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Spacetime discontinuous Galerkin methods for solving convection-diffusion systems*

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In this paper, we present two new approaches for solving systems of hyperbolic conservation laws with correct physical viscosity and heat conduction terms such as the compressible Navier-Stokes equations. Our methods are extensions of the spacetime discontinuous Galerkin method for hyperbolic conservation laws developed by Hiltebrand and Mishra [21]. Following this work, we use entropy variables as degrees of freedom and entropy stable fluxes. For the discretization of the diffusion term, we consider two different approaches: the interior penalty approach, resulting in the ST-SIPG and the ST-NIPG method, and a variant of the local discontinuous Galerkin method, resulting in the ST-LDG method. We show entropy stability of the ST-NIPG and the ST-LDG method when applied to the compressible Navier-Stokes equations. For the ST-SIPG method, this result holds under an assumption on the computed solution. All schemes incorporate shock capturing terms. Therefore, the schemes can handle both regimes of underresolved and fully resolved physical diffusion. We present a numerical comparison of the three methods in one dimension.

1. Introduction

In this paper, we present new methods for solving convection-diffusion systems. We focus on problems that can be written as a system of hyperbolic conservation laws with a physical diffusion term. In particular, we are interested in solving the compressible Navier-Stokes equations. Our goal is to develop new schemes that both satisfy theoretical stability estimates and that are robust and accurate. Therefore, we start with an existing method for conservation laws that has this property and extend it to convection-diffusion systems using two different approaches.

In [21], Hiltebrand and Mishra developed a spacetime (ST) discontinuous Galerkin (DG) method for solving systems of conservation laws based on using entropy variables instead of the standard conserved variables. The scheme uses entropy stable fluxes and features a streamline diffusion and a shock capturing term to handle the shocks and discontinuities occurring in the solution of the system. The scheme is (arbitrarily) high order in smooth flow, robust in the

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presence of shocks, and one can show a priori entropy stability estimates for the fully discrete scheme for systems of conservation laws.

In order to extend this scheme to convection-diffusion systems, we need to find suitable treatments of the viscous terms. We consider two different approaches:

1. an approach based on the interior penalty (IP) method introduced by Arnold [1] resulting in the ST-SIPG and the ST-NIPG method;

2. an approach in the spirit of the local discontinuous Galerkin (LDG) method introduced by Cockburn and Shu [8] resulting in the ST-LDG method.

For all three methods, we will deduce an entropy stability estimate for convection-diffusion systems. In particular, our result will imply entropy stability of the ST-NIPG and the ST-LDG method for solving the compressible Navier-Stokes equations. Under the assumption of uniform boundedness of the computed solutions, we can also show entropy stability of the ST-SIPG method for solving the compressible Navier-Stokes equations.

Furthermore, we like our methods to be robust for convection-dominated problems. The presence of the physical diffusion term adds a certain level of stability to the system compared to considering the pure system of conservation laws. Therefore, the use of the streamline diffusion and shock capturing terms that were an essential part of the method developed in [21] might seem unnecessary. However, if the physical diffusion is small, then – depending on the grid size used – the physical diffusion might be underresolved. In this case, the system will behave similar to the corresponding system of conservation laws (without physical diffusion terms). In particular, there will probably be oscillations around shocks and discontinuities if no discontinuity capturing operators are used. Especially in higher dimensions this will be a problem as it typically will not be possible to fully resolve all areas of the flow domain. Therefore, we incorporate the streamline diffusion and shock capturing terms used in [21] in our new methods and adjust them appropriately. This guarantees stability of our methods for a wide range of grid sizes.

Often the presence of shock capturing terms results in methods being only first- or second-order accurate, even for smooth flow. We note that for the artificial diffusion terms used in our new schemes, this is not the case. For the ST-LDG method, numerical results indicate for smooth flow an ideal convergence rate of $O(\Delta x^{k+1})$ for polynomials of degree k. For the ST-SIPG and the ST-NIPG method, we observe in worst case a decrease of the convergence order by one, resulting in convergence orders of at least $O(\Delta x^{k})$. Therefore, the new methods combine stability in an underresolved regime with good accuracy in smooth flow.

Finally, we note that our schemes are based on using entropy variables. This makes the methods more complicated to implement and somewhat more expensive due to the additional change of variables. In return, we can show a stability result (entropy stability) for the actual problem we like to solve (the compressible Navier-Stokes equations), whereas most other methods only come with theoretical results for simplified model problems.

In the literature, there are various DG methods for solving the compressible Navier-Stokes equations that are based on discretizing the conserved variables of the system, see, e.g., [4, 5, 6, 7, 8, 9, 10, 14, 18, 19, 25, 29] and the references cited therein. Most methods use either a variant of the IP or of the LDG approach for discretizing the diffusion term. However, to the best of our knowledge, theoretical stability results (if available) always concern the case of a model problem that has an elliptic diffusion operator, and not the actual compressible Navier-Stokes equations. A unified comparison of several of these methods for the case of elliptic operators can be found in [2]. Also, many of these methods do not include shock capturing terms that both guarantee stability in an underresolved regime and allow for high-order accuracy.

Though significantly less extensive, there is also some literature for solving the compressible Navier-Stokes equations using entropy variables as degrees of freedom. Early work has been done by Shakib et al. [31]. The authors use a finite element formulation that also incorporates a discontinuity capturing term and show entropy stability for their method. However, the authors use continuous elements in space. By using discontinuous elements in space, we guarantee local mass conservation, which is especially important in the underresolved scheme, for which shocks may occur. The possibility of choosing suitable numerical fluxes (which is not the case for continuous elements in space) is very favorable in this scenario as well. Barth [3] includes the compressible Navier-Stokes equations in his theoretical considerations and examines DG methods for solving the Euler equations, but he neither suggests a discretization for the physical diffusion term nor shows numerical results for the compressible Navier-Stokes equations. Tadmor and Zhong [33] have developed a difference scheme based on entropy variables. The scheme uses entropy conservative fluxes for the non-linear term and centered differences for the discretization of the dissipation term. The authors show entropy stability for the semi-discrete form. However, their method does not include shock capturing terms and is at most second-order accurate.

This paper is structured as follows: We start with a review of the original method for conservation laws [21] in section 2. Then, in section 3, we discuss the effect of switching to entropy variables on the diffusion matrix. In sections 4 and 5, we present our two different extensions, the ST-SIPG and the ST-NIPG method, and the ST-LDG method, and prove entropy stability estimates under suitable assumptions. In section 6, we show numerical results in one space dimension comparing all three methods. In particular, we solve the compressible Navier-Stokes equations. We conclude this work with a comparison of the three methods in section 7.

2. Review of the spacetime DG method for hyperbolic systems

In this section, we review the spacetime DG method developed for hyperbolic conservation laws [21] that our method is based on. We focus on systems of conservation laws in one dimension given by

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0. \tag{1}$$

Here, $\mathbf{U} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}$, $m \in \mathbb{N}$, is the vector of conserved variables and $\mathbf{F}$ is the flux vector. Assuming a strictly convex entropy function S, the map $\mathbf{U} \to \mathbf{V}$ is one-to-one, where $\mathbf{V} = S_{\mathbf{U}}(\mathbf{U})$ denotes the entropy variables. Therefore, we can equivalently write the system as

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = 0 \tag{2}$$

with $\mathbf{F}(\mathbf{V}) = \mathbf{F}(\mathbf{U}(\mathbf{V}))$ for simplicity. To discretize, we consider a spacetime grid with each spacetime element being a tensor-product of a spatial grid cell $K_i = [x_{i-1/2}, x_{i+1/2}] \subset \Omega$ and a time segment $I_n = [t_n, t_{n+1}] \subset [0, T]$. Then, approximations $\mathbf{V}^{\Delta x} = (v_1^{\Delta x}, \dots, v_m^{\Delta x})^T$ (we will use the superscript Δx when referring to discrete quantities) to the solution $\mathbf{V}$ are sought in the space

$$\mathbf{V}^{\Delta x} \in \mathcal{V}^k = \left\{ \mathbf{\Phi}^{\Delta x} \in (L^1(\Omega \times [0,T]))^m \ : \ \begin{array}{l} \mathbf{\Phi}^{\Delta x}\big|_{K_i \times I_n} \text{ is a polynomial} \\ \text{of degree } k \text{ in each component} \end{array} \right\}.$$

Multiplying (2) with test functions $\mathbf{\Phi}^{\Delta x}$, integrating in space and time, and doing integration by parts with using numerical fluxes where appropriate results in

$$
\begin{aligned}
&\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \\
&\quad -\sum_{n,i} \int_{I_n} \int_{K_i} \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}), \mathbf{\Phi}_t^{\Delta x} \rangle + \langle \mathbf{F}(\mathbf{V}^{\Delta x}), \mathbf{\Phi}_x^{\Delta x} \rangle \right) \, dx \, dt \\
&\quad + \sum_{n,i} \int_{K_i} \left(\langle \mathbb{U}(\mathbf{V}_{n+1,-}^{\Delta x}, \mathbf{V}_{n+1,+}^{\Delta x}), \mathbf{\Phi}_{n+1,-}^{\Delta x} \rangle - \langle \mathbb{U}(\mathbf{V}_{n,-}^{\Delta x}, \mathbf{V}_{n,+}^{\Delta x}), \mathbf{\Phi}_{n,+}^{\Delta x} \rangle \right) \, dx \\
&\quad + \sum_{n,i} \int_{I_n} \Big(\langle \mathbb{F}(\mathbf{V}_{i+1/2,L}^{\Delta x}, \mathbf{V}_{i+1/2,R}^{\Delta x}), \mathbf{\Phi}_{i+1/2,L}^{\Delta x} \rangle - \langle \mathbb{F}(\mathbf{V}_{i-1/2,L}^{\Delta x}, \mathbf{V}_{i-1/2,R}^{\Delta x}), \mathbf{\Phi}_{i-1/2,R}^{\Delta x} \rangle \Big) \, dt.
\end{aligned} \tag{3}
$$

Here, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product $\langle \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x} \rangle = \sum_{j=1}^m v_j^{\Delta x} \Phi_j^{\Delta x}$ and the indices $+/-$ and L/R denote the following limits

$$
v_{n+1,\pm}^{\Delta x}(x) = \lim_{h \to 0} v^{\Delta x}(x, t_{n+1} \pm h), \quad v_{i+1/2,R/L}^{\Delta x}(t) = \lim_{h \to 0} v^{\Delta x}(x_{i+1/2} \pm h, t).
$$

Furthermore, $\mathbb{U}$ and $\mathbb{F}$ denote the fluxes in time and space. In order to enable proper time marching, the upwind flux is chosen in time, i.e.,

$$
\mathbb{U}(\mathbf{V}_{n+1,-}^{\Delta x}, \mathbf{V}_{n+1,+}^{\Delta x}) = \mathbf{U}(\mathbf{V}_{n+1,-}^{\Delta x}).
$$

For the numerical flux in space either an entropy conservative flux or an entropy stable flux is used. Entropy conservative fluxes $\mathbb{F}^*$ have been examined in [32] and satisfy

$$
\langle b - a, \mathbb{F}^*(a, b) \rangle = \psi(b) - \psi(a) \tag{4}
$$

with $\psi = \langle \mathbf{V}, \mathbf{F} \rangle - Q$ being the entropy potential and (S,Q) being an entropy pair. (We will specify the entropy conservative fluxes that we use for numerical tests in the corresponding parts of section 6.)

Entropy stable fluxes are created out of entropy conservative fluxes by adding a diffusion term

$$
\mathbb{F}(\mathbf{V}_a^{\Delta x}, \mathbf{V}_b^{\Delta x}) = \mathbb{F}^*(\mathbf{V}_a^{\Delta x}, \mathbf{V}_b^{\Delta x}) - \frac{1}{2} \mathbb{D}(\mathbf{V}_a^{\Delta x}, \mathbf{V}_b^{\Delta x})(\mathbf{V}_b^{\Delta x} - \mathbf{V}_a^{\Delta x}) \tag{5}
$$

with

$$
\mathbb{D}(\mathbf{a}, \mathbf{b}) = \mathbf{R} \, \mathbf{P}(\boldsymbol{\Lambda}; \mathbf{a}, \mathbf{b}) \, \mathbf{R}^T.
$$

Here $\boldsymbol{\Lambda}$, $\mathbf{R}$ are the (real) eigenvalue and eigenvector matrices of the Jacobian $\partial_{\mathbf{U}} \mathbf{F}$ with $\mathbf{R}$ being scaled such that $\mathbf{R}\mathbf{R}^T = \mathbf{U}_{\mathbf{V}}$. In this work, we use the Rusanov diffusion operator given by

$$
\mathbf{P}(\boldsymbol{\Lambda}; \mathbf{a}, \mathbf{b}) = \max \{ \lambda_{\max}(\mathbf{a}), \lambda_{\max}(\mathbf{b}) \} \, \mathbf{I}_m
$$

with $\lambda_{\max}(\mathbf{U})$ denoting the maximum wave speed. Several other choices are possible, see [32, 11, 12, 21]. In the following we will refer by $\mathbb{F}$ both to entropy stable and entropy conservative fluxes.

