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ELECTROMAGNETIC WAVE SCATTERING BY RANDOM SURFACES: UNCERTAINTY QUANTIFICATION VIA SPARSE TENSOR BOUNDARY ELEMENTS

CARLOS JEREZ-HANCKES AND CHRISTOPH SCHWAB

Abstract. For time-harmonic scattering of electromagnetic waves from obstacles with uncertain geometry, we perform a domain perturbation analysis. Assuming as known both the scatterers' nominal geometry and its small-amplitude random perturbations statistics, we derive a tensorized boundary integral equation which describes, to leading order, the second order statistics, i.e. the two-point correlation of the randomly scattered electromagnetic fields. Perfectly conducting as well as homogeneous dielectric scatterers with random boundary and interface, respectively, are considered. Deterministic tensor equations for second-order statistics of both, Cauchy data on the nominal domain of the scatterer as well as of the far-field pattern are derived, generalizing the work by Harbrecht, Schneider and Schwab (*Numer. Math.*, 109(3):385–414, 2008) to electromagnetics and to interface problems, and being an instance of the general programme outlined by Chernov and Schwab (*Math. Comp.*, 82(284):1859–1888, 2013). The tensorized boundary integral equations are formulated on the surface of the known nominal scatterer. Sparse tensor Galerkin discretizations of these BIEs are proposed and analyzed based on the stability results by Hiptmair, Jerez-Hanckes and Schwab (*BIT*, 53(4):925-939); we show that they allow, to leading order, consistent Galerkin approximations of the complete second order statistics of the random scattered electric field, with computational work equivalent to that for the Galerkin solution of the nominal problem up to logarithmic terms.

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Contents

1. Introduction

The scattering of electromagnetic waves is an important problem in numerous areas of electrical engineering; an incomplete list comprises radar imaging, nondestructive testing, remote sensing, wireless communication networks. The governing equations for these problems are, to measurement accuracy, Maxwell equations. Often, in applications and numerical simulations, one is faced with the issue of *uncertainty* in the problem data, specifically, *uncertainty in the geometry of the scatterer*. Concretely, slight changes during manufacturing processes or at operation due to aging and/or parameter variations such as humidity, temperature, to name a few, generate small perturbations in the shape of the devices studied with significant variations in their performance. Hence the need for more robust prototyping simulation schemes capable of quantifying these uncertainties. Previous efforts from the engineering community have been focused on applying techniques such as polynomial chaos, reduced order modeling or the stochastic finite-element method [38, 4, 21].

In the present paper we perform, for time-harmonic scattering of electromagnetic waves, a first-order perturbation analysis of the scattered waves subject to random amplitude perturbations of the scatterers' geometry. The lack of precise information on the domain perturbation is modeled within a stochastic framework, i.e. by assuming that the domain perturbations are random, but of small amplitude almost surely. This naturally leads to the use of *shape gradients* for the first order analysis of the scattered fields. *Shape gradients*, or "domain derivatives" have been developed in the past decade, mainly in the context of shape optimization and for shape identification in acoustic and electromagnetic scattering; we refer to [28, 32, 33, 16, 18, 29] and references therein. Our *First Order Second statistical Moments* (FOSM) stochastic perturbation analysis is hence based on shape gradients of time harmonic solutions of Maxwell's equations from these references.

As already shown for acoustic and even more general equations in [17, 23], the shape gradients are governed to leading order, i.e. up to second order perturbations, by a set of *homogeneous* Maxwell equations in the nominal scatterer. The absence of volume sources in these equations motivates their *boundary reduction* via potentials of electric and magnetic surface currents which, in turn, satisfy *Boundary Integral Equations* (BIEs). In the context of electromagnetic scattering, one customarily finds the so-called *Electric or Magnetic Field Integral Equation*–EFIE and MFIE, respectively, for short– as well as their linear *Combined Field Integral Equation* (CFIE). The discretization of these boundary integral equations is effected by the *Boundary Element Method* (BEM) –a.k.a. Method of Moments– which relies on a Galerkin approximation of surface unknowns.

In what follows, we derive tensorized boundary integral equations for deterministic, first order approximations of the second moments of the random scattered electromagnetic field, due to small amplitude perturbations in the scatterers' geometry. We consider both perfectly conducting as well as homogeneous dielectric scatterers with uncertain interface location. As mentioned before, the approximations are of first order in the perturbation amplitude. We derive and show the well-posedness of tensorized BIEs which relate the second order statistics of the random scattered electromagnetic field to the second order statistics of the shape uncertainty, assumed to be known. We then address the efficient numerical solution of these tensorized BIEs by Galerkin discretization. We show, in particular, that *sparse tensor product Galerkin discretizations* provide numerical approximations with the same accuracy versus work –up to logarithmic terms– as the Galerkin discretization of one instance of the BIE in the deterministic case. We then address the implementation of the corresponding sparse tensor Galerkin algorithms. Based on sparse grid techniques, we present an algorithmic realization of the sparse tensor Galerkin discretization which does not require hierarchic bases and is able to access any available accelerated solver for the deterministic problem, in particular matrix vector multiplications based on the *Fast Multipole Methods* (FMM) (*cf.* [14] and references therein) and multilevel and Calderón-type preconditioning methods. We show that the computational complexity of the resulting algorithms to approximate the second order statistics of random scattered electromagnetic field is log-linear with respect to the corresponding number of degrees of freedom necessary for the deterministic problem.

The outline of this paper is as follows:

In Section 2, we present the time-harmonic, electromagnetic exterior scattering problem for two situations commonly encountered, specifically: (i) a perfectly conducting scatterer; and (ii) a dielectric scatterer. We present the mathematical formulation of each boundary value problem and briefly recapitulate existence and uniqueness results for them.

Section 3 reviews some of the elements of shape calculus from [37, 20], and present shape gradients for the scattered electric fields with respect to the geometry of the perfect conductor and dielectric scatterers, taken from [25] and, in particular, from [19, Sec. 6]. The connection to random shape variations is presented in Section 4, where we briefly show the general approach to first-order, k-th statistical moment analysis of parametric, nonlinear operator equations with random inputs from [17], and fix notation for the probabilistic formulation of the boundary uncertainty. With this, we can define the randomly perturbed surfaces versions of the perfect conductor and transmission problems in Sections 4.3 and 4.4, respectively. As in the case of time-harmonic wave propagation of acoustric waves, considered e.g. in [28, 23, 24] and consistent with Hadamard's theorem, the shape gradient of the scattered electric field is found to be a solution of the homogeneous, time-harmonic Maxwell equations in the nominal domain, denoted D_0 herein, subject to Cauchy data[1] being a linear functional of the solution of the nominal scattering problem.

In Section 5, electromagnetic scattering problems without source terms –typically appearing in first order, second moment analysis– are equivalently reduced to a (system of) BIE(s) on the boundary (resp. interface) of the nominal shape of the conductor (resp. dielectric interface). The direct method of boundary reduction of Maxwell's equation from [13] implies strong ellipticity in the natural trace spaces of the Cauchy data on the scattering surfaces.

Section 6 combines all the previous elements to derive sparse tensor Galerkin boundary element discretizations for the computation of FOSM. In other words, we are interested in obtaining two-point correlations of electric and magnetic surface currents and of far-field patterns of the scattered electric field. We then propose a sparse tensor Galerkin discretization of these tensorized BIEs and prove, by extending [26], their quasi-optimal convergence.

Section 7 develops computational aspects of the derived FOSM BIEs. A key issue is to avoid explicit formation of the matrix corresponding to the tensorized operator. We offer one way to do this, based on an abstraction of the so-called *sparse-grid combination technique*, and present results on error convergence and log-linear complexity based on the use of FMM for log-linear EFIE matrix-vector computations. Final remarks and conclusions are presented in Section 8.

2. Electromagnetic Scattering

We introduce the basic problem classes to be considered throughout; at this stage, we formulate models for a generic scatterer geometry denoted by D. More precisely, let $D \subset \mathbb{R}^3$ be an open, bounded Lipschitz domain with simply connected boundary $\Gamma := \partial D$ and set $D^c := \mathbb{R}^3 \backslash \overline{D}$.

For a constant angular frequency $\omega > 0$, we consider the time-harmonic propagation of electromagnetic waves; with ε and μ denoting the dielectric permittivity and magnetic permeability, respectively, assumed to be positive constants in D^c. Denoting as usual by $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields, respectively, the Maxwell equations without sources read[2]

$$\mathbf{curl}\,\mathbf{E} - \imath\omega\mu\mathbf{H} = 0\ , \quad \mathbf{curl}\,\mathbf{H} + \imath\omega\varepsilon\mathbf{E} = 0 \quad \text{in} \quad D \cup D^c\ . \tag{2.1}$$

Setting $\kappa := \omega\sqrt{\mu\varepsilon}$, the system (2.1) can be reduced to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U} - \kappa^2\mathbf{U} = 0 \quad \text{in} \quad D \cup D^c\ . \tag{2.2}$$

Here, the unknown $\mathbf{U} = \mathbf{E}$, and the magnetic flux density is $\mathbf{H} = \frac{1}{\imath\omega\mu}\,\mathbf{curl}\,\mathbf{U}$, which is computed *a posteriori*. In D^c, we impose the Silver-Müller radiation condition:

$$\left|\mathbf{curl}\,\mathbf{U}(\mathbf{R}) \times \frac{\mathbf{R}}{R} - \imath\kappa\mathbf{U}(\mathbf{R})\right| = \mathcal{O}\left(\frac{1}{R^2}\right)\ , \quad \mathbf{R} \in \mathbb{R}^3,\ R \to \infty\ , \tag{2.3}$$

[1] For Cauchy data we refer to the tangential components of the electric fields and of magnetic fluxes on the scatterers surface which are solution of the homogeneous Maxwell scattering problems considered.

[2] In the following, we denote scalars in simple typeface, vector fields with boldface. Quantities defined over volumes are written in capital letters and surface ones in lower case.

where $R := \|\mathbf{R}\|_2$ and $\|\cdot\|_2$ denotes the standard Euclidean norm.

Rather than developing the theory in the most general setting, we will consider simultaneously two particular time-harmonic electromagnetic wave scattering problems:

I. A **perfect conductor** which occupies D and leads to an exterior Dirichlet problem in D^c for (2.2) on the scattering surface Γ; and,

II. A **transmission problem** in $D \cup D^c$ where Γ is now a dielectric interface, with different material parameters in D and in D^c.

For the mathematical formulation of problems I and II as well as for the boundary reduction of the second moment equations, we need to precise our functional spaces setting as well as introduce *Boundary Integral Operators* (BIOs) on Γ set forth in Section 5.1.3.

2.1. **Functional spaces.** Let $d = 1, 2, 3$. For a domain $K \subseteq \mathbb{R}^d$, $C^m(K)$, with $m \in \mathbb{N}$, denotes the space of m-times differentiable scalar functions on K, and similarly for the space infinitely differentiable scalar continuous functions $C^\infty(K)$. Let $L^2(K)$ denote the class of square-integrable functions over K. Throughout, slanted boldface symbols for functional spaces represent vector-valued counterparts, e.g. $\boldsymbol{L}^2(K)$ is the space of vector valued functions with d components in $L^2(K)$.

Spaces on the closed surface $\Gamma \subset \mathbb{R}^3$ shall be defined in terms of the traces of the following vector functional spaces in the bounded Lipschitz domain $D \subset \mathbb{R}^3$:

$$\begin{aligned}
\boldsymbol{H}(\mathbf{curl}, D) &:= \{\mathbf{U} \in \boldsymbol{L}^2(D) \ : \ \mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2(D)\},\\
\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c) &:= \{\mathbf{U} \in \boldsymbol{L}^2_{\mathrm{loc}}(D^c) \ : \ \mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2_{\mathrm{loc}}(D^c)\},\\
\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D) &:= \{\, \mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \mid \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2(D)\,\},\\
\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl}, D^c) &:= \{\, \mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c) \mid \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2_{\mathrm{loc}}(D)\,\}\ .
\end{aligned}$$

Observe that if $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D)$ (resp. $\mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c)$) solves the homogeneous Maxwell equations, $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D)$ (resp. $\mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl}, D^c)$). We also introduce the following Hilbert spaces over the exterior domain [31, Section 5.3]:

$$\begin{aligned}
\boldsymbol{H}_\kappa(\mathbf{curl}, D^c) &:= \Big\{\mathbf{U} : \mathbf{U} \text{ satisfies (2.3)}, \mathbf{U}/R \in \boldsymbol{L}^2(D^c),\ \mathbf{curl}\,\mathbf{U}/R \in \boldsymbol{L}^2(D^c),\\
&\qquad \frac{\mathbf{U}\cdot\mathbf{R}}{R} \in L^2(D^c),\ \frac{\mathbf{curl}\,\mathbf{U}\cdot\mathbf{R}}{R} \in L^2(D^c)\Big\},\\
\boldsymbol{H}_\kappa(\mathbf{curl}\,\mathbf{curl}, D^c) &:= \Big\{\mathbf{U} \,:\, \mathbf{curl}\,\mathbf{U} \in \boldsymbol{H}_\kappa(D^c)\Big\}\ .
\end{aligned} \tag{2.4}$$

On the closed surface Γ, we use standard Sobolev spaces, $H^s(\Gamma)$, of complex-valued scalar functions on Γ endowed with standard norms $\|\cdot\|_s$ for $s \in [-1, 1]$ and with the standard notation $H^0(\Gamma) = L^2(\Gamma)$ [30, Chap. 3]. By γ, we denote the standard trace operator mapping $\gamma : H^{s+1/2}(D) \to H^s(\Gamma)$, $u \mapsto u|_\Gamma$, $s \in (0, 1)$, continuously. Similar considerations hold component-wise for vector spaces $\boldsymbol{H}^s(\Gamma)$. We define spaces of complex valued tangential vector fields as:

$$\boldsymbol{V}^s_\pi(\Gamma) := (\mathbf{n} \times \boldsymbol{H}^s(\Gamma)) \times \mathbf{n}, \qquad s \in [0, 1], \tag{2.5}$$

endowed with the induced operator norms $\|\cdot\|_{\boldsymbol{V}^s_\pi}$ and normal vector $\mathbf{n}$ on the boundary pointing from D to D^c.

We will be mainly concerned with the space $\boldsymbol{V}^{1/2}_\pi(\Gamma)$ for which we drop the superscript, $\boldsymbol{V}_\pi(\Gamma) \equiv \boldsymbol{V}^{1/2}_\pi(\Gamma)$ – compare with the notation adopted in [10]. We denote by $\boldsymbol{V}'_\pi(\Gamma)$ its dual space with $\boldsymbol{V}^0_\pi(\Gamma)$ as pivot space and by $\langle\cdot,\cdot\rangle_{\boldsymbol{V}'_\pi,\boldsymbol{V}_\pi}$ the corresponding duality pairing. Finally, we shall make use of first-order surface differential operators defined on Γ, div_Γ and $\mathbf{curl}_\Gamma$ [31, Chap. 2.5].

Definition 2.1. For $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D})$, the Dirichlet and Neumann traces on $\Gamma = \partial D$ are defined by

$$\gamma_{\mathrm{D}}\mathbf{U} := (\mathbf{n} \times \mathbf{U})|_\Gamma \ \text{ and } \ \gamma_{\mathrm{N}}\mathbf{U} := \kappa^{-1}\,(\mathbf{n} \times \mathbf{curl}\,\mathbf{U})|_\Gamma\ ,$$

respectively. We introduce the space:

$$\boldsymbol{X}(\Gamma) := \{\, \boldsymbol{\lambda} \in \boldsymbol{V}'_\pi(\Gamma) : \mathrm{div}_\Gamma\,\boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\,\}$$

with norm $\|\boldsymbol{\lambda}\|_{\boldsymbol{X}} = \|\boldsymbol{\lambda}\|_{\boldsymbol{V}'_\pi} + \|\mathrm{div}_\Gamma\,\boldsymbol{\lambda}\|_{-1/2}$.

Theorem 2.2. *([10, 12]) The operators γ_{D} and γ_{N} are linear and continuous from $\boldsymbol{C}^\infty(\overline{D})$ to $\boldsymbol{V}_\pi^0(\Gamma)$ and they can be extended to linear and continuous operators from $\boldsymbol{H}(\mathbf{curl}, D)$ and $\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D)$, respectively, to $\boldsymbol{X}(\Gamma)$. Moreover, they admit linear and continuous right inverses.*

For $\mathbf{U} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c)$, $\mathbf{V} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl}, D^c)$, or alternatively, $\mathbf{U} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$, $\mathbf{V} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$, we define $\gamma_{\mathrm{D}}^c\mathbf{U}$ and $\gamma_{\mathrm{N}}^c\mathbf{V}$ in the same way and similar mapping properties hold. We set then

$$\boldsymbol{H}_0(\mathbf{curl}, D) \;\; := \;\; \{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \; : \; \gamma_{\mathrm{D}}\mathbf{U} = 0\} \, .$$

By continuity of γ_{D}, $\boldsymbol{H}_0(\mathbf{curl}, D)$ is a closed subspace of $\boldsymbol{H}(\mathbf{curl}, D)$. The operator $\times\mathbf{n}$: $\boldsymbol{V}_\pi^0(\Gamma) \to \boldsymbol{V}_\pi^0(\Gamma)$ associated with the mapping $\mathbf{U} \mapsto \mathbf{U} \times \mathbf{n}$ can be extended to a linear and continuous isomorphism between $\boldsymbol{X}(\Gamma)$ and its dual [13, Thm. 3.3], where $\boldsymbol{X}(\Gamma)$ is shown to be self-dual under the $\mathsf{b}(\cdot,\cdot)$ pairing defined next [8].

Set $\boldsymbol{X}^2(\Gamma) := \boldsymbol{X}(\Gamma) \times \boldsymbol{X}(\Gamma)$. A crucial role in deriving BIEs is taken by the bilinear form:

$$\mathsf{b}(\mathbf{v}, \mathbf{w}) := \int_\Gamma \mathbf{v} \cdot (\mathbf{w} \times \mathbf{n}) = -\,\mathsf{b}(\mathbf{w}, \mathbf{v}) : \boldsymbol{X}^2(\Gamma) \to \mathbb{C} \, . \tag{2.6}$$

In the coercivity proofs of BIOs, as well as in the analysis of Galerkin boundary element methods, the following Hodge decomposition of $\boldsymbol{X}(\Gamma)$ is fundamental. It has been proved in [10] in the case of a simply connected manifold Γ and in [12] for multiply connected domains.

Theorem 2.3. *Define $\boldsymbol{W}(\Gamma) := \{\boldsymbol{\lambda} \in \boldsymbol{X}(\Gamma) \; : \; \mathrm{div}_\Gamma\, \boldsymbol{\lambda} = 0\}$ and*

$$\boldsymbol{V}(\Gamma) := \{\boldsymbol{\lambda} \in \boldsymbol{X}(\Gamma) \; : \; \int_\Gamma \boldsymbol{\lambda} \cdot \mathbf{w} = 0, \quad \forall\, \mathbf{w} \in \boldsymbol{W}(\Gamma) \cap \boldsymbol{V}_\pi^0(\Gamma)\} \, .$$

There holds $\boldsymbol{X}(\Gamma) = \boldsymbol{W}(\Gamma) \oplus \boldsymbol{V}(\Gamma)$, and $\boldsymbol{W}(\Gamma)$ can be decomposed as

$$\boldsymbol{W}(\Gamma) = \boldsymbol{W}_0(\Gamma) \oplus \mathbb{H}(\Gamma) \, , \quad \boldsymbol{W}_0(\Gamma) := \mathbf{curl}_\Gamma\, H^{\frac{1}{2}}(\Gamma) \; ; \; dim\{\mathbb{H}(\Gamma)\} = 2N_e \, , \tag{2.7}$$

where the space $\mathbb{H}(\Gamma)$ is composed of the direct sum of the tangential traces of the Neumann fields associated to D and D^c; N_e is the first Betti number associated with the domain D. Moreover, if $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \boldsymbol{V}(\Gamma)$, $\mathbf{w} \in \boldsymbol{W}(\Gamma)$, we have the following norm equivalences:

$$\begin{aligned} c_1(\|\mathbf{v}\|_{\boldsymbol{X}} + \|\mathbf{w}\|_{\boldsymbol{X}}) \leq \|\mathbf{u}\|_{\boldsymbol{X}} &\leq \|\mathbf{v}\|_{\boldsymbol{X}} + \|\mathbf{w}\|_{\boldsymbol{X}} \, , \\ \|\mathrm{div}_\Gamma\, \mathbf{u}\|_{-1/2} \leq \|\mathbf{v}\|_{\boldsymbol{X}} &\leq c_2 \, \|\mathrm{div}_\Gamma\, \mathbf{u}\|_{-1/2} \, , \end{aligned} \tag{2.8}$$

where c_1, c_2 are positive constants dependent on Γ, and $\boldsymbol{V}(\Gamma) \hookrightarrow \boldsymbol{V}_\pi^0(\Gamma)$ with compact injection.

