

# Nonsmooth Trust Region Algorithms for Locally Lipschitz Functions on Riemannian Manifolds

P. Grohs and S. Hosseini

Research Report No. 2014-20

August 2014

Latest revision: October 2014

Seminar für Angewandte Mathematik  
Eidgenössische Technische Hochschule  
CH-8092 Zürich  
Switzerland

# NONSMOOTH TRUST REGION ALGORITHMS FOR LOCALLY LIPSCHITZ FUNCTIONS ON RIEMANNIAN MANIFOLDS

P. GROHS\*, S. HOSSEINI\*

ABSTRACT. This paper presents a Riemannian trust region algorithm for unconstrained optimization problems with locally Lipschitz objective functions defined on complete Riemannian manifolds. To this end we define a function  $\Phi : TM \rightarrow \mathbb{R}$  on the tangent bundle  $TM$ , and at  $k$ -th iteration, using the restricted function  $\Phi|_{T_{x_k}M}$  where  $T_{x_k}M$  is the tangent space at  $x_k$ , a local model function  $Q_k$  that carries both first and second order information for the locally Lipschitz objective function  $f : M \rightarrow \mathbb{R}$  on a Riemannian manifold  $M$ , is defined and minimized over a trust region. We establish the global convergence of the proposed algorithm. Moreover, using the Riemannian  $\varepsilon$ -subdifferential, a suitable model function is defined. Numerical experiments illustrate our results.

## 1. INTRODUCTION

Most classical problems considered in optimization are formulated in Banach spaces, where the linear structure plays an important role. However, many problems in computer vision, robotics, signal processing and geometric mechanics, to name but a few, are more conveniently expressed as optimization problems on Riemannian manifolds [4, 34, 30, 40, 43, 46]. Therefore it is of eminent interest to develop useful computational and theoretical tools of optimization on manifolds. This paper is concerned with the numerical solution of optimization problems defined on Riemannian manifolds where the objective function may be nonsmooth.

Many algorithms for solving the following unconstrained optimization problem,

$$\min_{x \in \mathbb{R}^n} f(x)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable, have been proposed. Trust region methods are an important class of iterative methods due to their strong global convergence and fast local convergence; see [14]. In this class of iterative methods, a step to the  $k+1$ -th iterate is obtained by minimizing a model function  $Q_k$  defined by

$$Q_k(x_k, d) = f(x_k) + \nabla f(x_k)^T d + \frac{1}{2} d^T B_k d,$$

over a restricted region centered at the current iterate. It is worth pointing out that in this model function,  $B_k$  is adequately selected and the model function preserves the first and second order information of the objective function  $f$ . The

---

*Key words and phrases.* Riemannian manifolds, Lipschitz function, Trust region method, Clarke subdifferential.

*AMS Subject Classifications:* 49J52, 65K05, 58C05.

*\*ETH Zürich, Seminar for Applied Mathematics, Sälimstrasse 101, 8092 Zürich, Switzerland (pgrohs@sam.math.ethz.ch, seyedehsomyeh.hosseini@math.ethz.ch).*

so-called trust region ratio evaluates an agreement between the model and the actual objective reductions along the computed step. Considering the trust region ratio, one can decide whether the step is accepted or rejected. After that the trust region radius is updated and a new point is obtained.

The classical trust region methods for smooth problems cannot be used for nonsmooth ones since in general the gradient of the objective function at the current iterate does not exist. Several trust region methods for minimizing a nonsmooth objective function defined on a linear space have been presented and applied to the nonlinear equations problem, the nonlinear fitting problem, and the constrained optimization problems; see [37, 15, 3] and references therein. The most well-known nonsmooth trust region methods begin from a starting point  $x_1$  which may not be close to the minimum of the objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . They use a function  $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  to build at each iteration a model  $Q_k$  defined by

$$Q_k(x_k, d) = f(x_k) + \Phi(x_k, d) + \frac{1}{2}d^T B_k d,$$

which must be an approximation of  $f(x_k + d)$  for small  $d$ . Therefore, they should impose some conditions on the function  $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and the sequence of the symmetric matrices  $B_k$ . For instance, [37] is among the first works on trust region methods for unconstrained optimization problems with locally Lipschitz objective functions; in that article several conditions on  $B_k$  and  $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  are proposed, which insure that  $Q_k$  is an approximation for  $f(x_k + d)$  for small  $d$  and the algorithm is convergent.

Nonsmooth trust region algorithms approximately solve the subproblem

$$\min_{\{d \in \mathbb{R}^n : \|d\| \leq \delta_k\}} Q_k(x_k, d)$$

to obtain  $d_k$ . Using the trust region ratio, either the step is accepted or rejected.

The extension of nonsmooth trust region algorithms and their (global) convergence properties to Riemannian manifolds are the subject of the present paper. A manifold, in general, does not have a linear structure; hence the usual techniques, which are often used to study optimization problems on linear spaces cannot be applied and new techniques need to be developed.

The development of Riemannian nonsmooth optimization algorithms is primarily motivated by large-scale applications that have gained much popularity in recent years regarding the framework of  $l^1$  optimization, stochastic optimization, and statistical learning. Those applications include robust, sparse, structured principal component analysis, statistics on manifolds (e.g. median calculation of positive semidefinite tensors), and low-rank optimization (matrix completion, collaborative filtering, source separation); see [28, 47, 48, 46]. Furthermore, these algorithms have many applications in image processing, computer vision, nonsmooth constrained optimization problems on linear spaces; [5, 12, 18].

**Previous Work.** For the optimization of smooth objective functions many classical methods for unconstrained minimization, such as Newton-type and trust-region methods, have been successfully generalized to problems on Riemannian manifolds; see [1, 4, 16, 34, 39, 44, 45, 49]. The recent monograph [2] by Absil, Mahony and Sepulchre discusses, in a systematic way, the framework and many numerical first-order and second-order manifold-based algorithms for minimization problems on Riemannian manifolds with an emphasis on applications to numerical linear algebra; see [2].

As we discussed above, one of the most important methods in the unconstrained optimization of smooth functions is the trust region method due to its strong global convergence and fast local convergence. In [1], a Riemannian trust region method for smooth functions on Riemannian manifolds was introduced. Similar to Euclidean trust region methods, the Riemannian trust region method ensures global convergence properties while allowing superlinear local convergence. The trick in [1] was to define a retraction  $R$  on a Riemannian manifold  $M$  that defines for any  $x \in M$ , a one-to-one correspondence  $R_x$  between a neighborhood of  $x$  in  $M$  and a neighborhood of  $0_x$  in the tangent space  $T_x M$ . Using this retraction, the objective function  $f$  on  $M$  is lifted to an objective function  $\overline{f}_x = f \circ R_x$  on  $T_x M$ . Then a quadratic model of  $\overline{f}_x$  is defined, and a classical method on the Euclidean space  $T_x M$  is used to compute a minimizer of the model within a trust region around  $0_x \in T_x M$ . Afterward, the minimizer is lifted back to  $M$  to be a new candidate for the next iterate. The most noticeable point in the mentioned trust region method is that it does not deal with a unique objective function since in any iteration the retraction is changed. However, the authors proved that under some conditions, the nice properties of the classical trust region method are preserved in their Riemannian generalizations.

In considering optimization problems with nonsmooth objective functions on Riemannian manifolds, it is necessary to generalize concepts of nonsmooth analysis to Riemannian manifolds. In the past few years a number of results have been obtained on numerous aspects of nonsmooth analysis on Riemannian manifolds; see [6, 7, 23, 24, 25, 33].

Recently, some mathematicians have started developing nonsmooth optimization algorithms for manifold settings although their attempts are limited to generalizing some subgradient based and proximal point algorithms. In [8, 11, 19], constrained minimization problems on Hadamard manifolds are solved using a generalization of the proximal point method. There have also been some studies by Ferreira, Bento and Oliveira and their colleagues who generalized subgradient-type methods for convex and quasiconvex functions defined on Riemannian manifolds; see [9, 10, 17, 20, 36]. Finally, it is worth mentioning paper [18], which presents a survey on Riemannian geometry methods for smooth and nonsmooth constrained optimization as well as gradient and subgradient descent algorithms on a Riemannian manifold. In that paper, the methods are illustrated by applications from robotics and multi antenna communication.

**Contributions.** Our main contributions are twofold. First, we impose some conditions on a function  $\Phi : TM \rightarrow \mathbb{R}$ , where  $TM$  is the tangent bundle of a Riemannian manifold  $M$ , to insure that our model function

$$Q_k(x_k, \cdot) = f(x_k) + \Phi(x_k, \cdot) + \frac{1}{2} \langle B_k \cdot, \cdot \rangle : T_{x_k} M \rightarrow \mathbb{R}$$

preserves the first order information of the locally Lipschitz objective function  $f : M \rightarrow \mathbb{R}$ . Our proposed conditions are generalizations of the conditions in [37]. Our first main result, Theorem 3.6, states that, provided these assumptions are satisfied, then the Riemannian trust region scheme which iteratively (approximately) minimizes the model function  $Q_k$  on a ball in  $T_{x_k} M$  converges globally to a critical point of  $f$ .