In the simplest form of the scheme, the discrete numerical solution $\mathbf{V}^{\Delta x} \in \mathcal{V}^k$ to (2) is then found as the solution of the system

$$
\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = 0 \quad \forall \, \mathbf{\Phi}^{\Delta x} \in \mathcal{V}^k. \tag{6}
$$

For problems involving shocks or discontinuities, however, this scheme exhibits oscillations and overshoot. Therefore, in the scheme developed in [21], a streamline diffusion operator $\mathcal{B}_{\mathrm{SD}}$ and a shock capturing operator $\mathcal{B}_{\mathrm{SC}}$ are added. Then, the discrete solution $\mathbf{V}^{\Delta x} \in \mathcal{V}^k$ is found as the solution to the system

$$\mathcal{B}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{SD}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{SC}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = 0 \tag{7}$$

for all $\mathbf{\Phi}^{\Delta x} \in \mathcal{V}^k$.

A slightly improved version of the streamline diffusion term used in [21] is given by (compare [20])

$$\mathcal{B}_{\mathrm{SD}}(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{U}_{\mathbf{V}}(\mathbf{V}^{\Delta x}) \mathbf{W}_t^{\Delta x} + \mathbf{F}_{\mathbf{V}}(\mathbf{V}^{\Delta x}) \mathbf{W}_x^{\Delta x}, \mathbf{D}^{SD} \,\mathrm{Res} \rangle \, dx \, dt \tag{8}$$

with

$$\mathrm{Res} = \mathbf{U}(\mathbf{V}^{\Delta x})_t + \mathbf{F}(\mathbf{V}^{\Delta x})_x \tag{9}$$

and

$$\mathbf{D}^{SD} = C^{SD} \Delta t_n \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^{\Delta x}). \tag{10}$$

Typically, $C^{SD} = 10$ is used. The shock capturing term in [21, 20] uses both an inner and a boundary residual in order to adjust the amount of viscosity that is added to the systems. As the boundary residual is very complicated and has only fairly small impact, we will only use the inner residual for our extensions of the method. Therefore, we only present a reduced form of the shock capturing term used in [20], which is given by

$$\begin{aligned} &\mathcal{B}_{\mathrm{SC}}(\mathbf{V}^{\Delta x}, \mathbf{W}^{\Delta x}) \\ &\quad = \sum_{n,i} \int_{I_n} \int_{K_i} D_{n,i}^{SC} \left(\langle \mathbf{W}_t^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_t^{\Delta x} \rangle + \frac{\Delta x_i^2}{\Delta t_n^2} \langle \mathbf{W}_x^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_x^{\Delta x} \rangle \right) dx \, dt \end{aligned} \tag{11}$$

with

$$\tilde{\mathbf{V}}_{n,i} = \frac{1}{\mathrm{meas}(I_n \times K_i)} \int_{I_n} \int_{K_i} \mathbf{V}^{\Delta x}(x,t) \, dx \, dt$$

and

$$D_{n,i}^{SC} = \frac{\Delta t_n C^{SC} \overline{\mathrm{Res}}_{n,i}}{\sqrt{\int_{I_n} \int_{K_i} \langle \mathbf{V}_t^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_t^{\Delta x} \rangle + \frac{\Delta x_i^2}{\Delta t_n^2} \langle \mathbf{V}_x^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,K}) \mathbf{V}_x^{\Delta x} \rangle \, dx \, dt} + \varepsilon} \tag{12}$$

and

$$\overline{\mathrm{Res}}_{n,i} = \sqrt{\int_{I_n} \int_{K_i} \langle \mathrm{Res}, \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^{\Delta x}) \mathrm{Res} \rangle \, dx \, dt}.$$

Here, $\varepsilon = \frac{\Delta x^{3/2}}{\Delta t_n^{1/2} \mathrm{diam}(\Omega)}$ and typically $C^{SC} = 1$ is used.

Remark 2.1. *In a first attempt of extending the original scheme from conservation laws to convection-diffusion systems, we did not include the streamline diffusion and shock capturing operators $\mathcal{B}_{\mathrm{SD}}$ and $\mathcal{B}_{\mathrm{SC}}$. We expected the natural viscosity introduced by the additional diffusion term to take over the role of the streamline diffusion and shock capturing operators in terms of* *avoiding overshoot. Numerical tests indicated that this is the case if the grid size is chosen fine enough to resolve the additional viscosity in the system. If the grid is too coarse, however, this is not the case and we observed overshoot and oscillations. Therefore, to ensure robustness of our methods both in a fully resolved and in an underresolved regime, we also extend the streamline diffusion and shock capturing operators to convection-diffusion systems.*

Among other properties, one can show that the method described in (7) (as well as the method in (6)) produces entropy stable discrete solutions.

Theorem 2.1 (Part of Theorem 3.1 in [21])**.** *Consider the system of conservation laws* (1) *with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain Ω. Let the final time be denoted by t_N. Then the approximate solutions produced by* (7) *satisfy*

$$\int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx \leq \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx.$$

This concludes our summary of the method for hyperbolic conservation laws and of the features that are relevant for our new methods. We will present the details of our extensions ST-SIPG, ST-NIPG, and ST-LDG in sections 4 and 5. First, however, we will examine the effect of switching to entropy variables on the convection-diffusion systems that we consider.

3. Convection-diffusion systems written in entropy variables

We consider a system of conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$$

and add a diffusion matrix $\mathbf{D} : \mathbb{R}^m \to \mathbb{R}^m$, to be thought of as, e.g., physical viscosity and heat conduction terms resulting in

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = (\mathbf{D}(\mathbf{U})\mathbf{U}_x)_x.$$

We assume that we are given a strictly convex entropy function S with corresponding entropy flux Q. Using entropy variables $\mathbf{V} = S_{\mathbf{U}}(\mathbf{U})$ as degrees of freedom results in

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = (\mathbf{A}(\mathbf{V})\mathbf{V}_x)_x \tag{13}$$

with $\mathbf{A}(\mathbf{V}) = \mathbf{D}(\mathbf{U}(\mathbf{V}))\mathbf{U}_{\mathbf{V}}(\mathbf{V})$. It is well-known that such a change to entropy variables symmetrizes a hyperbolic system of conservation laws [15, 13, 26, 17]. Additionally, this change of variable can have a positive effect on the properties of the matrix $\mathbf{A}$.

3.1. The compressible Navier-Stokes equations

We shortly describe the situation for the compressible Navier-Stokes equations, our main application. We focus on introducing the entropy S that we use and on discussing the properties of $\mathbf{A}$. Further information for using compressible Navier-Stokes equations with entropy variables can be found in the literature [3, 16, 23, 31, 33].

The compressible Navier-Stokes equations in one space dimension are given by

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= \nu u_{xx}, \\ E_t + ((E+p)u)_x &= \nu \left(\frac{u^2}{2}\right)_{xx} + \kappa \theta_{xx}, \end{aligned} \tag{14}$$

with $\rho = \rho(x,t)$ denoting the density, $u = u(x,t)$ the velocity, $p = p(x,t)$ the pressure, and $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2$ being the total energy. We will also use $m = m(x,t) = \rho(x,t)u(x,t)$ for the momentum. Additionally, $\theta = \frac{p}{R\rho}$ refers to the temperature. We assume the viscosity ν and the conductivity κ to be constant. We further assume the relation between ν and κ/R to be given by the Prandtl number $P_r = 4\gamma/(9\gamma-5)$ via

$$\frac{\kappa}{R} = \frac{\gamma C_v}{R P_r}\nu = \frac{9\gamma - 5}{4(\gamma-1)}\nu.$$

Writing the right hand side of (14) in the form $(\mathbf{D}(\mathbf{U})\mathbf{U}_x)_x$ with $\mathbf{U} = (\rho, m, E)^T$ results in

$$\mathbf{D}(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 \\ -\nu\frac{m}{\rho^2} & \frac{\nu}{\rho} & 0 \\ -\nu\frac{m^2}{\rho^3} + \frac{\kappa}{R}(\gamma-1)\left(\frac{m^2}{\rho^3} - \frac{E}{\rho^2}\right) & \nu\frac{m}{\rho^2} - \frac{\kappa}{R}(\gamma-1)\frac{m}{\rho^2} & \frac{\kappa}{R}\frac{\gamma-1}{\rho} \end{pmatrix}.$$

We note that the matrix $\mathbf{D}(\mathbf{U})$ is *not symmetric*.

For the transformation to entropy variables, we use the physical entropy and the corresponding entropy flux given by

$$S = \frac{-\rho s}{\gamma - 1}, \quad Q = \frac{-\rho u s}{\gamma - 1}, \quad s = \log(p) - \gamma\log(\rho).$$

This results in the entropy variables (written in terms of primitive variables and s for simplicity)

$$\mathbf{V} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{\rho u^2}{2p}, \quad \frac{\rho u}{p}, \quad -\frac{\rho}{p}\right)^T.$$

The matrix $\mathbf{A}(\mathbf{V})$ is then given by

$$\mathbf{A}(\mathbf{V}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\nu\frac{1}{v_3} & \nu\frac{v_2}{v_3^2} \\ 0 & \nu\frac{v_2}{v_3^2} & -\nu\frac{v_2^2}{v_3^3} + \frac{\kappa}{R}\frac{1}{v_3^2} \end{pmatrix}. \tag{15}$$

For $\rho, p > 0$, this matrix is *symmetric positive semi-definite*. Furthermore, the reduced matrix

$$\tilde{\mathbf{A}}(\mathbf{V}) = \begin{pmatrix} -\nu\frac{1}{v_3} & \nu\frac{v_2}{v_3^2} \\ \nu\frac{v_2}{v_3^2} & -\nu\frac{v_2^2}{v_3^3} + \frac{\kappa}{R}\frac{1}{v_3^2} \end{pmatrix} \tag{16}$$

is *symmetric positive definite* with eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(-b \mp \sqrt{b^2 + 4\nu\frac{\kappa}{R}\frac{1}{v_3^3}}\right) > 0, \quad b = \nu\frac{v_2^2}{v_3^3} - \frac{\kappa}{R}\frac{1}{v_3^2} + \frac{\nu}{v_3}, \tag{17}$$

if $\nu, \kappa > 0$ and $\rho, p > 0$.

3.2. Assumptions on the matrix $\mathbf{A}$

For the methods that we will present in the following we need to make the following assumptions concerning the matrix $\mathbf{A}$ in (13) in order to prove entropy stability.

Assumption 3.1 (Ass. for ST-NIPG and ST-LDG)**.** *We assume that the matrix* $\mathbf{A}(\mathbf{V}) : \mathbb{R}^m \to \mathbb{R}^m$ *in* (13) *is symmetric positive semi-definite.*

We note that this is the case for the compressible Navier-Stokes equations. In other words, our results will imply that the ST-NIPG and the ST-LDG method are entropy stable for the compressible Navier-Stokes equations.

To prove entropy stability for the ST-SIPG method, we need a stronger assumption.

Assumption 3.2 (Ass. for ST-SIPG)**.** *We assume that the matrix* $\mathbf{A}(\mathbf{V}) : \mathbb{R}^m \to \mathbb{R}^m$ *in* (13) *is symmetric positive definite. We further assume that there are lower and upper bounds* (λ, Λ) *on the eigenvalues of* $\mathbf{A}$ *(corresponding to the discrete solution) such that* $0 < \lambda \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \Lambda$.

Based on our considerations above, this is *not* the case for the compressible Navier-Stokes equations. However, the matrix $\mathbf{A}$ for the compressible Navier-Stokes equations has the structure

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathbf{A}} \end{bmatrix}. \tag{18}$$

It is sufficient (see the explanation in Remark 4.3 after having presented the ST-SIPG method) to satisfy Assumption 3.2 for the positive definite matrix $\tilde{\mathbf{A}}$. This implies that we need uniform bounds on the eigenvalues $\lambda_{1,2}$ given in (17), for which we need the following assumption. We note that the uniform bounds in the following assumption only refer to the computed solution, which might be checked a posteriori.

Assumption 3.3. *We assume that there are uniform lower bounds* $\rho_0, p_0 > 0$ *such that* $\rho^{\Delta x} \geq \rho_0$ *and* $p^{\Delta x} \geq p_0$. *We further assume that there are uniform upper bounds* $\rho_M, u_M, p_M > 0$ *such that* $\rho^{\Delta x} \leq \rho_M$, $|u^{\Delta x}| \leq u_M$, *and* $p^{\Delta x} \leq p_M$.

Lemma 3.1. *Under Assumption 3.3, there exist* $\lambda(\rho_M, u_M, p_0)$ *and* $\Lambda(\rho_0, u_M, p_M)$ *such that* $0 < \lambda \leq \lambda_1 < \lambda_2 \leq \Lambda$ *for the eigenvalues of* $\tilde{\mathbf{A}}$.

Proof. The eigenvalues $\lambda_{1,2}$ of $\tilde{\mathbf{A}}$ are given by (17) with λ_2 corresponding to the '+' sign. We can bound λ_2 from above by $\lambda_2 \leq |b|$. Then, we can bound $|b|$ using Assumption 3.3 in terms of ρ_0, u_M, and p_M. For the lower bound, we write

$$\lambda_1 = \frac{1}{2}|b| \left(1 - \sqrt{1-\varepsilon}\right), \quad \varepsilon = -\frac{4\nu \frac{\kappa}{R} \frac{1}{v_3^3}}{b^2}.$$

We note that $0 < \varepsilon < 1$. We use the Taylor expansion of the square root function to bound

$$\sqrt{1-\varepsilon} \leq 1 - \frac{1}{2}\varepsilon.$$

This implies $\lambda_1 \geq \frac{1}{4}|b|\,\varepsilon$, which in turn can be bounded using Assumption 3.3 in terms of ρ_M, u_M, and p_0. $\square$

To summarize, if we assume Assumption 3.3 to be true, then our main application, the compressible Navier-Stokes equations, satisfies Assumption 3.2, which will be the main assumption for showing entropy stability for the ST-SIPG method.