2.2. **Scattering by a perfect conductor.** Assume D to be filled by a perfect conductor and D^c a purely dielectric exterior domain with real material constants $\mu, \varepsilon > 0$, $\kappa := \omega\sqrt{\mu\varepsilon}$ again with $\omega > 0$. Let us consider an incident field $\mathbf{U}^{\mathrm{inc}} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\mathrm{inc}} - \kappa^2\mathbf{U}^{\mathrm{inc}} = 0$. If $\mathbf{U}^{\mathrm{pc}}$ denotes the scattered field by the perfect conductor, the total electric field in the exterior domain $\mathbf{U}^{\mathrm{tot}} = \mathbf{U}^{\mathrm{inc}} + \mathbf{U}^{\mathrm{pc}}$ should satisfy $\gamma_D^c\mathbf{U}^{\mathrm{tot}} = 0$, which implies

$$\gamma_{\mathrm{D}}^c\mathbf{U}^{\mathrm{pc}} = -\gamma_{\mathrm{D}}^c\mathbf{U}^{\mathrm{inc}} \; := \boldsymbol{m}^{\mathrm{pc}} \quad \text{on } \Gamma, \tag{2.9}$$

and $\boldsymbol{m}^{\mathrm{pc}} \in \boldsymbol{X}(\Gamma)$. Hence, the scattering by a perfect conductor problem can be stated as follows: for a given $\boldsymbol{m}^{\mathrm{pc}} \in \boldsymbol{X}(\Gamma)$, we seek a scattered field $\mathbf{U}^{\mathrm{pc}}$ in D^c satisfying (2.2), (2.3) and (2.9) with

$$\mathbf{U}^{\mathrm{pc}} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c) \, . \tag{2.10}$$

Remark 2.4. For lossless materials, the corresponding homogeneous interior problem in general admits non-trivial solutions (*cf.* [13, Thm. 5.2]). In other words, assume that $\mathbf{U}^{\mathrm{pc}} \in \boldsymbol{H}(\mathbf{curl}, D)$ is a Maxwell solution in D with $\gamma_{\mathrm{D}}\mathbf{U}^{\mathrm{pc}} = 0$. Then $\mathbf{U}^{\mathrm{pc}} = 0$ unless $\kappa^2 \in S_{\mathrm{Dir}}$ where the set S_{Dir} of eigenvalues of the interior Maxwell problem with homogeneous Dirichlet boundary condition is countable and accumulates only at infinity (see Remark 6.2 ahead).

Remark 2.5. In the case of lossy materials, i.e. material parameters with non-zero imaginary values, our entire program will remain valid. However, for the sake of brevity we forgo this case both for perfect conducting and dielectric scattering problems.

2.3. **Dielectric scattering problem.** Given two dielectric media which occupy D and D^c, with real material parameters $\mu_1, \varepsilon_1 > 0$ in D and with $\mu_2, \varepsilon_2 > 0$ in D^c, and with $\kappa_i := \omega\sqrt{\mu_i\varepsilon_i}$, $i = 1, 2$, for a given incident field $\mathbf{U}^{\mathrm{inc}} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D^c)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\mathrm{inc}} - \kappa_2^2\mathbf{U}^{\mathrm{inc}} = 0$, we seek again time-harmonic solutions with common circular frequency ω. Specifically, we seek fields $\mathbf{U}^i$ such that

$$\mathbf{U}^1 \in \boldsymbol{H}(\mathbf{curl}, D)\ , \qquad \mathbf{U}^2 \in \boldsymbol{H}_{\kappa_2}(\mathbf{curl}, D^c)\ , \tag{2.11}$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^1 - \kappa_1^2\mathbf{U}^1 = 0 \text{ in } D\ , \qquad \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^2 - \kappa_2^2\mathbf{U}^2 = 0 \text{ in } D^c\ . \tag{2.12}$$

Observe that $\mathbf{U}^1$ is the total field in D while $\mathbf{U}^2$ is the exterior scattered field. Hence, the total exterior field is $\mathbf{U}^{\mathrm{tot},2} = \mathbf{U}^{\mathrm{inc}} + \mathbf{U}^2$ in D^c. The problem (2.11) - (2.12) is completed by *transmission conditions* on the interface Γ. The tangential components of the electric field and magnetic flux densities should be continuous across Γ. Denoting by γ_{D}^c and γ_{N}^c the operators in Definition 2.1 with respect to D^c. Define $\widehat{\gamma_{\mathrm{N}}}\mathbf{U} := \mathbf{n} \times \mathbf{curl}\,\mathbf{U}|_\Gamma$. Then the transmission conditions on Γ read

$$\begin{aligned} \gamma_{\mathrm{D}}\mathbf{U}^1 - \gamma_{\mathrm{D}}^c\mathbf{U}^2 &= \gamma_{\mathrm{D}}^c\mathbf{U}^{\mathrm{inc}} &&=: \boldsymbol{m}^{\mathrm{de}}, &(2.13)\\ \mu_1^{-1}\widehat{\gamma_{\mathrm{N}}}\mathbf{U}^1 - \mu_2^{-1}\widehat{\gamma_{\mathrm{N}}}^c\mathbf{U}^2 &= \mu_2^{-1}\widehat{\gamma_{\mathrm{N}}}^c\mathbf{U}^{\mathrm{inc}} &&=: \mathbf{j}^{\mathrm{de}}\ . &(2.14)\end{aligned}$$

Given data $\boldsymbol{m}^{\mathrm{de}}$, $\boldsymbol{j}^{\mathrm{de}} \in \boldsymbol{X}(\Gamma)$ we wish to find $\mathbf{U}^1, \mathbf{U}^2$ satisfying (2.11)–(2.12). The dielectric problem has at most one solution (*cf.* [13, Thm. 6.1]).

Remark 2.6. Existence and uniqueness of solutions for these exterior problems is established, for example, in the monographs [15] and [31, Thm. 5.4.8]. A key argument in the proof is Rellich's lemma for uniqueness in order to apply Fredholm alternative arguments. Observe the introduction of *weighted function spaces* $\boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$, which are Hilbert and impose radiation conditions essentially [31, 1]. This avoids the use of spaces of locally integrable functions on unbounded domains, which are no longer Hilbert but Fréchet in nature. Therefore, whenever discussing existence and uniqueness of solutions we will consider both types of spaces: Fréchet-spaces of locally integrable function which satisfy radiation conditions and Hilbert spaces with suitable weights at infinity. The latter, separable weighted Hilbert spaces $\boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$ for the formulation of the exterior problems will allow in Section 4 ahead to develop the stochastic setting based on Bochner integrals, and in particular obviates the use of more general notions of integration.

3. Shape Derivatives

As we are concerned with the scattered fields in the presence of *small random shape variations* of the boundary Γ, we perform a first-order perturbation analysis of the electromagnetic scattering problems introduced in Section 2. To simplify notation, we assume in what follows that in (2.7) holds $N_e = 0$ and $\mathbb{H} = \{0\}$; in the general case, the ensuing analysis applies upon localization to each connected component of Γ.

3.1. **Shape Calculus.** As in [28, 23], our analysis is based on the first derivative of the scattered fields with respect to the geometry Γ of the scatterer. This derivative is also known as *shape gradient*. For numerous applications in time-harmonic acoustic and electromagnetic scattering, such shape gradients have been developed in the past decade, mainly in the context of shape optimization and for the inverse problem of shape identification; we refer to [28, 32, 33, 16, 18] and associated bibliography. To define the shape gradients, we consider a one-parameter family $\{D_\delta\}_\delta$ of domains defined for $|\delta| < \delta_0$ as follows:

- **(S1)** for $\delta = 0$, we consider given a *nominal domain* D_0 with C^2-boundary Γ_0, unit normal vector $\mathbf{n}_0$ pointing from D_0 into D_0^c, and a Γ_0-transversal, C^2-unit perturbation field $\boldsymbol{\psi} : \Gamma_0 \mapsto \mathbb{R}^3$, i.e. $\mathbf{n}_0 \cdot \boldsymbol{\psi} \geq \zeta > 0$ and $\|\boldsymbol{\psi}\|_2 = 1$ on Γ_0, defining the space $C^2(\Gamma_0; \mathbb{R}^3)$; and,
- **(S2)** for $0 \leq |\delta| < \delta_0$, boundary surfaces Γ_δ are given by

$$\Gamma_\delta(\boldsymbol{\psi}) := T_{\delta\boldsymbol{\psi}}(\Gamma_0) = \{\mathbf{x} = T_{\delta\boldsymbol{\psi}}(\mathbf{x}_0) := (\mathsf{I} + \delta\boldsymbol{\psi})(\mathbf{x}_0) : \mathbf{x}_0 \in \Gamma_0\}\ , \tag{3.1}$$

where the operator I denotes identity.

Proposition 3.1. *Assume that in* (3.1) *the perturbation field* $\boldsymbol{\psi} \in C^2(\Gamma_0;\mathbb{R}^3)$ *satisfies* $\mathbf{n}_0 \cdot \boldsymbol{\psi} \geq \zeta > 0$ *and* $\|\boldsymbol{\psi}\|_2 = 1$ *on* Γ_0*. Then there exists* $\delta_0 > 0$ *depending on the maximum curvature of* Γ_0*, such that* (3.1) *uniquely defines a* C^2*-surface for every* $0 \leq |\delta| < \delta_0$*. For these* δ*, the transformation* $T_{\delta\boldsymbol{\psi}} := \mathsf{I} + \delta\boldsymbol{\psi} : \mathbf{x} \mapsto \mathbf{x} + \delta\boldsymbol{\psi}(\mathbf{x})$ *is a* C^2*-diffeomorphism from* Γ_0 *to* $\Gamma_\delta(\boldsymbol{\psi})$*. If, in particular,* Γ_0 *is a* C^3*-boundary, then the assertion remains valid with* $\boldsymbol{\psi} = \mathbf{n}_0$*, the unit normal exterior vector field to* Γ_0*.*

Proof. By compactness of the bounded nominal surface Γ_0 and by the continuity of the mapping $\Gamma_0 \ni \mathbf{x}_0 \to \mathbf{n}_0(\mathbf{x}_0) \cdot \boldsymbol{\psi}(\mathbf{x}_0)$, there exists $c_0 > 0$ such that $\min\{\mathbf{n}_0(\mathbf{x}_0) \cdot \boldsymbol{\psi}(\mathbf{x}_0) : \mathbf{x}_0 \in \Gamma_0\} \geq c_0 > 0$. Since Γ_0 is a C^2-boundary, for sufficiently small $\delta_0 > 0$, the set enclosed by the surfaces Γ_0 and $\Gamma_{\delta_0}(\boldsymbol{\psi})$ is a tubular neighborhood of Γ_0, i.e. for each $\mathbf{x}$ in this set, there exists a δ such that $\delta_0 > |\delta| \geq 0$ and a unique $\mathbf{x}_0 \in \Gamma_0$ such that the mapping $\mathbf{x} = T_{\delta\boldsymbol{\psi}}(\mathbf{x}_0) = \mathbf{x}_0 + \delta\boldsymbol{\psi}(\mathbf{x}_0)$ is a bijective $C^2(\Gamma_0)$-diffeomorphism. □

Up to now, the transformations $T_{\delta\boldsymbol{\psi}}$ were introduced on the nominal surface $\Gamma_0 = \partial D_0$ enclosing the *bounded nominal domain* $D_0 \subset \mathbb{R}^3$. For sufficiently small $|\delta|$, the transformations $T_{\delta\boldsymbol{\psi}}$ admit a unique C^2-extension to D_0 and D_0^c, still denoted by $T_{\delta\boldsymbol{\psi}}$, such that $T_{\delta\boldsymbol{\psi}}(\mathbf{R}) \equiv \mathsf{I}$ for $R \geq R_0 > 0$ (*cf.* (2.3) for $\mathbf{R}$ and R) sufficiently large, and such that

$$D_\delta(\boldsymbol{\psi}) := \operatorname{int}(\Gamma_\delta(\boldsymbol{\psi})) = T_{\delta\boldsymbol{\psi}}(D_0)\ , \qquad 0 \leq |\delta| < \delta_0\ , \tag{3.2}$$

is likewise a bijective C^2-diffeomorphism. In particular, its inverse, denoted by $T_{\delta\boldsymbol{\psi}}^{-1}$, maps $D_\delta(\boldsymbol{\psi})$ bijectively onto D_0. For each domain $D_\delta(\boldsymbol{\psi})$ and $D_\delta^c(\boldsymbol{\psi}) := \mathbb{R}^3\backslash\overline{D_\delta}(\boldsymbol{\psi})$ with $0 \leq |\delta| < \delta_0$, there exist unique solutions of the perfect conductor and dielectric problems, described in Sections 2.2 and 2.3, respectively. We denote these solutions by $\mathbf{U}_\delta^{\mathrm{pc}}$ and $\mathbf{U}_\delta^{\mathrm{de}} := (\mathbf{U}_\delta^1, \mathbf{U}_\delta^2)$, respectively. Evidently, $\mathbf{U}_\delta^{\mathrm{pc}}$ and $(\mathbf{U}_\delta^1, \mathbf{U}_\delta^2)$ depend also on $\boldsymbol{\psi}$; we shall write occasionally $\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$ and $(\mathbf{U}_\delta^1, \mathbf{U}_\delta^2)(\boldsymbol{\psi})$ to emphasize this dependence. Using the diffeomorphsim $T_{\delta\boldsymbol{\psi}} : D_0 \to D_\delta(\boldsymbol{\psi})$ in (3.1) and (3.2), we denote by

$$\check{\mathbf{U}}_\delta^{\mathrm{pc}} := \mathbf{U}_\delta^{\mathrm{pc}} \circ T_{\delta\boldsymbol{\psi}} \quad \text{and} \quad (\check{\mathbf{U}}_\delta^1, \check{\mathbf{U}}_\delta^2) := (\mathbf{U}_\delta^1 \circ T_{\delta\boldsymbol{\psi}}, \mathbf{U}_\delta^2 \circ T_{\delta\boldsymbol{\psi}})\ , \quad 0 \leq |\delta| < \delta_0 \tag{3.3}$$

the pullbacks of the solutions $\mathbf{U}_\delta^{\mathrm{pc}}$ and $(\mathbf{U}_\delta^1, \mathbf{U}_\delta^2)$ to their corresponding nominal domains D_0 and $D_0 \times D_0^c$. Due to $T_0 = \mathsf{I}$, $\check{\mathbf{U}}_0^{\mathrm{pc}} \equiv \mathbf{U}_0^{\mathrm{pc}}$ in D_0^c and $(\check{\mathbf{U}}_0^1, \check{\mathbf{U}}_0^2) \equiv (\mathbf{U}_0^1, \mathbf{U}_0^2)$ in $D_0 \times D_0^c$. The *shape derivatives in the direction* $\boldsymbol{\psi}$ are now defined as

$$\begin{aligned} d\mathbf{U}_0^{\mathrm{pc}}(\boldsymbol{\psi}) &= \lim_{\delta\to 0} \delta^{-1}\left(\check{\mathbf{U}}_\delta^{\mathrm{pc}}(\boldsymbol{\psi}) - \mathbf{U}_0^{\mathrm{pc}}\right) \ \text{ in } \boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)\ , \\ (d\mathbf{U}_0^1, d\mathbf{U}_0^2)(\boldsymbol{\psi}) &= \lim_{\delta\to 0} \delta^{-1}\left(\check{\mathbf{U}}_\delta^1(\boldsymbol{\psi}) - \mathbf{U}_0^1, \check{\mathbf{U}}_\delta^2(\boldsymbol{\psi}) - \mathbf{U}_0^2\right) \ \text{ in } \boldsymbol{H}(\mathbf{curl}, D_0) \times \boldsymbol{H}_{\kappa_2}(\mathbf{curl}, D_0^c)\ , \end{aligned}$$

with the shape derivative shorthand $d\mathbf{U}_0^{\mathrm{de}}(\boldsymbol{\psi}) := (d\mathbf{U}_0^1, d\mathbf{U}_0^2)(\boldsymbol{\psi})$.

The existence of these limits, to which we refer as “domain derivatives” or “shape derivatives”, has been established in [28, 19, 25, 33, 29] and the references there. Shape derivatives are solutions of *Maxwell equations without source terms* and, therefore, can be approached numerically by boundary reduction to (systems of) boundary integral equations on the nominal surfaces Γ_0. We next present the boundary value problems characterizing the shape gradients for Problems I and II.

3.2. **Shape Derivative of the Perfect Conductor Problem.** Following [28, 33], there holds the following characterization of the shape derivative $d\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$ of $\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$.

Proposition 3.2. *For the solution* $\mathbf{U}_\delta^{pc}(\boldsymbol{\psi})$ *of the perfect conductor scattering problem* (2.9)–(2.10) *with scatterer geometry* $D_\delta^c(\boldsymbol{\psi})$ *with boundary* $\Gamma_\delta(\boldsymbol{\psi})$*, the first shape derivative* $d\mathbf{U}_0^{pc}(\boldsymbol{\psi})$ *at the nominal boundary* Γ_0 *of the nominal domain* D_0 *in the direction* $\boldsymbol{\psi} \in C^2(\Gamma_0;\mathbb{R}^3)$ *also solves the perfect conductor scattering problem:*

$$\mathbf{curl}\,\mathbf{curl}\, d\mathbf{U}_0^{pc}(\boldsymbol{\psi}) - \kappa^2 d\mathbf{U}_0^{pc}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad D_0^c\ , \tag{3.4}$$

where $d\mathbf{U}_0^{pc}(\boldsymbol{\psi}) \in \boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)$ *and also satisfies the boundary condition:*

$$\gamma_{\mathrm{D}}^c d\mathbf{U}_0^{pc}(\boldsymbol{\psi}) = \mathbf{g}_{\mathrm{D}}^{pc}(\mathbf{U}_0^{pc}, \mathbf{U}^{inc}, \boldsymbol{\psi}) \quad \text{on} \quad \Gamma_0\ , \tag{3.5}$$

where $\mathbf{g}_{\mathrm{D}}^{pc} \in \boldsymbol{X}(\Gamma_0)$ is given in terms of the nominal solution $\mathbf{U}_0^{pc}$, the associated incident field $\mathbf{U}^{inc}$ and perturbation field $\boldsymbol{\psi}$ by

$$\mathbf{g}_{\mathrm{D}}^{pc}(\mathbf{U}_0^{pc}, \mathbf{U}^{inc}, \boldsymbol{\psi}) := (\boldsymbol{\psi}\cdot\mathbf{n}_0)\widehat{\gamma_{\mathrm{N}}}(\mathbf{U}_0^{pc}+\mathbf{U}^{inc})\times\mathbf{n}_0 - \mathbf{curl}_{\Gamma_0}\left((\boldsymbol{\psi}\cdot\mathbf{n}_0)(\mathbf{n}_0\cdot(\mathbf{U}_0^{pc}+\mathbf{U}^{inc}))\right). \quad (3.6)$$

Based on Proposition 3.2, we have the *shape Taylor expansion of the perfect conductor problem*:

$$\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\boldsymbol{\psi}) = \mathbf{U}_0^{\mathrm{pc}} + \delta d\mathbf{U}_0^{\mathrm{pc}}(\boldsymbol{\psi}) + \mathcal{O}(\delta^2), \quad 0 \le |\delta| < \delta_0, \quad (3.7)$$

with $\mathcal{O}(\delta^2)$ valid in $\boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)$.

3.3. **Shape Derivative of the Dielectric Interface Problem.** We now precise the notion of shape differentiability for the solution of the dielectric interface problem.

For a given incident field $\mathbf{U}^{\mathrm{inc}} \in \boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl}, D_\delta^c)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\mathrm{inc}} - \kappa_2^2\mathbf{U}^{\mathrm{inc}} = 0$, let $\mathbf{U}_\delta^{\mathrm{de}} = (\mathbf{U}_\delta^1, \mathbf{U}_\delta^2)$ be the solution pair of the dielectric scattering problem (2.11)–(2.13) with respect to the scatterer geometry $\Gamma_\delta(\boldsymbol{\psi})$.

