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OPTIMAL OPERATOR PRECONDITIONING FOR BOUNDARY ELEMENTS ON OPEN CURVES

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Abstract. Boundary value problems for the Poisson equation in the exterior of an open bounded Lipschitz curve $\mathcal{C}$ can be recast as first-kind boundary integral equations featuring weakly singular or hypersingular boundary integral operators (BIEs). Based on the recent discovery in [C. Jerez-Hanckes and J. Nédélec, *Explicit variational forms for the inverses of integral logarithmic operators over an interval*, SIAM Journal on Mathematical Analysis, 44 (2012), pp. 2666–2694.] of inverses of these BIEs for $\mathcal{C} = [-1, 1]$, we pursue operator preconditioning of the linear systems of equations arising from Galerkin-Petrov discretization by means of zeroth and first order boundary elements. The preconditioners rely on boundary element spaces defined on dual meshes and they can be shown to perform uniformly well independently of the number of degrees of freedom even for families of locally refined meshes.

Key words. Calderón preconditioning, screen problems, fracture problems, boundary integral operators

AMS subject classifications. 65R20 (65F35 65N22 65N38)

1. Introduction. We consider the following Dirichlet and Neumann boundary value problems (BVPs) in the exterior of a open curve $\mathcal{C} \subset \mathbb{R}^2$,

$$-\Delta U = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\mathcal{C}} \quad , \quad U = g \quad \text{or} \quad \frac{\partial U}{\partial \mathbf{n}} = f \quad \text{on } \mathcal{C} \, , \tag{1.1}$$

plus appropriate decay conditions at ∞, see [18, Thm. 8.9] and with suitable boundary data g or f. If $\mathcal{C}$ is a regular Lipschitz curve, then (1.1) possesses a unique weak solution in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{\mathcal{C}})$. Exterior BVPs like (1.1) play a central role in a number of mathematical models like crack models in elasticity [9] or dimensionally reduced antenna models in electromagnetics [23].

For the approximate numerical solution of boundary value problems like (1.1), posed on an unbounded homogeneous exterior domain, *boundary element methods* are an attractive option, because they respect the decay conditions at infinity and require a mesh of $\mathcal{C}$ only. They exploit the possibility that (1.1) can be converted into *first-kind boundary integral equations* (BIEs) for the unknown jump of the complementary boundary data on $\mathcal{C}$. These boundary integral equations, their variational formulation in suitable Sobolev spaces, and boundary element Galerkin discretization have been studied thoroughly, prominently by E. Stephan and coworkers. Please refer to [29, 32, 8, 31], and the textbook [22, Sect. 3.5.3].

Since we face first-kind BIEs, the spectral condition numbers of the linear systems of equations arising from low-order Galerkin boundary element methods (BEM) for (1.1) using the customary locally supported basis functions will grow like $O(h^{-1})$, where h is the size of the smallest cell of the mesh, see [22, Sect. 4.5]. Thus, effective preconditioning becomes indispensable when conjugate gradient type iterative solvers are used to compute BEM solutions on (locally) fine meshes.

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Admittedly, on curves satisfactory resolution can already be achieved with moderate numbers of degrees of freedom, which allows the assembly of the dense Galerkin matrices and the use of direct solvers. This is no longer the case for the three-dimensional counterpart of (1.1), where $\mathcal{C}$ has to be replaced with an oriented two-dimensional Lipschitz manifold. Then we encounter a genuine *screen problem*, for which we may have to resort to fine triangulations of $\mathcal{C}$, which, in turns, entails the use of matrix compression and iterative solvers. Then preconditioning becomes a key issue. Thus, this article with its focus on curves and numerical analysis, should be viewed as a first "proof of concept" for a preconditioning strategy that, we believe, can be extended to three dimensions.

A powerful preconditioning technique for BEM on closed surfaces is the so-called policy of *Calderón preconditioning*, which exploits Calderón identities, that is, the fact that certain products of boundary integral operators evaluate to the identity map plus a compact perturbation [22, Sect. 3.6]. It fits the more general strategy of *operator preconditioning* for Galerkin discretizations, introduced in [10], see also [17]. For low-order Galerkin BEM on closed surfaces, it takes pairs of primal and dual meshes to realize this approach to preconditioning, as has been discovered by Steinbach and Wendland in [28]. A very general perspective was developed by Buffa and Christiansen in [5] and it paved the way for the application of Calderón preconditioning to electromagnetic boundary integral equations. The new technique has quickly been adopted in computational engineering [2, 1, 6], which highlights its huge potential for practical simulations.

For open curves, analogues of Calderón identities had been elusive until recently, which hampered the adaption of Calderón preconditioning. One can still pursue a weaker version, the idea of preconditioning with operators of opposite order. This was done in by McLean and Steinbach [19], where the single layer operator provided a preconditioner for the discrete hypersingular BIE on an arc. Yet, this method is not asymptotically optimal in a strict sense, because the condition number of the preconditioned linear system still grows like $O(|\log h|)$. The reason is that on open curves the boundary integral operators have to be considered on Sobolev spaces that take into account special conditions at the endpoints. These spaces fail to provide the duality relationships that form the foundations of operator preconditioning.

Several approaches have been proposed in the literature to overcome this difficulty by extending the classical Calderón relations to the case of open surfaces. Recently, Bruno and Lintner in [15, 4] have developed a generalized Calderón formula for open surfaces. When combined with their high-order numerical methods, they observe excellent performance of their Calderón preconditioner for a wide range of geometries and wave propagation problems. However, no mathematical analysis of this method is available, let alone results about asymptotic optimality of the preconditioner.

In this article we propose the first provably asymptotically optimal Calderón preconditioning approach for low-order Galerkin BEM for the BIE arising from (1.1). This has been made possible by a breakthrough result achieved by Nédélec and one of the authors in [13, 12]. They have found explicit inverses for weakly singular and hypersingular integral operators on a line segment. These new relations are a perfect substitute for the conventional Calderón identities in the context of operator preconditioning, and in this article we are going to elaborate this rigorously.

Throughout we take pains to cover rather general *locally refined meshes* in our analysis. This is important, because we can expect pronounced singularities of the solutions of the BIEs at the endpoints. More precisely, they behave as $1/\sqrt{d}$ where d

is the distance to the endpoints [7, 20, 14]. For piecewise polynomial approximation spaces this entails using algebraically or geometrically graded meshes, for which cells adjacent to the endpoints are much smaller than those in the middle of $\mathcal{C}$.

Operator preconditioning. Awareness of the gist of operator preconditioning as presented in [10] is crucial for appreciating the considerations in the remainder of the article. Thus, we briefly recall the main result of [10].

THEOREM 1.1 (Theorem 2.1 [10]). *Let X, Y be reflexive Banach spaces, $X_h := \operatorname{span}\{\varphi_i\}_{i=0}^N \subset X$, $Y_h := \operatorname{span}\{\phi_j\}_{j=0}^M \subset Y$ finite-dimensional subspaces with bases $\{\varphi_i\}_{i=0}^N$ and $\{\phi_j\}_{j=0}^M$. Further, let $\mathsf{a} \in L(X \times X, \mathbb{C})$ and $\mathsf{b} \in L(Y \times Y, \mathbb{C})$ be continuous sesquilinear forms (with norms $\|\mathsf{a}\|$ and $\|\mathsf{b}\|$, resp.), each satisfying discrete inf-sup conditions with constants $c_A, c_B > 0$ on X_h and Y_h, respectively. If there is a continuous sesquilinear form $\mathsf{t} \in L(X \times Y, \mathbb{C})$ that also satisfies a discrete inf-sup condition on $X_h \times Y_h$ with constant $c_T > 0$, then the associated Galerkin matrices:*

$$\mathbf{A}_h := (\mathsf{a}(\varphi_i, \varphi_j))_{i,j=1}^N\,,\ \mathbf{B}_h := (\mathsf{b}(\phi_i, \phi_j))_{i,j=1}^M\,,\ \mathbf{T}_h := (\mathsf{t}(\varphi_i, \phi_j))_{i,j=1}^{N,M}\,,$$

satisfy

$$\kappa(\mathbf{T}_h^{-1}\mathbf{B}_h\mathbf{T}_h^{-H}\mathbf{A}_h) \leq \frac{\|\mathsf{a}\|\|\mathsf{b}\|\|\mathsf{t}\|^2}{c_A c_B c_T^2}\,, \tag{1.3}$$

where κ designates the spectral condition number.

As this theorem targets variational problems and Galerkin discretization, we will always focus on the weak form of boundary integral equations. Moreover, as explained in [10, Sect. 4], Calderón preconditioning boils down to an application of Theorem 1.1 where the spaces X and Y are dual to each other, with duality induced by the pairing sesquilinear form t. For the concrete trace spaces, on which the weak BIEs are posed, t will be an extension of the inner product in $L^2(\mathcal{C})$.

In light of Theorem 1.1, when confronted with a variational BIE $\mathsf{a}(u, v) = \ell(v)$, $v \in X$, $\ell \in X'$, on a trace space X, the key questions are,

(Q1) whether we can find another boundary integral operator that induces a bounded sesqui-linear form on $Y := X'$,

(Q2) what sub-spaces $X_h \subset X$ and $Y_h \subset Y$ furnish stable Galerkin discretizations,

(Q3) if the pairs X_h and Y_h allow an X/Y-stable L^2-pairing (, for which a necessary condition is $\dim X_h = \dim Y_h$).

Given positive answers to these questions and assuming that all inf-sup constants can be chosen independently of the (local) mesh width, Theorem 1.1 will permit us to conclude that the product $\mathbf{T}_h^{-1}\mathbf{B}_h\mathbf{T}_h^{-H}$ of Galerkin matrices represents an asymptotically optimal preconditioner for $\mathbf{A}_h$.

REMARK 1.2. *We stress that the assertion of Theorem 1.1 is valid for any choice of bases for X_h and Y_h and the associated Galerkin matrices. Thus, the focus can exclusively be on the construction of appropriate spaces X_h and Y_h.*

Outline. Next, in Section 2 we give a precise description of the relevant Sobolev spaces, and afterwards, we introduce the boundary integral operators, along with *elliptic* boundary integral equations in variational form. This will done on a straight line segment, but Section 2.4 will argue, why the case of a smooth open curve is fully covered. Theorems 2.1 and 2.5 will answer Questions (Q1) and (Q2). Piecewise polynomial boundary element spaces on primal and dual meshes are defined in Section 3. In Section 4 uniform inf-sup conditions for discrete L^2-duality pairings are established, thus verifying the last missing assumption of Theorem 1.1, see Theorem 4.3.

The proofs take the cue from the general technique developed by O. Steinbach in [26]. In the final section, a number of numerical experiments confirm the power and asymptotic optimality of the new preconditioner.

2. Boundary Integral Operators (BIO).

2.1. Sobolev spaces. We employ the usual notations for Sobolev spaces from [18, Ch. 3]; let $\mathcal{O} \subseteq \mathbb{R}^d$, with $d = 1, 2$, be open. For $s \in \mathbb{R}$, $H^s(\mathcal{O})$ denotes standard Sobolev spaces [18, 27]. If $s > 0$ and $\mathcal{O} \subset \mathbb{R}^d$ is a Lipschitz domain, $\widetilde{H}^s(\mathcal{O})$ stands for the space of distributions in $H^s(\mathcal{O})$ whose extension by zero to $\mathbb{R}^d$ belongs to $H^s(\mathbb{R}^d)$. We introduce

$$\widetilde{H}^{-1/2}(\mathcal{O}) \;\equiv\; (H^{1/2}(\mathcal{O}))' \qquad \text{and} \qquad H^{-1/2}(\mathcal{O}) \;\equiv\; (\widetilde{H}^{1/2}(\mathcal{O}))'. \tag{2.1}$$

Here and below primes designate dual spaces and duality pairings will be indicated by angular brackets $\langle \cdot, \cdot \rangle$. Using $L^2(\mathcal{O})$ as pivot space, this yields the Gelfand triples

$$H^{1/2}(\mathcal{O}) \subset L^2(\mathcal{O}) \subset \widetilde{H}^{-1/2}(\mathcal{O}) \quad , \quad \widetilde{H}^{1/2}(\mathcal{O}) \subset L^2(\mathcal{O}) \subset H^{-1/2}(\mathcal{O}) \; ,$$

with continuous and dense embeddings.

