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Finite elements with mesh refinement for elastic wave propagation in polygons

Fabian Müller[a]* and Christoph Schwab[a]

Error estimates for the space-semidiscrete Finite Element approximation of solutions to initial boundary value problems for linear, second-order hyperbolic systems in bounded polygons $G \subset \mathbb{R}^2$ with straight sides are presented. Using recent results on corner asymptotics of solutions of linear wave equations with time-independent coefficients in conical domains, it is shown that continuous, simplicial Lagrangian Finite Elements of uniform polynomial degree $p \geq 1$ with either suitably graded mesh refinement or with bisection tree mesh refinement towards the corners of G, achieve the (maximal) asymptotic rate of convergence $O(N^{-p/2})$, where N denotes the number of degrees of freedom spent for the Finite Element space semidiscretization. In the present analysis, Dirichlet, Neumann and mixed boundary conditions are considered. Numerical experiments which confirm the theoretical results are presented for linear elasticity.

Keywords: High order, Lagrangian Finite Elements, Wave equation, Elasticity, Elastodynamics, Regularity, Weighted Sobolev spaces, Method of lines, Local mesh refinement, Graded meshes, Newest vertex bisection, bisection tree

1. Introduction

The regularity of strongly elliptic, second order systems of partial differential equations in polygonal and polyhedral domains G has been studied for several decades, starting with the work by Kondrat'ev [22] and Maz'ya and Plamenevskiĭ [31]. We refer to Maz'ya and Rossmann [32] for a recent account of these results, and in particular also to [12], [3], [4], [33] for regularity shifts in polyhedral domains G in $\mathbb{R}^3$. There, also a comprehensive list of references to further results is provided. In essence, full elliptic regularity shifts in scales of Sobolev spaces with weights that vanish at the singular support of the solutions are now available in 2 and 3 spatial dimensions.

It is by now well-known that regularity results in scales of Sobolev spaces with weights allow to recover optimal convergence rates for Finite Element Methods (FEM) with local mesh refinement in the vicinity of corners; we refer to Raugel [37] and Babuška et al. [1, 2], and Băcuţă et al. [5] and to the references there for so-called *graded meshes*, and, more recently, to [16] and the references there for simplicial meshes with bisection tree refinements produced by Adaptive Finite Element Methods (AFEMs).

In the past decade, shift theorems in weighted Sobolev spaces for solutions of initial-boundary value problems for linear, second order Wave equations in polygons $G \subset \mathbb{R}^2$ and in certain domains $G \subset \mathbb{R}^3$ have been obtained by B. A. Plamenevskiĭ and collaborators in [36, 17, 18, 19] for the acoustic Wave equation and [20, 21, 30] for a more general class of linear, second-order hyperbolic PDEs.

Based on these results, we established optimal convergence rates for the FEM semi-discretization of the acoustic Wave equation in [34]. Using a "method of lines" approach with Lagrangian nodal Finite Elements of uniform polynomial degree $p \geq 1$ and on regular, simplicial meshes in G, we proved that with an appropriate local refinement towards all vertices of G, optimal convergence rates can be obtained for the space semi-discrete formulation. The proof in [34] exploited specific properties of the corner singularities of the Laplacian in two space dimensions. Generalizations of these analytical results for a wide class of linear, second order Wave equations with strongly elliptic spatial operator have been proved by Plamenevskiĭ et al. in [21, 20] and [30].

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Operators of this type typically turn up in the modelling of waves propagating in an elastic body. The allowed class of operators includes the general case of linearized elasticity, and also some dimension reduced models in elasticity.
The goal of the present article is to generalize our results on Finite Element convergence rates with mesh refinement in [34] to this wider class of operators. We would like to emphasize that we restrict ourselves to the two-dimensional case, i. e. $G \subseteq \mathbb{R}^2$. In three spatial dimensions, the necessary regularity results are only given for a few special cases of geometrical settings.
The outline of the present paper is as follows. In Section 2, we present structural conditions on the elliptic spatial differential and boundary operators considered in this article, and we also recall its weak formulation. The regularity results obtained by [30] are briefly reviewed in Section 3, and the results necessary in the present error analysis are collected. Finally, in Section 4, we introduce two classes of regular, simplicial mesh families with several types of local refinement at the corners of G. We show that either of these local mesh refinements yields, for space-semidiscrete Finite Element approximations, optimal convergence rates, i. e. the proposed mesh-refinements compensate for the lack of regularity due to corner singularities. We illustrate our theoretical results by numerical experiments in Section 5.
Notation which is used in the present paper is as follows: vector fields on the 2 dimensional domain G are always denoted by a bold face latin letter, e. g. $\mathbf{v}$, while their components are not written in bold face and carry indices, i. e. $\mathbf{v} := (v_1, v_2)$. Moreover, the differential operator ∂_t refers to the derivative w. r. to the time-variable t, while ∂_1, ∂_2 refer to the space variables x_1, x_2. Differential operators such as ∇ and Δ never involve time derivatives, and if Δ is applied to a vector field, it is understood to be applied componentwise.
Moreover, speaking of "convergence rates", we refer to the (largest) number $\varrho \geq 0$, such that as $N \to \infty$ (referring e. g. to the number of degrees of freedom used in the space-discretization or, equivalently, the number of nodes in Lagrangian Finite Elements), the error (in some norm to be specified) tends to 0 as $O(N^{-\varrho})$.

2. Problem formulation

2.1. Geometric setting

Throughout, we denote by $G \subset \mathbb{R}^2$ an open, connected, polygonal domain, with straight edges e_i and corner points $\mathbf{c}_i$, $i = 1, \ldots, M$, enumerated in counterclockwise orientation along ∂G (with cyclic repetition for $i > M$ and $i < 1$) and such that $\{\mathbf{c}_i\} = \overline{e_i} \cap \overline{e_{i-1}}$; here, subscripts i are understood modulo M. We collect all corners $\mathbf{c}_i$ in the set $\mathcal{C} = \bigcup_{i=1}^{M} \{\mathbf{c}_i\}$. We assume that the boundary ∂G is a union of two disjoint subsets $\bar{\Gamma}_D$ and $\bar{\Gamma}_N$ such that $\partial G = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, and such that each edge e_i entirely belongs either to $\bar{\Gamma}_D$ or to $\bar{\Gamma}_N$. This can be assumed wlog. It is moreover necessary to assume, since a change of boundary condition on a straight edge induces a singularity of the same type as a polygonal corner. For each $i = 1, \ldots, M$, let $\phi_i \in (0, 2\pi)$ denote the interior opening angle of G at $\mathbf{c}_i$. The analytic results which we use in the present Finite Element convergence analysis require that $\phi_i < 2\pi$. For $\phi_i = 2\pi$, which formally corresponds to a 'slit' domain (in which case the boundary ∂G is not locally Lipschitz at corner $\mathbf{c}_i$), the theoretical result required for our analysis does not hold anymore; in particular, a counterexample can be found in Section 3.1 of [21]. Nevertheless, as we show in the numerical experiments in §5, the general conclusions seem to extend also to this limiting case (possibly with logarithmic terms which are not resolved by the numerical studies which are reported in §5 of the present paper).
For the treatment of evolution equations, we introduce the (open) time integration interval $I := (0, T_{\max})$, with finite, given time-horizon $0 < T_{\max} < \infty$, as well as the corresponding open space-time cylinder $Q := G \times I$.

2.2. Strong formulation

We consider a linear, second order, hyperbolic system of PDEs in G. Specifically, the spatial differential operators are the linear, second-order operators $\mathcal{A}(\mathbf{x}) : \mathbf{v} \mapsto \mathcal{A}(\mathbf{x})\mathbf{v} := ((\mathcal{A}(\mathbf{x})\mathbf{v})_i)_{1 \leq i \leq 2}$, defined by

$$(\mathcal{A}(\mathbf{x})\mathbf{v})_i := \sum_{j,k,l=1}^{2} \partial_j \left(a_{ij}^{kl}(\mathbf{x}) \, \partial_l v_k \right) \, , \tag{1}$$

with *time-independent* coefficient functions $a_{ij}^{kl}(\mathbf{x})$ which are subject to the conditions

$$a_{ij}^{kl} \in C^\infty(\bar{G}) \quad \forall\, 1 \leq i, j, k, l \leq 2 \, , \tag{2}$$

$$a_{ij}^{kl} = a_{kl}^{ij} = a_{ji}^{kl} \quad \forall\, 1 \leq i, j, k, l \leq 2 \, , \tag{3}$$

$$\exists c > 0 \, : \forall\, 0 \neq \boldsymbol{\xi} = (\xi_{ij})_{1 \leq i,j \leq 2} = (\xi_{ji})_{1 \leq i,j \leq 2} : \quad \operatorname{essinf}_{\mathbf{x} \in G} \frac{\sum_{i,j,k,l} a_{ij}^{kl}(\mathbf{x}) \xi_{ij} \xi_{kl}}{\sum_{i,j} (\xi_{ij})^2} \geq c \, . \tag{4}$$

Condition (4) is referred to as *strong ellipticity of* $\mathcal{A}$ (see e. g. [40, Definition 10.4]). It is necessary in order to apply the results by Matyukevich and Plamenevskiĭ from [30], see the Equation (17) there. Given a vector field $\mathbf{f} = (f_1, f_2) \in C^0(\bar{I}; L^2(G)^2)$, as well as initial data $\mathbf{u}_0(\mathbf{x}) \in H^1(G)$ such that $\mathbf{u}_0(\mathbf{x})|_{\Gamma_D} = 0$ and $\mathbf{u}_1(\mathbf{x}) \in L^2(G)$, the complete strong formulation of the initial boundary value problem for the second order hyperbolic wave equation reads:

$$\left(\partial_t^2 - \mathcal{A}(\mathbf{x}) \right) \mathbf{u}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) \quad \forall (\mathbf{x}, t) \in Q, \tag{5}$$

supplied with initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \quad \forall \mathbf{x}\in G, \quad \partial_t \mathbf{u}(\mathbf{x},0)=\mathbf{u}_1(\mathbf{x}) \quad \forall \mathbf{x}\in G\ . \tag{6}$$

On $\partial G\times I$, we consider mixed boundary conditions. To specify these, we partition the boundary ∂G into two open, disjoint pieces Γ_D and Γ_N, as described in Section 2.1. We assume that $|\Gamma_D|\geq 0$, $|\Gamma_N|\geq 0$, i. e. we admit in particular the pure (homogeneous) Dirichlet and pure (homogeneous) Neumann boundary conditions. We consider the mixed boundary conditions

$$\begin{aligned} \mathbf{u}(\mathbf{x},t)&=0 \quad \forall \mathbf{x}\in\Gamma_D,\, t\in I,\\ \mathcal{N}\mathbf{u}(\mathbf{x},t)&=0 \quad \forall \mathbf{x}\in\Gamma_N,\, t\in I\ , \end{aligned} \tag{7}$$

where $\mathcal{N}$ denotes the Neumann boundary operator, defined formally by

$$(\mathcal{N}\mathbf{u}(\mathbf{x},t))_i := \sum_{j,k,l=1}^{2} \nu_j(\mathbf{x})\, a_{ij}^{kl}(\mathbf{x})\, \partial_l u_k(\mathbf{x},t)\ , \quad i=1,2\ , \tag{8}$$

where $\boldsymbol{\nu}(\mathbf{x})=(\nu_j(\mathbf{x}))_{j=1,2}$ is the outward unit normal vector defined for almost every $\mathbf{x}\in\partial G$.
The homogeneous Dirichlet boundary conditions in (7) are understood in the sense of trace, and the Neumann boundary conditions in (7) will be considered in the sense of distributions in a suitable weak formulation of the initial boundary-value problem. In (7), the Dirichlet (resp. Neumann) boundary condition are considered void in the case that $|\Gamma_D|=0$ (resp. $|\Gamma_N|=0$).

