Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

Anisotropic Multiscale Systems on Domains

P. Grohs and G. Kutyniok and J. Ma and P. Petersen

Research Report No. 2015-30
October 2015

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Anisotropic Multiscale Systems on Bounded Domains

Philipp Grohs Gitta Kutyniok Jackie Ma Philipp Petersen

October 15, 2015

Abstract

In this paper we provide a construction of multiscale systems on a bounded domain $\Omega \subset \mathbb{R}^2$ coined boundary shearlet systems, which satisfy several properties advantageous for applications to imaging science and numerical analysis of partial differential equations. More precisely, we construct boundary shearlet systems that form frames for $L^2(\Omega)$ with controllable frame bounds and admit optimally sparse approximations for functions, which are smooth apart from a curve-like discontinuity. Indeed, the constructed systems allow for boundary conditions, and characterize Sobolev spaces over Ω by their analysis coefficients. Finally, we demonstrate numerically that these systems also constitute a Gelfand frame for $(H^s(\Omega), L^2(\Omega), H^{-s}(\Omega))$ for $s \in \mathbb{N}$.

1 Introduction

In the past two decades there has been a flurry of work related to the design of novel representation systems for functions with the intent to build systems which are optimally suited for sparse approximation of various signal classes. The first breakthrough was the introduction of the multiscale system of wavelets [18] providing optimally sparse approximations of functions governed by point singularities while allowing a unified concept of the continuum and digital realm to enable faithful implementations. And indeed wavelets have nowadays become a standard tool for, in particular, imaging science and numerical analysis of partial differential equations.

However, as it is typical for *multivariate* functions, images as well as the solutions of various types of partial differential equations are in fact governed by singularities along hypersurfaces. Recently, significant progress has been achieved by the introduction of ridgelet systems [1], then curvelet systems [3], and shearlet systems [37], which are capable of optimally approximating certain classes of multivariate functions with singularities along hypersurfaces. One main drawback so far is the fact that all these systems are designed for $L^2(\mathbb{R}^2)$, whereas applications, in particular, adaptive schemes for partial differential equations, typically require systems defined on a bounded domain.

In this paper, we will introduce the first class of anisotropic multiscale systems, which can be adapted to very general domains in $\mathbb{R}^2$ while still exhibiting optimally sparse approximation properties as well as further properties specifically necessary for their application in numerical approximation of PDEs. These systems are constructed as hybrid systems combining wavelets and shearlets.

1.1 Anisotropic multiscale systems

The first anisotropic multiscale systems achieving optimally sparse approximations of the model class of so-called cartoon-like functions [20], which are roughly speaking compactly supported piecewise C^2-functions defined on $\mathbb{R}^2$ with a C^2 discontinuity curve, were curvelets. In fact, curvelets are capable of approximating such functions with a decay rate of the L^2-error of the best N-term approximation rate by N^{-1} up to logarithmic terms, whereas wavelets can only achieve a rate of $N^{-1/2}$. This result from 2002 [3] can be considered a milestone in the area of applied harmonic analysis.

However, curvelets had the drawback of being based on rotations to provide directional sensitivity, which prevents a unified treatment of the continuum and digital world. Hence faithful implementations are not available. Shearlets were introduced in 2006 to resolve this problem, leading to a class of systems which also provide optimally sparse approximations of so-called cartoon-like functions [32] – even for higher order derivatives [41] – while allowing faithful implementations [35]. Moreover, this concept allows a shearlet *frame* for $L^2(\mathbb{R}^2)$ with controllable frame bounds, consisting of compactly supported elements [31]. Let us stress that the notion of a frame allows for stable decomposition and reconstruction formulas, see [4].

1.2 The problem of bounded domains

Anisotropic multiscale systems are today extensively used in imaging science for tasks such as feature extraction or inpainting, see, e.g., [23, 21, 25]. Despite these successes, the use of these novel representation systems for the numerical approximation of PDEs is however still at its infancy – even though the solutions of a large class of PDEs do admit hypersurface singularities. As examples, we mention elliptic PDEs with discontinuous (or distributional) source terms or coefficients, boundary value problems on polygonal domains, or transport equations.

The main bottleneck in developing PDE solvers based on anisotropic multiscale systems is the fact that originally these systems are constructed as representation systems, or frames, for functions defined on $\mathbb{R}^d$, while most PDEs are defined on a finite domain $\Omega \subset \mathbb{R}^d$. Thus the development of effective PDE solvers crucially depends on the construction of anisotropic systems on finite domains, satisfying various boundary conditions. The attentive reader will have realized that in fact also images are defined on bounded domains. The fact that this issue did not cause a significant problem so far indicates that the handling of the boundary for imaging tasks is not that sensitive. However, certainly, also in this range of applications the fact that the data lives on a bounded domain must not be disregarded, also in the theoretical (continuum) analysis.

Even for wavelet systems, the adaption to general bounded domains is a challenging problem which is by now, after decades of research, quite well understood (see, for instance, [30],[6]), albeit still with several open questions remaining. The construction of anisotropic multiscale systems on bounded domains is even much more challenging. While the MRA structure of wavelet systems [18] yields a powerful tool for the construction of boundary wavelet frames, no such structure seems to be available for more general multiscale systems. In fact, one can imagine that the anisotropically shaped support can intersect the boundary to various degrees and at various angles requiring a careful adaption of each single element while the numerical complexity of associated transform algorithms has to remain at about the same level.

Concerning the construction of shearlet systems on bounded domains, a first attempt has been made in [33]. While the constructed system forms an L^2–frame for arbitrary domains and yields optimally sparse approximations for cartoon-like functions, its elements are not boundary adapted, destroying vanishing moment and smoothness properties. Thus no characterization of smoothness spaces or faithful treatment of boundary data is possible. Another approach was undertaken in [12, 13], which however does not form a frame nor is this system able to characterize Sobolev spaces.

1.3 Adaptive schemes and frames

Several important steps have already been made in the direction of utilizing representation systems from applied harmonic analysis for adaptive solvers for different types of partial differential equations. A breakthrough was achieved by Cohen, Dahmen, and DeVore in the seminal paper [7] by introducing a provably optimal adaptive wavelet based solver for elliptic PDEs whose solutions exhibit only point singularities. Several further steps could be achieved in the realm of elliptic PDEs, in particular, extending this concept to more flexible frames instead of orthonormal bases, see, e.g., [9, 43, 10].

The exploitation of anisotropic frame systems for the numerical solution of PDEs is a rather new topic of research, presumably due to the significantly more involved structure of those systems; and we already discussed the delicacy of adapting those to a bounded domain. Let us also briefly recall the key properties required from such a system to be suitable for this task using the example of elliptic PDEs. First, the system should allow boundary conditions to be imposed. Second, its transform coefficients should characterize Sobolev spaces and moreover even form a Gelfand frame for Sobolev spaces, since the discretization of an elliptic PDE using a Gelfand frame in a Sobolev space yields, after a simple diagonal preconditioning, a uniformly well-conditioned linear system which can then be solved numerically by iterative methods such as damped Richardson iteration or conjugate gradients [9]. Third, optimal sparse approximation properties of the solutions is crucial to ensure that those can be approximated at an optimal asymptotic ratio between computational work and accuracy. In order for this approximation rate to be realized by a numerical solver it is, fourth, furthermore needed that the resulting Galerkin matrix is compressible in the sense of [7].

However, some notable first steps towards anisotropic frame systems for the numerical solution of PDEs have already been taken in [24, 28], where optimal adaptive ridgelet-based solvers are constructed for linear advection equations and [12, 13], where a shearlet-based construction is used to solve general advection equations. Also related is the work [19, 2], where frames of wave atoms, respectively curvelets, are used for the efficient representation and computation of wave propagators. Despite these successes, none of the aforementioned work successfully addressed the essential issue of problem formulations on finite domains with non-periodic boundary conditions.

1.4 Our contribution

In this paper we present a significant first step in the construction of an anisotropic multiscale frame system on bounded domains $\Omega \subset \mathbb{R}^2$, which is able to optimally resolve curvilinear singularities. The novel system coined boundary shearlet system is a hybrid system, consisting of shearlet elements to provide the optimal approximation rate for anisotropic structures and wavelet elements for handling the boundary. More precisely, we start with a compactly supported shearlet frame for $L^2(\mathbb{R}^2)$ as constructed, for instance, in [31]. From this frame we only choose those elements whose support is fully contained in Ω. Since this is by far not a complete system for $L^2(\mathbb{R}^2)$ – and certainly cannot handle boundary conditions –, we augment it by boundary adapted wavelets as constructed, for instance, in [8]. This augmentation procedure has to be done very carefully, but if it is carried out correctly we wind up with a frame system which satisfies the aforementioned desirable properties:

- Spatially highly localized elements.
- Boundary conditions can be imposed.
- Frame for $L^2(\Omega)$ with controlled frame bounds.

- Characterization of Sobelev spaces with numerical verification of the Gelfand frame property.
- Optimally sparse approximation of an even extended class of cartoon-like functions, which also include singularity curves traversing the boundary.

In fact, we first carry out this construction on the unit square $\Omega = [0, 1]^2$. Then, in a further step, we lift it by a patchwise construction similar to [43] to more arbitrary domains with smooth boundary.

1.5 Expected impact

We anticipate our results to have the following impacts:

- *Numerical solution of partial differential equations.* Our work represents a step in a larger research program, namely designing and applying specifically designed frames for the numerical solution of partial differential equations. Specifically, we envisage the system constructed in this paper to lead to the first optimally convergent adaptive algorithms for elliptic PDEs whose solutions possess singularities along hypersurfaces, and eventually also for linear transport equations. At the same time it is important to emphasize that, while we feel that the results concerning the construction of directional frames presented in this paper represent a significant advance, let us mention a few of the open problems already in this realm that need to be addressed in the future. First, the compressibility in the sense of [7] of the matrix representation of elliptic PDEs in our constructed system needs to be studied. Second, our results only show that Sobolev norms can be characterized by weighted ℓ^2 norms on the transform coefficients. The construction of optimal adaptive PDE solvers in the spirit of [7, 9, 43] requires slightly more, namely that the representation system constitutes a Gelfand frame [9] or at least a Sobolev frame [43]. In Section 7 we are so far only able to verify these properties numerically. Third, also on the practical side several issues remain such as the development of efficient quadrature rules for the representation of elliptic operators, say, in our representation system.
- *Imaging sciences.* Images are naturally supported on a rectangular domain. A common approach so far was to theoretically analyze algorithmic procedures from applied harmonic analysis for, for instance, denoising, inpainting or feature extraction, see [23, 21, 25], in $L^2(\mathbb{R}^2)$ disregarding any boundary issues. And, consequently, also their digitization was not particularly designed to handle a bounded domain. Thus, with the construction of multiscale systems on bounded domains, we now open the door, first, to a unified concept of the continuum and digital realm for data on bounded domains, and, second, to the design of novel directional systems for images adapted to the bounded domain they live on.
- *Hybrid systems and sparse approximation.* While there already exist some work on hybrid constructions using systems from applied harmonic analysis [21, 22], approximation properties of hybrid systems, or more precisely, the design of hybrid systems aiming at prescribed approximation properties has not been studied before. Thus, one might also see our construction as one first step in a line of research introducing more flexible systems – in this case sparse approximation properties combined with boundary adaption – by combining already well-studied systems in a careful way, so that they inherit parts of their behavior.

1.6 Outline

The paper is organized as follows. Section 2 is devoted to reviewing the main definitions and results for boundary adapted wavelet systems and shearlet systems. In the same section, we also already set the stage by fixing the notion of admissible boundary wavelet and shearlet systems, which will play a crucial role later. The precise definition of our multiscale anisotropic directional systems on bounded domains, coined boundary shearlet systems, and the analysis of their frame properties can be found in Section 3. In Section 4, we show how to construct shearlet systems that characterize $H^s(\mathbb{R}^2)$ by their analysis coefficients and how this gives rise to shearlet systems on bounded domains that characterize $H^s([0,1]^2)$. In the next section, which is Section 5, it is shown that the newly constructed systems provide optimal approximation rates for a more general class of cartoon-like functions than in previous literature considered. The extension of the previous definitions and results to more general domains is discussed in Section 6. Finally, in Section 7 we demonstrate the Gelfand frame property of boundary shearlet systems and also show stability and compressibility properties numerically.

2 Review of wavelet and shearlet systems

Since wavelets on bounded domains and shearlets will be the key ingredient in the construction of our anisotropic multiscale system on bounded domains – coined boundary shearlets –, this section shall serve as a review of their main definitions and properties. ¿From Section 3 on, we will then first present the construction of boundary shearlets on the product domain $[0,1]^2$, and defer the definition on arbitrary domains to Section 6, utilizing the previous construction. Thus, in this review, we similarly also first focus on function systems defined on $[0,1]^2$.

2.1 Boundary adapted wavelet systems

We start by briefly recalling the definition and main structural properties of wavelets on $\mathbb{R}$ and continue by giving a construction on a bounded domain $[0,1]$.

Letting $\psi^1 \in L^2(\mathbb{R})$ (the upper index 1 indicating that it is a function on $\mathbb{R}^1$), a *wavelet system* is constructed from all translates and dyadically rescaled versions of ψ^1, i.e.,

$$\{\psi^1_{j,m} := 2^{j/2}\psi^1(2^j \cdot -m) : j \in \mathbb{Z}, m \in \mathbb{Z}\}.$$

In order to construct wavelet systems that yield orthonormal bases for $L^2(\mathbb{R})$ the method of *multiresolution analysis* (MRA) was introduced in [38] and [40]. An MRA is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $\mathbb{R}$ satisfying the following conditions:

(I) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$.

(II) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(III) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$.

(IV) There exists some $\phi^1 \in V_0$ such that $\{\phi^1_{0,m} = \phi^1(\cdot - m),\ m \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function ϕ^1 of (IV) is called *scaling function.* The spaces V_j are customarily called *scaling spaces*, and the spaces $W_j := V_{j+1} \ominus V_j$ are referred to as *wavelet spaces.* A direct consequence of this construction and (II) is that $\bigoplus_{j\in\mathbb{Z}} W_j$ is dense in $L^2(\mathbb{R})$. Thus, if $\psi^1 \in W_0$ is chosen such that $\{\psi^1_{0,m} : m \in \mathbb{Z}\}$ constitutes an orthonormal basis for W_0, then, by (III), the wavelet system corresponding to ψ^1 is an orthonormal basis of $L^2(\mathbb{R})$. In this case, the function ψ^1 is called *wavelet associated with* ϕ^1 and vice versa.

An important breakthrough was achieved by the introduction of compactly supported wavelets with arbitrarily many vanishing moments, which were first constructed in [17], see also [18]. The resulting scaling functions and wavelets are called *Daubechies scaling functions* and *Daubechies wavelets*. Daubechies wavelets can be used to derive boundary adapted wavelet systems on $[0,1]$. The aim of the upcoming subsection is to recall the construction by Cohen, Daubechies, and Vial [8] of a boundary adapted wavelet system.

2.1.1 Orthonormal basis of wavelets on the interval

In our work, we will use the construction by Cohen, Daubechies, and Vial [8], which was the first construction of such a wavelet orthonormal basis – and is maybe today also the most widely used – to fulfill certain desired properties simultaneously such as inheriting a multiresolution analysis structure, possessing a sufficient number of vanishing moments, and exhibiting a certain degree of smoothness. One main guiding principle in the construction is to consider 'interior' and 'boundary' or 'edge' type scaling functions and define two corresponding classes of wavelets as linear combinations from those.

To review the construction from [8] (for more details we refer to this paper and to the book [39]), let $\phi^1 \in L^2(\mathbb{R})$ be a compactly supported Daubechies scaling function associated with a wavelet $\psi^1 \in L^2(\mathbb{R})$ having $p \in \mathbb{N}$ vanishing moments, i.e.,

$$\int_{\mathbb{R}} x^l \psi^1(x)\, dx = 0, \qquad \text{for all } l \in \{1, \ldots, p-1\}.$$

It is known, [8], that the associated scaling function ϕ^1 has support length $2p-1$, thus, we can assume $\operatorname{supp} \phi^1 = [-p+1, p]$. In order to have wavelets whose supports are fully contained in $[0,1]$ the scale has to be chosen sufficiently large. More precisely, for $j \in \mathbb{N}$ such that $j > \log_2 p$ we define *interior scaling functions*, i.e., scaling functions which have support inside $[0,1]$, by

$$\phi^{\mathrm{b}}_{j,m} := \phi^1_{j,m} := 2^{j/2}\phi^1(2^j \cdot - m), \quad \text{for } p \le m < 2^j - p.$$

Moreover, there exist boundary functions $(\phi^{\mathrm{left}}_m)_{m=0,\ldots,p-1}$ and $(\phi^{\mathrm{right}}_m)_{n=0,\ldots,p-1}$ [8], [39], such that using the *p left boundary scaling functions*, defined as

$$\phi^{\mathrm{b}}_{j,m} := 2^{j/2}\phi^{\mathrm{left}}_m(2^j \cdot), \quad \text{for } 0 \le m < p,$$

and the *p right boundary scaling functions* defined by

$$\phi^{\mathrm{b}}_{j,m} := 2^{j/2}\phi^{\mathrm{right}}_{2^j-1-m}(2^j(\cdot - 1)), \quad \text{for } 2^j - p \le m < 2^j,$$

one can construct a multiresolution analysis, see Theorem 2.1 below. For the statement of the theorem, we set

$$V^{\mathrm{b}}_j := \overline{\operatorname{span}\,(\phi^{\mathrm{b}}_{j,n})}_{n=0,\ldots,2^j-1}.$$

We also remark that this construction of boundary adapted scaling functions leads to 2^j scaling functions in total, which is the number of original scaling functions $(\phi_{j,m})_m$, i.e., not boundary corrected, that intersect $[0,1]$.

