N\|_{L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})} \tag{3.35}$$

which, together with the main approximation estimate (3.31) yields the same rate also for the mean fields.

Corollary 6. *Let the operator family $\{G(\sigma) \in \mathcal{L}(\mathcal{X},\mathcal{X}') : \sigma \in \Gamma\}$ satisfy Assumption 1 for some $0 < \mathfrak{p} \leq 1$. Then, with the approximation q_N defined in Theorem 5, it holds*

$$\|\overline{q} - \overline{q_N}\|_{\mathcal{X}} \lesssim N^{-r}\,, \quad r = \frac{1}{\mathfrak{p}} - \frac{1}{2}. \tag{3.36}$$

As a consequence, if the parametric operator family $\{G(\sigma) \in \mathcal{L}(\mathcal{X},\mathcal{X}') : \sigma \in \Gamma\}$ satisfies Assumption 1 just for weakest case $\mathfrak{p} = 1$, our spectral approach achieves the same benchmark rate as Monte-Carlo methods. For any $\mathfrak{p} < 1$, the rate of the spectral Galerkin approximation will already outperform Monte-Carlo methods. In the extreme case that Assumption 1 holds for any $0 < \mathfrak{p} \leq 1$ as in the case of affine parameter dependence of Section 3.3, the above rate r is arbitrarily high.

In the next sections, we will establish the practical construction of the index sets in Theorem 5 and the computation of the Galerkin approximation q_N of (3.21). This will be achieved simultaneously with a finite-dimensional approximation of the Galerin approximation with respect to time and space. The starting point for these routines will be an equivalent matrix representation of the continuous problem (3.19) in terms of the basis $\boldsymbol{P}$ for the approximation of σ and an appropriate wavelet basis for $\mathcal{X}$ specified next, as in [GAS].

4. Wavelet bases on the physical domain. For the discretization of the operator G with respect to the physical variables, i.e., space and time, assume at our disposal wavelet bases for $\mathcal{X}$. These bases should be Riesz bases of $\mathcal{X}$ which, being the cartesian product of function spaces for state, costate and control y, u, p (see the abstract formulation (2.6)), will in turn be triplets of wavelet-type Riesz bases for the respective spaces: For all situations encountered in Section 2 ahead, the space $\mathcal{X}$ will be a cartesian product of Sobolev spaces (or intersections of Sobolev spaces). For each of these, constructions of Riesz bases were detailed in [GAS, SSt] and in [DK, GK, K] for corresponding control problems. Here, we state only those properties of the Riesz basis $\boldsymbol{\Psi}$ which are required in what follows.

Properties 7. *We call the collection $\boldsymbol{\Psi} := \{\psi_\nu : \nu \in \mathbb{I}\}$ a wavelet basis for $\mathcal{X}$ if*

(R) $\boldsymbol{\Psi}$ is a Riesz basis *for $\mathcal{X}$, i.e., every $v \in \mathcal{X}$ has a unique expansion in terms of $\boldsymbol{\Psi}$,*

$$v = \sum_{\nu\in\mathbb{I}} v_\nu \, \psi_\nu =: \mathbf{v}^\top \boldsymbol{\Psi}, \quad \mathbf{v} := (v_\nu)_{\nu\in\mathbb{I}}, \tag{4.1}$$

and its expansion coefficients satisfy the norm equivalence

$$c\|\mathbf{v}\|_{\ell_2(\mathbb{I})} \le \|\mathbf{v}^\top \boldsymbol{\Psi}\|_{\mathcal{X}} \le C\|\mathbf{v}\|_{\ell_2(\mathbb{I})}, \quad \mathbf{v} \in \ell_2(\mathbb{I}) \tag{4.2}$$

with some finite positive constants c, C*;*

(L) (Locality) *all functions* ψ_ν *are compactly supported;*

(CP) (Cancellation property) *There exists an integer* $\tilde{d}$ *such that*

$$|\langle v, \boldsymbol{\Psi}_\mu\rangle_{\mathcal{X}'\times\mathcal{X}}| \;\lesssim\; 2^{-|\mu|(n/2+\tilde{d})}\|v^{(\tilde{d})}\|_{L_\infty(\operatorname{supp}\,\psi_\lambda)}, \tag{4.3}$$

where n *is the dimension of the underlying domain and* $v^{(\tilde{d})}$ *denotes the weak* $\tilde{d}$*th weak derivative of* v*.*

5. Bi-infinite matrix equations for parametric saddle point problems. Recall from (3.30) that the *polynomial synthesis operator*

$$T_{\boldsymbol{P}} : \ell^2(\mathfrak{F}) \to L^2_{\boldsymbol{\pi}}(\Gamma), \qquad \mathbf{v} := \{v_\nu : \nu \in \mathfrak{F}\} \mapsto \sum_{\nu\in\mathfrak{F}} v_\nu P_\nu$$

is an orthogonal isomorphism. Moreover, the Riesz basis property (R) in Properties 7 means that the *wavelet synthesis operator*

$$T_{\boldsymbol{\Psi}} : \ell^2(\mathbb{I}) \to \mathcal{X}, \qquad \mathbf{v} := \{v_\mu : \mu \in \mathbb{I}\} \mapsto \sum_{\mu\in\mathbb{I}} v_\mu \psi_\mu \tag{5.1}$$

is boundedly invertible. Combining both, the *synthesis operator* $T_{\boldsymbol{P}\otimes\boldsymbol{\Psi}}$ *for the tensorized basis* $\boldsymbol{P}\otimes\boldsymbol{\Psi}$ equals the tensor product of the synthesis operators so that

$$T_{\boldsymbol{P}\otimes\boldsymbol{\Psi}} := T_{\boldsymbol{P}} \otimes T_{\boldsymbol{\Psi}} : \ell^2(\mathfrak{F}\times\mathbb{I}) \to L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X}) \simeq L^2_{\boldsymbol{\pi}}(\Gamma)\otimes\mathcal{X} \tag{5.2}$$

is an isomorphism of Hilbert spaces. In particular, $\boldsymbol{P}\otimes\boldsymbol{\Psi}$ is a Riesz basis for $L^2_{\boldsymbol{\pi}}(\Gamma)\otimes\mathcal{X}$ (which, we recall, is isometrically isomorphic to the Bochner space $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X})$ of strongly measurable functions which are square integrable in $\mathcal{X}$-norm). We shall use the notation $L^2_{\boldsymbol{\pi}}(\Gamma;\mathcal{X}) = L^2_{\boldsymbol{\pi}}(\Gamma)\otimes\mathcal{X}$ to underline the tensor-product structure of the space and the bases.