4. The ST-NIPG and the ST-SIPG method

In this section, we present our extensions ST-NIPG and ST-SIPG that are based on the interior penalty (IP) approach by Arnold [1] as well as on earlier work by Nitsche [27]. For more information about IP methods, we refer to Oden et al. [28] and Rivière [30]. For simplicity we assume uniform mesh width Δx – though this is not necessary. For both methods, we seek the discrete solution $\mathbf{V}^{\Delta x} \in \mathcal{V}^k$ that satisfies for all $\mathbf{\Phi}^{\Delta x} \in \mathcal{V}^k$ the quasilinear form

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \mathcal{B}_{\mathrm{bdy}}(\mathbf{\Phi}^{\Delta x}). \tag{19}$$

Here, $\mathcal{B}_{\mathrm{DG}}$ is given by (3), $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}$ and $\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}$ are modifications of $\mathcal{B}_{\mathrm{SD}}$ and $\mathcal{B}_{\mathrm{SC}}$ that will be described below, and $\mathcal{B}_{\mathrm{bdy}}$ incorporates a suitable discretization of boundary terms. Finally, $\mathcal{B}_{\mathrm{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x})$ – which depends on the parameter κ – represents the discretization of the diffusion term $-(\mathbf{A}(\mathbf{V})\mathbf{V}_x)_x$ and is given by

$$\begin{aligned}
\mathcal{B}_{\mathrm{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = &\sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{A}(\mathbf{V}^{\Delta x})\mathbf{V}_x^{\Delta x}, \mathbf{\Phi}_x^{\Delta x} \rangle \, dx \, dt \\
&+ \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2}, [\mathbf{\Phi}^{\Delta x}]_{i+1/2} \rangle \, dt \\
&- \kappa \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\})[\mathbf{V}^{\Delta x}]_{i+1/2}, \left\{\mathbf{\Phi}_x^{\Delta x}\right\}_{i+1/2} \rangle \, dt \\
&+ \sum_{n,i} \int_{I_n} \frac{\gamma}{\Delta x} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\})[\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{\Phi}^{\Delta x}]_{i+1/2} \rangle dt
\end{aligned} \tag{20}$$

for interior edges $i \pm 1/2$. We use the standard notation for average and jump given by

$$\left\{v^{\Delta x}\right\}_{i+1/2} = \frac{1}{2}\left(v_{i+1/2,R}^{\Delta x} + v_{i+1/2,L}^{\Delta x}\right), \quad [v^{\Delta x}]_{i+1/2} = v_{i+1/2,R}^{\Delta x} - v_{i+1/2,L}^{\Delta x}.$$

We refer to the new methods

- as ST-NIPG for $\kappa = 1$,
- and as ST-SIPG for $\kappa = -1$.

We note that the edge terms in the second line result from integration by parts using central fluxes. The edge terms in the third line are added to make the form (anti-)symmetric. Finally, the jump terms in the fourth line are stabilization terms that enforce coercivity of the form $\mathcal{B}_{\mathrm{IP}}$ for the ST-SIPG method, with the parameter γ being the penalty parameter.

Finally, we need to adjust the streamline diffusion and shock capturing terms given by (8) and (11) for the original method. We first adjust the definition of the residual. Instead of (9), we use

$$\mathrm{Res}^{\mathrm{IP}} = \mathbf{U}(\mathbf{V}^{\Delta x})_t + \mathbf{F}(\mathbf{V}^{\Delta x})_x - \mathbf{A}(\mathbf{V}^{\Delta x})_x \mathbf{V}_x^{\Delta x} - \mathbf{A}(\mathbf{V}^{\Delta x})\mathbf{V}_{xx}^{\Delta x}. \tag{21}$$

Note that now second derivatives need to be evaluated. This completes the changes for the shock capturing term. For the streamline diffusion term, an additional change is necessary. The form $\mathcal{B}_{\mathrm{SD}}$ given by (8) has been constructed such that the left hand side of the inner product corresponds to a linearized residual. Consequently, we make the following adjustment

$$\mathcal{B}_{\text{SD}}^{\text{IP}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{U}_\mathbf{V}(\mathbf{V}^{\Delta x}) \mathbf{\Phi}_t^{\Delta x} + \mathbf{F}_\mathbf{V}(\mathbf{V}^{\Delta x}) \mathbf{\Phi}_x^{\Delta x} \\ - \mathbf{A}(\mathbf{V}^{\Delta x})_x \mathbf{\Phi}_x^{\Delta x} - \mathbf{A}(\mathbf{V}^{\Delta x}) \mathbf{\Phi}_{xx}^{\Delta x}, \mathbf{D}^{SD} \operatorname{Res}^{\text{IP}} \rangle \, dx \, dt \quad (22)$$

We can show the following theorems for the ST-NIPG and ST-SIPG method.

Theorem 4.1 (Entropy stability for ST-NIPG)**.** *Let Assumption 3.1 hold true. Consider the system* (13) *with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain Ω. Let the final time be denoted by t_N. Then, the approximate solutions generated by the scheme* (19) *with $\kappa = 1$ and $\gamma \geq 0$ satisfy*

$$\int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx \leq \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx.$$

Theorem 4.2 (Entropy stability for ST-SIPG)**.** *Let Assumption 3.2 hold true. Consider the system* (13) *with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain Ω. Let the final time be denoted by t_N. Then, the approximate solutions generated by the scheme* (19) *with $\kappa = -1$ satisfy*

$$\int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx \leq \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx,$$

provided γ is chosen sufficiently large such that

$$\gamma > \frac{c_{\text{inv}} \Lambda}{\lambda} \quad (23)$$

where λ, Λ are defined in Assumption 3.2 and c_{inv} will be specified later in Lemma 4.5.

Remark 4.1. *In this work, which considers one-dimensional convection-diffusion systems, we will focus on the interior of the flow domain and not consider discretizations of boundary conditions. Finding a suitable discretization of inflow, outflow, and solid wall with no heat flux boundary conditions for the compressible Navier-Stokes equations in two and three dimensions is often not trivial. Incorporating the boundary conditions in the entropy considerations helps to single out physically relevant formulations and strengthens results such as Theorems 4.1 and 4.2. However, as there are no physically relevant boundary conditions for the compressible Navier-Stokes equations in one dimension, we postpone these considerations to future work about multi-dimensional extensions.*

Proof of Theorems 4.1 and 4.2. For proving the statements about entropy stability, we will make use of several lemmata that we will present (and prove) after this proof. Using the compact support of the discrete solution, we can drop the boundary terms $\mathcal{B}_{\text{bdy}}$ in (19). Then, testing with $\mathbf{\Phi}^{\Delta x} = \mathbf{V}^{\Delta x}$ results in

$$\mathcal{B}_{\text{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) + \mathcal{B}_{\text{SD}}^{\text{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) + \mathcal{B}_{\text{SC}}^{\text{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) + \mathcal{B}_{\text{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = 0.$$

We consider each of the four terms individually:

1. *Term* $\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x})$: According to Lemma 4.1, there holds

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx.$$

2. *Term* $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x})$: *Claim*: There holds

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0.$$

   Proof: We essentially follow the proof of Theorem 3.1 in [21]. Based on our new definition of the streamline diffusion term, there holds by chain rule

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathrm{Res}^{\mathrm{IP}}, \mathbf{D}^{SD} \, \mathrm{Res}^{\mathrm{IP}} \rangle \, dx \, dt.$$

   With the definition of $\mathbf{D}^{SD}$ given by (10) and due to the assumption that the entropy S is strictly convex, this implies $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0$. We note that our adjustment of the term $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}$ compared to the original term $\mathcal{B}_{\mathrm{SD}}$ was essential for proving this claim.

3. *Term* $\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x})$: *Claim*: There holds

$$\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0.$$

   Proof: There holds (compare (11))

$$\begin{aligned} &\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \\ &\quad = \sum_{n,i} \int_{I_n} \int_{K_i} D_{n,i}^{SC} \left(\langle \mathbf{V}_t^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_t^{\Delta x} \rangle + \frac{\Delta x^2}{(\Delta t_n)^2} \langle \mathbf{V}_x^{\Delta x}, \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_x^{\Delta x} \rangle \right) dx \, dt \end{aligned}$$

   with $D_{n,i}^{SC}$ being given by (12) but with $\overline{\mathrm{Res}}_{n,i}$ being based on $\mathrm{Res}^{\mathrm{IP}}$ instead of being based on Res. Due to the strict convexity of the entropy function S, both $\mathbf{U}_{\mathbf{V}}$ and $\mathbf{U}_{\mathbf{V}}^{-1}$ are strictly positive definite. This implies $D_{n,i}^{SC} \geq 0$. This also directly implies $\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0$.

4. *Term* $\mathcal{B}_{\mathrm{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x})$: Based on Lemma 4.2 and on Corollary 4.1 there holds for both the ST-NIPG and the ST-SIPG method (under the respective assumptions)

$$\mathcal{B}_{\mathrm{IP}}(\kappa; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0.$$

Summarizing the estimates for the four terms results in

$$0 \geq \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx + 0 + 0 + 0,$$

which concludes the proof. □

In the given proof we used the auxiliary results Lemma 4.1, Lemma 4.2, and Corollary 4.1. We will state and prove these results in the following.

Lemma 4.1. *Under the conditions of Theorem 4.1 and of Theorem 4.2, respectively, there holds*

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) \, dx - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0))) \, dx. \tag{24}$$

The proof follows directly from the proof of the original method given in [21] and is summarized in Appendix A.

Lemma 4.2. *For the ST-NIPG method, under the assumptions of Theorem 4.1, there holds*

$$\mathcal{B}_{\mathrm{IP}}(1; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0.$$

Proof. By definition and due to the symmetry of $\mathbf{A}$, there holds

$$\begin{aligned}\mathcal{B}_{\mathrm{IP}}(1; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{A}(\mathbf{V}^{\Delta x}) \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x} \rangle \, dx \, dt \\ + \sum_{n,i} \int_{I_n} \frac{\gamma}{\Delta x} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) [\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle dt.\end{aligned}$$

As $\mathbf{A}$ is assumed to be positive semi-definite, this implies the claim. □

This completes the proof of Theorem 4.1. In order to show Theorem 4.2 (entropy stability for the ST-SIPG method), we still need to show the estimate for Term 4 in the above proof.

Lemma 4.3. *For the ST-SIPG method, under the assumptions of Theorem 4.2, there holds*

$$\begin{aligned}\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq (\lambda - c_{\mathrm{inv}} \delta \Lambda) \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x} \rangle \, dx \, dt \\ + \frac{\gamma - \frac{1}{\delta}}{\Delta x} \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) [\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle \, dt,\end{aligned}$$

with c_{inv} being the constant from the inverse estimate from Lemma 4.5.

Corollary 4.1. *Under the assumptions of Theorem 4.2, there holds*

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0.$$

Proof. The proof follows directly from Lemma 4.3, the assumption that $\mathbf{A}$ is positive definite, and the assumption on γ given by (23) by defining $\delta = \frac{1}{\gamma}$. □

It remains to prove Lemma 4.3 in order to conclude the proof of Theorem 4.2. To do so, we will need the following lemma, which generalizes Young's inequality to matrices.

Lemma 4.4. *Let the matrix $\mathbf{C} : \mathbb{R}^m \to \mathbb{R}^m$ be symmetric positive definite. Then there holds for arbitrary vectors $v, w \in \mathbb{R}^m$ and $\delta > 0$*

$$2 w^T \mathbf{C} v \leq \delta w^T \mathbf{C} w + \frac{1}{\delta} v^T \mathbf{C} v.$$

Proof. The proof follows directly from

$$0 \leq \frac{1}{\delta}\left((\delta w - v)^T \mathbf{C}(\delta w - v)\right) = \delta w^T \mathbf{C} w - 2w^T \mathbf{C} v + \frac{1}{\delta} v^T \mathbf{C} v.$$

□

We also need the following inverse estimate, which can be shown using the equivalence of norms in finite dimensions.

Lemma 4.5. *For the discrete spacetime polynomials there holds*

$$\int_{I_n} (v_{x,B}^{\Delta x})^2 \, dt \leq \frac{c_{\text{inv}}}{\Delta x} \int_{I_n} \int_{K_i} (v_x^{\Delta x})^2 \, dx \, dt$$

with $v_{x,B}^{\Delta x} = v_{x,i+1/2,L}^{\Delta x}$ *or* $v_{x,B}^{\Delta x} = v_{x,i-1/2,R}^{\Delta x}$.

With these prerequisites, we can prove Lemma 4.3.