Below we are going to examine integral equations on a special curve, namely the straight line segment $(-1, 1) \times \{0\} \subset \mathbb{R}^2$. Thus, we abbreviate $\Gamma := (-1, 1)$. Based on the weight function

$$\omega(x) := \sqrt{1 - x^2}, \qquad x \in \Gamma \; ,$$

let us introduce the subspaces

$$\widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma) \;:=\; \left\{ \varphi \in \widetilde{H}^{-1/2}(\Gamma) \;:\; \langle 1 \,, \varphi \rangle_{\widetilde{H}^{-1/2}(\Gamma)} \;=\; 0 \right\}, \tag{2.2}$$

$$H^{1/2}_{*}(\Gamma) \;:=\; \left\{ g \in H^{1/2}(\Gamma) \;:\; \left\langle g \,, \omega^{-1} \right\rangle_{\widetilde{H}^{-1/2}(\Gamma)} \;=\; 0 \right\}. \tag{2.3}$$

Analogous to (2.1), we find duality with respect to the pivot space $L^2(\Gamma)$:

$$\widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma) = (H^{1/2}_{*}(\Gamma))' \; . \tag{2.4}$$

2.2. Boundary integral operators on a segment. Following the notation in [11], we introduce the standard weakly singular boundary integral operator (BIO) associated with the Laplacian $-\Delta$ as V, and recall that it is defined by

$$\mathsf{V}\,\varphi(x) \;:=\; \int_\Gamma \log \frac{1}{|x - y|} \varphi(y)\, dy, \quad x \in \Gamma, \quad \varphi \;\in\; C_0^\infty(\Gamma) \; .$$

Additionally, taking the cue from [12, Sec. 3], we define a modified version of the weakly singular BIO as

$$\bar{\mathsf{V}}\varphi(x) \;:=\; \int_\Gamma \log \frac{M(x, y)}{|x - y|} \varphi(y)\, dy, \quad x \in \Gamma, \quad \varphi \in C^\infty(\Gamma) \; ,$$

where

$$M(x, y) := \frac{1}{2} \left((y - x)^2 + (\omega(x) + \omega(y))^2 \right), \quad (x, y) \in \Gamma \times \Gamma \; .$$

Analogously, we define the Laplace standard hypersingular operator W and its modified version $\bar{\mathsf{W}}$ as

$$\mathsf{W} := -\left(\tfrac{d}{dx}\right)^* \circ \mathsf{V} \circ \tfrac{d}{dx}\,, \quad \bar{\mathsf{W}} := -\tfrac{d}{dx} \circ \bar{\mathsf{V}} \circ \left(\tfrac{d}{dx}\right)^* \,.$$

Here, $\frac{d}{dx}$ is the standard derivative on $C^1(\Gamma)$, which gives rise to a mapping $\frac{d}{dx}$: $\widetilde{H}^{1/2}(\Gamma) \to \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$ with continuous adjoint $\left(\frac{d}{dx}\right)^*$: $H^{1/2}_*(\Gamma) \to H^{-1/2}(\Gamma)$. The following fundamental result establishes key continuity properties of the integral operators.

Theorem 2.1 ([12, Prop. 3.1 and 3.3]). *The boundary integral operators introduced above can be extended to* bounded *operators*

$$\mathsf{V} : \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma) \to H^{1/2}_*(\Gamma)\,, \qquad \mathsf{W} : \widetilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

and

$$\bar{\mathsf{V}} : H^{-1/2}(\Gamma) \to \widetilde{H}^{1/2}(\Gamma)\,, \qquad \bar{\mathsf{W}} : H^{1/2}_*(\Gamma) \to \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma).$$

The significance of this theorem for operator preconditioning is evident: first of all, in light of (2.1) and (2.4), it confirms that every operator maps continuously from a space to its L^2-dual, which naturally relates them to bilinear forms. Secondly, we see that $\bar{\mathsf{W}}$ and $\bar{\mathsf{V}}$ induce continuous bilinear forms on the image spaces of V and W, respectively. In a sense, Theorem 2.1 answers Question (Q1) for the operators V and W.

We would like to point out that the pairs $\mathsf{V} \leftrightarrow \bar{\mathsf{W}}$ and $\mathsf{W} \leftrightarrow \bar{\mathsf{V}}$ of operators are even connected by a particularly simple "Calderón identity", expressed in the next theorem.

Theorem 2.2 ([12, Prop. 3.6]). *The following identities hold:*

$$\bar{\mathsf{V}} \circ \mathsf{W} = \mathsf{Id}_{\widetilde{H}^{1/2}(\Gamma)}, \qquad \mathsf{V} \circ \bar{\mathsf{W}} = \mathsf{Id}_{H^{1/2}_*(\Gamma)}, \tag{2.5a}$$

$$\bar{\mathsf{W}} \circ \mathsf{V} = \mathsf{Id}_{\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)}, \qquad \mathsf{W} \circ \bar{\mathsf{V}} = \mathsf{Id}_{H^{-1/2}(\Gamma)}\,. \tag{2.5b}$$

However, we emphasize that it is Theorem 2.1 that paves the way for operator preconditioning. The result of Theorem 2.2 merely bolsters confidence that excellent condition numbers can be achieved.

Remark 2.3. *We would like to alert the reader to the striking differences between the cases of closed (boundaries) and open curves. In the former the single layer and hypersingular operators map continuously back and forth between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. These spaces are in natural duality, so that operator preconditioning can rely on these operators alone. Conversely, on open curves the "$\tilde{\ }$-spaces" come into play and we need to modify the integral operators in order to ensure continuity on the L^2-duals of these $\tilde{\ }$-spaces.*

2.3. (Augmented) boundary integral equations. In line with the perspective of operator preconditioning, we introduce the weak form of the boundary integral equations. First, consider the variational problem for the weakly singular operator V: given $g \in H^{1/2}_*(\Gamma)$ find $\varphi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$ such that

$$\mathsf{a}_{\mathsf{V}}(\varphi, \psi) = \langle \mathsf{V}\,\varphi\,,\, \psi \rangle_{\widetilde{H}^{-1/2}(\Gamma)} = \langle g\,,\, \psi\rangle_{\widetilde{H}^{-1/2}(\Gamma)}\,, \quad \forall\, \psi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma). \tag{2.6}$$

This variational problem is connected with the Dirichlet problem of (1.1), when $\mathcal{C}$ is the line segment $\{0\} \times \Gamma$, see [30].

The variational problem for the hypersingular operator W (**Case B**) can be stated as: find $u \in \widetilde{H}^{1/2}(\Gamma)$ such that for $f \in H^{-1/2}(\Gamma)$

$$\mathsf{a}_{\mathsf{W}}(u,w) \;= \langle \mathsf{W}\,u\,,\,w\rangle_{\widetilde{H}^{1/2}(\Gamma)} = \langle f\,,\,w\rangle_{\widetilde{H}^{1/2}(\Gamma)}\,, \quad \forall\, w \in \widetilde{H}^{1/2}(\Gamma). \tag{2.7}$$

As demonstrated in [32], this variational problem is satisfied by the jump of the Dirichlet trace of the solution of the Neumann problem (1.1) in the exterior of the line segment.

The next two variational problems are not directly related to the boundary value problems (1.1). Nevertheless, we are going to discuss operator preconditioning also for them. For the modified weakly singular operator $\bar{\mathsf{V}}$ the associated variational problem reads as follows (**Case C**): for $g \in \widetilde{H}^{1/2}(\Gamma)$ find $\phi \in H^{-1/2}(\Gamma)$ such that

$$\mathsf{a}_{\bar{\mathsf{V}}}(\phi,\psi) \;= \left\langle \bar{\mathsf{V}}\phi\,,\,\psi\right\rangle_{H^{-1/2}(\Gamma)} = \langle g\,,\,\psi\rangle_{H^{-1/2}(\Gamma)} \quad \forall\, \psi \in H^{-1/2}(\Gamma). \tag{2.8}$$

Finally the variational problem for the modified hypersingular operator $\bar{\mathsf{W}}$ is: find $v \in H^{1/2}_*(\Gamma)$ such that for a given $f \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$, it holds

$$\mathsf{a}_{\bar{\mathsf{W}}}(v,w) \;:=\; \left\langle \bar{\mathsf{W}}v\,,\,w\right\rangle_{H^{1/2}(\Gamma)} = \langle f\,,\,w\rangle_{H^{1/2}(\Gamma)} \quad \forall\, w \in H^{1/2}_*(\Gamma). \tag{2.9}$$

Direct Galerkin discretization of $\mathsf{a}_{\mathsf{V}}(\varphi,\psi)$ and $\mathsf{a}_{\bar{\mathsf{W}}}(w,v)$ would require trial and test spaces to comply with the constraints in (2.2) and (2.3). In order to avoid this, we suppress the orthogonality restrictions and define two *augmented* bilinear forms. First introduce for $\alpha \in \mathbb{R}$

$$\widetilde{\mathsf{a}}_{\mathsf{V}}[\alpha](\varphi,\psi) \;:=\; \langle \mathsf{V}\,\varphi\,,\,\psi\rangle + \alpha\,\langle 1\,,\,\varphi\rangle\,\langle 1\,,\,\psi\rangle\,, \quad \varphi,\psi \in \widetilde{H}^{-1/2}(\Gamma)\,, \tag{2.10}$$

with duality pairings $\langle\cdot\,,\,\cdot\rangle$ on $\widetilde{H}^{-1/2}(\Gamma)$. Obviously, $\widetilde{\mathsf{a}}_{\mathsf{V}}[\alpha] : \widetilde{H}^{-1/2}(\Gamma)\times\widetilde{H}^{-1/2}(\Gamma) \to \mathbb{C}$ is continuous for any $\alpha \in \mathbb{R}$. Similarly, define for $\beta \in \mathbb{R}$

$$\widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}[\beta](v,w) \;:=\; \left\langle \bar{\mathsf{W}}v\,,\,w\right\rangle + \beta\left\langle v\,,\,\omega^{-1}\right\rangle\left\langle w\,,\,\omega^{-1}\right\rangle, \quad v,w \in H^{1/2}(\Gamma)\,, \tag{2.11}$$

where $\widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}[\beta] \;:\; H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{C}$ is bounded for any $\beta \in \mathbb{R}$. Now we consider $\alpha > 0$ and $\beta > 0$ fixed and usually drop $[\alpha]$ and $[\beta]$ from the notation for the bilinear forms. To begin with we note that augmentation does not change the solutions of the variational problems. The proof is given in Appendix A.

THEOREM 2.4. *The variational problem* (2.6) *is equivalent to the augmented variational problem (**Case A**): find $\varphi \in \widetilde{H}^{-1/2}(\Gamma)$ such that*

$$\widetilde{\mathsf{a}}_{\mathsf{V}}(\varphi,\psi) = \langle g\,,\,\psi\rangle\,, \quad \forall\, \psi \in \widetilde{H}^{-1/2}(\Gamma)\,, \tag{2.12}$$

and the variational problem (2.9) *is equivalent to the augmented variational problem (**Case D**): find $v \in H^{1/2}(\Gamma)$ such that*

$$\widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}(v,w) = \langle f\,,\,w\rangle_{H^{1/2}(\Gamma)}\,, \quad \forall\, w \in H^{1/2}(\Gamma)\,. \tag{2.13}$$

The next result essentially confirms the unique solvability of all (augmented) variational problems. Its proof relies on the $\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$-ellipticity of V and the $H^{1/2}_*(\Gamma)$-ellipticity of $\bar{\mathsf{W}}$, both established in [12, Prop. 3.1].

THEOREM 2.5. *For any $\alpha, \beta \in \mathbb{R}_+$ the (augmented) bilinear forms $\widetilde{\mathsf{a}}_{\mathsf{V}}$, a_{W}, $\mathsf{a}_{\bar{\mathsf{V}}}$, and $\widetilde{\mathsf{a}}_{\bar{\mathsf{V}}}$ are bounded and* elliptic *on $\widetilde{H}^{-1/2}(\Gamma)$, $\widetilde{H}^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$, and $H^{1/2}(\Gamma)$, respectively.*

Thanks to the Lax-Milgram lemma Theorem 2.5 gives a positive answer to Question (Q2) for *any* conforming choice of trial/test spaces for the Galerkin discretization of the variational problems (2.7), (2.8), (2.12), and (2.13): throughout the ellipticity constants will supply possible constants in the inf-sup conditions and those will obviously be independent of the finite dimensional spaces.

2.4. Generalizations. We argue that the setting of the line segment $\{0\} \times \Gamma$ is sufficiently general for the discussion of operator preconditioning, because the variational problems (2.6)– (2.9) can be lifted to an open curve $\mathcal{C}$ defined by a C^2-parameterization $\mathbf{s} : \Gamma \to \mathcal{C}$ with $\|\dot{\mathbf{s}}(\tau)\| = 1$ for all $\tau \in \Gamma$. For instance, the bilinear form associated with the weakly singular integral operator on $\mathcal{C}$ reads

$$\mathsf{a}_{\mathsf{V},\mathcal{C}}(\phi,\psi) := \int_{-1}^{1}\int_{-1}^{1} \log \frac{1}{\|\mathbf{s}(x)-\mathbf{s}(y)\|}\phi(\mathbf{s}(x))\psi(\mathbf{s}(y))\,dy\,dx\ , \quad \phi,\psi \in \widetilde{H}^{-1/2}(\mathcal{C})\ .$$

It can be pulled back to Γ, which yields

$$\mathsf{a}_{\mathsf{V},\mathcal{C}}(\phi,\psi) := \int_{-1}^{1}\int_{-1}^{1} \log \frac{1}{\|\mathbf{s}(x)-\mathbf{s}(y)\|}\phi(x)\psi(y)\,dy\,dx\ , \quad \phi,\psi \in \widetilde{H}^{-1/2}(\Gamma)\ .$$

Slightly abusing notation, we have kept the same symbol $\mathsf{a}_{\mathsf{V},\mathcal{C}}$. Analogous considerations apply to the other bilinear forms $\mathsf{a}_{\mathsf{W},\mathcal{C}}$, $\mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}}$, and $\mathsf{a}_{\bar{\mathsf{W}},\mathcal{C}}$ defined for functions on $\mathcal{C}$. We point out that it is exactly the bilinear forms $\mathsf{a}_{\mathsf{V},\mathcal{C}}$ and $\mathsf{a}_{\mathsf{W},\mathcal{C}}$ that occur in the variational boundary integral equations associated with (1.1).