As the second main contribution of this paper, in Section 4 we propose several choices for a suitable function  $\Phi$ , the simplest one being the Clarke generalized

directional derivative of  $f$  as a function  $TM \rightarrow \mathbb{R}$ . However, this function is only practical if the subdifferential of  $f$  can be computed explicitly, which might not be the case. Therefore, based on an approximation scheme for the so-called  $\varepsilon$ -subdifferential first introduced in [22] for the Riemannian case, we also present a numerical approximation of the subdifferential which is efficiently computable and which still yields a globally convergent algorithms. This is carried out in Section 4.

To the best of our knowledge, the resulting algorithm is the first practical non-smooth trust region algorithm for locally Lipschitz functions defined on Riemannian manifolds. In Section 5 we present some numerical experiments. Our proposed algorithm is implemented in MATLAB and applied to some nonsmooth problems with locally Lipschitz objective functions. Numerical results show that the proposed algorithm has a far better performance in some problems in comparison with existing first order methods, such as subgradient descent.

## 2. PRELIMINARIES

Throughout this paper,  $M$  is an  $n$ -dimensional complete manifold endowed with a Riemannian metric  $\langle \cdot, \cdot \rangle$  on the tangent space  $T_x M$ ; see [29]. As usual  $B(x, \delta)$  denotes the open ball centered at  $x$  with radius  $\delta$ . For the point  $x \in M$ ,  $\exp_x : U_x \rightarrow M$  defines the exponential function at  $x$ , where  $U_x$  is an open subset of  $T_x M$ . For a minimizing geodesic  $\gamma : [0, l] \rightarrow M$  connecting  $x$  to  $y$  in  $M$ , and for a vector  $v \in T_x M$  there is a unique parallel vector field  $P$  along  $\gamma$  such that  $P(0) = v$ , this is called the parallel translation of  $v$  along  $\gamma$ . The mapping  $T_x M \ni v \mapsto P(l) \in T_y M$  is a linear isometry from  $T_x M$  onto  $T_y M$ . This map is denoted by  $L_{xy}$ . An easy consequence of the definition of the parallel translation along a curve as a solution to an ordinary linear differential equation implies that the mapping

$$(2.1) \quad C : TM \rightarrow T_{x_0} M, \quad C(x, \xi) = L_{xx_0}(\xi),$$

when  $x$  is in a neighborhood  $U$  of  $x_0$ , is well defined and continuous at  $(x_0, \xi_0)$ ; that is, if  $(x_n, \xi_n) \rightarrow (x_0, \xi_0)$  in  $TM$  then  $L_{x_n x_0}(\xi_n) \rightarrow L_{x_0 x_0}(\xi_0) = \xi_0$ , for every  $(x_0, \xi_0) \in TM$ ; see [6, Remark 6.11].

Note that  $i_M(x)$  denotes the injectivity radius of  $M$  at  $x$ ; that is the supremum of the radius  $r$  of all balls  $B(0_x, r)$  in  $T_x M$  for which  $\exp_x$  is a diffeomorphism from  $B(0_x, r)$  onto  $B(x, r)$ . If  $U$  is a compact subset of a Riemannian manifold  $M$  and  $i(U) := \inf\{i_M(x) : x \in U\}$ , then  $0 < i(U)$ ; see [26].

A set  $S$  in a Riemannian manifold  $M$  is said to be convex if every two points  $p_1, p_2 \in S$  can be joined by a unique geodesic whose image belongs to  $S$ .

A real valued function  $f$  is said to be locally Lipschitz on  $M$  if  $f$  is Lipschitz near  $x$ , for every  $x \in M$ ; that is for every  $x \in M$ , there exist an open neighborhood  $B(x, \delta)$  and a real number  $k > 0$  such that  $|f(z) - f(y)| \leq k \text{dist}(z, y)$  for every  $z, y \in B(x, \delta)$ , where  $\text{dist}$  is the Riemannian distance on  $M$ .

It is easy to see that the metric on  $TM$  can be defined as follows; see [13]. Assume that  $u, v \in TM$ ,  $\pi : TM \rightarrow M$  is the projection map, and let  $\gamma$  be a piecewise smooth path from  $x = \pi(u)$  to  $y = \pi(v)$ , whose derivative is never zero. Let  $L_{xy}^\gamma$  be the parallel translation along  $\gamma$ . We define the square of the distance from  $u$  to  $v$  as the infimum over all paths  $\gamma$  from  $\pi(u)$  to  $\pi(v)$  of

$$\text{dist}_\gamma(u, v)^2 = \|L_{xy}^\gamma(u) - v\|^2 + \text{length}(\gamma)^2.$$

To justify the above assertion, note that over a small piece of  $\gamma$ , parallel translation gives us a canonical identification of any two tangent spaces, and hence the tangent bundle is metrically the product of an interval and  $\mathbb{R}^n$ .

If  $x$  and  $y$  are two points in the same convex neighbourhood in  $M$ , let  $\gamma$  be the geodesic joining them and  $L_{xy}$  be the parallel translation along  $\gamma$  from the tangent space at  $x$  to the tangent space at  $y$ . Then

$$\text{dist}_{TM}(u, v)^2 \leq \|L_{xy}(u) - v\|^2 + \text{dist}(x, y)^2.$$

Moreover,

$$\text{dist}(x, y) \leq \text{dist}_{TM}(u, v).$$

### 3. A NONSMOOTH TRUST REGION METHOD ON RIEMANNIAN MANIFOLDS

Assume that  $f : M \rightarrow \mathbb{R}$  is a locally Lipschitz function on a complete Riemannian manifold  $M$ .

The trust region algorithm proposed in the present section, which is based on [37], relies on the choice of a function  $\Phi : TM \rightarrow \mathbb{R}$  (modeling the derivative of  $f$ ) and a sequence  $\{B_k : k = 1, 2, \dots\}$  of  $n \times n$  symmetric matrices (modeling the Hessian of  $f$ ) from which we build a sequence of model functions

$$Q_k : \begin{cases} T_{x_k}M & \rightarrow & \mathbb{R} \\ (x_k, d) & \mapsto & f(x_k) + \Phi(x_k, d) + \frac{1}{2}\langle B_k d, d \rangle \end{cases}$$

analogous to a second order Taylor expansion in the Euclidean case.

Then the proposed trust region scheme iteratively computes approximative minima of these model functions over a trust region as follows:

Let

$$(3.1) \quad d_k^* = \text{argmin}\{Q_k(x_k, d_k) = f(x_k) + \Phi(x_k, d_k) + 1/2\langle B_k d_k, d_k \rangle : d_k \in T_{x_k}M, \|d_k\| \leq \delta_k\}.$$

Then in each iteration step we compute  $d_k^*$  or, if this turns out to be not practical, an approximation  $\bar{d}_k$  of  $d_k^*$  in the sense that

$$(3.2) \quad f(x_k) - Q_k(x_k, \bar{d}_k) \geq c_0[f(x_k) - Q_k(x_k, d_k^*)] \quad \text{and} \quad \|\bar{d}_k\| \leq \delta_k,$$

for a fixed constant  $0 < c_0 \leq 1$ .

Then, depending on the trust ratio in each step we either reduce the trust region by reducing the value  $\delta_k$  and solving (3.1), or we update

$$(3.3) \quad x_{k+1} := \exp_{x_k}(\bar{d}_k),$$

see Algorithm 1.

**Remark 3.1.** *A natural choice for the function  $\Phi$  would be the Clarke generalized directional derivative of  $f$  as defined below in (4.1).*

*In case the Clarke generalized directional derivative of  $f$  is not given explicitly (as is the case for instance in [18, Sections 3.2 and 3.3] and [27]), in Section 4 we also propose a numerical approximation which only solves (3.1). The choice of matrices  $\{B_k\}$  is quite general; in our results they simply need to be uniformly bounded but even this property can be weakened; see for instance [37]. In our algorithms we have used a Riemannian generalization of the BFGS method to iteratively update the matrices  $B_k$ ; see also [38].*