Proof of Lemma 4.3. Due to the solutions having compact support, there holds with $\kappa = -1$

$$\begin{aligned}\mathcal{B}_{\text{IP}}(-1; \mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = &\sum_{n,i} \int_{I_n} \int_{K_i} \underbrace{\langle \mathbf{A}(\mathbf{V}^{\Delta x}) \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x} \rangle}_{\Gamma_1} \, dx \, dt \\ &+ 2 \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle \, dt \\ &+ \sum_{n,i} \int_{I_n} \underbrace{\frac{\gamma}{\Delta x} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) [\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle}_{\Gamma_2} \, dt.\end{aligned}$$

Based on Assumption 3.2, the matrix $\mathbf{A}$ is positive definite. Applying Lemma 4.4 with arbitrary $\delta > 0$ to the middle term on the right hand side results in

$$\begin{aligned}&2 \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle \\ &\leq \underbrace{\delta \Delta x \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2}, \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2} \rangle}_{\Pi_1} + \underbrace{\frac{1}{\delta \Delta x} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) [\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle}_{\Pi_2}.\end{aligned}$$

This implies

$$\sum_{n,i} \int_{I_n} \Pi_2 \, dt + \sum_{n,i} \int_{I_n} \Gamma_2 \, dt \geq \frac{\gamma - \frac{1}{\delta}}{\Delta x} \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\}) [\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2} \rangle \, dt.$$

Let us consider the remaining terms. By assumption, the eigenvalues of the matrix $\mathbf{A}$ are uniformly bounded, i.e., there holds $0 < \lambda \leq \lambda_1 \leq \ldots \leq \lambda_m \leq \Lambda$. Therefore,

$$\begin{aligned}&\sum_{n,i} \int_{I_n} \int_{K_i} \Gamma_1 \, dx \, dt + \sum_{n,i} \int_{I_n} \Pi_1 \, dt \geq \\ &\qquad \sum_{n,i} \int_{I_n} \int_{K_i} \lambda \langle \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x} \rangle \, dx \, dt - \sum_{n,i} \int_{I_n} \delta \Delta x \Lambda \langle \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2}, \left\{\mathbf{V}_x^{\Delta x}\right\}_{i+1/2} \rangle \, dt.\end{aligned}$$

As

$$\langle \{\mathbf{V}_x^{\Delta x}\}_{i+1/2}, \{\mathbf{V}_x^{\Delta x}\}_{i+1/2}\rangle = \sum_{j=1}^{m} \Big(\{(v_j^{\Delta x})_x\}_{i+1/2}\Big)^2,$$

we can apply an inverse estimate to each component. Using

$$\Big(\{v_x^{\Delta x}\}_{i+1/2}\Big)^2 = \left(\frac{1}{2}(v_{x,i+1/2,L}^{\Delta x} + v_{x,i+1/2,R}^{\Delta x})\right)^2 \leq \frac{1}{2}(v_{x,i+1/2,L}^{\Delta x})^2 + \frac{1}{2}(v_{x,i+1/2,R}^{\Delta x})^2$$

and Lemma 4.5, we get

$$\sum_{n,i} \int_{I_n} \delta \Delta x \Lambda \langle \{\mathbf{V}_x^{\Delta x}\}_{i+1/2}, \{\mathbf{V}_x^{\Delta x}\}_{i+1/2}\rangle \, dt \leq \sum_{n,i} \int_{I_n} \int_{K_i} c_{\mathrm{inv}} \delta \Lambda \langle \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x}\rangle \, dx \, dt.$$

This implies

$$\sum_{n,i} \int_{I_n} \int_{K_i} \Gamma_1 \, dx \, dt + \sum_{n,i} \int_{I_n} \Pi_1 \, dt \geq (\lambda - c_{\mathrm{inv}} \delta \Lambda) \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x}\rangle \, dx \, dt.$$

Summarizing all results, there holds

$$\begin{aligned} \mathcal{B}_{\mathrm{IP}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq (\lambda - c_{\mathrm{inv}} \delta \Lambda) &\sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{V}_x^{\Delta x}, \mathbf{V}_x^{\Delta x}\rangle \, dx \, dt \\ &+ \frac{\gamma - \frac{1}{\delta}}{\Delta x} \sum_{n,i} \int_{I_n} \langle \mathbf{A}(\{\mathbf{V}^{\Delta x}\})[\mathbf{V}^{\Delta x}]_{i+1/2}, [\mathbf{V}^{\Delta x}]_{i+1/2}\rangle \, dt, \end{aligned}$$

which concludes the proof. $\square$

Remark 4.2. *For solving the scalar equation $u_t + f(u)_x = (au_x)_x$ with $0 < \underline{a} \leq a \leq \overline{a}$ and bounds $m \leq S_{uu} \leq M$, the condition* (23) *on γ reduces to*

$$\gamma > \frac{c_{\mathrm{inv}}(\overline{a}/m)}{\underline{a}/M}. \tag{25}$$

Remark 4.3. *We now like to return to our claim that under Assumption 3.3 (uniform bounds on the computed solutions), the ST-SIPG method is entropy stable for the compressible Navier-Stokes equations if γ is chosen sufficiently large. Based on the structure of $\mathbf{A}$ given by* (18) *and the fact that all terms in the form $\mathcal{B}_{\mathrm{IP}}$ contain the matrix $\mathbf{A}$, there holds for $\widetilde{\mathbf{V}}^{\Delta x} = (v_2^{\Delta x}, v_3^{\Delta x})$ and $\widetilde{\mathbf{\Phi}}^{\Delta x} = (\Phi_2^{\Delta x}, \Phi_3^{\Delta x})$,*

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \widetilde{\mathcal{B}_{\mathrm{IP}}}(-1; \widetilde{\mathbf{V}}^{\Delta x}, \widetilde{\mathbf{\Phi}}^{\Delta x}) \quad \forall\, \mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x} \in \mathcal{V}^k,$$

with $\widetilde{\mathcal{B}_{\mathrm{IP}}}$ defined as $\mathcal{B}_{\mathrm{IP}}$ (compare (20)*) but with $\mathbf{A}$ replaced by $\tilde{\mathbf{A}}$ and the scalar product being taken only over 2 components. Due to this equivalence, it is sufficient to implement the shortened form $\widetilde{\mathcal{B}_{\mathrm{IP}}}$. For entropy stability, one now needs to show*

$$\widetilde{\mathcal{B}_{\mathrm{IP}}}(-1; \widetilde{\mathbf{V}}^{\Delta x}, \widetilde{\mathbf{V}}^{\Delta x}) \geq 0.$$

In order to apply the proof of Lemma 4.3, Assumption 3.2 needs to be satisfied for the matrix $\tilde{\mathbf{A}}$. This necessitates uniform bounds on the eigenvalues of $\tilde{\mathbf{A}}$, which can be deduced from Assumption 3.3.

5. The ST-LDG method

Our second approach for extending the original spacetime DG method is a variant of the LDG method introduced by Cockburn and Shu [8]. For the ST-LDG method, we take Assumption 3.1 for granted (which assumes that $\mathbf{A}$ is positive semi-definite). This implies that there exists a symmetric positive semi-definite matrix $\mathbf{B}(\mathbf{V})$ such that $\mathbf{B}^2 = \mathbf{A}$.

Remark 5.1. *We refer to Appendix B for a description of the matrix* $\mathbf{B}$ *that we use in our numerical algorithm for solving the compressible Navier-Stokes equations.*

Using this definition, we now consider the convection-diffusion system

$$\mathbf{U}_t + \mathbf{F}(\mathbf{V})_x = (\mathbf{B}^2(\mathbf{V})\mathbf{V}_x)_x.$$

We define $\mathbf{P} = \mathbf{B}(\mathbf{V})\mathbf{V}_x$ and rewrite the second-order system as

$$\begin{aligned} \mathbf{U}_t + \mathbf{F}(\mathbf{V})_x &= (\mathbf{B}(\mathbf{V})\mathbf{P})_x, \\ \mathbf{P} &= \mathbf{B}(\mathbf{V})\mathbf{V}_x. \end{aligned}$$

We note that we keep the second set of equations in *non-conservative* form. In the original LDG method, all equations are written in conservation form. To achieve this, an auxiliary function $g(\mathbf{V})$ is introduced with the property that $g(\mathbf{V})_x = -\mathbf{B}(\mathbf{V})\mathbf{V}_x$. This is very practical in terms of theory. Numerically, however, it is very complicated to evaluate g if the matrix $\mathbf{B}$ depends non-linearly on $\mathbf{V}$, which is the case for us. Therefore, we use the non-conservative form. Note though that the first set of equations, which contains the conserved quantity $\mathbf{U}$, is written in conservation form.

Let the discrete solution be given by the pair $\mathbf{W}^{\Delta x} = (\mathbf{V}^{\Delta x}, \mathbf{P}^{\Delta x})$. To deduce the variational formulation, we multiply the first set of equations with a test function $\mathbf{\Phi}^{\Delta x}$, integrate over spacetime elements, and do integration by parts using central fluxes for the diffusion terms. This results in

$$\begin{aligned} \mathcal{B}^1(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = {} & \mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) \\ & + \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{B}(\mathbf{V}^{\Delta x})\mathbf{P}^{\Delta x}, \mathbf{\Phi}_x^{\Delta x} \rangle \, dx \, dt \\ & - \sum_{n,i} \int_{I_n} \langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i+1/2} \left\{\mathbf{P}^{\Delta x}\right\}_{i+1/2}, \mathbf{\Phi}^{\Delta x}_{i+1/2,L} \rangle \, dt \\ & + \sum_{n,i} \int_{I_n} \langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i-1/2} \left\{\mathbf{P}^{\Delta x}\right\}_{i-1/2}, \mathbf{\Phi}^{\Delta x}_{i-1/2,R} \rangle \, dt \end{aligned} \tag{26}$$

with $\{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i+1/2} = \left(\mathbf{B}(\mathbf{V}^{\Delta x}_{i+1/2,L}) + \mathbf{B}(\mathbf{V}^{\Delta x}_{i+1/2,R})\right)/2$. For the discretization of the non-conservative equations $\mathbf{P} = \mathbf{B}(\mathbf{V})\mathbf{V}_x$, we multiply both sides with a test function $\mathbf{\Psi}^{\Delta x}$, integrate over spacetime elements, and then add stability terms resulting in

$$
\begin{aligned}
\mathcal{B}^2(\mathbf{W}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) = &\sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{P}^{\Delta x}, \mathbf{\Psi}^{\Delta x} \rangle \, dx \, dt \\
&- \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{B}(\mathbf{V}^{\Delta x}) \mathbf{V}^{\Delta x}_x, \mathbf{\Psi}^{\Delta x} \rangle \, dx \, dt \\
&- \frac{1}{2} \sum_{n,i} \int_{I_n} \langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i+1/2} [\mathbf{V}^{\Delta x}]_{i+1/2}, \mathbf{\Psi}^{\Delta x}_{i+1/2,L} \rangle \, dt \\
&- \frac{1}{2} \sum_{n,i} \int_{I_n} \langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i-1/2} [\mathbf{V}^{\Delta x}]_{i-1/2}, \mathbf{\Psi}^{\Delta x}_{i-1/2,R} \rangle \, dt.
\end{aligned}
\tag{27}
$$

We note that we did not apply integration by parts in this case. The method without streamline diffusion and shock capturing terms then reads: find $\mathbf{W}^{\Delta x} = (\mathbf{V}^{\Delta x}, \mathbf{P}^{\Delta x}) \in \mathcal{V}^k \times \mathcal{V}^k$ such that

$$
\mathcal{B}^1(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}^2(\mathbf{W}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) = 0 \tag{28}
$$

for all $\mathbf{\Sigma}^{\Delta x} = (\mathbf{\Phi}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) \in \mathcal{V}^k \times \mathcal{V}^k$.

Remark 5.2. *The method* (28) *is* consistent *for solving equation* (13) *if the true solution* $\mathbf{V}$ *is continuous and if* $\mathbf{B}(\mathbf{V})$ *depends continuously on* $\mathbf{V}$*.*

It remains to adjust $\mathcal{B}_{\mathrm{SD}}$ and $\mathcal{B}_{\mathrm{SC}}$ appropriately. We first adjust the residual (given by (9) for the original method) to

$$
\mathrm{Res}^{\mathrm{LDG}} = \mathbf{U}(\mathbf{V}^{\Delta x})_t + \mathbf{F}(\mathbf{V})_x - \mathbf{B}(\mathbf{V}^{\Delta x})_x \mathbf{P}^{\Delta x} - \mathbf{B}(\mathbf{V}^{\Delta x}) \mathbf{P}^{\Delta x}_x.
$$

For the shock capturing term $\mathcal{B}_{\mathrm{SC}}^{\mathrm{LDG}}$, which is only added to the first set of equations, no further adjustment is necessary. The streamline diffusion term $\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG}}$ is split up in two parts:

1. in the first set of equations, we use

$$
\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},1}(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{U}_{\mathbf{V}}(\mathbf{V}^{\Delta x}) \mathbf{\Phi}^{\Delta x}_t + \mathbf{F}_{\mathbf{V}}(\mathbf{V}^{\Delta x}) \mathbf{\Phi}^{\Delta x}_x, \mathbf{D}^{SD} \, \mathrm{Res}^{\mathrm{LDG}} \rangle \, dx \, dt,
$$

2. in the second set of equations, we use

$$
\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},2}(\mathbf{W}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle -\mathbf{B}(\mathbf{V}^{\Delta x})_x \mathbf{\Psi}^{\Delta x} - \mathbf{B}(\mathbf{V}^{\Delta x}) \mathbf{\Psi}^{\Delta x}_x, \mathbf{D}^{SD} \, \mathrm{Res}^{\mathrm{LDG}} \rangle \, dx \, dt.
$$

Then, the complete ST-LDG method is given by: find $\mathbf{W}^{\Delta x} = (\mathbf{V}^{\Delta x}, \mathbf{P}^{\Delta x}) \in \mathcal{V}^k \times \mathcal{V}^k$ such that

$$
\begin{aligned}
\mathcal{B}_{\mathrm{ST-LDG}}(\mathbf{W}^{\Delta x}, \mathbf{\Sigma}^{\Delta x}) = &\mathcal{B}^1(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}^2(\mathbf{W}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) \\
&+ \mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},1}(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},2}(\mathbf{W}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) + \mathcal{B}_{\mathrm{SC}}^{\mathrm{LDG}}(\mathbf{W}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = 0
\end{aligned}
\tag{29}
$$

for all $\mathbf{\Sigma}^{\Delta x} = (\mathbf{\Phi}^{\Delta x}, \mathbf{\Psi}^{\Delta x}) \in \mathcal{V}^k \times \mathcal{V}^k$.