2.3. Linear Elastodynamics

Linear operators of the form (1) satisfying (2-4) describe wave propagation in a linearly elastic body. In the particular case where the reference configuration occupies a 2-dimensional, bounded domain, the coefficients a_{ij}^{kl} of the operator coincide with the coefficients of the compliance tensor which appears in Hooke's law as the factor relating the linearized stress and strain tensors. By the usual assumptions made in linearized elasticity, (3) and (4) are fulfilled. However, (2) obviously fails to hold for heterogeneous media, where the a_{ij}^{kl} may have discontinuities.
For details about the deduction of the governing equations in linearized elasticity, we refer to [9], [25], or [7, Chapter VI] and [39, Chapter 6], where the discretization by Finite Elements of these equations is reviewed.

2.4. Weak formulation

Following, for example, [13, Chap. XVIII, §5], we state a variational formulation of the initial boundary value problem which is to be discretized. To present it, we assume that $|\Gamma_D|>0$. The homogeneous Dirichlet boundary conditions then motivate the definition of the following space:

$$V := \{\mathbf{v}\in H^1(G)^2\ :\ \mathbf{v}|_{\Gamma_D}\equiv 0\}\ . \tag{9}$$

The continuity of the trace operator implies that $V\subset H^1(G)^2$ is a closed, linear subspace. With (9), and setting $H=L^2(G)^2$, we consider the abstract evolution triplet $V\subset H\simeq H^*\subset V^*$, i. e. *we identify H with its own dual space H^*.* A formal application of the second Green's formula for the elliptic spatial operator $\mathcal{A}$ yields the following weak variational formulation of the initial boundary value problem (5 - 7) which is a particular case of the abstract "Problème (P_1)" in [13, Chap XVIII, Par. 5.1.3].

$$\begin{aligned} &\text{Find } \mathbf{u}\in H^1(I;V)\cap H^2(I;V^*),\ \text{such that for all } t\in I \text{ and } \forall v\in V\\ &\qquad (\partial_t^2\mathbf{u}(\cdot,t),\mathbf{v})+a(\mathbf{u}(\cdot,t),\mathbf{v}) = (\mathbf{f}(\cdot,t),\mathbf{v})_H\\ &\qquad (\mathbf{u}(\cdot,0),\mathbf{v})_H=(\mathbf{u}_0,\mathbf{v})_H\ ,\\ &\qquad \partial_t(\mathbf{u}(\cdot,0),\mathbf{v})_H=(\mathbf{u}_1,\mathbf{v})_H\ , \end{aligned} \tag{10}$$

where $a(\mathbf{u}(\cdot,t),v):=\int_G\sum_{i,j,k,l=1}^2 a_{ij}^{kl}(\mathbf{x})\partial_j u_i(\mathbf{x},t)\,\partial_l v_k(\mathbf{x})\,d\mathbf{x}$. Here, the time derivatives are to be understood in the sense of distributions. Well-posedness of (10) follows from [13, Ch. XVIII, §5.1-4], under the assumptions that $a(\cdot,\cdot)$ is bilinear (which is obvious in our case), bounded (follows by the Cauchy-Schwarz inequality) and that there exist constants $c_1,c_2\in\mathbb{R}$, $c_1>0$, such that the Gårding inequality holds:

$$\forall \mathbf{v}\in V: \quad a(\mathbf{v},\mathbf{v})\geq c_1\|\mathbf{v}\|_V^2 - c_2\|\mathbf{v}\|_H^2\ . \tag{11}$$

For $a(\cdot,\cdot)$ as defined in (10), this last requirement is satisfied thanks to [23, Theorem 2].
By the strong ellipticity (4), we even have V-coercivity of $a(\cdot,\cdot)$, i. e. $c_2=0$: for every $\mathbf{w}\in V$, there holds

$$\begin{aligned} a(\mathbf{w},\mathbf{w}) &= \int_G \sum_{i,j,k,l=1}^{2} a_{ij}^{kl}(\mathbf{x})\partial_j w_i(\mathbf{x})\,\partial_l w_k(\mathbf{x})\,d\mathbf{x}\\ &\overset{(3)}{\geq} \frac{1}{4}\int_G\sum_{ijkl} a_{ij}^{kl}(\mathbf{x})\,(\partial_i w_j(\mathbf{x})+\partial_j w_i(\mathbf{x}))\,(\partial_k w_l(\mathbf{x})+\partial_l w_k(\mathbf{x}))\ d\mathbf{x}\\ &\overset{(4)}{\geq} \frac{c}{4}\int_G\sum_{ij}(\partial_j w_i+\partial_i w_j)^2\,d\mathbf{x} \geq cC(G,\Gamma_D)\|\nabla\mathbf{w}\|_{L^2(G)}^2\ , \end{aligned} \tag{12}$$

where in the last step, we have used the first Korn inequality with a constant $C(G, \Gamma_D) > 0$. Since $|\Gamma_D| > 0$, we obtain V-coercivity of $a(\cdot,\cdot)$ using the first Poincaré inequality [40, Theorem 7.6]. In the case that $|\Gamma_D| = 0$, Korn's inequality is still valid in a quotient space of $H^1(G)^2$: it is well-known that for $\mathbf{v} \in H^1(G)^2$, $a(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v} \in \mathcal{R}$, where

$$\mathcal{R} := \{G \ni \mathbf{x} \mapsto \mathbf{M}\mathbf{x} + \mathbf{b} \,:\, \mathbf{M} = -\mathbf{M}^\top,\ \mathbf{b} \in \mathbb{R}^2\} ,$$

which is, for displacements in in space dimension $d = 2$, a 3-dimensional, linear space. Accordingly, we introduce the factorspace

$$V := H^1(G)^2 \cap \mathcal{R}^\perp \simeq H^1(G)^2/\mathcal{R} ,$$

and the so-called second inequalities of Korn and of Poincaré, respectively, (see, eg., [10, Theorem 2.3]) imply once more the V-coercivity of $a(\cdot,\cdot)$.

2.5. An abstract regularity result

In the case of a coercive, symmetric and bounded bilinear form $a(\cdot,\cdot)$, a regularity shift in the t-variable can be obtained by separation of variables. It is a generalization of a result by Domínguez and Sayas who investigated the case where $\mathcal{A} = \Delta$ and of homogeneous initial data, in [14, Appendix A].
As we require this regularity for various function spaces in G, we work in an abstract framework which contains the 'usual' weak formulation as a special case. To this end, let $V \subseteq H \subseteq V^*$ denote an evolution triplet over $\mathbb{R}$ in the sense of [40, Definition 17.1], i. e. V is a reflexive real Banach space, H is a real Hilbert space and there is a continuous inclusion $\iota : V \hookrightarrow H$ such that $\iota(V) \subseteq H$ is dense. As stated before, we identify $H \simeq H^*$.

Theorem 2.1 *Let the inclusion $V \subseteq H$ be compact, and let $a : V \times V \to \mathbb{R}$ be a bounded, symmetric, coercive bilinear form on V. Moreover, let $f \in C^0(\bar{I}; H)$ and $u^0 \in V, u^1 \in H$ be a-priori given data. Then, the (abstract) evolution equation*

$$\begin{aligned} &\textit{Find } u : \bar{I} \to H, \textit{ s. t.} \\ \partial_t^2(u(t), v)_H + a(u(t), v) &= (f(t), v)_H \quad \forall v \in V, t \in \bar{I} , \\ u(0) &= u^0 , \\ \partial_t u(0) &= u^1 , \end{aligned} \tag{13}$$

admits a unique solution u which lies in the space

$$C^2(\bar{I}; V^*) \cap C^1(\bar{I}; H) \cap C^0(\bar{I}; V) . \tag{14}$$

Proof. Let $\|\cdot\|$ denote the norm in H and $|v| := \sqrt{a(v,v)}$ for $v \in V$ the seminorm in V induced by $a(\cdot,\cdot)$.
By [40, Theorem 17.11], under the assumptions we made, there exist countably many eigenpairs $(\lambda_n, w_n)_{n\in\mathbb{N}}$, $\lambda_n \in \mathbb{R}$, $w_n \in V$ of $a(\cdot,\cdot)$, which are solutions to the spectral problem

$$a(v, w_n) = \lambda_n (v, w_n)_H \quad \forall v \in V . \tag{15}$$

By the spectral theorem for compact operators, we can assume the real eigenvalues to be enumerated according to $0 < \lambda_1 \le \lambda_2 \le \ldots$, with the only accumulation point of $(\lambda_n)_{n\in\mathbb{N}}$ at infinity, and that $(w_n)_{n\in\mathbb{N}}$ form an orthonormal basis of H so that $\left(\frac{w_n}{\sqrt{\lambda_n}}\right)_{n\in\mathbb{N}}$ constitutes an orthonormal basis of V endowed with the scalar product $a(\cdot,\cdot)$.
We consider the expansions of u, f, u^0, u^1 in $\{w_n\}_{n\in\mathbb{N}}$:

$$u(t) = \sum_{n\in\mathbb{N}} \alpha_n(t) w_n,\ f(t) = \sum_{n\in\mathbb{N}} f_n(t) w_n,\ u^0 = \sum_{n\in\mathbb{N}} \alpha_n^0 w_n,\ u^1 = \sum_{n\in\mathbb{N}} \alpha_n^1 w_n . \tag{16}$$