---

**Algorithm 1** A nonsmooth trust region algorithm on Riemannian manifolds
 

---

- 1: **Data:** An  $n$ -dimensional complete Riemannian manifold  $(M, g)$ ; a real valued locally Lipschitz function  $f$  on  $M$ .
  - 2: **Parameters:**  $\delta_0 > 0$ ,  $\delta_0 > \delta_1 > 0$ ,  $c_0, c_1, c_2, c_3, c_4 > 0$ ,  $c_2 < c_1 < 1$ ,  $c_0 \leq 1$ .
  - 3: **Input:** initial iterate  $x_1 \in M$ , and  $B_1 \in S(n)$ .
  - 4: **Output:** sequence of iterates  $\{x_k\}$ .
  - 5: **for**  $k = 1, 2, \dots$  **do** find
    - (3.4)  $d_k^* = \operatorname{argmin}\{Q_k(x_k, d_k) = f(x_k) + \Phi(x_k, d_k) + 1/2 \langle B_k d_k, d_k \rangle : d_k \in T_{x_k} M, \|d_k\| \leq \delta_k\}$ .
    - where  $\Phi : TM \rightarrow \mathbb{R}$  is a given function.
    - 6: Assume  $\bar{d}_k$  is an inexact solution of 3.4 in the sense that
 
$$f(x_k) - Q_k(x_k, \bar{d}_k) \geq c_0 [f(x_k) - Q_k(x_k, d_k^*)]$$
 and  $\|\bar{d}_k\| \leq \delta_k$ .
    - 7: **if**  $\bar{d}_k = 0$  **then**, Stop.
    - 8: **else**
    - 9:
 
$$r_k = \frac{f(x_k) - f(\exp_{x_k}(\bar{d}_k))}{f(x_k) - Q_k(x_k, \bar{d}_k)},$$
    - 10: **if**  $c_2 < r_k$ , **then**  $x_{k+1} = \exp_{x_k}(\bar{d}_k)$  and update  $B_k$ .
    - 11: **end if**
    - 12: **if**  $r_k \leq c_2$ , **then**  $x_{k+1} = x_k$ ,  $\delta_{k+1} = c_3 \delta_k$ .
    - 13: **else**
    - 14: **if**  $c_2 < r_k \leq c_1$ , **then**  $\delta_{k+1} = \delta_k$ .
    - 15: **else**  $\delta_{k+1} = \min\{c_4 \delta_k, \delta_0\}$ .
    - 16: **end if**
    - 17: **end if**
    - 18: **end if**
    - 19: **end for**
- 

**Remark 3.2.** *Instead of using the exponential map to update  $x_k$ , we can choose a retraction  $R : TM \rightarrow M$ . The notion of retraction on a manifold, includes all first-order approximations to the Riemannian exponential; see [1]. The retraction can be used to take a step in the direction of a tangent vector. Using a good retraction amounts to finding an approximation of the exponential mapping that can be computed with low computational cost while not adversely affecting the behavior of the optimization algorithm.*

In the remainder of this section we formulate assumptions on the  $\Phi$  to ensure global convergence of the sequence  $(x_k)_k$  to a critical point of  $f$ , in the following sense.

**Definition 3.3.** *With  $f : M \rightarrow \mathbb{R}$  and  $\Phi : TM \rightarrow \mathbb{R}$ , define*

$$(3.5) \quad \psi(x, \delta) = \sup\{-\Phi(x, d) : d \in T_x M, \|d\| \leq \delta\}.$$

*The point  $x \in M$  is called a critical point with respect to  $\Phi$  of the objective function  $f$  if there exists  $\delta > 0$  such that  $\psi(x, \delta) = 0$ .*

In the Euclidean case the notion of critical point as defined in Definition 3.3 is common in the context of nonsmooth trust region methods; see [37, 15]. Under

weak assumptions a critical point corresponds to a so-called ‘Dini stationary point’ [37].

One can show that a local minimizer  $x$  of a locally Lipschitz function  $f : M \rightarrow \mathbb{R}$  is always a critical point, provided that the function  $\Phi$  satisfies some natural assumption. This will be done in Lemma 3.4 below.

**3.1. Convergence Conditions.** Let  $\Phi : TM \rightarrow \mathbb{R}$  and  $I = \{1, 2, \dots\}$ ,  $N = \{x \in M : f(x) \leq f(x_1)\}$  where  $x_1$  is a starting point for Algorithm 1. Let  $\{B_k : k = 1, 2, \dots\}$  be a sequence of  $n \times n$  symmetric matrices. In this section, we make some assumptions on  $\Phi$  and  $B_k$  and show that these assumptions ensure global convergence of Algorithm 1 to a critical point of  $f$  in the sense of Definition 3.3. We extend results of [37] to the Riemannian case.

We start with the following assumptions on the function  $\Phi$ .

**Assumption 3.1.** *Let  $\Phi : TM \rightarrow \mathbb{R}$ . Assume that*

$$(3.6) \quad \Phi(x, 0_x) = 0 \quad \forall x \in M,$$

$$(3.7) \quad \Phi(x, \alpha d) \leq \alpha \Phi(x, d), \quad \forall (x, d) \in TM, 0 \leq \alpha \leq 1,$$

$$(3.8) \quad \text{for all } x \in M, \Phi|_{T_x M} \text{ is lower semi continuous,}$$

for any  $(x, d) \in TM$  it holds that

$$(3.9) \quad f(\exp_x(d)) - f(x) \leq \Phi(d) + o(\|d\|),$$

and there exists  $\delta_*$  such that

$$(3.10) \quad \text{for all } \delta < \delta_* \text{ the function } \psi(\cdot, \delta) \text{ is lower semi continuous,}$$

where  $\psi$  is defined in (3.5) and the implicit constant in the  $o$ -term is uniform over compact sets.

First we show that, provided  $\Phi$  satisfies Assumption 3.1, every local minimum of a locally Lipschitz function  $f : M \rightarrow \mathbb{R}$  is a critical point in the sense of Definition 3.3.

**Lemma 3.4.** *Suppose that  $f : M \rightarrow \mathbb{R}$  and  $\Phi : TM \rightarrow \mathbb{R}$  such that Assumption 3.1 holds. Then every local minimizer of  $f$  is a critical point in the sense of Definition 3.3.*

In order to establish this result we utilize the following simple lemma which is a straightforward extension of [37, Lemma 3.1].

**Lemma 3.5.** *Suppose that  $\Phi$  satisfies Assumption 3.1 and let  $\psi$  be defined by (3.5). Then for any  $x \in M$ , the function  $d \mapsto \psi(x, d)$ ,  $d \in T_x M$ , is nonnegative and nondecreasing and for any  $\alpha \in [0, 1]$ ,*

$$\psi(x, \alpha d) \geq \alpha \psi(x, d),$$

and for any  $\delta > 0$ ,

$$\psi(x, \delta) = 0 \text{ if and only if } \psi(x, 1) = 0.$$

We can now proceed to the

*Proof of Lemma 3.4.* Suppose that  $\bar{x}$  is a local minimizer of  $f$  on some neighborhood  $B(\bar{x}, \delta)$  which is not a critical point, then Lemma 3.5 implies that  $\psi(\bar{x}, 1) > 0$ . Therefore, there exists  $d_{\bar{x}}$  with  $\|d_{\bar{x}}\| \leq 1$  such that  $\Phi(\bar{x}, d_{\bar{x}}) < 0$ . Assume that  $t < 1$  is small enough, hence by assumptions (3.9) and (3.7)

$$f(\exp_{\bar{x}}(td_{\bar{x}})) - f(\bar{x}) \leq \Phi(\bar{x}, td_{\bar{x}}) + o(\|td_{\bar{x}}\|) \leq t\Phi(\bar{x}, d_{\bar{x}}) + o(\|td_{\bar{x}}\|).$$

Therefore, if  $t$  is small enough

$$\frac{f(\exp_{\bar{x}}(td_{\bar{x}})) - f(\bar{x})}{t} \leq \Phi(\bar{x}, d_{\bar{x}})/2 < 0,$$

which means that  $\bar{x}$  is not a local minimizer of  $f$  and this is a contradiction.  $\square$

Now we want to establish conditions for the convergence of Algorithm 1 to a critical point of  $f$ .

We need to make more assumptions on the starting point and the sequence  $(B_k)_k$ .

**Assumption 3.2.** Recall  $N = \{x \in M : f(x) \leq f(x_1)\}$  where  $x_1$  is the starting point of Algorithm 1. Assume that  $N$  is bounded. Furthermore assume that there exists  $C > 0$  such that  $\|B_k\| \leq C$ , for all  $k = 1, 2, \dots$

The main theorem of this section reads as follows.

**Theorem 3.6.** Suppose that  $\Phi$  and  $(B_k)_k$  are such that Assumptions 3.1 and 3.2 hold true. If  $\bar{x}$  is an accumulation point of  $\{x_k\}$ , generated by Algorithm 1, then  $\bar{x}$  is a critical point of  $f$  in the sense of Definition 3.3.

In order to establish Theorem 3.6 we require the following result which can be proved exactly in the same way as Lemma 3.2 in [37].

**Lemma 3.7.** Let  $x \in M$  and  $\delta > 0, 0 < c_0 \leq 1$ . Define

$$d^* := \operatorname{argmin}\{Q_x(d) := f(x) + \Phi(x, d) + 1/2\langle Bd, d \rangle : d \in T_x M, \|d\| \leq \delta\}$$

and let  $\bar{d}$  be an approximate solution of the above problem in the sense that

$$f(x) - Q_x(\bar{d}) \geq c_0[f(x) - Q_x(d^*)] \quad \text{and} \quad \|\bar{d}\| \leq \delta.$$

Then, for all  $\nu \geq \delta$  it holds that

$$f(x) - Q_x(\bar{d}) \geq \frac{c_0}{2\nu} \psi(x, \nu) \min\{\delta, \psi(x, \nu)/(\|B\|\nu)\},$$

where the second term in the min notation is understood as  $\infty$  if  $B = 0$ .