Using the analysis and synthesis operators, we reformulate the Galerkin system (3.19) equivalently as bi-infinite matrix-vector equation in the sequence space $\ell^2(\mathfrak{F}\times\mathbb{I})$. Define

$$\mathbf{q} := T^{-1}_{\boldsymbol{P}\otimes\boldsymbol{\Psi}} q \in \ell^2(\mathfrak{F}\times\mathbb{I}), \qquad \mathbf{g} := T^*_{\boldsymbol{P}\otimes\boldsymbol{\Psi}} g \in \ell^2(\mathfrak{F}\times\mathbb{I}) \tag{5.3}$$

and

$$\boldsymbol{G} := T^*_{\boldsymbol{P}\otimes\boldsymbol{\Psi}} G T_{\boldsymbol{P}\otimes\boldsymbol{\Psi}} : \ell^2(\mathfrak{F}\times\mathbb{I}) \to \ell^2(\mathfrak{F}\times\mathbb{I})\,. \tag{5.4}$$

Owing to the isomorphism properties of $T_{\boldsymbol{P}}, T_{\boldsymbol{\Psi}}$, the stochastic Galerkin formulation (3.19) of the infinite-dimensional parametric control problem is then equivalent to the bi-infinite linear system

$$\boldsymbol{G}\boldsymbol{q} = \boldsymbol{g}\,, \tag{5.5}$$

and the solution $q \in L^2_{\boldsymbol{\pi}}(\Gamma)\otimes\mathcal{X}$ can be reconstructed from the expansion coefficients $\mathbf{q} \in \ell^2(\mathfrak{F}\times\mathbb{I})$ via

$$q = T_{\boldsymbol{P}\otimes\boldsymbol{\Psi}}\mathbf{q} = \sum_{(\nu,\mu)\in\mathfrak{F}\otimes\mathbb{I}} q_{\nu,\mu}\, P_\nu \otimes \psi_\mu\,. \tag{5.6}$$

In the case of affine parameter dependence as in Section 3.3, the structure of the operator G in terms of $\mathcal{A}, \mathcal{B}$, see (2.6), and these, in turn, of the differential and Riesz operators, can be provided explicitly also as tensor products, separating the temporal and the spatial coordinates, see [Gi1, GAS].

6. Adaptive stochastic Galerkin methods. After having established the existence of a sequence of nested index sets $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathfrak{F}$ for which the a-priori estimates (3.31), (3.36) hold, we will describe next how to construct these index sets and how to practically compute the Galerkin approximation q_N in (3.22). This will be achieved in an adaptive a-posteriori fashion, following the ideas in [Gi2, SG] and the realizations in [EGSZ, GAS] for a single elliptic or parabolic PDE. Essentially, the approximation of the bi-infinite system (5.5) follows ideas known from adaptive wavelet discretizations, see, e.g., [St2] for an extensive survey covering specifically [CDD1, CDD2], applied to both discretizations in terms of orthonormal polynomials and to wavelets.

6.1. Basic ideas and routines. Once the bi-infinite system (5.5) is derived, we can employ routines which have been established in [CDD1, CDD2] for solving deterministic stationary linear operator equations, discretized in terms of wavelets. These techniques were extended in [SSt] to adaptive space-time Galerkin discretizations of parabolic PDEs. We will abbreviate $\ell^2 := \ell^2(\mathfrak{F}\times\mathbb{I})$ and $\|\cdot\| := \|\cdot\|_{\ell^2}$ for the corresponding sequence norm.

Given an arbitrary vector of coefficients $\mathbf{v} \in \ell^2$, the most economical approximations for $\mathbf{v}$ with N coefficients are provided by those vectors with support of length N which contain N largest (in modulus) coefficients from $\mathbf{v}$, yielding a best N-term approximation of $\mathbf{v}$ in ℓ^2. We denote any such best N-term approximation of $\mathbf{v}$ in ℓ^2 by $\Pi_N\mathbf{v}$.

We first describe the main ingredients as black-box solvers. The first is a routine $\textsc{Rhs}_{\mathbf{g}}[\varepsilon] \to \mathbf{g}_\varepsilon$ which, for given tolerance $\varepsilon > 0$, produces a finitely supported approximation of the right hand side $\mathbf{g}$ such that $\|\mathbf{g} - \mathbf{g}_\varepsilon\| \le \varepsilon$. We require

$$\#\operatorname{supp}\mathbf{g}_\varepsilon \le \inf_{N\in\mathbb{N}}\{\|\mathbf{g} - \Pi_N\mathbf{g}\| \lesssim \varepsilon\}\,. \tag{6.1}$$

The vector $\mathbf{g}_\varepsilon$ is supposed to be of minimal support prescribed tolerance ε, and the number of arithmetic operations to compute $\mathbf{g}_\varepsilon$ is bounded by some absolute multiple of $\#\operatorname{supp}\mathbf{g}_\varepsilon + 1$. This is achieved by a routine

$$\text{Apply}_{\boldsymbol{G}}[\boldsymbol{w}, \epsilon] \mapsto \boldsymbol{z} \tag{6.2}$$

which, for any $\epsilon > 0$ and any finitely supported vector $\boldsymbol{w}$, constructs a finitely supported vector $\boldsymbol{z}$ with $\|\boldsymbol{G}\boldsymbol{w} - \boldsymbol{z}\|_{\ell^2} \le \epsilon$.

The concrete format of the routine $\text{Apply}_{\boldsymbol{G}}$ has been specified for the control problem derived from the Dirichlet control problem in Section 2.2 in [DK] and for the control problem constrained by a parabolic PDE mentioned in Section 2.3 in [GK]. In both cases, the routine $\text{Apply}_{\boldsymbol{G}}$ is based on a gradient method for the control variable $\mathbf{u}$ (the condensed equation) which is updated by applying a routine $\text{Apply}_{\mathbf{A}}$ (or $\text{Apply}_{\boldsymbol{B}}$ in the parabolic case) to update the current approximation $\mathbf{y}$, followed by applying a routine $\text{Apply}_{\mathbf{A}^*}$ (or $\text{Apply}_{\boldsymbol{B}^*}$ in the parabolic case) to update the current approximation $\mathbf{p}$. Here $\mathbf{A}, \boldsymbol{B}$ denote the bi-infinite representations of the elliptic and parabolic differential operators appearing in (2.7) and Section 2.3, with respect to the basis for $\ell^2(\mathfrak{F}\times\mathbb{I})$ in analogy to $\boldsymbol{G}$ defined in (5.4). Similarly, the Riesz operators appearing in (2.11) or Section 2.3 can be written as bi-infinite matrices $\mathbf{R}$ for which corresponding routines $\text{Apply}_{\mathbf{R}}$ can be designed as well.