We can show the following result concerning entropy stability.

Theorem 5.1 (Entropy stability for ST-LDG)**.** *Let Assumption 3.1 hold true. Consider the system* (13) *with strictly convex entropy function S and entropy flux function Q. For simplicity, also assume that the exact and approximate solutions have compact support inside the spatial domain. Let the final time be denoted by t_N. Then, the approximate solutions generated by the scheme* (29) *satisfy*

$$\int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x,t_{N,-})))\,dx + \int_{t_0}^{t_N}\int_\Omega (\mathbf{P}^{\Delta x})^2\,dx\,dt \le \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x,t_0)))\,dx.$$

Proof. We set $(\mathbf{\Phi}^{\Delta x},\mathbf{\Psi}^{\Delta x}) = (\mathbf{V}^{\Delta x},\mathbf{P}^{\Delta x})$ to get

$$\begin{aligned}
0 &= \mathcal{B}_{\text{ST}-\text{LDG}}(\mathbf{W}^{\Delta x},\mathbf{W}^{\Delta x})\\
&= \mathcal{B}_{\text{SD}}^{\text{LDG},1}(\mathbf{W}^{\Delta x},\mathbf{V}^{\Delta x}) + \mathcal{B}_{\text{SD}}^{\text{LDG},2}(\mathbf{W}^{\Delta x},\mathbf{P}^{\Delta x}) + \mathcal{B}_{\text{SC}}^{\text{LDG}}(\mathbf{W}^{\Delta x},\mathbf{V}^{\Delta x})\\
&\quad + \mathcal{B}_{\text{DG}}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}) + \sum_{n,i}\int_{I_n}\int_{K_i}\langle \mathbf{P}^{\Delta x},\mathbf{P}^{\Delta x}\rangle\,dx\,dt\\
&\quad + \sum_{n,i}\int_{I_n}\int_{K_i}\big(\langle \mathbf{B}(\mathbf{V}^{\Delta x})\mathbf{P}^{\Delta x},\mathbf{V}^{\Delta x}_x\rangle - \langle \mathbf{B}(\mathbf{V}^{\Delta x})\mathbf{V}^{\Delta x}_x,\mathbf{P}^{\Delta x}\rangle\big)\,dx\,dt\\
&\quad - \sum_{n,i}\int_{I_n}\langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i+1/2}\,\{\mathbf{P}^{\Delta x}\}_{i+1/2},\mathbf{V}^{\Delta x}_{i+1/2,L}\rangle\,dt\\
&\quad + \sum_{n,i}\int_{I_n}\langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i-1/2}\,\{\mathbf{P}^{\Delta x}\}_{i-1/2},\mathbf{V}^{\Delta x}_{i-1/2,R}\rangle\,dt\\
&\quad - \frac{1}{2}\sum_{n,i}\int_{I_n}\langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i+1/2}[\mathbf{V}^{\Delta x}]_{i+1/2},\mathbf{P}^{\Delta x}_{i+1/2,L}\rangle\,dt\\
&\quad - \frac{1}{2}\sum_{n,i}\int_{I_n}\langle \{\mathbf{B}(\mathbf{V}^{\Delta x})\}_{i-1/2}[\mathbf{V}^{\Delta x}]_{i-1/2},\mathbf{P}^{\Delta x}_{i-1/2,R}\rangle\,dt.
\end{aligned}$$

We examine the single terms, starting from below. Taking the symmetry of $\mathbf{B}$ and the compact support of the discrete solution into account, the boundary terms in the last four lines cancel each other. Also, the domain terms in the line above the boundary terms cancel each other. For the remaining terms, there holds

1. for the streamline diffusion term

$$\mathcal{B}_{\text{SD}}^{\text{LDG},1}(\mathbf{W}^{\Delta x},\mathbf{V}^{\Delta x}) + \mathcal{B}_{\text{SD}}^{\text{LDG},2}(\mathbf{W}^{\Delta x},\mathbf{P}^{\Delta x}) = \sum_{n,i}\int_{I_n}\int_{K_i}\langle \text{Res}^{\text{LDG}},\mathbf{D}^{SD}\,\text{Res}^{\text{LDG}}\rangle\,dx\,dt \ge 0,$$

2. for the shock capturing term

$$\mathcal{B}_{\text{SC}}^{\text{LDG}}(\mathbf{W}^{\Delta x},\mathbf{V}^{\Delta x}) \ge 0,$$

3. and for the $\mathcal{B}_{\text{DG}}$ term due to Lemma 4.1

$$\mathcal{B}_{\text{DG}}(\mathbf{V}^{\Delta x},\mathbf{V}^{\Delta x}) \ge \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x,t_{N,-})))\,dx - \int_\Omega S(\mathbf{U}(\mathbf{V}^{\Delta x}(x,t_0)))\,dx.$$

This then directly implies the claim. □

6. Numerical results

In this section we present various numerical results, comparing the ST-NIPG, the ST-SIPG, and the ST-LDG method. In all our tests, the choice of the time step is purely based on the convection term, i.e., it does not take the presence of a diffusion term into account. We note that for stability reasons it is not necessary to restrict the time step due to the spacetime DG approach. But for accuracy reasons, we typically use the CFL condition

$$\Delta t_n \leq C_{\mathrm{CFL}} \min_{x\in\Omega} \frac{\Delta x}{\lambda_{\max}(\mathbf{U}^{\Delta x}(x,t_n))}$$

with Δt_n denoting the time step from t_n to t_{n+1} and $C_{\mathrm{CFL}} = 0.5$. We also use equidistant grid cells in our tests, but this is not necessary.

Our code is an extension of the one-dimensional version of SPARCCLE - the software package developed for the original scheme for hyperbolic conservation laws [21]. We refer to [21, 22] for more detailed information concerning the implementation and only give a short summary here.

The approximate solution $\mathbf{V}^{\Delta x} = (v_1^{\Delta x}, \ldots, v_m^{\Delta x})^T$ is sought in $\mathcal{V}^k$ with each component $v_j^{\Delta x}$ being of the form

$$v_j^{\Delta x} = \sum_{n,i,l} \hat{v}_{i,j,l}^n \phi_{i,l}^n$$

with n indicating the time segment, i the spatial cell, and $1 \leq l \leq n_f$ the degree of freedom depending on the choice of $\mathcal{V}^k$. Further, $\phi_{i,l}^n$ are basis functions with finite support given by

$$\phi_{i,l}^n\big|_{K_i\times I_n} = \left(\frac{t-t_{n+1}}{\Delta t_n}\right)^{k_{t,l}} \left(\frac{x-x_i}{\Delta x}\right)^{k_{x,l}}$$

with x_i denoting the centroid of the spatial cell i, and $k_{t,l}+k_{x,l} \leq k$. All spacetime and boundary integrals appearing in the numerical methods are evaluated using Gaussian quadrature formulae of the appropriate order.

While the form $\mathcal{B}_{\mathrm{DG}}$ is non-linear in the discrete solution, it is linear in the test function. The same holds true for $\mathcal{B}_{\mathrm{SD}}$ and $\mathcal{B}_{\mathrm{SC}}$. Therefore, it is sufficient to satisfy (7), the method for conservation laws, for all basis functions of $\mathcal{V}^k$. Due to the choice of the upwind flux in time, one can solve each time step separately. All in all, in each time step, one needs to solve a non-linear system with $N_c \times n_f \times m$ unknowns, N_c denoting the number of spatial cells. Newton method with an analytically computed Jacobian is used for this purpose. For test problems in one dimension, it is typically sufficient to use a sparse LU decomposition in order to solve the linear problem in each Newton iteration. For the two-dimensional version of the code, suitable preconditioners have been developed to efficiently solve these subproblems [22].

In order to extend the code for conservation laws to convection-diffusion systems, some adjustments were necessary. Analogous to the original method, it is sufficient to satisfy equations (19) and (29) for all basis functions of $\mathcal{V}^k$. For the ST-NIPG and the ST-SIPG method, additional terms needed to be added in the evaluation of the residual and of the Jacobian of the non-linear system. For the ST-LDG method, the changes were more substantial due to the introduction of the auxiliary variables. We have not yet examined whether it is possible to eliminate the auxiliary variables in the implementation, which would result in a system corresponding to the size of the original variables (compare also the discussion in section 7).

In the following, we will first show results for the scalar linear advection diffusion equation. Then, in section 6.2, we will present results for the compressible Navier-Stokes equations.

6.1. Numerical results for the linear advection diffusion equation

We start with the linear advection diffusion equation

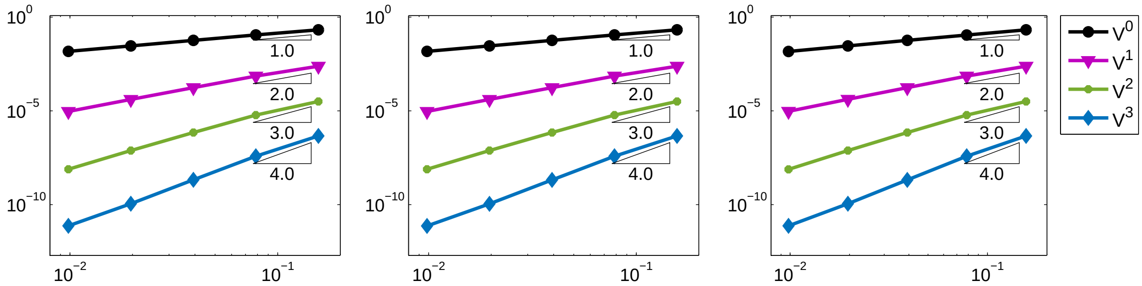
$$\partial_t u + c\partial_x u = a\partial_x^2 u$$

with c and a constant (compare [8]). The initial data are chosen as

$$u(t=0,x) = \sin(x)$$

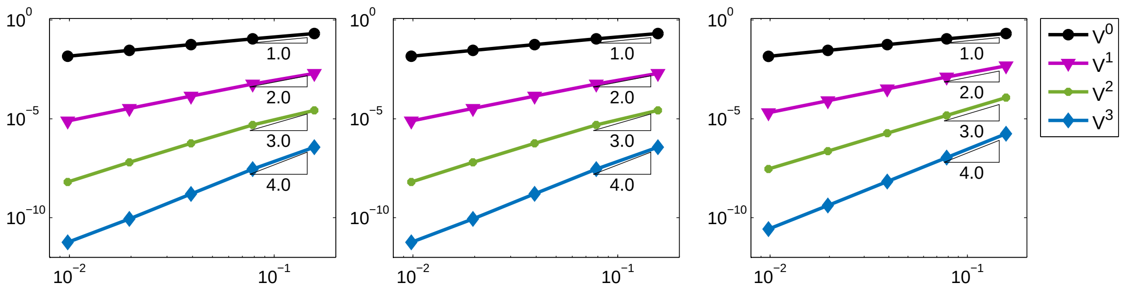
on the domain $[0{,}2\pi]$ with periodic boundary conditions. The exact solution is $u(t,x) = e^{-at}\sin(x-ct)$. Since we are interested in convection-dominated problems, we use the parameters $c=1$ and $a=10^{-5}$. We compute the solution at $T=2$.

We use the quadratic entropy function $S(u) = \frac{1}{2}u^2$ and central flux for the entropy conservative flux $\mathbb{F}^*$, i.e., $\mathbb{F}^*(a,b) = \frac{1}{2}(a+b)$. We add Rusanov diffusion, which results in the numerical flux $\mathbb{F}$ corresponding to standard upwind flux.



(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

Figure 1: Lin. adv. diff. eqn. for $a = 10^{-5}$ *without* $\mathcal{B}_{\text{SD}}^{(\cdot)}$ and $\mathcal{B}_{\text{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.

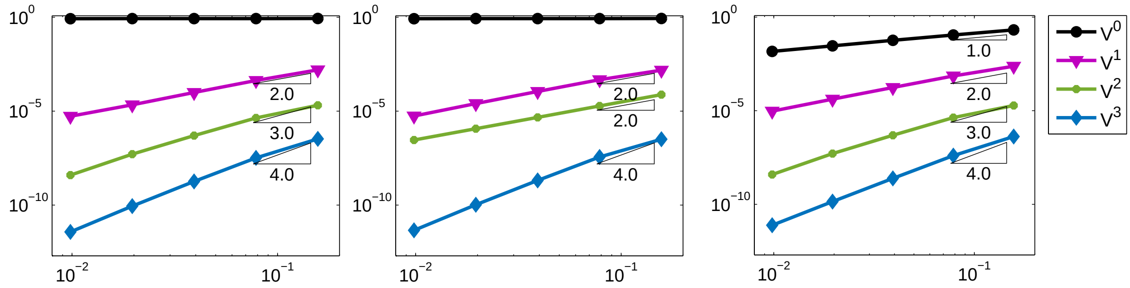


(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

Figure 2: Lin. adv. diff. eqn. for $a = 10^{-5}$ *with* $\mathcal{B}_{\text{SD}}^{(\cdot)}$ and $\mathcal{B}_{\text{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.