PROPOSITION 2.6. *The following bilinear forms are compact*

$$\begin{aligned}
&\mathsf{a}_{\mathsf{V}} - \mathsf{a}_{\mathsf{V},\mathcal{C}} : \widetilde{H}^{-1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma) \to \mathbb{R}\ , \qquad &&\mathsf{a}_{\mathsf{W}} - \mathsf{a}_{\mathsf{W},\mathcal{C}} : \widetilde{H}^{1/2}(\Gamma) \times \widetilde{H}^{1/2}(\Gamma) \to \mathbb{R}\ ,\\
&\mathsf{a}_{\bar{\mathsf{V}}} - \mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}} : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}\ , \qquad &&\mathsf{a}_{\bar{\mathsf{W}}} - \mathsf{a}_{\bar{\mathsf{W}},\mathcal{C}} : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{R}\ .
\end{aligned}$$

Proof. We focus on the modified weakly singular integral operator $\bar{\mathsf{V}}$ and note that

$$(\mathsf{a}_{\bar{\mathsf{V}}} - \mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}})(\phi,\psi) = \int_{-1}^{1}\int_{-1}^{1}\left(\log\frac{\|\mathbf{s}(x)-\mathbf{s}(y)\|}{|x-y|} + \log\frac{M_{\mathcal{C}}(\mathbf{s}(x),\mathbf{s}(y))}{M(x,y)}\right)\phi(x)\psi(y)\,dy\,dx\ ,$$

where $M_{\mathcal{C}}(\mathbf{x},\mathbf{y}) := \frac{1}{2}\left(\|\mathbf{x}-\mathbf{y}\|^2 + (\mathrm{dist}(\mathbf{x},\partial\mathcal{C}) + \mathrm{dist}(\mathbf{y},\partial\mathcal{C}))^2\right)$. By Taylor expansion about $x = y$ and using $\|\dot{\mathbf{s}}\| = 1$, we find for $x \approx y$

$$\begin{aligned}
\log\frac{\|\mathbf{s}(x)-\mathbf{s}(y)\|}{|x-y|} &= \log(1 + \dot{\mathbf{s}}(x)\cdot\ddot{\mathbf{s}}(x)(x-y) + O(|x-y|^2)) = O(|x-y|)\ ,\\
\log\frac{M_{\mathcal{C}}(\mathbf{s}(x),\mathbf{s}(y))}{M(x,y)} &= \log\frac{\|\mathbf{s}(x)-\mathbf{s}(y)\|^2 + (\omega(x)+\omega(y))^2}{|x-y|^2 + (\omega(x)+\omega(y))^2} \le O(|x-y|)\ .
\end{aligned}$$

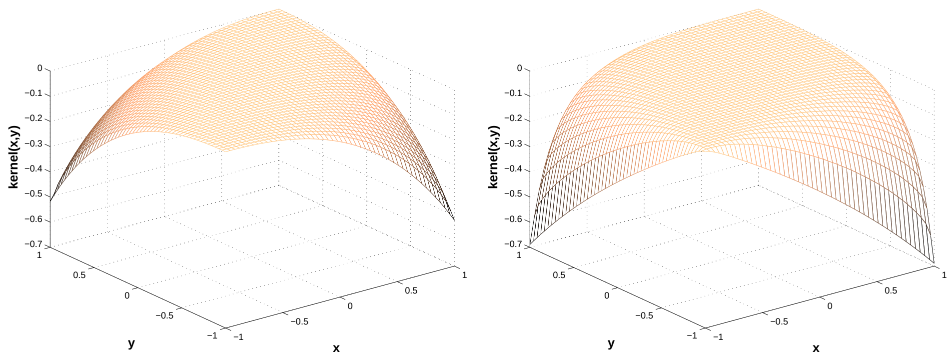


Fig. 2.1: Plot of kernel δk associated with $\mathsf{a}_{\bar{\mathsf{V}}} - \mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}}$ for $\mathbf{s}(t) = \binom{\sin(t)}{\cos(t)}$ (arc curve, left) and $\mathbf{s}(t) = \frac{t+0.01}{\sqrt{2}}\binom{\cos(\log(\frac{t+0.01}{\sqrt{2}}))}{\sin(\log(\frac{t+0.01}{\sqrt{2}}))}$ (spiral, right). The kernels are piecewise smooth and continuous.

Hence, the difference $\mathsf{a}_{\bar{\mathsf{V}}} - \mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}}$ of the bilinear forms is induced by an integral operator $\delta\,\mathsf{V}$ with a piecewise smooth and globally Lipschitz continuous kernel $\delta k = \delta k(x, y)$, see Figure 2.1 for plots of two specimens.

In particular, the kernel δk belongs to $W^{1,1}(\Gamma\times\Gamma)$, the Sobolev space of functions in $L^1(\Gamma\times\Gamma)$ such that its first order weak derivatives are also $L^1(\Gamma\times\Gamma)$. Recalling that integral operators with L^1-kernels induce compact mappings $L^2(\Gamma)\to L^2(\Gamma)$, we conclude that a kernel in $W^{1,1}(\Gamma\times\Gamma)$ spawns a compact operator $L^2(\Gamma)\to H^1(\Gamma)$. Hence, after subtracting a linear function, which is a simple compact modification, we end up with a compact mapping $L^2(\Gamma)\to H^1_0(\Gamma)$. Thanks to the symmetry of the kernel, it agrees with its adjoint (modulo a compact perturbation), which will be a compact mapping $H^{-1}(\Gamma)\to L^2(\Gamma)$. By interpolation between $L^2(\Gamma)/H^{-1}(\Gamma)$ and $H^1(\Gamma)/L^2(\Gamma)$ we finally infer that $\delta\,\mathsf{V}$ is an integral operator mapping compactly $H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$.

Similar arguments apply to the other differences of bilinear forms. As regards $\mathsf{a}_{\mathsf{V}} - \mathsf{a}_{\mathsf{V},\mathcal{C}}$ and $\mathsf{a}_{\mathsf{W}} - \mathsf{a}_{\mathsf{W},\mathcal{C}}$ we can simply appeal to the continuous embeddings $\widetilde{H}^{-1/2}(\Gamma)\subset H^{-1/2}(\Gamma)$ and $\widetilde{H}^{1/2}(\Gamma)\subset H^{1/2}(\Gamma)$. $\square$

A first conclusion we can draw from this theorem is that the stability of Galerkin discretizations of the bilinear forms on $\mathcal{C}$ can be inferred from Theorem 2.5, provided that the resolution of the trial and test spaces is large enough, *cf.* [22, Sect. 4.2.3].

Further, as compact perturbations of a bilinear form do not affect the asymptotic performance of operator preconditioning, this result confirms that $\mathsf{a}_{\bar{\mathsf{W}}}$ is suitable for preconditioning $\mathsf{a}_{\mathsf{V},\mathcal{C}}$, $\mathsf{a}_{\bar{\mathsf{V}}}$ spawns a preconditioner for $\mathsf{a}_{\mathsf{W},\mathcal{C}}$, and so on. Of course, the constants will depend on the shape of $\mathcal{C}$.

The building blocks of operator preconditioning as they have been assembled so far, are listed in Table 2.1. The missing pieces, namely the families of boundary element spaces X_h and Y_h, will be specified in the next section.

Remark 2.7. *As another generalization of the variational problems studied in Section 2.3 we may consider the boundary integral operators associated with boundary value problems for the Helmholtz equation $-\Delta u - k^2u = 0$ with wave number $k > 0$. For the line segment the corresponding weakly singular and hypersingular operators,*

	a	b	X	Y
Case A:	$\widetilde{\mathsf{a}}_{\mathsf{V},\mathcal{C}}$, *cf.* (2.12)	$\mathsf{a}_{\bar{\mathsf{W}},\mathcal{C}}$	$\widetilde{H}^{-1/2}(\mathcal{C})$	$H^{1/2}(\mathcal{C})$
Case B:	$\mathsf{a}_{\mathsf{W},\mathcal{C}}$, *cf.* (2.7)	$\mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}}$	$\widetilde{H}^{1/2}(\mathcal{C})$	$H^{-1/2}(\mathcal{C})$
Case C:	$\mathsf{a}_{\bar{\mathsf{V}},\mathcal{C}}$, *cf.* (2.8)	$\mathsf{a}_{\mathsf{W},\mathcal{C}}$	$H^{-1/2}(\mathcal{C})$	$\widetilde{H}^{1/2}(\mathcal{C})$
Case D:	$\widetilde{\mathsf{a}}_{\bar{\mathsf{W}},\mathcal{C}}$, *cf.* (2.13)	$\mathsf{a}_{\mathsf{V},\mathcal{C}}$	$H^{1/2}(\mathcal{C})$	$\widetilde{H}^{-1/2}(\mathcal{C})$

Table 2.1: (Partial) summary of operator preconditioning strategy for variational boundary integral equations on an open curve $\mathcal{C}$. For notations see Theorem 1.1.

V^k *and* W^k*, read*

$$\begin{aligned} \mathsf{V}^k\,\varphi(x) &:= \frac{i}{4}\int_\Gamma H_0^{(1)}(k\,|x-y|)\varphi(y)\,dy\ , \quad x\in\Gamma\ , \quad \varphi\in C_0^\infty(\Gamma)\ , \\ \mathsf{W}^k &:= -\left(\tfrac{d}{dx}\right)^*\circ\mathsf{V}^k\circ\tfrac{d}{dx}+k^2\,\mathsf{V}^k\ , \end{aligned} \tag{2.14}$$

where $H_0^{(1)}(\xi)$ *stands for the Hankel function of the first kind [22, Eq. (3.3)]. Since the following representation holds [23, Sect. 2.3.1]*

$$H_0^{(1)}(k\,|x-y|) = \frac{1}{\pi}\log\frac{1}{|x-y|} + K(k\,|x-y|)\ , \tag{2.15}$$

one can write the operators as

$$\mathsf{V}^k = \frac{i}{4\pi}\,\mathsf{V}+\mathsf{H}^k, \quad \textit{and} \quad \mathsf{W}^k = \frac{i}{4\pi}\left(\mathsf{W}+k^2\,\mathsf{V}\right)+\mathsf{L}^k\ , \tag{2.16}$$

with compact operators $\mathsf{H}^k : \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)\to H^{1/2}_*(\Gamma)$ *and* $\mathsf{L}^k : \widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$*, as the convolution kernel* $K(\cdot)$ *is piecewise smooth and continuous, and integration is over a bounded domain [21]. As a consequence, the bilinear forms for the Helmholtz counterparts of the variational problems* (2.6) *and* (2.7) *will be compact perturbations of* a_V *and* a_W*, respectively,* cf. *[22, Lemma 3.9.8].*

3. Boundary Element Spaces. We employ low-order mapped piecewise polynomial conforming boundary element spaces $X_h\subset X$ and $Y_h\subset Y$ for the Galerkin discretization of the various bilinear forms a and b as listed in Table 2.1 and defined in Section 2.4. These spaces a are built upon partitions of Γ and, by virtue of the mapping approach outlined in Section 2.4, all considerations can be confined to Γ.

3.1. Primal and dual meshes. First we construct primal and dual meshes of Γ as explained in [26, Sect. 2.2], [10, Section 4], and [5]. We introduce a *primal mesh* Γ_h of the interval Γ and denote its N nodes by $-1 =: x_1 < x_2 < \cdots < x_{N-1} < x_N := 1$, $N\in\mathbb{N}$.

Based on Γ_h we build a *dual mesh* $\hat{\Gamma}_h$ of Γ, whose nodes are the midpoints of intervals of Γ_h plus the points -1 and $+1$. More explicitly, the $N+1$ nodes η_i, $i=0,\ldots,N$, of the dual mesh $\hat{\Gamma}_h$ are given by

$$\eta_0 := -1\ , \quad \eta_i := \tfrac{1}{2}(x_i+x_{i+1})\ , \quad i=1,\ldots,N-1\ , \qquad \eta_N := 1\ . \tag{3.1}$$

3.2. Dual pairs of spaces. A positive answer to Question (Q3) entails a judicious construction of dual pairs of spaces $X_h \subset X$, $Y_h \subset Y$ in each of the four cases. They will be based on pairs of primal and dual meshes. Throughout, we write $\mathbb{P}_m$ for the space of uni-variate polynomials of degree $\leq m$.