The discrete operator $\boldsymbol{G}$ is called s^*-admissible for a given $s^* \in (0,\infty]$ if, for all $s \in (0,s^*)$, $\#\operatorname{supp} \boldsymbol{z} \lesssim \epsilon^{-1/s}\|w\|^{1/s}_{\mathcal{A}^s_\infty(\ell^2)}$ and the number of arithmetic operations and storage locations used by $\mathrm{Apply}_{\boldsymbol{G}}[\boldsymbol{w},\epsilon]$ is bounded by (6.4) up to a fixed constant independent of $\boldsymbol{w}$. Here $\mathcal{A}^s_\infty(\ell^2)$ denotes the subspace of ℓ^2 defined by the quasi-norm (6.3) We recall from (6.2) the routine $\textsc{Apply}_{\boldsymbol{G}}[\mathbf{v},\varepsilon] \to \mathbf{v}_\varepsilon$ which determines for a given tolerance $\varepsilon > 0$ and a finitely supported input vector $\mathbf{v}$ a finitely supported sequence $\mathbf{v}_\varepsilon$ such that (6.2) holds.

In order to quantify the computational complexity when designing Apply, we employ the following notions. We collect in one class all vectors $\mathbf{v}$ whose best N-term approximations $\Pi_N\mathbf{v}$ converge to $\mathbf{v}$ in ℓ^2 at rate $s > 0$.

To this end, we introduce the approximation class $\mathcal{A}^s_\infty(\ell^2)$, the subset of ℓ^2 of all sequences for which the quasi-norm

$$\|\boldsymbol{v}\|_{\mathcal{A}^s_\infty(\ell^2)} := \sup_{N\in\mathbb{N}_0} (N+1)^s \|\boldsymbol{v} - P_N(\boldsymbol{v})\|_{\ell^2}\,, \tag{6.3}$$

is finite, where $P_N(\boldsymbol{v})$ is any best N-term approximation of $\boldsymbol{v}$ in ℓ^2.

If $\mathbf{q}$ is the solution of an infinite matrix-vector equation like (5.5), best N-term approximations $P_N(\mathbf{q})$ require an infinite search, and are therefore not computationally accessible.

The main issue of the construction of adaptive schemes is to design a practical method that yields approximations $P_N(\mathbf{q})$ of $\mathbf{q}$ which, if $\mathbf{q} \in \mathcal{A}^s$ for some $s > 0$, converge to $\mathbf{q}$ at this rate s with *linear computational complexity*. To realize this means to find an adaptive scheme which is asymptotically not more expensive than an absolute, fixed multiple of the support size N of the best N-term approximation $P_N(\mathbf{q})$, and assuming that the vector $\mathbf{q}$ was known.

The operator $\boldsymbol{G}$ is called s^*-*admissible* for a given $s^* \in (0,\infty]$ if, for all $s \in (0,s^*)$, the output vector satisfies $\operatorname{supp}\mathbf{v}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s}$ and the amount of arithmetic operations and storage locations used in $\textsc{Apply}_{\boldsymbol{G}}[\mathbf{v},\varepsilon]$ is bounded by

$$\varepsilon^{-1/s}\|\mathbf{v}\|^{1/s}_{\mathcal{A}^s} + \operatorname{supp}\mathbf{v} + 1, \tag{6.4}$$

up to a fixed constant independent of $\mathbf{v}$.

In all the cases where $\boldsymbol{G}$ stems from a saddle point problem as in Section 2, $\boldsymbol{G}$ is symmetric but not coercive. The formulation in [KS], eliminating the control, yields a coercive but nonsymmetric system (for weight parameter $\omega \in (0,1]$ in the control functional (2.8)). In both situations we can work with an Apply routine which realizes applications to a finitely supported vector for the (bi-infinite) normal equations

$$\boldsymbol{G}^*\boldsymbol{G}\mathbf{q} = \boldsymbol{G}^*\mathbf{g}. \tag{6.5}$$

Routines which are valid within the general setting of [CDD2, GHS] are

$$\begin{aligned} \textsc{Apply}_{\boldsymbol{G}^*\boldsymbol{G}}[\mathbf{v},\varepsilon] &:= \textsc{Apply}_{\boldsymbol{G}^*}[\textsc{Apply}_{\boldsymbol{G}}[\mathbf{v},\varepsilon(2\|\boldsymbol{G}\|)^{-1},\varepsilon/2]] \\ \textsc{Rhs}_{\boldsymbol{G}^*\mathbf{g}}[\varepsilon] &:= \textsc{Apply}_{\boldsymbol{G}^*}\left[\textsc{Rhs}_{\mathbf{g}}[\varepsilon(2\|\boldsymbol{G}\|)^{-1}],\varepsilon/2\right] \end{aligned} \tag{6.6}$$

where $\varepsilon > 0$ is arbitrary, and where $\textsc{Apply}_{\boldsymbol{G}^*}$ is the equivalent of $\textsc{Apply}_{\boldsymbol{G}}$ when exchanging $\boldsymbol{G}$ by $\boldsymbol{G}^*$.

6.2. Approximate application of bi-infinite operators. The construction of $\textsc{Apply}_{\boldsymbol{G}}$ hinges on the ability to approximate the bi-infinite matrix $\boldsymbol{G}$ by sparsely populated matrices with controlled error versus accuracy. Any bi-infinite matrix $\mathbf{C} \in \mathcal{L}(\ell^2, \ell^2)$ is called *n-sparse* if each column of $\mathbf{C}$ contains at most n nonzero entries. $\boldsymbol{G}$ is called *s^*-compressible* for some $s^* \in (0, \infty]$ if there exists a sequence $\{\boldsymbol{G}^{(j)}\}_{j\in\mathbb{N}}$ in $\mathcal{L}(\ell^2, \ell^2)$ such that $\boldsymbol{G}^{(j)}$ is n_j-sparse with strictly increasing sequence $(n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ which satisfies $\sup_{j\in\mathbb{N}} n_{j+1}/n_j < \infty$ and such that for every $s \in (0, s^*)$, the sparse $\boldsymbol{G}^{(j)}$ converge to $\boldsymbol{G}$ in norm with rate essentially s^* with respect to n_j, i.e.,

$$\sup_{j\in\mathbb{N}} (n_j)^s \, \|\boldsymbol{G} - \boldsymbol{G}^{(j)}\|_{\ell^2\to\ell^2} < \infty \, . \tag{6.7}$$

The definition of s^*-compressibility involves the parameter n_j which is proportional to the cost of accessing one column of $\boldsymbol{G}^{(j)}$. This, however, does not include the cost of assembling this column. A stronger notion is *s^*-computability* (eg. [GHS, St2]): The bi-infinite matrix operator $\boldsymbol{G}$ is called *s^*-computable* for some $s^* \in (0, \infty]$ if it is s^*-compressible and if the number of arithmetic operations and storage locations used to construct an arbitrary column of $\boldsymbol{G}^{(j)}$ is bounded by an absolute, fixed multiple of n_j for all $j \in \mathbb{N}$.