Theorem 2.1 ([8])**.** *Retaining the notations from this subsection, the sequence of spaces* $(V^{\mathrm{b}}_j)_{j\in\mathbb{N}_0}$ *is nested, i.e.,*

$$V^{\mathrm{b}}_0 \subset \ldots \subset V^{\mathrm{b}}_j \subset V^{\mathrm{b}}_{j+1} \subset \ldots$$

Moreover, for all $j \in \mathbb{N}$ such that $2^j \geq 2p$, the system $(\phi^{\mathrm{b}}_{j,m})_{m=0,\ldots,2^j-1}$ constitutes an orthonormal basis for V^{b}_j and the respective sequence of spaces V^{b}_j is complete, i.e.

$$\overline{\bigcup_{j>\log_2 p} V^{\mathrm{b}}_j} = L^2([0,1]).$$

Let now W^{b}_j denote the *wavelet space at level* j, i.e., the orthogonal complement of V^{b}_j in V^{b}_{j+1}. Then an orthonormal basis of wavelets for each space W^{b}_j can be constructed as follows. Let ϕ^1 be a compactly supported scaling function with supp $\psi^1 = [-p+1, p]$, and let ψ^1 be the corresponding wavelet possessing p vanishing moments. Similar to the construction of the boundary scaling functions previously discussed, we can construct wavelets, which are fully supported in $[0,1]$. Again, there exist boundary adapted wavelets $(\psi^{\mathrm{left}}_n)_n$ and $(\psi^{\mathrm{right}}_n)_n$ [8, 39], leading to $2^j - 2p$ *interior wavelets*

$$\psi^{\mathrm{b}}_{j,m} := \psi^1_{j,m} := 2^{j/2}\psi^1(2^j \cdot -m), \quad \text{for } p \leq m < 2^j - p,$$

p *left boundary wavelets*

$$\psi^{\mathrm{b}}_{j,m} := 2^{j/2}\psi^{\mathrm{left}}_m(2^j \cdot), \quad \text{for } 0 \leq m < p,$$

and p *right boundary wavelets*

$$\psi^{\mathrm{b}}_{j,m} := 2^{j/2}\psi^{\mathrm{right}}_{2^j-1-m}(2^j(\cdot - 1)), \quad \text{for } 2^j - p \leq m < 2^j.$$

This set of wavelets satisfies the following properties.

Theorem 2.2 ([8]). *Retaining the notations from this subsection, for $J \in \mathbb{N}$ with $2^J \geq 2p$, the following properties hold:*

i) $(\psi^{\mathrm{b}}_{J,m})_{m=0,\ldots,2^J-1}$ *is an orthonormal basis for* W^{b}_J.

ii) $L^2([0,1])$ *can be decomposed as*

$$L^2([0,1]) = V^{\mathrm{b}}_J \oplus W^{\mathrm{b}}_J \oplus W^{\mathrm{b}}_{J+1} \oplus W^{\mathrm{b}}_{J+2} \oplus \ldots = V^{\mathrm{b}}_J \oplus \bigoplus_{j=J}^{\infty} W^{\mathrm{b}}_j.$$

iii) $\left\{(\phi^{\mathrm{b}}_{J,m})_{m=0,\ldots,2^J-1}, (\psi^{\mathrm{b}}_{j,m})_{j\geq J, m=0,\ldots,2^j-1}\right\}$ *is an orthonormal basis for* $L^2([0,1])$.

iv) If $\phi^1, \psi^1 \in C^r([0,1])$, *then* $\left\{(\phi^{\mathrm{b}}_{j,m})_{m=0,\ldots,2^J-1}, (\psi^{\mathrm{b}}_{j,m})_{j\geq J, m=0,\ldots,2^j-1}\right\}$ *is an unconditional basis for* $C^s([0,1])$ *for all* $s < r$.

A two dimensional (2D) boundary adapted wavelet system can be obtained by tensor products of 1D boundary adapted wavelets, as we will see in the next subsection.

2.1.2 Orthonormal basis of wavelets on $[0,1]^2$

We briefly comment on the construction of 2D boundary adapted wavelets by tensor products of 1D boundary adapted wavelets. For this, let ϕ^1 be a 1D compactly supported Daubechies scaling function. Further, let ψ^1 be the corresponding wavelet to ϕ^1 with p vanishing moments, and let $(\phi^{\mathrm{b}}_{j,m})_{j,m}$ and $(\psi^{\mathrm{b}}_{j,m})_{j,m}$ be the scaling functions and wavelets as described in Subsection 2.1.1, respectively.

If $J \in \mathbb{N}$ denotes the smallest number such that $2^J \geq 2p$, then the 2D scaling functions can be obtained by

$$\omega_{J,(m_1,m_2),0} := \phi^{\mathrm{b}}_{J,m_1} \otimes \phi^{\mathrm{b}}_{J,m_2}, \quad (m_1, m_2) \in \mathbb{Z}^2,\ 0 \leq m_1, m_2 \leq 2^J - 1.$$

The corresponding 2D wavelet functions are defined by the tensor products given by

$$\omega_{j,(m_1,m_2),\upsilon} := \begin{cases} \phi^{\mathrm{b}}_{j,m_1} \otimes \psi^{\mathrm{b}}_{j,m_2}, & j \geq J, \upsilon = 1, \\ \psi^{\mathrm{b}}_{j,m_1} \otimes \psi^{\mathrm{b}}_{j,m_2}, & j \geq J, \upsilon = 2, \\ \psi^{\mathrm{b}}_{j,m_1} \otimes \phi^{\mathrm{b}}_{j,m_2}, & j \geq J, \upsilon = 3, \end{cases}$$

for $0 \leq m_1, m_2 \leq 2^j - 1$ with $j \geq J$. This gives rise to the notion of boundary wavelet system defined as follows.

Definition 2.3. *Let ϕ^1 be a 1D compactly supported Daubechies scaling function, and let ψ^1 be the corresponding wavelet to ϕ^1. Then*

$$\mathcal{W}(\phi^1) := \{\omega_{j,m,\upsilon} \,:\, (j,m,\upsilon) \in \Delta\}$$

is called boundary wavelet system associated with ϕ^1, *with indexing set given by*

$$\Delta := \left\{(J,(m_1,m_2),0) : 0 \leq m_1, m_2 \leq 2^J - 1\right\} \\ \cup \left\{(j,m,\upsilon) : j \geq J, 0 \leq m_1, m_2 \leq 2^j - 1, \upsilon \in \{1,2,3\}\right\}.$$

Similar as before, we define scaling spaces V^{b}_J and wavelet spaces W^{b}_j, $j \geq J$, yielding the following result.

Theorem 2.4 ([39])**.** *Retaining the notations from this subsection, for each $j \geq J$, the system $(\omega_{j,(m_1,m_2),\upsilon})_{(m_1,m_2),\upsilon}$ forms an orthonormal basis for W^{b}_j, and $L^2([0,1]^2)$ can be decomposed as*

$$L^2([0,1]^2) = V^{\mathrm{b}}_J \oplus W^{\mathrm{b}}_J \oplus W^{\mathrm{b}}_{J+1} \oplus W^{\mathrm{b}}_{J+2} \oplus \ldots = V^{\mathrm{b}}_J \oplus \bigoplus_{j=J}^{\infty} W^{\mathrm{b}}_j.$$

2.1.3 Characterization of $H^s([0,1]^2)$ using boundary wavelets

One key property of wavelets is the fact that they are able to characterize Sobolev spaces $H^s([0,1]^2)$. The system presented in the previous subsection can possess this property, in fact, the existence of a scaling function ϕ^1 such that the associated boundary wavelet system yields such a characterization was proved in [5, Cor. 29.2]. More precisely, for any $s > 0$ there exist $0 < C_{\mathrm{s}} \leq D_{\mathrm{s}} < \infty$ and $p \in \mathbb{N}$ such that with the boundary wavelet system $\mathcal{W}(\phi^1)$ constructed from the Daubechies scaling function Φ^1 and Daubechies wavelet ψ^1 with p vanishing moments, for all $s' \leq s$, we have

$$C_{\mathrm{s}} \|f\|^2_{H^{s'}([0,1]^2)} \leq \sum_{(j,m,\upsilon)\in\Delta} 2^{2js'} |\langle f, \omega_{j,m,\upsilon}\rangle|^2 \leq D_{\mathrm{s}} \|f\|^2_{H^{s'}([0,1]^2)},$$

for all $f \in H^{s'}([0,1]^2)$.

2.1.4 Admissible boundary wavelets

The system of boundary wavelets constructed in Subsection 2.1.2 on $\Omega = [0,1]^2$ exhibits certain properties, which will be crucial for our later construction of boundary shearlets. Therefore, we will call a system with those properties an *admissible boundary wavelet system.*

More precisely, we are interested in the following list of desiderata, a boundary wavelet system $\mathcal{W}(\phi^1)$ as defined in Definition 2.3 can satisfy on $\Omega := [0,1]^2$:

(W1) $\mathcal{W}(\phi^1)$ constitutes an orthonormal basis for $L^2(\Omega)$.

(W2) The interior wavelets $\omega^1 = \psi^1 \otimes \phi^1$, $\omega^2 = \psi^1 \otimes \psi^1$, and $\omega^3 = \phi^1 \otimes \psi^1$ obey a moment condition of the form

$$|\widehat{\omega^v}(\xi)| \lesssim \frac{\min\{1, |\xi_i|^\alpha\}}{\max\{1, |\xi_1|^\beta\} \max\{1, |\xi_2|^\beta\}},$$

for at least one $i \in \{1, 2\}$ and $\alpha, \beta > 0$.

(W3) For $s \in \mathbb{N}$, the wavelet system $\mathcal{W}(\phi^1)$ characterizes $H^s(\Omega)$, i.e., there exists $0 < C_\mathrm{s} \leq D_\mathrm{s} < \infty$ such that, for all $s' \leq s$ and $f \in H^{s'}(\Omega)$,

$$C_\mathrm{s} \|f\|^2_{H^{s'}(\Omega)} \leq \sum_{(j,m,v)\in\Lambda} 2^{2js'} |\langle f, \omega_{j,m,v}\rangle|^2 \leq D_\mathrm{s} \|f\|^2_{H^{s'}(\Omega)}.$$

Note that (W2) is always satisfied as soon as the generating wavelet is sufficiently smooth and possesses sufficiently many vanishing moments, which can always be achieved by the construction of Subsection 2.1 when we choose p large enough. This gives rise to the following definition of admissible wavelet systems which will then be one key ingredient in our construction of anisotropic multiscale systems on bounded domains.

Definition 2.5. *A boundary wavelet system $\mathcal{W}(\phi^1)$ that admits properties (W1), (W2), and (W3) with $s \in \mathbb{N}$ and $\alpha, \beta > 0$ is called an (s, α, β)-*admissible boundary wavelet system*.*

2.2 Shearlets

Shearlet systems are designed to provide optimally sparse approximations of a model class of functions which are governed by curvilinear singularities while allowing a faithful implementation with their construction being based on the framework of affine systems. They were first introduced by Guo, Labate, Lim, Weiss, and one of the authors in [29, 37]. Since we aim to use shearlets as interior elements for the construction of our anisotropic multiscale systems on bounded domains, in the sequel we present the definition and properties of (cone-adapted) shearlet systems. In fact, as already discussed in the introduction, only very preliminary attempts have been made yet to define a shearlet system for bounded domains.

2.2.1 Construction of (cone-adapted) shearlet systems

The construction of shearlet systems is based on *anisotropic scaling* and *shearing*. To state the precise definition, for $j \in \mathbb{N}, k \in \mathbb{Z}$, we denote the *anisotropic scaling matrices* A_j and the *shear matrices* S_k by

$$A_j := \mathrm{diag}(2^j, 2^{\frac{j}{2}}) = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\frac{j}{2}} \end{pmatrix}, \quad S_k := \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Then a *(cone-adapted) shearlet system* is defined as follows:

Definition 2.6 ([31]). *Let $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$, $c = [c_1, c_2]^T \in \mathbb{R}^2$ with $c_1, c_2 > 0$. Then the* (cone-adapted) shearlet system *is defined by*

$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) = \Phi(\phi, c_1) \cup \Psi(\psi, c) \cup \tilde{\Psi}(\tilde{\psi}, c),$$

where

$$\Phi(\phi, c_1) = \left\{\phi(\cdot - c_1 m) : m \in \mathbb{Z}^2\right\},$$

$$\Psi(\psi, c) = \left\{ \psi_{j,k,m} = 2^{\frac{3j}{4}} \psi(S_k A_j \cdot - M_c m) : j \in \mathbb{N}, |k| \leq \left\lceil 2^{\frac{j}{2}} \right\rceil, m \in \mathbb{Z}^2 \right\},$$
$$\tilde{\Psi}(\tilde{\psi}, c) = \left\{ \tilde{\psi}_{j,k,m} = 2^{\frac{3j}{4}} \tilde{\psi}(S_k^T \tilde{A}_j \cdot - M_{\tilde{c}} m) : j \in \mathbb{N}, |k| \leq \left\lceil 2^{\frac{j}{2}} \right\rceil, m \in \mathbb{Z}^2 \right\},$$

with $M_c := \mathrm{diag}(c_1, c_2)$*,* $M_{\tilde{c}} = \mathrm{diag}(c_2, c_1)$*, and* $\tilde{A}_{2^j} = \mathrm{diag}(2^{\frac{j}{2}}, 2^j)$*.*

We shorten the notation by defining

$$\psi_{j,k,m,\iota} := \begin{cases} \psi_{j,k,m} & \text{if } \iota = 1, \\ \phi(\cdot - c_1 m) & \text{if } \iota = 0, \\ \widetilde{\psi}_{j,k,m} & \text{if } \iota = -1, \end{cases}$$

and denote the index set for the full shearlet system by

$$\Lambda := \left\{ (j, k, m, \iota) \,:\, \iota \in \{-1, 0, 1\} :\; |\iota| j = j \geq 0,\; |k| \leq |\iota| \left\lceil 2^{\frac{j}{2}} \right\rceil,\; m \in \mathbb{Z}^2 \right\}.$$

2.2.2 Frames of shearlet systems

Cone-adapted shearlet systems are redundant systems, hence cannot form an orthonormal basis. Nonetheless they still exhibit stability properties in the sense of constituting a frame for $L^2(\mathbb{R}^2)$ with controllable frame bounds. Recall that a family of elements $(\varphi_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ forms a *frame* for $\mathcal{H}$, if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{n\in\mathbb{N}} |\langle \varphi_n, f\rangle_{\mathcal{H}}|^2 \leq B\|f\|_{\mathcal{H}}^2, \qquad \text{for all } f \in \mathcal{H}.$$

If only the second inequality holds, then the system $(\varphi_n)_{n\in\mathbb{N}}$ is called a *Bessel sequence*. Associated to every Bessel sequence $(\varphi_n)_{n\in\mathbb{N}}$ is the *analysis operator* T given by

$$T : \mathcal{H} \to \ell^2(\mathbb{N}), \quad f \mapsto (\langle f, \varphi_n\rangle)_{n\in\mathbb{N}}.$$

The inner products $\langle f, \varphi_n\rangle$, $n \in \mathbb{N}$ are sometimes also termed *analysis coefficients*; in contrast to the coefficients of an expansion of f in the system $(\varphi_n)_{n\in\mathbb{N}}$ being referred to as *synthesis coefficients*. If $(\varphi_n)_{n\in\mathbb{N}}$ constitutes even a frame for $\mathcal{H}$, it can be shown that the *frame operator* $S := T^*T$ is a bounded invertible operator with bounded inverse [4]. Aiming to reconstruct any f from Tf, one defines yet another frame $(\varphi_n^d)_{n\in\mathbb{N}}$, the so-called *canonical dual frame* of $(\varphi_n)_{n\in\mathbb{N}}$, by

$$\varphi_n^d := S^{-1}\varphi_n.$$

This leads to the formulae

$$f = \sum_{n\in\mathbb{N}} \langle f, \varphi_n\rangle \varphi_n^d = \sum_{n\in\mathbb{N}} \langle f, \varphi_n^d\rangle \varphi_n, \qquad \text{for all } f \in \mathcal{H},$$

of which the first part is the desired reconstruction formula, and the second can be regarded as an expansion of f in terms of the frame $(\varphi_n)_{n\in\mathbb{N}}$.