Shape differentiability of the solution pair $\mathbf{U}_\delta^{\mathrm{de}}$ of the dielectric scattering problem (2.11)–(2.13) was shown in [19, Section 6], where the following result was proved:

Proposition 3.3. *The first derivative $d\mathbf{U}_0^{de}(\boldsymbol{\psi}) = (d\mathbf{U}_0^1, d\mathbf{U}_0^2)(\boldsymbol{\psi})$ of the solution pair $\mathbf{U}_\delta^{de}(\boldsymbol{\psi}) = \left(\mathbf{U}_\delta^1, \mathbf{U}_\delta^2\right)(\boldsymbol{\psi})$ of the dielectric scattering problem (2.11)–(2.13) with respect to the scatterer geometry $\Gamma_\delta(\boldsymbol{\psi})$ at the nominal boundary Γ_0 in the direction $\boldsymbol{\psi} \in C^2(\Gamma_0; \mathbb{R}^3)$ solves the transmission problem:*

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,d\mathbf{U}_0^1(\boldsymbol{\psi}) - \kappa_1^2 d\mathbf{U}_0^1(\boldsymbol{\psi}) &= 0 \quad \text{in} \quad D_0, \\ \mathbf{curl}\,\mathbf{curl}\,d\mathbf{U}_0^2(\boldsymbol{\psi}) - \kappa_2^2 d\mathbf{U}_0^2(\boldsymbol{\psi}) &= 0 \quad \text{in} \quad D_0^c, \end{aligned} \quad (3.8)$$

with $d\mathbf{U}_0^2(\boldsymbol{\psi}) \in \boldsymbol{H}_{\kappa_2}(\mathbf{curl}, D^c)$ and interface conditions on Γ_0:

$$\begin{aligned} \gamma_{\mathrm{D}} d\mathbf{U}_0^1(\boldsymbol{\psi}) - \gamma_{\mathrm{D}}^c d\mathbf{U}_0^2(\boldsymbol{\psi}) &= \mathbf{g}_{\mathrm{D}}^{de}(\mathbf{U}_0^{de}, \mathbf{U}^{inc}, \boldsymbol{\psi}), \\ \mu_1^{-1}\widehat{\gamma_{\mathrm{N}}} d\mathbf{U}_0^1(\boldsymbol{\psi}) - \mu_2^{-1}\widehat{\gamma_{\mathrm{N}}}^c d\mathbf{U}_0^2(\boldsymbol{\psi}) &= \mathbf{g}_{\mathrm{N}}^{de}(\mathbf{U}_0^{de}, \mathbf{U}^{inc}, \boldsymbol{\psi}), \end{aligned} \quad (3.9)$$

where the data $\mathbf{g}_{\mathrm{D}}^{de}$ and $\mathbf{g}_{\mathrm{N}}^{de}$ belong to $\boldsymbol{X}(\Gamma_0)$ and are given in terms of the associated incident field $\mathbf{U}^{inc}$, the nominal solution $\mathbf{U}_0^{de}$ and of the perturbation field $\boldsymbol{\psi}$ by

$$\begin{aligned} \mathbf{g}_{\mathrm{D}}^{de}(\mathbf{U}_0^{de}, \mathbf{U}^{inc}, \boldsymbol{\psi}) \;&:=\; -(\boldsymbol{\psi}\cdot\mathbf{n}_0)\left(\hat{\gamma}_{\mathrm{N}}\mathbf{U}_0^1 - \hat{\gamma}_{\mathrm{N}}(\mathbf{U}_0^2+\mathbf{U}^{inc})\right)\times\mathbf{n}_0 \\ &\quad + \mathbf{curl}_{\Gamma_0}\left((\boldsymbol{\psi}\cdot\mathbf{n}_0)(\mathbf{n}_0\cdot\mathbf{U}_0^1 - \mathbf{n}_0\cdot(\mathbf{U}_0^2+\mathbf{U}^{inc}))\right), \\ \mathbf{g}_{\mathrm{N}}^{de}(\mathbf{U}_0^{de}, \mathbf{U}^{inc}, \boldsymbol{\psi}) \;&:=\; -(\boldsymbol{\psi}\cdot\mathbf{n}_0)\left(\frac{\kappa_1^2}{\mu_1}\gamma_{\mathrm{D}}\mathbf{U}_0^1 - \frac{\kappa_2^2}{\mu_2}\gamma_{\mathrm{D}}(\mathbf{U}_0^2+\mathbf{U}^{inc})\right) \\ &\quad + \mathbf{curl}_{\Gamma_0}\left((\boldsymbol{\psi}\cdot\mathbf{n}_0)(\mu_1^{-1}\,\mathbf{curl}_{\Gamma_0}\mathbf{U}_0^1 - \mu_2^{-1}\,\mathbf{curl}_{\Gamma_0}(\mathbf{U}_0^2+\mathbf{U}^{inc}))\right), \end{aligned}$$

with $\widehat{\gamma_{\mathrm{N}}}$ as defined in (2.14).

Based on Proposition 3.3, we have the *shape Taylor expansion of the dielectric interface problem*:

$$\breve{\mathbf{U}}_\delta^{\mathrm{de}}(\boldsymbol{\psi}) = \mathbf{U}_0^{\mathrm{de}} + \delta d\mathbf{U}_0^{\mathrm{de}}(\boldsymbol{\psi}) + \mathcal{O}(\delta^2), \quad 0 \le |\delta| < \delta_0, \quad (3.10)$$

in $\boldsymbol{H}(\mathbf{curl}, D_0) \times \boldsymbol{H}_{\kappa_2}(\mathbf{curl}, D_0^c)$. We remark that (3.10) can be transported by $T_{\delta\boldsymbol{\psi}(\omega)}^{-1}$ to $D_\delta(\omega) \cup D_\delta^c(\omega)$. For later use in the FOSM analysis, we note in passing that the shape derivative $d\mathbf{U}_0^{\mathrm{de}}(\boldsymbol{\psi})$ depends on the perturbation field $\boldsymbol{\psi}$ linearly and only through Cauchy data on Γ_0, which holds generically, due to Hadamard's theorem (*cf.* [37]): $d\mathbf{U}_0^{\mathrm{de}}(\boldsymbol{\psi})$ is solution of the linear operator equation (3.8)–(3.9) and the boundary data $\mathbf{g}^{\mathrm{de}} := (\mathbf{g}_{\mathrm{D}}^{\mathrm{de}}, \mathbf{g}_{\mathrm{N}}^{\mathrm{de}})$ in (3.9) are linear in $\boldsymbol{\psi}$.

4. First Order Second Moment (FOSM) Analysis

We considered the shape differentiability of the model problems and presented, in Sections 3.2 and 3.3, the boundary value problems which characterize the corresponding shape gradients for a *given domain perturbation field* $\boldsymbol{\psi}$. In the present section, we consider *random domain perturbation fields* $\boldsymbol{\psi}$; we show that the solutions of the model problems on the perturbed domains become random variables taking values in suitable function spaces. We also derive, using the shape gradients derived in Sections 3.2 and 3.3, deterministic expressions for first-order approximations of the second moments of the random electric fields. For this, our main argument hinges on the linear dependence of the shape gradients $d\mathbf{U}_0^{\mathrm{pc}}(\boldsymbol{\psi})$ and $d\mathbf{U}_0^{\mathrm{de}}(\boldsymbol{\psi})$ on the direction field $\boldsymbol{\psi}$.

4.1. **Random Fields.** We introduce parts of the theory presented in [39]. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, where, as customary, Ω denotes the set of all elementary events, $\mathcal{A}$ a σ-algebra of events and $\mathbb{P}$ a probability measure. We define a random field g with values in a generic, separable Hilbert space X as a strongly measurable mapping $g : \Omega \to X$ which maps events $E \in \mathcal{A}$ to Borel sets in X. This induces a measure $\tilde{\mathbb{P}}$ on X.

Let $k \in \mathbb{N}$. We say that a random variable $g : \Omega \to X$ is in the Bochner space $L^k(\Omega, \mathbb{P}; X)$ if $\omega \mapsto \|g(\omega)\|_X^k$ is measurable and integrable so that $\|g\|_{L^k(\Omega,\mathbb{P};X)} := \left(\int_\Omega \|g(\omega)\|_X^k \,\mathrm{d}\,\mathbb{P}(\omega)\right)^{1/k}$ is finite. If so, for $k = 1$, the mathematical expectation:

$$\mathbb{E}[g] \ := \ \int_\Omega u(\omega) \,\mathrm{d}\,\mathbb{P}(\omega) \ \in \ X \tag{4.1}$$

exists as a Bochner integral and it holds

$$\|\mathbb{E}[g]\|_X \ \leq \ \|g\|_{L^1(\Omega,\mathbb{P};X)} \ . \tag{4.2}$$

Let B denote a continuous linear mapping from X into another separable Hilbert space Y. For a random variable g in $L^k(\Omega, \mathbb{P}; X)$ one constructs another random variable $h(\omega) = \mathsf{B}\, g(\omega) \in L^k(\Omega, \mathbb{P}; Y)$ and

$$\|\mathsf{B}\, g\|_{L^k(\Omega,\mathbb{P};Y)} \ \leq \ \|g\|_{L^k(\Omega,\mathbb{P};X)} \,. \tag{4.3}$$

Furthermore,

$$\mathsf{B} \int_\Omega g(\omega) \,\mathrm{d}\,\mathbb{P}(\omega) \ = \ \int_\Omega \mathsf{B}\, g(\omega) \,\mathrm{d}\,\mathbb{P}(\omega) \,. \tag{4.4}$$

In order to define statistical moments, we introduce for $k \in \mathbb{N}$ and for a separable Hilbert space X the k-fold tensor product space:

$$X^{(k)} \ := \ \underbrace{X \otimes \cdots \otimes X}_{k\text{-times}} \tag{4.5}$$

equipped with the natural norm $\|\cdot\|_{X^{(k)}}$, which is a cross-norm, i.e.

$$\|g_1 \otimes \cdots \otimes g_k\|_{X^{(k)}} \ = \ \|g_1\|_X \cdots \|g_k\|_X \ , \tag{4.6}$$

for all $g_1, \ldots, g_k$ in X. We refer to [35, Chap. II.4] and the references there for these and further results on tensor products of separable Hilbert spaces. Let now X and Y be separable Hilbert spaces. For B in $\mathcal{L}(X, Y)$, the space of linear continuous mappings from X to Y, there is a unique linear, continuous tensor product operator:

$$\mathsf{B}^{(k)} \ := \ \underbrace{\mathsf{B} \otimes \cdots \otimes \mathsf{B}}_{k\text{-times}} \ \in \ \mathcal{L}(X^{(k)}, Y^{(k)}) \,. \tag{4.7}$$

For a random field $u \in L^k(\Omega, \mathbb{P}; X)$, consider the k-fold simple tensor product $u^{(k)} := u(\omega) \otimes \cdots \otimes u(\omega)$. Then, $u^{(k)} \in L^1(\Omega, \mathbb{P}; X^{(k)})$. For $u \in L^k(\Omega, \mathbb{P}; X)$ with $k \in \mathbb{N}$, the kth moment of $u(\omega)$ is defined by

$$\mathcal{M}^k u \ = \ \mathbb{E}\big[\underbrace{u \otimes \cdots \otimes u}_{k\text{-times}}\big] \ = \ \int_{\omega\in\Omega} \underbrace{u(\omega) \otimes \cdots \otimes u(\omega)}_{k\text{-times}} \,\mathrm{d}\,\mathbb{P}(\omega) \,. \tag{4.8}$$

Nonetheless, in the present work we will just focus on first and second order moments, i.e. $k = 1, 2$.

Remark 4.1. For a vector space[3] X and for $k \in \mathbb{N}$, we denote by $X^k = X \times ... \times X$ the k-fold cartesian product of X, with graph norm given by the sum of k components, which is not to be confused with k-fold tensor product $X^{(k)}$ in (4.5).

[3]All spaces are understood over the coefficient field $\mathbb{C}$, unless explicitly stated otherwise.

4.2. **Random Surfaces.** For the analysis of *shape uncertainty* in the scattering problems, we assume that the scatterers are given as ensembles of surfaces $\Gamma_\delta(\boldsymbol{\psi})$ with nominal geometry Γ_0 *parametrized by random perturbation fields* $\boldsymbol{\psi}$, which we assume to be $C^2(\Gamma_0;\mathbb{R}^3)$-random fields over a probability space $(\Omega,\mathcal{A},\mathbb{P})$. Denote by $\mathcal{B}(C^2(\Gamma_0;\mathbb{R}^3))$ the σ-algebra of Borel sets on the separable Banach space $C^2(\Gamma_0;\mathbb{R}^3)$. Then a random domain perturbation field is an $(\mathcal{A},\mathcal{B})$-measurable mapping $\boldsymbol{\psi}:\Omega\to C^2(\Gamma_0;\mathbb{R}^3)$. With this mapping, and with $\delta_0>0$ sufficiently small, for all $\delta_0>|\delta|\geq 0$, we associate a one-parameter family of random surfaces via the mapping:

$$\Omega\ni\omega\mapsto\Gamma_\delta(\boldsymbol{\psi}(\omega))=T_{\delta\boldsymbol{\psi}(\omega)}(\Gamma_0)=(\mathsf{I}+\delta\boldsymbol{\psi}(\omega))(\Gamma_0)\;, \tag{4.9}$$

where the collection $\{T_{\delta\boldsymbol{\psi}(\omega)}=\mathsf{I}+\delta\boldsymbol{\psi}(\omega):\omega\in\Omega\}$ of diffeomorphisms is measurable with respect to the topology generated by the open neighborhoods in the set of all diffeomorphisms from $\Gamma_0\to\Gamma_\delta$.

We shall confine ourselves in the remainder of this work to normal perturbation fields[4],

$$\boldsymbol{\psi}(\mathbf{x}_0,\omega)=\eta(\mathbf{x}_0,\omega)\mathbf{n}_0(\mathbf{x})\;,\quad \mathbf{x}_0\in\Gamma_0\;,\quad \eta\in L^2(\Omega,\mathbb{P};C^2(\Gamma_0;\mathbb{R}))\;. \tag{4.10}$$

As before, $\mathbf{n}_0$ denotes the exterior unit normal vector to the nominal boundary $\Gamma_0=\partial D_0$ and $\eta\in L^2(\Omega,\mathbb{P};C^2(\Gamma_0))$ a scalar, random perturbation amplitude, i.e. a measurable mapping such that $\|\eta\|_{L^2(\Omega,\mathbb{P};C^2(\Gamma_0))}\leq 1$. If Γ_0 is a C^3-boundary, for $|\delta|>0$ sufficiently small and for a $\mathbb{P}$-a.s. realization $\eta(\cdot,\omega)\in C^2(\Gamma_0)$, conditions **(S1)** and **(S2)** hold and the corresponding realization:

$$\Gamma_\delta(\omega)=T_{\delta\boldsymbol{\psi}(\omega)}(\Gamma_0)=(\mathsf{I}+\delta\boldsymbol{\psi}(\omega))(\Gamma_0)=(\mathsf{I}+\delta\eta(\cdot,\omega)\mathbf{n}_0)(\Gamma_0) \tag{4.11}$$

is then $\mathbb{P}$-a.s. a C^2-manifold embedded into $\mathbb{R}^3$; as before, we denote each realization of the perturbed domain by $D_\delta(\omega)=\mathrm{int}(\Gamma_\delta(\omega))=T_{\delta\boldsymbol{\psi}(\omega)}(D_0)$ and $D_\delta^c(\omega):=\mathbb{R}^3\backslash\overline{D_\delta(\omega)}$.

For a given, random perturbation field $\boldsymbol{\psi}(\omega)$ and sufficiently small $|\delta|\geq 0$, and $\mathbb{P}$-a.e. ω, the model electromagnetic scattering problems I and II from Section 2 in the stochastically perturbed domains $D_\delta(\omega)$ will admit unique solutions, denoted $\mathbf{U}_\delta^{\mathrm{pc}}(\omega)$ and $\mathbf{U}_\delta^{\mathrm{de}}(\omega)$, respectively – the reader should note the implicit dependence on $\boldsymbol{\psi}$. The collections $\{\mathbf{U}_\delta^{\mathrm{pc}}(\omega):\omega\in\Omega\}$ and $\{\mathbf{U}_\delta^{\mathrm{de}}(\omega):\omega\in\Omega\}$ of *individual solutions for each realization of the geometry* $D_\delta(\omega)$ are candidates for random solutions. As maps from $(\Omega,\mathcal{A})$ into the appropriate functional spaces for the corresponding Maxwell problems I and II, these collections must verify strong measurability.

4.3. **Random Shape Perfect Conductor.** Let $\mathbf{U}^{\mathrm{inc}}\in\boldsymbol{H}_{\mathrm{loc}}(\mathbf{curl},D_\delta^c(\omega))$ be a given, deterministic incident field such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}^{\mathrm{inc}}-\kappa^2\mathbf{U}^{\mathrm{inc}}=0$ in $D_\delta^c(\omega)$. The random shape perfect conductor time-harmonic scattering problem can be stated as follows: we seek a scattered field $\mathbf{U}_\delta^{\mathrm{pc}}(\omega)$ in $D_\delta^c(\omega)$ such that (2.2) and (2.3) hold in $D_\delta^c(\omega)$, with

$$\mathbf{U}_\delta^{\mathrm{pc}}\in\boldsymbol{H}_\kappa(\mathbf{curl},D_\delta^c(\omega))\;, \tag{4.12}$$

and Dirichlet boundary condition:

$$\gamma_{\mathrm{D}}^c\mathbf{U}_\delta^{\mathrm{pc}}=-\gamma_{\mathrm{D}}^c\mathbf{U}^{\mathrm{inc}}=:\boldsymbol{m}_\delta^{\mathrm{pc}}(\omega)\quad\text{on}\quad\Gamma_\delta(\omega)\;. \tag{4.13}$$

Since for $0\leq|\delta|<\delta_0$ with sufficiently small δ_0, all realizations $\Gamma_\delta(\omega)$ of the random boundary are closed C^2-surfaces in $\mathbb{R}^3$, for $\mathbb{P}$-a.s. perturbation amplitude $\eta(\cdot,\omega)$ in (4.10), the problem (4.12)–(4.13) admits a unique solution $\mathbf{U}_\delta^{\mathrm{pc}}(\omega)\in\boldsymbol{H}_\kappa(\mathbf{curl},D_\delta^c(\omega))$ which satisfies (2.2) and (2.3) in $D_\delta^c(\omega)$.

Proposition 4.2. *Assume hypotheses* (4.10), (4.11). *Then, the following statements hold:*

(i) *There exists $\delta_0>0$ depending on Γ_0 and on κ, but independent of $\mathbf{U}^{inc}$ such that, for every $0\leq|\delta|<\delta_0$, the family $\{\breve{\mathbf{U}}_\delta^{pc}(\omega):\omega\in\Omega\}$ of solutions to the perfect conductor problem is a $\boldsymbol{H}_\kappa(\mathbf{curl},D_0^c)$-valued random variable on $(\Omega,\mathcal{A},\mathbb{P})$, to which we shall refer as random solution of the perfect conductor problem.*

(ii) *For sufficiently small domain perturbation amplitudes $0\leq|\delta|<\delta_0$, for $\mathbb{P}$-a.s. ω the shape Taylor expansion of the random solution field of the perfect conductor problem is*

$$\breve{\mathbf{U}}_\delta^{pc}(\omega)=\mathbf{U}_0^{pc}+\delta d\mathbf{U}_0^{pc}(\eta(\cdot,\omega)\mathbf{n}_0)+\mathcal{O}(\delta^2)\;\;\mathbb{P}\text{-}a.s.\;in\;\;\boldsymbol{H}_\kappa(\mathbf{curl},D_0^c)\;. \tag{4.14}$$

[4]Similar considerations remain valid for more general random domain perturbation fields, $\boldsymbol{\psi}(\omega)$, as long as the outgoing condition is preserved, i.e. property **(S1)** in Section 3.1 is fulfilled.

(iii) *The corresponding shape gradient* $d\mathbf{U}_0^{pc}(\eta(\cdot,\omega)\mathbf{n}_0)$ *defined in* (3.4)–(3.5) *belongs to the space* $L^2(\Omega,\mathbb{P};\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}}))$.

Proof. Assuming (4.10), (4.11), we first prove (i). For each realization ω and for every $0\le|\delta|<\delta_0$, the corresponding C^2-diffeomorphism $T_{\delta\boldsymbol{\psi}}$ is bijective from D_0 to $D_\delta(\boldsymbol{\psi}(\omega))$ and from D_0^{c} to $D_\delta^{\mathrm{c}}(\boldsymbol{\psi}(\omega))$. For each ω, there exists a unique solution of the perfect conductor problem in $D_\delta^{\mathrm{c}}(\boldsymbol{\psi}(\omega))$, $\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi}(\omega))\in\boldsymbol{H}_\kappa(\mathbf{curl},D_\delta^{\mathrm{c}}(\boldsymbol{\psi}(\omega)))$. Therefore, the collection $\{\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi}(\omega)):\omega\in\Omega\}$ is well-defined, and thus, for every ω, the pullback $\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega)=\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi}(\omega))\circ T_{\delta\boldsymbol{\psi}(\omega)}$ belongs to $\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}})$.

The existence of the shape derivative in direction $\boldsymbol{\psi}$ implies continuous dependence of $\mathbf{U}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$ on the scatterers' shape $\boldsymbol{\psi}$: the map $\boldsymbol{\psi}\mapsto\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$ is continuous from the space of perturbation fields $\{\boldsymbol{\psi}\in C^2(\Gamma_0;\mathbb{R}^3):(\mathbf{S1})\ \&\ (\mathbf{S2})\text{ hold}\}$ to $\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}})$. The collection $\{\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega):\omega\in\Omega\}$ is, therefore, the composition of a continuous map $\boldsymbol{\psi}\mapsto\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\boldsymbol{\psi})$ with the strongly measurable map $\omega\mapsto\boldsymbol{\psi}(\omega)$. As compositions of strongly measurable and continuous maps are strongly measurable, the map $\omega\mapsto\breve{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega)$ is strongly measurable from $(\Omega,\mathcal{A})$ to $(\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}}),\mathcal{B}(\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}}))$, and therefore a $\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}})$-valued random function.

Point (ii) is deduced by observing that the shape Taylor expansion (4.14) follows from (3.7) upon noting that the assumptions of Proposition 3.2 hold $\mathbb{P}$-a.s.