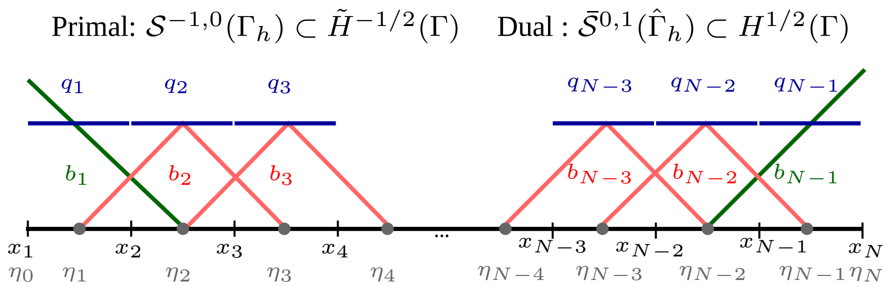


Fig. 3.1: **Case A**: $X := \widetilde{H}^{-1/2}$, $Y := H^{1/2}$, piecewise constant basis functions q_j for $X_h := \mathcal{S}^{-1,0}(\Gamma_h)$ in blue, piecewise linear basis functions ("tent functions") b_j for $Y_h := \bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h)$ in red/green. Note the extended "ramp functions" (in green) supported in the two leftmost and rightmost intervals of the dual mesh.

Case A: Discrete spaces for $X := \widetilde{H}^{-1/2}$, $Y := H^{1/2}$ (Row 1 of Table 2.1). This case addresses the variational equation (2.12) with $\mathsf{a} = \widetilde{\mathsf{a}}_{\mathsf{V}}$ using $\mathsf{b} = \widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}$ as preconditioning bilinear form, *cf.* Theorem 1.1. The primal and dual boundary element spaces are given by

$$X_h := \mathcal{S}^{-1,0}(\Gamma_h) = \{\varphi_h \in L^2(\Gamma) : \varphi_h|_{[x_j,x_{j+1}]} \in \mathbb{P}_0,\ j = 1, \ldots, N-1\} \subset X \ ,$$
$$Y_h := \bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h) := \left\{ v_h \in C^0(\overline{\Gamma}) : \begin{array}{l} v_h|_{[\eta_{j-1},\eta_j]} \in \mathbb{P}_1,\ j = 3, \ldots, N-2 \\ v_h|_{[\eta_0,\eta_2]},\ v_h|_{[\eta_{N-2},\eta_N]} \in \mathbb{P}_1 \end{array} \right\} \subset Y \ .$$

By means of their canonical basis functions the spaces are visualized in Figure 3.1. Obviously, they have the *same dimension*, which is N.

Case B: Discrete spaces for $X := \widetilde{H}^{1/2}$, $Y := H^{-1/2}$ (Row 2 of Table 2.1). This setting arises from the variational equation (2.7) where $\mathsf{a} = \mathsf{a}_{\mathsf{W}}$. Thus, we precondition a by the bilinear form $\mathsf{b} = \mathsf{a}_{\bar{\mathsf{V}}}$, *cf.* Theorem 1.1. The primal and dual boundary element spaces are defined as follows

$$X_h := \mathcal{S}_0^{0,1}(\Gamma_h) := \{v_h \in C^0(\overline{\Gamma}), v_h|_{[x_{j-1},x_j]} \in \mathbb{P}_1,\ j = 2, \ldots, N-1\} \subset X \ ,$$
$$Y_h := \bar{\mathcal{S}}^{-1,0}(\hat{\Gamma}_h) := \left\{ \varphi_h \in L^2(\Gamma) : \begin{array}{l} \varphi_h|_{[\eta_j,\eta_{j+1}]} \in \mathbb{P}_0,\ j = 2, \ldots, N-3 \\ \varphi_h|_{[\eta_0,\eta_2]}, \varphi_h|_{[\eta_{N-2},\eta_N]} \in \mathbb{P}_0 \end{array} \right\} \subset Y \ .$$

Figure 3.2 shows these spaces' representation in terms of their canonical basis functions. Note that both spaces have dimension $N-2$.

Case C: Discrete spaces for $X := H^{-1/2}$, $Y := \widetilde{H}^{1/2}$ (Row 3 of Table 2.1). In order to perform operator preconditioning for the variational equation (2.8), we employ $\mathsf{b} = \mathsf{a}_{\mathsf{W}}$ as preconditioning bilinear form. The primal and dual boundary

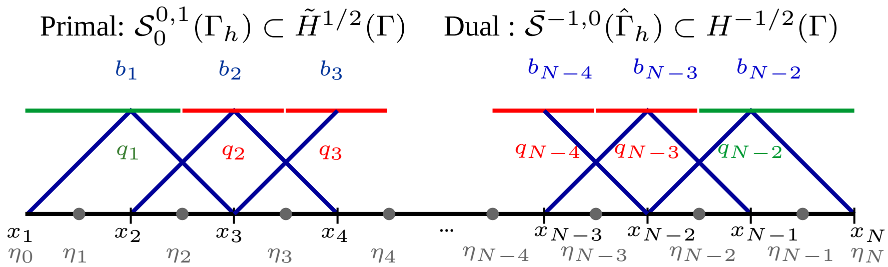


Fig. 3.2: **Case B**: $X := \widetilde{H}^{1/2}$, $Y := H^{-1/2}$, piecewise linear basis functions ("tent functions") b_j for $X_h := \mathcal{S}_0^{0,1}(\Gamma_h)$ in blue, piecewise constant basis functions q_j for $Y_h := \bar{\mathcal{S}}^{-1,0}(\hat{\Gamma}_h)$ in red/green. Note the extended "characteristic functions" (in green) for the two leftmost and rightmost intervals of the dual mesh.

element spaces are given by

$$X_h := \mathcal{S}^{-1,0}(\Gamma_h) = \{\varphi_h \in L^2(\Gamma) : \varphi_{h\,|[x_j, x_{j+1}]} \in \mathbb{P}_0,\ j = 1, \dots, N-1\} \subset X \ ,$$

$$Y_h := \mathcal{S}_0^{0,1}(\hat{\Gamma}_h) := \left\{ v_h \in C^0(\overline{\Gamma}) : \begin{array}{l} v_h|_{[\eta_{j-1}, \eta_j]} \in \mathbb{P}_1,\ j = 1, \dots, N, \\ v_h(\eta_0) = v_h(\eta_N) = 0 \end{array} \right\} \subset Y \ .$$

This yields dimension $N-1$ in both cases, as the reader may see from Figure 3.3, where the boundary element spaces are illustrated using their canonical basis functions.

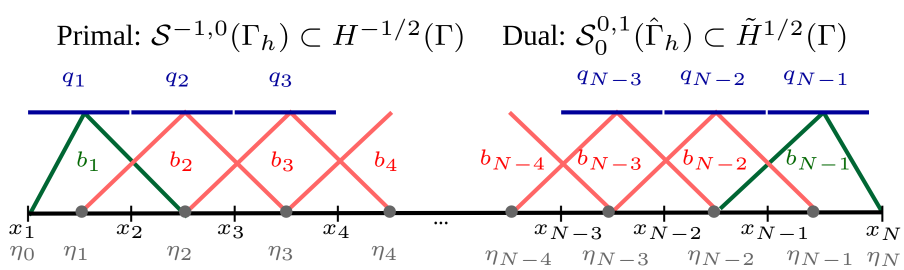


Fig. 3.3: **Case C**: $X := H^{-1/2}$,$Y := \widetilde{H}^{1/2}$, piecewise constant basis functions q_j for $X_h := \mathcal{S}^{-1,0}(\Gamma_h)$ in blue, piecewise linear basis functions ("tent functions") b_j for $Y_h := \mathcal{S}_0^{0,1}(\hat{\Gamma}_h)$ in red/green. Note that no basis functions are assigned to η_0 and η_N.

Case D: Discrete spaces for $X := H^{1/2}$, $Y := \widetilde{H}^{-1/2}$ (Row 4 of Table 2.1). This last case corresponds to the variational equation (2.9) where $\mathsf{a} = \widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}$. Hence, we use bilinear form $\mathsf{b} = \widetilde{\mathsf{a}}_{\mathsf{V}}$ to build the preconditioner. One can define the primal and dual boundary element spaces as follows

$$X_h := \mathcal{S}^{0,1}(\Gamma_h) = \{v_h \in C^0(\overline{\Gamma}),\ v_h|_{[x_{j-1}, x_j]} \in \mathbb{P}_1,\ j = 2, \dots, N\} \subset X \ ,$$

$$Y_h := \mathcal{S}^{-1,0}(\hat{\Gamma}_h) := \{\varphi_h \in L^2(\Gamma) : \varphi_h|_{[\eta_{j-1}, \eta_j]} \in \mathbb{P}_0,\ j = 1, \dots, N\} \subset Y \ .$$

As in the previous cases, we provide their canonical representation in Figure 3.4. Observe both spaces have dimension N.

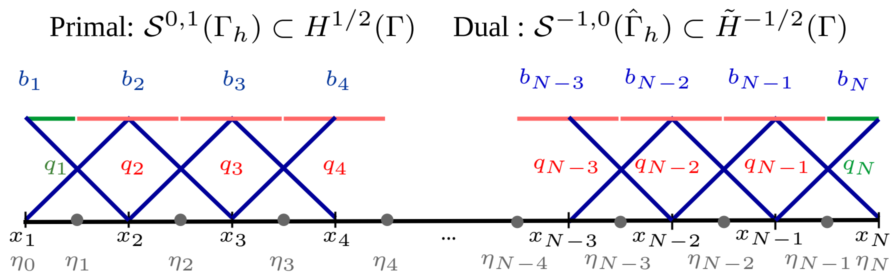


Fig. 3.4: **Case D**: $X := H^{1/2}$, $Y := \widetilde{H}^{-1/2}$, piecewise linear basis functions ("tent functions") b_j for $X_h := \mathcal{S}^{0,1}(\Gamma_h)$ in blue, basis functions q_j for $Y_h := \mathcal{S}^{-1,0}(\hat{\Gamma}_h)$, the characteristic functions of the dual mesh intervals, in red/green.

The choices for the discrete spaces X_h and Y_h are summarized in Table 3.1, whose rows correspond to those of Table 2.1.

Table 3.1: Summary of operator preconditioning strategy for variational boundary integral equations on the interval Γ. For notations see Theorem 1.1.

	Continuous				Discrete	
	a	b	X	Y	X_h	Y_h
Case A	$\widetilde{\mathsf{a}}_{\mathsf{V}}$	$\widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}$	$\widetilde{H}^{-1/2}(\Gamma)$	$H^{1/2}(\Gamma)$	$\mathcal{S}^{-1,0}(\Gamma_h)$	$\bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h)$
Case B	a_{W}	$\mathsf{a}_{\bar{\mathsf{V}}}$	$\widetilde{H}^{1/2}(\Gamma)$	$H^{-1/2}(\Gamma)$	$\mathcal{S}_0^{0,1}(\Gamma_h)$	$\bar{\mathcal{S}}^{-1,0}(\hat{\Gamma}_h)$
Case C	$\mathsf{a}_{\bar{\mathsf{V}}}$	a_{W}	$H^{-1/2}(\Gamma)$	$\widetilde{H}^{1/2}(\Gamma)$	$\mathcal{S}^{-1,0}(\Gamma_h)$	$\mathcal{S}_0^{0,1}(\hat{\Gamma}_h)$
Case D	$\widetilde{\mathsf{a}}_{\bar{\mathsf{W}}}$	$\widetilde{\mathsf{a}}_{\mathsf{V}}$	$H^{1/2}(\Gamma)$	$\widetilde{H}^{-1/2}(\Gamma)$	$\mathcal{S}^{0,1}(\Gamma_h)$	$\mathcal{S}^{-1,0}(\hat{\Gamma}_h)$

4. Stability of Discrete Duality Pairings. Now we tackle Question (Q3) for the pairs (X_h, Y_h) of discrete spaces defined for Cases A–D in Section 3.2. We closely follow the policy developed by O. Steinbach in [24, 25, 26] for the case when $X = H^{1/2}(\Gamma)$ (Case D), and we are going to extend his results to the other remaining cases.

Since we aim for mesh-uniform stability results, we consider an infinite family of meshes $\{\Gamma_h\}_{h\in\mathbb{H}}$ of Γ, whose members are labelled by h from the index set $\mathbb{H}$ and serve as primal meshes. Concrete specimens of such families will be presented in Section 4.3. All of the constants introduced below can be chosen independently of h. Suppressing the dependence on h we continue using the notations x_i and η_j to designate the nodes of the primal mesh Γ_h and its associated dual mesh $\hat{\Gamma}_h$, see Section 3.1. We also keep N for the total number of nodes of Γ_h.

4.1. Assumptions on mesh geometry. The stability results will hinge on certain assumptions on local properties of the meshes Γ_h, $h \in \mathbb{H}$. From the elaborations of Section 4.2 it will become clear that cases A, D, and B,C are connected by duality.

Therefore, only two sets of assumptions on the geometry of the meshes will suffice, corresponding to the cases A/D and B/C.

Below, we are going to use the same notations for entities that will be different for different cases. The concrete meaning should always be clear from the context of the current case being discussed. Moreover, in what follows, many notations are borrowed from [26]. In particular, we designate by $\tau_l := (x_l, x_{l+1})$ a mesh interval of the primal mesh Γ_h with length $h_l := x_{l+1} - x_l$, $l = 1, \dots, N-1$.

Case D (Case A). In this case we set $\hat{h}_1 := \frac{1}{3}h_1$, $\hat{h}_k := \frac{1}{3}(h_{k-1} + h_k)$, $k = 2, \dots, N-1$, $\hat{h}_N := h_{N-1}$. Then we define the following 2×2 matrices associated with the intervals of the primal mesh

$$\tilde{G}_l = \frac{h_l}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \quad , \quad G_l = \frac{h_l}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad , \quad H_l = \begin{pmatrix} \hat{h}_l^{1/2} & 0 \\ 0 & \hat{h}_{l+1}^{1/2} \end{pmatrix} , \tag{4.1}$$

for $l = 1, \dots, N-1$.