The following lemma proves that if Algorithm 1 generates a sequence  $\{x_k\}$  with  $x_k = \bar{x}$  for all large  $k$ , then  $\bar{x}$  is a critical point of  $f$ .

**Lemma 3.8.** Suppose that  $\bar{x}$  is an accumulation point of  $\{x_k\}$  which is not a critical point. Then there exist  $\epsilon > 0$  and  $\beta > 0$  such that for all  $k$  satisfying

$$(3.11) \quad \operatorname{dist}(x_k, \bar{x}) < \epsilon, \quad 0 < \delta_k < \beta, \quad \|B_k\| \leq C,$$

we have

$$r_k = \frac{f(x_k) - f(\exp_{x_k}(\bar{d}_k))}{f(x_k) - Q_k(x_k, \bar{d}_k)} > c_2,$$

where  $x_k, \delta_k, c_2$  are the same as in algorithm 1.

*Proof.* Since  $\bar{x}$  is not a critical point, Lemma 3.5 implies that  $\psi(\bar{x}, \bar{\delta}) > 0$  for every  $\bar{\delta} > 0$ . Using Assumption 3.10, let  $\delta_*$  be such that for all  $0 \leq \delta \leq \delta_*$ ,  $\psi(\cdot, \delta)$  is lower semi continuous. Therefore, there exist  $\theta > 0$  and  $\epsilon > 0$  such that  $\psi(\bar{x}, \delta_*) = 2\theta > 0$  and

$$\psi(x_k, \delta_*) > \theta \text{ provided } \text{dist}(x_k, \bar{x}) < \epsilon.$$

Assuming that  $\delta_k < \beta < \delta_*$ , then by Assumption 3.9 and Lemma 3.7

$$(3.12) \quad \frac{f(\exp_{x_k}(\bar{d}_k)) - Q_k(x_k, \bar{d}_k)}{f(x_k) - Q_k(x_k, \bar{d}_k)} \leq \frac{o(\|\bar{d}_k\|) - 1/2\langle B_k \bar{d}_k, \bar{d}_k \rangle}{(c_0/2\delta_*)\psi(x_k, \delta_*) \min\{\delta_k, \psi(x_k, \delta_*)/(\|B_k\|\delta_*)\}} \\ \leq \frac{o(\|\delta_k\|)}{(c_0/2\delta_*)\theta \min\{\delta_k, \theta/(C\delta_*)\}}.$$

Assuming that  $\beta$  is small enough, then

$$\frac{f(\exp_{x_k}(\bar{d}_k)) - Q_k(x_k, \bar{d}_k)}{f(x_k) - Q_k(x_k, \bar{d}_k)} < 1 - c_2,$$

which completes the proof.  $\square$

We can now proceed to the

*Proof of Theorem 3.6.* Note that  $M$  is a complete Riemannian manifold and  $N$  is compact. We assume that an infinite subsequence  $\{x_k : k \in I_*\}$  converges to some  $\bar{x}$  which is not a critical point. Let  $I_0 = \{k : r_k > c_2\}$  and  $I_* \subset I_0$ . Therefore, by Lemma 3.7

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq c_2(f(x_k) - Q_k(x_k, d_k)) \\ &\geq \frac{c_0 c_2}{2\delta_0} \psi(x_k, \delta_0) \min\{\delta_k, \psi(x_k, \delta_0)/(\|B_k\|\delta_0)\} \\ &\geq \frac{c_0 c_2}{2\delta_0} \psi(x_k, \delta_0) \min\{\delta_k, \psi(x_k, \delta_0)/(C\delta_0)\}. \end{aligned}$$

Therefore,

$$(3.13) \quad +\infty > f(x_0) - f(\bar{x}) \geq \frac{c_0 c_2}{2\delta_0} \sum_{k \in I_0} \psi(x_k, \delta_0) \min\{\delta_k, \psi(x_k, \delta_0)/(C\delta_0)\}.$$

By Lemma 3.5, we may find  $\beta$  and  $\epsilon_0$  such that for all  $0 < \epsilon \leq \epsilon_0$  and  $x_k \in \text{cl}B(\bar{x}, \epsilon)$ ,

$$\psi(x_k, \delta_0) \geq \psi(x_k, \delta_*) \geq \beta > 0.$$

Set  $I_1 = \{k \in I_0 : \delta_k \geq \frac{\psi(x_k, \delta_0)}{C\delta_0}\}$  and  $I_2 = I_0 \setminus I_1$ . Then (3.13) shows that

$\sum_{k \in I_1} (\psi(x_k, \delta_0))^2 < +\infty$ , and  $\sum_{k \in I_2} \psi(x_k, \delta_0)\delta_k < +\infty$ . Hence, we may assume that there is  $N(\epsilon) > 0$  such that for all  $k > N(\epsilon)$  and  $k \in I_*$ , we have  $\text{dist}(x_k, \bar{x}) \leq \frac{\epsilon}{2}$

and  $\sum_{k \in I_2, k > N(\epsilon)} \psi(x_k, \delta_0)\delta_k \leq \frac{\epsilon}{2}$ ; for more details see [37].

By induction we can prove that for all  $k$  with  $k_0 < k \leq k_1$ , where  $k_0$  is an index in  $I_*$  with  $k_0 \geq N(\epsilon)$  and  $k_1$  is the first index in  $I_1$  greater than  $k_0$ ,

$$\text{dist}(x_k, \bar{x}) \leq \epsilon,$$

and therefore

$$\psi(x_k, \delta_0) \geq \psi(x_k, \delta_*) \geq \beta > 0.$$

This means that  $I_1$  must be finite since otherwise  $\sum_{k \in I_1} (\psi(x_k, \delta_0))^2$  is not finite. Therefore, we have  $\lim_{k \rightarrow \infty} x_k = \bar{x}$  for  $k \in I_0$  and  $\sum_{k \in I_0} \delta_k < \infty$ . This contradicts the fact that  $\delta_{k+1} \geq \delta_k$  for all large  $k \in I_0$  and the proof is complete.  $\square$

#### 4. A SUITABLE MODEL FUNCTION

In the previous Section 3, we have developed a general trust region method and established global convergence, provided that some criteria on the function  $\Phi$  and the sequence  $(B_k)_k$  hold true. The present section presents several choices for  $\Phi$  which lead to convergent algorithms.

**4.1. The Clarke generalized directional derivative.** If  $f$  were a smooth function, then clearly the function  $\Phi$  would simply be the directional derivative of  $f$ . It is clear that since the function is not necessarily differentiable, we cannot use the differential of the objective function, however we might be able to use generalized directional derivatives instead. Let us continue with the definition of the Clarke generalized directional derivative for locally Lipschitz functions on Riemannian manifolds; see [23, 25].

**Definition 4.1** (Clarke generalized directional derivative). *Suppose  $f : M \rightarrow \mathbb{R}$  is a locally Lipschitz function on a Riemannian manifold  $M$ . Let  $\phi_x : U_x \rightarrow T_x M$  be an exponential chart at  $x$ . Given another point  $y \in U_x$ , consider  $\sigma_{y,v}(t) := \phi_y^{-1}(tw)$ , a geodesic passing through  $y$  with derivative  $w$ , where  $(\phi_y, y)$  is an exponential chart around  $y$  and  $d(\phi_x \circ \phi_y^{-1})(0_y)(w) = v$ . Then, the Clarke generalized directional derivative of  $f$  at  $x \in M$  in the direction  $v \in T_x M$ , denoted by  $f^\circ(x; v)$ , is defined as*

$$f^\circ(x, v) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(\sigma_{y,v}(t)) - f(y)}{t}.$$

If  $f$  is differentiable in  $x \in M$ , we define the gradient of  $f$  as the unique vector  $\text{grad } f(x) \in T_x M$ , which satisfies

$$\langle \text{grad } f(x), \xi \rangle = df(x)(\xi) \quad \text{for all } \xi \in T_x M.$$

**Definition 4.2** (Subdifferential). *We define the subdifferential of  $f$ , denoted by  $\partial f(x)$ , as the subset of  $T_x M$  whose support function is  $f^\circ(x; \cdot)$ . It can be proved [23] that*

$$\partial f(x) = \text{conv} \left\{ \lim_{i \rightarrow \infty} \text{grad } f(x_i) : \{x_i\} \subseteq \Omega_f, x_i \rightarrow x \right\},$$

where  $\Omega_f$  is a dense subset of  $M$  on which  $f$  is differentiable.<sup>1</sup>

Therefore, we have a good candidate for the function  $\Phi : TM \rightarrow \mathbb{R}$ , which can be defined by

$$(4.1) \quad \Phi(x, d) := f^\circ(x, d) = \sup \{ \langle \xi, d \rangle : \xi \in \partial f(x) \},$$

see [23].

The resulting model function leads to a convergent algorithm as the following result shows.