By imposing some weak conditions on the generators ϕ, ψ, and $\tilde{\psi}$, the system $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c)$ forms a frame for $L^2(\mathbb{R}^2)$. In fact, the following result has been proved in [31].

Theorem 2.7 ([31])**.** *Let* $\phi, \psi \in L^2(\mathbb{R}^2)$ *such that*

$$|\widehat{\phi}(\xi_1, \xi_2)| \leq C_1 \min\{1, |\xi_1|^{-\beta}\} \min\{1, |\xi_2|^{-\beta}\}$$

and
$$|\widehat{\psi}(\xi_1,\xi_2)| \leq C_2 \min\{1,|\xi_1|^{\alpha}\} \min\{1,|\xi_1|^{-\beta}\} \min\{1,|\xi_2|^{-\beta}\},$$
for some constants $C_1, C_2 > 0, \alpha > \beta > 3$ *and almost every* $(\xi_1,\xi_2) \in \mathbb{R}^2$. *Then for any* $c = (c_1,c_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ *the cone-adapted shearlet system* $\mathcal{SH}(\phi,\psi,\widetilde{\psi},c)$ *forms a Bessel sequence for* $L^2(\mathbb{R}^2)$. *Further, let* $\widetilde{\psi}(x_1,x_2) = \psi(x_2,x_1)$ *and assume there exists a positive constant* $A > 0$ *such that*
$$|\widehat{\phi}(\xi)|^2 + \sum_{j\in\mathbb{N}} \sum_{|k|\leq\lceil 2^{j/2}\rceil} |\widehat{\psi}(S_k^T(A_j)^{-1}\xi)|^2 + \sum_{j\in\mathbb{N}} \sum_{|k|\leq\lceil 2^{j/2}\rceil} |\widehat{\widetilde{\psi}}(S_k(\widetilde{A}_j)^{-1}\xi)|^2 > A$$
holds almost everywhere. Then there exists $c = (c_1,c_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ *such that the cone-adapted shearlet system* $\mathcal{SH}(\phi,\psi,\widetilde{\psi},c)$ *forms a frame for* $L^2(\mathbb{R}^2)$.

In view of the results for boundary wavelets in the preceding Subsection 2.1 one can pose the question, whether a characterization of Sobolev spaces is also possible with shearlet frames. While there are some embedding results of Besov spaces into shearlet spaces and vice versa, see [36], [11], we are unaware of a precise characterization of Sobolev spaces $H^s(\mathbb{R}^2)$ by compactly supported shearlets. In Subsection 4.1 we outline how such a construction can be achieved.

2.2.3 Admissible shearlet systems

Similarly to the wavelet construction, shearlet systems satisfying the following properties will play a crucial role in the construction of boundary shearlet systems.

(S1) The functions $\phi,\psi,\widetilde{\psi} \in L^2(\mathbb{R}^2)$ are compactly supported functions and the corresponding shearlet system $\mathcal{SH}(\phi,\psi,\widetilde{\psi},c)$ of Definition 2.6 constitutes a frame for $L^2(\mathbb{R}^2)$.

(S2) The decay conditions
$$|\widehat{\psi}(\xi_1,\xi_2)| \lesssim \frac{\min\{1,|\xi_1|^{\alpha}\}}{\max\{1,|\xi_1|^{\beta}\}\max\{1,|\xi_2|^{\beta}\}}$$
and
$$|\widehat{\widetilde{\psi}}(\xi_1,\xi_2)| \lesssim \frac{\min\{1,|\xi_2|^{\alpha}\}}{\max\{1,|\xi_1|^{\beta}\}\max\{1,|\xi_2|^{\beta}\}}$$
are obeyed for all $(\xi_1,\xi_2) \in \mathbb{R}^2$, for some $\alpha,\beta > 0$.

(S3) The shearlet system $\mathcal{SH}(\phi,\psi,\tilde{\psi},c)$ characterizes Sobolev spaces up to order $s \in \mathbb{N}$, i.e., there exist $0 < A_{\mathrm{s}}, B_{\mathrm{s}} < \infty$ such that
$$A_{\mathrm{s}}\|f\|^2_{H^{s'}(\mathbb{R}^2)} \leq \sum_{(j,k,m,\iota)\in\Lambda} 2^{2js'} |\langle f,\psi_{j,k,m,\iota}\rangle|^2 \leq B_{\mathrm{s}}\|f\|^2_{H^{s'}(\mathbb{R}^2)},$$
for all $f \in H^{s'}(\mathbb{R}^2),\ s' \leq s$.

This gives rise to the following class of shearlet systems.

Definition 2.8. *A shearlet system that admits properties (S1), (S2), and (S3) with* $s \in \mathbb{N}$ *and* $\alpha,\beta > 0$ *is called* (s,α,β)-admissible shearlet system. *If the parameters* (s,α,β) *do not play any role, we simply write* admissible shearlet system.

Let us emphasize that by the results in this subsection, an abundance of $(0,\alpha,\beta)$-admissible shearlet systems do exist. And in fact, this will be a common requirement in later results. Let us also remind the reader again that the existence of shearlet systems admitting properties (S1), (S2) in combination with (S3) will be outlined in Subsection 4.1. Nevertheless we stress that the focus of this paper is not the construction of admissible shearlet systems but the (much more challenging) problem of obtaining an adaption of such a system to a finite domain.

2.2.4 Approximation properties

The target property shearlet systems were designed to satisfy is optimal approximation of cartoon-like functions. This class of functions was first introduced by Donoho in [20] as a suitable model for natural images, and consists – roughly speaking – of compactly supported functions which are C^2 apart from a C^2-discontinuity curve.

One key ingredient in the definition of cartoon-like functions is the class of sets $STAR^2(\nu)$, which are star-shaped sets with C^2 boundary and curvature bounded by $\nu > 0$. Then the definition of cartoon-like functions reads as follows.

Definition 2.9. *For $\nu > 0$, let $\mathcal{E}^2(\nu)$ denote the set of functions $f \in L^2(\mathbb{R}^2)$, for which there exist some $D \in STAR^2(\nu)$ and $f_i \in C^2(\mathbb{R}^2)$ with compact support in $[0,1]^2$ as well as $\|f_i\|_{C^2} \leq 1$ for $i = 1, 2$, such that*

$$f = f_1 + \chi_D f_2.$$

The elements $f \in \mathcal{E}^2(\nu)$ are called cartoon-like functions.

In the same paper [20], Donoho presented the first optimality result concerning the approximation rate for this class of functions by more or less arbitrary representation systems. In fact, in [20] it was shown that for any representation system $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^2)$, the minimally achievable asymptotic approximation error for $f \in \mathcal{E}^2(\nu)$ is

$$\|f - f_N\|_2^2 = O(N^{-2}) \quad \text{as } N \to \infty,$$

where f_N denotes the non-linear best N-term approximation of f obtained by choosing the N largest coefficients through *polynomial depth search*. The term polynomial depth search means that the i-th term in the expansion can only be chosen in accordance with a selection rule $\sigma(i, f)$, which obeys $\sigma(i, f) \leq \pi(i)$ for a fixed polynomial $\pi(i)$, see [20]; thereby avoiding artificial representation systems such as those being dense in $L^2(\mathbb{R}^2)$.

In [32], one of the authors together with Lim proved that there do indeed exist shearlet systems that can achieve this optimal rate up to a log factor. More precisely, the following theorem was shown.

Theorem 2.10 ([32]). *Let $c > 0$, and let $\phi, \psi, \widetilde{\psi} \in L^2(\mathbb{R}^2)$ be compactly supported. Suppose that, in addition, for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ the shearlet ψ satisfies*

(i) $|\hat{\psi}(\xi)|^2 \leq C_1 \cdot \min(1, |\xi_1|^\alpha) \cdot \min(1, |\xi_1|^{-\beta}) \cdot \min(1, |\xi_2|^{-\beta})$ *and*

(ii) $|\frac{\partial}{\partial x_2}\hat{\psi}(\xi)| \leq |h(\xi_1)| \cdot \left(1 + \frac{|\xi_2|}{|\xi_1|}\right)^{-\beta}$,

where $\alpha > 5$, $\beta \geq 4$, $h \in L^1(\mathbb{R})$, and $C_1 > 0$, and suppose that $\widetilde{\psi}$ satisfies (i) and (ii) with ξ_1 and ξ_2 interchanged. Further suppose that $\mathcal{SH}(\phi, \psi, \widetilde{\psi}, c)$ forms a frame for $L^2(\mathbb{R}^2)$. Then for any $\nu > 0$, the shearlet frame $\mathcal{SH}(\phi, \psi, \widetilde{\psi}, c)$ provides (almost) optimally sparse approximations of functions *$f \in \mathcal{E}^2(\nu)$, i.e.*

$$\sum_{n \geq N} \theta_n(f) \lesssim N^{-2} \log(N)^3, \;\; \textit{as } N \to \infty,$$

where $\theta_n(f)$ denotes the n-th entry of the non-increasing rearrangement of the coefficient sequence $(|\langle f, \psi_{j,k,m,\iota}\rangle|^2)_{(j,k,m,\iota)\in\Lambda}$.

We wish to emphasize that all presented results of shearlet systems in this section only hold for $\mathbb{R}^2$. The main objective of this paper is to introduce a suitable multiscale anisotropic directional system which allows to transfer these results to preferably any bounded (polygonal shaped) domain $\Omega \subseteq \mathbb{R}^2$.

3 Domain adapted multiscale anisotropic directional systems

The key idea for the construction of domain-adapted multiscale anisotropic directional systems on a domain $\Omega = [0,1]^2$ which we will coin *boundary shearlet systems* consists in combining two different frames, i.e., to employ hybrid systems. To be more precise, we will on the one hand use boundary wavelet elements to handle the boundary $\partial\Omega$, since these systems are already adapted to such a boundary. On the other hand, compactly supported shearlet elements will be used for the interior of Ω to achieve the desired optimal approximation rates of cartoon-like functions. To this end we fix an admissible boundary wavelet system $\mathcal{W}(\phi^1)$ for $\Omega = [0,1]^2$ and an admissible shearlet system $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c)$.

Certainly, those elements of each system used for the hybrid construction have to be carefully selected. This will be done by defining a tubular region $\Gamma_{\gamma(j)}$ around the boundary with $\gamma(j)$ depending on the scales j and selecting those wavelet elements non-trivially intersecting these regions for each scale. As for the selection of the elements from a compactly supported shearlet system, we choose all those whose support is completely contained inside Ω. It is conceivable that the tubular region $\Gamma_{\gamma(j)}$ needs to be defined depending on properties of the shearlets, since the frame property of the hybrid systems requires a small cross-localization of elements from both systems.

To define the tubular regions, let q be chosen such that supp $\psi_{j,k,0,\iota} \subseteq B_{\frac{q}{2}2^{-j/2}}(0)$ for all $(j,k,0,\iota) \in \Lambda$, where $B_r(0) := \{x \in \mathbb{R}^2 : |x_1|, |x_2| \le r\}$. For $r \in \mathbb{R}$, we now define the tubular region Γ_r by

$$\Gamma_r := \{x \in \Omega : d(x, \partial\Omega) < q2^{-r}\}, \tag{1}$$

i.e., as the part of Ω that has distance less than $q2^{-r}$ from $\partial\Omega$. This gives rise to the following definition.

Definition 3.1. *Let $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) = (\psi_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$ be an admissible shearlet system, let $\tau > 0$ and $t > 0$. Further, let $\mathcal{W}(\phi^1)$ be an admissible boundary wavelet system, and set*

$$\mathcal{W}_{t,\tau}(\phi^1) := \{\omega_{j,m,\upsilon} \in \mathcal{W}(\phi^1) : (j,m,\upsilon) \in \Delta_{t,\tau}\},$$

where

$$\Delta_{t,\tau} := \{(j,m,\upsilon) \in \Delta : \text{supp } \omega_{j,m,\upsilon} \cap \Gamma_{\tau(j-t)} \neq \emptyset\}.$$

Further, let

$$\Lambda_0 := \{(j,k,m,\iota) \in \Lambda : \text{supp } \psi_{j,k,m,\iota} \subseteq \Omega\}.$$

Then, the boundary shearlet system with offsets t and τ *is defined as*

$$\mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c) := \{\psi_{j,k,m,\iota} : (j,k,m,\iota) \in \Lambda_0\} \cup \mathcal{W}_{t,\tau}(\phi^1).$$

This definition of a boundary shearlet system mimics precisely the program we just intuitively described before. The reader should notice that as $j \to \infty$, the size of the tubular region shrinks accordingly. Concerning the two offsets, the parameter t acts as a shift for the dependence on the scale, whereas the parameter τ is merely an overall factor controlling the speed of decay.

Indeed this choice of wavelets versus shearlets provides us with a cross localization property, which will be crucial for, for instance, the proofs of Theorems 3.3 and 4.4. Since the proof is quite lengthy and technical, we defer it to Subsection 3.2. We denote $\Lambda_0^c := \Lambda \setminus \Lambda_0$ and $\Delta_{\tau,t}^c := \Delta \setminus \Delta_{t,\tau}$.

Proposition 3.2. *Let $\alpha > 1$, $\beta > \alpha+1$, $\tau > 0$ and $\epsilon > 0$ such that $((1-\epsilon)/\tau-2)\alpha > \frac{5}{2}$. Further, assume that $\mathcal{W}(\phi^1)$ is a $(0, \alpha, \beta)$-admissible boundary wavelet system on Ω and $\mathcal{SH}(\phi, \psi, \widetilde{\psi}, c)$ is a $(0, 0, \beta')$-admissible shearlet system with $\beta'-\alpha > 1$. Then there exists a constant C (dependent on $\phi^1, \phi, \psi, \widetilde{\psi}, c, \tau, \epsilon$) such that, for all $t > 0$,*

$$\sum_{(j_s,k,m_s,\iota)\in\Lambda_0^c} \sum_{(j_w,m_w)\in\Delta_{\tau,t}^c} |\langle \omega_{j_w,m_w,\upsilon}, \psi_{j_s,k,m_s,\iota}\rangle_{L^2(\Omega)}|^2 \leq C \cdot 2^{-2(1-\epsilon)\alpha t}.$$

3.1 Frame property of boundary shearlet systems

We now turn to analyzing the frame properties of boundary shearlet systems. The following result provides weak sufficient conditions for these systems to form a frame for $L^2(\Omega)$.

Theorem 3.3. *Let $\mathcal{W}(\phi^1)$ be a $(0, \alpha, \beta)$-admissible wavelet system, let $\mathcal{SH}(\phi, \psi, \widetilde{\psi}, c)$ be a $(0, 0, \beta')$-admissible shearlet system and let $\alpha, \beta, \beta', \tau$ and ϵ be as in Proposition 3.2. Let $t > 0$ be such that*

$$A' := \frac{A - C2^{-2(1-\epsilon)\alpha t+1}}{2B} > 0, \tag{2}$$

where A and B are the frame bounds of the full shearlet system restricted to $L^2(\Omega)$ and C the constant from Proposition 3.2. Then the boundary shearlet system $\mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$ yields a frame for $L^2(\Omega)$. Furthermore, a lower and an upper frame bound are given by A' and $B + 1$, respectively.

Proof. We first observe that

$$L^2(\Omega) = \operatorname{span}(\mathcal{W}(\phi^1) \setminus \mathcal{W}_{t,\tau}(\phi^1)) \oplus \operatorname{span}(\mathcal{W}_{t,\tau}(\phi^1)) =: \Xi_1 \oplus \Xi_2.$$

Now let $f_1 := P_{\Xi_1} f$ and $f_2 := P_{\Xi_2} f$, where P_{Ξ_i} denotes the orthogonal projection onto the spaces Ξ_i, $i = 1, 2$.

We first focus on proving the existence of a lower frame bound. For every $f \in L^2(\Omega)$, the frame property of the full shearlet system restricted to $L^2(\Omega)$, see [33], implies that

$$\|f\|^2_{L^2(\Omega)} \leq \frac{1}{A} \sum_{(j,k,m,\iota)\in\Lambda} |\langle f, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2,$$

where A is the lower frame bound of the shearlet frame. Splitting the right hand side appropriately, yields

$$\begin{aligned}\|f\|^2_{L^2(\Omega)} \leq \frac{1}{A}\Bigg(&\sum_{(j,k,m,\iota)\in\Lambda_0} |\langle f, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 + 2 \sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle f_1, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \\ &+ 2 \sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle f_2, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \Bigg) =: T_1 + T_2 + T_3. \end{aligned} \tag{3}$$

According to the construction of the boundary shearlet system, we only need to further study T_2 and T_3.