Case B (Case C). In this case we define $\hat{h}_k := \frac{1}{3}(h_k + h_{k+1})$, $k = 1, \dots, N-2$. Further, for $l = 2, \dots, N-2$, the 2×2-matrices $\tilde{G}_l$, G_l, and H_l are defined exactly as in (4.1). Besides, for mesh intervals adjacent to the endpoints -1 and 1 these matrices reduce to the following 1×1-matrices (numbers)

$$\tilde{G}_l = \frac{h_l}{2} \quad , \quad D_l = \frac{h_l}{3} \quad , \quad H_l = \hat{h}_l \ , \quad \text{for } l = 1, N-1 \ . \tag{4.2}$$

Using the notations just introduced, we now state geometric assumptions on the meshes Γ_h valid for all cases. Throughout, l runs through the maximum possible index interval, and $M_l \in \{1, 2\}$ designates the size of the matrices $\tilde{G}_l, G_l$, and H_l, from (4.1) and (4.2), respectively.

Assumption 4.1 (Assumption 1.1 in [26]). *There are constants $c_1^G, c_2^G > 0$ independent of h and l such that*

$$c_1^G(D_l\mathbf{x}_l, \mathbf{x}_l) \le (G_l\mathbf{x}_l, \mathbf{x}_l) \le c_2^G(D_l\mathbf{x}_l, \mathbf{x}_l), \quad \forall \mathbf{x}_l \in \mathbb{R}^{M_l} , \tag{4.3}$$

where $D_l := \mathrm{diag}(D_l)$.

Assumption 4.2 (Assumption 2 in [26]). *We can find a constant $c_0 > 0$ such that*

$$(H_l\tilde{G}_l^T H_l^{-1}\mathbf{x}_l, \mathbf{x}_l) \ge c_0 \cdot (D_l\mathbf{x}_l, \mathbf{x}_l), \quad \forall \mathbf{x}_l \in \mathbb{R}^{M_l} \tag{4.4}$$

for all l and h.

4.2. Stability Results. Now we establish the crucial stability results for the four cases A-D. Their proof will be elaborated in several steps throughout this section.

Theorem 4.3. *Let Assumptions 4.1 and 4.2 be satisfied. Then, for the following combinations of discrete spaces*

$$\begin{array}{lll} \textit{Case A:} & X_h = \mathcal{S}^{-1,0}(\Gamma_h) \subset X = \widetilde{H}^{-1/2}(\Gamma), & Y_h = \bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h) \subset Y = H^{1/2}(\Gamma), \\ \textit{Case B:} & X_h = \mathcal{S}_0^{0,1}(\Gamma_h) \subset X = \widetilde{H}^{1/2}(\Gamma), & Y_h = \bar{\mathcal{S}}^{-1,0}(\hat{\Gamma}_h) \subset Y = H^{-1/2}(\Gamma), \\ \textit{Case C:} & X_h = \mathcal{S}^{-1,0}(\Gamma_h) \subset X = H^{-1/2}(\Gamma), & Y_h = \mathcal{S}_0^{0,1}(\hat{\Gamma}_h) \subset Y = \widetilde{H}^{1/2}(\Gamma), \\ \textit{Case D:} & X_h = \mathcal{S}^{0,1}(\Gamma_h) \subset X = H^{1/2}(\Gamma), & Y_h = \mathcal{S}^{-1,0}(\hat{\Gamma}_h) \subset Y = \widetilde{H}^{-1/2}(\Gamma), \end{array}$$

the discrete inf-sup condition:

$$\sup_{v_h \in Y_h} \frac{|\langle w_h\,,\, v_h \rangle|}{\|v_h\|_Y} \geq \frac{1}{c_s} \|w_h\|_X \,, \qquad \forall\, w_h \,\in\, X_h. \tag{4.5}$$

holds with a positive constant c_s independent of h.

Given the assertion of this theorem, all the abstract assumptions of Theorem 1.1 have now been verified. Theorem 2.1 provides the continuity of the bilinear forms, Theorem 2.5 uniform stability of the (discrete) variational problems, and, finally, Theorem 4.3 the stability of the discrete duality pairing. Hence, for all the concrete choices listed in Table 3.1 operator preconditioning will yield preconditioners that achieve bounded condition numbers *independently of the resolution of the mesh*: they are asymptotically optimal.

We split the proof of Theorem 4.3 into the individual cases. Moreover, inherent dependencies suggest to treat them in the order to D-B-C-A.

Proof of Theorem 4.3 for Case D. As stated before, this case follows as a Corollorary of [26, Theorem 2.1 and 2.2].

Proof of Theorem 4.3 for Case B. In order to prove this case, we extend the stability results developed for Case D [26, Theorem 2.2], using an analogous policy. We start with an assertion of L^2-stability of the discrete pairing, see Appendix B for a proof using elementary local estimates.

Lemma 4.4. *The L^2-stability holds*

$$\sup_{\psi_h \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h)} \frac{|\langle \psi_h\,,\, w_h \rangle|}{\|\psi_h\|_{L^2(\Gamma)}} \geq c_{st} \|w_h\|_{L^2(\Gamma)} \,, \quad \forall\, w_h \,\in\, \mathcal{S}_0^{0,1}(\Gamma_h),\ \forall\, h \in \mathbb{H}, \tag{4.6}$$

where $c_{st} = \frac{1}{2}$.

Next, consider the standard Galerkin L^2-Projection $Q_h : L^2(\Gamma) \to \mathcal{S}_0^{0,1}(\Gamma_h)$, and the generalized Galerkin L^2-Projection $\tilde{Q}_h : L^2(\Gamma) \to \mathcal{S}_0^{0,1}(\Gamma_h)$, for a given $u \in L^2(\Gamma)$ defined according to

$$\langle Q_h u\,,\, v_h \rangle_{L^2(\Gamma)} = \langle u\,,\, v_h \rangle_{L^2(\Gamma)} \,, \qquad \forall v_h \,\in\, \mathcal{S}_0^{0,1}(\Gamma_h), \tag{4.7}$$

$$\left\langle \tilde{Q}_h u\,,\, \phi_h \right\rangle_{L^2(\Gamma)} = \langle u\,,\, \phi_h \rangle_{L^2(\Gamma)} \,, \qquad \forall \phi_h \,\in\, \mathcal{S}^{-1,0}(\hat{\Gamma}_h)\,. \tag{4.8}$$

Lemma 4.4 ensures that $\tilde{Q}_h$ is well-defined, because it guarantees unique solvability of (4.8). It also furnishes the stability estimate

$$\left\| \tilde{Q}_h u \right\|_{L^2(\Gamma)} \leq \frac{1}{c_{st}} \|u\|_{L^2(\Gamma)} \,, \qquad \text{for all } \ u \in L^2(\Gamma)\,. \tag{4.9}$$

Additionally, we want to prove the H^1-stability of $\tilde{Q}_h$ and, following [26, Section 1.5], resort to a quasi-interpolation operator. For the sake of clarity, the proof of the following will also be provided in Appendix B.

PROPOSITION 4.5. *Let Assumptions 4.1 and 4.2 be satisfied. Then the L^2-projection $\tilde{Q}_h : H_0^1(\Gamma) \to X_h = \mathcal{S}_0^{0,1}(\Gamma_h)$ defined in* (4.8) *satisfies*

$$\left\| \tilde{Q}_h u \right\|_{H^1(\Gamma)} \leq \tilde{c}_{st} \left\| u \right\|_{H^1(\Gamma)}, \qquad \forall\, u \in H_0^1(\Gamma), \tag{4.10}$$

with $\tilde{c}_{st}$ a positive constant independent of h.

Now we are in a position to proof the key stability results for Case B.

Proof of Theorem 4.3 for Case B $\widetilde{H}^{1/2}(\Gamma)$ can be obtained by interpolating between $L^2(\Gamma)$ and $H_0^1(\Gamma)$, see [16, Thm. 11.7]. Thus, by interpolation of bounded linear operators we obtain from (4.9) and Proposition 4.5 that

$$\left\| \tilde{Q}_h u \right\|_{\widetilde{H}^{1/2}(\Gamma)} \leq c_B \left\| u \right\|_{\widetilde{H}^{1/2}(\Gamma)}, \qquad \forall\, u \in \widetilde{H}^{1/2}(\Gamma),\ h \in \mathbb{H}\,. \tag{4.11}$$

Introduce the projection operators $\Pi_h : \widetilde{H}^{1/2}(\Gamma) \to \mathcal{S}^{-1,0}(\hat{\Gamma}_h) \subseteq H^{-1/2}(\Gamma)$, satisfying

$$\langle \Pi_h u\,,\, w_h \rangle_{L^2(\Gamma)} = \langle u\,,\, w_h \rangle_{\widetilde{H}^{1/2}(\Gamma)}\,, \qquad \forall\, w_h \,\in\, \mathcal{S}_0^{0,1}(\Gamma_h),\ h \in \mathbb{H} \tag{4.12}$$

where $\langle u\,,\, w_h \rangle_{\widetilde{H}^{1/2}(\Gamma)}$ denotes the $\widetilde{H}^{1/2}(\Gamma)$-inner product. By the dual norm definition and continuity of $\tilde{Q}_h$, one can derive

$$\left\| \Pi_h u \right\|_{H^{-1/2}(\Gamma)} \leq c_B \left\| u \right\|_{\widetilde{H}^{1/2}(\Gamma)}, \quad \forall\, u \,\in \widetilde{H}^{1/2}(\Gamma),\ h \in \mathbb{H}.$$

Finally, for any $w_h \in \mathcal{S}_0^{0,1}(\Gamma_h)$ by the above inequality we obtain the assertion

$$\begin{aligned} \left\| w_h \right\|_{\widetilde{H}^{1/2}(\Gamma)} &= \frac{\left| \langle w_h\,,\, w_h \rangle_{\widetilde{H}^{1/2}(\Gamma)} \right|}{\left\| w_h \right\|_{\widetilde{H}^{1/2}(\Gamma)}} = \frac{\left| \langle w_h\,,\, \Pi_h w_h \rangle_{L^2(\Gamma)} \right|}{\left\| w_h \right\|_{\widetilde{H}^{1/2}(\Gamma)}} \\ &\leq c_B\, \frac{\left| \langle w_h\,,\, \Pi_h w_h \rangle_{L^2(\Gamma)} \right|}{\left\| \Pi_h w_h \right\|_{H^{-1/2}(\Gamma)}} \leq c_B \sup_{0 \neq v_h \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h)} \frac{\left| \langle w_h\,,\, v_h \rangle_{L^2(\Gamma)} \right|}{\left\| v_h \right\|_{H^{-1/2}(\Gamma)}}\,. \end{aligned}$$

□

Proof of Theorem 4.3 for Case C. We appeal to an analogue of Lemma 4.4 to define $\tilde{Q}_h^2 : L^2(\Gamma) \to \mathcal{S}_0^{0,1}(\hat{\Gamma}_h)$ for a given $u \in L^2(\Gamma)$ as solution of the variational problem

$$\left\langle \tilde{Q}_h^2 u\,,\, \phi_h \right\rangle_{L^2(\Gamma)} = \langle u\,,\, \phi_h \rangle_{L^2(\Gamma)}\,, \quad \forall\, \phi_h \,\in\, \mathcal{S}^{-1,0}(\Gamma_h)\,. \tag{4.13}$$

Along the same lines as above one can prove that

$$\left\| \tilde{Q}_h^2 u \right\|_{\widetilde{H}^{1/2}(\Gamma)} \leq c_C \left\| u \right\|_{\widetilde{H}^{1/2}(\Gamma)}, \quad \forall\, u \,\in \widetilde{H}^{1/2}(\Gamma),\ h \in \mathbb{H}\,. \tag{4.14}$$

The arguments are very similar to those employed in the proof of Proposition 4.5 and (4.9) for $\tilde{Q}_h$. Next, using the dual norm definition and (4.14), we have for all

$v_h \in \mathcal{S}^{-1,0}(\Gamma_h)$ that

$$\begin{aligned}\|v_h\|_{H^{-1/2}(\Gamma)} &= \sup_{0\neq w\in \widetilde{H}^{1/2}(\Gamma)} \frac{\left|\langle v_h\,,\,w\rangle_{L^2(\Gamma)}\right|}{\|w\|_{\widetilde{H}^{1/2}(\Gamma)}} = \sup_{0\neq w\in \widetilde{H}^{1/2}(\Gamma)} \frac{\left|\left\langle v_h\,,\,\tilde{Q}_h^2 w\right\rangle_{L^2(\Gamma)}\right|}{\|w\|_{\widetilde{H}^{1/2}(\Gamma)}} \\ &\leq c_C \sup_{0\neq w\in \widetilde{H}^{1/2}(\Gamma)} \frac{\left|\left\langle v_h\,,\,\tilde{Q}_h^2 w\right\rangle_{L^2(\Gamma)}\right|}{\left\|\tilde{Q}_h^2 w\right\|_{\widetilde{H}^{1/2}(\Gamma)}} \leq c_C \sup_{0\neq w_h\in \widetilde{\mathcal{S}}^{0,1}(\hat{\Gamma}_h)} \frac{\left|\langle v_h\,,\,w_h\rangle_{L^2(\Gamma)}\right|}{\|w_h\|_{\widetilde{H}^{1/2}(\Gamma)}}\,,\end{aligned}$$

which amounts to the assertion in Case C.