<sup>1</sup>Note that  $\lim_{i \rightarrow \infty} \text{grad } f(x_i)$  in this definition is obtained as follows. Let  $\xi_i \in T_{x_i} M$ ,  $i = 1, 2, \dots$  be a sequence of tangent vectors of  $M$  and  $\xi \in T_x M$ . We say  $\xi_i$  converges to  $\xi$ , denoted by  $\lim \xi_i = \xi$ , provided that  $x_i \rightarrow x$  and, for any smooth vector field  $X$ ,  $\langle \xi_i, X(x_i) \rangle \rightarrow \langle \xi, X(x) \rangle$ .

**Theorem 4.3.** *Let  $f : M \rightarrow \mathbb{R}$  be locally Lipschitz and define  $\Phi$  as in (4.1). Then the function  $\Phi$  satisfies Assumption 3.1. In particular, Algorithm 1 converges globally to a critical point of  $f$ .*

*Proof.* The fact that  $\Phi$  satisfies Assumption 3.1 follows directly from Theorem 2.4 and Theorem 2.9 of [23]. Therefore, by Theorem 3.6 the resulting trust region algorithm converges.  $\square$

**4.2. The  $\varepsilon$ -subdifferential.** A crucial observation is that the computation of  $\Phi$  as in (4.1) can be impractical in case that no explicit expression for the subdifferential  $\partial f(x)$  is available. Using an approximation of the Clarke subdifferential, we overcome this problem and define a local model that is practically and efficiently implementable.

The following definition presents an approximation of the subdifferential which can be computed approximately; see [22].

**Definition 4.4** ( $\varepsilon$ -subdifferential). *Let  $f : M \rightarrow \mathbb{R}$  be a locally Lipschitz function on a Riemannian manifold  $M$ ,  $\varepsilon < i_M(x)$ . We define the  $\varepsilon$ -subdifferential of  $f$  at  $x$  denoted by  $\partial_\varepsilon f(x)$  as follows;*

$$\partial_\varepsilon f(x) = \text{conv}\{d \exp_x^{-1}(y)(\partial f(y)) : y \in \text{cl}B(x, \varepsilon)\}.$$

The following result has been proven in [21].

**Lemma 4.5.** *Let  $U$  be a compact subset of  $M$  and  $\varepsilon < i(U)$ ; then for every open neighborhood  $W$  in  $U$ , the set valued mapping  $\partial_\varepsilon f : W \rightarrow TM$  is upper semi continuous.*

Now we construct a suitable model function for the following unconstrained optimization problem,

$$\min_{x \in M} f(x),$$

where  $f : M \rightarrow \mathbb{R}$  is a locally Lipschitz function. Assume that  $D$  is a bounded open subset of  $M$  and  $\varepsilon < i(\text{cl}(D))$ . We define  $\Phi : TD \rightarrow \mathbb{R}$  by

$$(4.2) \quad \Phi(x, d) := \sup\{\langle \xi, d \rangle : \xi \in \partial_\varepsilon f(x)\}, \text{ for every } x \in D.$$

We now show that this function  $\Phi$  satisfies Assumption 3.1.

**Theorem 4.6.** *The function  $\Phi : TD \rightarrow \mathbb{R}$  defined by (4.2) satisfies Assumption 3.1.*

*Proof.* It is easy to prove that  $\Phi$  satisfies assumptions 3.6 and 3.8. Moreover,  $\Phi|_{T_x M}$  is upper semi continuous, to show this for every  $\epsilon > 0$ , we set  $\delta < \epsilon/K$  where  $K$  is the Lipschitz constant of  $f$  on a neighborhood of  $x$ . Now if  $\|d - w\| < \delta$ , then

$$\Phi(x, w) \leq \Phi(x, w - d) + \Phi(x, d) \leq K\|w - d\| + \Phi(x, d) < \epsilon + \Phi(x, d),$$

therefore the claim is proved.

To prove that  $\Phi$  satisfies Assumption 3.9, note that for  $(x, d) \in TM$ ,

$$f(\exp_x(d)) - f(x) \leq f^\circ(x, d) + o(\|d\|) \leq \Phi(x, d) + o(\|d\|).$$

Now we prove that  $\Phi$  satisfies Assumption 3.10. To this end, we first prove that: For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for  $x \in B(x_0, \delta) \subset D$

$$L_{xx_0}(\partial_\varepsilon f(x)) \subseteq \partial_\varepsilon f(x_0) + \epsilon B_{T_{x_0} M},$$

where  $B_{T_{x_0}M}$  is the unit ball of  $T_{x_0}M$ . To see this; note that for  $\epsilon > 0$  the set  $\partial_\epsilon f(x_0) + \epsilon B_{T_{x_0}M}$  is an open neighborhood of  $L_{x_0x_0}(\partial_\epsilon f(x_0)) = \partial_\epsilon f(x_0)$ . It follows from the continuity of (2.1) that there exists an open neighborhood  $V \subset TM$  of  $\partial_\epsilon f(x_0)$  such that

$$C(V) \subseteq \partial_\epsilon f(x_0) + \epsilon B_{T_{x_0}M}.$$

By the upper semi continuity of  $\partial_\epsilon f$ , there exists a neighborhood  $V'$  of  $x_0$  such that for each  $x \in V'$ , we have  $\partial_\epsilon f(x) \subseteq V$ . Now let  $x \in V'$ , then

$$L_{xx_0}(\partial_\epsilon f(x)) \subseteq \partial_\epsilon f(x_0) + \epsilon B_{T_{x_0}M},$$

as required. Now we claim that there exists  $\delta_* > 0$  such that for all  $x \in D$  and  $d_x \in T_xM$  with  $\|d_x\| < \delta_*$ , we have  $\Phi$  is upper semi continuous at  $d_x$ .

To prove the claim; assuming that  $d \exp_x^{-1}$  on  $\text{cl}B(x, \epsilon)$  is bounded by some  $m_x \geq 0$ , from the Lipschitzness of  $f$  on  $\text{cl}B(x, \epsilon)$ , Theorem 2.9 of [23] implies that for every  $\xi \in \partial_\epsilon f(x)$ ,  $\|\xi\| \leq m_x K_x$ . Since  $\text{cl}(D)$  is compact, there exists a finite number of neighborhoods  $B(x_i, \epsilon)$  such that  $D \subset \bigcup_{i=1}^n B(x_i, \epsilon)$ . Assume that  $m_1 K := \min\{m_{x_i} K_{x_i} : i = 1, \dots, n\}$ ,  $\delta_* := m_1 K$ . Let  $x$  and  $d_x$  be, respectively, arbitrary elements of  $D$  and  $T_xM$  with  $\|d_x\| < \delta_*$ .

We prove that for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\Phi(y, d_y) < \Phi(x, d_x) + \epsilon, \text{ provided } \text{dist}_{TM}(d_x, d_y) < \delta.$$

Assume that  $0 < \epsilon < 1$ , then there exists  $\delta_1 > 0$  such that for each  $y \in B(x, \delta_1) \subset D$ ,

$$L_{yx}(\partial_\epsilon f(y)) \subseteq \partial_\epsilon f(x) + \frac{\epsilon}{3Km_1} B_{T_xM}.$$

Since  $\Phi|_{T_xM}$  is upper semi continuous and  $\Phi(x, 0_x) = 0$ , hence there exists  $\delta_3 > 0$  such that  $\|w_x\| < \delta_3$  implies that  $\Phi(x, w_x) < \epsilon/3$ . Assume that  $\sigma := \min\{\delta_3, \frac{m_1 K \epsilon}{3}\}$ , then the continuity of (2.1) implies that there exists  $\delta_2 > 0$ ,

$$\|L_{yx}(d_y) - d_x\| < \sigma, \text{ provided } \text{dist}_{TM}(d_x, d_y) < \delta_2.$$

Let  $\delta < \min\{\delta_1, \delta_2, \delta_3\}$  be such that  $B(x, \delta)$  is convex. We suppose that  $\text{dist}_{TM}(d_x, d_y) < \delta$ , then  $\text{dist}(x, y) < \delta$  and  $y \in B(x, \delta) \subset D$ . Therefore

$$\begin{aligned} \Phi(y, d_y) &= \sup_{\xi} \{\langle \xi, d_y \rangle : \xi \in \partial_\epsilon f(y)\} \\ &= \sup \{\langle L_{yx}\xi, L_{yx}d_y \rangle : \xi \in \partial_\epsilon f(y)\} \\ &\leq \sup_{\eta} \{\langle \eta, L_{yx}d_y \rangle : \eta \in \partial_\epsilon f(x) + \frac{\epsilon}{3Km_1} B_{T_xM}\} \\ (4.3) \quad &= \sup_{\theta, v} \{\langle \theta + v, d_x + w_x \rangle : \theta \in \partial_\epsilon f(x), v \in \frac{\epsilon}{3Km_1} B_{T_xM}, \|w_x\| < \sigma\} \\ &\leq \sup_{\theta} \{\langle \theta, d_x \rangle + \langle \theta, w_x \rangle : \theta \in \partial_\epsilon f(x), \|w_x\| < \sigma\} + \frac{\epsilon^2}{9} + \frac{\epsilon}{3} \\ &\leq \Phi(x, d_x) + \frac{\epsilon^2}{9} + \frac{\epsilon}{3} + \Phi(x, w_x) \leq \Phi(x, d_x) + \epsilon, \end{aligned}$$