We start by estimating term T_2. Employing the Parseval identity and then the Cauchy Schwarz inequality yields

$$\sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle f_1, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2$$

$$= \sum_{(j,k,m,\iota)\in\Lambda_0^c} \left| \sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} \langle f_1, \omega_{j',m',\upsilon}\rangle \langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)} \right|^2$$

$$\leq \sum_{(j,k,m,\iota)\in\Lambda_0^c} \left(\sum_{(j',m',\upsilon)\in\Delta_{\tau,t}^c} |\langle f_1, \omega_{j',m',\upsilon}\rangle|^2 \sum_{(j',m',\upsilon)\in\Delta_{\tau,t}^c} |\langle \omega_{j',m',\upsilon} \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \right)$$

$$\leq \|f_1\|_{L^2(\Omega)}^2 \sum_{(j,k,m,\iota)\in\Lambda_0^c} \sum_{(j',m',\upsilon)\in\Delta_{\tau,t}^c} |\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2.$$

Hence, by Proposition 3.2,

$$\sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle f_1, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \leq C\|f_1\|_{L^2(\Omega)}^2 2^{-(2\alpha(1-\epsilon))t} \tag{4}$$

for some constant $C > 0$.

With B being the upper frame bound of the shearlet frame, term T_3 can be estimated as

$$\sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle f_2, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \leq B\|f_2\|_{L^2(\Omega)}^2.$$

Further, since $f_2 \in \Xi_2$,

$$\|f_2\|_{L^2(\Omega)}^2 = \sum_{(j,m,\upsilon)\in\Delta_{\tau,t}} |\langle f_2, \omega_{j,m,\upsilon}\rangle_{L^2(\Omega)}|^2 = \sum_{(j,m,\upsilon)\in\Delta_{\tau,t}} |\langle f, \omega_{j,m,\upsilon}\rangle_{L^2(\Omega)}|^2. \tag{5}$$

Applying (4) and (5) to (3) yields that A' is a lower frame bound for the boundary shearlet system, and t was chosen such that $A' > 0$.

Now, applying the upper frame inequality for the shearlet frame and Parseval's equality for the wavelet orthonormal basis on the respective terms yields an upper frame bound of $B + 1$ for the boundary shearlet system $\mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$. □

3.2 Localization of shearlet and wavelet frames

We now turn to the proof of Proposition 3.2. For this, we will require the following technical lemma.

Lemma 3.4. *Let $\psi \in L(\mathbb{R}^2)$ be such that there exists $C > 0$ with*

$$|\widehat{\psi}(\xi_1, \xi_2)| \leq C \frac{\min\{1, |\xi_1|^\alpha\}}{\max\{1, |\xi_1|^\beta\} \max\{1, |\xi_2|^\beta\}}, \quad \textit{for a.e. } (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where $\beta/2 > \alpha > 1$. Then, for $\iota = -1, 1$,

$$\sum_{|k|\leq 2^{j/2}} |(\psi_{j,k,m,\iota})^\wedge(\xi_1, \xi_2)| \leq 2^{-3/4j} C' \frac{1}{\max\{1, |2^{-j}\xi_1|^{\beta/2}\}} \frac{1}{\max\{1, |2^{-j}\xi_2|^{\beta/2}\}},$$

for a.e. $(\xi_1, \xi_2) \in \mathbb{R}^2$ and a constant C'.

Proof. We only present the proof for the case $\iota = -1$. The other case can be shown analogously. We first develop simple estimates in two different cases. If on the one hand $|\xi_1| \geq |\xi_2|/2$, then we have

$$\max\{1, |\xi_1|^\beta\} \max\{1, |\xi_2|^\beta\} \geq \max\{1, |\xi_1|^{\frac{\beta}{2}}\} \max\{1, |\xi_1|^{\frac{\beta}{2}}\} \tag{6}$$

$$\geq 2^{-\frac{\beta}{2}} \max\{1, |\xi_1|^{\frac{\beta}{2}}\} \max\{1, |\xi_2|^{\frac{\beta}{2}}\}.$$

If on the other hand $|\xi_1| \leq |\xi_2|/2$, then

$$\begin{aligned}
\max\{1, |k2^{-j}\xi_1 + 2^{-j/2}\xi_2|^\beta\} &\geq \max\{1, (|2^{-j/2}\xi_2| - |k2^{-j}\xi_1|)^\beta\} \\
&\geq \max\{1, (|2^{-j/2}\xi_2| - |2^{-j/2}\xi_1|)^\beta\} \\
&\geq \max\{1, (|2^{-j/2}\xi_2/2|)^\beta\} \\
&\geq 2^{-\beta} \max\{1, (|2^{-j/2}\xi_2|)^\beta\}, \qquad (7)
\end{aligned}$$

where we have used that $|k2^{-j/2}| \leq 1$ in the second inequality. Using the estimates (6) and (7) the claim follows by the following computation. First we notice that

$$\sum_{|k| \leq 2^{j/2}} \frac{\min\{1, |2^{-j}\xi_1|^\alpha\}}{\max\{1, |k2^{-j}\xi_1 + 2^{-j/2}\xi_2|^\beta\}} \leq \sup_{(\xi_1,\xi_2)\in\mathbb{R}^2} \sum_{k\in\mathbb{Z}} \frac{\min\{1, |\xi_1|^\alpha\}}{\max\{1, |k\xi_1 + \xi_2|^\beta\}} < C',$$

for some constant C'. For $|\xi_1| \geq |\xi_2|/2$, we conclude that

$$\begin{aligned}
&\sum_{|k|\leq 2^{j/2}} |(\psi_{j,k,m,\iota})^\wedge(\xi_1,\xi_2)| \\
&\leq C 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \sup_{(\xi_1,\xi_2)\in\mathbb{R}^2} \sum_{|k|\leq 2^{j/2}} \frac{\min\{1, |2^{-j}\xi_1|^\alpha\}}{\max\{1, |k2^{-j}\xi_1 + 2^{-j/2}\xi_2|^\beta\}} \\
&\leq C'' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \\
&\leq C'' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^{\beta/2}\} \max\{1, |2^{-j}\xi_2|^{\beta/2}\}},
\end{aligned}$$

where we used (6) in the last estimate. For $0 < |\xi_1| \leq |\xi_2|/2$, we derive by employing (7)

$$\begin{aligned}
&\sum_{|k|\leq 2^{j/2}} |(\psi_{j,k,m,\iota})^\wedge(\xi_1,\xi_2)| \\
&\leq C 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \sum_{|k|\leq 2^{j/2}} \frac{|2^{-j}\xi_1|^\alpha}{\max\{1, |k2^{-j}\xi_1 + 2^{-j/2}\xi_2|^\beta\}} \\
&\leq C' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \sum_{|k|\leq 2^{j/2}} \frac{|2^{-j}\xi_1|^\alpha}{\max\{1, |2^{-j/2}\xi_2|^\beta\}} \\
&\leq C' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \frac{|2^{-j/2}\xi_1|^\alpha}{\max\{1, |2^{-j/2}\xi_2|^\beta\}} \\
&\leq C'' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \frac{|2^{-j/2}\xi_2|^\alpha}{\max\{1, |2^{-j/2}\xi_2|^\beta\}} \\
&\leq C'' 2^{-3/4j} \frac{1}{\max\{1, |2^{-j}\xi_1|^\beta\}} \frac{1}{\max\{1, |2^{-j/2}\xi_2|^{\beta-\alpha}\}}.
\end{aligned}$$

The lemma is proven. $\square$

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. We start with the observation that for a fixed scale j_s and fixed shearing parameter k the number of shearlet translates $(\psi_{j_s,k,m,\iota})_m$ that have nontrivial intersection with the support of one fixed $\omega_{j_w,m_w,\upsilon}$ is bounded by a constant independent of j_w, m_w, j_s and k. Second, we notice that the support of an arbitrary $\omega_{j_w,m_w,\upsilon} \in \mathcal{W}(\phi^1)$ has at least a distance of $q2^{\tau(t-j_w)}$ to the boundary. Further, the support of a shearlet $\psi_{j_s,k,m_s,\iota}$ with $(j_s,k,m_s,\iota) \in \Lambda_0^c$ is not fully contained in Ω and has at most length $q2^{j_s/2}$. Hence, for all $(j_s,k,m_s,\iota) \in \Lambda_0^c$ and $j_w > 1/(2\tau)j_s+t$, each $\omega_{j_w,m_w,\upsilon} \in \mathcal{W}(\phi^1)\backslash\mathcal{W}_{t,\tau}(\phi^1)$ satisfies $\operatorname{supp}\omega_{j_w,m_w,\upsilon} \cap \operatorname{supp}\psi_{(j_s,k,m_s,\iota)} = \emptyset$.

We now assume w.l.o.g. that $\upsilon = 1$. For $\upsilon = 2,3$, the following computations can be made in a similar manner with ξ_1 and ξ_2 interchanged. Also note that, by the same argument as above, for $\upsilon = 0$ we have $\langle \omega_{J_0,m',0}, \psi_{j,k,m,\iota}\rangle = 0$.

Note that the total number of wavelet translates for a fixed level j_w is of order 2^{2j_w}. Thus, using the previous observations and Parseval's identity, we obtain

$$\sum_{(j',m')\in\Delta_{\tau,t}^c} \sum_{(j,k,m,\iota)\in\Lambda_0^c} |\langle \omega_{j',m',1}, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2$$
$$\lesssim \sum_{j_w=0}^{\infty} \sum_{j_s=0}^{(2\tau)(j_w-t)} \sum_{|k|\le 2^{j_s/2}} 2^{2j_w} \max_{m_s,m_w} |\langle \widehat{\omega_{j_w,m,1}}, \widehat{\psi_{j_s,k,m_s,\iota}}\rangle_{L^2(\Omega)}|^2. \quad (8)$$

Exploiting next the frequency decay of the corresponding shearlet atoms and applying Lemma 3.4, using (W2) of the admissible wavelet system as well as applying the substitution $\xi \mapsto 2^{j_s}\xi$ yields

$$\begin{aligned}
&\sum_{j_s=0}^{(2\tau)(j_w-t)} \sum_{|k|\le 2^{j_s/2}} 2^{2j_w} \max_{m_w,m_s} |\langle \widehat{\omega_{j_w,m,1}}, \widehat{\psi_{j_s,k,m_s,\iota}}\rangle_{L^2(\Omega)}|^2 \\
&\quad\lesssim \sum_{j_s=0}^{(2\tau)(j_w-t)} 2^{-3/2j_s} \Bigg(\int_{\mathbb{R}^2} \frac{\min\{1,|2^{-j_w}\xi_1|^\alpha\}}{\max\{1,|2^{-j_w}\xi_1|^\beta\}\max\{1,|2^{-j_w}\xi_2|^\beta\}} \\
&\qquad \cdot \frac{1}{\max\{1,|2^{-j_s}\xi_1|^{\beta'}\}\max\{1,|2^{-j_s}\xi_2|^{\beta'}\}}\, d\xi\Bigg)^2 \\
&\quad\lesssim \sum_{j_s=0}^{(2\tau)(j_w-t)} 2^{5/2j_s} \Bigg(\int_{\mathbb{R}^2} \frac{\min\{1,|2^{j_s-j_w}\xi_1|^\alpha\}}{\max\{1,|2^{j_s-j_w}\xi_1|^\beta\}\max\{1,|2^{j_s-j_w}\xi_2|^\beta\}} \\
&\qquad \cdot \frac{1}{\max\{1,|\xi_1|^{\beta'}\}\max\{1,|\xi_2|^{\beta'}\}}\, d\xi\Bigg)^2. \\
&\quad\lesssim \sum_{j_s=0}^{\infty} 2^{5/2j_s} \Bigg(\int_{\mathbb{R}^2} \frac{\min\{1,|2^{j_s-j_w}\xi_1|^\alpha\}}{\max\{1,|\xi_1|^{\beta'}\}\max\{1,|\xi_2|^{\beta'}\}}\, d\xi\Bigg)^2 \\
&\quad\lesssim \sum_{j_s=0}^{\infty} 2^{5/2j_s+2\alpha(j_s-j_w)} \Bigg(\int_{\mathbb{R}^2} \frac{|\xi_1|^\alpha}{\max\{1,|\xi_1|^{\beta'}\}\max\{1,|\xi_2|^{\beta'}\}}\, d\xi\Bigg)^2.
\end{aligned}$$

Since $\beta' - \alpha > 1$ we obtain that the integral above is finite and hence we obtain

$$\sum_{j_s=0}^{(2\tau)(j_w-t)} \sum_{|k|\le 2^{j_s/2}} 2^{2j_w} \max_{m_w,m_s} |\langle \widehat{\omega_{j_w,m,1}}, \widehat{\psi_{j_s,k,m_s,\iota}}\rangle_{L^2(\Omega)}|^2 \lesssim \sum_{j_s=0}^{(2\tau)(j_w-t)} 2^{5/2j_s+2\alpha(j_s-j_w)}. \quad (9)$$

We rewrite the last sum above as

$$\sum_{j_s=0}^{(2\tau)(j_w-t)} 2^{5/2j_s+2\alpha(j_s-j_w)} = 2^{-2\alpha\epsilon j_w} \sum_{j_s=0}^{(2\tau)(j_w-t)} 2^{5/2j_s+2\alpha(j_s-(1-\epsilon)j_w)}. \quad (10)$$

Since $j_w > 1/(2\tau)j_s + t$, we can now estimate

$$\sum_{j_s=0}^{\infty} 2^{5/2j_s+2\alpha(j_s-(1-\epsilon)j_w)} \lesssim 2^{-2\alpha(1-\epsilon)t} \sum_{j_s}^{\infty} 2^{5/2j_s+2\alpha(j_s-(1-\epsilon)(1/(2\tau)j_s)}.$$

Since $\alpha((1-\varepsilon)/\tau - 2) > 5/2$ by assumption, the latter sum is finite. This leads to the estimate

$$\sum_{j_s=0}^{\infty} 2^{5/2j_s+2\alpha(j_s-(1-\epsilon)j_w)} \lesssim 2^{-2\alpha(1-\epsilon)t}. \tag{11}$$

Now (11) in combination with (10) and (9) implies together with (8) that

$$\sum_{(j,k,m,\iota)\in\Lambda_0^c} \sum_{(j',m')\in\Delta_{\tau,t}^c} |\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\rangle_{L^2(\Omega)}|^2 \lesssim \sum_{j_w=0}^{\infty} 2^{-2\alpha\epsilon j_w} 2^{-2\alpha(1-\epsilon)t} \lesssim 2^{-2\alpha(1-\epsilon)t}.$$

The proof is complete. □

4 Characterization of Sobolev spaces

It is well known, that wavelets can characterize Sobolev spaces [39]. In this section, we will show that similar results hold for shearlets. We will start by proving the latter result on $\mathbb{R}^2$ first.

4.1 Characterization of $H^s(\mathbb{R}^2)$

As already previously announced, we will provide a construction of (s, α, β)-admissible shearlet systems for arbitrary $s \in \mathbb{N}$, thereby showing that indeed also condition (S3) can be simultaneously fulfilled. For the convenience of the reader, let us state that we use the Bessel potential characterization

$$H^s(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) : (1 + |\cdot|^2)^{\frac{s}{2}} \hat{f} \in L^2(\mathbb{R}^2)\}, \quad s \in \mathbb{N},$$

of the classical *Sobolev spaces.*

For the construction, we first require two lemmata. We start with a lemma showing one side of the required inclusion.

Lemma 4.1. *Let $s \in \mathbb{N}$, and $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c)$ be $(0, \alpha, \beta)$-admissible with $\beta > 3$ and $\alpha > \beta + s$. Then*

$$\left\| \left(2^{js}\langle f, \psi_{j,k,m,\iota}\rangle\right)_{(j,k,m,\iota)\in\Lambda} \right\|_{\ell^2(\Lambda)} \lesssim \|f\|_{H^s(\mathbb{R}^2)}, \quad \textit{for all } f \in H^s(\mathbb{R}^2).$$

Proof. Since ψ obeys (S2) and has $\alpha > \beta + s$ vanishing moments, there exists some function $\theta \in L^2(\mathbb{R}^2)$ such that

$$\psi = \frac{\partial^s}{\partial x_1}\theta$$

with θ obeying (S2) for all $\alpha > \beta > 3$. Then Theorem 2.7 implies that $\mathcal{SH}(\phi, \theta, \tilde{\theta}, c)$ is a Bessel sequence for $L^2(\mathbb{R}^2)$.