Proof of Theorem 4.3 for Case A. The assertion for Case A follows in a similar way from Case D as Case C from Case B.

4.3. Local mesh conditions. For important families of meshes that feature local refinement towards the endpoints of Γ, we demonstrate that the constraints imposed by Assumption 4.2 are not severe. As explained above, we need consider only the cases D and B. In the interest of reducing this mesh condition to an eigenvalue problem, we use the symmetric form of Assumption 4.2 as in [26, Assumption 2.1]: with

$$G_l^S := \frac{1}{2}\left[H_l \tilde{G}_l^T H_l^{-1} + H_l^{-1}\tilde{G}_l H_l\right]\,, \tag{4.15}$$

(4.4) becomes equivalent to

$$(G_l^S \mathbf{x}_l, \mathbf{x}_l) \geq c_0 \cdot (D_l \mathbf{x}_l, \mathbf{x}_l), \quad \text{for all } \mathbf{x}_l \in \mathbb{R}^{M_l}\,. \tag{4.16}$$

We will study the following non-uniform meshes, all of which are symmetric to $x = 0$:

- For odd N and $q > 1$ geometric meshes whose nodes are

$$x_k := \begin{cases} -1 & ,\ \text{for } k = 1\,, \\ -1 + q^{-\frac{N-1}{2}+k-1} & ,\ \text{for } k = 2,\dots,\frac{N-1}{2}+1\,, \\ 1 - q^{\frac{N-1}{2}-k+1} & ,\ \text{for } k = \frac{N-1}{2}+2,\dots,N-1\,, \\ 1 & ,\ \text{for } k = N\,. \end{cases} \tag{4.17}$$

- For even N and grading factor $\alpha > 0$ algebraically graded meshes defined by

$$x_k := \begin{cases} -1 + \left(2\,\frac{k-1}{N-1}\right)^\alpha, & \text{for } k = 1,\dots,\frac{N}{2}\,, \\ 1 - \left(2 - 2\,\frac{k-1}{N-1}\right)^\alpha, & \text{for } k = \frac{N}{2}+1,\dots,N\,. \end{cases} \tag{4.18}$$

- Chebychev meshes with nodes

$$x_k := -\cos\left(\pi\,\frac{k-1}{N-1}\right)\,, \quad k = 1,\dots,N\,. \tag{4.19}$$

Case D. As can easily be deduced from the definition of the matrices from (4.1), Assumption 4.2 boils down to

$$\lambda_{\min}\begin{pmatrix} 6 & \sqrt{\frac{\hat{h}_l}{\hat{h}_{l+1}}}+\sqrt{\frac{\hat{h}_{l+1}}{\hat{h}_l}} \\ \sqrt{\frac{\hat{h}_l}{\hat{h}_{l+1}}}+\sqrt{\frac{\hat{h}_{l+1}}{\hat{h}_l}} & 6 \end{pmatrix}$$
$$= 6-\left(\sqrt{\frac{\hat{h}_l}{\hat{h}_{l+1}}}+\sqrt{\frac{\hat{h}_{l+1}}{\hat{h}_l}}\right) \geq \frac{16}{3}c_0 \quad \text{for } l=1,\dots,N-1\,.$$

Hence, if this minimal eigenvalue is positive, we can set

$$c_0 = \min_{l=1,\dots,N-1}\frac{3}{16}\left(6-\left(\sqrt{\frac{\hat{h}_l}{\hat{h}_{l+1}}}+\sqrt{\frac{\hat{h}_{l+1}}{\hat{h}_l}}\right)\right)\,. \tag{4.20}$$

Note that the minimum will be attained for extremal values of the ratios $\hat{h}_l : \hat{h}_{l+1}$.

Case B. As mentioned in Section 4.1, we get the same matrices (4.1) for $l = 2,\dots,N-2$. Hence, for the internal mesh intervals, we obtain the same formulas as in Case D. For the terminal intervals we merely have to compare the numbers from (4.2) and find that this just means $c_0 \leq \frac{3}{2}$.

Hence, for all cases we have to check the existence of $c_0 > 0$ given by (4.20). We introduce the abbreviation $r_l := \frac{\hat{h}_l}{\hat{h}_{l+1}}$ and note that thanks to symmetry, only the mesh intervals in $[-1,0]$ have to be examined.

For the geometrically graded mesh (4.17) we find $h_l = q^{-\frac{N-1}{2}+l-1}(q-1)$, $k = 1,..,\frac{N-1}{2}$, and end up with

$$r_l = \begin{cases} 1+q & \text{for } l=1\,, \\ q & \text{for } l=2,\dots,\frac{N-1}{2}+1\,. \end{cases}$$

From this we conclude

$$c_0 = \frac{3}{16}\left(6-\sqrt{1+q}-\sqrt{1+q}^{-1}\right) > 0\,, \quad \text{if} \quad q < 16+12\sqrt{2} \approx 32.9\,.$$

For the algebraically graded mesh (4.18) we obtain

$$r_l = \begin{cases} 2^\alpha & \text{for } l=1\,, \\ \frac{(l+1)^\alpha-(l-1)^\alpha}{(l)^\alpha-(l-2)^\alpha} & \text{for } l=2,\dots,\frac{N}{2}\,. \end{cases}$$

Here r_l attains its extremal value for $l=1$ and we find

$$c_0 = \frac{3}{16}(6-2^{\alpha/2}-\frac{1}{2^{\alpha/2}}) > 0\,, \quad \text{if} \quad \alpha < 2\frac{\log(3+\sqrt{2})}{\log 2} \approx 4.28\,.$$

For the Chebychev meshes, we find

$$r_l = \begin{cases} \frac{\sin^2(\frac{\pi}{2(N-1)})}{\sin^2(\frac{\pi}{N-1})} & \text{for } l=1,N-1\,, \\ \frac{\sin(\frac{\pi l}{N-1})}{\sin(\frac{\pi(l-1)}{N-1})} & \text{for } l=2,\dots,N-2\,, \end{cases}$$

which attains its extremal values for $l=1,N-1$. Furthermore, $\lambda_{\min} \searrow \frac{7}{2}$ as $N\to\infty$ and we can chose $c_0 = \frac{21}{32}$.

5. Numerical Experiments. Theory merely gives estimates with undetermined constants. In order to get clues of their sizes, we report the practical performance of operator preconditioning for the line segment, for all cases listed in Table 3.1, and three different families of meshes: (i) uniform meshes with equidistant nodes, (ii) Chebychev meshes according to (4.19), and (iii) algebraically graded meshes with grading factor $\alpha = 3$ as specified in (4.18). As elaborated in Section 4.3, all these meshes meet the geometric constraints of Assumptions 4.1, 4.2. The parameters α and β in the augmented bilinear forms from (2.10) and (2.11) were simply set to 1 throughout.

As stipulated by Theorem 1.1 the matrix $\mathbf{M}_h := \mathbf{T}_h^{-1}\mathbf{B}_h\mathbf{T}_h^{-H}$ was used as a preconditioner for the Galerkin matrix $\mathbf{A}_h$; please refer to Theorem 1.1 for the definition of the matrices. The spaces X_h and Y_h in Cases A-D are chosen as defined in Section 3, see Table 3.1 for a summary. Throughout the locally supported canonical basis functions illustrated in Figures 3.1– 3.4 were used for the computations. We add that t agrees with the inner product in $L^2(\Gamma)$, which renders the $\mathbf{T}_h$'s sparse "primal-dual mass matrices". Semi-analytic formulas were used for the computations of the entries of the Galerkin matrices for integral operators with singular kernels. For numerical integration we used (tensor product) Gauss-Legendre quadrature of order eight[1].

In the numerical experiments, we monitor the spectral condition numbers $\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$ and $\kappa(\mathbf{M}_h\mathbf{A}_h)$ for sequences of meshes with increasing number of nodes. Here $\mathbf{D}_h$ stands for the diagonal part of $\mathbf{A}_h$. In addition, we recorded the number of iterations it took the preconditioned conjugate gradient method[2] to achieve a reduction of the residual norm by a factor 10^{10}. Initial guess was zero and the right hand side vectors hat entries $+1$ in its upper half, -1 for the remaining components. In some cases we also plot the spectrum of $\mathbf{M}_h\mathbf{A}_h$ for different meshes.

5.1. Weakly singular operator (Case A, row 1 of Table 3.1). In this case $\mathbf{A}_h$ is related to the weakly singular operator V, whereas $\mathbf{B}_h$ arises from the modified hypersingular operator $\bar{\mathsf{W}}$. The results are documented in Table 5.1 and reveal that the new operator preconditioning strategy achieves condition numbers that are essentially independent of the resolution of the meshes. Moreover, in Figure 5.1 we observe pronounced and mesh-independent clustering of the eigenvalues of the preconditioned matrices.

For comparison we include results obtained for operator preconditioning with "operators of opposite order" in the spirit of [19]. There $\mathbf{B}_h$ is replaced with Galerkin matrices associated with the *unmodified* hypersingular operator W. For its discretization we used two different boundary element spaces:

1. We may choose the trial and test space $\mathcal{S}_0^{0,1}(\widehat{\Gamma}_h)$ on the dual mesh (from Case **C**, see Figure 3.3) and obtain the Galerkin matrix $\widehat{\mathbf{B}}_h^C$. This spawns the preconditioning matrix $\widehat{\mathbf{M}}_h^C := \mathbf{T}_h^{-1}\widehat{\mathbf{B}}_h^C\mathbf{T}_h^{-T}$.
2. We may also refrain from enforcing zero boundary conditions and use the space $\bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h)$ on the dual mesh (from Case **A**, see Figure 3.1). However, by doing this, we will end up with a singular Galerkin matrix $\widehat{\mathbf{B}}_h^A$. Consequently, we have to regularize it by adding a rank-1 correction that removes the kernel. To state it, we write $\mathbf{g}_1$ and $\mathbf{g}_\omega$ for the column vectors that arise from

[1] The numerical experiments presented in this section were performed with MATLAB R2013a, 64-bit.

[2] We used the implementation of the preconditioned conjugate gradient method provided by MATLAB's `pcg` function.

the Galerkin discretization of the linear forms $v \mapsto \langle 1\,,\, v\rangle$ and $v \mapsto \langle \omega\,,\, v\rangle$, respectively, on $\bar{\mathcal{S}}^{0,1}(\hat{\Gamma}_h)$. Then the correction can be implemented by adding the matrices $\mathbf{g}_1\mathbf{g}_1^T$ or $\mathbf{g}_\omega\mathbf{g}_\omega^T$. This yields the following matrix representations of the preconditioners:

$$\widehat{\mathbf{M}}_{1h}^A := \mathbf{T}_h^{-1}\left(\widehat{\mathbf{B}}_h^A + \mathbf{g}_1\mathbf{g}_1^T\right)\mathbf{T}_h^{-T}, \quad \widehat{\mathbf{M}}_{\omega h}^A := \mathbf{T}_h^{-1}\left(\widehat{\mathbf{B}}_h^A + \mathbf{g}_\omega\mathbf{g}_\omega^T\right)\mathbf{T}_h^{-T},$$

For all these preconditioners we still expect a logarithmic growth of $\kappa(\widehat{\mathbf{M}}_{*h}^*\mathbf{A}_h)$. The measured condition numbers are listed in Table 5.2 and display the expected moderate growth as the meshes are refined. Obviously, judged by the condition numbers, our new preconditioner $\mathbf{M}_h$ is superior to any $\widehat{\mathbf{M}}_{*h}^*$. The gain in terms of speed of convergence of the CG iteration is not as impressive.

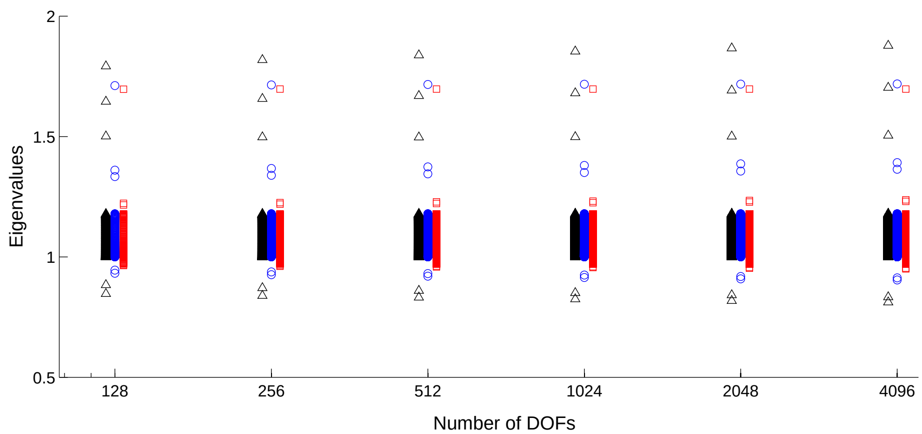
Table 5.1: Performance of preconditioners for V_h (Case A, row 1 of Table 3.1).