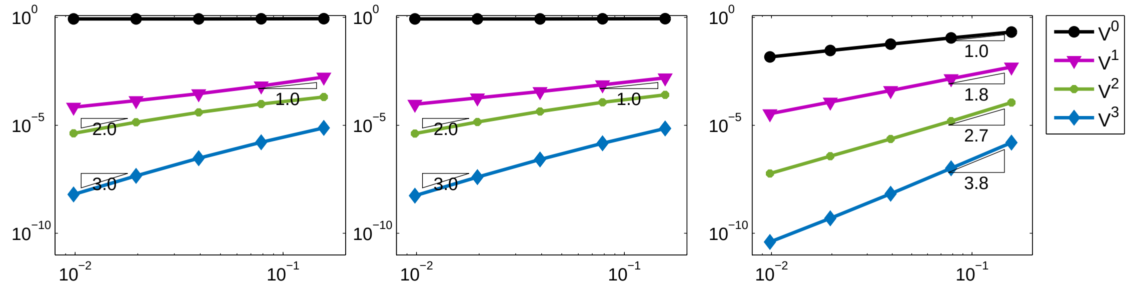
As artificial diffusion terms are known to typically decrease the accuracy of a smooth solution, we drop the streamline diffusion and shock capturing terms in our methods. Figure 1 shows the results for the ST-SIPG method, the ST-NIPG method (both using $\gamma = 10$), and the ST-LDG for varying spaces $\mathcal{V}^k$, $k = 0, 1, 2, 3$. The result is (almost) identical for all three methods. We observe convergence orders of $O(\Delta x^{k+1})$ for all three methods.

Next, we repeat the test with the full methods, i.e., we also use the streamline diffusion and shock capturing terms even though they are not necessary for this smooth test. The result is shown in Figure 2. Even with the terms $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ included, all methods show convergence rates of $O(\Delta x^{k+1})$.



(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

Figure 3: Lin. adv. diff. eqn. for $a = 0.1$ *without* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.



(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

Figure 4: Lin. adv. diff. eqn. for $a = 0.1$ *with* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.

The fact that the result is very insensitive to the choice of the method indicates that the diffusion term might be too small relative to the mesh width. Indeed, on the finest grid tested, we use the mesh width $\Delta x \approx 10^{-2}$ which is significantly bigger than the diffusion coefficient $a = 10^{-5}$. Instead of using extremely fine grids, we repeat the test with $a = 0.1$. For this test, the different discretizations of the diffusion operator should make a difference.

Figure 3 shows the results for the ST-SIPG method, the ST-NIPG method (both using $\gamma = 10$), and the ST-LDG for dropping again the artificial viscosity terms. For the ST-LDG method we observe the same convergence orders as before: the method converges with $O(\Delta x^{k+1})$ for the spaces $\mathcal{V}^k$. Both the ST-SIPG and the ST-NIPG method do not converge for

$\mathcal{V}^0$. This is consistent with the fact that for piecewise constant polynomials, most terms in the $\mathcal{B}_{\mathrm{IP}}$ form drop out and only the penalty term is left. In addition, the ST-NIPG method shows the fairly well-known phenomenon (see Rivière [30]) of a reduced convergence order $O(\Delta x^k)$ for k even.

Finally, we repeat the test with $a = 0.1$ using the complete methods, i.e., with streamline diffusion and shock capturing terms. The result is shown in Figure 4. For the ST-LDG method, we observe almost ideal convergence rates. For both the ST-SIPG and the ST-NIPG method, we observe suboptimal convergence rates of $O(\Delta x^k)$. We believe that this is due to the fact that $\mathrm{Res}^{\mathrm{IP}}$ (different to Res and $\mathrm{Res}^{\mathrm{LDG}}$) now involves second derivatives, see (21). This reduces the convergence order of the residual. Note that the complete ST-NIPG method (with $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}$ and $\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}$ included) leads to almost identical results as the complete ST-SIPG method for this test.

To conclude this test, we examine the dependence of the ST-SIPG method on the penalty parameter γ for $a = 0.1$ without using $\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}$ and $\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}$. The stability condition for scalar equations is given by (25). In this fairly simple test, there holds $\underline{a} = \overline{a} = a$ and $m = M = 1$. Therefore, the stability condition requires

$$\gamma > c_{\mathrm{inv}}.$$

Table 1: Lin. adv. diff. eqn.: Influence of γ on the stability of the ST-SIPG method. '×' denotes an unstable test, '✓' a stable one.

	$\mathcal{V}^1$	$\mathcal{V}^2$	$\mathcal{V}^3$
$\gamma = 0.1$	×	×	×
$\gamma = 1$	✓	×	×
$\gamma = 10$	✓	✓	✓

Numerically, we observe the results shown in Table 6.1. This is consistent with our theoretical stability considerations that for $\mathcal{V}^1$ polynomials the inverse estimate holds for our polynomials with $c_{\mathrm{inv}} = 1$, whereas for $\mathcal{V}^2$ and $\mathcal{V}^3$ the constant c_{inv} cannot be smaller than 3. In terms of accuracy, the influence of γ seems to be very small: in most of our tests there was only a small dependence of the accuracy on the parameter γ. Overall, choosing $\gamma = 10$ seemed to be a good default choice for the ST-SIPG method.

6.2. Numerical results for the compressible Navier-Stokes equations

In this section, we solve the compressible Navier-Stokes equations in one dimension given by (14) with Dirichlet boundary conditions and $\gamma = 1.4$. (Here, γ refers to the adiabatic exponent in the compressible Navier-Stokes equations.) We follow [24] for the entropy conservative flux $\mathbb{F}^*$ for the convection terms. Entropy stable flux is attained by adding the Rusanov diffusion operator.

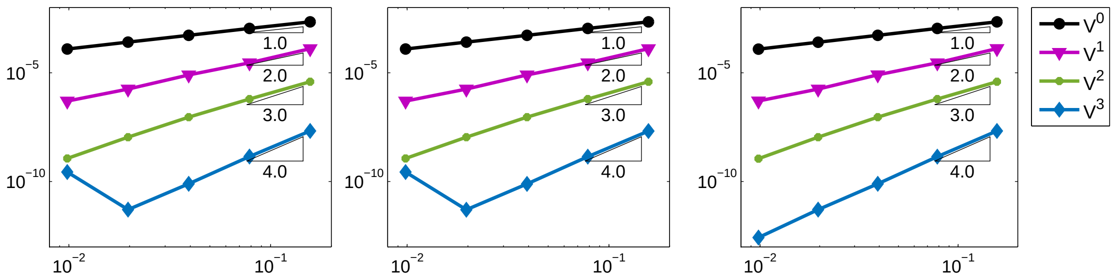
Remark 6.1. *In the following tests, we will use $\gamma = 10$ for the ST-SIPG and the ST-NIPG method.*

6.2.1. Manufactured solution

We start with a test that has a manufactured solution. To assess the accuracy of our methods, we like the solution to be given by the smooth functions

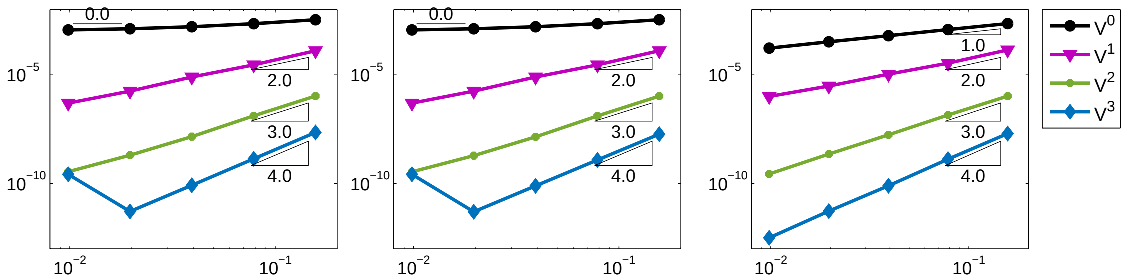
$$\begin{aligned} \rho(x,t) &= \sin(x^2 + 5t) + 1.5, \\ u(x,t) &= 2\left[\sin(x^2 + 5t) + 0.1\right], \\ e(x,t) &= 3\left[\cos(x^2 + 5t) + 1.5\right]. \end{aligned}$$

We insert this solution into the compressible Navier-Stokes equations and compute the corresponding source terms that need to be added on the right hand side of the equations to render the above triple (ρ, u, e) a solution of the resulting equations. For this test, we do not use our artificial viscosity terms as they are not built to deal with source terms. We use the viscosity coefficient $\nu = 2 \cdot 10^{-5}$ and entropy stable flux. The test domain is given by $\Omega = [-0.1; 0.9]$ and the final time is $T = 0.05$.



(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

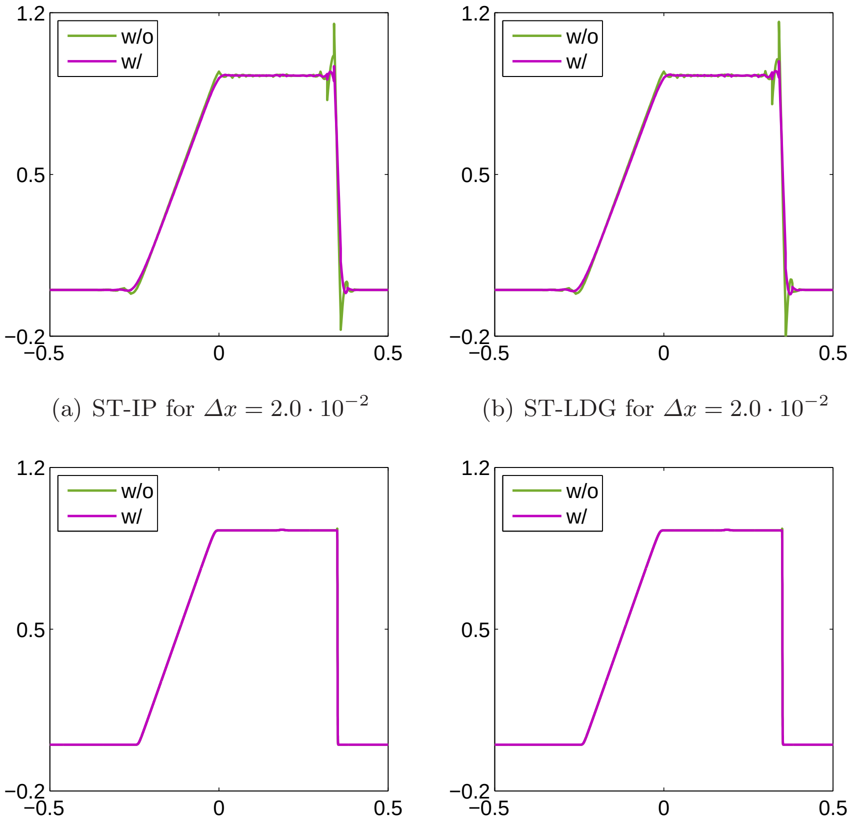
Figure 5: Manufactured solution for $\nu = 2.0 \cdot 10^{-5}$ *without* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.



(a) ST-SIPG (b) ST-NIPG (c) ST-LDG

Figure 6: Manufactured solution for $\nu = 0.02$ *without* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms: The $x-$axis denotes the mesh width Δx, the $y-$axis the L^1 error. All plots use the same axes scaling for better comparison.

Figure 5 shows the error in density. Momentum and energy behave qualitatively the same. We observe optimal convergence rates $O(\Delta x^{k+1})$ for all methods. This indicates that the viscosity was not resolved. Therefore, we repeat the test with $\nu = 0.02$. The result for the error in density is shown in Figure 6. For this choice of viscosity coefficient we observe the expected convergence stalling for the ST-SIPG and the ST-NIPG method for $\mathcal{V}^0$. Surprisingly though the ST-NIPG method converges with third order for $\mathcal{V}^2$. A comparison with the error in momentum and energy shows that for the ST-NIPG method the error in energy only converges with second order for $\mathcal{V}^2$. Therefore, the results for this test are consistent with the results for the linear advection diffusion equation.



(a) ST-IP for $\Delta x = 2.0 \cdot 10^{-2}$

(b) ST-LDG for $\Delta x = 2.0 \cdot 10^{-2}$

(c) ST-IP for $\Delta x = 5.0 \cdot 10^{-4}$

(d) ST-LDG for $\Delta x = 5.0 \cdot 10^{-4}$

Figure 7: Modified Sod test: Comparison for the velocity component for the ST-IP method and the ST-LDG method for $\mathcal{V}^2$. (As the results for the ST-SIPG and the ST-NIPG method are very similar, we show them together and refer to the method as 'ST-IP' method.) The plots show the solution for using the methods both *without* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms ('w/o') and *with* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms ('w/').