Now let $f \in H^s(\mathbb{R}^2)$, and let $\Lambda^{(i)} := \{(j, k, m, \iota) \in \Lambda : \iota = i\}$ for $i = -1, 0, 1$. Then, invoking the Plancherel identity, we have

$$\|(2^{js}\langle f, \psi_{j,k,m,\iota}\rangle)_{(j,k,m,\iota)\in\Lambda}\|_{\ell^2(\Lambda)}^2$$

$$
\begin{aligned}
&= \|(\langle \hat{f}, 2^{js}\widehat{\psi}_{j,k,m,\iota}\rangle)_{(j,k,m,\iota)\in\Lambda}\|^2_{\ell^2(\Lambda)} \\
&= \|(\langle \hat{f}, 2^{js}\widehat{\psi}_{j,k,m,-1}\rangle)_{(j,k,m,\iota)\in\Lambda^{(-1)}}\|^2_{\ell^2(\Lambda^{(-1)})} + \|(\langle \hat{f}, \widehat{\psi}_{j,k,m,0}\rangle)_{(j,k,m,\iota)\in\Lambda^{(0)}}\|^2_{\ell^2(\Lambda^{(0)})} + \\
&\quad + \|(\langle \hat{f}, 2^{js}\widehat{\psi}_{j,k,m,1}\rangle)_{(j,k,m,\iota)\in\Lambda^{(1)}}\|^2_{\ell^2(\Lambda^{(1)})}.
\end{aligned}
$$

The relations $2^{js}\widehat{\psi}_{j,k,m,1}(\xi) = \xi_1^s\hat{\theta}_{j,k,m}(\xi)$ and $2^{js}\widehat{\psi}_{j,k,m,-1}(\xi) = \xi_2^s\hat{\tilde{\theta}}_{j,k,m}(\xi)$ now lead to

$$
\begin{aligned}
&\|(2^{js}\langle f, \psi_{j,k,m,\iota}\rangle)_{(j,k,m,\iota)\in\Lambda}\|^2_{\ell^2(\Lambda)} \\
&\quad = \|(\langle \xi_1^s\hat{f}, \hat{\tilde{\theta}}_{j,k,m}\rangle)_{(j,k,m,-1)\in\Lambda}\|^2_{\ell^2(\Lambda^{(-1)})} + \|(\langle \hat{f}, \widehat{\psi}_{j,k,m,0}\rangle)_{(j,k,m,0)\in\Lambda^{(0)}}\|^2_{\ell^2(\Lambda^{(0)})} + \\
&\qquad + \|(\langle \xi_2^s\hat{f}, \hat{\theta}_{j,k,m}(\xi)\rangle)_{(j,k,m,1)\in\Lambda^{(1)}}\|^2_{\ell^2(\Lambda^{(1)})}.
\end{aligned}
$$

Further, $f \in H^s(\mathbb{R}^2)$ implies that $\xi_i^s\hat{f} \in L^2(\mathbb{R}^2)$ for $i = 1, 2$. Noting that for $\iota = 0$ the functions $\psi_{j,k,m,0}$ are not affected by weights and thus can directly be bounded by $B\|f\|_{L^2(\mathbb{R}^2)} \leq B\|f\|_{H^s(\mathbb{R}^2)}$ due to the frame property. Using the simple fact that subsets of Bessel sequences are again Bessel sequences, we can conclude that

$$
\|(2^{js}\langle f, \psi_{j,k,m,\iota}\rangle))_{(j,k,m,\iota)\in\Lambda}\|^2_{\ell^2(\Lambda)} \leq B\|f\|^2_{H^s(\mathbb{R}^2)}
$$

for some positive constant B. $\square$

Lemma 4.1 yields the upper bound in (S3). Proving the lower bound in (S3) requires a bit more work. Therefore we start with the following lemma, which provides us with a characterization of $H^s(\mathbb{R}^2)$ via the dual of a shearlet frame.

Lemma 4.2. *Let $\phi, \psi, \tilde{\psi} \in L^2(\mathbb{R}^2)$ such that for $s \in \mathbb{N}$ and all $0 \leq |\alpha| \leq s$ the shearlet system $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c)$ obeys*

$$
\left\| \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota}(D^\alpha\psi)_{j,k,m,\iota} \right\|_2^2 \lesssim \|c\|^2_{\ell^2} \quad \text{for all } c \in \ell^2(\Lambda). \tag{12}
$$

Then, for all $f \in H^s(\mathbb{R}^2)$,

$$
\sum_{(j,k,m,\iota)\in\Lambda} 2^{2js}\left|\left\langle f, \psi^d_{j,k,m,\iota}\right\rangle\right|^2 \gtrsim \|f\|^2_{H^s(\mathbb{R}^2)}.
$$

Proof. Denoting by $(\psi^d_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$ the dual elements of $(\psi_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$, we have

$$
\begin{aligned}
\|f\|_{H^s(\mathbb{R}^2)} \leq &\left\| \sum_{(j,k,m,-1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,-1}\right\rangle \psi_{j,k,m,-1} \right\|_{H^s(\mathbb{R}^2)} + \left\| \sum_{(j,k,m,0)\in\Lambda} \left\langle f, \psi^d_{j,k,m,0}\right\rangle \psi_{j,k,m,0} \right\|_{H^s(\mathbb{R}^2)} \\
&+ \left\| \sum_{(j,k,m,1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,1}\right\rangle \psi_{j,k,m,1} \right\|_{H^s(\mathbb{R}^2)}.
\end{aligned} \tag{13}
$$

We only estimate the term $\|\sum_{(j,k,m,1)\in\Lambda}\langle f, \psi^d_{j,k,m,1}\rangle\psi_{j,k,m,1}\|$, since the other terms can be estimated similarly. By definition of the Sobolev norm,

$$
\left\| \sum_{(j,k,m,1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,1}\right\rangle \psi_{j,k,m,1} \right\|_{H^s(\mathbb{R}^2)} \lesssim \sum_{|\alpha|\leq s} \left\| \sum_{(j,k,m,1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,1}\right\rangle \xi^\alpha\widehat{\psi}_{j,k,m,1} \right\|_{L^2(\mathbb{R}^2)}.
$$

Hence, for any fixed $|\alpha| \leq s$, it follows that

$$\left\| \sum_{(j,k,m,1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,1} \right\rangle \psi_{j,k,m,1} \right\|_{H^s(\mathbb{R}^2)} \lesssim \left\| \sum_{(j,k,m,1)\in\Lambda} \left\langle f, \psi^d_{j,k,m,1} \right\rangle 2^{j|\alpha|} \widehat{(D^\alpha \psi)}_{j,k,m,1} \right\|_{L^2(\mathbb{R}^2)}.$$

Inserting this estimate as well as similar ones for the other terms into (13) and using (12), proves the claim. □

Remark 4.1. *We assumed with* (12) *that* $\| \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota}(D^\alpha\psi)_{j,k,m,\iota}\|_2^2 \lesssim \|c\|_{\ell^2}^2$ *for a shearlet system* $(\psi_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$. *It is certainly not obvious that there exist many shearlet systems satisfying this condition. But we will argue in the sequel that this estimate will indeed hold under very weak conditions on the shearlet system. One instance of this is are shearlet systems, for which* $(D^\alpha\psi)_{(j,k,m,\iota)\in\Lambda}$ *satisfies*

$$|\langle (D^\alpha\psi)_\mu, (D^\alpha\psi)_\eta\rangle| \leq w(\mu,\eta)^{-N} \tag{14}$$

for some appropriate distance function w *on* Λ *and some* $N \in \mathbb{N}$. *Then we have*

$$\begin{aligned}\left\| \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota}(D^\alpha\psi)_{j,k,m,\iota} \right\|_2^2 &= \int \sum_{\mu\in\Lambda}\sum_{\eta\in\Lambda} c_\mu \overline{c_\eta} (D^\alpha\psi)_\mu(\xi)\overline{(D^\alpha\psi)_\eta(\xi)} d\xi \\ &= \sum_{\mu\in\Lambda}\sum_{\eta\in\Lambda} c_\mu\overline{c_\eta} \langle (D^\alpha\psi)_\mu, (D^\alpha\psi)_\eta\rangle \\ &\lesssim \sum_{\mu\in\Lambda}\sum_{\eta\in\Lambda} |c_\mu||c_\eta| w(\mu,\eta)^{-N} = \langle |c|, T|c|\rangle, \end{aligned} \tag{15}$$

where $(Tc)_\mu := \sum_\lambda w(\mu,\lambda)^{-N} c_\lambda$. *Since* $\langle |c|, T|c|\rangle\rangle \leq \|c\|\|T(|c|)\|$, *boundedness of* T *would imply*

$$\langle |c|, T|c|\rangle \lesssim \|c\|_2^2.$$

Combining this estimate with (15) *proves the desired estimate.*

It remains to discuss condition (14) *as well as boundedness of* T. *In fact, [27, Lem. 4.4] implies that, provided* $(\psi_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$ *forms a* system of parabolic molecules *of sufficiently high order, there exists a distance function* w *such that* (14) *holds and such that* $T = (w(\mu,\lambda)^{-N})_{\mu,\lambda}$ *is a bounded operator from* $\ell^2 \to \ell^2$ *for all* $N > 1$. *The notion of a system of parabolic molecules, introduced in [27], can be regarded as a general framework including various anisotropic systems such as curvelets and shearlets. Since this framework is very involved and technical and since also our main goal in this subsection is just to show the existence of shearlet systems characterizing* $H^s(\mathbb{R}^2)$, *we decided not to include the precise, but lengthy definition.*

If the shearlet frame has the additional property that

$$\left\| \left(\langle \psi_{j,k,m,\iota}, f\rangle_{j,k,m,\iota\in\Lambda} \right) \right\|_{\ell^{w,p}} \sim \left\| \left(\left\langle \psi^d_{j,k,m,\iota}, f \right\rangle \right)_{j,k,m,\iota\in\Lambda} \right\|_{\ell^{w,p}}, \tag{16}$$

for the weight $w(j,k,m,\iota) = 2^{js}$, then we are able to connect the two Lemmata 4.1 and 4.2 to obtain a characterization of $H^s(\mathbb{R}^2)$ by the primal frame.

In particular, if $(\psi_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda}$ is *intrinsically localized* - a property studied extensively in [26] - the results of [26, Sec. 3.2], in particular, [26, Prop. 3.5] applied to [26, Eq. 49], describe when (16) holds.

With Lemmata 4.1 and 4.2 we now obtain the following theorem on existence and construction of shearlet systems satisfying (S3).

Theorem 4.3. *Let $s \in \mathbb{N}$, and $\mathcal{SH}(\phi, \psi, \tilde{\psi}, c)$ be $(0, \alpha, \beta)$-admissible with $\beta > 3$ and $\alpha > \beta + s$, which obeys* (12) *and* (16). *Then*

$$\|f\|^2_{H^s(\mathbb{R}^2)} \sim \sum_{(j,k,m,\iota)\in\Lambda} 2^{2js} \left|\langle f, \psi_{j,k,m,\iota}\rangle\right|^2 \sim \sum_{(j,k,m,\iota)\in\Lambda} 2^{2js} \left|\left\langle f, \psi^d_{j,k,m,\iota}\right\rangle\right|^2. \tag{17}$$

Proof. Lemmata 4.1 and 4.2 imply the upper bound for $\sum_{(j,k,m,\iota)\in\Lambda} 2^{2js} \left|\langle f, \psi_{j,k,m,\iota}\rangle\right|^2$ and the lower bound for $\sum_{(j,k,m,\iota)\in\Lambda} 2^{2js} \left|\left\langle f, \psi^d_{j,k,m,\iota}\right\rangle\right|^2$. The result then follows from (16). □

4.2 Characterization of $H^s(\Omega)$

We now use the results on $\mathbb{R}^2$ to obtain a characterization of $H^s(\Omega)$ by the analysis coefficients with respect to a shearlet system on Ω. At the end of this section, we briefly remark on a characterization of $H^s(\Omega)$ by dual frame coefficients, with a bit more elaborate discussion in Subsection 7.3.

The following theorem states the main result of this subsection. Notice that it uses the existence of an (s, α', β')-admissible shearlet system for $s \in \mathbb{N}$ as a hypothesis, which is now guaranteed by Theorem 4.3.

Theorem 4.4. *Let $\alpha, \beta, \alpha', \beta', \tau, \epsilon$ obey the assumptions of Proposition 3.2. Let further $s \in \mathbb{N}\backslash\{0\}$, $\mathcal{SH}(\phi, \psi, \widetilde{\psi}, c)$ be an (s, α', β')-admissible shearlet system, and $\mathcal{W}(\phi^1)$ be an (s, α, β)-admissible boundary wavelet system. Then there exists some $T > 0$ such that, for any $t \geq T$, the boundary shearlet system $\mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$ obeys (S3).*

Proof. We start by proving the upper bound in (S3). First, by the admissibility assumption on the shearlet system, there exist constants $0 < A_{\mathrm{s}} \leq B_{\mathrm{s}} < \infty$ such that

$$A_{\mathrm{s}} \|f\|^2_{H^s(\Omega)} \leq \sum_{(j,k,m,\iota)\in\Lambda} 2^{2js} |\langle f, \psi_{j,k,m,\iota}\rangle|^2 \leq B_{\mathrm{s}} \|f\|^2_{H^s(\Omega)}.$$

Moreover, the wavelet system $\mathcal{W}(\phi^1)$ obeys

$$C_{\mathrm{s}} \|f\|^2_{H^s(\Omega)} \leq \sum_{(j,m,\upsilon)\in\Delta} 2^{2js} |\langle f, \omega_{j,m,\upsilon}\rangle|^2 \leq D_{\mathrm{s}} \|f\|^2_{H^s(\Omega)}$$

for some $0 < C_{\mathrm{s}} \leq D_{\mathrm{s}} < \infty$. Thus setting $(\varphi_n)_{n\in\mathbb{N}} := \mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} 2^{2js} |\langle f, \varphi_n\rangle|^2 \\
&\quad = \sum_{(j,k,m,\iota)\in\Lambda_0} 2^{2js} |\langle f, \psi_{j,k,m,\iota}\rangle|^2 + \sum_{(j,m,\upsilon)\in\Delta_{t,\tau}} 2^{2js} |\langle f, \omega_{j,m,\upsilon}\rangle|^2 \\
&\quad \leq (B_{\mathrm{s}} + D_{\mathrm{s}}) \|f\|^2_{H^s(\Omega)},
\end{aligned}$$

which shows the existence of an upper bound.

Now we turn to lower bound in (S3). To this end, we set

$$\Xi_1 \oplus \Xi_2 := \operatorname{span}(\mathcal{W}(\phi^1) \setminus \mathcal{W}_{t,\tau}(\phi^1)) \oplus \operatorname{span}(\mathcal{W}_{t,\tau}(\phi^1)) = L^2(\Omega),$$

and for $f \in L^2(\Omega)$ we define $f_1 := P_{\Xi_1} f$ and $f_2 := P_{\Xi_2} f$, where P_{Ξ_i} denotes the orthogonal projection onto the space Ξ_i, $i = 1, 2$. Due to the characterization of Sobolev spaces by the wavelet systems we have that $\|f\|_{H^s(\Omega)} \sim \|f_1\|_{H^s(\Omega)} + \|f_2\|_{H^s(\Omega)}$ for all $f \in H^s(\Omega)$.

One main step will be to extend results from $H^s(\mathbb{R}^2)$ to $H^s(\Omega)$. For this, we recall that since Ω is a Lipschitz domain, there exists a bounded linear extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^2)$ such that $E(f)_{|\Omega} = f$ for all $f \in H^s(\Omega)$ and $\|E\|_{H^s(\Omega)\to H^s(\mathbb{R}^2)} \le M_{ext}$ for some $M_{ext} > 0$ (cf. [42]).

Based on this, we define the following modified extension operator

$$\tilde{E} : H^s(\Omega) \to H^s(\mathbb{R}^2), \qquad f \mapsto \begin{cases} E(f_2) + f_1 & \text{on } \Omega, \\ E(f_2) & \text{on } \mathbb{R}^2 \setminus \Omega. \end{cases}$$

We first notice that $\tilde{E}(f)_{|\Omega} = f$. To obtain well definedness of $\tilde{E}$ we next prove the boundedness of $\tilde{E}$. By interpreting the interior wavelets of $\mathcal{W}(\phi^1) \setminus \mathcal{W}_{t,\tau}(\phi^1)$ as part of a wavelet orthonormal basis on $L^2(\mathbb{R}^2)$ and employing the characterization of $H^s(\mathbb{R}^2)$ (see [5, Cor. 29.2]), we obtain that indeed $\|\tilde{E}(f_1)\|_{H^s(\mathbb{R}^2)} \lesssim \|f_1\|_{H^s(\Omega)}$ for all $f \in H^s(\Omega)$. The boundedness of the operator now follows from the fact that

$$\begin{aligned} \|\tilde{E}(f)\|_{H^s(\mathbb{R}^2)} &\lesssim \|E(f_2)\|_{H^s(\mathbb{R}^2)} + \|f_1\|_{H^s(\Omega)} \\ &\lesssim \|f_2\|_{H^s(\Omega)} + \|f_1\|_{H^s(\Omega)} \lesssim \|f\|_{H^s(\Omega)}, \end{aligned}$$

where we have used the boundedness of E in the second inequality.