N	Uniform mesh			Chebychev mesh			Algebraic mesh		
	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$
	Spectral Condition numbers κ								
128	272.8	26.56	2.117	314.2	57	1.836	392.7	94.62	1.760
256	547.7	30.51	2.167	660.8	64.53	1.852	845.6	108.2	1.765
512	1098	33.95	2.210	1375	72.03	1.865	1794	122.1	1.771
1024	2198	37.11	2.248	2841	79.62	1.878	3765	136.2	1.776
2048	4397	40.13	2.282	5837	87.34	1.890	7843	150.7	1.781
4096	8797	43.08	2.313	11950	95.17	1.901	16240	165.4	1.785
	Numbers of PCG iterations								
128	56	11	10	67	13	10	63	15	8
256	77	12	10	98	14	10	90	15	8
512	106	12	10	140	14	11	127	15	8
1024	145	12	10	205	14	11	177	16	8
2048	201	12	10	290	15	11	249	16	8
4096	273	12	10	417	15	11	347	16	8

Table 5.2: Results for operator preconditioning of V_h (Case A) with different (regularized) discrete versions of the unmodified hypersingular operator W

N	Uniform mesh			Chebychev mesh			Algebraic mesh		
	$\widehat{\mathbf{M}}_{1h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_{\omega h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$	$\widehat{\mathbf{M}}_{1h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_{\omega h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$	$\widehat{\mathbf{M}}_{1h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_{\omega h}^A\mathbf{A}_h$	$\widehat{\mathbf{M}}_h^C\mathbf{A}_h$
	Spectral Condition numbers κ								
128	7.113	7.035	26.56	9.661	9.638	57	16.18	16.17	94.62
256	7.714	7.648	30.51	11.61	11.59	64.53	20.38	20.38	108.2
512	8.370	8.311	33.95	13.75	13.75	72.03	25.08	25.08	122.1
1024	9.082	9.028	37.11	16.12	16.12	79.62	30.29	30.29	136.2
2048	9.845	9.796	40.13	18.68	18.67	87.34	36.01	36.01	150.7
4096	10.66	10.61	43.08	21.43	21.42	95.17	42.25	42.25	165.4
	Numbers of PCG iterations								
128	11	11	11	13	12	13	15	14	15
256	11	11	12	14	13	14	16	15	15
512	12	11	12	14	14	14	16	16	15
1024	12	11	12	15	14	14	17	16	16
2048	12	12	12	16	15	15	18	17	16
4096	12	12	12	16	15	15	18	17	16

Fig. 5.1: Plot of spectrum for V_h preconditioned by $\mathbf{M}_h$ (Case A, row 1 of Table 3.1) for our three meshes. We use black triangles to show the eigenvalues when using an uniform mesh, blue circles for the Chebychev mesh, and red squares for the algebraically graded mesh.



5.2. Hypersingular operator (Case B, row 2 of Table 3.1). Now, $\mathbf{A}_h$ is the Galerking matrix corresponding to the hypersingular operator W, and the modified weakly singular operator $\bar{\mathsf{V}}$ gives rise to $\mathbf{B}_h$. As before, we also compare with the sub-optimal "opposite order" preconditioner $\widehat{\mathbf{M}}_h$ obtained by replacing $\bar{\mathsf{V}}$ with the unmodified operator V.

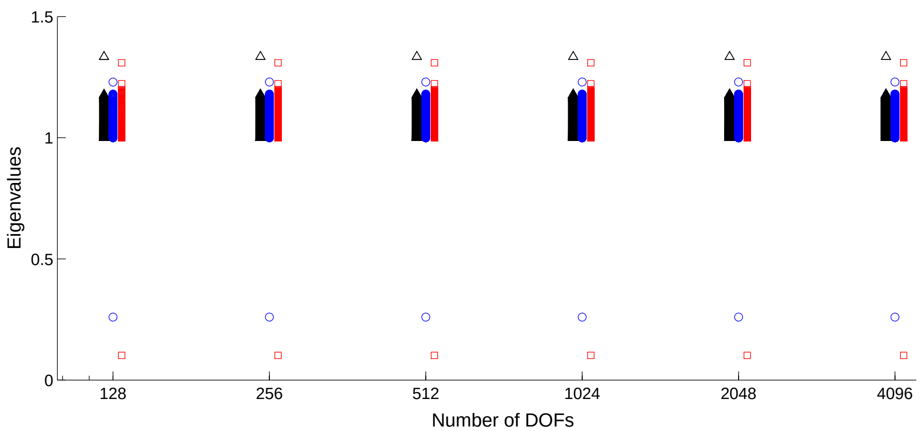
Mesh-independent performance of the new preconditioner and its superiority to other approaches is confirmed by the data of Table 5.3. These results are consistent with the plot of the spectrum for $\mathbf{M}_h\mathbf{A}_h$, which is shown in Figure 5.2.

Table 5.3: Performance of preconditioners for for W_h (Case B, row 2 of Table 3.1).

	Uniform mesh			Chebychev mesh			Algebraic mesh		
N	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$	$\mathbf{D}_h^{-1}\mathbf{A}_h$	$\widehat{\mathbf{M}}_h\mathbf{A}_h$	$\mathbf{M}_h\mathbf{A}_h$
128	62.1	6.16	1.335	46.46	13.27	4.729	42.07	54.72	12.89
256	124.8	7.003	1.335	93.28	16.36	4.731	84.51	69.87	12.89
512	250.2	7.902	1.335	186.9	19.79	4.732	169.7	86.95	12.89
1024	500.9	8.861	1.335	374.2	23.58	4.732	341.3	105.8	12.90
2048	1002	9.879	1.335	748.8	27.71	4.732	682.8	126.7	12.90
4096	2006	10.96	1.335	1499	32.2	4.732	1366	149.6	12.90
	Numbers of PCG iterations								
128	28	9	9	26	13	9	27	16	10
256	40	9	8	37	14	9	39	17	10
512	58	10	8	52	15	9	57	17	10
1024	84	10	8	74	16	9	110	18	10
2048	119	12	8	101	16	9	156	18	10
4096	169	11	8	136	16	9	222	20	10

5.3. Modified weakly singular operator (Case C, row 3 of Table 3.1). Although one would not try to solve an equation associated to $\bar{\mathsf{V}}$, it is interesting to also study the preconditioning strategy related to the case when $\mathbf{A}_h$ and $\mathbf{B}_h$ arise

Fig. 5.2: Plot of spectrum for W_h preconditioned by $\mathbf{M}_h$ (Case B, row 2 of Table 3.1) for our three meshes. We use black triangles to show the eigenvalues when using an uniform mesh, blue circles for the Chebychev mesh, and red squares for the algebraically graded mesh.



from $\bar{\mathsf{V}}$ and W, respectively. For this reason we just show the obtained condition numbers in Table 5.4, where we observe the growth of the condition number is once again minimal.

Table 5.4: Spectral condition numbers obtained for $\bar{\mathsf{V}}_h$ (Case C, row 3 of Table 3.1).

	Uniform mesh		Chebychev mesh		Algebraic mesh	
N	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$
128	210	8.726	154	11.78	111.5	13.01
256	420.4	9.195	308.5	11.82	223.8	13.01
512	840	9.442	616.7	11.83	448.1	13.01
1024	1677	9.569	1232	11.83	895.9	13.01
2048	3348	9.634	2459	11.83	1812	13.01
4096	6684	9.666	4911	11.84	3675	13.01

5.4. Modified hypersingular operator (Case D, row 4 of Table 3.1). For this final case, $\mathbf{A}_h$ is related to $\bar{\mathsf{W}}$ and $\mathbf{B}_h$ comes from V. Once again, since the operator is not related to a BVP, we are just interested in studying the obtained condition numbers. The results in Table 5.5 support that our preconditioner is performs excellently independent of the (locally refined) mesh.

5.5. Boundary integral operators for Helmholtz equation. In order to precondition the Galerkin matrices spawned by V_h^k and W_h^k from (2.14), we use the splittings (2.16) and pursue the same strategy that we used for V_h (case A) and W_h (case B), respectively. For the sake of clarity, we will denote by $\mathbf{M}_h$ the preconditioner arising from $\bar{\mathsf{W}}_h$, and $\widetilde{\mathbf{M}}_h$ the one related to $\bar{\mathsf{V}}_h$. Table 5.6 gives measurements of numbers of GMRES iterations for the diagonally scaled and operator preconditioned Helmholtz operators using *uniform* mesh and for different wave numbers k.

Table 5.5: Spectral condition numbers obtained for $\bar{\mathsf{W}}_h$ (Case D, row 4 of Table 3.1).

	Uniform mesh		Chebychev mesh		Algebraic mesh	
N	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$	$\kappa(\mathbf{D}_h^{-1}\mathbf{A}_h)$	$\kappa(\mathbf{M}_h\mathbf{A}_h)$
128	54.17	1.695	51.53	1.693	72.14	1.741
256	108.9	1.694	103.3	1.693	144.2	1.744
512	218.4	1.694	206.8	1.693	288.2	1.744
1024	437.3	1.693	413.9	1.693	576.4	1.745
2048	875.2	1.693	828.1	1.693	1153	1.745
4096	1751	1.693	1657	1.693	2305	1.745

Observe that for fixed k, the number of iterations for each operator becomes almost independent of N, when our new operator preconditioning approach is applied.

Table 5.6: GMRES iteration counts for Helmholtz operators using $k = 1, 4, 8$ as wave numbers.

	Case A						Case B					
N	V_h^1	$\mathbf{M}_h\mathsf{V}_h^1$	V_h^4	$\mathbf{M}_h\mathsf{V}_h^4$	V_h^8	$\mathbf{M}_h\mathsf{V}_h^8$	W_h^1	$\widetilde{\mathbf{M}}_h\mathsf{W}_h^1$	W_h^4	$\widetilde{\mathbf{M}}_h\mathsf{W}_h^4$	W_h^8	$\widetilde{\mathbf{M}}_h\mathsf{W}_h^8$
128	43	10	44	14	44	19	26	9	24	12	21	16
256	55	10	55	14	56	19	38	8	36	12	31	16
512	69	10	70	14	70	19	56	8	51	12	44	16
1024	87	11	87	14	87	19	79	8	73	12	63	15
2048	109	11	109	14	109	19	112	7	104	12	89	15
4096	136	11	136	14	136	19	158	7	147	11	126	15

6. Conclusions. We have demonstrated, both through rigorous analysis and numerical tests, the great potential of the Jerez-Nédélec Calderón-type identities for boundary integral operators on the line segment to induce asymptotically optimal operator preconditioners for low-order Galerkin BEM for the BIE arising from (1.1) on any smooth open curve. In particular, highly non-uniform meshes were covered. Forthcoming work will extend this preconditioning strategy to two-dimensional screens in three dimensions.

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Appendix A. Proof of Proposition 2.4.

We first introduce the following auxiliary lemma

Lemma A.1. *The following equalities hold*

$$\left\|\omega^{-1}\right\|_{\widetilde{H}^{-1/2}(\Gamma)} = \|1\|_{H^{1/2}(\Gamma)} = 1. \tag{A.1}$$

Proof. From [12], if $\psi(x) \in \widetilde{H}^{-1/2}(\Gamma)$, it can be expanded as

$$\psi(x) = \sum_{n=0}^{\infty} \psi_n \frac{T_n(x)}{\omega(x)}, \qquad x \in [-1,1].$$

and its norm is given by

$$\|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} = \psi_0^2 + \sum_{n=1}^{\infty} \frac{1}{n}\psi_n^2.$$

Since $\omega^{-1}(x) = 1\dfrac{T_0(x)}{\omega(x)}$, we have shown the first assertion. Similarly, from [12], if $g(x) \in H^{1/2}(\Gamma)$, the following expansion follows

$$g(x) = \sum_{n=0}^{\infty} g_n T_n(x), \qquad x \in [-1,1].$$

and its norm is given by

$$\|g\|_{H^{1/2}(\Gamma)} = g_0^2 + \sum_{n=1}^{\infty} n g_n^2.$$

Thus, $1(x) = 1T_0(x)$, from were we conclude our proof. ☐

Proof of Proposition 2.4. Recall we first want to prove that the augmented operators pencil $\widetilde{\mathsf{V}}[\alpha]$ and $\widetilde{\widetilde{\mathsf{W}}}[\beta]$ are bounded and elliptic in $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively, for $\alpha, \beta \in \mathbb{R}_+$ bounded.