Finally, the square integrability with respect to $\mathbb{P}$, i.e. $\eta\in L^2(\Omega,\mathbb{P};C^2(\Gamma_0))$, implies the finiteness of second statistical moments of the random Cauchy data $\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}(\mathbf{U}_0^{\mathrm{pc}},\mathbf{U}^{\mathrm{inc}},\eta(\cdot,\omega)\mathbf{n}_0)$. This is a consequence of the linear dependence of $\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}$ in (3.6) on the domain perturbation field $\boldsymbol{\psi}(\omega)=\eta(\cdot,\omega)\mathbf{n}_0$. Since the perfect conductor problem (3.4)–(3.6) is linear and $\eta\in L^2(\Omega,\mathbb{P};C^2(\Gamma_0))$, this linear dependence of the data on η implies $d\mathbf{U}_0^{\mathrm{pc}}(\eta(\cdot,\omega)\mathbf{n}_0)\in L^2(\Omega,\mathbb{P};\boldsymbol{H}_\kappa(\mathbf{curl},D_0^{\mathrm{c}}))$ as stated in (iii). □

We remark that the shape Taylor expansion (4.14) is formulated on the ω-independent nominal domains D_0^{c} and $\Gamma_0=\partial D_0$. Transporting (4.14) via the diffeomorphism $T^{-1}_{\delta\boldsymbol{\psi}(\omega)}$ to $\Gamma_\delta(\omega)$ results in

$$\mathbf{U}_\delta^{\mathrm{pc}}(\omega)=(\mathbf{U}_0^{\mathrm{pc}}+\delta d\mathbf{U}_0^{\mathrm{pc}}(\eta(\cdot,\omega)\mathbf{n}_0))\circ T^{-1}_{\delta\boldsymbol{\psi}(\omega)}+\mathcal{O}(\delta^2)\ \text{ in }\ \boldsymbol{H}_\kappa(\mathbf{curl},D_\delta^{\mathrm{c}}(\omega))\,.$$

4.4. **Random Dielectric Interface.**

Proposition 4.3. *Let* (4.10), (4.11) *hold. Then,*

(i) *There exists* $\delta_0>0$ *such that, for every* $0\le|\delta|<\delta_0$, *the mapping* $\Omega\ni\omega\mapsto\breve{\mathbf{U}}_\delta^{de}(\omega)$ *is measurable from* $(\Omega,\mathcal{A})$ *to*

$$(\boldsymbol{H}(\mathbf{curl},D_0)\times\boldsymbol{H}_{\kappa_2}(\mathbf{curl},D_0^{\mathrm{c}}),\mathcal{B}(\boldsymbol{H}(\mathbf{curl},D_0))\times\mathcal{B}(\boldsymbol{H}_{\kappa_2}(\mathbf{curl},D_0^{\mathrm{c}})))\ .$$

(ii) *For* $0\le|\delta|<\delta_0$, *and for* $\mathbb{P}$*-a.e.* ω *holds the shape Taylor expansion over the space* $\boldsymbol{H}(\mathbf{curl},D_0)\times\boldsymbol{H}_{\kappa_2}(\mathbf{curl},D_0^{\mathrm{c}})$:

$$\breve{\mathbf{U}}_\delta^{de}(\omega)=\mathbf{U}_0^{de}+\delta d\mathbf{U}_0^{de}(\eta(\cdot,\omega)\mathbf{n}_0)+\mathcal{O}(\delta^2)\ . \tag{4.15}$$

(iii) *The random Cauchy data* $\mathbf{g}^{de}(\mathbf{U}_0^{de},\mathbf{U}^{inc},\eta(\cdot,\omega)\mathbf{n}_0)$ *defining the corresponding shape gradients* $d\mathbf{U}_0^{de}(\eta(\cdot,\omega)\mathbf{n}_0)$ *via* (3.8), (3.9), *have finite second moments, and*

$$d\mathbf{U}_0^{de}(\eta(\cdot,\omega)\mathbf{n}_0)\in L^2(\Omega;\mathbb{P};\boldsymbol{H}(\mathbf{curl},D_0)\times\boldsymbol{H}_{\kappa_2}(\mathbf{curl},D_0^{\mathrm{c}}))\ .$$

Proof. We take the cue from the perfect conductor case and repeat similar arguments to prove (i). Similarly for its shape derivative: assertion (ii) follows from (3.10) by inserting $\boldsymbol{\psi}=\eta(\cdot,\omega)\mathbf{n}_0$ into the expressions $\mathbf{g}_{\mathrm{D}}^{\mathrm{de}}$ and $\mathbf{g}_{\mathrm{N}}^{\mathrm{de}}$ for the dielectric scatterer Cauchy data in the boundary condition (3.9). Lastly, statement (iii), concerning the square integrability with respect to $\mathbb{P}$ and implying the finiteness of second statistical moments of the random Cauchy data, $\mathbf{g}_{\mathrm{D}}^{\mathrm{de}}(\mathbf{U}_0^{\mathrm{de}},\mathbf{U}^{\mathrm{inc}},\boldsymbol{\psi})$, and $\mathbf{g}_{\mathrm{N}}^{\mathrm{de}}(\mathbf{U}_0^{\mathrm{de}},\mathbf{U}^{\mathrm{inc}},\boldsymbol{\psi})$, comes from the linear dependence of $\mathbf{g}_{\mathrm{D}}^{\mathrm{de}}$ and $\mathbf{g}_{\mathrm{N}}^{\mathrm{de}}$ on the domain perturbation field $\boldsymbol{\psi}$ and from the assumption that $\eta\in L^2(\Omega,\mathbb{P};C^2(\Gamma_0))$. □

5. Boundary Reduction and Galerkin Boundary Element Approximation

The boundary value problems (3.4)–(3.5) and (3.8)–(3.9), characterizing the shape gradients of the model problems are governed by the *homogeneous Maxwell equations* with Cauchy data prescribed on the nominal boundary $\Gamma_0 = \partial D_0$. The Cauchy data of these problems depend only on boundary values of solutions to the nominal problem, which is a consequence of Hadamard's theorem characterizing shape gradients as Radon measures supported on Γ_0 (see, e.g., [37] for a discussion). In computing these shape gradients, for homogeneous media it is therefore suggestive to reduce these problems to boundary integral equations on the nominal boundaries, which are equivalent to the variational problems (3.4)–(3.5) and (3.8)–(3.9) that characterize the shape gradients. This procedure of boundary reduction is well known (e.g. [30, 36, 31] and references therein) and widely used in computational electromagnetics. This boundary reduction is not unique; in the so-called *indirect method of boundary reduction*, the electric and magnetic fields in the volume domain are represented as potentials of unknown current densities on the scatterer though lacking physical meaning; for our shape-sensitivity analysis, we propose the *direct method of boundary reduction* as developed in [13]. This results in BIEs where the unknown layer densities on the scatterer are the Cauchy data of the domain problems. Furthermore, as we showed above, the second moment analysis requires these Cauchy data, and therefore, BIEs obtained by the direct method of boundary reduction are in a sense natural in our context. We add that any other, indirect, boundary reduction is equally applicable.

We review the corresponding boundary reduction in Section 5.1 ahead, establish properties of the corresponding boundary integral operators, and finally generalize the reasoning presented in [39], for the Helmholtz equation, to the FOSM analysis of the presently considered model problems. Thus, one derives in *tensorized, variational BIEs* on $\Gamma_0 \times \Gamma_0$, which, as we show, are well posed in suitable tensor products of trace spaces. We develop the BIEs for a generic bounded domain D with bounded, connected C^2 boundary $\Gamma = \partial D$.

5.1. **Variational Boundary Integral Equations.**

5.1.1. *First Green formula.* From [10], we know that

$$\forall\, \mathbf{U}\,, \mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, D): \int_D (\mathbf{curl}\,\mathbf{U} \cdot \mathbf{V} - \mathbf{U} \cdot \mathbf{curl}\,\mathbf{V})\, \mathrm{d}D = \mathsf{b}\left(\gamma_{\mathrm{D}}\mathbf{V}, \gamma_{\mathrm{D}}\mathbf{U}\right)\ . \tag{5.1}$$

We define the bilinear form $\Phi_D \colon \boldsymbol{H}(\mathbf{curl}, D) \times \boldsymbol{H}(\mathbf{curl}, D) \to \mathbb{C}$ by

$$\Phi_D(\mathbf{U}, \mathbf{V}) := \int_D \left(\kappa^{-1}\,\mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\mathbf{V} - \kappa \mathbf{U} \cdot \mathbf{V}\right) \mathrm{d}D. \tag{5.2}$$

For $\mathbf{U}, \mathbf{V} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$ we define correspondingly $\Phi_{D^c}(\mathbf{U}, \mathbf{V})$.

Assume that $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D)$ is a Maxwell solution in D and $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, D)$. Then

$$\Phi_D(\mathbf{U}, \mathbf{V}) = \mathsf{b}\left(\gamma_{\mathrm{D}}\mathbf{V}, \gamma_{\mathrm{N}}\mathbf{U}\right). \tag{5.3}$$

Remark 5.1. The following symmetry between electric and magnetic field quantities takes place: assume that $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D)$ is a Maxwell solution in D and let $\tilde{\mathbf{U}} := \kappa^{-1}\,\mathbf{curl}\,\mathbf{U}$. Then, we have

$$\tilde{\mathbf{U}} = \kappa^{-1}\,\mathbf{curl}\,\mathbf{U}, \qquad \mathbf{U} = \kappa^{-1}\,\mathbf{curl}\,\tilde{\mathbf{U}},$$
$$\gamma_{\mathrm{D}}\tilde{\mathbf{U}} = \gamma_{\mathrm{N}}\mathbf{U}, \qquad \gamma_{\mathrm{D}}\mathbf{U} = \gamma_{\mathrm{N}}\tilde{\mathbf{U}}.$$

5.1.2. *Potentials.* We recall the three-dimensional fundamental solution G^κ for the Helmholtz equation:

$$G^\kappa(\mathbf{x}, \mathbf{y}) = \frac{e^{i\kappa\|\mathbf{x}-\mathbf{y}\|_2}}{4\pi\ \|\mathbf{x}-\mathbf{y}\|_2}\ , \tag{5.4}$$

where $\kappa \geq 0$ denotes the wavenumber. The corresponding scalar *single layer potential* reads

$$(\Psi^\kappa v)(\mathbf{x}) := \int_\Gamma G^\kappa(\mathbf{x}, \mathbf{y}) v(\mathbf{y}) dS_{\mathbf{y}}\ , \quad \mathbf{x} \in D \cup D^c\ . \tag{5.5}$$

When applied to tangent vectors so as to yield volume potentials, we will simply use boldface $\boldsymbol{\Psi}^\kappa$. The following mapping and coercivity properties of the potentials will be required.

Proposition 5.2. *For every $\kappa \geq 0$, the mappings:*

$$\Psi^\kappa : H^{-\frac{1}{2}+\sigma}(\Gamma) \to H^{1+\sigma}(D) \times H^{1+\sigma}_{loc}(D^c) , \quad \boldsymbol{\Psi}^\kappa : \boldsymbol{V}'_\pi(\Gamma) \to \boldsymbol{H}^1(D) \times \boldsymbol{H}^1_{loc}(D^c)$$

are linear and continuous for any $\sigma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, there are positive constants such that

$$\forall\, u \in H^{-1/2}(\Gamma) : \ \left\langle \overline{u}, \gamma_{\mathrm{D}} \Psi^0 u \right\rangle_{-\frac{1}{2},\frac{1}{2}} \gtrsim \|u\|^2_{-\frac{1}{2}} \ , \qquad \forall\, \boldsymbol{\lambda} \in \boldsymbol{V}'_\pi(\Gamma) : \ \mathsf{b}\left(\overline{\boldsymbol{\lambda}}, \gamma_{\mathrm{D}} \boldsymbol{\Psi}^0 \boldsymbol{\lambda}\right) \gtrsim \|\boldsymbol{\lambda}\|^2_{\boldsymbol{V}'_\pi} \ .$$

The electric potential $\boldsymbol{\Psi}^\kappa_E$ generated by the electric surface current $\mathbf{j} \in \boldsymbol{X}(\Gamma)$ is defined by

$$\boldsymbol{\Psi}^\kappa_E \mathbf{j} := \kappa \boldsymbol{\Psi}^\kappa \mathbf{j} + \kappa^{-1} \nabla \Psi^\kappa \operatorname{div}_\Gamma \mathbf{j} = \kappa^{-1} \,\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\Psi}^\kappa \mathbf{j} \ . \tag{5.6}$$

Analogously, we define a magnetic version $\boldsymbol{\Psi}^\kappa_M$ of $\boldsymbol{\Psi}^\kappa_E$ generated by $\boldsymbol{m} \in \boldsymbol{X}(\Gamma)$ as

$$\boldsymbol{\Psi}^\kappa_M \boldsymbol{m} := \mathbf{curl}\, \boldsymbol{\Psi}^\kappa \boldsymbol{m} \ . \tag{5.7}$$

These potentials are solutions of the Maxwell equations in $D \cup D^c$ without sources, satisfying

$$\kappa^{-1}\, \mathbf{curl}\, \boldsymbol{\Psi}^\kappa_E = \boldsymbol{\Psi}^\kappa_M, \qquad \kappa^{-1}\, \mathbf{curl}\, \boldsymbol{\Psi}^\kappa_M = \boldsymbol{\Psi}^\kappa_E \ . \tag{5.8}$$

With the mapping properties of Ψ^κ, $\boldsymbol{\Psi}^\kappa$, we deduce that the potentials $\boldsymbol{\Psi}^\kappa_E, \boldsymbol{\Psi}^\kappa_M$ are continuous from $\boldsymbol{X}(\Gamma)$ to $\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D) \times \boldsymbol{H}_\kappa(\mathbf{curl}\,\mathbf{curl}, D^c)$ and similarly for equations (5.6), (5.7), (5.8).

5.1.3. *Boundary Integral Operators.* The traces $\gamma_{\mathrm{D}}, \gamma^c_{\mathrm{D}}, \gamma_{\mathrm{N}}, \gamma^c_{\mathrm{N}}$ can be applied to $\boldsymbol{\Psi}^\kappa_E$, $\boldsymbol{\Psi}^\kappa_M$, and yield continuous mappings from $\boldsymbol{X}(\Gamma)$ to $\boldsymbol{X}(\Gamma)$. For the interior trace operators of the potentials, it holds

$$\gamma_{\mathrm{N}} \boldsymbol{\Psi}^\kappa_E = \gamma_{\mathrm{D}} \boldsymbol{\Psi}^\kappa_M \ , \qquad \gamma_{\mathrm{N}} \boldsymbol{\Psi}^\kappa_M = \gamma_{\mathrm{D}} \boldsymbol{\Psi}^\kappa_E \ , \tag{5.9}$$

with similar relations valid for exterior traces γ^c_{D} and γ^c_{N}.

The direct method of boundary reduction is formulated in terms of Dirichlet and Neumann jumps $[\gamma_{\mathrm{D}}] := \gamma_{\mathrm{D}} - \gamma^c_{\mathrm{D}}$ and $[\gamma_{\mathrm{N}}] := \gamma_{\mathrm{N}} - \gamma^c_{\mathrm{N}}$, respectively. With I denoting the identity mapping, these jumps satisfy the following *jump relations* for $\boldsymbol{\Psi}^\kappa_E$ (see [9]):

$$[\gamma_{\mathrm{D}}]\boldsymbol{\Psi}^\kappa_E = 0 \ , \quad [\gamma_{\mathrm{N}}]\boldsymbol{\Psi}^\kappa_E = -\,\mathsf{I} \ , \quad [\gamma_{\mathrm{D}}]\boldsymbol{\Psi}^\kappa_M = -\,\mathsf{I} \ , \quad [\gamma_{\mathrm{N}}]\boldsymbol{\Psi}^\kappa_M = 0 \ .$$

Assume that $\mathbf{U}|_D \in \boldsymbol{H}(\mathbf{curl}, D)$ is a Maxwell solution in D, and that $\mathbf{U}|_{D^c} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$ is a Maxwell solution in D^c. Then we have with $\mathbf{j} := [\gamma_{\mathrm{N}}]\mathbf{U}$, $\boldsymbol{m} := [\gamma_{\mathrm{D}}]\mathbf{U}$ the *integral representation formula* [9]:

$$\mathbf{U} = -\boldsymbol{\Psi}^\kappa_E \mathbf{j} - \boldsymbol{\Psi}^\kappa_M \boldsymbol{m} = -\begin{pmatrix} \boldsymbol{\Psi}^\kappa_E \\ \boldsymbol{\Psi}^\kappa_M \end{pmatrix}^\top \begin{pmatrix} \mathbf{j} \\ \boldsymbol{m} \end{pmatrix} \quad \text{on} \quad D \cup D^c \ . \tag{5.10}$$

Applying the symmetric parts $\{\gamma_{\mathrm{D}}\} := \frac{1}{2}(\gamma_{\mathrm{D}} + \gamma^c_{\mathrm{D}})$, $\{\gamma_{\mathrm{N}}\} := \frac{1}{2}(\gamma_{\mathrm{N}} + \gamma^c_{\mathrm{N}})$ of the traces to (5.10), yields for $\mathbf{j}, \boldsymbol{m} \in \boldsymbol{X}(\Gamma)$,

$$\mathsf{A}_\kappa \begin{pmatrix} \boldsymbol{m} \\ \mathbf{j} \end{pmatrix} := \begin{pmatrix} \{\gamma_{\mathrm{D}}\} \\ \{\gamma_{\mathrm{N}}\} \end{pmatrix} \left(-\boldsymbol{\Psi}^\kappa_E \mathbf{j} - \boldsymbol{\Psi}^\kappa_M \boldsymbol{m}\right) , \tag{5.11}$$

where we introduce the matrix boundary integral operator A_κ:

$$\mathsf{A}_\kappa := \begin{pmatrix} \mathsf{M}_\kappa & \mathsf{C}_\kappa \\ \mathsf{C}_\kappa & \mathsf{M}_\kappa \end{pmatrix} : \boldsymbol{X}^2(\Gamma) \to \boldsymbol{X}^2(\Gamma) \ , \tag{5.12}$$

with entries defined as

$$\mathsf{C}_\kappa := -\{\gamma_{\mathrm{D}}\}\boldsymbol{\Psi}^\kappa_E = -\{\gamma_{\mathrm{N}}\}\boldsymbol{\Psi}^\kappa_M \ , \quad \mathsf{M}_\kappa := -\{\gamma_{\mathrm{N}}\}\boldsymbol{\Psi}^\kappa_E = -\{\gamma_{\mathrm{D}}\}\boldsymbol{\Psi}^\kappa_M \ . \tag{5.13}$$

With these elements, we can construct interior and exterior *Calderón projectors*, defined respectively by

$$\mathsf{P}_\kappa := \frac{1}{2}\,\mathsf{I} + \mathsf{A}_\kappa \quad \text{and} \quad \mathsf{P}^c_\kappa := \frac{1}{2}\,\mathsf{I} - \mathsf{A}_\kappa,$$

mapping interior and exterior Cauchy data of Maxwell solutions into themselves. Note that the range of P_κ coincides with the kernel of P^c_κ.

In the proof of coercivity of the electric field operator C_κ and of the matrix operator A_κ, and also in the stability proofs of the sparse tensor product Galerkin discretization, the following *sign-flipping isomorphism* is known to play an essential role. Let $\mathbf{u} \in \boldsymbol{X}(\Gamma)$ admit a stable Hodge decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \boldsymbol{V}(\Gamma)$ and $\mathbf{w} \in \boldsymbol{W}(\Gamma)$ as indicated in Theorem 2.3. We denote by

$\Theta : \boldsymbol{X}(\Gamma) \to \boldsymbol{X}(\Gamma)$ the sign-flipping isomorphism associated with the mapping $\mathbf{u} = \mathbf{v}+\mathbf{w} \mapsto \overline{\mathbf{v}-\mathbf{w}}$ (*cf.* [13, Def. 3.11]).

Proposition 5.3. *(*[13, Thm. 3.12]*) Let* $\mathbf{u} \in \boldsymbol{X}(\Gamma)$ *admit a decomposition* $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \boldsymbol{V}(\Gamma)$ *and* $\mathbf{w} \in \boldsymbol{W}(\Gamma)$. *Denote by* $\Theta : \boldsymbol{X}(\Gamma) \to \boldsymbol{X}(\Gamma)$ *the sign-flipping isomorphism associated with the mapping* $\mathbf{u} = \mathbf{v} + \mathbf{w} \mapsto \overline{\mathbf{v}-\mathbf{w}}$. *Then, there exists a compact operator* $\mathsf{T}\colon \boldsymbol{X}^2(\Gamma) \to \boldsymbol{X}^2(\Gamma)$ *and* $\alpha > 0$ *such that*

$$\operatorname{Re}\left\{\mathsf{B}\left((\mathsf{A}_\kappa + \mathsf{T})\begin{pmatrix}\boldsymbol{m}\\ \mathbf{j}\end{pmatrix}, \begin{pmatrix}\Theta\boldsymbol{m}\\ \Theta\mathbf{j}\end{pmatrix}\right)\right\} \geq \alpha \left\|\begin{pmatrix}\boldsymbol{m}\\ \mathbf{j}\end{pmatrix}\right\|_{\boldsymbol{X}^2}^2 \quad \forall\, \boldsymbol{m}, \mathbf{j} \in \boldsymbol{X}(\Gamma)\,, \tag{5.14}$$

where the bilinear form $\mathsf{B}\colon \boldsymbol{X}^2(\Gamma) \times \boldsymbol{X}^2(\Gamma) \to \mathbb{C}$ *is defined as follows:*

$$\mathsf{B}\left(\begin{pmatrix}\boldsymbol{m}\\ \mathbf{j}\end{pmatrix}, \begin{pmatrix}\boldsymbol{m}'\\ \mathbf{j}'\end{pmatrix}\right) := \mathsf{b}\left(\boldsymbol{m}, \boldsymbol{m}'\right) + \mathsf{b}\left(\mathbf{j}, \mathbf{j}'\right). \tag{5.15}$$

5.2. **Galerkin BEM.** We recapitulate convergence results of abstract Galerkin discretizations of the BIE reformulations of the nominal problems from [10, 13, 7].