Using the operator $\tilde{E}(f)$, we obtain that

$$\begin{aligned} A_{\mathrm{s}}\|f\|^2_{H^s(\Omega)} &\le A_{\mathrm{s}}\|\tilde{E}(f)\|^2_{H^s(\mathbb{R}^2)} \\ &\le \sum_{(j,k,m,\iota)\in\Lambda} 2^{2js}|\langle \tilde{E}(f), \psi_{j,k,m,\iota}\rangle|^2 \\ &= \sum_{(j,k,m,\iota)\in\Lambda_0} 2^{2js}|\langle f, \psi_{j,k,m,\iota}\rangle|^2 + \sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js}|\langle \tilde{E}(f), \psi_{j,k,m,\iota}\rangle|^2 \\ &\le \sum_{(j,k,m,\iota)\in\Lambda_0} 2^{2js}|\langle f, \psi_{j,k,m,\iota}\rangle|^2 + \\ &\quad +2\left(\sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js}|\langle \tilde{E}(f) - \tilde{E}(f_2), \psi_{j,k,m,\iota}\rangle|^2 + \sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js}|\langle \tilde{E}(f_2), \psi_{j,k,m,\iota}\rangle|^2 \right) \\ &= \sum_{(j,k,m,\iota)\in\Lambda_0} 2^{2js}|\langle f, \psi_{j,k,m,\iota}\rangle|^2 + 2\,(\mathrm{I} + \mathrm{II})\,. \end{aligned} \tag{18}$$

We next estimate I and II, starting with II. By using hypothesis (S3), we immediately obtain the required estimate

$$\mathrm{II} \le B_{\mathrm{s}}\|\tilde{E}(f_2)\|^2_{H^s(\mathbb{R}^2)} \le M_{\mathrm{ext}}B_{\mathrm{s}}\|f_2\|^2_{H^s(\Omega)} \le \frac{M_{\mathrm{ext}}B_{\mathrm{s}}}{C_{\mathrm{s}}} \sum_{(j,m,\upsilon)\in\Delta_{\tau,t}} 2^{2js}|\langle f, \omega_{j,m,\upsilon}\rangle|^2.$$

The existence of a positive lower bound follows by subtracting 2I on both sides of the inequality (18), provided that we can show

$$\mathrm{I} \le A_s/2\|f\|^2_{H^s(\Omega)}. \tag{19}$$

The lower frame bound is then given by

$$(A_s - 2\mathrm{I})C_s/(2M_{\mathrm{ext}}B_s). \tag{20}$$

Since by construction, $\tilde{E}(f) - \tilde{E}(f_2) = \tilde{E}(f - f_2) = \tilde{E}(f_1)$, we can compute

$$
\begin{aligned}
\mathrm{I} &= \sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js} \left|\left\langle \tilde{E}(f) - \tilde{E}(f_2), \psi_{j,k,m,\iota}\right\rangle\right|^2 \\
&= \sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js} \left|\left\langle \tilde{E}(f_1), \psi_{j,k,m,\iota}\right\rangle\right|^2 \\
&= \sum_{(j,k,m,\iota)\in\Lambda_0^c} 2^{2js} \left| \sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} \left\langle f_1, \omega_{j',m',\upsilon}\right\rangle \left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle \right|^2 \\
&= \sum_{(j,k,m,\iota)\in\Lambda_0^c} \left| \sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} 2^{j's} \left\langle f_1, \omega_{j',m',\upsilon}\right\rangle 2^{(j-j')s} \left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle \right|^2 .
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality then yields

$$
\begin{aligned}
\mathrm{I} \leq& \sum_{(j,k,m,\iota)\in\Lambda_0^c} \left(\sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} 2^{2sj'} \left|\left\langle \tilde{E}(f_1), \omega_{j',m',\upsilon}\right\rangle\right|^2 \right) \cdot \\
&\cdot \left(\sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} 2^{2(j-j')s} \left|\left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle\right|^2 \right) \\
\leq& D_{\mathrm{s}} \|\tilde{E}(f_1)\|^2_{H^s(\mathbb{R}^2)} \sum_{(j,k,m,\iota)\in\Lambda_0^c} \sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} 2^{2(j-j')s} \left|\left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle\right|^2 . \qquad (21)
\end{aligned}
$$

The first term in (21) can be estimated using

$$
\|\tilde{E}(f_1)\|_{H^s(\mathbb{R}^2)} = \|f_1\|_{H^s(\Omega)} \leq \frac{D_s}{C_s} \|f\|_{H^s(\Omega)}. \qquad (22)
$$

Then notice that, by the construction of the boundary shearlet system, for $(j', m', \upsilon) \in \Delta_{t,\tau}^c$ and $(j, k, m, \iota) \in \Lambda_0^c$ the inner products $\left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle$ equal 0 for all $j' < j$. Thus we can assume $2^{2(j-j')s} \leq 1$. By Proposition 3.2, there exists a sufficiently large offset t such that

$$
\sum_{(j,k,m,\iota)\in\Lambda_0^c} \sum_{(j',m',\upsilon)\in\Delta_{t,\tau}^c} 2^{2(j-j')s} \left|\left\langle \omega_{j',m',\upsilon}, \psi_{j,k,m,\iota}\right\rangle\right|^2 < \frac{A_{\mathrm{s}} C_{\mathrm{s}}}{2D_{\mathrm{s}}^2}. \qquad (23)
$$

Applying (22) and (23) to (21) proves (19), thereby completing the proof. □

In addition to the characterization by analysis coefficients in Theorem 4.4 one might also be interested in the synthesis coefficients. However, this then turns into a statement that concerns the dual frame, which is not available in our setting. A concrete construction of the dual is not even available for the standard shearlet systems from Subsection 2.2. The only known first construction can be found in [34]. However the resulting system has a different structure than a standard shearlet system and is, in particular, highly redundant. Hence it is not clear how to even obtain characterizations of Sobolev spaces with the primal frame of the system of [34]. Therefore presenting such a characterization with dual frame coefficients is beyond the scope of this paper. We do, however, include a numerical analysis on this property in Subsection 7.3.

5 Approximation properties

Finally, we discuss approximation properties of the boundary shearlet systems. In Subsection 2.2.4 it was discussed that shearlet systems on $\mathbb{R}^2$ yield optimally sparse approximations of cartoon-like functions. To obtain a similar result for the newly introduced boundary shearlet systems, we first need to specify what we actually mean by cartoon-like functions on bounded domains. Since in Section 6, we require this definition for general domains, we immediately define it for the general situation.

The attentive reader will have noticed that in this case the definition of cartoon-like functions, i.e., functions in $\mathcal{E}^2(\nu)$, already focuses on functions with compact support in $[0,1]^2$. For our purposes and according to anticipated applications in imaging science and numerical analysis of partial differential equations, this definition is too restrictive. In fact, it does not include discontinuity curves, which not only touch the boundary of the domain, but actually intersect it, in particular, producing a point singularity on the boundary. This situation shall now be included.

The following definition makes these thoughts precise – now for general Ω –, and generalizes the previous notion of cartoon-like function from Definition 2.9 even for the special case $\Omega = [0,1]^2$, see also Figure 1 for an illustration.

Definition 5.1. *Let $\nu > 0$, $\Omega \subset \mathbb{R}^2$, $D \subset \mathbb{R}^2$, and $f = f_1 + \chi_D f_2$ with $f_i \in C^2(\mathbb{R}^2)$ and* supp $f_i \subset [-c_{supp}, c_{supp}]^2$ *for some $c_{supp} > 0$ and $i = 1,2$ such that $f(2c_{supp} \cdot -(1/2,1/2)) \in \mathcal{E}^2(2c_{supp} \cdot \nu)$. Further, let $|\partial D \cap \partial\Omega| \leq M$ for some $M \in \mathbb{N}$. Then we call $P_\Omega f$ a* cartoon-like function on Ω, *and denote the set of* cartoon-like functions on Ω *by $\mathcal{E}^2(\nu,\Omega)$.*

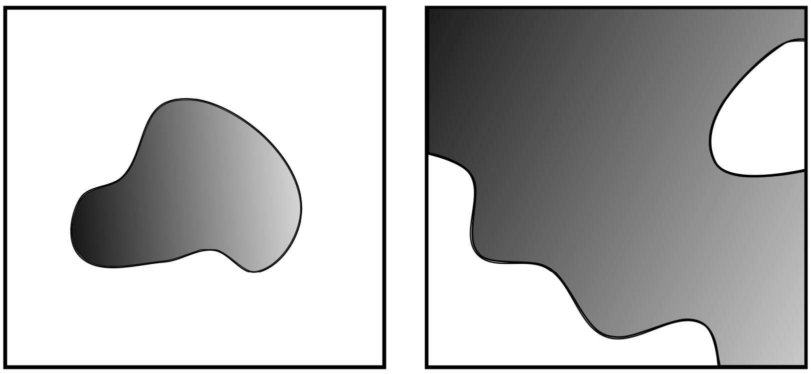

Figure 1: Cartoon-like functions in $\mathcal{E}^2(\nu,[0,1]^2)$ for some $\nu > 0$.

As announced, let us first restrict to $\Omega = [0,1]^2$. To estimate the error of the best N-term approximation, we use the following well known approach. Let $\mathcal{BSH}_{t,\tau}(\phi^1;\phi,\psi,\widetilde{\psi},c) =: (\varphi_n)_n$ be a boundary shearlet system which forms a frame for $L^2(\Omega)$, let $f \in \mathcal{E}^2(\nu,\Omega)$, and let $(\theta_n(f))_{n\in\mathbb{N}}$ be the non-increasing rearrangement of $(|\langle \varphi_n, f\rangle|^2)_{n\in\mathbb{N}}$. Then, by the frame inequality, we have

$$\|f - f_N\|_2^2 \lesssim \sum_{n\geq N} \theta_n(f) \quad \text{for all } N \in \mathbb{N}.$$

Hence to obtain the optimal best N-term approximation rate of Theorem 2.10, we require the estimate

$$\sum_{n\geq N} \theta_n(f) \lesssim N^{-2}\log(N)^3 \text{ as } N \to \infty.$$

The reader will certainly have noticed that it is not initially clear that this is also the optimally possible rate for the extended class of cartoon-like functions on Ω from Definition 5.1. But it is easy to see that a best $N-$term approximation rate *faster* than N^{-2} violates the optimality result of [20] for functions in $\mathcal{E}^2(\nu)$, since each system on $[0,1]^2$ can be extended by 0 to yield a system on $\mathbb{R}^2$. Recall that functions in $\mathcal{E}^2(\nu)$ vanish outside $[0,1]^2$, hence the extended system would imply a faster than optimal approximation rate for functions in $\mathcal{E}^2(\nu)$, a contradiction. Thus, the rate cannot be faster, but it is not clear whether the rate of N^{-2} (up to a log-factor) can actually be achieved. Theorem 5.2 shows that this is indeed the case.

To proceed we need to first assemble some results for the approximation rates of wavelets and shearlets. The optimal approximation rate of shearlets is guaranteed by Theorem 2.10. For $s \in \mathbb{N}$, an (s,α,β)-admissible boundary wavelet system $(\omega_{j,m,\upsilon})_{j,m,\upsilon}$ admits

$$\|f\|_{H^s(\Omega)}^2 \sim \sum_{j,m} 2^{2js} |\langle f, \omega_{j,m,\upsilon}\rangle|^2.$$

The result [5, Thm. 39.2] then implies that, for any $f \in H^2(\Omega)$ and $(\theta^\omega(f)_n)_{n\in\mathbb{N}}$ denoting the non-increasing rearrangement of $|\langle f, \omega_{j,m,\upsilon}\rangle|^2$,

$$\sum_{n\geq N} \theta_n^\omega(f) \lesssim N^{-2}. \tag{24}$$

Based on these approximation results, we obtain the following theorem for the approximation rate of cartoon-like functions on Ω by the hybrid system of boundary shearlets.

Theorem 5.2. *Let $\Omega = [0,1]^2$. Further, let $\phi, \psi, \widetilde{\psi}$ fulfill the assumptions of Theorem 2.10, and let $\mathcal{W}(\phi^1)$ be an $(s,0,0)$-admissible boundary wavelet system. Further let $t > 0$, $\tau > 1/3$, and let $\mathcal{BSH}_{t,\tau}(\phi^1;\phi,\psi,\widetilde{\psi},c) =: (\varphi_n)_{n\in\mathbb{N}}$ be a boundary shearlet system, which forms a frame for $L^2(\Omega)$. Then $\mathcal{BSH}_{t,\tau}(\phi^1;\phi,\psi,\widetilde{\psi},c)$ yields almost optimally sparse approximation for cartoon-like functions on Ω, i.e., for all $f \in \mathcal{E}^2(\nu,\Omega)$,*

$$\|f - f_N\|_{L^2(\Omega)}^2 \lesssim N^{-2}\log(N)^3 \quad \textit{for } N \to \infty,$$

where $f_N = \sum_{n\in I_n} \langle f, \varphi_n\rangle \varphi_n^d$ with I_n containing the N largest coefficients $\langle f, \varphi_n\rangle$ in modulus and $(\varphi_n^d)_{n\in\mathbb{N}}$ is the canonical dual frame of $(\varphi_n)_{n\in\mathbb{N}}$.

Proof. Let $f = P_\Omega(f_1 + \chi_D f_2) \in \mathcal{E}^2(\nu,\Omega)$ and consider

$$\|f - f_N^*\|_2^2 \leq \sum_{n\geq N} \theta_n(f) \leq \sum_{n\geq \frac{2N}{3}} \theta_n^\omega(f) + \sum_{n\geq\frac{N}{3}} \theta_n^\psi(f) =: T_1 + T_2, \tag{25}$$

where $\theta_n^\omega(f)$ is the non-increasing rearrangement of $|\langle f, \varphi_n\rangle|^2_{\varphi_n\in\mathcal{W}_{t,\tau}(\phi^1)}$ and $\theta_n^\psi(f)$ the non-increasing rearrangement of $|\langle f,\varphi_n\rangle|^2_{\varphi_n\in\mathcal{S}_0}$, where $\mathcal{S}_0 = \{\psi_{j,k,m,\iota}, (j,k,m,\iota) \in \Lambda_0\}$.

We now estimate T_1 and T_2. By Theorem 2.10,

$$T_2 \lesssim N^{-2}\log(N)^3. \tag{26}$$

The sum T_1 corresponding to the wavelet part can be split into two parts again. First, we denote by $\theta_n^\omega(f)^{(s)}$ the part of $(\theta_n^\omega)_{n\in\mathbb{N}}$ such that $|\langle\varphi_m, f\rangle|^2 = \theta_n^\omega$ and $\operatorname{supp}\varphi_m \cap \partial D \neq \emptyset$. These are the wavelet elements corresponding to the smooth part of the function f. Second, we label the remaining elements by $\theta_n^\omega(f)^{(ns)}$ for $n \in \mathbb{N}$. Using similar arguments about the set of largest coefficients as before, we obtain that

$$T_1 \leq \sum_{n\geq\frac{N}{3}} \theta_n^\omega(f)^{(s)} + \sum_{n\geq\frac{N}{3}} \theta_n^\omega(f)^{(ns)}. \tag{27}$$

By (24),

$$\sum_{n\geq \frac{N}{3}} \theta_n^{\omega}(f)^{(s)} \lesssim N^{-2} \quad \text{as } N \to \infty. \tag{28}$$

The wavelet coefficients corresponding to the non-smooth part of f can be estimated by observing that, since the boundary curve of D intersects $\partial\Omega$ only finitely often, due to the construction of the wavelet system $\mathcal{W}_{t,\tau}(\phi^1)$ for a small enough resolution we obtain that only $\sim 2^{(1-\tau)j} < 2^{(2/3-\epsilon)j}$ wavelets intersect the boundary of D where $0 < \varepsilon < \tau - 1/3$.