THEOREM 8. *[CDD1, CDD2, GHS] If $\boldsymbol{G}$ is symmetric positive definite and s^*-admissible, then for any $\varepsilon > 0$, the adaptive wavelet methods from [CDD1, CDD2, GHS] construct an approximation $\boldsymbol{q}_\varepsilon$ of $\boldsymbol{q}$ with $\|\boldsymbol{q} - \boldsymbol{q}_\varepsilon\|_{\ell^2} \le \varepsilon$. If $\boldsymbol{q} \in \mathcal{A}^s_\infty(\ell^2)$ for any $s > 0$, then $\#\operatorname{supp} \boldsymbol{q}_\varepsilon \lesssim \varepsilon^{-1/s}\|\boldsymbol{q}\|_{\mathcal{A}^s_\infty(\ell^2)}$. If $s < s^*$, then the number of arithmetic operations and storage locations used to compute $\boldsymbol{q}_\varepsilon$ is bounded by an affine function of $\varepsilon^{-1/s}\|\boldsymbol{q}\|_{\mathcal{A}^s_\infty(\ell^2)}$.*

REMARK 9. *The estimate $\#\operatorname{supp} \boldsymbol{q}_\varepsilon \lesssim \varepsilon^{-1/s}\|\boldsymbol{q}\|_{\mathcal{A}^s_\infty(\ell^2)}$ can be interpreted as follows: if $\|\boldsymbol{q} - P_N(\boldsymbol{q})\|_{\ell^2} \lesssim N^{-s}$, then $\|\boldsymbol{q}\|_{\mathcal{A}^s_\infty(\ell^2)}$ is finite, and consequently $\|\boldsymbol{q} - \boldsymbol{q}_\varepsilon\|_{\ell^2} \le \varepsilon \lesssim (\#\operatorname{supp} \boldsymbol{q}_\varepsilon)^{-s}$. Thus adaptive Galerkin methods recover the optimal convergence rate in terms of the support size, albeit with a larger constant in the error estimate. In the case $s < s^*$, the same estimate holds for the computational cost and, in this sense, the solvers have optimal complexity.*

REMARK 10. *Replacing $\boldsymbol{G}$ by $\boldsymbol{G}^*\boldsymbol{G}$ and $\boldsymbol{f}$ by $\boldsymbol{G}^*\boldsymbol{g}$, Theorem 8 applies to the normal equations for nonsymmetric or to symmetric, but indefinite systems such as those which arise from the saddle point formulation* (2.6)*, see [DK, GK] for details. In this case, valid* Apply *and* RHS *routines are given by*

$$\mathrm{Apply}_{\boldsymbol{G}^*\boldsymbol{G}}[\boldsymbol{w}, \varepsilon] := \mathrm{Apply}_{\boldsymbol{G}^*}[\mathrm{Apply}_{\boldsymbol{G}}[\boldsymbol{w}, \varepsilon/(2\|\boldsymbol{G}\|)], \varepsilon/2] \tag{6.8}$$

and

$$\mathrm{RHS}_{\boldsymbol{G}^*\boldsymbol{g}}[\varepsilon] := \mathrm{Apply}_{\boldsymbol{G}^*}[\mathrm{RHS}_{\boldsymbol{g}}[\varepsilon/(2\|\boldsymbol{G}\|)], \varepsilon/2] \, , \tag{6.9}$$

where $\|\boldsymbol{G}\|$ denotes the operator norm of $\boldsymbol{G}$ in $\ell^2(\mathfrak{F} \times \mathbb{I})$. The product $\boldsymbol{G}^*\boldsymbol{G}$ is s^*-admissible if both $\boldsymbol{G}$ and $\boldsymbol{G}^*$ are s^*-admissible. A slightly weaker but sufficient variant of (6.6) *is required to hold for* $\mathrm{RHS}_{\boldsymbol{G}^*\boldsymbol{g}}$.

7. Tensor structure of bi-infinite matrix equation. We now apply the foregoing, general concepts to the first order necessary conditions of the symmetric saddle point formulations, ie. to (2.6) and its instances (2.11) of optimal control problems for *affine* parametric operator eqations, ie. to the case where the operator A in the equation of state depends affinely on the parameter sequence σ. This affine dependence implies, in turn, that the operator G in (2.6) and its instances will take the form

(3.9) *with self-adjoint* (but indefinite) operator G_0. To apply the general approach, we convert the parametric control problem (2.6), (3.9) to a bi-infinite matrix equation (5.5). Due to the symmetry of G in (2.6), the bi-infinite matrix $\boldsymbol{G}$ is symmetric, ie $\boldsymbol{G}^* = \boldsymbol{G}$.

7.1. Tensor structure of the bi-infinite operator. We separate the spatial and temporal discretization from that in the parameter sequence $\sigma \in \Gamma$. Here and in the following, we restrict to the setting of Section 3.3, ie. $G(\sigma)$ has the form (3.9) with affine dependence on $\sigma \in \Gamma$.