Let us denote $\xi_n := \sqrt{\lambda_n}$. Using these expansions and orthonormality, Equation (13) can formally be written as a system of ordinary differential equations in $\alpha_n(t)$ (we abbreviate $\dot{\phi}(t) := \partial_t \phi(t)$ and $\ddot{\phi}(t) := \partial_t^2 \phi(t)$, where $\phi(t)$ is any function in the t-variable):

$$\ddot{\alpha}_n(t) + \lambda_n \alpha(t) = f_n(t) , \quad \alpha_n(0) = \alpha_n^0 , \quad \dot{\alpha}_n(0) = \alpha_n^1 . \tag{17}$$

It is straightforward to see that the unique solution of (17) is given by

$$\alpha_n(t) = \alpha_n^0 \cos(\xi_n t) + \xi_n^{-1} \alpha_n^1 \sin(\xi_n t) + \int_0^t \xi_n^{-1} \sin(\xi_n (t-\tau)) f_n(\tau)\, d\tau , \tag{18}$$

and by orthonormality, $f_n(t) = (f(t), w_n)_H$ and $\alpha_n(t) = (u(t), w_n)_H$.
The following formulae for even and odd higher derivatives of α_n are formal (since the higher derivatives of $f_n(t)$ do not necessarily exist) and are straightforward to prove by induction over all $m \in \mathbb{N}$:

$$\begin{aligned}\partial_t^{2m}\alpha_n(t) = \quad & (-1)^m \xi_n^{2m} \Bigg(\alpha_n^0 \cos(\xi_n t) + \xi_n^{-1}\alpha_n^1 \sin(\xi_n t) \\ & + \int_0^t \xi_n^{-1} \sin(\xi_n(t-\tau)) f_n(\tau)\, d\tau + \sum_{j=0}^{m-1} (-1)^{m+1+j} \xi_n^{-2(1+j)} \partial_t^{2j} f_n(t) \Bigg) ,\end{aligned} \tag{19}$$

$$\begin{aligned}\partial_t^{2m+1}\alpha_n(t) = \quad & (-1)^m \xi_n^{2m} \Bigg(-\xi_n \alpha_n^0 \sin(\xi_n t) + \alpha_n^1 \cos(\xi_n t) \\ & + \int_0^t \cos(\xi_n(t-\tau)) f_n(\tau)\, d\tau + \sum_{j=0}^{m-1} (-1)^{m+1+j} \xi_n^{-2(1+j)} \partial_t^{2j+1} f_n(t) \Bigg) .\end{aligned} \tag{20}$$

As an immediate consequence, $f_n \in C^m(\bar{I}; \mathbb{R})$ for some $m \in \mathbb{N}_0$ implies that $u_n(t) := \alpha_n(t) w_n \in C^{m+2}(\bar{I}; V)$.
It remains to show that $u(t) = \sum_{n\in\mathbb{N}} u_n(t)$ converges uniformly as series in the norms of the spaces which appear in in (14).
For $n, m \in \mathbb{N}$, $n \neq m$, we have $a(u_n(t), u_m(t)) = 0$. Moreover,

$$\begin{aligned}|u_n(t)|^2 &= a(u_n(t), u_n(t)) = \lambda_n \alpha_n(t)^2 \\ &\le \lambda_n \left(\alpha_n^0 + \alpha_n^1 \xi_n^{-1} + \xi_n^{-1} \int_0^T \sin(\xi_n(t-\tau)) f_n(\tau)\, d\tau \right)^2 \\ &\le 3 \underbrace{\left[\lambda_n \left(\alpha_n^0\right)^2 + \left(\alpha_n^1\right)^2 + T \int_0^T f_n(\tau)^2\, d\tau \right]}_{=:M_n} .\end{aligned} \tag{21}$$

By (15) and the orthonormality of $\{w_n\}_{n\in\mathbb{N}}$,

$$|u^0|^2 = \sum_{n\in\mathbb{N}} \lambda_n \left(\alpha_n^0\right)^2 , \quad \|u^1\|^2 = \sum_{n\in\mathbb{N}} \left(\alpha_n^1\right)^2 , \quad \|f(t)\|^2 = \sum_{n\in\mathbb{N}} f_n(t)^2 .$$

Using Beppo Levi's Theorem, we obtain from (21)

$$\sum_{n\in\mathbb{N}} |u_n(t)|^2 \le 3 \sum_{n\in\mathbb{N}} M_n \le 3 \left[|u_0|^2 + \|u_1\|^2 + T \|f; L^2(I; V)\|^2 \right] < \infty ,$$

which allows us to conclude as in [14, Lemma A.1] that $u = \sum_{n\in\mathbb{N}} u_n$ converges uniformly as series and its limit belongs to the space $C^0(\bar{I}; V)$. The uniform convergence implies $u(0) = u^0$ in H.
Next,

$$\begin{aligned}\|\dot{u}_n(t)\|^2 &= (\dot{u}_n(t), \dot{u}_n(t)) = \dot{\alpha}_n(t)^2 \\ &\le \left(\xi_n \alpha_n^0 + \alpha_n^1 + \int_0^T \cos(\xi_n(t-\tau)) f_n(\tau)\, d\tau \right)^2 \\ &\le 3 \left[\lambda_n \left(\alpha_n^0\right)^2 + \left(\alpha_n^1\right)^2 + T \int_0^T f_n(\tau)^2\, d\tau \right] = 3M_n ,\end{aligned} \tag{22}$$

hence, by the same argument, the sequence of partial sums $s_N := \sum_{n=1}^N \dot{u}_n$ converges uniformly to some continuous function $s \in C^0(\bar{I}; H)$. Now, since the sequence of functions $C^1(\bar{I}; H) \ni r_N(t) := \sum_{n=1}^N u_n(t)$ converges pointwise to $u(t)$ and the sequence $s_N = \dot{r}_N$ converges *uniformly* (for every $t \in I$ and in the norm of V) to s, we have $u \in C^1(\bar{I}; H)$ and $\dot{u} = s$ in $C^0(\bar{I}; H)$. Additionally, $\dot{u}(0) = u^1$ (with equality in H), by uniform convergence.
We observe that $a(u(t), v) = \sum_{n\in\mathbb{N}} a(u_n(t), v)$ for all $v \in V$, $t \in \bar{I}$, hence

$$-a(u(t), v) + (f(t), v)_H = \sum_{n\in\mathbb{N}} -a(u_n(t), v) + (f_n(t) w_n, v)_H .$$

Now, since

$$\sum_{n=1}^N (\ddot{u}_n(t), v) = \sum_{n=1}^N -a(u_n(t), v) + (f(t), v)_H \overset{N\to\infty}{\longrightarrow} -a(u(t), v) + (f(t), v)_H ,$$

the series $\sum_n (\ddot{u}_n(t), v)_H$ converges uniformly with respect to $t \in \bar{I}$ to a continuous function with values in V^*.
Since $\sum_n \dot{u}_n$ is convergent in H uniformly w. r. to t, it is convergent in V uniformly w. r. to t by the continuous inclusion $V \hookrightarrow H$,

and it follows that $(\ddot{u}(t), v) + a(u(t), v) = (f(t), v)_H$ and $u \in C^2(\bar{I}; V^*)$.
Finally, the fact that the solution $u(t)$ to the equation (13) is unique follows from the linearity of the equation and well-known energy arguments, see e. g. [15, Theorem 7.3.4] ■
In order to apply Theorem 2.1 on our case, we need to see that the situation described in Section 2 satisfies all assumptions made in 2.1. It is well-known that in polygonal domains, elliptic operators should be seen as operating in weighted Sobolev spaces. In that sense, the results we will cite do not provide the full regularity in the time-variable t. However, by a uniqueness argument, we can deduce from Theorem 2.1 using the Gelfand triplet $V \hookrightarrow L^2(G) \hookrightarrow H^{-1}(G)$ with compact inclusions by Rellich's Theorem that $u(\cdot, t)$ has C^2-regularity with values in $H^{-1}(G)$.
This will be necessary to establish the semidiscrete convergence result.
From now on, the space V will denote again the space defined by (9), and will not be used as an abstract notation in an evolution triplet, if not explicitly stated.

3. Corner asymptotics of u in polygons

Our goal is to recover quasi-optimal convergence rates for the space-semidiscrete wave equation, discretized in space by the Lagrangian Finite Elements on locally refined meshes in G. Similar to our analysis for the linear wave equation in [34], and similar to what is done in the elliptic setting, one possible approach to the analysis of Finite Element convergence rates is based on asymptotic expansions of the solution in the vicinity of the corners. The corner asymptotics which are required in our context were obtained in recent years by Plamenevskiĭ and coworkers. The case of interest here is covered in [30]. We review it in a form tailored to our requirements. At each corner $\mathbf{c}_i$, we choose a local neighborhood $G_i := B_{R_i}(\mathbf{c}_i) \cap G$, where R_i is chosen such that

$$R_i < \min_{i' \neq i} |\mathbf{c}_i - \mathbf{c}_{i'}|/4,$$

so that the G_i are disjoint, and where $B_R(\mathbf{x})$ denotes the open disc of radius $R > 0$ centered at $\mathbf{x}$.
The analysis of elliptic corner singularities is based on local polar coordinates (r_i, ϑ_i) in G_i centered at $\mathbf{c}_i$. Specifically, by $(r_i(\mathbf{x}), \vartheta_i(\mathbf{x}))$, we denote polar coordinates centered at corner $\mathbf{c}_i$, with $\vartheta_i \in [0, \phi_i]$ oriented counterclockwise, so that $e_i = \{x \in \partial G \,|\, \vartheta_i(\mathbf{x}) = 0\}$ and $e_{i-1} = \{x \in \partial G \,|\, \vartheta_i(\mathbf{x}) = \phi_i\}$. To extend this local definition from the vicinity G_i of $\mathbf{x}_i$ to the entire domain G, observe that for sufficiently small, fixed $\varepsilon > 0$, $r_i(\mathbf{x}) \in H^k(G_i \backslash B_\varepsilon(\mathbf{x}_i)))$ for all $k \in \mathbb{N}_0$. For all $k \in \mathbb{N}_0$, it can be extended from G_i to a function (denoted again r_i) which is smooth in $\overline{G \backslash G_i}$, bounded from above and away from zero. Next, we define the corner distance function

$$\Psi(\mathbf{x}) := \prod_{i=1}^{M} r_i(\mathbf{x}) \ ,$$

which is equivalent to the function $\mathbf{x} \mapsto r_i(\mathbf{x})$ in the corner vicinity G_i, for any $1 \le i \le M$. Let $\tilde{\chi} : \mathbb{R}_+ \to [0, 1]$ be a smooth cut-off function, such that

$$\tilde{\chi}(r) = \begin{cases} 1 & \text{if } r \le \frac{1}{2} \\ 0 & \text{if } r \ge 1. \end{cases}$$