6.2.2. Modified Sod test

Our next test is a modified Sod problem, similar to the test in [33]. We consider initial data

$$(\rho, m, E) = \begin{cases} (1.0, 0.0, 2.5) & \text{if } x < 0, \\ (0.125, 0.0, 0.25) & \text{if } x > 0, \end{cases}$$

on the domain $\Omega = [-0.5, 0.5]$. The viscosity $\nu = 2.5 \cdot 10^{-5}$ is fairly small.

Figure 7 shows the result for velocity using entropy stable flux and $\mathcal{V}^2$ with the final time $T = 0.2$. We first consider the case of a coarse grid width $\Delta x = 2.0 \cdot 10^{-2}$. In this case, the small viscosity $\nu = 2.5 \cdot 10^{-5}$ cannot be resolved and the solution shows oscillations around the shock if no streamline diffusion and shock capturing terms are used. When these terms are present, the oscillations are almost gone.

For a fine mesh width $\Delta x = 5.0 \cdot 10^{-4}$, the artificial diffusion terms are not necessary: The diffusion from the physical viscosity and heat conduction term is sufficient for reducing the

Table 2: Comparison of the ST-NIPG, the ST-SIPG, and the ST-LDG method.

	ST-NIPG	ST-SIPG	ST-LDG
Entropy stability:			
• Ass. on matrix $\mathbf{A}$ in (13)	Ass. 3.1	Ass. 3.2	Ass. 3.1
• applies to compr. NS eqns	✓	Ass. 3.3 & (23)	✓
Num. results for $\mathcal{V}^k$, $k \geq 1$:			
• smooth flow			
∘ w/o $\mathcal{B}^x_{\mathrm{SD}}$ & $\mathcal{B}^x_{\mathrm{SC}}$	$O(\Delta x^k)$ / $O(\Delta x^{k+1})$	$O(\Delta x^{k+1})$	$O(\Delta x^{k+1})$
∘ w/ $\mathcal{B}^x_{\mathrm{SD}}$ & $\mathcal{B}^x_{\mathrm{SC}}$	$O(\Delta x^k)$	$O(\Delta x^k)$	$O(\Delta x^{k+1})$
• shock problem	—— very comparable ——		

oscillations around the shock. But the presence of the streamline diffusion and shock capturing terms also does not deteriorate the solution. This confirms that our new methods are equally applicable to the fully resolved and the underresolved regime. Overall, we observe very similar behavior for all three methods (ST-SIPG, ST-NIPG, ST-LDG) for this test.

6.2.3. Modified Shu-Osher test

Finally, to test the robustness of our scheme, we use a modified Shu-Osher test. We consider initial data

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.33333) & \text{if } x < -4.0, \\ (1 + 0.2\sin(5.0x), 0.0, 1.0) & \text{if } x > -4.0, \end{cases}$$

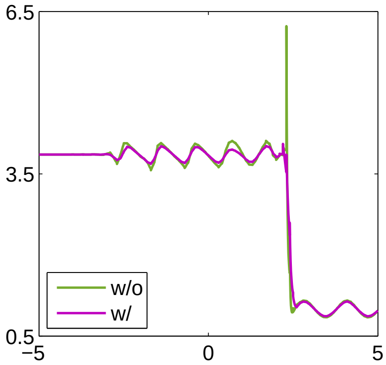
on the domain $\Omega = [-5.0, 5.0]$ with final time $T = 1.8$. The viscosity is chosen as $\nu = 4.0 \cdot 10^{-3}$ and we use entropy stable flux.

Figures 8(a)-8(d) show the computed solutions for density for two different grid sizes using $\mathcal{V}^1$ polynomials. Again, the results for all three methods (ST-NIPG, ST-SIPG, ST-LDG) are very similar. On the coarse grid, we observe big overshoot when using the methods without discontinuity capturing terms – including the $\mathcal{B}^{(\cdot)}_{\mathrm{SD}}$ and $\mathcal{B}^{(\cdot)}_{\mathrm{SC}}$ terms mostly eliminates the overshoot. On the fine grid, the physical diffusion is much better resolved and therefore also the versions without artificial diffusion lead to quite good results. Finally, Figures 8(e) and 8(f) show the results for using coarse grids but $\mathcal{V}^2$ polynomials. Using higher order polynomials leads to much better results for this test. (We note that Figures 8(e) and 8(f) seem to indicate that there is barely any overshoot for $\mathcal{V}^2$ polynomials when not including the $\mathcal{B}^{(\cdot)}_{\mathrm{SD}}$ and $\mathcal{B}^{(\cdot)}_{\mathrm{SC}}$ terms. We did observe though significant overshoot during intermediate times of the simulation.)

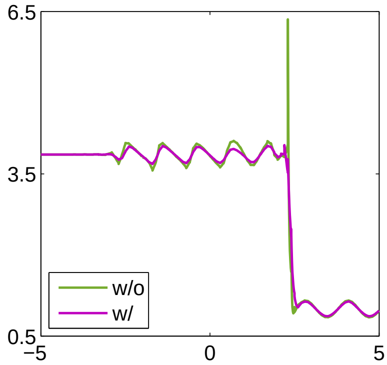
This completes the presentation of our numerical results. We conclude this work with a direct comparison of the ST-IP-DG and the ST-LDG method.

7. Comparison of the ST-NIPG, the ST-SIPG, and the ST-LDG method

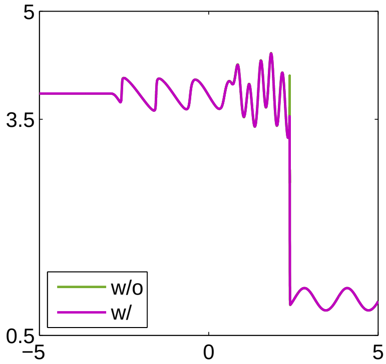
We summarize our main findings in Table 7. Both ST-IP methods have a drawback for at least one of the criteria considered. The ST-LDG method satisfies all criteria listed in Table 7 in an optimal way, which makes the ST-LDG method the 'winner' of this comparison. However,



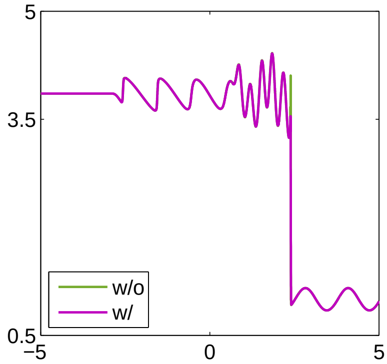
(a) ST-IP for $\Delta x = 1.0 \cdot 10^{-1}$, $\mathcal{V}^1$



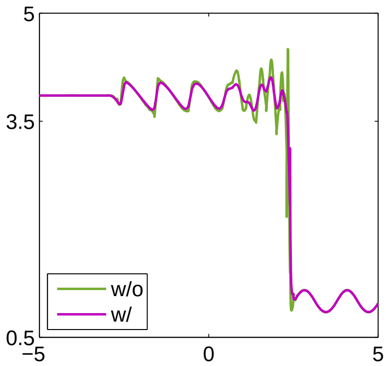
(b) ST-LDG for $\Delta x = 1.0 \cdot 10^{-1}$, $\mathcal{V}^1$



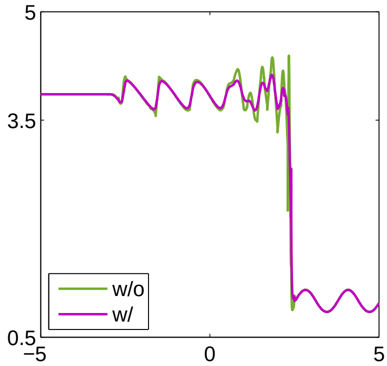
(c) ST-IP for $\Delta x = 4.0 \cdot 10^{-3}$, $\mathcal{V}^1$



(d) ST-LDG for $\Delta x = 4.0 \cdot 10^{-3}$, $\mathcal{V}^1$



(e) ST-IP for $\Delta x = 1.0 \cdot 10^{-1}$, $\mathcal{V}^2$



(f) ST-LDG for $\Delta x = 1.0 \cdot 10^{-1}$, $\mathcal{V}^2$

Figure 8: Modified Shu-Osher test: Comparison for the density component for the ST-IP method and the ST-LDG method. (As the results for the ST-SIPG and the ST-NIPG method are very similar, we show them together and refer to the method as 'ST-IP' method.) The plots show the solution for using the methods both *without* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms ('w/o') and *with* $\mathcal{B}_{\mathrm{SD}}^{(\cdot)}$ and $\mathcal{B}_{\mathrm{SC}}^{(\cdot)}$ terms ('w/').

the ST-IP methods have an advantage when it comes to the ease of implementation and the computational cost. For extending an existing code for conservation laws to convection-diffusion systems, the ST-IP methods only necessitate the implementation of some additional terms. For the ST-LDG method, one needs to introduce additional variables $\mathbf{P}$, which may lead to a change in the code structure. Also, the number of variables increases, e.g., from 3 to 5 for the compressible Navier-Stokes equations in one dimension (due to the special structure of the matrix $\mathbf{A}$). In each iteration of the Newton method, a linear system of the size $(m \times n_f \times N_c)^2$ must be solved. Depending on the solver used in this step, the increase in number of variables will result in a multiplied increase in cost. Furthermore, a suitable matrix $\mathbf{B}$ needs to be found for the method to work.

Even the comparison between ST-SIPG and ST-NIPG does not clearly favor one of them. The ST-NIPG needs less assumptions for entropy stability and avoids having to choose a suitable value for the penalty parameter γ. In terms of numerical results, the methods are comparable if artificial diffusion terms are used. Otherwise, the ST-SIPG method has a better order of convergence for even polynomial degrees. The ST-SIPG method also has the additional advantage of being symmetric, which is often helpful, e.g., for showing error estimates or in the context of optimal control problems.

At this stage, all of the three examined methods have their pros and cons – which will probably continue to hold true for the extensions to two and three dimensions. The proofs of entropy stability of the ST-NIPG and the ST-LDG methods should extend in a fairly straightforward way to higher dimensions. For the proof of entropy stability of the ST-SIPG method additional work will be necessary as the physical diffusion matrix $\mathbf{A} \in \mathbb{R}^{8\times 8}$ in two dimensions only has rank 5. But exploiting the structure of $\mathbf{A}$, we are very positive that it will be possible to transfer the proofs from one dimension to two. In terms of the practical aspects of the methods, the implementation of the ST-NIPG and the ST-SIPG will be fairly straightforward. This will not be the case for the ST-LDG method: besides solving the challenging task of finding a suitable matrix $\mathbf{B}$, it will be essential to find a way of eliminating the auxiliary variables as otherwise the non-linear solves will turn out to be too expensive.

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A. Proof of Lemma 4.1

Proof. The proof of (24) can be transferred from the proof of the entropy stability of the original scheme. Therefore, we do not give all details here. The full proof can be found in [21, Thm 3.1].

Step 1: Define the spatial part

$$\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = -\sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{F}(\mathbf{V}^{\Delta x}), \mathbf{\Phi}^{\Delta x}_x \rangle \, dx \, dt$$
$$+ \sum_{n,i} \int_{I_n} \Big(\langle \mathbb{F}(\mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbf{V}^{\Delta x}_{i+1/2,R}), \mathbf{\Phi}^{\Delta x}_{i+1/2,L} \rangle - \langle \mathbb{F}(\mathbf{V}^{\Delta x}_{i-1/2,L}, \mathbf{V}^{\Delta x}_{i-1/2,R}), \mathbf{\Phi}^{\Delta x}_{i-1/2,R} \rangle \Big) \, dt.$$

One can show that $\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq 0$: Due to the definition of the entropy flux function Q, there holds for the entropy potential $\psi_x = \langle \mathbf{V}_x, \mathbf{F} \rangle$. This implies

$$\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = -\sum_{n,i} \int_{I_n} \int_{K_i} \psi(\mathbf{V}^{\Delta x})_x \, dx \, dt$$
$$+ \sum_{n,i} \int_{I_n} \Big(\langle \mathbb{F}(\mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbf{V}^{\Delta x}_{i+1/2,R}), \mathbf{V}^{\Delta x}_{i+1/2,L} \rangle - \langle \mathbb{F}(\mathbf{V}^{\Delta x}_{i-1/2,L}, \mathbf{V}^{\Delta x}_{i-1/2,R}), \mathbf{V}^{\Delta x}_{i-1/2,R} \rangle \Big) \, dt.$$

Evaluating the (spatial) integral over ψ_x and using the definition of the flux $\mathbb{F}$ from (5) give

$$\begin{aligned}
\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = &\sum_{n,i} \int_{I_n} \Big(\langle \mathbb{F}^*(\mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbf{V}^{\Delta x}_{i+1/2,R}), \mathbf{V}^{\Delta x}_{i+1/2,L} \rangle - \psi(\mathbf{V}^{\Delta x}_{i+1/2,L}) \Big) \, dt \\
&- \sum_{n,i} \int_{I_n} \Big(\langle \mathbb{F}^*(\mathbf{V}^{\Delta x}_{i-1/2,L}, \mathbf{V}^{\Delta x}_{i-1/2,R}), \mathbf{V}^{\Delta x}_{i-1/2,R} \rangle - \psi(\mathbf{V}^{\Delta x}_{i-1/2,R}) \Big) \, dt \\
&- \frac{1}{2} \sum_{n,i} \int_{I_n} \langle \mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbb{D}(\mathbf{V}^{\Delta x}_{i+1/2,R} - \mathbf{V}^{\Delta x}_{i+1/2,L}) \rangle \, dt \\
&+ \frac{1}{2} \sum_{n,i} \int_{I_n} \langle \mathbf{V}^{\Delta x}_{i-1/2,R}, \mathbb{D}(\mathbf{V}^{\Delta x}_{i-1/2,R} - \mathbf{V}^{\Delta x}_{i-1/2,L}) \rangle \, dt.
\end{aligned}$$

Reordering the sum and exploiting the compact support of the approximate solutions result in

$$\begin{aligned}
\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = -\sum_{n,i} \int_{I_n} \big(&\langle \mathbb{F}^*(\mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbf{V}^{\Delta x}_{i+1/2,R}), \mathbf{V}^{\Delta x}_{i+1/2,R} - \mathbf{V}^{\Delta x}_{i+1/2,L} \rangle \\
&- (\psi(\mathbf{V}^{\Delta x}_{i+1/2,R}) - \psi(\mathbf{V}^{\Delta x}_{i+1/2,L})) \big) \, dt \\
&+ \frac{1}{2} \sum_{n,i} \int_{I_n} \langle \mathbf{V}^{\Delta x}_{i+1/2,R} - \mathbf{V}^{\Delta x}_{i+1/2,L}, \mathbb{D}(\mathbf{V}^{\Delta x}_{i+1/2,R} - \mathbf{V}^{\Delta x}_{i+1/2,L}) \rangle \, dt.
\end{aligned}$$

The terms in the first sum cancel due to (4), the terms in the second sum are non-negative due to the definition of the diffusion operator $\mathbb{D}$.