To this end, we construct an approximation to the sesquilinear form $\mathcal{G}$ in (3.1) which is independent of the parameter $\sigma \in \Gamma$. We interpret G_0 also as an operator $G_0 \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$. Similarly, we interpret G_m as a bounded linear map from $\mathcal{X}$ to $\mathcal{X}^*$. Then, due to the affine parameter dependence (3.9), for $w \in \mathcal{X}$ and for $v \in \mathcal{X}$,

$$\mathcal{G}(w,v) = \int_\Gamma \left(\mathcal{G}_0(w,v) + \sum_{m=1}^\infty \sigma_m \mathcal{G}_m(w,v) \right) \mathrm{d}\boldsymbol{\pi}(\sigma) \,. \tag{7.1}$$

Define for $m \geq 0$ the operators $\boldsymbol{G}_m := T_{\boldsymbol{\Psi}}^* G_m T_{\boldsymbol{\Psi}}$. These operators are bounded linear maps from $\ell^2(\mathbb{I})$ to $\ell^2(\mathbb{I})$ and can be represented as bi-infinite matrices. To capture the dependence on the parameter sequence $\sigma \in \Gamma$, we consider the representation of

$$K_m \colon L^2_{\boldsymbol{\pi}}(\Gamma) \to L^2_{\boldsymbol{\pi}}(\Gamma) \,, \quad v(\sigma) \mapsto \sigma_m v(\sigma) \tag{7.2}$$

with respect to the polynomial basis $\boldsymbol{P}$. Solving for the term ξP^m_{n-1} in (3.24), it follows that for all $m \in \mathbb{N}$, $\boldsymbol{K}_m := T^*_{\boldsymbol{P}} K_m T_{\boldsymbol{P}} \in \mathcal{L}(\ell^2(\mathfrak{F}))$ has the form

$$(\boldsymbol{K}_m \boldsymbol{c})_\mu = \beta^m_{\mu_m+1} c_{\mu+\epsilon_m} + \alpha^m_{\mu_m} c_\mu + \beta^m_{\mu_m} c_{\mu-\epsilon_m} \,, \quad \mu \in \mathfrak{F} \,, \tag{7.3}$$

for $\boldsymbol{c} = (c_\mu)_{\mu\in\mathfrak{F}} \in \ell^2(\mathfrak{F})$, where $c_\mu := 0$ if $\mu_m < 0$ for any $m \in \mathbb{N}$, and where ϵ_m denotes the Kronecker sequence $(\epsilon_m)_n = \delta_{mn}$. Also, let $\boldsymbol{I}$ be the identity on $\ell^2(\mathfrak{F})$.

Inserting the above basis representations into (5.4) yields, with convergence in $\mathcal{L}(\ell^2(\mathfrak{F}\times\mathbb{I}), \ell^2(\mathfrak{F}\times\mathbb{I}))$, the representation

$$\boldsymbol{G} = \boldsymbol{I} \otimes \boldsymbol{G}_0 + \sum_{m=1}^\infty \boldsymbol{K}_m \otimes \boldsymbol{G}_m \,. \tag{7.4}$$

7.2. Tensor structure of the bi-infinite normal equations. Since $\boldsymbol{G}$ is symmetric but in general not positive definite since it results from a saddle point problem, we consider the discrete normal equations

$$\boldsymbol{G}^* \boldsymbol{G} \boldsymbol{q} = \boldsymbol{G}^* \boldsymbol{g} \,, \tag{7.5}$$

where the discrete adjoint operator $\boldsymbol{G}^* = T^*_{\boldsymbol{P}\times\boldsymbol{\Psi}} G^* T_{\boldsymbol{P}\times\boldsymbol{\Psi}} \colon \ell^2(\mathfrak{F}\times\mathbb{I}) \to \ell^2(\mathfrak{F}\times\mathbb{I}) = \boldsymbol{G}$ due to the symmetry of G in (2.6). Since $\boldsymbol{G}$ is bijective, (7.5) is equivalent to (5.5), and $\boldsymbol{G}^*\boldsymbol{G}$ is positive definite. The KKT system (2.6) being symmetric, there holds $\boldsymbol{G} = \boldsymbol{G}^*$ which implies that we have for the bi-infinite matrices $\boldsymbol{G}_0 = \boldsymbol{G}_0{}^*$ and $\boldsymbol{G} = \boldsymbol{G}_m^*$. Since $\boldsymbol{I}$ and $\boldsymbol{K}_m$ are likewise symmetric,

$$\boldsymbol{G}^* = \boldsymbol{I} \otimes \boldsymbol{G}_0 + \sum_{m=1}^\infty \boldsymbol{K}_m \otimes \boldsymbol{G}_m \,. \tag{7.6}$$

8. Adaptive Stochastic Galerkin discretization of parametric saddle point problems.

8.1. Approximate application of discrete operators. The construction of a routine $\mathrm{Apply}_{\boldsymbol{G}}$ hinges on the ability to approximate $\boldsymbol{G}$ by sparse matrices. We call a bi-infinite matrix $\boldsymbol{G} \in \mathcal{L}(\ell^2)$ *n-sparse* if each column of $\boldsymbol{G}$ contains at most n nonzero entries. It is *s^*-compressible* for an $s^* \in (0,\infty]$ if there exists a sequence $(\boldsymbol{G}_j)_{j\in\mathbb{N}}$ in $\mathcal{L}(\ell^2)$ such that $\boldsymbol{G}_j$ is n_j-sparse with the increasing sequence $(n_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ satisfying

$$\sup_{j\in\mathbb{N}} \frac{n_{j+1}}{n_j} < \infty \tag{8.1}$$

and such that for every $s \in (0, s^*)$, (6.7) holds. Condition (6.7) states that the sparse operators $\boldsymbol{G}_j$ converge to $\boldsymbol{G}$ with a rate of essentially s^* with respect to n_j.

By (8.1), the sequence $(n_j)_{j\in\mathbb{N}}$ grows at most geometrically. Consequently, for any $r > 0$, there is a $j(r)$ such that $n_{j(r)} \le r$ and $\sup_{r>0} r^s \|\boldsymbol{G} - \boldsymbol{G}_{j(r)}\| < \infty$ for all $s \in (0, s^*)$. Here, we extend the sequence of approximations by $\boldsymbol{G}_0 := \boldsymbol{0}$ with $n_0 = 0$. In particular, we may assume without loss of generality that $n_j = j$, as is done eg in [GHS, SSt].

In the definition of s^*-compressibility, n_j is proportional to the cost of accessing one column of $\boldsymbol{G}_j$. In order to capture also the assembly cost, we introduce following [GHS], somewhat stronger notion of s^*-computability: a bi-infinite matrix $\boldsymbol{G} \in \mathcal{L}(\ell^2)$ is *s^*-computable* if it is s^*-compressible and if the number of arithmetic operations and storage locations used to construct an arbitrary column of $\boldsymbol{G}_j$ is bounded by a multiple of n_j for all $j \in \mathbb{N}$. The following result has been shown in [CDD1].

Proposition 11. *If $\boldsymbol{G} \in \mathcal{L}(\ell^2)$ is s^*-computable for some $s^* \in (0,\infty]$ then it is s^*-admissible.* This result was given in [CDD1] for wavelet discretizations by constructing a suitable routine $\textsc{Apply}_{\boldsymbol{G}}$ for the case of $\boldsymbol{G}$ being a deterministic stationary linear elliptic operator. Since the appearance of [CDD1], several variants of the (constructive) proof of [CDD1] have been obtained with subsequent refinements in [Gi2, MP, St2].