At each corner $\mathbf{c}_i$, $i = 1, \ldots, M$, we define local cut-offs $\chi_i(\mathbf{x}) := \tilde{\chi}\left(r_i(\mathbf{x})/R_i\right)$, whose supports are fully contained in G_i.
The regularity theory of [21, 30] is based upon a reformulation of the Initial-Boundary value problem (IBVP) (5) as the boundary value problem (BVP) in a cylinder $(\mathbf{x}, t) \in G \times \mathbb{R}$:

$$\left(\partial_t^2 - \mathcal{A}(\mathbf{x})\right) \tilde{\mathbf{u}}(\mathbf{x}, t) = \tilde{\mathbf{f}}(\mathbf{x}, t) \quad \forall (\mathbf{x}, t) \in G \times \mathbb{R}, \tag{23}$$

supplied with the boundary conditions (7). Throughout Section 3, we will denote $\tilde{\mathbf{u}}$ by $\mathbf{u}$ and $\tilde{\mathbf{f}}$ by $\mathbf{f}$. As the actual IBVP (5-7) will not be considered there, this will not generate a conflict of notation.
As is described e.g. in [17, Proof of Thm. 4.1], the corner asymptotics of $\tilde{\mathbf{u}}$ and $\mathbf{u}$ are equivalent. Therefore, the regularity theory of (23) will automatically lead to the corresponding results about solutions of (5-7).
In order to determine the regularity of (23), we need to Fourier transform the PDE in the time variable $t \in \mathbb{R}$. Let $\gamma > 0$ be an arbitrarily chosen, yet fixed, positive real number. Given a vector function $\mathbf{w}(\mathbf{x}, t)$, $\mathbf{x} \in \bar{G}$, $t \in \mathbb{R}$, its Fourier transformation with respect to the time variable t, shifted along the imaginary axis to the line $\mathbb{R} - i\gamma \ni \tau$ is defined to be

$$\hat{\mathbf{w}}(\mathbf{x}, \tau) := \mathcal{F}_{t \mapsto \tau}[\mathbf{w}(\mathbf{x}, t)](\mathbf{x}, \tau) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} \mathbf{w}(\mathbf{x}, t)\, dt \ .$$

If $\mathbf{u}(\mathbf{x}, t)$ is a solution of (23), $\hat{\mathbf{u}}(\mathbf{x}, \tau)$ is a solution of the transformed equation:

$$\left[-\tau^2 - \mathcal{A}\right] \hat{\mathbf{u}}(\mathbf{x}, t) = \hat{\mathbf{f}}(\mathbf{x}, t) \ , \tag{24}$$

and the same boundary conditions (7). As by now, all ingredients of the theory are introduced, we can define the weighted Sobolev spaces that accomodate a regularity shift for solutions to (5) in polygonal domains.

Definition 3.1 *Let $\gamma > 0$ and $\omega \in \mathbb{R}$ be given weight parameters, and let $s \in \mathbb{N}_0$ be an integer. Let $\tau = \sigma - i\gamma$, $\sigma \in \mathbb{R}$, denote the Fourier parameter defined above.*
For $s \in \mathbb{N}_0$, we define the spaces $H^s_\omega(G)^2$ and $H^s_\omega(G, |\tau|)^2$ as completions of $C_0^\infty\left(\bar{G} \setminus \mathcal{C}\right)^2$ with respect to the norms $\|\mathbf{v}; H^s_\omega(G)\|$ and $\|\mathbf{v}; H^s_\omega(G, |\tau|)\|$ which are given by

$$\|\mathbf{v}; H^s_\omega(G)\|^2 = \sum_{k=0}^{s} \int_G \Psi(\mathbf{x})^{2(\omega+k-s)} \left|\mathrm{D}^k \mathbf{v}(\mathbf{x})\right|^2 d\mathbf{x}\,, \quad \|\mathbf{v}; H^s_\omega(G, |\tau|)\|^2 = \sum_{j=0}^{s} |\tau|^{2j} \|\mathbf{v}; H^{s-j}_\omega(G)\|^2\,. \tag{25}$$

In the case when $s = 0$, we write $L^2_\omega(G) := H^0_\omega(G)$.
In the analysis of the homogeneous Dirichlet problem, one often uses the notation of the (nonweighted) Sobolev space $H^1_0(G) := \{v \in H^1(G) \,:\, v|_{\partial G} \equiv 0\}$. Here, this would give rise to a notational conflict with the choice of $\omega = 0, s = 1$ in the space $H^s_\omega(G)$. Therefore, we write $H^1_{\omega=0}(G)$ in that case in order to denote the weighted space in the sense of (25).
Furthermore, for $s \in \mathbb{N}_0$ and $\omega \in \mathbb{R}$, we define the space $V^s_\omega(Q;\gamma)^2$ as completion of $\left(C_0^\infty\left((\bar{G} \setminus \mathcal{C}) \times \bar{I}\right)\right)^2$ with respect to the following norm:

$$\|\mathbf{w}, V^s_\omega(Q;\gamma)\| := \left(\int_{\mathbb{R}} \|\hat{\mathbf{w}}(\cdot,\tau); H^s_\omega(G;|\tau|)\|^2 \, d\sigma\right)^{1/2}\,. \tag{26}$$

We will use an embedding from [34, Prop. 3.3], adapted to vector-valued functions.

Proposition 3.2 *Let $q, s, s' \in \mathbb{N}_0$ and $G \subseteq \mathbb{R}^2$ be a bounded polygonal domain.*
If, moreover, $q + 1 \geq s + s'$, then for all $\gamma > 0$ and $\omega \leq -q$ the following inclusion is continuous:

$$V^{q+1}_{\omega+q}(Q;\gamma)^2 \hookrightarrow H^s(I; H^{s'}(G)^2). \tag{27}$$

The formulation of the results on corner asymptotics of solutions $\mathbf{u}$ of (1) - (8) in the vicinity of $\mathbf{c}_i$ involves an extension operator X^i which is defined by

$$X^i \mathbf{w}(\mathbf{x}, t) := \mathcal{F}^{-1}_{\tau \mapsto t} \chi_i(|\tau|\,\mathbf{x})\, \mathcal{F}_{t' \mapsto \tau} \mathbf{w}(\mathbf{x}, t')\,, \quad i = 1, 2, ..., M\,. \tag{28}$$

Here, by χ_i we denote the cut-off corresponding to $\mathbf{c}_i$. These (nonlocal) operators appear in the corner-singularity analysis of parametric elliptic problems. They can be interpreted as smoothing extension operators outside the corner; see [35, Chap. 11.3] or the remarks in [30, p. 500] after [30, Thm. 7.5] for details. The weighted spaces $V^{q+1}_{\omega+q}(Q;\gamma)$ appear in the regularity theory of (5) and also in the following spaces, which have been introduced in [20, §5]. In [30, §7], these spaces have been introduced only in the case $q = 0$. The step to higher integer order $q > 0$ can be done by induction as described before Theorem 3.5 below.

Definition 3.3 *Let $\gamma > 0$, $\omega \in \mathbb{R}$ and $q \in \mathbb{N}_0$. For sufficiently smooth functions w of $\mathbf{x}$ and of t, we define*

$$\begin{aligned}\|\mathbf{u}; \mathrm{DV}^N_{\omega,q}(Q;\gamma)\|^2 := \sum_{i=1}^{M} \Bigg\{ \int_{\mathbb{R}} \left\| r_i(\mathbf{x})^{\omega-1}\, \partial_t \left[e^{-\gamma t} X^i \mathbf{u}(\cdot, t)\right]; L^2(G)^2 \right\|^2 + \\ \sum_{k=1}^{q+2} \int_{\mathbb{R}} \left\| r_i(\mathbf{x})^{\omega-2+k} e^{-\gamma t}\, \mathrm{D}^k \left[X^i \mathbf{u}\right](\mathbf{x}, t); L^2(G)^2 \right\|^2 dt \Bigg\} + \gamma^2 \|\mathbf{u}; V^{q+1}_{\omega+q}(Q;\gamma)^2\|^2,\end{aligned}$$

and the space $DV_{\omega,q}(Q;\gamma)$ as completion of $\left(C_0^\infty\left((\bar{Q} \setminus \mathcal{C}) \times \bar{I}\right)\right)^2$ w. r. to the norm defined by

$$\|\mathbf{w}; DV_{\omega,q}(Q;\gamma)\|^2 := \begin{cases} \sum_i \|X^i \mathbf{w}; V^{q+2}_{\omega+q}(Q;\gamma)^2\|^2 + \|\mathbf{w}; V^{q+1}_{\omega+q}(Q;\gamma)^2\|^2 & \text{if } |\Gamma_D| > 0, \\ \|\mathbf{w}; \mathrm{DV}^N_{\omega,q}(Q;\gamma)\|^2 & \text{if } \Gamma = \Gamma_N\,. \end{cases} \tag{29}$$

Analogously, for $q \in \mathbb{N}_0$ we introduce the space $RV_{\omega,q}(Q;\gamma)$ as completion with respect to the norm $\|f; RV_{\omega,q}(Q;\gamma)\|$ given by

$$\begin{aligned}\|\mathbf{f}; RV_{\omega,q}(Q;\gamma)\|^2 \;\; := \;\; & \sum_{j=0}^{q} \gamma^{-2j} \|\partial_t^j \mathbf{f}; V^{q-j}_{\beta+q-j}(Q,\gamma)^2\|^2 \\ & + \gamma^{-2(q+1)} \left\| \mathcal{F}^{-1}_{\tau\mapsto t}\left[|\tau|^{1-\omega+q} \mathcal{F}_{t\mapsto\tau} \mathbf{f}\right]; V^0_0(Q,\gamma)^2 \right\|^2\,.\end{aligned} \tag{30}$$

These spaces have been introduced in [20, §7], [18, §7].

Regularity shifts for second order linear wave equations in polygonal and polyhedral domains have been proved in [30] for pure homogeneous Dirichlet or Neumann conditions. These results can be extended to mixed boundary conditions, i. e. the case where $|\Gamma_D| > 0$ and $|\Gamma_N| > 0$, with an approach which is completely analogous to the techniques in [30] (and in the references therein). The regularity results in [30] are based on elliptic corner asymptotics of the boundary value problem (1) - (8), upon Fourier-transforming this problem with respect to the time variable. This way, the elastic Wave equation is transformed to the parametric

problem (24) with elliptic principal part $-\mathcal{A}$ and with a parameter $-\tau^2 \in \mathbb{C}$. For such elliptic problems and for a fixed parameter τ, the corner asymptotics are well-known, see e.g. the monograph [32] and the references there. The work by Plamenevskiĭ et al., in particular [30], provides uniform bounds of the asymptotics w. r. to the Fourier parameter in the scale of the weighted Sobolev spaces from Definition 3.1. This allowed them to conclude on the corner asymptotics of the time-dependent problem (5) by inverse-transformation of the results in the Fourier domain.
For the sake of readability, we restrict ourselves to providing only a short review of the regularity results and notions which we need for our purposes. However, we would like to stress the fact that the presence of the a-priori chosen parameter $\gamma > 0$ is necessary for the results of Plamenevskiĭ et al..
A crucial role for the asymptotical behaviour of $\mathbf{u}(\cdot, t)$ near the vertex $\mathbf{c}_i$ is played by the solutions of a Sturm-Liouville eigenvalue problem in the angular coordinate ϑ_i in the vicinity of $\mathbf{c}_i$.