5.2.1. *Abstract convergence theorem.* Consider a Hilbert space $\boldsymbol{X}$ with stable decomposition $\boldsymbol{X} = \boldsymbol{V} \oplus \boldsymbol{W}$, i.e. for every $u \in \boldsymbol{X}$ there exists a unique decomposition $\mathbf{u} = \mathbf{v}+\mathbf{w}$ with $\|\mathbf{v}\|_{\boldsymbol{X}} + \|\mathbf{w}\|_{\boldsymbol{X}} \lesssim \|\mathbf{u}\|_{\boldsymbol{X}}$. With this decomposition, we associate the mapping $\Theta\colon \boldsymbol{X} \to \boldsymbol{X}$ with $\mathbf{u} = \mathbf{v} + \mathbf{w} \mapsto \overline{\mathbf{v}-\mathbf{w}}$. Consider further a nested sequence $\{\boldsymbol{X}_h\}_{h\geq 0}$ of closed subspaces $\boldsymbol{X}_h \subset \boldsymbol{X}$ with *discrete decompositions* $\boldsymbol{X}_h = \boldsymbol{V}_h \oplus \boldsymbol{W}_h$ which satisfy the following assumptions:

(A1) The family $\{\boldsymbol{X}_h\}_{h\geq 0}$ is dense in the space $\boldsymbol{X}$, i.e.

$$\overline{\bigcup_{h\geq 0} \boldsymbol{X}_h} = \boldsymbol{X}\ .$$

(A2) for all h, it holds $\boldsymbol{W}_h \subset \boldsymbol{W}$ and

$$\forall\, \mathbf{v}_h \in \boldsymbol{V}_h : \qquad \inf_{\mathbf{v}\in\boldsymbol{V}} \|\mathbf{v}_h - \mathbf{v}\|_{\boldsymbol{X}} \leq \delta_h \|\mathbf{v}\|_{\boldsymbol{X}}\ , \tag{5.16}$$

with $\delta_h \to 0$ for $h \to 0$.

Proposition 5.4. *[*[27, 13]*] Assume that* $A\colon \boldsymbol{X} \to \boldsymbol{X}'$ *is continuous and that there exist a compact operator* $T\colon \boldsymbol{X} \to \boldsymbol{X}'$ *and a constant* $\alpha > 0$ *such that*

$$\operatorname{Re}\left\{\langle (A+T)\mathbf{u}, \Theta\mathbf{u}\rangle\right\} \geq \alpha \|\mathbf{u}\|_{\boldsymbol{X}}^2\,, \quad \forall\, \mathbf{u} \in \boldsymbol{X},$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $\boldsymbol{X}'$ *and* $\boldsymbol{X}$. *Assume further that* A *is one-to-one. Let* $\{\boldsymbol{X}_h\}_{h\geq 0}$ *denote a sequence of subspaces of* $\boldsymbol{X}$ *satisfying* **(A1)** *and* **(A2)**. *Then there exists a discretization level* $h_0 > 0$ *depending on* Γ_0 *and on* κ *such that for all* $\mathbf{f} \in \boldsymbol{X}'$ *and for all* $h_0 \geq h > 0$ *the Galerkin equations:*

$$\langle \mathsf{A}\,\mathbf{u}_h, \mathbf{v}_h\rangle = \langle \mathbf{f}, \mathbf{v_h}\rangle \qquad \forall\, \mathbf{v}_h \in \boldsymbol{X}_h\ ,$$

admit a unique solution $\mathbf{u}_h \in \boldsymbol{X}_h$. *The sequence* $\{\mathbf{u}_h\}_{h\leq h_0}$ *converges quasi-optimally: there exists* $C > 0$ *independent of* h *such that for* $\mathbf{u} = A^{-1}\mathbf{f} \in \boldsymbol{X}$ *it holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\boldsymbol{X}} \leq C \inf_{\mathbf{v}_h\in\boldsymbol{X}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\boldsymbol{X}}\ .$$

5.2.2. *Galerkin BEM.* We recall that the surface Γ is assumed to be a closed C^2 surface. For a dense, one parameter family $\{\boldsymbol{X}_h\}_h$ of finite dimensional subspaces of $\boldsymbol{X}(\Gamma)$, we introduce discrete Hodge-decompositions:

$$\boldsymbol{W}_h := \{\mathbf{w}_h \in \boldsymbol{X}_h\ :\ \operatorname{div}_\Gamma \mathbf{w}_h = 0\}\,, \quad \boldsymbol{V}_h := \{\mathbf{v}_h \in \boldsymbol{X}_h\ :\ \int_\Gamma \mathbf{v}_h \cdot \mathbf{w}_h = 0\ \forall\, \mathbf{w}_h \in \boldsymbol{W}_h\}\,. \tag{5.17}$$

By construction, $\boldsymbol{W}_h \subset \boldsymbol{W}$, but generally $\boldsymbol{V}_h \not\subset \boldsymbol{V}$.

Let now $\mathcal{T}_h$ be a family of regular triangulations decomposing Γ. Both *Raviart-Thomas* (RT) and the *Brezzi-Douglas-Marini* (BDM) finite elements (see e.g., [6] for definitions and properties) can be defined on $\mathcal{T}_h$ and are conforming approximations of the space $\boldsymbol{X}^0(\Gamma) := \{\mathbf{u} \in \boldsymbol{V}_\pi^0 : \operatorname{div}_\Gamma \mathbf{u} \in \boldsymbol{L}^2(\Gamma)\}$ endowed with norm:

$$\|\mathbf{u}\|_{\boldsymbol{X}^0} := \|\mathbf{u}\|_{\boldsymbol{V}_\pi^0} + \|\operatorname{div}_\Gamma \mathbf{u}\|_0 \ .$$

Note that the injection $\boldsymbol{X}^0(\Gamma) \subset \boldsymbol{X}(\Gamma)$ is dense. We denote by $\boldsymbol{X}_h$ the approximation of $\boldsymbol{X}(\Gamma)$ generated either by RT or by BDM finite elements of order $k \geq 0$ or $k \geq 1$, respectively.

Let R_h be the standard interpolation operator [6] from regular vectors on Γ_0 onto $\boldsymbol{X}_h$. We recall that this interpolation operator is obtained by defining the degrees of freedom on the reference triangle (or square) $\hat{T}$ and then transforming vectors by the standard Piola transform [6, Section III.1.3]. Moreover, moments up to order k of the normal component to the edges are among the degrees of freedom. We refer to [27] for a suitable definition on curved nominal boundaries Γ_0. The following properties on the projectors R_h have been identified in [13, 7] as sufficient for stability and consistency of Galerkin BEM:

(P1): For any $s > 0$, $\mathsf{R}_h : \boldsymbol{X}^0(\Gamma) \cap \boldsymbol{V}_\pi^s(\Gamma) \to \boldsymbol{X}_h$ is linear and uniformly continuous in h [6, Formula (3.40)] and there exists a function depending on s, denoted by δ_s, such that $\delta_s : \mathbb{R}^+ \to \mathbb{R}^+$ with $\delta_s(h) \to 0$ when $h \to 0$ and

$$\|\mathbf{u} - \mathsf{R}_h \mathbf{u}\|_{\boldsymbol{X}^0} \leq C\delta_s(h)\big(\|\mathbf{u}\|_{\boldsymbol{X}^0} + \|\mathbf{u}\|_{\boldsymbol{V}_\pi^s} \big) \ .$$

For $\tilde{\boldsymbol{V}}_h := \{\mathbf{v} \in \boldsymbol{X}(\Gamma) \ : \operatorname{div}_\Gamma \mathbf{v} \in \operatorname{div}_\Gamma(\boldsymbol{X}_h)\}$ and for $s > 0$ there holds

$$\mathbf{u} \in \tilde{\boldsymbol{V}}_h \cap \boldsymbol{V}_\pi^s(\Gamma)\,, \quad \|\mathbf{u} - \mathsf{R}_h \mathbf{u}\|_{\boldsymbol{V}_\pi^0} \leq C\delta_s(h)\, \|\mathbf{u}\|_{\boldsymbol{V}_\pi^s} \ .$$

(P2): Let L_h denote the L^2-orthogonal projection from $\boldsymbol{L}^2(\Gamma)$ onto the space $\operatorname{div}_\Gamma(\boldsymbol{X}_h)$. For RT finite elements of order k and for the BDM family of finite elements of order $k-1$, the space $\operatorname{div}_\Gamma(\boldsymbol{X}_h)$ consists of piecewise polynomials of degree k [6]. Then, for any $s > 0$ there holds [6, Proposition 3.7]:

$$\mathbf{u} \in \boldsymbol{X}^0(\Gamma) \cap \boldsymbol{V}_\pi^s(\Gamma)\,, \qquad \operatorname{div}_\Gamma(\mathsf{R}_h \mathbf{u}) = \mathsf{L}_h(\operatorname{div}_\Gamma \mathbf{u}) \ .$$

In [7, 27], (**P1**) and (**P2**) have been verified for RT and BDM boundary elements of any order k, which thus satisfy assumptions (**A1**) and (**A2**). RT_0 elements coincide in particular with the so-called RWG (Rao-Wilton-Glisson) boundary elements [34].

5.3. **Electromagnetic scattering at a perfect conductor.** In Section 2.2 we defined the perfect conductor problem. For non-smooth boundaries, the problem was considered in [9, 27, 7], where an indirect boundary element method was used. Here we use the so-called direct method where the unknown in the boundary integral equation is the Neumann trace of the domain solution. Both approaches lead to BIEs with the EFIE operator C_κ. The inf-sup condition for this operator follows as a special case from Proposition 5.3.

5.3.1. *Boundary Reduction.* We consider the unknown Neumann data:

$$\mathbf{j}^{\mathrm{pc}} = \gamma_{\mathrm{N}}^{c} \mathbf{U}^{\mathrm{pc}} \quad \text{on } \Gamma \ . \tag{5.18}$$

The properties of the Calderón projector imply

$$\left(\tfrac{1}{2}\,\mathsf{I} + \mathsf{A}_\kappa\right) \begin{pmatrix} \boldsymbol{m}^{\mathrm{pc}} \\ \mathbf{j}^{\mathrm{pc}} \end{pmatrix} = \mathbf{0} \tag{5.19}$$

if and only if there exists $\mathbf{U}^{\mathrm{pc}}$ satisfying (2.9) and (5.18). The first row of (5.19) implies

$$\mathsf{C}_\kappa\, \mathbf{j}^{\mathrm{pc}} = -(\tfrac{1}{2}\,\mathsf{I} + \mathsf{M}_\kappa)\boldsymbol{m}^{\mathrm{pc}} \quad \text{on } \Gamma \ . \tag{5.20}$$

which is known as the EFIE.

Proposition 5.5. *Assume $\kappa^2 \notin S_{Dir}$. Then the boundary integral equation* (5.20) *holds if and only if there exists $\mathbf{U}^{pc}$ satisfying* (2.9) *and* (5.18).

For a proof of Proposition 5.5, we refer to [13]. With the isomorphism Θ introduced in Proposition 5.3, we have coercivity for C_κ:

Proposition 5.6. *(*[13, Thm 5.4]*) There is a compact operator* $\mathsf{T}^{pc}\colon \boldsymbol{X}(\Gamma) \to \boldsymbol{X}(\Gamma)$ *and a constant* $\alpha^{pc} > 0$ *such that*

$$\forall \mathbf{j} \in \boldsymbol{X}(\Gamma): \qquad \operatorname{Re}\{\mathsf{b}(\Theta\mathbf{j}, (\mathsf{C}_\kappa + \mathsf{T}^{pc})\mathbf{j})\} \geq \alpha^{pc} \|\mathbf{j}\|_{\boldsymbol{X}}^2 \ . \tag{5.21}$$

For $\kappa^2 \notin S_{Dir}$*, the BIE* (5.20) *admits, for every* $\boldsymbol{m}^{pc} \in \boldsymbol{X}(\Gamma)$*, a unique solution* $\mathbf{j}^{pc} \in \boldsymbol{X}(\Gamma)$*.*

If $\kappa^2 \in S_{Dir}$*, then also* $\mathbf{j}^{pc} = \gamma_{\mathrm{N}}^{c}\mathbf{U}^{pc}$ *is a solution of* (5.20)*. The solution of* (5.20) *has the form* $\mathbf{j}^{pc} = \gamma_{\mathrm{N}}^{c}\mathbf{U}^{pc} + \gamma_{\mathrm{N}}\tilde{\mathbf{U}}^{pc}$ *where* $\tilde{\mathbf{U}}^{pc}$ *is a Maxwell solution in* D *with* $\gamma_{\mathrm{D}}\tilde{\mathbf{U}}^{pc} = 0$*. Using this* $\mathbf{j}^{pc}$ *and* $\boldsymbol{m}^{pc}$ *in the representation formula* (5.10) *gives valid Maxwell solutions according to*

$$\boldsymbol{\Psi}_E^\kappa \mathbf{j}^{pc} + \boldsymbol{\Psi}_M^\kappa \boldsymbol{m}^{pc} = \begin{cases} \mathbf{U}^{pc} & \text{in } D^c \ , \\ -\tilde{\mathbf{U}}^{pc} & \text{in } D \ . \end{cases}$$

Based on (5.21), we introduce the *bilinear forms* $\mathsf{c}_\kappa(\cdot,\cdot)$ *and* $\boldsymbol{\mu}_\kappa(\cdot,\cdot)$ *corresponding to the "conductive, electric" and "magnetic" operators in the BIE* (5.20):

$$\mathsf{c}_\kappa(\tilde{\mathbf{j}}, \mathbf{j}) := \mathsf{b}\left(\tilde{\mathbf{j}}, \mathsf{C}_\kappa \mathbf{j}\right) \ , \quad \boldsymbol{\mu}_\kappa(\tilde{\mathbf{j}}, \boldsymbol{m}) := \mathsf{b}\left(\tilde{\mathbf{j}}, (\tfrac{1}{2}\,\mathsf{I} + \mathsf{M}_\kappa)\boldsymbol{m}\right) \ , \ \mathbf{j}, \tilde{\mathbf{j}}, \boldsymbol{m} \in \boldsymbol{X}(\Gamma) \ . \tag{5.22}$$

5.3.2. *Galerkin Discretization.* Denote by a subscript $\ell \in \mathbb{N}$ the discretization level corresponding to meshwidth $h_\ell = 2^{-\ell}h_0$. We choose a family $\{\boldsymbol{X}_\ell\}_\ell$ of finite dimensional subspaces of $\boldsymbol{X}(\Gamma)$ satisfying Assumptions **(A1)** and **(A2)** of Section 5.2.1 and consider the Galerkin discretization:

$$\text{Find } \mathbf{j}_\ell^{\mathrm{pc}} \in \boldsymbol{X}_\ell: \quad \mathsf{c}_\kappa(\tilde{\mathbf{j}}_\ell, \mathbf{j}_\ell^{\mathrm{pc}}) = -\boldsymbol{\mu}_\kappa(\tilde{\mathbf{j}}_\ell, \boldsymbol{m}^{\mathrm{pc}}) \qquad \text{for all } \tilde{\mathbf{j}}_\ell \in \boldsymbol{X}_\ell. \tag{5.23}$$

This allows one to define linear mappings $\Pi_\ell^{\mathrm{pc}} : \boldsymbol{X}(\Gamma) \to \boldsymbol{X}_\ell$, the *Galerkin projectors* associated to the perfect conductor problem, with ranges $\boldsymbol{X}_\ell$, for $\ell \geq L_0$ with a certain $L_0 > 0$. We combine Propositions 5.6 and 5.4 to obtain the next result.

Theorem 5.7. *Assume* $\kappa^2 \notin S_{Dir}$ *and let* $\mathbf{j}^{pc} \in \boldsymbol{X}(\Gamma)$ *denote the solution of* (5.20)*. Then there exists* $L_0 > 0$ *and* $C^{pc} > 0$*, both dependent on* κ *and* Γ_0*, such that, for all* $\ell \geq L_0$*, the discretized problem* (5.23) *admits a unique Galerkin solution* $\mathbf{j}_\ell^{pc} \in \boldsymbol{X}_\ell$*, which converges quasi-optimally:*

$$\|\mathbf{j}^{pc} - \mathbf{j}_\ell\|_{\boldsymbol{X}} \leq C^{pc} \inf_{\tilde{\mathbf{j}}_\ell \in \boldsymbol{X}_\ell} \left\|\mathbf{j}^{pc} - \tilde{\mathbf{j}}_\ell\right\|_{\boldsymbol{X}} \ .$$

Also, the family of projectors $\{\Pi_\ell^{pc}\}_{\ell \geq L_0}$ *is stable, i.e. their norms are uniformly bounded with respect to* $\ell \geq L_0$*.*

5.4. **Electromagnetic scattering at a dielectric interface.** We now consider the transmission problem between two dielectric media with different electromagnetic properties in the two domains D and D^c and derive a boundary integral formulation. Recall the definitions given in Section 2.3. The coercivity property of the BIOs in this formulation follows from the coercivity property of the operator A_κ defined in (5.12). Here, and unlike the case of the perfect conductor problem, we need the general case of this theorem which requires the compactness property of M.

5.4.1. *Boundary integral reformulation.* Following Section 5.1.3 and simplifying notation for operators $\mathsf{A}_i, \mathsf{C}_i, \mathsf{M}_i$ for $i = 1, 2$ where subscripts refer to wavenumbers κ_i, we also introduce

$$\hat{\mathsf{A}}_i := \begin{pmatrix} 1 & 0 \\ 0 & \kappa_i\mu_i^{-1} \end{pmatrix} \mathsf{A}_i \begin{pmatrix} 1 & 0 \\ 0 & \kappa_i^{-1}\mu_i \end{pmatrix} = \begin{pmatrix} \mathsf{M}_i & \kappa_i^{-1}\mu_i\,\mathsf{C}_i \\ \kappa_i\mu_i^{-1}\,\mathsf{C}_i & \mathsf{M}_i \end{pmatrix} \ .$$

Let us recall the Cartesian product space $\boldsymbol{X}^2(\Gamma) := \boldsymbol{X}(\Gamma) \times \boldsymbol{X}(\Gamma)$ so that data $\boldsymbol{\xi}^{\mathrm{de}} := (\boldsymbol{m}^{\mathrm{de}}, \mathbf{j}^{\mathrm{de}}) \in \boldsymbol{X}^2(\Gamma)$. We consider Cauchy unknown defined by

$$\boldsymbol{\xi}^1 := \begin{pmatrix} \gamma_{\mathrm{D}} \\ \mu_1^{-1}\widehat{\gamma_{\mathrm{N}}} \end{pmatrix} \mathbf{U}^1, \quad \boldsymbol{\xi}^2 := \begin{pmatrix} \gamma_{\mathrm{D}}^c \\ \mu_2^{-1}\widehat{\gamma_{\mathrm{N}}}^c \end{pmatrix} \mathbf{U}^2, \quad \boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \boldsymbol{X}^2(\Gamma) \ . \tag{5.24}$$

The projection properties of the Calderón operator imply

$$\left(\tfrac{1}{2}\,\mathsf{I} - \hat{\mathsf{A}}_1\right)\boldsymbol{\xi}^1 = 0, \qquad \left(\tfrac{1}{2}\,\mathsf{I} + \hat{\mathsf{A}}_2\right)\boldsymbol{\xi}^2 = 0, \qquad \boldsymbol{\xi}^1 - \boldsymbol{\xi}^2 = \boldsymbol{\xi}^{\mathrm{de}} \ , \tag{5.25}$$

if and only if there exists $\mathbf{U}^{\mathrm{de}}$ satisfying (2.12) and (5.24). To derive an equivalent BIE we write $\boldsymbol{\xi}^1 = \boldsymbol{\xi}^2 + \boldsymbol{\xi}^{\mathrm{de}}$ and subtract the first equation from the second one in (5.25) to obtain the BIEs:

$$\left(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2\right) \boldsymbol{\xi}^2 = \left(\tfrac{1}{2}\,\mathsf{I} - \hat{\mathsf{A}}_1\right) \boldsymbol{\xi}^{\mathrm{de}} , \tag{5.26}$$

$$\left(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2\right) \boldsymbol{\xi}^1 = \left(\tfrac{1}{2}\,\mathsf{I} + \hat{\mathsf{A}}_2\right) \boldsymbol{\xi}^{\mathrm{de}} . \tag{5.27}$$

The BIEs (5.26)–(5.27) hold if and only if there exist $(\mathbf{U}^1, \mathbf{U}^2)$ satisfying (2.12) and (5.24) (*cf.* [13, Thm. 6.2]). The boundary integral operator $\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2$ which appears in (5.26) and (5.27) is coercive with $\Theta\colon \boldsymbol{X}(\Gamma) \to \boldsymbol{X}(\Gamma)$ from Proposition 5.3 (*cf.* [13, Thm. 6.3]).

Theorem 5.8. *There exists a compact operator $\mathsf{T}^{de}\colon \boldsymbol{X}^2(\Gamma) \to \boldsymbol{X}^2(\Gamma)$ and a constant $\alpha^{de} > 0$ such that, with the isomorphism Θ in Proposition 5.3,*

$$\forall \boldsymbol{\xi} \in \boldsymbol{X}^2(\Gamma): \qquad \operatorname{Re}\left\{ \mathsf{B}\left(\left(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2 + \mathsf{T}^{de}\right) \boldsymbol{\xi}, \Theta\boldsymbol{\xi}\right)\right\} \geq \alpha^{de} \, \|\boldsymbol{\xi}\|^2_{\boldsymbol{X}^2} . \tag{5.28}$$

with the bilinear form B as defined in (5.15).