9. Approximations of deterministic operators.

9.1. Compressibility of discrete elliptic and parabolic operators. The wavelets in Section 4 were chosen to ensure that the deterministic operators appearing in the series expansions (7.4) and (7.6) of $\boldsymbol{G}$ and $\boldsymbol{G}^*$ be s^*-computable.

Proposition 12. *For sufficiently smooth $\tilde{a}$ and a_m, $m \in \mathbb{N}$, the bi-infinite matrices $\boldsymbol{G}_0$, $\boldsymbol{G}_m$ are s^*-computable with $s^* = \min(\tilde{d}_t, \tilde{d}_x)$.*

A proof of Proposition 12 is given in [SSt, Sec. 8]. Proposition 12 implies that there is a sequence $(\boldsymbol{G}_{0j})_{j\in\mathbb{N}}$ of bi-infinite matrices such that $\boldsymbol{G}_{0j}$ is $n_{0,j}$-sparse with $(n_{0,j})_{j\in\mathbb{N}}$ increasing and satisfying (6.6), and

$$\|\boldsymbol{G}_0 - \boldsymbol{G}_{0j}\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} \lesssim n_{0,j}^{-s} \qquad \forall s \in (0, s^*)\,. \tag{9.1}$$

Furthermore, the number of arithmetic operations and storage locations required to compute any column of $\boldsymbol{G}_{0j}$ is an affine function of $n_{0,j}$. We extend these sequences by $\boldsymbol{G}_{00} := \boldsymbol{0}$ and $n_{0,0} := 0$. By the self-ajointness of G_m, $m \ge 0$, analogous properties hold for $\boldsymbol{G}_0{}^*$, $\boldsymbol{G}_m$ and $\boldsymbol{G}_m^*$. We denote the sequences of sparse approximations by $(\boldsymbol{G}_{0j}^*)_{j\in\mathbb{N}_0}$, $(\boldsymbol{G}_{m,j})_{j\in\mathbb{N}_0}$ and $(\boldsymbol{G}_{m,j}^*)_{j\in\mathbb{N}_0}$, and the corresponding sparsity indices by $(n_{0,j}^*)_{j\in\mathbb{N}_0}$, $(n_{m,j})_{j\in\mathbb{N}_0}$ and $(n_{m,j}^*)_{j\in\mathbb{N}_0}$, respectively. The matrices $\boldsymbol{G}_0$ and $\boldsymbol{G}_{m,j}$ resulting from the KKT conditions (2.6) for controlling affine-parametric operators $G(\sigma)$ as in (3.9) are symmetric. Thus $\boldsymbol{G}_{m,j}^* = \boldsymbol{G}_{m,j}$ for all $m \ge 0$.

9.2. Compressibility of discrete Riesz maps. Recall that the Riesz operators appear in the definition of the Lagrangian (2.10) as the representation of the chosen norms in the optimization functional (2.1). In (2.8), these have been chosen as the natural norms for the Dirichlet problem with distributed control. For the particular case $Y = H^1_0(\Omega)$, R would coincide with Laplace's operator in weak formulation. It was observed already in [DK] that the choice of the norms in the functional allows for a large amount of ambiguity. One can essentially reduce all the cases which lead to a well-posed control problem to Riesz operators which can be represented as a diagonally scaled version of a mass matrix. Thus, $\mathbf{R}$ as well as its inverse is compressible with the same properties as for the elliptic and parabolic PDE operator, see [GK, Prop. 4.2] and [P, Sect. 2.2.6].

9.3. Numerical approximation of error bounds. For constructing sparse approximations of $\boldsymbol{G}$, we require explicit knowledge of the constants in the estimates (9.1) and similar estimates for $\boldsymbol{G}_m$ or, more precisely, we require numerical sequences $(\tilde{e}_{m,j})_{j\in\mathbb{N}_0}$, $m \in \mathbb{N}_0$, such that

$$\|\boldsymbol{G}_0 - \boldsymbol{G}_{0j}\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} \le \tilde{e}_{0,j} \quad \text{and} \quad \|\boldsymbol{G}_m - \boldsymbol{G}_{m,j}\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} \le \tilde{e}_{m,j}\,. \tag{9.2}$$

Optimal values of $\tilde{e}_{0,j}$ are given by the square roots of the spectral radii of the positive symmetric operators $(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})^*(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})$ since

$$\|\boldsymbol{G}_0 - \boldsymbol{G}_{0j}\|^2_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} = \sup_{\|\boldsymbol{v}\|_{\ell^2(\mathbb{I})}=1} |\boldsymbol{v}^\top(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})^\top(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})\boldsymbol{v}|\,, \tag{9.3}$$

and similarly for $\tilde{e}_{m,j}$. Following [CJG, Sec. 6], we approximate these bounds by a power iteration with suitably approximated matrix-vector multiplies.

The primary component of this power iteration is the repeated approximate application of the operators $(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})^*$ and $\boldsymbol{G}_0 - \boldsymbol{G}_{0j}$ to finitely supported vectors. This is achieved using the sparse approximations $\boldsymbol{G}_{0j+k} - \boldsymbol{G}_{0j}$, $k \in \mathbb{N}$, of $\boldsymbol{G}_0 - \boldsymbol{G}_{0j}$ and $\boldsymbol{G}^*_{0j+k} - (\boldsymbol{G}_{0j})^*$ of $(\boldsymbol{G}_0 - \boldsymbol{G}_{0j})^*$ in routines

$$\mathrm{NApply}_{\boldsymbol{G}-\boldsymbol{G}_j}[\boldsymbol{v}, N] \mapsto \boldsymbol{z} \quad \text{and} \quad \mathrm{NApply}_{(\boldsymbol{G}-\boldsymbol{G}_j)^\top}[\boldsymbol{w}, N] \mapsto \boldsymbol{z} \tag{9.4}$$

similar to $\mathrm{Apply}_{\boldsymbol{G}}$ from (6.6), but with a prescribed maximal support size $\#\operatorname{supp}\boldsymbol{z} \le N$ instead of a target accuracy ε. These routines combine to

$$\mathrm{NApply}_{(\boldsymbol{G}-\boldsymbol{G}_j)^\top(\boldsymbol{G}-\boldsymbol{G}_j)}[\boldsymbol{v}, N] := \mathrm{NApply}_{(\boldsymbol{G}-\boldsymbol{G}_j)^\top}[\mathrm{NApply}_{\boldsymbol{G}-\boldsymbol{G}_j}[\boldsymbol{v}, N], N]\,. \tag{9.5}$$

All vectors appearing within these routines are ensured to have support size not larger than N.