Step 2: Define the temporal part

$$\mathcal{B}^t_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{\Phi}^{\Delta x}) = -\sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{U}(\mathbf{V}^{\Delta x}), \mathbf{\Phi}^{\Delta x}_t \rangle \, dx \, dt$$
$$+ \sum_{n,i} \int_{K_i} \big(\langle \mathbf{U}(\mathbf{V}^{\Delta x}_{n+1,-}), \mathbf{\Phi}^{\Delta x}_{n+1,-} \rangle - \langle \mathbf{U}(\mathbf{V}^{\Delta x}_{n,-}), \mathbf{\Phi}^{\Delta x}_{n,+} \rangle \big) \, dx.$$

Set $\mathbf{\Phi}^{\Delta x} = \mathbf{V}^{\Delta x}$ and use integration by parts with respect to time. The boundary terms evaluated at $t_{n+1,-}$ cancel resulting in

$$\mathcal{B}^t_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = \sum_{n,i} \int_{I_n} \int_{K_i} \langle \mathbf{U}(\mathbf{V}^{\Delta x})_t, \mathbf{V}^{\Delta x} \rangle \, dx \, dt \\ + \sum_{n,i} \int_{K_i} \left(\langle \mathbf{U}(\mathbf{V}^{\Delta x}_{n,+}), \mathbf{V}^{\Delta x}_{n,+} \rangle - \langle \mathbf{U}(\mathbf{V}^{\Delta x}_{n,-}), \mathbf{V}^{\Delta x}_{n,+} \rangle \right) \, dx.$$

By the definition of the entropy function, $\langle \mathbf{U}(\mathbf{V}^{\Delta x})_t, \mathbf{V}^{\Delta x} \rangle = S(\mathbf{U}(\mathbf{V}^{\Delta x}))_t$. Evaluating the time integral and adding a zero-sum involving $S(\mathbf{U}(\mathbf{V}^{\Delta x}_{n,-}))$, this implies

$$\mathcal{B}^t_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = \sum_{n,i} \int_{K_i} \left(S(\mathbf{U}(\mathbf{V}^{\Delta x}_{n+1,-})) - S(\mathbf{U}(\mathbf{V}^{\Delta x}_{n,-})) \right) \, dx \\ + \sum_{n,i} \int_{K_i} \left(S(\mathbf{U}(\mathbf{V}^{\Delta x}_{n,-})) - S(\mathbf{U}(\mathbf{V}^{\Delta x}_{n,+})) \right) \, dx + \sum_{n,i} \int_{K_i} \langle \mathbf{U}(\mathbf{V}^{\Delta x}_{n,+}) - \mathbf{U}(\mathbf{V}^{\Delta x}_{n,-}), \mathbf{V}^{\Delta x}_{n,+} \rangle \, dx.$$

The first sum corresponds to a telescope sum. For the second and third sum, the change of variables $\mathbf{V}(\theta) = \theta \mathbf{V}_{n,-} + (1-\theta)\mathbf{V}_{n,+}$ is used. This results in

$$\mathcal{B}^t_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) = \int_{\Omega} \left(S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-}))) - S(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{0,-}))) \right) \, dx \\ + \sum_{n,i} \int_{K_i} \int_0^1 \theta \, \langle \mathbf{V}_{n,-} - \mathbf{V}_{n,+}, \mathbf{U}_{\mathbf{V}}(\theta)(\mathbf{V}_{n,-} - \mathbf{V}_{n,+}) \rangle d\theta \, dx.$$

Due to S being strictly convex, the terms in the second line are positive, implying

$$\mathcal{B}^t_{\mathrm{DG}}(\mathbf{V}^{\Delta x}, \mathbf{V}^{\Delta x}) \geq \int_{\Omega} S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_{N,-})) \right) \, dx - \int_{\Omega} S\left(\mathbf{U}(\mathbf{V}^{\Delta x}(x, t_0)) \right) \, dx.$$

This concludes the proof. □

B. Using the ST-LDG method for solving the compressible Navier-Stokes equations

The ST-LDG method is based on the existence of a positive semi-definite matrix $\mathbf{B}(\mathbf{V})$ such that $\mathbf{B}^2 = \mathbf{A}$ with $\mathbf{A}$ given by (13). In the following we describe the matrix $\mathbf{B}$ that we use in our numerical tests for the compressible Navier-Stokes equations.

Instead of decomposing the matrix $\mathbf{A}$ we decompose the reduced matrix $\tilde{\mathbf{A}}$ given by (16). We can write $\tilde{\mathbf{A}} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}^{-1}$ with the matrix $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2)$ containing the positive eigenvalues of $\tilde{\mathbf{A}}$ and the columns of $\mathbf{C}$ containing the corresponding eigenvectors. We then define $\tilde{\mathbf{B}} = \mathbf{C}\mathbf{\Lambda}^{1/2}\mathbf{C}^{-1}$. In our tests we use

$$\mathbf{C} = \begin{pmatrix} \frac{1}{N_1}\left(\lambda_1 + \nu \frac{v_2^2}{v_3^3} - \frac{\kappa}{R}\frac{1}{v_3^2}\right) & \frac{1}{N_2} \cdot \nu \frac{v_2}{v_3^2} \\ \frac{1}{N_1} \cdot \nu \frac{v_2}{v_3^2} & \frac{1}{N_2}\left(\lambda_2 + \frac{\nu}{v_3}\right) \end{pmatrix}$$

with N_1 and N_2 representing the appropriate normalization factors given by

$$N_1 = \sqrt{\left(\lambda_1 + \nu \frac{v_2^2}{v_3^3} - \frac{\kappa}{R}\frac{1}{v_3^2}\right)^2 + \left(\nu \frac{v_2}{v_3^2}\right)^2} \quad \text{and} \quad N_2 = \sqrt{\left(\nu \frac{v_2}{v_3^2}\right)^2 + \left(\lambda_2 + \frac{\nu}{v_3}\right)^2}.$$

References

[1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.

[2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2002.

[3] T. J. Barth. Numerical method for gasdynamic systems on unstructured grids. In D. Kröener, M. Ohlberge, and C. Rohde, editors, *An introduction to recent developments in theory and numerics of conservation laws*, volume 5 of *Lecture Notes in Compututational Science and Engineering*, pages 195–285. Springer, 1999.

[4] F. Bassy and S. Rebay. A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. *J. Comput. Phys.*, 131:267–279, 1997.

[5] F. Bassy, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini. A high-order accurate discontinuous finite element method for inviscid turbomachinery flows. In *Proceedings of the 2nd European Conference on Turbomachinery Fluid Dynamics and Thermodynamics*, pages 99–108, 1997. Antwerp, Belgium.

[6] C. E. Baumann and J. T. Oden. A discontinuous *hp* finite element method for the euler and navier-stokes equations. *Int. J. Numer. Methods Fluids*, 31:79–95, 1999.

[7] S. Brdar, A. Dedner, and R. Klöfkorn. Compact and stable discontinuous Galerkin methods for convection-diffusion problems. *SIAM J. Sci. Comput.*, 34:A263–A282, 2012.

[8] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *Siam J. Numer. Anal.*, 35:2440–2463, 1998.

[9] V. Dolejši. On the discontinuous Galerkin method for the numerical solution of the Navier-Stokes equations. *Int. J. Numer. Methods Fluids*, 45:1083–1106, 2004.

[10] V. Dolejši. Semi-implicit interior penalty discontinuous Galerkin methods for viscous compressible flows. *Commun. Comput. Phys.*, 4:231–274, 2008.

[11] U. S. Fjordholm, S. Mishra, and E. Tadmor. Energy preserving and energy stable schemes for the shallow water equations. In *Foundations of Computational Mathematics*, pages 93–139, 2009. London Math. Soc. Lecture Notes Ser. 363.

[12] U. S. Fjordholm, S. Mishra, and E. Tadmor. Arbitrarily high order accurate entropy stable essentially non-oscillatory schemes for systems of conservation laws. *SIAM J. Numer. Anal.*, 50:544–573, 2012.

[13] K. O. Friedrichs and P. D. Lax. Systems of conservation laws with a convex extension. *Proc. Nat. Acad. Sci. U.S.A.*, 68:1686–1688, 1971.

[14] G. Gassner, F. Lörcher, and C.-D. Munz. A discontinuous Galerkin scheme based on a space-time expansion II. Viscous flow equations in multi dimensions. *J. Sci. Comput.*, 34:260–286, 2008.

[15] S. K. Godunov. An interesting class of quasilinear systems. *Dokl. Acad. Nauk. SSSR*, 139:521–523, 1961.

[16] A. Harten. On the symmetric form of systems of conservation laws with entropy. *J. Comput. Phys.*, 49:151–164, 1983.

[17] A. Harten and P. D. Lax. A random choice finite difference scheme for hyperbolic conservation laws. *SIAM J. Numer. Anal.*, 18:289–315, 1981.

[18] R. Hartmann and P. Houston. Symmetric interior penalty DG methods for the compressible Navier-Stokes equations I: Method formulation. *Int. J. Numer. Anal. Model*, 3:1–20, 2006.

[19] R. Hartmann and P. Houston. An optimal order interior penalty discontinuous Galerkin discretization of the compressible Navier-Stokes equations. *J. Comput. Phys.*, 227:9670–9685, 2008.

[20] A. Hiltebrand. *Entropy-stable discontinuous Galerkin finite element methods with streamline diffusion and shock-capturing for hyperbolic systems of conservation laws.* PhD thesis, Seminar for Applied Mathematics, ETH Zurich, 2014.

[21] A. Hiltebrand and S. Mishra. Entropy stable shock capturing space-time discontinuous Galerkin schemes for systems of conservation laws. *Numer. Math.*, 126:103–151, 2014.

[22] A. Hiltebrand and S. Mishra. Efficient preconditioners for a shock capturing space-time discontinuous Galerkin method for systems of conservation laws. *Commun. Comput. Phys.*, 17:1360–1387, 2015.

[23] T. J. R. Hughes, L. P. Franca, and M. Mallet. A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics. *Comput. Methods Appl. Mech. Engrg.*, 54:223–234, 1986.

[24] F. Ismail and P. L. Roe. Affordable, entropy-consistent Euler flux functions II: Entropy production at shocks. *J. Comput. Phys.*, 228:5410–5436, 2009.

[25] C. M. Klaij, J. J. W. van der Vegt, and H. van der Ven. Space-time discontinuous Galerkin method for the compressible Navier-Stokes equations. *J. Comput. Phys.*, 217:589–611, 2006.

[26] M. S. Mock. Systems of conservations laws of mixed type. *J. Differential Equations*, 37:70–88, 1980.

[27] J.A. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abh. Math. Sem. Univ. Hamburg*, 36:9–15, 1971.

[28] J. T. Oden, I. Babuška, and C. E. Baumann. A discontinuous *hp* finite element method for diffusion problems. *J. Comput. Phys.*, 146:491–519, 1998.

[29] P.-O. Persson and J. Peraire. Newton-GMRES preconditioning for discontinuous Galerkin discretizations of the Navier-Stokes equations. *SIAM J. Sci. Comput.*, 30:2709–2733, 2008.

[30] B. Rivière. *Discontinuous Galerkin Methods for solving elliptic and parabolic equations: Theory and implementation.* SIAM, 2008.

[31] F. Shakib, T. J. R. Hughes, and Z. Johan. A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 89:141–219, 1991.

[32] E. Tadmor. The numerical viscosity of entropy stable schemes for systems of conservation laws. *Math. Comp.*, 49:91–103, 1987.

[33] E. Tadmor and W. Zhong. Entropy stable approximations of Navier-Stokes equations with no artificial numerical viscosity. *J. Hyperbolic Differ. Equ.*, 3:529–559, 2006.

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