□

A key property of the  $\epsilon$ -subdifferential is that it can be approximated efficiently. In our implementations, we substitute the  $\epsilon$ -subdifferential of the objective function  $f$  with its approximation presented in [21]. Indeed, to approximate the  $\epsilon$ -subdifferential at  $x_k$ , we start with the gradient of an arbitrary point nearby  $x_k$  and move the gradient to the tangent space in  $x_k$  via the derivative of the logarithm

mapping, and in every subsequent iteration, the gradient of a new point nearby  $x_k$  is computed and moved to the tangent space in  $x_k$  to add to the working set to improve the approximation of  $\partial_\varepsilon f(x_k)$ . Indeed, we do not want to provide a description of the entire  $\varepsilon$ -subdifferential set at each iteration; what we do is approximate  $\partial_\varepsilon f(x_k)$  by the convex hull of its elements. In this way, let  $W_l := \{v_1, \dots, v_l\} \subseteq \partial_\varepsilon f(x_k)$ ; then we define

$$w_l := \operatorname{argmin}_{v \in \operatorname{conv} W_l} \|v\|.$$

Now if we have

$$(4.4) \quad f(\exp_{x_k}(\varepsilon g_l)) - f(x_k) \leq -c\varepsilon \|w_l\|, \quad c \in (0, 1)$$

where  $g_l = -\frac{w_l}{\|w_l\|}$ , then we can say  $\operatorname{conv} W_l$  is an acceptable approximation for  $\partial_\varepsilon f(x_k)$ . Otherwise, we add a new element of  $\partial_\varepsilon f(x_k) \setminus \operatorname{conv} W_l$  to  $W_l$ . See [21] for further details on how to algorithmically realize this approximation procedure.

## 5. NUMERICAL EXPERIMENTS

The Riemannian nonsmooth trust region algorithm presented in the previous section was implemented in Matlab. To the best of our knowledge, our algorithm is the first practical nonsmooth trust region algorithm for locally Lipschitz functions defined on Riemannian manifolds. In our implementation, we update the sequence of matrices  $\{B_k\}$  by using the BFGS method; see [38]. Moreover, we compare the nonsmooth trust region algorithm with Riemannian subgradient descent algorithms presented in [12, 22, 18]. The number of function evaluations is used as a measure efficiency for the algorithms. The parameters are initialized similar to the smooth version of the classical trust method. We set the parameters as  $c_1 = 0.75$ ,  $c_2 = 0$ ,  $c_3 = 0.5$ ,  $c_4 = 2$ ,  $c = 10^{-4}$ ,  $\varepsilon = 10^{-6}$ ,  $\delta = 10^{-1}$ ,  $\delta_1 = 10^{-1}$ . As in the examples, our manifold is a unit sphere, we assume that  $\delta_0 = 7$ .

The unit sphere  $S^2$  is the smooth compact manifold

$$S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\},$$

and the global coordinates on  $S^2$  are naturally given by this embedding into  $\mathbb{R}^3$ . The tangent space at a point  $x \in S^2$  is

$$T_x S^2 = \{v \in \mathbb{R}^3 : \langle x, v \rangle = 0\}.$$

The inner product on  $T_x S^2$  is defined by

$$\langle v, w \rangle_{T_x S^2} = \langle v, w \rangle_{\mathbb{R}^3}.$$

The exponential map

$$\exp_x : T_x S^2 \rightarrow S^2$$

is defined by

$$\exp_x(v) = \cos(\|v\|)x + \sin(\|v\|)\frac{v}{\|v\|}.$$

Moreover, if  $x \in S^2$ , then

$$\exp_x^{-1} : S^2 \rightarrow T_x S^2$$

is defined by

$$\exp_x^{-1}(y) = \frac{\theta}{\sin(\theta)}(y - x \cos(\theta)),$$

where  $\theta = \arccos\langle x, y \rangle$ . The Riemannian distance between two points  $x, y$  in  $S^2$  is given by

$$\text{dist}(x, y) = \arccos\langle x, y \rangle.$$

Let  $t \rightarrow \gamma(t)$  be a geodesic on  $S^2$ , and let  $u = \frac{\gamma^\circ(0)}{\|\gamma^\circ(0)\|}$ . The parallel translation of a vector  $v \in T_{\gamma(0)}S^2$ , along the geodesic  $\gamma$ , is given by [2]

$$L_{\gamma(0)\gamma(t)}(v) = -\gamma(0) \sin(\|\gamma^\circ(0)\|t)u'v + u \cos(\|\gamma^\circ(0)\|t)u'v + (I - uu')v.$$

**5.1. Denoising on a sphere.** First, we are going to solve the one dimensional total variation problem for functions which map into a two dimensional sphere  $S^2$ . Assuming that  $M$  is a manifold, consider the minimization problem

$$(5.1) \quad \min_{u \in BV([0,1];M)} \{F(u) := \text{dist}_2(f, u)^2 + \lambda \|\nabla u\|_1\}$$

where  $f : [0, 1] \rightarrow M$  is the given (noisy) function,  $u$  is a function of bounded variation from  $[0, 1]$  to  $M$ ,  $\text{dist}_2$  is the distance on the function space  $L^2([0, 1]; M)$  and  $\lambda > 0$  is a Lagrangian parameter, [41]. Note that for every  $w \in [0, 1]$ ,  $\nabla u(w) : \mathbb{R} \rightarrow T_{u(w)}M$  and  $\|\nabla u\|_1 = \int_{[0,1]} \|\nabla u(w)\| dw$ . Now we can formulate a discrete version of the problem (5.1) by restricting the space of functions to  $V_h^M$  which is the space of all geodesic finite element functions for  $M$  associated with a regular grid on  $[0, 1]$ ; see [42, 21]. We refer to [42] for the definition of geodesic finite element spaces  $V_h^M$ .

Using the nodal evaluation operator  $\varepsilon : V_h^M \rightarrow M^n$ ,  $(\varepsilon(v_h))_i = v_h(x_i)$ , where  $x_i$  is the  $i$ -th vertex of the simplicial grid on  $[0, 1]$ , one can find an equivalent problem defined on  $M^n$  as follows,

$$(5.2) \quad \min_{u \in M^n} \{F_*(u) := \text{dist}_*(\varepsilon(f), u)^2 + \lambda \|\nabla(\varepsilon^{-1}(u))\|_1\}$$

where  $\text{dist}_*$  is the Riemannian distance on  $M^n$ .

Let  $\varepsilon(f) = (p_1, \dots, p_n)$ , then  $F_* : M^n \rightarrow \mathbb{R}$  can be defined by

$$F_*(u_1, \dots, u_n) = \sum_{i=1}^n \text{dist}(p_i, u_i)^2 + \lambda \sum_{i=1}^{n-1} \text{dist}(u_i, u_{i+1}),$$

where  $\text{dist}$  is the Riemannian distance on  $M$ .

Now we assume that  $M = S^2$ . First, we need to define a function from  $[0, 1]$  to  $S^2$  to get the original image. Afterward, we add a gaussian noise to the image to get the noisy image. Finally, we apply Algorithm 1 to the function  $F_*$  defined on  $M^{100}$  to get the denoised image; see Figure 1.

Table 1 provides the numerical results for the TV regularization on  $S^2$  using the nonsmooth trust region method and  $\varepsilon$ -subdifferential method. In this table, “nfval” stands for the number of function evaluations, we also have presented the minimum value of the function. The number of function evaluations is considered as a measure of efficiency for the trust region method.

**5.2. Riemannian geometric median on a sphere.** Our second numerical experiment is concerned with the Riemannian geometric median on  $S^2$ . Let  $M$  be a Riemannian manifold. Given points  $p_1, \dots, p_m$  in  $M$  and corresponding positive real weights  $w_1, \dots, w_m$ , with  $\sum_{i=1}^m w_i = 1$ , define the weighted sum of distance functions

$$f(q) = \sum_{i=1}^m w_i \text{dist}(p_i, q),$$

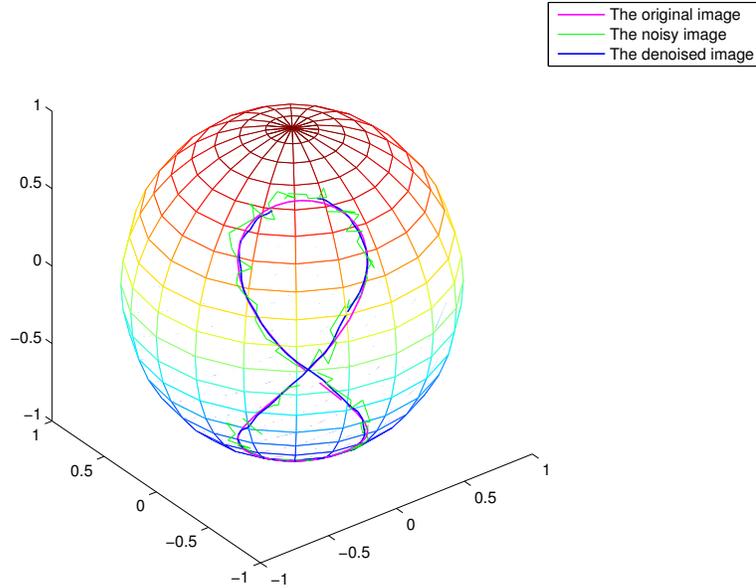


FIGURE 1. TV regularization on  $S^2$ .