Furthermore, due to the boundedness of f, we have $| \langle \omega_{j,m,\upsilon}, f \rangle|^2 \lesssim 2^{-2j}$. Hence, we obtain

$$\sum_n (\theta_n^{\omega}(f)^{(ns)})^{\frac{1}{3}} \lesssim \sum_{j\in\mathbb{N}} 2^{(2/3-\epsilon)j}(2^{-2j})^{\frac{1}{3}} < \infty.$$

Consequently, $(\theta_n^{\omega}(f)^{(ns)})_{n\in\mathbb{N}} \in \ell^{\frac{1}{3}}$ which, by the Stechkin Lemma, yields

$$\sum_{n\geq N} (\theta_n^{\omega}(f)^{(ns)}) \lesssim N^{-2} \text{ for } N \to \infty. \tag{29}$$

Applying (26)–(29) to (25) proves the claim. □

6 Arbitrary domains

The preceding sections were devoted to the introduction and analysis of a directional anisotropic frame, coined boundary shearlet system, on the domain $\Omega = [0,1]^2$. However, in many applications, in particular, for the adaptive solution of partial differential equations more complicated domains are of interest.

A common approach to construct systems for more general domains is to lift the construction from $[0,1]^2$ to 2-manifolds by using smooth charts $\kappa_i : [0,1]^2 \to \Omega_i$, $i \leq M \in \mathbb{N}$. In this way domains of the form

$$\Omega = \bigcup_{i\leq M} \Omega_i, \text{ where } \Omega_i = \kappa_i([0,1]^2) \tag{30}$$

can be handled. The constructions fall into two categories, namely overlapping and non-overlapping constructions. Non-overlapping constructions, i.e., where the sets Ω_i are assumed to be disjoint, were introduced in [16] and further developed in [14] with an analysis of the characterization of smoothness spaces presented in [15] and [30]. In these cases it is possible to construct biorthogonal bases on $L^2(\Omega)$. If one is though not necessarily interested in a basis, but a frame on $L^2(\Omega)$ is sufficient, constructions with presumably overlapping smooth images $\Omega_i = \kappa_i([0,1]^d)$ can be used. This approach has been introduced in [43] and further developed in [9].

Returning to boundary shearlet systems on arbitrary domains, we aim to use a construction similar to those wavelet constructions. Is is evident that overlapping constructions are sufficient in our case, since we can not hope to obtain a basis. In the sequel we follow the construction of [9] adapted to the shearlet system case, and discuss some of its properties. Since many arguments are quite straightforward, we settle for describing the main construction steps and argumentation steps.

We start by assuming we are given C^k-diffeomorphisms $\kappa_i : [0,1]^2 \to \Omega_i$, $k \in \mathbb{N}$, $i \leq M \in \mathbb{N}$, as in (30), where

$$0 < c_1 \leq |\det (D\kappa_i)| \leq c_2 < \infty, \quad \text{for all } i \leq M.$$

To obtain a frame on Ω for $L^2(\Omega)$, we require in addition a smooth partition of unity $(\sigma_i)_{i\leq M}$ on Ω subordinate to the patches $(\Omega_i)_{i\leq M}$ such that, for all $i \in M$, (P1) supp $\sigma_i \subset \Omega_i$ and

(P2) $\sum_{i\leq M} \|\sigma_i f\|_{H^s(\Omega_i)} \sim \|f\|_{H^s(\Omega)}$ for all $f \in H^s(\Omega)$.

Certainly, these requirements restrict the set of possible domains Ω. However, it has been noted in [9], that the class of polyhedral domains is still covered.

Given a boundary shearlet system $(\varphi_n^\square)_{n\in\mathbb{N}} = \mathcal{BSH}_{t,\tau}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$, say, on the unit square $[0,1]^2$ as constructed in Definition 3.1, its elements can be lifted to Ω by

$$\varphi_{i,n}(x) := \begin{cases} \sigma_i(x) \frac{\varphi_n^\square(\kappa_i^{-1}(x))}{|\det (D\kappa_i)(\kappa_i^{-1}(x))|^{1/2}} & , \text{ for } x \in \Omega_i, \\ 0 & , \text{ else,} \end{cases}$$

i.e., $(\varphi_{i,n})_{i\leq M, n\in N}$ is the constructed *boundary shearlet system on* Ω.

Using the fact that each κ_i is a diffeomorphism, for all $0 \leq r \leq s$ and $f \in H^r(\Omega_i)$, we obtain

$$\|f\|^2_{H^r(\Omega_i)} \sim \|\sigma_i(\kappa_i(x)) f(\kappa_i(x))\|^2_{H^r([0,1]^2)}.$$

By Theorems 3.3 and 4.4, we can find conditions such that, for all $0 \leq r \leq s$,

$$\|\sigma_i(\kappa_i(x)) f(\kappa_i(x))\|^2_{H^r([0,1]^2)} \sim \sum_{n\in\mathbb{N}} 2^{2rj_n} |\langle \sigma_i(\kappa_i(x)) f(\kappa_i(x)), \varphi_n^\square(x)\rangle|^2 \sim \sum_{n\in\mathbb{N}} 2^{2rj_n} |\langle f, \varphi_{i,n}\rangle_{L^2(\Omega)}|^2.$$

Consequently, the conditions on the partition of unity imply

$$\|f\|^2_{H^r(\Omega)} \sim \sum_{n\in\mathbb{N}, i\leq M} 2^{2tj_n} |\langle f, \varphi_{i,n}\rangle_{L^2(\Omega)}|^2.$$

This yields the frame property and the characterization of Sobolev spaces by analysis coefficients for the boundary shearlet systems on Ω.

The second – maybe even first – important feature that the boundary shearlet systems should have is optimally sparse approximation of cartoon-like functions on Ω. Notice that the general definition for cartoon-like functions on Ω was already stated in Definition 5.1.

For this we require an additional property of the partition of unity $(\sigma_i)_{i\leq M}$, which states that each maps σ_i should vanish on a neighborhood of the boundary of Ω_i, but not necessarily near the boundary of the entire domain Ω. More precisely, we require the following condition:

(P3) There exists $\delta > 0$ such that $\sigma_i = 0$ on $(\partial\Omega_i + B_\delta(0)) \setminus (\partial\Omega + B_\delta(0))$ for all $i \leq M$.

Given a cartoon-like function f on Ω, by definition the singularity curve of f touches the boundary curve of Ω at most finitely many times. By assuming condition (P3), this implies that the singularity curve of $\sigma_i f$ touches the boundary curve of Ω_i only finitely many times. Since in addition the charts κ_i are smooth, we can conclude that $\sigma_i f \circ \kappa_i$ is a cartoon-like function on $[0,1]^2$ as in Definition 5.1. Hence we can use the techniques of the proof of Theorem 5.2 to obtain the desired decay of the coefficients $(\langle f, \varphi_{i,n}\rangle)_n$ for increasing resolution. Finally, since there are only finitely many Ω_i's, with the same arguments as in Theorem 5.2 this yields the approximation rate of $O(N^{-2}\log(N)^3)$ for any $f \in \mathcal{E}^2(\nu, \Omega)$.

7 Numerical experiments

We now numerically analyze some of the properties of boundary shearlet systems. Since estimates for frame bounds as derived in Theorem 3.3 are typically far from being tight, in Subsection 7.1 we numerically compute the frame bounds. In Subsections 7.2 and 7.3 we then analyze the localization properties of the Gramian and the Gelfand frame property, which are features of boundary shearlet systems whose theoretical analysis was far beyond the scope of this paper.

For all numerical experiments, we choose a digitized version $\mathbf{\Omega}$ of the domain $\Omega = [0,1]^2$ as an $n \times n$ pixel image. We will specify the number n at the relevant points later. Our implementation of boundary shearlet systems then uses the MATLAB toolboxes `WaveLab` from `http://statweb.stanford.edu/~wavelab/` and `ShearLab` from `http://www.shearlab.org` for the implementation of the analysis and synthesis operator of boundary shearlet systems. In `WaveLab` and `ShearLab`, the wavelet and shearlet elements are not normalized. Since this is crucial for the setting of bounded domains, we normalize all these functions. For later use, let $\mathbf{T}_{\mathbf{\Phi_w}}$ and $\mathbf{T}_{\mathbf{\Phi_s}}$ denote the implementation of the analysis operators of the wavelet and shearlet systems after normalization.

The definition of boundary shearlet systems requires a hybrid system consisting of a subset of the wavelet system and a subset of the shearlet system. Concerning the wavelet elements, we only choose those which are close to the boundary. Depending on the offset of the boundary shearlet system, we construct a mask, $\mathbf{M_w}$, for the wavelet system that restricts the analysis operator to a subset of the full wavelet system. In the sequel we will always choose $\tau = 1/3$ and only vary the offset t. Similarly, we need to subsample the shearlet system provided by `ShearLab`. In fact, `ShearLab` provides a non-subsampled shearlet transform, i.e., it computes the shearlet coefficients using the full system

$$\left\{ \psi_{j,k,(S_k A_j m),\iota} \,:\, j \leq J,\ \iota \in \{-1,0,1\},\ |k| \leq |\iota| 2^{\lfloor j/2 \rfloor},\ m \in c\mathbb{Z}^2, \right\}.$$

On the other hand the theory requires us to restrict to the shearlet system

$$\left\{ \psi_{j,k,m,\iota} \,:\, j \leq J,\ \iota \in \{-1,0,1\},\ |k| \leq |\iota| 2^{\lfloor j/2 \rfloor},\ m \in c\mathbb{Z}^2, \right\}.$$

Furthermore, we exclude shearlets from our system that intersect the boundary of Ω. We incorporate all of these requirements in a mask $\mathbf{M_s}$.

The analysis operator of the combined system is now given by

$$\mathbf{T}_{\mathbf{\Phi}} := \begin{pmatrix} \mathbf{M_w} \mathbf{T}_{\mathbf{\Phi_w}} \\ \mathbf{M_s} \mathbf{T}_{\mathbf{\Phi_s}} \end{pmatrix}. \tag{31}$$

Using these operators, we derive an implementation of the synthesis operator of boundary shearlet systems by using

$$\mathbf{T}^*_{\mathbf{\Phi}} = \mathbf{M_w} \mathbf{T}^*_{\mathbf{\Phi_w}} + \mathbf{M_s} \mathbf{T}^*_{\mathbf{\Phi_s}}.$$

The implementation of the frame operator is given by

$$\mathbf{S} := \mathbf{T}^*_{\mathbf{\Phi_w}} \mathbf{M_w} \mathbf{T}_{\mathbf{\Phi_w}} + \mathbf{T}^*_{\mathbf{\Phi_s}} \mathbf{M_s} \mathbf{T}_{\mathbf{\Phi_s}}.$$

To apply inverse frame operator $\mathbf{S^{-1}}$, we use MATLAB's build-in *conjugate gradients method*, `pcg`.

7.1 Frame properties

We now compute the frame bounds of a boundary shearlet system for various offsets of the wavelet part. We pick an 256×256 pixel domain as a digitization of Ω. The wavelet and shearlet systems are computed using 3 scales. Since the optimal frame bounds A and B are the extremal points of the spectrum of the frame operator of the system, we numerically compute them for this boundary shearlet system by computing the smallest and largest eigenvalues of $\mathbf{S}$.

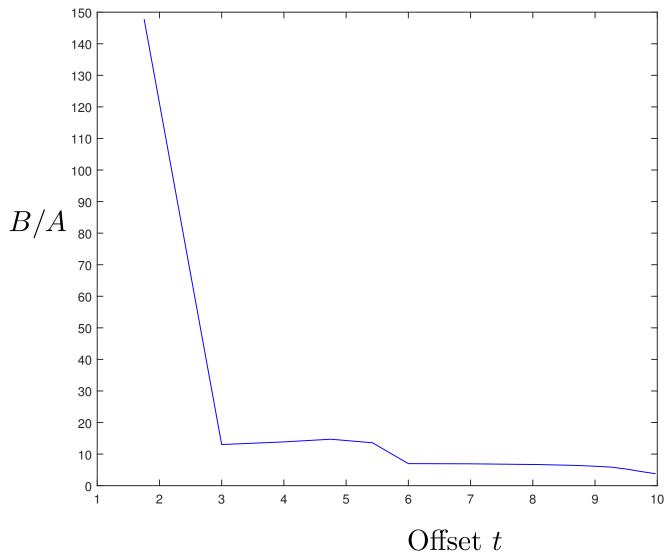

Figure 2: Quotient of the frame bounds for varying offset. One observes that for high offset the quotient becomes stable and explodes for decreasing offset.

For this task we used MATLAB's build-in method `eigs`. In Figure 2, we depict the quotient B/A for varying offset.

We observe that for larger offset the ratio of the frame bounds is somehow not too far from 1, which provides us with reasonably good condition numbers for the computation of $\mathbf{S}^{-1}$. In fact, the values of these quotients are comparable with those of the full shearlet system used in `ShearLab`, [35]. As expected, the frame property breaks down, when the offset becomes too small. This is in accordance with Theorem 3.3.

7.2 Localization of the Gramian

Using the analysis operator as defined in (31), the Gramian of the boundary shearlet system is given by

$$\mathbf{G} := \mathbf{T}_{\mathbf{\Phi}} \mathbf{T}_{\mathbf{\Phi}}^*.$$

The linear operators $\mathbf{T}_{\mathbf{\Phi}}$ and $\mathbf{T}_{\mathbf{\Phi}}^*$ are implemented using the `Spot` Toolbox, which is available at `http://www.cs.ubc.ca/labs/scl/spot/index.html`. The matrix representation of the Gramian is shown in Figure 3. It is clearly visible, that the Gramian of the boundary shearlet system has diagonal structure.

The figures were produced for a 256×256 digitization of Ω, 4 scales in the boundary shearlet system with number of directions being [1 1 2 2].

7.3 Gelfand property

As we already mentioned in the introduction, the Gelfand frame property is of particular interest for the solvability of elliptic PDE's. Thus, we now aim to check whether, for a fixed $s > 0$, the boundary shearlet system $(\varphi_n)_{n\in\mathbb{N}} := \mathcal{BSH}_{t,1/3}(\phi^1; \phi, \psi, \widetilde{\psi}, c)$ yields a Gelfand frame for the Gelfand triple $(H^s(\Omega), L^2(\Omega), H^{-s}(\Omega))$. Recall that this means that for $(\varphi_n^d)_{n\in\mathbb{N}}$ denoting the

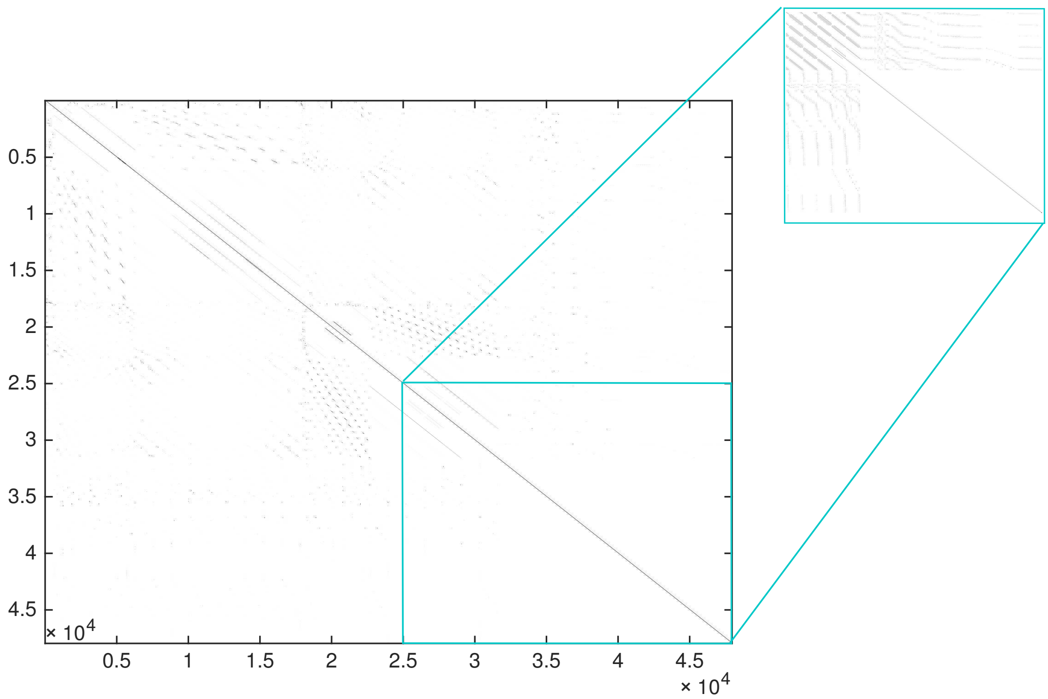

Figure 3: Gramian of the boundary shearlet system. The part zoomed region is shown with changed contrast for better visualization of the different sparsity patterns of shear-shear, shear-wave and wave-wave.

canonical dual of $(\varphi_n)_{n\in\mathbb{N}}$, we require

$$\left\| \sum_{n\in\mathbb{N}} c_n \varphi_n \right\|_{H^s(\Omega)}^2 \lesssim \left\| (2^{j_n s} c_n)_{n\in\mathbb{N}} \right\|_{\ell^2}^2, \qquad \text{for all } c \in \ell^2 \tag{GFA1}$$

and

$$\left\| (2^{j_n s} \langle f, \varphi_n^d \rangle)_{n\in\mathbb{N}} \right\|_{\ell^2}^2 \lesssim \|f\|_{H^s(\Omega)}^2, \qquad \text{for all } f \in H^s(\Omega). \tag{GFA2}$$

Intuitively, a Gelfand frame seeks to describe the mapping properties of the synthesis operator.