Definition 3.4 ([30], Sec. 6.1, pp. 492) *For each corner $_i$, we define the operator pencil*

$$\mathbb{C} \ni \lambda \mapsto \mathfrak{A}_i(\lambda)[\boldsymbol{\psi}] := \left\{ r_i^{2-\mathrm{i}\lambda} \mathcal{A} \left[r_i^{\mathrm{i}\lambda} \boldsymbol{\psi} \right], \mathcal{B}_i \boldsymbol{\psi} \right\} ,$$

where $\mathcal{B}_i$ denotes the boundary operator at the edges e_{i-1}, e_i induced by (7).
It is straightforward to compute a representation of $\mathfrak{A}_i(\lambda)$ which is independent of the angular variable ϑ_i.
Eigenpairs $(\lambda, \boldsymbol{\psi})$ of $\mathfrak{A}_i$ consist of complex numbers λ and smooth functions $\psi(\vartheta_i)$, s.t. $\mathfrak{A}_i(\lambda)[\boldsymbol{\psi}] = 0$.
We refer to [30] and to [32] and, in particular in the two-dimensional case, to [11] and references therein for details. It is shown in these references that $\mathfrak{A}_i$ admits a countable number of isolated, in general complex eigenvalues (e. g. [32] and the references there). We refer to [24] for a general survey of such spectral problems, where $\mathfrak{A}_i$ is induced by an elliptic operator $\mathcal{A}$.
Moreover, it has been shown that $\{\Re(\mathrm{i}\lambda_{\mu,i})\}_\mu$ forms a set of positive (real) numbers. Therefore, we can assume that the eigenvalues are arranged in a way such that $0 < \Re\mathrm{i}\lambda_{1,i} \leq \Re\mathrm{i}\lambda_{2,i} \leq \cdots \to \infty$.
In the case that $e_{i-1} \cup e_i \subseteq \Gamma_N$, an eigenvalue $= 0$ is possible, but does not appear explicitly in the asymptotics, see [21, 30].

Corner asymptotics of solutions to the elliptic operator $\mathcal{A}$ with boundary conditions (7) at $\mathbf{c}_i$ are power-logarithmic asymptotic expansions of the form

$$\mathbf{v}(\mathbf{x}) := r_i^{\mathrm{i}\lambda} \sum_{q=0}^{k} \frac{1}{q!} \left(\mathrm{i} \ln r_i\right)^q \boldsymbol{\psi}^{(k-q)}(\vartheta_i) , \tag{31}$$

which is a formal solution of the homogeneous boundary value problem

$$\mathcal{A}\mathbf{v}(\mathbf{x}) \equiv 0 \quad \mathbf{x} \in G_i , \quad v(\mathbf{x}) \equiv 0 \quad \mathbf{x} \in \partial G_i \cap \Gamma_D , \quad \mathcal{N}[\mathbf{v}](\mathbf{x}) \equiv 0 \quad \mathbf{x} \in \partial G_i \cap \Gamma_N , \tag{32}$$

if and only if λ is an eigenvalue of $\mathfrak{A}_i$ and $\{\boldsymbol{\psi}^{(0)}, \ldots, \boldsymbol{\psi}^{(k-q)}\}$ is a corresponding Jordan chain.
It is important to mention that Plamenevskiĭ et al. in [30, 36, 17] replace the derivatives w. r. to the cartesian coordinate functions ∂_{x_i} by a scaled version $-\mathrm{i}\partial_{x_i}$. As an outcome, the eigenvalues of the respective pencils will be multiplied by i. Therefore, our singular exponents will be denoted by $\mathrm{i}\lambda$ instead of λ, as is the convention in e. g. [12].
Let $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_K \in \mathbb{N}$ be partial multiplicities of an (generally complex) eigenvalue $\lambda_{\mu,i}$ and let $\{\boldsymbol{\psi}_i^{(0,j)}, \ldots, \boldsymbol{\psi}_i^{(\kappa_j-1,j)}\}$ be a canonical system of Jordan chains associated to this eigenvalue. At each corner $\mathbf{c}_i$, let $(r_i(\mathbf{x}), \vartheta_i(\mathbf{x}))$ denote the polar coordinates centered at $\mathbf{c}_i$ which were introduced above, corresponding to the cartesian coordinate function $\mathbf{x}$. Then the functions

$$\mathbf{u}_{\mu,i}^{(k,j)}(\mathbf{x}) := r_i(\mathbf{x})^{\mathrm{i}\lambda_{\mu,i}} \sum_{q=0}^{k} \frac{1}{q!} \left(\mathrm{i} \ln r_i(\mathbf{x})\right)^q \boldsymbol{\psi}_i^{(k-q,j)}(\vartheta_i(\mathbf{x})) \tag{33}$$

form a basis of power-logarithmic, formal particular solutions of the kind (31) of the homogeneous problem in the infinite sector generated by the edges meeting at $\mathbf{x}_i$, which correspond to $\lambda_{\mu,i}$. For each $i = 1, \ldots, M$, let $1 > \omega_{1,i} > \omega_{2,i} > \ldots$ be real numbers in $(-\infty, 1)$, such that every line

$$\{\lambda \in \mathbb{C} \,|\, \Im\lambda = \omega_{k,i} - 1\}$$

contains at least one eigenvalue $\lambda_{\mu,i}$ of $\mathfrak{A}_i$. The regularity result we need for our purposes is contained in [30, Theorem 7.5]. There, the result is only proved for $q = 0$. For $q > 0$, in the Dirichlet case $|\Gamma_N| = 0$, it has been proved in [20, Theorem 7.5], [19, Theorem 8.1]. For the case $\mathcal{A} = \Delta$, the result for the Neumann case is contained in [18, Theorem 7.4]. In both of these cases, an induction over $q \in \mathbb{N}$ leads from the case where $q = 0$ to the case where $q > 0$.
The same induction argument can be used to obtain the result from [30] for $q > 0$. The proof of the induction step is analogous to the proofs of [20, Theorem 7.5] and [18, Theorem 7.4]. For the sake of simplicity, we do not include the proof here.

Theorem 3.5 *Let G be a bounded polygonal domain with M corners and interior opening angles $\phi_i \in (0, 2\pi)$, $i = 1, \ldots, M$. Let $\gamma > 0$, $q \in \mathbb{N}_0$, and $\omega < 1$, such that there is no $\lambda_{\mu,i}$ with $\Im(\lambda_{\mu,i}) = \omega - 1$. For each $1 \leq i \leq M$ and q, we define*

$$\Lambda_{q,i} := \{\lambda_{\mu,i} \,|\, \Im\lambda_{\mu,i} = \omega_{q,i} - 1\} ,$$

which is a finite set, see [30, p. 479] and [36, Prop. 5.2, 2)]. If the data satisfies $\mathbf{u}_0 \in V, \mathbf{u}_1 \in L^2(G)^2$, and if $\mathbf{f} \in RV_{\omega,q}(Q;\gamma)^2$, then the solutions of (5) admit a decomposition of the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_r^{\omega,q}(\mathbf{x},t) + \sum_{i=1}^{M} \chi_i(r_i)\mathbf{u}_{s,i}^{\omega,q}(\mathbf{x},t), \tag{34}$$

where the singular functions $\mathbf{u}_{s,i}^{\omega,q}(\mathbf{x},t)$ are given by

$$\mathbf{u}_{s,i}^{\omega,q}(\mathbf{x},t) := \sum_{\mu\in\Lambda_{q,i}} \sum_{j=1}^{K} \sum_{k=0}^{\kappa_j-1} \left(X_i \check{d}_n^{(k,j)}(t)\right) \mathbf{u}_{\mu,i}^{(k,j)}(\mathbf{x}),$$

with X_i as in (28)*, where the regular part satisfies $\mathbf{u}_r^{\omega,q} \in DV_{\omega,q}(Q;\gamma)$, and $\check{d}_n^{(k,j)}(t) \in L^2(I)$.*

Remark 3.6 *If $\mathbf{f}$ is not a smooth function in time, $\check{d}_n^{(k,j)} \notin H^1(I)$, at least in the nonweighted scale, see [30], or [21, Sec. 5].*

Proposition 3.2 yields the following interpretation of this result: choosing suitable parameters ω, q, the solution is represented as sum of a "sufficiently smooth" function $\mathbf{u}_r^{\omega,q} \in H^s(I; H^{s'}(G)^2)$ and, for each finite order s' of Sobolev regularity, a finite number (depending on that regularity order) of singular terms. If $s+s'$ is increased, or if $\min_\mu(\Re\lambda_{\mu,i}) \to 0$, the parameters ω and q need to be changed (this can happen if either the interior opening angle increases up to 2π, or the material properties modelled by the operator coefficients $a_{ij}^{kl}(\mathbf{x})$ are modified), such that $\mathbf{u}_{s,i}^{\omega,q}$ contains more (but still finitely many) terms.
With a suitable choice of an evolution triplet, this result can be combined with Theorem 2.1 to conclude higher regularity of the $\check{d}_n^{(k,j)}(t)$.

4. Convergence Rates of Space-semidiscretization

In this section, we show that quasi-optimal convergence rates of FEM can be obtained using the solution decomposition furnished by Theorem 3.5. We recall two ways of mesh refinements, one by a-priori graded refinement and one that results in a hierarchical mesh family from a bisection-tree refinement algorithm presented in [16]. Comparisons of adaptive and graded mesh refinements were carried out recently in the paper [27].