TABLE 1. Numerical results in terms of number of function evaluations and the final obtained value of the function for TV regularization.

No.	f in trust region method	nfeval	f in $\varepsilon$ -subdifferential method	nfeval
1	3.2266	365	3.2362	934
2	3.2512	327	3.2579	976
3	3.2902	424	3.3145	3316
4	3.2794	509	3.2896	3097
5	3.2366	365	3.2472	934
6	3.2212	327	3.2572	971
7	3.2432	424	3.3066	3216
8	3.2884	509	3.2976	1097
9	3.2187	325	3.2472	1100
10	3.2012	327	3.2972	1906
11	3.2302	424	3.3136	1001
12	3.2394	509	3.2576	1097
13	3.2234	365	3.2365	4934
14	3.2456	327	3.2542	3976
15	3.2912	424	3.3298	3316

where  $\text{dist}$  is the Riemannian distance function on  $M$ . We define the weighted geometric median  $x$  as the minimizer of  $f$ . When all the weights are equal,  $w_i = 1/m$ , we call  $x$  simply the geometric median. Now, we assume that  $M = S^2$ .

TABLE 2. Numerical results in terms of number of function evaluations and the final obtained value of the function for the geometric median on  $S^2$

No.	f in trust region method	nfeval	f in $\varepsilon$ -subdifferential method	nfeval
1	0.4306	8	0.4365	192
2	0.4302	11	0.4305	67
3	0.4101	2	0.4200	108
4	0.4338	13	0.4390	110
5	0.4206	13	0.4273	93
6	0.4312	5	0.4326	97
7	0.4302	11	0.4321	105
8	0.4394	11	0.4398	109
9	0.4316	11	0.4321	98
10	0.4212	8	0.4237	97
11	0.4302	9	0.4388	128
12	0.4334	10	0.4399	167
13	0.4266	6	0.4280	211
14	0.4292	5	0.4300	145
15	0.4372	4	0.4384	316

Table 2 provides the numerical results for finding the geometric median on  $S^2$  with  $m = 5000$ , using the nonsmooth trust region method and the  $\varepsilon$ -subdifferential method. The starting point and the points  $p_i$  are chosen randomly. As before, “nfeval” stands for the number of function evaluations; we also have presented the minimum value of the function. The number of function evaluations is considered as a measure of efficiency for the trust region method.

**5.3. Rayleigh quotients on a sphere.** Now we are going to compare our nonsmooth trust region algorithm with the Riemannian subgradient descent presented in [18]. To this end, we consider the maximum of  $m$  Rayleigh quotients on the sphere  $S^{n-1}$ , i.e.,

$$(5.3) \quad f(x) = \max_{i=1, \dots, m} \frac{1}{2} x' A_i x,$$

where  $A_i \in \mathbb{R}^{n \times n}$  is symmetric. Our aim is to find a minimum of  $f$ . Table 3 provides the numerical results for finding the minimum of  $f$  on  $S^2$  with  $m = 20$ , using the nonsmooth trust region method and the subgradient descent method.

Moreover, we say that an algorithm solves a problem successfully if the following condition is satisfied:

$$\frac{|f_{opt} - f^*|}{|f_{opt} + 1|} \leq \epsilon,$$

where  $f_{opt}$  is the minimum value of the function and  $f^*$  is the minimum value obtained by the algorithm, and  $\epsilon = 10^{-4}$  is an accuracy level. Figure 2 shows that the nonsmooth trust region algorithm solve the problem.

## 6. CONCLUSIONS

We have presented a practical algorithm in the context of trust region methods for nonsmooth problems on Riemannian manifolds. To the best of our knowledge,

TABLE 3. Numerical results in terms of number of function evaluations and the final obtained value of the function for Rayleigh quotients on  $S^2$

No.	f in trust region method	nfeval	f in subdifferential method	nfeval
1	0.6529	128	1.7832	501
2	0.3294	116	0.3341	3614
3	0.7321	116	0.7996	5501
4	0.5211	123	0.5458	2123
5	0.5156	117	0.4962	509
6	0.6321	156	0.6578	1504
7	0.4321	167	0.4567	1350
8	0.6421	127	0.6432	680
9	0.7691	117	0.7761	1384
10	0.4021	116	0.4187	890
11	0.6531	170	0.6520	1230
12	0.5090	175	0.5021	2012
13	0.5401	149	0.5721	1890
14	0.3489	110	0.3901	2056
15	0.3209	113	0.3009	1105

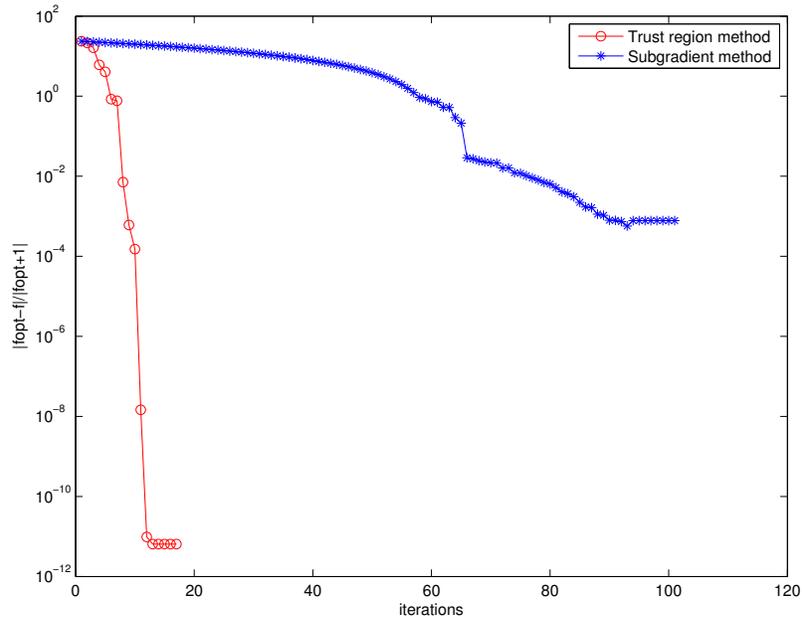


FIGURE 2. Rayleigh quotients on  $S^2$ . We compare the trust region method introduced in this paper with the Riemannian subgradient algorithm from [18].

this is the first paper on nonsmooth trust region method on Riemannian manifolds. We also introduce a practical local model in our trust region scheme for locally Lipschitz functions. We have seen that the use of exponential map yields trust region subproblems expressed in Euclidean spaces  $T_x M$ . Therefore, all the classical methods for solving the trust region subproblem can be applied. In our implementation, we use the approach based on the Cauchy point and the CG-Steihaug methods; see [3]. The main result is the global convergence property of our trust region method which is stated in Theorem 3.6.

An implementation of our proposed trust region algorithm, along with the subgradient and  $\varepsilon$ -subgradient methods, is given in Matlab environment and tested on some problems. Numerical results of the considered algorithms show that comparing with the  $\varepsilon$ -subgradient algorithm, the nonsmooth trust region algorithm has a better performance in terms of the number of function evaluations. Moreover, comparing with the subgradient algorithm, the nonsmooth trust region method gives us a better approximation of the minimum value of the function.