In fact, the property (GFA1) can be easily proved for a boundary shearlet system $(\varphi_n)_{n\in\mathbb{N}}$ satisfying (17). Equation (17) implies that

$$\begin{aligned} &\left\| \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota} \psi_{j,k,m,\iota} \right\|_{H^s(\Omega)} \\ \lesssim &\left\| \left(\langle \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota} \psi_{j,k,m,\iota}, \psi^d_{j',k',m',\iota'} \rangle \right)_{(j',k',m',\iota')\in\Lambda} \right\|_{\ell^{2,w}(\Lambda)} . \end{aligned}$$

Defining $(Gc)_{j',k',m',\iota'} := \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota} \langle \psi_{j,k,m,\iota}, \psi^d_{j',k',m',\iota'} \rangle$ we get

$$\|Gc\|_{\ell^{2,w}(\Lambda)} \leq \|G\|_{\ell^{2,w}(\Lambda)\to\ell^{2,w}(\Lambda)} \|c\|_{\ell^{2,w}(\Lambda)},$$

and $\|G\|_{\ell^{2,w}(\Lambda)\to\ell^{2,w}(\Lambda)}$ is bounded if G has sufficient localization by [26, Sec. 3.2]. This yields

$$\left\| \sum_{(j,k,m,\iota)\in\Lambda} c_{j,k,m,\iota}\psi_{j,k,m,\iota} \right\|^2_{H^s(\Omega)} \lesssim \left\| (2^{js}c_{j,k,m,\iota})_{(j,k,m,\iota)\in\Lambda} \right\|^2_{\ell^2(\Lambda)},$$

which is (GFA1). As already discussed at the end of Subsection 4.2, a theoretical analysis of property (GFA2) is to date out of reach due to the non-availability of a concrete construction of a dual shearlet system. Therefore we now numerically analyze and in fact show that the constructed boundary shearlet systems also satisfy property (GFA2).

We first require a numerically computable discretization of property (GFA2). For this, notice that employing the characterization of $H^s(\Omega)$ by a wavelet orthonormal basis (see Subsection 2.1.3) and an appropriate weight in the sense that

$$\|T_{\Phi_w}c\|^2_{H^s(\Omega)} \sim \|c\|^2_{\ell^{2,w}},$$

we can obtain the following property which is equivalent to (GFA2):

$$\|\langle T_{\Phi_w}c, \varphi_n^d\rangle\|^2_{\ell^{2,w}} \lesssim \|c\|^2_{\ell^{2,w}}, \qquad \text{for all } c \in \ell^{2,w}. \tag{32}$$

This property, however, involves the dual frame, whose analysis is – as just mentioned – intractable. To derive a discrete analogue of (32), we first let

$$W : \ell^{2,w} \to \ell^2, (x_k)_k \mapsto (w_k \cdot x_k)_k$$

be the canonicial isometry. Furthermore, since $\varphi_n^d = S^{-1}\varphi_n$, it follows that

$$\langle T_{\Phi_w}c, \varphi_n^d\rangle = \langle S^{-1}T_{\Phi_w}c, \varphi_n\rangle.$$

Using the canonical discretization of W as a diagonal matrix, we obtain two matrices $\mathbf{W}$ and $\mathbf{W_w}$ adapted to the indexing of the boundary shearlet system and the full wavelet system, respectively. The discrete analogue of (32) now takes the form

$$\|\mathbf{W}\mathbf{T}_{\boldsymbol{\Phi}}(\mathbf{S^{-1}}\mathbf{T}_{\Phi_w}c)\|^2 \lesssim \|\mathbf{W_w}c\|^2, \qquad \text{for all } c \in \mathbb{R}^{n^2}.$$

In order to examine this bound and check its validity for our boundary shearlet system in the discrete setting, we estimate

$$\max_{\|c\|=1} \|\mathbf{W}\mathbf{T}_{\boldsymbol{\Phi}}\mathbf{S^{-1}}\mathbf{T}_{\Phi_w}\mathbf{W_w}^{-1}c\|_2^2$$

by computing the square-root of the largest eigenvalue of

$$\mathbf{W_w}^{-1}\mathbf{T}_{\Phi_w}\mathbf{S^{-1}}\mathbf{T}^*_{\boldsymbol{\Phi}}\mathbf{W}^2\mathbf{T}_{\boldsymbol{\Phi}}\mathbf{S^{-1}}\mathbf{T}_{\Phi_w}\mathbf{W_w}^{-1}. \tag{33}$$

In Figure 4, we depict the square-root of the largest eigenvalue of the operator (33) with different weights $\mathbf{W}, \mathbf{W_w}$ and different offset for $n = 512$. The precise values can be found in Table 1. In this numerical experiment, the weights are chosen as 2^{js}, where j describes the scale of the frame element, both of wavelet and shearlet, and s is a parameter that takes values between 0 and 1.5. The shearlet and wavelet systems were constructed with 4 scales.

In Figure 4 as well as Table 1, one can observe that, although the largest eigenvalues of (33) increase with growing Sobolev parameter, they do so remarkably slow if the offset is sufficiently high. Thus we conclude that our experiments demonstrate the proper mapping properties of the dual frame.

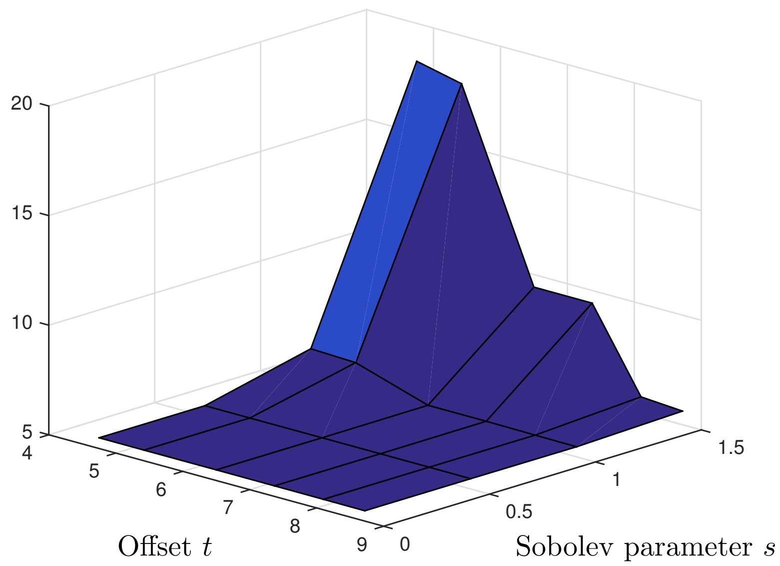

Figure 4: Largest singular value of $\mathbf{W}\mathbf{T}_{\mathbf{\Phi}}\mathbf{S}^{-1}\mathbf{T}_{\Phi_w}\mathbf{W_w}^{-1}$ with varying weights $\mathbf{W}, \mathbf{W_w}$ and varying offset.

Offset	s = 0	s = 0.5	s= 1	s= 1.5
8.72	5.47	5.47	5.46	5.62
8.10	5.48	5.48	5.47	5.76
7.37	5.48	5.48	5.50	9.42
6.50	5.49	5.48	5.50	9.43
5.42	5.49	5.48	6.55	17.81
4.75	5.49	5.49	6.62	18.27

Table 1: Largest eigenvalues of (33) for varying offset and Sobolev parameter s.

Acknowledgements

P. Petersen would like to thank Kristof Schröder and Massimo Fornasier for various discussions on related topics. Furthermore, P. Petersen expresses his gratitude to Massimo Fornasier and the Technische Universität München for the hospitality during P. Petersen's research visit. Parts of this work was also done when J. Ma visited the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge, and he is grateful for its hospitality. J. Ma and P. Petersen acknowledge support from the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics"; they are also supported by the Berlin Mathematical School. G. Kutyniok acknowledges support by the Einstein Foundation Berlin, by the Einstein Center for Mathematics Berlin (ECMath), by Deutsche Forschungsgemeinschaft (DFG) SPP 1798, by the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics", and by the DFG Research Center Matheon "Mathematics for key technologies" in Berlin. Parts of this work was accomplished while G. Kutyniok was visiting ETH Zürich. She is grateful to the Institute for Mathematical Research (FIM) and the Seminar for Applied Mathematics (SAM) for their hospitality and support during this visit.

References

[1] E. J. Candès. *Ridgelets: theory and applications*. PhD thesis, Stanford University, 1998.

[2] E. J. Candès and L. Demanet. The curvelet representation of wave propagators is optimally sparse. *Comm. Pure Appl. Math.*, 58(11):1472–1528, 2005.

[3] E. J. Candès and D. L. Donoho. New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities. *Comm. Pure Appl. Math.*, 57(2):219–266, 2004.

[4] O. Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003.

[5] A. Cohen. Wavelet methods in numerical analysis. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 417–711. North-Holland, Amsterdam, 2000.

[6] A. Cohen, W. Dahmen, and R. DeVore. Multiscale decompositions on bounded domains. *Trans. Amer. Math. Soc.*, 352(8):3651–3685, 2000.

[7] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations: convergence rates. *Math. Comp.*, 70(233):27–75, 2001.

[8] A. Cohen, I. Daubechies, and P. Vial. Wavelets on the interval and fast wavelet transforms. *Appl. Comput. Harmon. Anal.*, 1(1):54–81, 1993.

[9] S. Dahlke, M. Fornasier, and T. Raasch. Adaptive frame methods for elliptic operator equations. *Adv. Comput. Math.*, 27(1):27–63, 2007.

[10] S. Dahlke, T. Raasch, M. Werner, M. Fornasier, and R. Stevenson. Adaptive frame methods for elliptic operator equations: the steepest descent approach. *IMA J. Numer. Anal.*, 27(4):717–740, 2007.

[11] S. Dahlke, G. Steidl, and G. Teschke. Shearlet coorbit spaces: compactly supported analyzing shearlets, traces and embeddings. *J. Fourier Anal. Appl.*, 17(6):1232–1255, 2011.

[12] W. Dahmen, C. Huang, G. Kutyniok, W.-Q Lim, C. Schwab, and G. Welper. Efficient resolution of anisotropic structures. In *Extraction of Quantifiable Information from Complex Systems*, pages 25–51. Springer, 2014.

[13] W. Dahmen, C. Huang, C. Schwab, and G. Welper. Adaptive petrov–galerkin methods for first order transport equations. *SIAM J. Numer. Anal.*, 50(5):2420–2445, 2012.

[14] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal spline wavelets on the interval-stability and moment conditions. *Appl. Comput. Harmon. Anal.*, 6(2):132 – 196, 1999.

[15] W. Dahmen and R. Schneider. Wavelets with complementary boundary conditions—function spaces on the cube. *Results Math.*, 34(3-4):255–293, 1998.

[16] W. Dahmen and R. Schneider. Wavelets on manifolds I: Construction and domain decomposition. *SIAM J. Math. Anal.*, 31(1):184–230, 1999.

[17] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41(7):909–996, 1988.

[18] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[19] L. Demanet and L. Ying. Wave atoms and time upscaling of wave equations. *Numer. Math.*, 113(1):1–71, 2009.

[20] D. Donoho. Sparse components of images and optimal atomic decompositions. *Constr. Approx.*, 17(3):353–382, 2001.

[21] D. L. Donoho and G. Kutyniok. Microlocal analysis of the geometric separation problem. *Comm. Pure Appl. Math.*, 66:1–47, 2013.

[22] G. Easley, D. Labate, and P. Negi. 3D data denoising using combined sparse dictionaries. *Math. Model. Nat. Phenom.*, 8(1):60–74, 2013.

[23] M. Elad, J.-L. Starck, P. Querre, and D. L. Donoho. Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA). *Appl. Comput. Harmon. Anal.*, 19:340–358, 2005.

[24] S. Etter, P. Grohs, and A. Obermeier. FFRT - a fast finite ridgelet transform for radiative transport. *Multiscale Model. Simul.*, 13(1), 2014.

[25] M. Genzel and G. Kutyniok. Asymptotic analysis of inpainting via universal shearlet systems. *SIAM J. Imaging Sci.*, 7:2301–2339, 2014.

[26] P. Grohs. Intrinsic localization of anisotropic frames. *Appl. Comput. Harmon. Anal.*, 35(2):264–283, 2013.

[27] P. Grohs and G. Kutyniok. Parabolic molecules. *Found. Comput. Math.*, 14(2):299–337, 2014.

[28] P. Grohs and A. Obermeier. Optimal adaptive ridgelet schemes for linear transport equations. *Appl. Comput. Harmon. Anal.* to appear.

[29] K. Guo, G. Kutyniok, and D. Labate. Sparse multidimensional representations using anisotropic dilation and shear operators. In *Wavelets and splines: Athens 2005*, Mod. Methods Math., pages 189–201. Nashboro Press, Brentwood, TN, 2006.

[30] H. Harbrecht and R. Stevenson. Wavelets with patchwise cancellation properties. *Math. Comp.*, 75(256):1871–1889, 2006.

[31] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Construction of compactly supported shearlet frames. *Constr. Approx.*, 35(1):21–72, 2012.

[32] G. Kutyniok and W.-Q Lim. Compactly supported shearlets are optimally sparse. *J. Approx. Theory*, 163(11):1564–1589, 2011.

[33] G. Kutyniok and W.-Q Lim. Shearlets on bounded domains. In *Approximation theory XIII: San Antonio 2010*, volume 13 of *Springer Proc. Math.*, pages 187–206. Springer, New York, 2012.

[34] G. Kutyniok and W.-Q Lim. Dualizable shearlet frames and sparse approximation. 2015. preprint, arXiv:1411.2303.

[35] G. Kutyniok, W.-Q Lim, and R. Reisenhofer. ShearLab 3D: Faithful digital shearlet transforms based on compactly supported shearlets. *ACM Trans. Math. Software.* to appear.

[36] D. Labate, L. Mantovani, and P. Negi. Shearlet smoothness spaces. *J. Fourier Anal. Appl.*, 19(3):577–611, 2013.

[37] D. Labate, W.-Q Lim, G. Kutyniok, and G. Weiss. Sparse multidimensional representation using shearlets. Wavelets XI (San Diego, CA, 2005), 254-262, SPIE Proc. 5914, SPIE, Bellingham, WA, 2005.

[38] S. Mallat. A theory for multiresolution signal decomposition: The wavelet representation. *IEEE Trans. Pattern Anal. Mach. Intell.*, 11(7):674–693, 1989.

[39] S. Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way.* Academic Press, 3rd edition, 2008.

[40] Y. Meyer. Wavelets and operators. In *Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987)*, volume 137 of *London Math. Soc. Lecture Note Ser.*, pages 256–365. Cambridge Univ. Press, Cambridge, 1989.

[41] P. Petersen. Shearlet approximation of functions with discontinuous derivatives. 2015. preprint, arXiv:1508.00409.

[42] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[43] R. Stevenson. Adaptive solution of operator equations using wavelet frames. *SIAM J. Numer. Anal*, 41(3):1074–1100, 2003.

Recent Research Reports

Nr.	Authors/Title
2015-20	R. Hiptmair and A. Moiola and I. Perugia A Survey of Trefftz Methods for the Helmholtz Equation
2015-21	P. Chen and Ch. Schwab Sparse-Grid, Reduced-Basis Bayesian Inversion: Nonaffine-Parametric Nonlinear Equations
2015-22	F. Kuo and R. Scheichl and Ch. Schwab and I. Sloan and E. Ullmann Multilevel Quasi-Monte Carlo Methods for Lognormal Diffusion Problems
2015-23	C. Jerez-Hanckes and Ch. Schwab Electromagnetic Wave Scattering by Random Surfaces: Uncertainty Quantification via Sparse Tensor Boundary Elements
2015-24	V. Kazeev and Ch. Schwab Quantized tensor-structured finite elements for second-order elliptic PDEs in two dimensions
2015-25	S. Etter Parallel ALS Algorithm for the Hierarchical Tucker Representation
2015-26	C. Hafner and R. Hiptmair and P. Souzangar Data Sparse Numerical Models for SNOM Tips
2015-27	F. Leonardi and S. Mishra and Ch. Schwab Numerical approximation of statistical solutions of incompressible flow
2015-28	P. Chen and Ch. Schwab Model Order Reduction Methods in Computational Uncertainty Quantification
2015-29	G. S. Alberti and S. Dahlke and F. De Mari and E. De Vito and S. Vigogna Continuous and discrete frames generated by the evolution flow of the Schrödinger equation