4.1. A-priori graded, regular simplicial meshes

Let $K_0 = \mathrm{conv}\{(1,0),(0,0),(0,1)\}$ be the unit triangle. On K_0, we construct a parametric family of meshes which are graded towards the vertex $(0,0)$ so as to ensure an optimal rate of convergence of Lagrange interpolating Finite Elements of order $p \geq 1$. Later, the polygonal domain G can be written as a finite union of isometric, orientation-preserving images of K_0, hence the graded mesh can be mapped isometrically to G, with a local refinement towards the corners (we refer to [34] for a more detailed description).
Given an integer $m \geq 2$ and a so-called grading parameter $\beta \geq 1$, define

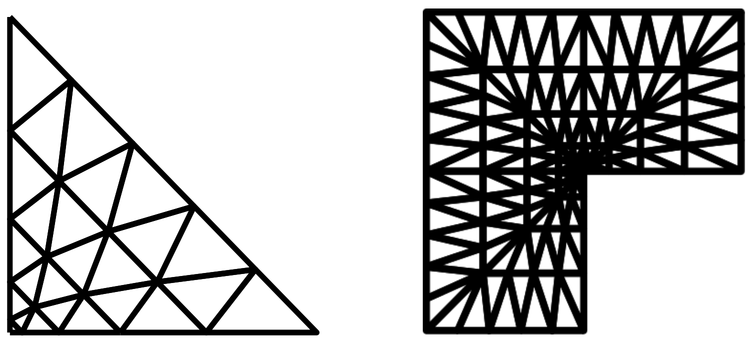

Figure 1. Graded meshes with parameters $n = 5$ and $\beta = 2$. Left: The mesh on the reference patch K_0. Right: Mesh graded towards the reentrant corner on the L-shaped domain which is obtained by composition of six affine images of the reference mesh.

$$z_l := \left(\frac{l}{n}\right)^{\beta} \in [0,1], \quad l = 0,1,\ldots,m\,.$$

The nodes of the mesh that lie on the rectangular edges of K_0 are $(z_l, 0)$ and $(0, z_l)$, $l = 0, \dots, m$. Then, being d_l the diagonal joining $(z_l, 0)$ and $(0, z_l)$, we divide d_l uniformly into $l+1$ points. This defines all the nodes of a so-called β-graded mesh $\mathcal{T}_{m,\beta}(K_0)$ on K_0.
We assume that $\{\mathcal{T}_N : N \in \mathcal{N} \subseteq \mathbb{N}\}$ denotes a regular family of simplicial meshes with meshwidth h. Given a polynomial degree $p \in \mathbb{N}$, we obtain a sequence of conforming finite element spaces, denoted by

$$V_N := V \cap S^{p,1}(G; \mathcal{T}_N)^2 .$$

The dimension $\dim(V_N)$ is finite and tends to ∞, as $h \to 0$. Function classes that afford optimal approximation rates on this class of meshes are weighted Sobolev spaces which are defined as follows:

Definition 4.1 *Let $G \subseteq \mathbb{R}^2$ be a bounded polygon with a finite number M of corners $\mathbf{c}_i \in \partial G$.*
Let $\delta \in \mathbb{R}$, and $s \geq s_0 \in \mathbb{N}_0$. We define the weighted Sobolev space $H^{s,s_0}_\delta(G)$ as the completion of $C^\infty(\bar{G} \setminus \mathcal{C})$ with respect to the norm

$$\|v; H^{s,s_0}_\delta(G)\|^2 := \|v; H^{s_0-1}(G)\|^2 + \underbrace{\sum_{k=s_0}^{s} \int_G \Psi(\mathbf{x})^{2(\delta+k-s_0)} |\mathrm{D}^k v(\mathbf{x})|^2 \, d\mathbf{x}}_{=:|v; H^{s,s_0}_\delta(G)|^2} \tag{35}$$

The case $s_0 = 2$ and $0 \leq \delta < 1$ is of particular interest for spatial approximation with continuous, piecewise linear Finite Elements on regular triangulations of G. We cite two properties of $H^{s,2}_\delta(G)$, proved in [2] and [39], respectively.

Proposition 4.2 *Let G, r and ϑ be as in Definition 4.1, $\delta \in [0, 1)$, and $s \geq 2$. Then there hold the following assertions.*

1. *The inclusion $H^{s,2}_\delta(G) \hookrightarrow C^0(\bar{G})$ is continuous.*
2. *Let $T_0 \in \mathbb{R}^2$ be a nondegenerate triangle with $(0, 0)$ as a vertex and meshwidth h_{T_0}. Then, for every $0 \leq \delta < 1$, there exists a constant $C(\delta) > 0$ such that $\forall v \in H^{2,2}_\delta(T_0)$:*

$$\|v - I_1 v; H^1(T_0)\| \leq C h^{1-\delta}_{T_0} \|v; H^{2,2}_\delta(T_0)\| , \tag{36}$$

where I_1 denotes the linear, nodal interpolant in the three vertices of T_0.

In our case, the singularities (for a fixed time t and up to the finite Sobolev regularity $p+1$ accessed by the Finite Element approximation) at a fixed corner $\mathbf{c}_i$ are decribed by finite sums of vector fields with coefficients of the form $r_i^\lambda (\log r_i)^q \Phi(\vartheta_i)$, where $\Re\lambda > 0$. Hence, the λ with smallest real part $\Re\lambda$ appearing in the sum dictates the regularity of the solution. Now suppose that (r, ϑ) are the polar coordinates centered in $(0, 0)$. A straightforward computation shows that the function $r^\lambda \Phi(\vartheta)$ lies in $H^{p+1,2}_\delta(T_0)$, for $\delta > 1 - \Re(\lambda)$ for all $p \in \mathbb{N}$. Hence, we can use the following result in K_0 proved in [34]:

Proposition 4.3 *Let $\delta \in [0, 1)$, $p \in \mathbb{N}$, $v \in H^{p+1,2}_\delta(K_0)$, and*

$$\beta > \max\left\{1, \frac{p}{1-\delta}\right\} .$$

Consider a β-graded mesh family $\mathcal{T}_{m,\beta}(K_0)$ on the reference patch K_0 with total number N of vertices behaving as $N := \#\mathcal{T}_{m,\beta}(K_0) \overset{m\to\infty}{\longrightarrow} \infty$.

Then, there exists a constant $C > 0$ which is independent of v and N, such that

$$\min_{w \in S^{p,1}(K_0, \mathcal{T}_{m,\beta}(K_0))} \|v - w; H^1(K_0)\| \leq C N^{-p/2} \, \|v; H^{p+1,2}_\delta(K_0)\|, \; N \to \infty . \tag{37}$$

From these observations, we immediately obtain the following result.

Theorem 4.4 *Let $p \in \mathbb{N}$, $G \subseteq \mathbb{R}^2$ be a bounded polygon and let*

$$\delta_i > 1 - \Re(\mathrm{i}\lambda_{\mu,i}), \; i = 1, \dots, M, \; \text{and} \; \delta := \max_i \delta_i ,$$

for all μ such that $\mathbf{u}^{\omega,q}_{\mu,i} \notin H^{p+1}(G)$ and where the $\lambda_{\mu,i}$ are as given in Definition 3.4. Given mesh grading parameters $\beta_i > \max\left\{1, \frac{p}{1-\delta_i}\right\}$ and $m \in \mathbb{N}$, we construct a sequence $(\mathcal{T}_{m,\boldsymbol{\beta}}(G))_{m\in\mathbb{N}}$ of $\boldsymbol{\beta}$-graded meshes which is conforming, shape-regular, and consists

1. *at each corner $\mathbf{c}_i$ of finitely many copies of β_i-graded meshes graded towards $\mathbf{c}_i$,*
2. *in $G \setminus \bigcup_i G_i$ of a quasi-uniform mesh, conforming with the graded mesh in the corner patches G_i at the patch boundaries $\partial G_i \cap G$.*

For each $0 \leq \delta < 1$ and $v \in H^{p+1,2}_\delta(G)$, there is a constant $C > 0$, independent of v and $N := \#\mathcal{T}_{m,\boldsymbol{\beta}}(G)$, such that

$$\min_{w \in S^{p,1}(G, \mathcal{T}_{m,\boldsymbol{\beta}}(G))} \|v - w; H^1(G)\| \leq C N^{-p/2} \|v; H^{p+1,2}_\delta(G)\| .$$

Note that $N \to \infty$ and $\min_{T \in \mathcal{T}_{m,\boldsymbol{\beta}}(G)} h(T) \to 0$, as $m \to \infty$.

4.2. Regular simplicial meshes with binary tree structure

One major disadvantage of a-priori β-graded meshes is that they are not nested, in general. As an alternative, one can proceed similarly as in the construction of adaptive meshes produced by AFEM in course of local refinement of a given initial mesh. As we do not want to elaborate on adaptive FEM here, we use the a-priori knowledge of singularity structure of the solutions to design an approximation process. Specifically, we consider a local refinement procedure which produces by mesh-bisection typically used in adaptive Finite Elements a sequence of bisection tree refinements which yields optimal convergence rates in space. This question has been considered by Gaspoz and Morin in [16] and we briefly review their results here: denote by $\mathtt{REFINE\_MARKED}(\mathcal{T},\mathcal{M})$ a procedure that bisects all "marked" elements contained in a list $\mathcal{M}$ of the initial triangulation $\mathcal{T}$.
Given a nondegenerate simplex (z_0, z_1, z_2) in a regular, initial triangulation of G to be refined, we proceed by recursive bisection, i. e. the edge connecting z_0 and z_2 is bisected (let $\bar{z}$ be its midpoint) and a new edge $\overline{(z_1\,\bar{z})}$ is introduced. If $\bar{z}$ is a hanging node, we bisect the adjacent simplex by the same procedure until it comes to an end. The local vertex numberings of the children of the simplex (z_0, z_1, z_2) are defined as $(\bar{z}, z_0, z_1)$ and $(z_1, z_2, \bar{z})$ such that for consecutive application of the bisection procedure on the children, it is always the recently introduced edge $\overline{(z_1\,\bar{z})}$ which is bisected again. Therefore, this refinement procedure is called the "newest vertex bisection". Complexity estimates have been proved, as well as the fact that the output of this procedure has the same shape-regularity constant as the input, i. e. the initial mesh. Moreover, one can analyze this procedure via binary trees, which is a powerful tool for e. g. AFEM, see [16] and references therein.