Corollary 5.9. *The boundary integral equation* (5.27) *admits a unique solution $\boldsymbol{\xi}^1 \in \boldsymbol{X}^2(\Gamma)$.*

5.4.2. *Galerkin discretization.* We choose a family $\{\boldsymbol{X}_\ell\}_{\ell\geq 1}$ of finite dimensional subspaces of $\boldsymbol{X}(\Gamma)$ satisfying assumptions **(A1)** and **(A2)** of Section 5.2.1 and consider the Galerkin discretization: find $\boldsymbol{\xi}^1_\ell \in \boldsymbol{X}^2_\ell := \boldsymbol{X}_\ell \times \boldsymbol{X}_\ell$ such that

$$\mathsf{B}\left(\left(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2\right) \boldsymbol{\xi}^1_\ell, \tilde{\boldsymbol{\xi}}_\ell\right) = \mathsf{B}\left(\left(\tfrac{1}{2}\,\mathsf{I} + \hat{\mathsf{A}}_2\right) \boldsymbol{\xi}^{\mathrm{de}}, \tilde{\boldsymbol{\xi}}_\ell\right) \qquad \text{for all } \tilde{\boldsymbol{\xi}}_\ell \in \boldsymbol{X}^2_\ell . \tag{5.29}$$

The above induces mappings $\Pi^{\mathrm{de}}_\ell : \boldsymbol{X}^2(\Gamma) \to \boldsymbol{X}^2_\ell : \boldsymbol{\xi}^2 \mapsto \boldsymbol{\xi}^2_\ell$, which constitute as before a family Galerkin projectors associated to the dielectric problem for $\ell \geq L_0$, with L_0 as indicated below. Propositions 5.8 and 5.4 immediately yield the next result:

Theorem 5.10. *(*[13, Thm. 6.5]*) Let $\boldsymbol{\xi}^1 \in \boldsymbol{X}^2(\Gamma)$ denote the solution of* (5.27)*. There exists $L_0 > 0$ and C^{de}, dependent on κ_i, $i = 1, 2$, such that for all $\ell \geq L_0$ the Galerkin equations* (5.29) *admit a unique solution $\boldsymbol{\xi}^1_\ell \in \boldsymbol{X}^2_\ell$.*

The family $\{\Pi^{de}_\ell\}_\ell$ is uniformly bounded with respect to $\ell \geq L_0$. The approximate solutions $\boldsymbol{\xi}^1_\ell = \Pi^{de}_\ell \boldsymbol{\xi}^1$ converge quasi-optimally:

$$\left\|\boldsymbol{\xi}^1 - \boldsymbol{\xi}^1_\ell\right\|_{\boldsymbol{X}^2} \leq C_0 \inf_{\tilde{\boldsymbol{\xi}}_\ell \in \boldsymbol{X}^2_\ell} \left\|\boldsymbol{\xi}^1 - \tilde{\boldsymbol{\xi}}_\ell\right\|_{\boldsymbol{X}^2} .$$

6. Sparse Tensor Second Moment Galerkin Approximation

The BIEs arising from the direct method of boundary reduction of problems defining the shape gradients portray the generic form $Au = Bf$. In the present section, we present the boundary reduction of the nominal problems for the model problems considered in Section 2. We indicate the strong ellipticity of the resulting BIOs, following [13], and quasi-optimality of their Galerkin discretizations.

6.1. Tensor BIEs.

The FOSM perturbation analysis of the model problems in Sections 2.2 and 2.3 leads to tensorized BIEs on the nominal boundaries Γ_0; we are then faced with tensorized versions of the abstract BIE $Au = Bf$ with deterministic operators A and B, but with random right-hand side f. Our FOSM analysis is based on a deterministic equation for the second moment of the random solution u.

Proposition 6.1. *Assume given $A \in \mathcal{L}(X, Z)$, $B \in \mathcal{L}(Y, Z)$ for three Hilbert spaces X, Y, Z, with A boundedly invertible. Then, for $f \in L^2(\Omega, \mathbb{P}; Y)$, the solution of the operator stochastic equation:*

$$Au(\omega) = Bf(\omega) \tag{6.1}$$

admits a unique solution $u \in L^2(\Omega, \mathbb{P}; X)$ whose second moment, $u^{(2)} := \mathbb{E}[u \otimes u] \in X^{(2)}$, is the unique solution of the deterministic tensor operator equation:

$$(A \otimes A) u^{(2)} = (B \otimes B) \mathcal{M}^2 f \quad \text{in} \quad Z^{(2)} , \tag{6.2}$$

where $\mathcal{M}^2 f := \mathbb{E}[f \otimes f] \in Y^{(2)}$.

Proof. The bounded invertibility of A and the linearity of equation (6.1) imply $u(\omega) = A^{-1}Bf(\omega)$ and, since A and B are deterministic,

$$u^{(2)} = \mathbb{E}[u \otimes u] = \mathbb{E}\left[(A^{-1}Bf) \otimes (A^{-1}Bf)\right] = (A \otimes A)^{-1}(B \otimes B)\mathbb{E}[f \otimes f]\,, \tag{6.3}$$

as claimed. □

The preceding abstract result will apply in particular to the boundary integral reformulations of the Maxwell problems (3.4)–(3.5) and (3.8)–(3.9) which characterize the shape gradients for our model problems. We next specify it for these problems, indicating in each case the particular choices for A, B and f.

6.1.1. *Perfect Conductor Problem.* Comparing (3.4)–(3.5) with (2.2)–(2.9), the shape derivative $d\mathbf{U}_0^{\mathrm{pc}}(\eta(\cdot,\omega)\mathbf{n}_0)$ admits the integral representation (5.10). Specifically, we seek a random electrical current[5] $d\mathbf{j}^{\mathrm{pc}}(\omega)$ that originates the solution $d\mathbf{U}_0^{\mathrm{pc}}(\eta(\cdot,\omega)\mathbf{n}_0) \in \boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)$ by satisfying the BIE (5.20):

$$\mathsf{C}_\kappa\, d\mathbf{j}^{\mathrm{pc}}(\omega) = -\left(\frac{1}{2}\mathsf{I} + \mathsf{M}_\kappa\right)\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}(\mathbf{U}_0^{\mathrm{pc}}, \mathbf{U}^{\mathrm{inc}}, \eta(\cdot,\omega)\mathbf{n}_0)\,, \tag{6.4}$$

where $\mathbf{U}_0^{\mathrm{pc}}$ and $\mathbf{U}^{\mathrm{inc}}$ are the scattered and incident volume electric fields for the nominal boundary problem as described in Section 3.2. The BIE (6.4) is of the general form (6.1) with the instances[6]

$$X, Y, Z \leftarrow \boldsymbol{X}(\Gamma_0)\,, \quad A \leftarrow \mathsf{C}_\kappa\,, \quad B \leftarrow -\left(\frac{1}{2}\mathsf{I} + \mathsf{M}_\kappa\right)\,, \quad f \leftarrow \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}(\mathbf{U}_0^{\mathrm{pc}}, \mathbf{U}^{\mathrm{inc}}, \eta(\cdot,\omega)\mathbf{n}_0)\,. \tag{6.5}$$

By Proposition 4.2, $\eta \in L^2(\Omega, \mathbb{P}; C^2(\Gamma_0))$ implies $\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}} \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}(\Gamma_0))$. Let $\kappa \notin S_{\mathrm{Dir}}$, Theorem 5.6 establishes the bounded invertibility of C_κ and, together with its linearity as well as that of $\frac{1}{2}\mathsf{I} + \mathsf{M}_\kappa$, one obtains that $\mathbb{P}$-a.s. there exists a unique solution $d\mathbf{j}^{\mathrm{pc}} \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}(\Gamma_0))$ of the BIE (6.4). By the representation formula (5.10), it holds

$$d\mathbf{U}_0^{\mathrm{pc}}(\eta(\cdot,\omega)\mathbf{n}_0) = -\begin{pmatrix}\boldsymbol{\Psi}_E^\kappa \\ \boldsymbol{\Psi}_M^\kappa\end{pmatrix}^\top \begin{pmatrix} d\mathbf{j}^{\mathrm{pc}}(\omega) \\ \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}(\omega)\end{pmatrix} \in L^2(\Omega, \mathbb{P}; \boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)) \quad \text{in} \quad D_0^c,$$

due to $d\mathbf{j}^{\mathrm{pc}}$, $\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}(\omega)$ in $L^2(\Omega, \mathbb{P}; \boldsymbol{X}(\Gamma_0))$ and due to the linearity of the representation formula.

From the shape Taylor expansion (4.14) and again from (5.10), we find the *first order deterministic approximation for the second moment of the scattered field in the volume D_0^c by tensorizing the representation formula* (5.10): we evaluate the tensor product expectation of $(\check{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega) - \mathbf{U}_0^{\mathrm{pc}})\big|_{D_0^c}$ as follows

$$\begin{aligned}\mathbb{E}\left[(\check{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega) - \mathbf{U}_0^{\mathrm{pc}}) \otimes (\check{\mathbf{U}}_\delta^{\mathrm{pc}}(\omega) - \mathbf{U}_0^{\mathrm{pc}})\right] &= \delta^2 \mathbb{E}[d\mathbf{U}_0^{\mathrm{pc}} \otimes d\mathbf{U}_0^{\mathrm{pc}}] + \mathcal{O}(\delta^3) \\ &= \delta^2 \begin{pmatrix}\boldsymbol{\Psi}_E^\kappa \\ \boldsymbol{\Psi}_M^\kappa\end{pmatrix}^\top \otimes \begin{pmatrix}\boldsymbol{\Psi}_E^\kappa \\ \boldsymbol{\Psi}_M^\kappa\end{pmatrix}^\top \mathbb{E}\left[\begin{pmatrix} d\mathbf{j}^{\mathrm{pc}} \\ \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}\end{pmatrix} \otimes \begin{pmatrix} d\mathbf{j}^{\mathrm{pc}} \\ \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}\end{pmatrix}\right] + \mathcal{O}(\delta^3)\end{aligned} \tag{6.6}$$

which holds in $\boldsymbol{H}_\kappa(\mathbf{curl}, D_0^c)^{(2)}$.

By Proposition 6.1 with (6.5), the second moment $(d\mathbf{j}^{\mathrm{pc}})^{(2)} := \mathbb{E}[d\mathbf{j}^{\mathrm{pc}} \otimes d\mathbf{j}^{\mathrm{pc}}] \in \boldsymbol{X}^{(2)}(\Gamma_0)$ solves the tensorized, deterministic BIE: find $(d\mathbf{j}^{\mathrm{pc}})^{(2)} \in \boldsymbol{X}^{(2)}(\Gamma_0)$ such that, by (6.5), it holds

$$\mathsf{C}_\kappa^{(2)}(d\mathbf{j}^{\mathrm{pc}})^{(2)} = \left(\frac{1}{2}\mathsf{I} + \mathsf{M}_\kappa\right)^{(2)} \mathcal{M}^2 \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}\,, \tag{6.7}$$

where $\mathcal{M}^2\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}} = \mathbb{E}\left[\mathbf{g}_{\mathrm{D}}^{\mathrm{pc}} \otimes \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}}\right] \in \boldsymbol{X}^{(2)}(\Gamma_0)$ and tensor operators $\mathsf{C}_\kappa^{(2)}$ and $\left(\frac{1}{2}\mathsf{I} + \mathsf{M}_\kappa\right)^{(2)}$ are defined as in (4.7). Observe that the minus sign on the right-hand side of (6.4) is cancelled in the tensorization.

By the bounded invertibility of the boundary integral operator C_κ on $\boldsymbol{X}(\Gamma_0)$ which follows from $\kappa \notin S_{\mathrm{Dir}}$ and from Theorem 5.6, the tensorized operator $\mathsf{C}_\kappa^{(2)}$, appearing in (6.7) is likewise

[5]Observe that $d\mathbf{j}^{\mathrm{pc}}$ is not necessarily the shape derivative of current $\mathbf{j}^{\mathrm{pc}}$. In fact, it can only be interpreted as a density originating the field shape derivative at Γ_0.

[6]We specify the boundary on which $\boldsymbol{X}$ is built, in this case $\boldsymbol{X}(\Gamma_0)$.

boundedly invertible so that (6.7) admits a unique solution $(d\mathbf{j}^{\mathrm{pc}})^{(2)}$ which coincides with the second moment of the random Cauchy data $d\mathbf{j}^{\mathrm{pc}} \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}(\Gamma_0))$.

Remark 6.2. If $\kappa \in S_{\mathrm{Dir}}$, the spectrum of the interior Dirichlet problem *in the nominal domain* D_0, the BIE still provides the correct solution in the exterior domain D_0^c; in D_0, however, the solution is unique only up to "resonant modes" (*cf.* comments on [13, p. 480]). We emphasize that this nonuniqueness is a consequence of the presently adopted, direct method of boundary reduction for the shape gradients presented in Propositions 4.2, 4.3. These "volume Maxwell problems" admit unique solutions, for all admissible shapes, due to the Silver-Müller conditions. An alternative boundary reduction of the problems in Propositions 4.2, 4.3 which does not exhibit resonances is via the so-called *Combined Field Integral Equation (CFIE) approach* in [11]. As this approach involves the same integral operators as the presently considered direct method of boundary reduction, the ensuing FoSM analysis and their sparse tensor Galerkin discretization can be performed along the same lines, with analogous error bounds and complexity.

6.1.2. *Dielectric Interface Problem.* Comparison between Problems (3.8)–(3.9) and (2.12)–(2.13) reveals that the shape derivatives $d\mathbf{U}_0^{\mathrm{de}}(\eta(\cdot,\omega)\mathbf{n}_0) = (d\mathbf{U}_0^1, d\mathbf{U}_0^2)(\eta(\cdot,\omega)\mathbf{n}_0)$ also admit the integral representation (5.10) in terms of random Cauchy data[7] $d\boldsymbol{\xi}^1(\omega)$ of the solution $d\mathbf{U}_0^1(\eta(\cdot,\omega)\mathbf{n}_0) \in \boldsymbol{H}(\mathbf{curl}, D_0)$. The reformulation of (3.8)–(3.9) in terms of the BIE (5.27) then reads

$$(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2) d\boldsymbol{\xi}^1(\omega) = \left(\frac{1}{2}\mathsf{I} + \hat{\mathsf{A}}_2\right) \mathbf{g}^{\mathrm{de}}(\mathbf{U}_0^{\mathrm{de}}, \mathbf{U}^{\mathrm{inc}}, \eta(\cdot,\omega)\mathbf{n}_0)\ . \tag{6.8}$$

This equation is of the general form (6.1) with the instances:

$$X, Y, Z \leftarrow \boldsymbol{X}^2(\Gamma_0)\ , \quad A \leftarrow \hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2\ , \quad B \leftarrow \frac{1}{2}\mathsf{I} + \hat{\mathsf{A}}_2\ , \quad f \leftarrow \mathbf{g}^{\mathrm{de}}(\mathbf{U}_0^{\mathrm{de}}, \mathbf{U}^{\mathrm{inc}}, \eta(\cdot,\omega)\mathbf{n}_0)\ . \tag{6.9}$$

By Proposition 4.3, $\eta \in L^2(\Omega, \mathbb{P}; C^2(\Gamma_0))$ implies $f \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}^2(\Gamma_0))$. The bounded invertibility of $\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2$ established in Theorem 5.8 and the linearity of $\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2$ and of $\frac{1}{2}\mathsf{I} + \hat{\mathsf{A}}_2$ then implies that there exists a unique solution $d\boldsymbol{\xi}^1 \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}^2(\Gamma_0))$ of the BIE (6.8). By (5.24) and the representation formula (5.10), we also have

$$d\mathbf{U}_0^1(\eta(\cdot,\omega)\mathbf{n}_0) = -\begin{pmatrix} \boldsymbol{\Psi}_E^{\kappa_1} \\ \boldsymbol{\Psi}_M^{\kappa_1} \end{pmatrix}^\top d\boldsymbol{\xi}^1(\omega) \in \boldsymbol{L}^2(\Omega, \mathbb{P}; \boldsymbol{H}(\mathbf{curl}, D_0)) \quad \text{in} \quad D_0\ ,$$

with an analogous representation of $d\mathbf{U}_0^2(\eta(\cdot,\omega)\mathbf{n}_0)$ in terms of $d\boldsymbol{\xi}^2(\omega)$. Here, the square integrability of $d\mathbf{U}_0^1$ follows from $d\boldsymbol{\xi}^1(\omega) \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}^2(\Gamma_0))$ and the linearity of the representation formula.

From the shape Taylor expansion (4.15), we find the first order deterministic approximation for the second moment of $(\check{\mathbf{U}}_\delta^1(\omega) - \mathbf{U}_0^1)|_{D_0}$, valid in $\boldsymbol{H}(\mathbf{curl}, D_0)^{(2)}$:

$$\begin{aligned} \mathbb{E}\left[(\check{\mathbf{U}}_\delta^1(\omega) - \mathbf{U}_0^1) \otimes (\check{\mathbf{U}}_\delta^1(\omega) - \mathbf{U}_0^1)\right] &= \delta^2 \mathbb{E}[d\mathbf{U}_0^1 \otimes d\mathbf{U}_0^1] + O(\delta^3) \\ &= \delta^2 \begin{pmatrix} \boldsymbol{\Psi}_E^{\kappa_1} \\ \boldsymbol{\Psi}_M^{\kappa_1} \end{pmatrix}^\top \otimes \begin{pmatrix} \boldsymbol{\Psi}_E^{\kappa_1} \\ \boldsymbol{\Psi}_M^{\kappa_1} \end{pmatrix}^\top \mathbb{E}[d\boldsymbol{\xi}^1 \otimes d\boldsymbol{\xi}^1] + O(\delta^3)\ . \end{aligned}$$

An analogous expression holds for $(\check{\mathbf{U}}_\delta^2(\omega) - \mathbf{U}_0^2)|_{D_0^c}$ in $H_{\kappa_2}(\mathbf{curl}, D_0^c)^{(2)}$.

By Proposition 6.1 with (6.9), the second moment $(d\boldsymbol{\xi}^1)^{(2)} := \mathbb{E}[d\boldsymbol{\xi}^1 \otimes d\boldsymbol{\xi}^1] \in (\boldsymbol{X}^2)^{(2)}(\Gamma_0)$ solves the tensorized, deterministic BIE: find $(d\boldsymbol{\xi}^1)^{(2)} \in (\boldsymbol{X}^2)^{(2)}(\Gamma_0)$ such that

$$(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2)^{(2)} (d\boldsymbol{\xi}^1)^{(2)} = \left(\frac{1}{2}\mathsf{I} + \hat{\mathsf{A}}_2\right)^{(2)} \mathcal{M}^2 \mathbf{g}^{\mathrm{de}}\ . \tag{6.10}$$

By the bounded invertibility of $\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2$ on $\boldsymbol{X}^2(\Gamma_0)$, the tensorized operator $(\hat{\mathsf{A}}_1 + \hat{\mathsf{A}}_2)^{(2)}$, defined by (4.7), appearing in (6.10) is likewise boundedly invertible on $(\boldsymbol{X}^2)^{(2)}(\Gamma_0)$ so that the tensorized BIE (6.10) admits a unique solution $(d\boldsymbol{\xi}^1)^{(2)}$ which coincides with the two-point correlation of the random Cauchy data $\boldsymbol{\xi}^1(\omega) \in L^2(\Omega, \mathbb{P}; \boldsymbol{X}^2(\Gamma_0))$.

[7] As before, $d\boldsymbol{\xi}^1(\omega)$ is not the shape derivative of the Cauchy data $\boldsymbol{\xi}^1(\omega)$ but the solution pair originating the field $d\mathbf{U}^1$ in Ω_0.

6.2. **Sparse Tensor Galerkin BEM.** Stable Galerkin discretizations of the tensorized BIEs (6.2), (6.7) and (6.10), are now obtained as in [26]. Since we will henceforth work over the nominal domain surface Γ_0, we simply write $\boldsymbol{X} \equiv \boldsymbol{X}(\Gamma_0)$. Consider a sequence of nested, regular triangulations $\{\mathcal{T}_\ell\}_{\ell\geq 0}$ of the nominal boundary Γ_0, and set $h_\ell = \max_{K\in\mathcal{T}_\ell} \operatorname{diam}(K)$. Taking $h_\ell \sim O(2^{-\ell})$, we indicate the nested triangulations by writing $\mathcal{T}_0 \preceq \mathcal{T}_1 \preceq ... \preceq \mathcal{T}_\ell \preceq ....$ On the sequence $\{\mathcal{T}_\ell\}_{\ell\geq 0}$ of triangulations of the nominal boundary Γ_0, we introduce a family $\{\boldsymbol{X}_{h_\ell}\}_{h_\ell}$ of finite dimensional subspaces of $\boldsymbol{X}$ satisfying Assumptions **(A1)** and **(A2)** of Section 5.2.1 and in particular the discrete decompositions introduced in Section 5.2.