The approximate power iteration for the computation of $\tilde{e}_{0,j}$ consists of repeated application of $\mathrm{NApply}_{(\boldsymbol{G}-\boldsymbol{G}_j)^\top(\boldsymbol{G}-\boldsymbol{G}_j)}$ and normalization of the resulting vector. The approximations of $\tilde{e}_{0,j}$ are given by the scalar products

$$\tilde{e}_{0,j} \approx \tilde{e}^n_{0,j} := \frac{\boldsymbol{v}_n^\top \boldsymbol{v}_{n+1}}{\boldsymbol{v}_n^\top \boldsymbol{v}_n}\,, \quad \boldsymbol{v}_{n+1} := \mathrm{NApply}_{(\boldsymbol{G}-\boldsymbol{G}_j)^\top(\boldsymbol{G}-\boldsymbol{G}_j)}[\boldsymbol{v}_n, N]\,, \tag{9.6}$$

where $\boldsymbol{v}_0$ is chosen randomly and N is fixed. We use analogous iterations to approximate $\tilde{e}_{m,j}$ as well as the bounds in

$$\|\boldsymbol{G}_0{}^\top - \boldsymbol{G}_{0j}^\top\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} \le \tilde{e}^*_{0,j} \quad \text{and} \quad \|\boldsymbol{G}_m^\top - \boldsymbol{G}_{m,j}^\top\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})} \le \tilde{e}^*_{m,j} \tag{9.7}$$

used to construct sparse approximations of $\boldsymbol{G}^*$.

Convergence of (a somewhat idealized variant of) (9.6) is shown in [CJG, Thm. 6.3].

10. Approximations of countably-parametric KKT operators for stochastic control problems. We discuss the existence and construction of sparse approximations of the operators $\boldsymbol{G}_m$ which appear in the expansion (7.4).

In fact, the saddle point operator $\boldsymbol{G}$ derived from (2.6) contains operators $\mathbf{A}$ in the case of the elliptic PDE, $\boldsymbol{B}$ for control problems constrained by parabolic PDEs, their adjoints, for some cases mass matrices, and diagonal matrices. For the PDE operators and their adjoints, s-sparsity has been established in [CDD1, CDD2] and [SSt], respectively. As detailed in [DK, GK], the application of $\boldsymbol{G}$ is reduced to a consecutive application of the solution of each of the single block systems in the discrete version of (2.6).

10.1. Sparse approximation of discrete random operators. We construct sparse approximations of $\boldsymbol{G}$ by truncating the series (7.4) and by replacing the remaining bi-infinite matrices $\boldsymbol{G}_0$ and $\boldsymbol{G}_m$ by appropriate sparse approximations $\boldsymbol{G}_{0\,j_0}$ and $\boldsymbol{G}_{m,j_m}$.

To this end, we assume that sequences $(\tilde{e}_{m,j})_{j\in\mathbb{N}_0}$ are available for all $m \in \mathbb{N}_0$ such that (9.2) holds. These can be computed numerically as described in Section 9.3 and [CJG, Sec. 6] or derived analytically as in [St1, BDD]. By switching to a subsequence, we assume without loss of generality that $(\tilde{e}_{m,j})_{j\in\mathbb{N}_0}$ is nonincreasing for all $m \in \mathbb{N}_0$ and, if $i \geq j$, then

$$\frac{-(\tilde{e}_{m,i+1} - \tilde{e}_{m,i})}{n_{m,i+1} - n_{m,i}} \leq \frac{-(\tilde{e}_{m,j+1} - \tilde{e}_{m,j})}{n_{m,j+1} - n_{m,j}}\,, \tag{10.1}$$

where $\boldsymbol{G}_{0\,j,0}$ is $n_{0,j}$-sparse and $\boldsymbol{G}_{m,j}$ is $n_{m,j}$-sparse.

For all finitely supported sequences $\boldsymbol{j} := (j_m)_{m\in\mathbb{N}_0}$ in $\mathbb{N}_0$, we define

$$\boldsymbol{G}_{\boldsymbol{j}} := \boldsymbol{I} \otimes \boldsymbol{G}_{0\,j_0} + \sum_{m=1}^{\infty} \boldsymbol{K}_m \otimes \boldsymbol{G}_{m,j_m}\,. \tag{10.2}$$

Since $\boldsymbol{j}$ is finitely supported and since $\boldsymbol{G}_{m,0} = \mathbf{0}$ for all m, the sum in (10.2) is finite, and no convergence issues arise. By the triangle inequality,

$$\|\boldsymbol{G} - \boldsymbol{G}_{\boldsymbol{j}}\|_{\ell^2(\mathfrak{F}\times\mathbb{I})\to\ell^2(\mathfrak{F}\times\mathbb{I})} \leq \sum_{m=0}^{\infty} \tilde{e}_{m,j_m} =: \tilde{e}_{\boldsymbol{j}}\,. \tag{10.3}$$

By (7.3), $\boldsymbol{K}_m$ is σ_m-sparse with $\sigma_m = 2$ if the distribution $\boldsymbol{\pi}_m$ is symmetric, and $\sigma_m = 3$ in general. Consequently, $\boldsymbol{K}_m \otimes \boldsymbol{G}_{m,j}$ is $\sigma_m n_{m,j}$-sparse. We set $\sigma_0 := 1$ such that $\boldsymbol{I} \otimes \boldsymbol{G}_{0\,j}$ is $\sigma_0 n_{0,j}$-sparse for all $j \in \mathbb{N}_0$. Then the total number of nonzero elements in any column of $\boldsymbol{G}_{\boldsymbol{j}}$ is at most

$$N_{\boldsymbol{j}} := \sum_{m=0}^{\infty} \sigma_m n_{m,j_m}\,, \tag{10.4}$$

and, assuming that entries of $\boldsymbol{G}_{0\,j}$ and $\boldsymbol{G}_{m,j}$ can be computed in unit time, $N_{\boldsymbol{j}}$ is also a bound for the cost of constructing any column of $\boldsymbol{G}_{\boldsymbol{j}}$.