Definition 4.5 *Let $G \subseteq \mathbb{R}^2$ be a bounded polygonal domain with a given conforming triangulation $\mathcal{T}_0$. Let $p \in \mathbb{N}$. Let $\lambda > 0$ be a positive real number (it corresponds to the real parts of $\mathrm{i}\lambda_{\mu,i}$). Then, the following algorithm takes as input a regular initial triangulation of G, an upper bound $\varepsilon > 0$ for the local meshsize, a singular exponent $\lambda > 0$, the local polynomial degree $p \in \mathbb{N}$, and returns a shape-regular, conforming simplicial refinement of $\mathcal{T}_0$.*

$\mathcal{T}_\varepsilon = \mathtt{REFINE}(\mathcal{T}_0, \varepsilon, \lambda, p)$
  $\mathcal{T} := \mathcal{T}_0$
  $\mathcal{M} := \{T \in \mathcal{T};\ h_T > \varepsilon\}$
  % global refinement
  WHILE $\mathcal{M} \neq \emptyset$
    $\mathcal{M} := \{T \in \mathcal{T};\ h_T > \varepsilon\}$
    $\mathcal{T} = \mathtt{REFINE\_MARKED}(\mathcal{T},\mathcal{M})$
  END
  % local graded refinement
  $K := \lceil -\frac{p+1}{\lambda}\log_2(\varepsilon) - 1\rceil$
  $l := 1$
  WHILE $l < 2K+1$
    $\mathcal{M} := \left\{T \in \mathcal{T};\ h_T > \varepsilon 2^{-\frac{l(p+1-\lambda)}{2(p+1)}} \text{ and } \min_i \operatorname{dist}(\mathbf{c}_i, \bar{T}) \leq \sqrt{2}^{-l}\right\}$
    $\mathcal{T} = \mathtt{REFINE\_MARKED}(\mathcal{T},\mathcal{M})$
    $l := l+1$
  END
RETURN $\mathcal{T}_\varepsilon := \mathcal{T}$

Remark 4.6 *The preceding approximation algorithm was presented in [16, Section 4.2]. For details, we refer to this reference and references therein.*
The first part of REFINE *bisects all elements T whose width h_T is greater than the threshold ε. Afterwards, this quasi-uniform refinement is graded towards each corner.*
The refinement process is steered by p and λ in the formula for the number of refinements K. The integer K can be viewed as refinement intensity. It becomes larger as p increases, or as ε or $\lambda \searrow 0$.

The main result about this class of meshes is

Theorem 4.7 *([16, Theorem 5.1]) Let $G \subseteq \mathbb{R}^2$ be a bounded, polygonal domain with interior opening angles $\phi_i \in (0, 2\pi]$, $i = 1, \ldots, M$ at the corners $\mathbf{c}_i$ of the polygon G. Let $p \in \mathbb{N}$, $\lambda := \min\{\Re(\lambda_{\mu,i})\}$, and let $v(\mathbf{x})$ be a function which can be decomposed into $v = v_0 + \sum_{i=1}^{M}\sum_{k=1}^{K} v_k^{(i)}$, such that $v_0 \in H^{p+1}(G)$ and such that the following conditions are satisfied for all $i = 1, \ldots, M$:*

$$\forall \mathbf{x} \in G_i : \qquad |\mathrm{D}^k v_k^{(i)}(\mathbf{x})| \lesssim \Psi(\mathbf{x})^{\lambda-k} \quad \text{for} \quad k \in \{0, 1, \ldots, p+1\}\ . \tag{38}$$

Then, there exists a constant $C > 0$ depending only on the conforming initial triangulation $\mathcal{T}_0$ and on the domain G, such that for any tolerance $\varepsilon > 0$ as in Definition 4.5 and for $\mathcal{T}_\varepsilon := \mathtt{REFINE}(\mathcal{T}_0, \varepsilon, \lambda, p)$,

$$\|v - I_p v; H^1(G)\| \leq C(\#\mathcal{T}_\varepsilon - \mathcal{T}_0)^{-p/2} .$$

4.3. Quasioptimal convergence rates for the space-semidiscrete problem

Including both types of mesh refinements cited in the two preceding subsections, let $\{\mathcal{T}_N\}_N$ be a shape-regular, conforming simplicial triangulation family of G such that for a finite set (to be determined) of singular functions $\mathbf{u}_{s,i}^{\omega,q}$, we have quasi-optimal approximation w. r. to the spatial variable, i. e. there exists a constant $C > 0$ such that for all N, there holds

$$\|\mathbf{u}_{s,i}^{\omega,q} - I_p \mathbf{u}_{s,i}^{\omega,q}; H^1(G)^2\| \leq CN^{-\frac{p}{2}} , \tag{39}$$

being I_p the Clément-interpolant of local polynomial degree p. We know from Proposition 3.2 and Theorem 3.5, that we can choose (ω, q) such that $\mathbf{u}_r^{\omega,q} \in H^s(I; H^{s'}(G)^2)$. Let $\mathbf{u}(\mathbf{x}, t)$ be the solution of (10). We consider solutions $\mathbf{u}_N(\mathbf{x}, t)$ of the following space semidiscrete initial boundary value problem of (10) which results from the classical "Method-of-Lines" approach to time-domain wave propagation problems, and which is given by

$$\begin{aligned} &\text{Find } \mathbf{u}_N \in C^0(\bar{I}; V_N) \text{ such that } \forall \mathbf{v} \in V_N \text{ and } t \in I : \\ &\partial_t^2 (\mathbf{u}_N(\cdot, t), \mathbf{v}) + a(\mathbf{u}_N(\cdot, t), \mathbf{v}) = (\mathbf{f}(\cdot, t), \mathbf{v}) , \quad (\mathbf{u}_N, \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}) , \quad \partial_t (\mathbf{u}_N(\cdot, 0), \mathbf{v}) = (\mathbf{u}_1, \mathbf{v}) , \end{aligned} \tag{40}$$

where $V_N := S_0^p(G, \mathcal{T}_N) \cap V$ denotes the finite-dimensional space of continuous piecewise polynomials of total degree at most p on the regular, simplicial triangulation $\mathcal{T}_N$ of the domain G. At this point, we can collect the results obtained so far in our main result:

Theorem 4.8 *Let $d = 2$, $p \in \mathbb{N}, \gamma > 0$, and assume that $\mathbf{f} \in RV_{\omega,q}(Q; \gamma) \cap C^0(\bar{I}; L^2(G))$, and $\mathbf{u}_0 \in V, \mathbf{u}_1 \in L^2(G)$. Moreover, let $\mathbf{u}(\mathbf{x}, t)$ be the solution of (5) with $|\Gamma_D| > 0$, and $\mathbf{u}_N(\mathbf{x}, t)$ the solution of (40) obtained on a mesh family $\{\mathcal{T}_N\}_N$ that satisfies (39), and let (ω, q) such that $DV_{\omega,q}(Q; \gamma) \hookrightarrow L^2_\omega(\bar{I}; H^{p+1}(G))$.*
Then, there exists a constant $C > 0$ (depending on T and on p), such that for every $t \in \bar{I}$ holds

$$\begin{aligned} \|\mathbf{u}(\cdot, t) - \mathbf{u}_N(\cdot, t); H^1(G)^2\| &+ \|\partial_t \mathbf{u}(\cdot, t) - \partial_t \mathbf{u}_N(\cdot, t); L^2(G)^2\| \\ &\leq C \Bigg\{ \|\mathbf{u}_0 - \mathbf{u}_{0,h}; H^1(G)^2\| + \|\mathbf{u}_1 - \mathbf{u}_{1,h}; L^2(G)^2\| \\ &+ N^{-p/2} \Bigg[\|\mathbf{u}(\cdot, t); H^{p+1}(G)^2\| + \|\partial_t \mathbf{u}(\cdot, t); H^{p+1}(G)^2\| \\ &\quad + \int_0^t \|\partial_t^2 \mathbf{u}(\cdot, s); H^{p+1}(G)^2\| \, ds \Bigg] \Bigg\} . \end{aligned} \tag{41}$$

Proof. The proof is analogous to that of Theorem 5.4 in [34].
Given $p \in \mathbb{N}$, we decompose $\mathbf{u}$ into $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_r^{\omega,q}(\mathbf{x}, t) + \sum_{i=1}^M \chi_i(r_i) \mathbf{u}_{s,i}^{\omega,q}(\mathbf{x}, t)$ using Theorem 3.5, where $\mathbf{u}_r(\cdot, t) \in H^{p+1}(D)$ at each time $t \in I$, and the finite sum of $\chi_i \mathbf{u}_{s,i}^{\omega,q}$ is approximated with optimal convergence order using either Theorem 4.4 or Theorem 4.7.
Let us assume that $\mathbf{u} \in C^2(\bar{I}; L^2_{\omega+q}(G))$ and postpone the proof of this regularity for a moment. An argument similar to [38] shows that if $\mathbf{u}$ is a C^2-function in time as under this assumption, the convergence rate in N only depends on the maximal $\varrho > 0$ such that there exists a constant $c > 0$ such that, as $N \to \infty$, there holds

$$\inf_{\mathbf{v}_N \in V_N} \|\mathbf{u}_{\mu,i}^{(k,j)}(\mathbf{x}) - \mathbf{v}_N\| \leq CN^{-\varrho/2} .$$

By the assumption made on the mesh family $(\mathcal{T}_N)_N$, either Theorem 4.4 or Theorem 4.7 implies that $\varrho = p$.
It remains to show that the regularity of $\mathbf{u}$ in the t-variable is $C^2(\bar{I})$. To this end, let us check the requirements of Theorem 2.1. For $\omega > 0$, $L^2_\omega(G)$ is a Hilbert space with scalar product $(w, v)_\omega := \int_G \Psi^\omega(\mathbf{x})^2 w(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x}$.
By [6, Theorem 4.2], for all $\omega, q > 0$, the embeddings $H^s_\omega(G) \hookrightarrow L^2_\omega(G)$ are compact for all $s \geq 0$. Hence, $H^{q+1}_{\omega+q}(G) \hookrightarrow L^2_{\omega+q}(G) \simeq L^2_{\omega+q}(G)^* \hookrightarrow H^{q+1}_{\omega+q}(G)^*$ is a valid choice for the evolution triplet. Moreover, since $|\Gamma_D| > 0$, $a(\cdot, \cdot)$ is coercive, Equation (3) implies that $a(\cdot, \cdot)$ is symmetric and $a(\cdot, \cdot)$ is obviously bilinear. Therefore, we can use Theorem 2.1 and the assumption that $\mathbf{f} \in C^0(\bar{I}; L^2_{\omega+q}(G)^2)$ to conclude that $\mathbf{u} \in C^2(\bar{I}; H^{q+1}_{\omega+q}(G)^*)^2$. ■

5. Numerical Experiments

To verify the performance of the FEM semidiscretization on locally refined meshes numerically, we use a space-time discretization with continuous, piecewise linear FEM in space, combined with the (unconditionally stable) Crank-Nicolson scheme with constant

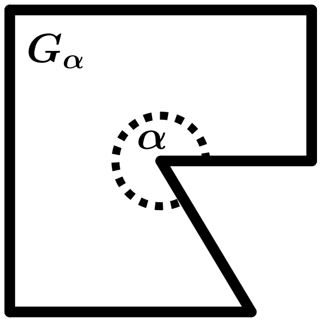

Figure 2. An example of a domain type on which our tests have been performed.