#### Acknowledgments

*The first author wish to thank R. Yousefpour for useful discussions.*

#### REFERENCES

- [1] P. A. Absil, C. G. Baker, *Trust-region methods on Riemannian manifolds*, Found. Comput. Math., 7 (2007), pp. 303-330.
- [2] P. A. Absil, R. Mahony, R. Sepulchre, *Optimization Algorithm on Matrix Manifolds*, Princeton University Press, 2008.
- [3] Z. Akbari, R. Yousefpour, M. Reza Peyghami, *A new nonsmooth trust region algorithm for locally Lipschitz unconstrained optimization problems*, To appear in J. Optim. Theory Appl.
- [4] R. L. Adler, J. P. Dedieu, J. Y. Margulies, M. Martens, M. Shub, *Newton's method on Riemannian manifolds and a geometric model for the human spine*, IMA J. Numer. Anal., 22 (2002), pp. 359-390.
- [5] B. Afsari, R. Tron, R. Vidal, *On the convergence of gradient descent for finding the Riemannian center of mass*, SIAM J. Control Optim., 51 (2013), pp. 2230-2260.
- [6] D. Azagra, J. Ferrera, F. López-Mesas, *Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds*, J. Funct. Anal., 220 (2005), pp. 304-361.
- [7] D. Azagra, J. Ferrera, *Applications of proximal calculus to fixed point theory on Riemannian manifolds*, Nonlinear. Anal., 67 (2007), pp. 154-174.
- [8] G. C. Bento, O. P. Ferreira, P. R. Oliveira, *Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds*, Nonlinear Anal., 73 (2010), pp. 564-572.
- [9] G. C. Bento, O. P. Ferreira, P. R. Oliveira, *Unconstrained steepest descent method for multicriteria optimization on Riemannian manifolds*, J. Optim. Theory Appl., 154 (2012), pp. 88-107.
- [10] G. C. Bento, J. G. Melo, *A subgradient method for convex feasibility on Riemannian manifolds*, J. Optim. Theory Appl., 152 (2012), pp. 773-785.
- [11] G. C. Bento, O. P. Ferreira, P. R. Oliveira, *Proximal point method for a special class of nonconvex functions on Hadamard manifolds*, Optimization (2012), pp. 1-31.
- [12] P. B. Borckmans, S. Easter Selvan, N. Boumal, P. A. Absil, *A Riemannian subgradient algorithm for economic dispatch with valve-point effect*, J. Comput. Appl. Math., 255 (2014), pp. 848-866.
- [13] R. D. Canary, D. B. A. Epstein, A. Marden, *Fundamentals of Hyperbolic Geometry: Selected Expositions*, Cambridge University Press, Cambridge (2006).
- [14] A. R. Conn, N. I. M. Gould, Ph. L. Toint, *Trust Region Methods*, SIAM Series on Optimization, SIAM, Philadelphia, 2000.
- [15] J. E. Dennis, S. B. Li, R. A. Tapia, *A unified approach to global convergence of trust region methods for nonsmooth optimization*, Math. Program., 68 (1995), pp. 319-346.

- [16] J. X. da Cruz Neto, L. L. Lima, P. R. Oliveira, *Geodesic Algorithm in Riemannian Manifolds*, Balkan J. Geom. Appl., 2 (1998), pp. 89 - 100.
- [17] J.X. da Cruz Neto, O. P. Ferreira and L. R. Lucambio Perez, *A proximal regularization of the steepest descent method in Riemannian manifold*, Balkan J. Geom. Appl., 2 (1999), pp. 1-8.
- [18] G. Dirr, U. Helmke, C. Lageman, *Nonsmooth Riemannian optimization with applications to sphere packing and grasping*, In Lagrangian and Hamiltonian Methods for Nonlinear Control 2006: Proceedings from the 3rd IFAC Workshop, Nagoya, Japan, 2006, Lecture Notes in Control and Information Sciences, Vol. 366, Springer Verlag, Berlin, 2007.
- [19] O. P. Ferreira, P. R. Oliveira, *Proximal point algorithm on Riemannian manifolds*, Optimization. 51 (2002), pp. 257-270.
- [20] O. P. Ferreira, P. R. Oliveira, *Subgradient algorithm on Riemannian manifolds*, J. Optim. Theory Appl., 97 (1998), pp. 93-104.
- [21] P. Grohs, H. Hardering, O. Sander, *Optimal a priori discretization error bounds for geodesic finite elements*, SAM Report 2013-16, ETH Zürich, (2013), Submitted.
- [22] P.Grohs, S. Hosseini,  $\varepsilon$ -subgradient algorithms for locally Lipschitz functions on Riemannian manifolds, Submitted.
- [23] S. Hosseini, M. R. Pouryayevali, *Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds*, Nonlinear Anal., 74 (2011), pp. 3884-3895.
- [24] S. Hosseini, M. R. Pouryayevali, *Euler characterization of epi-Lipschitz subsets of Riemannian manifolds*, J. Convex. Anal., 20 (2013), No. 1, pp. 67-91.
- [25] S. Hosseini, M. R. Pouryayevali, *On the metric projection onto prox-regular subsets of Riemannian manifolds*, Proc. Amer. Math. Soc., 141 (2013), pp. 233-244.
- [26] W. Klingenberg, *Riemannian Geometry*, Walter de Gruyter Studies in Mathematics, Vol. 1, Walter de Gruyter, Berlin, New York, 1995.
- [27] C. Lageman, U. Helmke, *Optimization Methods for the Sphere Packing Problem on Grassmannians*, Taming Heterogeneity and Complexity of Embedded Control, (2013), 393-408.
- [28] D. Kressner, M. Steinlechner, B. Vandereycken, *Low-rank tensor completion by Riemannian optimization*, BIT Numer. Math., 54 (2014), 447-468.
- [29] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics, Vol. 191, Springer, New York, 1999.
- [30] P. Y. Lee, *Geometric Optimization for Computer Vision*, PhD thesis, Australian National University, 2005.
- [31] C. Lemarechal, *Nondifferentiable optimization*, In: Handbook in Operations Research and Management Science (G. L. Nemhauser et al., Eds.), Vol. 1, North Holland, Amsterdam (1989), pp. 529-572.
- [32] M. M. Makela, P. Neittaanmaki, *Nonsmooth Optimization*, World Scientific (1992).
- [33] C. Li, B. S. Mordukhovich, J. Wang, J. C. Yao, *Weak sharp minima on Riemannian manifolds*, SIAM J. Optim., 21(4) (2011), pp. 1523-1560.
- [34] R. E. Mahony, *The constrained Newton method on a Lie group and the symmetric eigenvalue problem*, Linear Algebra. Appl., 248 (1996), pp. 67-89.
- [35] N. Mahdavi-Amiri, R. Yousefpour, *An effective nonsmooth optimization algorithm for locally Lipschitz functions*, J. Optim. Theory Appl., 155 (2012), pp. 180-195.
- [36] E. A. Papa Quiroz, E. M. Quispe, P. R. Oliveira, *Steepest descent method with a generalized Armijo search for quasiconvex functions on Riemannian manifolds*, J. Math. Anal. Appl., 341 (2008), pp. 467-477.
- [37] L. Qi, J. Sun, *A trust region algorithm for minimization of locally Lipschitzian functions*, Math. Program., 66 (1994), pp. 25-43.
- [38] C. Qi, K. A. Gallivan, P. A. Absil, *Riemannian BFGS algorithm with applications*, Recent Advances in Optimization and its Applications in Engineering, (2010), 183-192.
- [39] W. Ring, B. Wirth, *Optimization methods on Riemannian manifolds and their application to shape space*, SIAM J. Optim., 22(2) (2012), pp. 596-627.
- [40] R. C. Riddell, *Minimax problems on Grassmann manifolds. Sums of eigenvalues*, Adv. Math., 54 (1984), pp. 107-199.
- [41] L. Rudin, S. Osher, E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D., 60 (1) (1992), pp. 259-268.
- [42] O. Sander, *Geodesic finite elements for Cosserat rods*, Internat. J. Numer. Methods Engrg., 82 (2010), pp. 1645-1670.

- [43] M. Sarkis, K. Diepold, *Camera-pose estimation via projective Newton optimization on the manifold*, IEEE Trans. Image Processing., 21 (2012), pp. 1729-1741.
- [44] S. T. Smith, *Optimization techniques on Riemannian manifolds*, Fields Institute Communications, 3 (1994), pp. 113-146.
- [45] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1994.
- [46] K. Usevich, I. Markovsky, *Optimization on a Grassmann manifold with application to system identification*, Submitted, (<http://homepages.vub.ac.be/~imarkovs/t-abstracts.html>)
- [47] B. Vandereycken, S. Vandewalle, *A Riemannian optimization approach for computing low-rank solutions of Lyapunov equations*, SIAM. J. Matrix Anal. Appl., 31(5) (2010), 2553-2579.
- [48] B. Vandereycken, *Low-rank matrix completion by Riemannian optimization*, SIAM J. Optim., 23(2) (2013), 1214-1236.
- [49] L. H. Zhang, *Riemannian Newton method for the multivariate eigenvalue problem*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2972-2996.

## Recent Research Reports

Nr.	Authors/Title
2014-10	P. Grohs and S. Vigogna Intrinsic Localization of Anisotropic Frames II: $\alpha$ -Molecules
2014-11	S. Etter and P. Grohs and A. Obermeier FFRT - A Fast Finite Ridgelet Transform for Radiative Transport
2014-12	A. Paganini Approximate Shape Gradients for Interface Problems
2014-13	E. Fonn and P. Grohs and R. Hiptmair Polar Spectral Scheme for the Spatially Homogeneous Boltzmann Equation
2014-14	J. Dick and F.Y. Kuo and Q.T. Le Gia and Ch. Schwab Multi-level higher order QMC Galerkin discretization for affine parametric operator equations
2014-15	Ch. Schwab Exponential convergence of simplicial $hp$ -FEM for $H^1$ -functions with isotropic singularities
2014-16	P. Grohs and S. Keiper and G. Kutyniok and M. Schaefer $\alpha$ -Molecules
2014-17	A. Hildebrand and S. Mishra Efficient computation of all speed flows using an entropy stable shock-capturing space-time discontinuous Galerkin method
2014-18	D. Conus and A. Jentzen and R. Kurniawan Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients
2014-19	J. Doelz and H. Harbrecht and Ch. Schwab Covariance regularity and H-matrix approximation for rough random fields