We use a greedy algorithm to select a sequence $(\boldsymbol{j}_k)_{k\in\mathbb{N}_0}$, and define $\boldsymbol{G}_k := \boldsymbol{G}_{\boldsymbol{j}_k}$, which is an approximation of $\boldsymbol{G}$ with error at most $\tilde{e}_k := \tilde{e}_{\boldsymbol{j}_k}$, and containing at most $N_k := N_{\boldsymbol{j}_k}$ nonzero elements per column.

We initialize the adaptive Galerkin iteration with $\boldsymbol{G}_0 = \boldsymbol{0}$, with $\boldsymbol{j}_0 := \boldsymbol{0}$. In the adaptive refinement $\boldsymbol{j}_k = (j_{k,m})_{m\in\mathbb{N}_0}$ to $\boldsymbol{j}_{k+1}$, the entry $j_{k,m}$ for which m maximizes

$$\frac{-(\tilde{e}_{m,j_{k,m}+1} - \tilde{e}_{m,j_{k,m}})}{\sigma_m(n_{m,j_{k,m}+1} - n_{m,j_{k,m}})} \tag{10.5}$$

is incremented by one. If this m is not unique, one maximum is selected, eg. the smallest natural number m that maximizes (10.5).

In order to ensure optimality of this greedy algorithm, we assume that the sequence $(\tilde{e}_{m,0})_{m\in\mathbb{N}}$ is in ℓ^1 and $\sigma_m^{-1}n_{m,1}^{-1}(\tilde{e}_{m,1} - \tilde{e}_{m,0})$ is nonincreasing in m. The following optimality property of the sparse approximations $\boldsymbol{G}_k$ is [CJG, Cor. 7.2].

THEOREM 13. *For all $k \in \mathbb{N}_0$, $\boldsymbol{j}_k$ minimizes the error bound $\tilde{e}_{\boldsymbol{j}}$ among all finitely supported sequences $\boldsymbol{j}$ in $\mathbb{N}_0$ with $N_{\boldsymbol{j}} \le N_k$. Furthermore, if $\tilde{e}_k \neq 0$, then $\boldsymbol{j}_k$ minimizes $N_{\boldsymbol{j}}$ among all $\boldsymbol{j}$ with $\tilde{e}_{\boldsymbol{j}} \le \tilde{e}_k$.*

10.2. Compressibility and computability. In order to derive s^*-compressibility of $\boldsymbol{G}$, we assume that the estimate (8.1) holds uniformly for all $(n_{m,j})_{j\in\mathbb{N}}$, ie.

$$\sup_{m\in\mathbb{N}_0}\sup_{j\in\mathbb{N}}\frac{n_{m,j+1}}{n_{m,j}} < \infty\,. \tag{10.6}$$

The following theorem is the first case of [CJG, Thm. 8.4]. All unspecified norms refer to operator norms between sequence spaces ℓ^2 for the appropriate index sets.

THEOREM 14. *If* (10.6) *holds, and if*

$$\sum_{m=1}^{\infty}\big(\sup_{j\in\mathbb{N}} n_{m,j}^s\|\boldsymbol{G}_m - \boldsymbol{G}_{m,j}\|\big)^{\frac{1}{s+1}} < \infty \tag{10.7}$$

for all $s \in (0,\bar{s})$, then $\boldsymbol{G}$ is s^*-compressible for $s^* = \min(\tilde{d}_t, \tilde{d}_x, \bar{s})$ and $(\boldsymbol{G}_k)_{k\in\mathbb{N}_0}$ from Sec. 10.1 is a valid sequence of sparse approximations, satisfying

$$N_k^s\|\boldsymbol{G} - \boldsymbol{G}_k\| \le \left(\big(\sup_{j\in\mathbb{N}} n_{0,j}^s\|\boldsymbol{G}_0 - \boldsymbol{G}_{0j}\|\big)^{\frac{1}{s+1}} + \sum_{m=1}^{\infty}\big(\sup_{j\in\mathbb{N}} n_{m,j}^s\|\boldsymbol{G}_m - \boldsymbol{G}_{m,j}\|\big)^{\frac{1}{s+1}}\right)^{s+1}, \tag{10.8}$$

for all $s \in (0, s^*)$, where $\boldsymbol{G}_k$ is N_k-sparse. Compressibility of $\boldsymbol{G}$ can also be derived if (10.7) does not hold, as in the following theorem, which is the second case of [CJG, Thm. 8.4].

THEOREM 15. *[Compressibility for parabolic PDE operator] If* (10.6) *holds and if*

$$\sum_{m=1}^{\infty}\|\boldsymbol{G}_m\|^{\frac{1}{s+1}} < \infty \tag{10.9}$$

for all $s \in (0,\bar{s}_0)$, and

$$\sup_{M\in\mathbb{N}} M^{-\tau}\sum_{m=1}^{M}\big(\sup_{j\in\mathbb{N}} n_{m,j}^s\|\boldsymbol{G}_m - \boldsymbol{G}_{m,j}\|_{\ell^2(\mathbb{I})\to\ell^2(\mathbb{I})}\big)^{\frac{1}{s+1}} < \infty \tag{10.10}$$

for all $s \in (0,\hat{s})$, then $\boldsymbol{G}$ is s^*-compressible for

$$s^* := \frac{\min(\tilde{d}_t, \tilde{d}_x, \hat{s})}{1 + \tau/\bar{s}_0} \tag{10.11}$$

and $(\boldsymbol{G}_k)_{k\in\mathbb{N}_0}$ *from Section 10.1 is a valid sequence of sparse approximations.*

In case of the elliptic operator, the compressiblity indices need to be adjusted: choose in (4.3) the constant $\tilde{d}$ as $\tilde{d} = \max\{\tilde{d}_t, \tilde{d}_x\}$.

REMARK 16. *A numerical algorithm for constructing an arbitrary column of* $\boldsymbol{G}_k$ *is provided in [CJG, Sec. 7.2]. It assumes that either* $(\boldsymbol{j}_k)$ *are precomputed, or that the operators* $\boldsymbol{G}_k$ *are accessed sequentially, such that only one step of the greedy optimization needs to be performed the first time* $\boldsymbol{G}_k$ *is accessed. If the work for this precomputation step is disregarded,* s^**-computability, and thus* s^**-admissibility, of* $\boldsymbol{G}$ *follow from Theorems 14 and 15.*

The above discussion carries over to show s^*-computability of $\boldsymbol{G}^*$, and s^*-admissibility of $\boldsymbol{G}^*\boldsymbol{G}$ follows as in Remark 10. In particular, Theorem 8 applies, showing optimality of adaptive Galerkin discretizations based on tensor products of Legendre polynomial chaos with wavelet bases on the space-time domain applied to the parametric operator equation (7.5).

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