timestep. The timestep Δt is chosen sufficiently small, such that the contribution of the time discretization error to the total discretization error (which was numerically estimated in the presented numerical results) in the Bochner Norm of $L^2(I; H^1(G)^2)$ is negligible.
We simulate the propagation of elastic waves for a series of meshes with decreasing meshsize. For each run, we compute a reference solution obtained by four more refinements and the same Δt. This allows us to compute a viable estimator for the space discretization error in the $L^2(I; H^1(G)^2)$-Norm. Once the errors are collected for all meshes, we compute an empirical convergence rate by a linear fit through the natural logarithms of the data sets containing the numbers of d.o.f. and the errors.
Given an angle $\alpha \in [\pi, 2\pi)$, the polygon $G = G_\alpha$ is chosen to be a conical subset of the unit square $(0,1)^2$ with vertex $\mathbf{c}_1 := (\frac{1}{2}, \frac{1}{2})$, such that one edge of the cone goes through $(1, \frac{1}{2})$. The other edge abutting at $\mathbf{c}_1$ is chosen s.t. the interior opening angle at $\mathbf{c}_1$ equals α. Hence, we compute on a domain with one non-convex and five convex corners. See Figure 2 for an illustration.
On the edges abutting at $\mathbf{c}_1$, we impose homogeneous Dirichlet conditions, while on the other edges, homogeneous Neumann conditions are set.
The volume forcing function, as well as the initial condition on the first time derivative are set to be zero: $\mathbf{f} \equiv 0 \equiv \mathbf{u}^1$, whereas the initial condition on $\mathbf{u}(\cdot, 0)$ is set to be

$$(\mathbf{u}^0)_{1,2}(x_1, x_2) = \psi_{(\frac{1}{4},\frac{1}{2});0.2}(x_1)\psi_{(\frac{1}{4},\frac{1}{2});0.2}(x_2) ,$$

with ψ given by $\psi_{x_0;r} := \exp\left(-\left(1-\left(\frac{x-x_0}{r}\right)^2\right)^{-1}\right) \mathbb{1}_{[x_0-r,x_0+r]}(x)$, denoting $\mathbb{1}_S$ the characteristic function on a subset $S \subseteq \mathbb{R}$.
In other words, we observe the evolution of a compactly supported smooth wave in the absence of body forces acting on the body G, which is fixed at the two edges where Dirichlet boundaries are given.
Solutions are computed on a uniform mesh family and on a mesh family obtained by bisection refinement as described in Theorem 4.7.
As discussed in Remark 4.6, the number of refinement layers towards any vertex can be measured by K and tends to infinity as $\Re \mathrm{i}\lambda_{\mu,i} \searrow 0$. For a general operator $\mathcal{A}$ meeting the assumptions (2-4), there is no closed-form formula for the singular exponents $\mathrm{i}\lambda_{\mu,i}$. In [11], Costabel and Dauge proposed an efficient method to compute the $\Re \mathrm{i}\lambda_{\mu,i}$ by numerical computation of the zeros of a certain determinant.
In our experiments, we have not computed the values of the singular exponents. The convergence tests on locally mesh families obtained by calls of $\mathtt{REFINE}(\mathcal{T}_0, \varepsilon, \lambda, p)$ were carried out with various input parameters λ. After computation of the empirical convergence rate, we compared it with the expected optimal rate and run the test again with a mesh obtained by passing a smaller λ to REFINE. This is repeated until we have reached a λ^* for which a call of REFINE with smaller $\lambda < \lambda^*$ does not improve the empirical convergence rate.
The material is assumed to be homogeneous (i. e. the elastic moduli are assumed to be constant in G) and isotropic, i. e.

$$\mathcal{A}[\mathbf{u}] = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) ,$$

where $\lambda, \mu \in \mathbb{R}$ such that $\min(\mu, \lambda + \mu) > 0$, are the Lamé-constants.
The numerical experiments have been performed using the DOLFIN [29] and FEniCS [28] libraries in Python.

5.1. Test 1: Influence of α on the convergence rates

We run the procedure described above with piecewise linear Finite Elements (i .e., $p = 1$) on a series of $(G_{\alpha_j})_{j=1,2,3}$ with $\alpha_1 = \pi$, $\alpha_2 = \frac{\pi}{2}$, $\alpha_3 := 0.9 \cdot 2\pi$ and investigate the convergence rates which result on locally refined meshes and a quasi-uniform mesh family.
On a uniformly refined mesh family, we get suboptimal convergence rates as depicted in Figure 3. However, on the locally refined meshes obtained as in Section 4.2, the optimal convergence order $O(N^{-1/2})$ (see e. g. [8, Thm. 4.4.4]) is obtained for all the angles. The resulting convergence rates are listed in Table 1. As is well-known from the elliptic setting (see, e. g., [37, 2, 5]), the

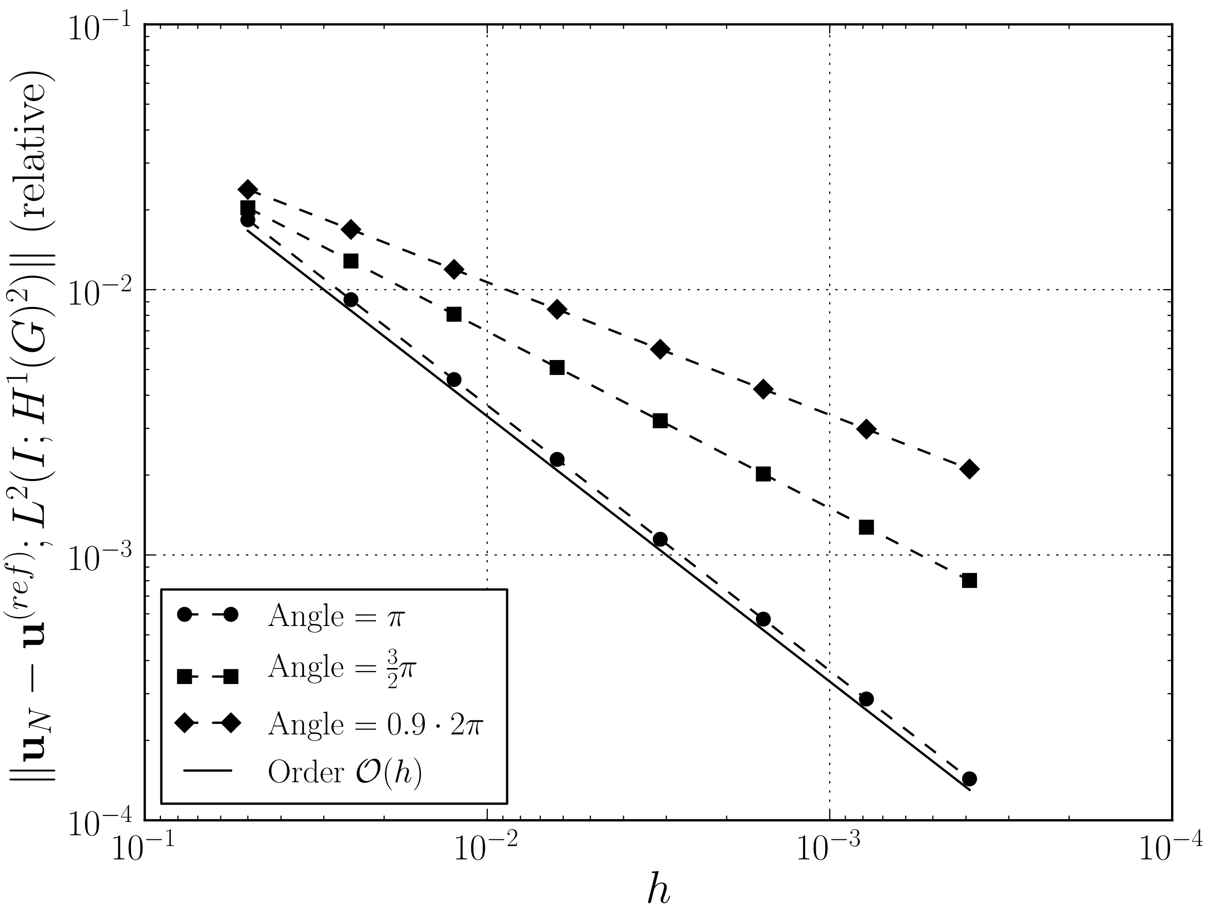

Figure 3. Numerical results for the test problem on uniform meshes: using the same compliance tensor, the convergence rate decreases as α tends to 2π. For $\alpha = \pi$, G_α is convex and therefore, $u(\cdot, t) \in H^2(G)$ at, all times t. This is reflected in the experiment in the fact that the convergence rates are already optimal for a uniform mesh family. The *relative error in the norm* $\| \circ \|_{L^2(I;H^1(G)^2)}$ *is plotted versus the uniform mesh size h, which behaves asymptotically as* $O(N^{-\frac{1}{2}})$. *In the numerical results which are reported here, we used a fixed timestep of size* $\Delta t = 10^{-7}$ *and* $T_{\max} = 0.5$.

Table 1. Results of Test 1: As the optimal convergence rate is 0.5, the experiment confirms that judicious local mesh grading towards the corners of the domain can recover the optimal convergence rates afforded by continuous, piecewise linear Finite Element spatial discretization.

α	empirical convergence rate
1.25π	0.5036
1.5π	0.5155
1.75π	0.4988
1.98π	0.4955

mesh-grading must be stronger for increasing interior corner angles. For a complexity estimate of the Finite Element Method in the presence of such local refinement, and for the complexity of the bisection tree refinement itself, we refer again to [16] and the references there.

5.2. Test 2: Convergence rates on graded Meshes

As a second test, we repeat the procedure of Test 1 with piecewise linear FEM on a graded mesh family obtained by so-called κ-refinement, as introduced by Băcuţă et al. in [5].
These mesh families are related to the idea of grading towards the corners we briefly reviewed here in Section 4.1. The construction of the mesh depends on various parameters.

Table 2. Results of Test 2.

α	empirical convergence rate
1.25π	0.4918
1.5π	0.5003
1.75π	0.4832
1.98π	0.4891

It has been proven in [5, Thm. 4.2] that choosing the "right" set of parameters, a function $v(\mathbf{x}) \in H_\delta^{p+1,2}(G)$ gets approximated with optimal convergence rate $\mathcal{O}(h^p) = \mathcal{O}(N^{-p/2})$ (as $h \to 0$ or $N \to \infty$) by Finite Elements with piecewise polynomial degree p [†].
An efficient implementation of linearly graded meshes on polygonal domains has been made available by Li and Nistor under the name LNG_FEM, see [26] for a detailed description of the software.
We repeat the previous experiment on mesh families obtained by $\kappa = 0.1$ towards all the vertices. As expected, the optimal convergence rates are obtained also using these meshes. The empirical convergence rates are listed in Table 2.
The discretization in time is done identically as in Subsection 5.1 for Test 1.

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