In the construction of sparse tensor product spaces from [26], the explicit dependence of the subspace on the discretization level ℓ in the sequence $\{\mathcal{T}_\ell\}_{\ell\geq 0}$ will be important; to indicate it, we write from now on $\{\boldsymbol{X}_\ell\}_{\ell\geq 0}$ in place of $\{\boldsymbol{X}_{h_\ell}\}_{h_\ell}$. Due to the nestedness $\mathcal{T}_0 \preceq \mathcal{T}_1 \preceq ... \preceq \mathcal{T}_\ell \preceq ...$, we obtain $\boldsymbol{X}_0 \subset \boldsymbol{X}_1 \subset ....$ It follows readily from Theorems 5.7 and 5.10 that for either the tensor BIEs (6.7) or (6.10) –the FOSM shape uncertainty of the perfect conductor and dielectric interface problems, respectively– for $\ell \geq L_0$, where L_0 is as in Theorems 5.7 and 5.10, the Galerkin solutions converge quasi-optimally. Being tensor products of boundary element spaces $\boldsymbol{X}_\ell$ (resp. $\boldsymbol{X}_\ell^{(2)}$) entails quadratic complexity of $\dim(\boldsymbol{X}_\ell)$.

Based on our previous work for the EFIE in [26], we present a so-called sparse tensor product Galerkin discretization of the BIEs (6.7), (6.10). To this end, introduce the index set:

$$\Lambda(L, L_0) := \{(\ell, \ell') \in \mathbb{N}_0 : 0 \leq \ell, \ell' \leq L_0 + L,\ \ \ell + \ell' \leq 2L_0 + L\}\ , \tag{6.11}$$

for discretization levels $L, L_0 \geq 0$, with $L_0 > 0$ to be specified, and recall from [39, 26], the *sparse tensor product spaces*:

$$\widehat{\boldsymbol{X}_\ell^{(2)}} \subset \boldsymbol{X}_\ell^{(2)} \quad \text{and} \quad \widehat{(\boldsymbol{X}_\ell^2)^{(2)}} \subset (\boldsymbol{X}_\ell^2)^{(2)},$$

defined by the non-direct sums:

$$\widehat{\boldsymbol{X}_L^{(2)}} := \sum_{(\ell,\ell')\in\Lambda(L,L_0)} \boldsymbol{X}_\ell \otimes \boldsymbol{X}_{\ell'}\ , \quad \widehat{(\boldsymbol{X}_L^2)^{(2)}} := \sum_{(\ell,\ell')\in\Lambda(L,L_0)} \boldsymbol{X}_\ell^2 \otimes \boldsymbol{X}_{\ell'}^2\ . \tag{6.12}$$

Here, we used the convention $\boldsymbol{X}_\ell := \{0\}$ and $\boldsymbol{X}_\ell^2 := \{0\} \times \{0\}$ for $\ell < 0$. Note that for every $L \geq 0$ we have

$$\boldsymbol{X}_{L_0}^{(2)} \subset \widehat{\boldsymbol{X}_L^{(2)}}\ , \quad (\boldsymbol{X}_{L_0}^2)^{(2)} \subset \widehat{(\boldsymbol{X}_L^2)^{(2)}}\ . \tag{6.13}$$

Based on these spaces, the *sparse tensor product Galerkin BEM* are defined next:

I. For the *perfect conductor BIE* (6.7) we seek $\widehat{(d\mathbf{j}^{\rm pc})_L^{(2)}} \in \widehat{\boldsymbol{X}_L^{(2)}}$ such that

$$\mathsf{c}_\kappa^{(2)}\left(\tilde{\mathbf{j}}_L^{(2)}, \widehat{(d\mathbf{j}^{\rm pc})_L^{(2)}}\right) = \boldsymbol{\mu}_\kappa^{(2)}\left(\tilde{\mathbf{j}}_L^{(2)}, \mathcal{M}^2\mathbf{g}_{\rm D}^{\rm pc}\right) \quad \forall\, \tilde{\mathbf{j}}_L^{(2)} \in \widehat{\boldsymbol{X}_L^{(2)}}, \tag{6.14}$$

where the bilinear forms $\mathsf{c}_\kappa^{(2)}$ and $\boldsymbol{\mu}_\kappa^{(2)}$ mapping $\boldsymbol{X}^{(2)} \times \boldsymbol{X}^{(2)} \to \mathbb{C}$ are defined as follows: for dyads $\mathbf{j}_1 \otimes \mathbf{j}_2, \tilde{\mathbf{j}}_1 \otimes \tilde{\mathbf{j}}_2 \in \boldsymbol{X}^{(2)}$, $\mathsf{c}_\kappa^{(2)}\left(\tilde{\mathbf{j}}_1 \otimes \tilde{\mathbf{j}}_2, \mathbf{j}_1 \otimes \mathbf{j}_2\right) := \mathsf{c}_\kappa(\tilde{\mathbf{j}}_1, \mathbf{j}_1)\mathsf{c}_\kappa(\tilde{\mathbf{j}}_2, \mathbf{j}_2)$. This definition extends by bilinearity to finite linear combinations of dyads. As $\boldsymbol{X}$ is a separable Hilbert space, any $\mathbf{j}^{(2)} \in \boldsymbol{X}^{(2)}$ is the limit, in the norm $\|\cdot\|_{\boldsymbol{X}^{(2)}}$, of finite linear combinations of dyads. By the continuity of $\mathsf{c}_\kappa(\cdot,\cdot)$ on $\boldsymbol{X} \times \boldsymbol{X}$, we may pass to the limit:

$$\forall \mathbf{j}^{(2)}, \tilde{\mathbf{j}}^{(2)} \in \boldsymbol{X}^{(2)} : \mathsf{c}_\kappa^{(2)}\left(\tilde{\mathbf{j}}^{(2)}, \mathbf{j}^{(2)}\right) = \lim_{I\to\infty} \sum_{i,i'=1}^{I} c_i c_{i'} \mathsf{c}_\kappa^{(2)}\left(\tilde{\mathbf{j}}_{1,i'} \otimes \tilde{\mathbf{j}}_{2,i'}, \mathbf{j}_{1,i} \otimes \mathbf{j}_{2,i}\right)$$

and obtain a continous bilinear form $\mathsf{c}_\kappa^{(2)}(\cdot,\cdot)$ on $\boldsymbol{X}^{(2)} \times \boldsymbol{X}^{(2)}$. The definitions of $\boldsymbol{\mu}_\kappa^{(2)}(\cdot,\cdot)$ and $\mathsf{B}^{(2)}(\cdot,\cdot)$ for $\mathsf{B}(\cdot,\cdot)$ as in (5.15) are analogous.

II. For the corresponding *dielectric tensor product BIE* (6.10): find $\widehat{(d\boldsymbol{\xi}^1)_L^{(2)}} \in \widehat{(\boldsymbol{X}_L^2)^{(2)}}$ such that for all $\tilde{\boldsymbol{\xi}}_L^{(2)} \in \widehat{(\boldsymbol{X}_L^2)^{(2)}}$ holds

$$\mathsf{B}^{(2)}\left(\tilde{\boldsymbol{\xi}}_L^{(2)},\left(\hat{\mathsf{A}}_1+\hat{\mathsf{A}}_2\right)^{(2)}\widehat{(d\boldsymbol{\xi}^1)_L^{(2)}}\right)=\mathsf{B}^{(2)}\left(\tilde{\boldsymbol{\xi}_L}^{(2)},\left(\frac{1}{2}\mathsf{I}-\hat{\mathsf{A}}_2\right)^{(2)}\begin{pmatrix}\mathcal{M}^2\mathbf{g}_{\mathrm{D}}^{\mathrm{de}}\\ \mathcal{M}^2\mathbf{g}_{\mathrm{N}}^{\mathrm{de}}\end{pmatrix}\right), \tag{6.15}$$

A counting argument shows that there exist constants $C_1, C_2 > 0$, depending on L_0 but not on L, such that it holds

$$\dim(\widehat{\boldsymbol{X}_L^{(2)}}) \le C_1 \dim(\boldsymbol{X}_L)\log(\dim(\boldsymbol{X}_L)) , \quad \dim(\widehat{(\boldsymbol{X}_L^2)^{(2)}}) \le C_2 \dim(\boldsymbol{X}_L)\log(\dim(\boldsymbol{X}_L)). \tag{6.16}$$

Sparse tensor products of boundary edge element spaces are stable [26, Thm. 5.1].

Theorem 6.3. *There exists a level L_0 sufficiently large, depending on D_0 and on the wavenumbers κ, κ_1, κ_2, and positive constants C^{pc}, C^{de}, such that for all $L \ge 0$ hold* discrete inf-sup conditions*:*

$$\inf_{0\neq \mathbf{j}_L^{(2)}\in\widehat{\boldsymbol{X}_L^{(2)}}}\ \sup_{0\neq\tilde{\mathbf{j}}_L^{(2)}\in\widehat{\boldsymbol{X}_L^{(2)}}} \frac{\mathsf{c}_\kappa^{(2)}\left(\tilde{\mathbf{j}}_L^{(2)},\mathbf{j}_L^{(2)}\right)}{\left\|\mathbf{j}_L^{(2)}\right\|_{\boldsymbol{X}^{(2)}}\left\|\tilde{\mathbf{j}}_L^{(2)}\right\|_{\boldsymbol{X}^{(2)}}} \ge C^{pc}, \tag{6.17}$$

and

$$\inf_{0\neq \boldsymbol{\xi}_L^{(2)}\in\widehat{(\boldsymbol{X}_L^2)^{(2)}}}\ \sup_{0\neq\tilde{\boldsymbol{\xi}}_L^{(2)}\in\widehat{(\boldsymbol{X}_L^2)^{(2)}}} \frac{\mathsf{B}^{(2)}\left(\tilde{\boldsymbol{\xi}}_L^{(2)},\left(\hat{\mathsf{A}}_1+\hat{\mathsf{A}}_2\right)^{(2)}\boldsymbol{\xi}_L^{(2)}\right)}{\left\|\tilde{\boldsymbol{\xi}}_L^{(2)}\right\|_{(\boldsymbol{X}^2)^{(2)}}\left\|\boldsymbol{\xi}_L^{(2)}\right\|_{(\boldsymbol{X}^2)^{(2)}}} \ge C^{de}. \tag{6.18}$$

In particular, with this choice of L_0, for every $L \ge 0$ the sparse tensor Galerkin boundary integral equations (6.14), (6.15) *admit unique solutions $\widehat{(d\mathbf{j}^{pc})_L^{(2)}} \in \widehat{\boldsymbol{X}_L^{(2)}}$ and $\widehat{(d\boldsymbol{\xi}^1)_L^{(2)}} \in \widehat{(\boldsymbol{X}_L^2)^{(2)}}$ which converge quasi-optimally in $\boldsymbol{X}^{(2)}$ and in $(\boldsymbol{X}^2)^{(2)}$, respectively.*

7. Algorithmic Realization, Error Analysis and Complexity

Theorem 6.3 implies that the Galerkin projections onto the sparse tensor product spaces $\widehat{\boldsymbol{X}_L^{(2)}}$ and $\widehat{(\boldsymbol{X}_L^2)^{(2)}}$ are well-defined and stable. We address the computational realization of the sparse tensor Galerkin projections from Theorems 5.7 and 5.10. These projections are easily realized when bases for the "detail spaces" $\boldsymbol{X}_\ell \cap \boldsymbol{X}_{\ell-1}^{\perp}$ are available; such bases with good condition properties are furnished, for example, by multi-resolution wavelet bases of $\boldsymbol{X}$. Unfortunately, the construction of stable, piecewise polynomial bases for general surfaces Γ_0 does not appear to be available, currently (see, however, [3] for an "almost stable" construction).

Even if available, their implementation would mandate modification of the existing EFIE Galerkin codes which are, as a rule, based on the RWG (Rao-Wilton-Glisson) boundary elements [34] (which correspond on regular triangulations $\mathcal{T}$ of Γ_0 to RT elements of order zero and satisfy conditions (A1) and (A2)). We therefore outline in Section 7.1 an alternative approach for the computational realization of the Galerkin projections for $L \ge L_0$ which is not based on the explicit availability of such bases, but involves only the 'usual' one-scale RWG type basis on each level, and is based on the so-called *combination technique*. Its use in the context of sparse tensor product Galerkin discretizations for potential and Helmholtz problems was proposed recently in [22]. Here, we briefly review this approach, and indicate the essential modifications entailed by the more complicated structure of the index set $\Lambda(L, L_0)$ defined in (6.11).

7.1. **Realization of Tensorized Boundary Element Basis.**

From Theorems 5.7 and 5.10 we recall the corresponding Galerkin projectors Π_ℓ^{pc} and Π_ℓ^{de} with ranges in $\boldsymbol{X}_\ell$ and $\boldsymbol{X}_\ell^2$, respectively. We only detail the analysis for the perfect conductor case, as similar considerations apply for the dielectric case. Accordingly, in the following we set $\Pi_\ell \equiv \Pi_\ell^{\mathrm{pc}}$ to ease notation. Also recall $\mathsf{c}_\kappa(\tilde{\mathbf{j}},\mathbf{j}) : \boldsymbol{X}\times\boldsymbol{X}\to\mathbb{C}$ the bilinear form corresponding to the electric field integral operator C_κ from (5.22). The definition of the Galerkin projection Π_ℓ implies the *Galerkin orthogonality*:

$$\forall\, \mathbf{j}\in\boldsymbol{X}: \qquad \mathsf{c}_\kappa(\tilde{\mathbf{j}}_\ell, \mathbf{j}-\Pi_\ell\mathbf{j}) = 0 \qquad \forall\, \tilde{\mathbf{j}}_\ell\in\boldsymbol{X}_\ell\,. \tag{7.1}$$

By Theorem 6.3, the sparse tensor product Galerkin projectors $\widehat{\Pi_L^{(2)}}$ are, for $L \geq L_0$, well defined, stable –i.e. bounded independent of L– and admit, for $L \geq L_0$, the representation:

$$\widehat{\Pi_L^{(2)}} = \Pi_{L_0}^{(2)} + \sum_{(\ell,\ell')\in\hat{\Lambda}(L,L_0)} (\Pi_\ell - \Pi_{\ell-1}) \otimes (\Pi_{\ell'} - \Pi_{\ell'-1}) \ . \tag{7.2}$$

Here, for $0 \leq L_0 < L$, we defined the index set:

$$\hat{\Lambda}(L,L_0) = \{(\ell,\ell') \in \mathbb{N}_0^2 \ : \ L_0 < \ell,\ell' \leq L, \ \ell+\ell' \leq 2L_0 + L\} \subset \Lambda(L,L_0) \ . \tag{7.3}$$

With the help of Galerkin projectors $\Pi_\ell : \boldsymbol{X} \to \boldsymbol{X}_\ell$, we define the spaces:

$$\boldsymbol{Y}_\ell := (\Pi_\ell - \Pi_{\ell-1})\boldsymbol{X}_\ell \ , \quad \ell \geq L_0 + 1 \ . \tag{7.4}$$

Then, for every $L > 0$, holds the multi-level splitting via direct sums:

$$\boldsymbol{Y}_\ell \subset \boldsymbol{X}_\ell \ , \ \ell \geq L_0 + 1 \quad \text{and} \quad \boldsymbol{X}_{L_0+L} = \boldsymbol{X}_{L_0+L-1} \oplus \boldsymbol{Y}_{L_0+L} = ... = \boldsymbol{X}_{L_0} \oplus \bigoplus_{\ell=L_0+1}^{L_0+L} \boldsymbol{Y}_\ell \ . \tag{7.5}$$

The following decomposition of the sparse tensor product space $\widehat{\boldsymbol{X}_L^{(2)}}$ defined in (6.12) for $L, L_0 \geq 0$ corresponds to (7.2):

$$\begin{aligned}
\widehat{\boldsymbol{X}_L^{(2)}} &= \boldsymbol{X}_{L_0}^{(2)} \oplus \bigoplus_{(\ell,\ell')\in\hat{\Lambda}(L,L_0)} \boldsymbol{Y}_\ell \otimes \boldsymbol{Y}_{\ell'} \\
&= \boldsymbol{X}_{L_0}^{(2)} \oplus \left(\bigoplus_{\{L_0+1\leq \ell,\ell'\leq L\}\cap\{\ell+\ell'\leq 2L_0+L\}} \boldsymbol{Y}_\ell \otimes \boldsymbol{Y}_{\ell'} \right) \\
&\quad \oplus \boldsymbol{X}_{L_0} \otimes \left(\boldsymbol{X}_{L_0} \oplus \bigoplus_{\ell=L_0+1}^{L_0+L} \boldsymbol{Y}_\ell \right) \oplus \left(\boldsymbol{X}_{L_0} \oplus \bigoplus_{\ell=L_0+1}^{L_0+L} \boldsymbol{Y}_\ell \right) \otimes \boldsymbol{X}_{L_0} \\
&= \boldsymbol{X}_{L_0}^{(2)} \oplus \left(\bigoplus_{\ell=L_0+1}^{L_0+L} \ \bigoplus_{\ell'=L_0+1}^{2L_0+L-\ell} \boldsymbol{Y}_\ell \otimes \boldsymbol{Y}_{\ell'} \right) \oplus \boldsymbol{X}_{L_0} \otimes \boldsymbol{X}_{L_0+L} \oplus \boldsymbol{X}_{L_0+L} \otimes \boldsymbol{X}_{L_0} \\
&\subseteq \boldsymbol{X}^{(2)},
\end{aligned} \tag{7.6}$$

where, for $L = 0$ and for $L_0 > 0$, this reduces to the full tensor product space at level L_0. Note also that for $L_0 = 0$, due to $\boldsymbol{X}_0 := \{\mathbf{0}\}$, (7.6) reduces to the "usual" combination technique representation of the standard sparse tensor product space presented for example in [22, Sec. 4.2].

We next derive a suitable decomposition of the tensorized Galerkin formulation which uses only the regular Galerkin bases. To this end, we recall the definition of the bilinear form $\mathsf{c}_\kappa^{(2)}(\cdot,\cdot)$ of the tensorized EFIE operator $\mathsf{C}_\kappa^{(2)}$ defined in (6.14).

The Galerkin orthogonality (7.1) and Fubini's theorem imply the following result which generalizes [22, Lem. 6] to $L_0 > 0$: for arbitrary indices $\ell, \ell' \in [L_0+1, L_0+L]$ with $L_0, L \geq 1$ it holds

$$\forall \ \widehat{\mathbf{j}_\ell^{(2)}} \in \boldsymbol{Y}_\ell \otimes \boldsymbol{X}_{2L_0+L-\ell}, \ \ \widehat{\mathbf{j}_{\ell'}^{(2)}} \in \boldsymbol{Y}_{\ell'} \otimes \boldsymbol{X}_{2L_0+L-\ell'} : \quad \mathsf{c}_\kappa^{(2)}(\widehat{\mathbf{j}_\ell^{(2)}}, \widehat{\mathbf{j}_{\ell'}^{(2)}}) = 0 \qquad \text{if} \quad \ell \neq \ell' \ . \tag{7.7}$$

We recall the variational tensorized EFIE (6.14): find

$$\widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}} \in \widehat{\boldsymbol{X}_L^{(2)}} : \qquad \mathsf{c}_\kappa^{(2)} \left(\tilde{\mathbf{j}}_L^{(2)}, \widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}} \right) = \boldsymbol{\mu}_\kappa^{(2)} \left(\tilde{\mathbf{j}}_L^{(2)}, \mathcal{M}^2 \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}} \right) \quad \forall \ \tilde{\mathbf{j}}_L^{(2)} \in \widehat{\boldsymbol{X}_L^{(2)}} \ .$$

Recall definition (6.12) of the space $\widehat{\boldsymbol{X}_L^{(2)}}$, $L \geq 0$, which depends implicitly on $L_0 \geq 0$.

The orthogonality (7.7) implies with (7.6) the representation $\widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}} = \sum_{\ell=L_0}^{L_0+L} \widehat{\mathbf{v}}_\ell$, where the $\widehat{\mathbf{v}}_\ell \in \boldsymbol{Y}_\ell \otimes \boldsymbol{X}_{2L_0+L-\ell}$ are determined by the $L - L_0 + 1$ independent Galerkin problems:

$$\widehat{\mathbf{v}}_\ell \in \boldsymbol{Y}_\ell \otimes \boldsymbol{X}_{2L_0+L-\ell} : \quad \mathsf{c}_\kappa^{(2)} \left(\tilde{\mathbf{j}}_\ell^{(2)}, \widehat{\mathbf{v}}_\ell \right) = \boldsymbol{\mu}_\kappa^{(2)} \left(\tilde{\mathbf{j}}_\ell^{(2)}, \mathcal{M}^2 \mathbf{g}_{\mathrm{D}}^{\mathrm{pc}} \right) \quad \forall \ \tilde{\mathbf{j}}_\ell^{(2)} \in \boldsymbol{Y}_\ell \otimes \boldsymbol{X}_{2L_0+L-\ell} \ , \tag{7.8}$$

for $\ell = L_0, ..., L$.

Using (7.4), i.e. $\boldsymbol{Y}_\ell = (\Pi_\ell - \Pi_{\ell-1})\boldsymbol{X}$ for $\ell = L_0+1, ..., L$, we find the *one-scale representation* or *combination formula*:

$$\widehat{(d\mathbf{j}^{\mathrm{pc}})}^{(2)}_L = \sum_{\ell=L_0}^{L_0+L} \widehat{\mathbf{v}}_\ell = \sum_{\ell=L_0}^{L_0+L} (\boldsymbol{p}_{\ell,2L_0+L-\ell} - \boldsymbol{p}_{\ell-1,2L_0+L-\ell}) \,, \tag{7.9}$$

where the elements $\boldsymbol{p}_{\ell,2L_0+L-\ell} \in \boldsymbol{X}_\ell \otimes \boldsymbol{X}_{2L_0+L-\ell}$, to which we shall refer as *solution details*, satisfy the *one-scale tensor Galerkin problems*: for $L_0 < \ell, \ell' \leq L_0 + L$, we seek

$$\boldsymbol{p}_{\ell,\ell'} \in \boldsymbol{X}_\ell \otimes \boldsymbol{X}_{\ell'} : \quad \mathsf{c}^{(2)}_\kappa(\boldsymbol{q}_{\ell,\ell'}, \boldsymbol{p}_{\ell,\ell'}) = \boldsymbol{\mu}^{(2)}_\kappa(\boldsymbol{q}_{\ell,\ell'}, \mathcal{M}^2 \mathbf{g}^{\mathrm{pc}}_{\mathrm{D}}) \quad \forall\, \boldsymbol{q}_{\ell,\ell'} \in \boldsymbol{X}_\ell \otimes \boldsymbol{X}_{\ell'} \,. \tag{7.10}$$

In addition, $\boldsymbol{p}_{L_0,L_0} \in \boldsymbol{X}^{(2)}_{L_0}$ denotes the full tensor Galerkin solution, being well-defined by Theorem 6.3, since all discretization levels in (7.10) are greater or equal than L_0.

7.2. **Error Analysis.** We now investigate the asymptotic –when L tends to infinity– convergence rate of the sparse tensor Galerkin BEM discretizations (6.12), (6.14). To this end, we assume given a one parameter family $\{\boldsymbol{X}^s\}_{s\geq 0}$ of *smoothness spaces* $\boldsymbol{X} = \boldsymbol{X}^0 \supset \boldsymbol{X}^1 \supset ...$ such that the boundary integral operators C_κ and M_κ are isomorphisms from $\boldsymbol{X}^s$ onto $\boldsymbol{X}^s$, and $\mathsf{A}_\kappa : (\boldsymbol{X}^s)^2 \mapsto (\boldsymbol{X}^s)^2$ isomorphically. On smooth nominal boundaries Γ_0, this holds for $\boldsymbol{X}^s = \boldsymbol{H}^{-1/2+s}(\mathrm{div}_\Gamma;\Gamma_0)$ (*cf.* [31]). However, if Γ_0 is non-smooth, such as polyhedra or screens, analogous mapping properties hold true in these spaces only in a restricted range of s or for certain families $\boldsymbol{X}^s$ of weighted Sobolev spaces [9].

For the boundary element subspace $\boldsymbol{X}_\ell = \boldsymbol{X}_{h_\ell}$, $\ell = 0, 1, 2, ...$ defined in Section 6.2, denote by $\mathsf{P}_\ell : \boldsymbol{X} \mapsto \boldsymbol{X}_\ell$ the $\boldsymbol{X}$-orthogonal projection onto $\boldsymbol{X}_\ell$ with $\boldsymbol{X}^\perp_\ell$ denoting the complement space. We assume that for every $\mathbf{j} \in \boldsymbol{X}^s$ the boundary element spaces $\boldsymbol{X}_\ell$ satisfy the *approximation property*:

$$\|\mathbf{j} - \mathsf{P}_\ell\,\mathbf{j}\|_{\boldsymbol{X}} \lesssim \inf_{\mathbf{j}'_\ell \in \boldsymbol{X}_\ell} \|\mathbf{j} - \mathbf{j}'_\ell\|_{\boldsymbol{X}} \lesssim 2^{-s\ell}\, \|\mathbf{j}\|_{\boldsymbol{X}^s} \,, \quad \ell = 0, 1, 2, ... \tag{7.11}$$

Theorem 7.1. *Assume that the boundary integral operators* C_κ *and* M_κ *are isomorphisms from* $\boldsymbol{X}^s$ *onto* $\boldsymbol{X}^s$ *with* $\boldsymbol{X}^s = \boldsymbol{H}^{-1/2+s}(\mathrm{div}_\Gamma;\Gamma_0)$. *Then, the tensorized boundary integral operators* $\mathsf{C}^{(2)}_\kappa, \mathsf{M}^{(2)}_\kappa$ *constitute isomorphisms from* $(\boldsymbol{X}^s)^{(2)}$ *to* $(\boldsymbol{X}^s)^{(2)}$. *The FOSM BIE* (6.7) *then admits, for smooth* $\mathbf{U}^{pc}_0$, *two-point correlations* $(d\mathbf{j}^{pc})^{(2)}$ *and* $\mathcal{M}^2\mathbf{g}^{pc}_{\mathrm{D}}$ *in* (6.7) *which belong* $(\boldsymbol{X}^s)^{(2)}$.

Due to Theorem 6.3, for $L_0 \geq 0$ sufficiently large and depending on the wavenumber and on D_0, but fixed independently of L, the sparse tensor Galerkin BEM solutions exist and are quasi-optimal: for the EFIE, with the sparse Galerkin projector (7.2), there holds

$$\left\|(d\mathbf{j}^{\mathrm{pc}})^{(2)} - \widehat{(d\mathbf{j}^{\mathrm{pc}})}^{(2)}_L\right\|_{\boldsymbol{X}^{(2)}} \lesssim \left\|(d\mathbf{j}^{\mathrm{pc}})^{(2)} - \widehat{\mathsf{P}^{(2)}_L}(d\mathbf{j}^{\mathrm{pc}})^{(2)}\right\|_{\boldsymbol{X}^{(2)}} \,. \tag{7.12}$$

Here, the sparse tensor projector $\widehat{\mathsf{P}^{(2)}_L}$ is defined as in (7.2), (7.3), with P_ℓ in place of Π_ℓ.

Denoting $\mathsf{P}_{-1}\,\mathbf{j} = \mathbf{0}$ the projector with range $\{\mathbf{0}\} \subset \boldsymbol{X}$, we may introduce for every $\ell \geq 0$ the *detail projector*:

$$\mathsf{Q}_\ell := \mathsf{P}_\ell - \mathsf{P}_{\ell-1} : \boldsymbol{X} \to \boldsymbol{X}_\ell \cap \boldsymbol{X}^\perp_{\ell-1} \,, \quad \ell = 0, 1, 2, ... \tag{7.13}$$

The approximation property (7.11) implies the *detail estimate*:

$$\forall\, \ell \in \mathbb{N}_0,\ \forall\, \mathbf{j} \in \boldsymbol{X}^s : \quad \|\mathsf{Q}_\ell\,\mathbf{j}\|_{\boldsymbol{X}} \leq \|\mathbf{j} - \mathsf{P}_\ell\,\mathbf{j}\|_{\boldsymbol{X}} + \|\mathbf{j} - \mathsf{P}_{\ell-1}\,\mathbf{j}\|_{\boldsymbol{X}} \lesssim 2^{-s\ell}\, \|\mathbf{j}\|_{\boldsymbol{X}^s} \,. \tag{7.14}$$

The cross-norm property (4.6) implies with (7.14) then

$$\forall\, \ell, \ell' \in \mathbb{N}_0,\ \forall\, \mathbf{j}^{(2)} \in \boldsymbol{X}^s \otimes \boldsymbol{X}^{s'} : \quad \left\|(\mathsf{Q}_\ell \otimes \mathsf{Q}_{\ell'})\mathbf{j}^{(2)}\right\|_{\boldsymbol{X}^{(2)}} \lesssim 2^{-(s\ell + s'\ell')} \left\|\mathbf{j}^{(2)}\right\|_{\boldsymbol{X}^s \otimes \boldsymbol{X}^{s'}} \,. \tag{7.15}$$

The density of the boundary element spaces $\{\boldsymbol{X}_\ell\}_{\ell \geq 0}$ defined in Section 6.2 in the space $\boldsymbol{X}$ (*cf.* [13]) implies the norm-convergent expansions:

$$\forall\, \mathbf{j} \in \boldsymbol{X} : \quad \mathbf{j} = \sum_{\ell \geq 0} \mathsf{Q}_\ell\,\mathbf{j} \,, \quad \forall\, \mathbf{j}^{(2)} \in \boldsymbol{X}^{(2)} : \quad \mathbf{j}^{(2)} = \sum_{\ell, \ell' \geq 0} (\mathsf{Q}_\ell \otimes \mathsf{Q}_{\ell'})\mathbf{j}^{(2)} \,. \tag{7.16}$$

From (7.16), the quasi-optimality (7.12) and from the definition (7.3) of the index set $\hat{\Lambda}(L, L_0)$, we may bound the right hand side of (7.12) for $(d\mathbf{j}^{\mathrm{pc}})^{(2)} \in (\boldsymbol{X}^s)^{(2)}$ by

$$
\begin{aligned}
\left\| (d\mathbf{j}^{\mathrm{pc}})^{(2)} - \widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}} \right\|_{\boldsymbol{X}^{(2)}} &\lesssim \Big\| \sum_{\ell,\ell'\geq 0} (\mathsf{Q}_\ell \otimes \mathsf{Q}_{\ell'})(d\mathbf{j}^{\mathrm{pc}})^{(2)} - \sum_{(\ell,\ell')\in\hat{\Lambda}(L,L_0)} (\mathsf{Q}_\ell \otimes \mathsf{Q}_{\ell'})(d\mathbf{j}^{\mathrm{pc}})^{(2)} \Big\|_{\boldsymbol{X}^{(2)}} \\
&\lesssim \sum_{\ell+\ell'\geq 2L_0+L} \left\| (\mathsf{Q}_\ell \otimes \mathsf{Q}_{\ell'})(d\mathbf{j}^{\mathrm{pc}})^{(2)} \right\|_{\boldsymbol{X}^{(2)}} \\
&\lesssim \sum_{\ell+\ell'\geq 2L_0+L} 2^{-s(\ell+\ell')} \left\| (d\mathbf{j}^{\mathrm{pc}})^{(2)} \right\|_{(\boldsymbol{X}^s)^{(2)}} \\
&\simeq \left(\sum_{\ell=0}^{2L_0+L} \sum_{\ell'+\ell\geq 2L_0+L}^{\infty} 2^{-s(\ell+\ell')} + \sum_{\ell>2L_0+L} 2^{-s\ell} \sum_{\ell'\geq 0} 2^{-s\ell'} \right) \left\| (d\mathbf{j}^{\mathrm{pc}})^{(2)} \right\|_{(\boldsymbol{X}^s)^{(2)}} \\
&\simeq (2L_0+L+1) 2^{-s(2L_0+L)} \left\| (d\mathbf{j}^{\mathrm{pc}})^{(2)} \right\|_{(\boldsymbol{X}^s)^{(2)}} .
\end{aligned} \tag{7.17}
$$

7.3. **Computational Complexity.** We consider the perfect conductor case as similar results hold for the dielectric case. To estimate the asymptotic computational complexity, we note that the stiffness matrix of (7.10) is, in fact, a Kronecker product of the EFIE one-scale Galerkin stiffness matrices i.e. $\mathbf{C}_{\ell'} \otimes \mathbf{C}_\ell$ given by $\mathbf{C}_\ell = (\mathsf{c}_\kappa(\psi_{j'}^\ell, \psi_j^\ell))_{1\leq j,j'\leq N_\ell}$ with $N_\ell = \dim(\boldsymbol{X}_\ell)$, and with $\boldsymbol{X}_\ell$ given in terms of the one-scale basis (e.g. the RWG basis) at discretization level ℓ as $\boldsymbol{X}_\ell = \operatorname{span}\{\psi_j^\ell : 1 \leq j \leq N_\ell\}$. The Kronecker product $\mathbf{C}_{\ell'} \otimes \mathbf{C}_\ell$ is never formed explicitly, but only *approximately accessed factor-wise in (multipole-accelerated) matrix-vector products* during iterative solves of (7.10). Due to the structure of the sparse tensor product space, the total number of degrees of freedom (DOFs) involved in computing *all* details $\{\boldsymbol{p}_{\ell,\ell'}\}_{L_0<\ell,\ell'\leq L_0+L}$ in the combination formula (7.9) is $\mathcal{O}(N_L(\log N_L)^b)$ for some $b > 0$ with a constant depending on L_0 and on the wavenumber, as well as on D_0.

Using a preconditioner for the RWG EFIE, either of multilevel type developed in [3], or of Calderón-type as proposed in [2] for the matrices $\mathbf{C}_\ell$ and fast matrix-vector multiplication by the FMM (see e.g. [14] and references therein), approximate matrix-vector multiplications $\widetilde{\mathbf{C}}_\ell$ of the single-level Galerkin matrices $\mathbf{C}_\ell$ are realized in $\mathcal{O}(N_\ell(\log N_\ell)^b)$ work and memory. This results in a log-linear with respect to the number N_L of DOF on Γ_0 solution procedure of the sparse tensor Galerkin discretizations (6.14), (6.15) and, in particular, of the FOSM BIEs (6.7), (6.10) for the deterministic approximation of the second order statistics of the random scattered field. We observe that *for this particular solution procedure* only approximations $\widetilde{\mathbf{C}}_\ell$ of the one-scale Galerkin stiffness matrices $\mathbf{C}_\ell$ of the boundary integral operators have to be realized numerically, however on all levels $\ell = 0, ..., L$.

Once the approximate covariance $\widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}}$ of the random surface current in $\widehat{\boldsymbol{X}_L^{(2)}}$ has been computed, the covariance of the far-field is approximated by inserting $\widehat{(d\mathbf{j}^{\mathrm{pc}})_L^{(2)}}$ into the tensorized representation formula (6.6). In order to evaluate the resulting expression numerically to an accuracy on the order of the discretization error in log-linear complexity with respect to N_L, the number of degrees of freedom on Γ_0, clustering techniques must again be used as in, e.g. [5], together with tensorization by the combination formula (7.9).

8. Concluding Remarks

Domain and shape differentiability for electromagnetic and acoustic scattering have been well investigated in the context of shape optimization and inverse problems. We only refer to [28, 33, 16] and the references there. Usually, in these works explicit expressions for the first –and in certain cases second– domain derivatives for a number of boundary problems have been obtained. The first order domain derivatives of these problems can, with the methodology of the present paper, be used for a FOSM analysis of scattering in the presence of domain uncertainty. The resulting linear, tensorized boundary value problems characterize, to leading order, the second order statistics of the scattered field in terms of the statistics of the domain variation. Upon boundary reduction, tensorized BIEs result which allows sparse tensor Galerkin discretization with RWG boundary element as developed in the present paper.

Although the sparse tensor Galerkin discretization developed in Section 6 of the present work for boundary integral reformulations of the Maxwell equations is well-defined and stable on Lipschitz surfaces as arise in most engineering applications, *for such surfaces, the shape calculus and the present FOSM analysis are not valid, as* the FOSM perturbation analysis presented here assumed C^2-continuity of the surfaces. The normal direction field $\boldsymbol{\psi} = \eta(\cdot, \omega)\mathbf{n}_0$ used in (4.10), for example, for the shape calculus in Section 4.2, has to be reinterpreted in the presence of corners and edges in Γ_0. While for polyhedral surfaces we still expect the shape gradients to be solutions of the homogeneous Maxwell equations (3.4)–(3.5) and (3.8)–(3.9) in the nominal domain D_0, the Radon measure supported on Γ_0 which characterizes the shape gradient will not admit a bounded density with respect to the surface measure dS on Γ_0. The Radon measure characterizing the shape gradient provided by Hadamard's theorem will still be supported on Γ_0. The shape gradient contains in general a singular part supported at corners and edges of Γ_0. Extending the present FoSM boundary element analysis will require *nonstandard boundary integral formulations which accomodate measure-valued input data.* Details will be presented elsewhere.

References

[1] C. Amrouche, V. Girault, and J. Giroire. Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces. *J. Math. Pures Appl. (9)*, 76(1):55–81, 1997.

[2] Francesco P. Andriulli, Kristof Cools, Hakan Bağci, Femke Olyslager, Annalisa Buffa, Snorre Christiansen, and Eric Michielssen. A multiplicative Calderon preconditioner for the electric field integral equation. *IEEE Trans. Antennas and Propagation*, 56(8, part 1):2398–2412, 2008.

[3] Francesco P. Andriulli, Anita Tabacco, and Giuseppe Vecchi. Solving the EFIE at low frequencies with a conditioning that grows only logarithmically with the number of unknowns. *IEEE Trans. Antennas and Propagation*, 58(5):1614–1624, 2010.

[4] K. Beddek, Y. Le Menach, S. Clenet, and O. Moreau. 3-d stochastic spectral finite-element method in static electromagnetism using vector potential formulation. *Magnetics, IEEE Transactions on*, 47(5):1250–1253, May 2011.

[5] Steffen Börm and Lars Grasedyck. Low-rank approximation of integral operators by interpolation. *Computing*, 72(3-4):325–332, 2004.

[6] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[7] A. Buffa and S. H. Christiansen. The electric field integral equation on Lipschitz screens: definitions and numerical approximation. *Numer. Math.*, 94(2):229–267, 2003.

[8] A. Buffa and P. Ciarlet, Jr. On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Methods Appl. Sci.*, 24(1):31–48, 2001.

[9] A. Buffa, M. Costabel, and C. Schwab. Boundary element methods for Maxwell's equations on non-smooth domains. *Numer. Math.*, 92(4):679–710, 2002.

[10] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002.

[11] A. Buffa and R. Hiptmair. A coercive combined field integral equation for electromagnetic scattering. *SIAM J. Numer. Anal.*, 42(2):621–640, 2004.

[12] Annalisa Buffa. Trace theorems on non-smooth boundaries for functional spaces related to Maxwell equations: an overview. In *Computational electromagnetics (Kiel, 2001)*, volume 28 of *Lect. Notes Comput. Sci. Eng.*, pages 23–34. Springer, Berlin, 2003.

[13] Annalisa Buffa, Ralf Hiptmair, Tobias von Petersdorff, and Christoph Schwab. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95(3):459–485, 2003.

[14] Cris Cecka and Eric Darve. Fourier-based fast multipole method for the Helmholtz equation. *SIAM J. Sci. Comput.*, 35(1):A79–A103, 2013.

[15] Michel Cessenat. *Mathematical methods in electromagnetism*, volume 41 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. Linear theory and applications.

[16] Simon N. Chandler-Wilde and Roland Potthast. The domain derivative in rough-surface scattering and rigorous estimates for first-order perturbation theory. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 458(2028):2967–3001, 2002.

[17] Alexey Chernov and Christoph Schwab. First order k-th moment finite element analysis of nonlinear operator equations with stochastic data. *Math. Comp.*, 82(284):1859–1888, 2013.

[18] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.
[19] Martin Costabel and Frédérique Le Louër. Shape derivatives of boundary integral operators in electromagnetic scattering. Part II: Application to scattering by a homogeneous dielectric obstacle. *Integral Equations Operator Theory*, 73(1):17–48, 2012.
[20] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 22 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2011. Metrics, analysis, differential calculus, and optimization.
[21] Yang Du, Yingliang Luo, and Jin-Au Kong. Electromagnetic scattering from randomly rough surfaces using the stochastic second-degree method and the sparse matrix/canonical grid algorithm. *Geoscience and Remote Sensing, IEEE Transactions on*, 46(10):2831–2839, Oct 2008.
[22] Helmut Harbrecht, Michael Peters, and Markus Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *J. Comput. Phys.*, 252:128–141, 2013.
[23] Helmut Harbrecht, Reinhold Schneider, and Christoph Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numer. Math.*, 109(3):385–414, 2008.
[24] F. Hettlich and W. Rundell. A second degree method for nonlinear inverse problems. *SIAM J. Numer. Anal.*, 37(2):587–620, 2000.
[25] Frank Hettlich. The domain derivative of time-harmonic electromagnetic waves at interfaces. *Math. Methods Appl. Sci.*, 35(14):1681–1689, 2012.
[26] Ralf Hiptmair, Carlos Jerez-Hanckes, and Christoph Schwab. Sparse tensor edge elements. *BIT*, 53(4):925–939, 2013.
[27] Ralf Hiptmair and Christoph Schwab. Natural boundary element methods for the electric field integral equation on polyhedra. *SIAM J. Numer. Anal.*, 40(1):66–86, 2002.
[28] A. Kirsch. The domain derivative and two applications in inverse scattering theory. *Inverse Problems*, 9(1):81–96, 1993.
[29] Peijun Li. An inverse cavity problem for Maxwell's equations. *J. Differential Equations*, 252(4):3209–3225, 2012.
[30] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.
[31] Jean-Claude Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Integral representations for harmonic problems.
[32] Roland Potthast. Fréchet differentiability of boundary integral operators in inverse acoustic scattering. *Inverse Problems*, 10(2):431–447, 1994.
[33] Roland Potthast. Domain derivatives in electromagnetic scattering. *Math. Methods Appl. Sci.*, 19(15):1157–1175, 1996.
[34] S.M. Rao, D.R. Wilton, and A.W. Glisson. Electromagnetic scattering by surfaces of arbitrary shape. *IEEE Trans. Antennas and Propagation*, 30(3):409–418, 1982.
[35] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.
[36] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.
[37] Jan Sokolowski and Jean-Paul Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992. Shape sensitivity analysis.
[38] O. O. Sy, M. C. van Beurden, B. L. Michielsen, J. A. H. M. Vaessen, and A. G. Tijhuis. Second-order statistics of complex observables in fully stochastic electromagnetic interactions: Applications to emc. *Radio Science*, 45(4):n/a–n/a, 2010.
[39] Tobias von Petersdorff and Christoph Schwab. Sparse finite element methods for operator equations with stochastic data. *Appl. Math.*, 51(2):145–180, 2006.

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