Notice that $\widetilde{\mathsf{a}}_{\mathsf{V}}[\alpha]$ is well defined by continuity and linearity of both V and the duality product, thus showing the boundedness of $\widetilde{\mathsf{V}}[\alpha]$. One can derive

$$\begin{aligned} \langle 1\,,\,\varphi\rangle &\leq \|1\|_{H^{1/2}(\Gamma)} \|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,, \\ \text{(Lemma } A.1) \quad &\leq \|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,. \end{aligned}$$

From this we obtain the following bound

$$\begin{aligned} |\langle \mathsf{V}\,\varphi\,,\,\phi\rangle + \alpha\,\langle 1\,,\,\varphi\rangle\,\langle 1\,,\,\phi\rangle| &\leq c_2^{\mathsf{V}}\,\|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,\|\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \\ &\quad + \alpha\,\|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,\|\phi\|_{\widetilde{H}^{-1/2}(\Gamma)} \\ &= (c_2^{\mathsf{V}} + \alpha)\,\|\varphi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,\|\phi\|_{\widetilde{H}^{-1/2}(\Gamma)}\,, \end{aligned}$$

and we define the continuity constant

$$c_2^{\widetilde{\mathsf{V}}}(\alpha) \;:=\; c_2^{\mathsf{V}} + \alpha. \tag{A.2}$$

Now, consider the unique decomposition for $\varphi \in \widetilde{H}^{-1/2}(\Gamma)$:

$$\varphi = \tilde{\varphi} + \zeta\omega^{-1}, \qquad \text{where} \quad \widetilde{\varphi} \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma) \qquad \text{and} \qquad \zeta = \langle 1\,,\,\varphi\rangle\,. \tag{A.3}$$

Given $\widetilde{\varphi} \in \widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma)$, one can prove that $\widetilde{\mathsf{V}}[\alpha]\widetilde{\varphi} = \mathsf{V}\widetilde{\varphi} \in H^{1/2}_*(\Gamma)$ in the weak sense, whence $\left\langle \widetilde{\mathsf{V}}[\alpha]\widetilde{\varphi}\,,\, \omega^{-1} \right\rangle$ is equal to zero. From this we deduce

$$
\begin{aligned}
\left\langle \widetilde{\mathsf{V}}[\alpha]\varphi\,,\, \varphi \right\rangle &= \left\langle \widetilde{\mathsf{V}}[\alpha](\widetilde{\varphi} + \zeta\omega^{-1})\,,\, \widetilde{\varphi} + \zeta\omega^{-1} \right\rangle \\
&= \left\langle \widetilde{\mathsf{V}}[\alpha]\widetilde{\varphi}\,,\, \widetilde{\varphi} \right\rangle + 2\zeta \left\langle \widetilde{\mathsf{V}}[\alpha]\widetilde{\varphi}\,,\, \omega^{-1} \right\rangle + \zeta^2 \left\langle \widetilde{\mathsf{V}}[\alpha]\omega^{-1}\,,\, \omega^{-1} \right\rangle \\
&= \langle \mathsf{V}\,\widetilde{\varphi}\,,\, \widetilde{\varphi} \rangle + \zeta^2 \left\langle \widetilde{\mathsf{V}}[\alpha]\omega^{-1}\,,\, \omega^{-1} \right\rangle \\
&\geq c_1^{\mathsf{V}} \, \|\widetilde{\varphi}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \zeta^2 \left\langle \widetilde{\mathsf{V}}[\alpha]\omega^{-1}\,,\, \omega^{-1} \right\rangle \\
&\geq \min\left\{ c_1^{\mathsf{V}}, \left\langle \widetilde{\mathsf{V}}[\alpha]\omega^{-1}\,,\, \omega^{-1} \right\rangle \right\} \left(\|\widetilde{\varphi}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \zeta^2 \right),
\end{aligned}
\tag{A.4}
$$

where c_1^{V} is the ellipticity constant of V [12]. Since $\mathsf{V}\,\omega^{-1} = \pi \log 2 = C_\omega$, C_ω being a positive constant, the duality product on the second term on the right-hand side in (A.4) is equal to

$$
\begin{aligned}
\left\langle \widetilde{\mathsf{V}}[\alpha]\omega^{-1}\,,\, \omega^{-1} \right\rangle &= \left\langle \mathsf{V}\,\omega^{-1}\,,\, \omega^{-1} \right\rangle + \alpha \left\langle 1\,,\, \omega^{-1} \right\rangle^2 \\
&= \pi^2(\log 2 + \alpha) > 0,
\end{aligned}
\tag{A.5}
$$

by Chebychev's polynomial's properties on I. Futhermore, the decomposition (A.3) satisfies

$$
\begin{aligned}
\|\varphi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} &= \left\| \widetilde{\varphi} + \zeta\omega^{-1} \right\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\leq \left(\|\widetilde{\varphi}\|_{\widetilde{H}^{-1/2}(\Gamma)} + \zeta \left\|\omega^{-1}\right\|_{\widetilde{H}^{-1/2}(\Gamma)} \right)^2 \\
&\leq 2 \left(\|\widetilde{\varphi}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \zeta^2 \left\|\omega^{-1}\right\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \right) \\
\text{(by Lemma A.1)} &\leq 2 \left(\|\widetilde{\varphi}\|^2_{\widetilde{H}^{-1/2}(\Gamma)} + \zeta^2 \right).
\end{aligned}
\tag{A.6}
$$

Then, combining (A.5), (A.4), and (A.6), ellipticity follows with constant

$$
c_1^{\widetilde{\mathsf{V}}}(\alpha) := \frac{\min\left\{ c_1^{\mathsf{V}}, \pi^2 \left(\log 2 + \alpha\right) \right\}}{2}, \qquad \text{for all } \alpha > 0. \tag{A.7}
$$

On the other hand, using similar steps to those employed in (A.2), the sesquilinear form $\mathsf{a}_{\bar{\mathsf{W}}}[\beta](w, v)$ has a continuity constant equal to

$$
c_2^{\widetilde{\bar{\mathsf{W}}}}(\beta) \;:=\; c_2^{\bar{\mathsf{W}}} + \beta, \tag{A.8}
$$

for β fixed. Moreover, for $v \in H^{1/2}(\Gamma)$, the following decomposition holds

$$
v = v_* + \eta \qquad \text{where} \quad v_* \in H^{1/2}_*(\Gamma) \qquad \text{and} \qquad \eta = \left\langle v\,,\, \omega^{-1} \right\rangle. \tag{A.9}
$$

Since $\bar{\mathsf{W}}\rho = 0$ for any constant ρ, we consider $\mathrm{Ker}\bar{\mathsf{W}} = \mathrm{span}\{1\}$. In order to show ellipticity, we recall $g \in \widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma)$ and use $v_0 = 1$ as a test function. Consequently,

$$
\begin{aligned}
\left\langle \widetilde{\bar{\mathsf{W}}} v_*\,,\, 1 \right\rangle &= \left\langle \bar{\mathsf{W}} v_*\,,\, 1 \right\rangle + \beta \left\langle v_*\,,\, \omega^{-1} \right\rangle \left\langle 1\,,\, \omega^{-1} \right\rangle \\
&= \left\langle v_*\,,\, \bar{\mathsf{W}} 1 \right\rangle + \beta \left\langle v_*\,,\, \omega^{-1} \right\rangle \left\langle 1\,,\, \omega^{-1} \right\rangle = 0.
\end{aligned}
$$

Thus, as in (A.5), we observe

$$\begin{aligned}\left\langle \widetilde{\bar{\mathsf{W}}} v\,,\, v\right\rangle &= \left\langle \widetilde{\bar{\mathsf{W}}}(v_* + \eta)\,,\, v_* + \eta\right\rangle \\ &\geq \min\left\{c_1^{\bar{\mathsf{W}}}, \left\langle \widetilde{\bar{\mathsf{W}}} 1\,,\, 1\right\rangle\right\}\left(\|v_*\|^2_{H^{1/2}(\Gamma)} + \eta^2\right),\end{aligned}$$

by ellipticity of $\bar{\mathsf{W}}$ with constant $c_1^{\bar{\mathsf{W}}}$, and derive

$$\left\langle \widetilde{\bar{\mathsf{W}}} 1\,,\, 1\right\rangle = \left\langle \bar{\mathsf{W}} 1\,,\, 1\right\rangle + \beta\left\langle 1\,,\, \omega^{-1}\right\rangle\left\langle 1\,,\, \omega^{-1}\right\rangle = \pi^2\beta > 0.$$

Then, considering $\|1\|_{H^{1/2}(\Gamma)} = 1$, one can prove in an analogous way to (A.6) that

$$\begin{aligned}\|v\|^2_{H^{1/2}(\Gamma)} &= \|v_* + \eta\|^2_{H^{1/2}(\Gamma)} \\ &\leq 2\left(\|v_*\|^2_{H^{1/2}(\Gamma)} + \eta^2\right),\end{aligned}$$

and so ellipticity of $\widetilde{\bar{\mathsf{W}}}[\beta]$ follows with constant

$$c_1^{\widetilde{\bar{\mathsf{W}}}}(\beta) := \frac{\min\left\{c_1^{\bar{\mathsf{W}}}, \pi^2\beta\right\}}{2}, \quad \text{for all } \beta > 0. \tag{A.10}$$

In what follows, we are interested in showing that the augmented variational problems are equivalent to their original problems. Our first departure problem is to find $\phi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$ such that (2.6) is satisfied for all $\psi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$. Instead of solving problem (2.6) with a constraint, consider the following saddle point problem: find $(\phi, \lambda) \in \widetilde{H}^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\langle \mathsf{V}\,\phi\,,\, \psi\rangle + \lambda\langle 1\,,\, \psi\rangle = \langle f\,,\, \psi\rangle, \qquad \forall \psi \in \widetilde{H}^{-1/2}(\Gamma), \tag{A.11a}$$

$$\langle 1\,,\, \phi\rangle = 0. \tag{A.11b}$$

Again, since $\mathsf{V}\,\omega^{-1} = C_\omega$, $C_\omega > 0$, we have $\left\langle \mathsf{V}\,\omega^{-1}\,,\, \psi\right\rangle = 0$ for all $\psi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$. On the other hand, as it is shown in [12, Proposition 3.1] $f \in \mathrm{Im}_{\widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)}(\mathsf{V}) \equiv H^{1/2}_*(\Gamma)$. Hence, we can use ω^{-1} as a test function in (A.11a) to derive

$$\left\langle \phi\,,\, \mathsf{V}\,\omega^{-1}\right\rangle + \lambda\left\langle 1\,,\, \omega^{-1}\right\rangle = 0,$$

where we have used the symmetry of V. Finally, by (A.11b) we obtain

$$\lambda\left\langle 1\,,\, \omega^{-1}\right\rangle = 0,$$

and therefore $\lambda \equiv 0$. Consequently, the saddle point problem (A.11) is equivalent to finding $(\phi, \lambda) \in \widetilde{H}^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned}\langle \mathsf{V}\,\phi\,,\, \psi\rangle + \lambda\langle 1\,,\, \psi\rangle &= \langle f\,,\, \psi\rangle, \qquad \forall\, \psi \in \widetilde{H}^{-1/2}(\Gamma), \\ \langle 1\,,\, \phi\rangle \quad - \lambda/\alpha &= 0,\end{aligned} \tag{A.12}$$

with $\alpha \in \mathbb{R}_+$ a parameter to be chosen later. We obtain a new augmented variational problem by eliminating λ from (A.12): find $\phi \in \widetilde{H}^{-1/2}(\Gamma)$ such that

$$\widetilde{\mathsf{a}}_{\mathsf{V}}[\alpha](\phi, \psi) = \langle f\,,\, \psi\rangle, \qquad \forall\, \psi \in \widetilde{H}^{-1/2}(\Gamma), \quad \alpha > 0, \tag{A.13}$$

where $\widetilde{\mathsf{a}}_{\mathsf{V}}[\alpha] : \widetilde{H}^{-1/2}(\Gamma) \times \widetilde{H}^{-1/2}(\Gamma) \to \mathbb{C}$ is the sesquilinear form pencil associated to the augmented weakly singular operator $\widetilde{\mathsf{V}}[\alpha]$ defined as

$$\left\langle \widetilde{\mathsf{V}}[\alpha]\varphi\,,\,\psi \right\rangle := \langle \mathsf{V}\,\varphi\,,\,\psi\rangle + \alpha\,\langle 1\,,\,\varphi\rangle\,\langle 1\,,\,\psi\rangle\,, \qquad \forall\,\varphi,\psi \in \widetilde{H}^{-1/2}(\Gamma), \tag{A.14}$$

with clear duality pairing.

Lastly, ellipticity allows the use of the Lax-Milgram lemma to guarantee uniqueness and existence of solutions for (A.13). Now, since $f \in H_*^{1/2}(\Gamma)$, when testing (A.13) with $\psi = \omega^{-1}$ we obtain

$$C_\omega\,\langle 1\,,\,\varphi\rangle + \alpha\,\langle 1\,,\,\varphi\rangle \left\langle 1\,,\,\omega^{-1}\right\rangle = 0, \qquad \text{with} \quad \left\langle 1\,,\,\omega^{-1}\right\rangle > 0, \tag{A.15}$$

and consequently, $\varphi \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$, thereby proving the equivalence between the augmented problem (A.13) and the original one (2.6).

Now we focus on $\widetilde{\bar{\mathsf{W}}}[\beta]$. Instead of solving problem (2.9) with a constraint, we consider the following saddle point problem: find $(w, \Lambda) \in H^{1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \left\langle \bar{\mathsf{W}}w\,,\,v\right\rangle + \Lambda\left\langle \omega^{-1}\,,\,v\right\rangle &= \langle g\,,\,v\rangle\,, \qquad \forall\,v \in H^{1/2}(\Gamma), && \text{(A.16a)}\\ \left\langle w\,,\,\omega^{-1}\right\rangle &= 0. && \text{(A.16b)} \end{aligned}$$

The proof follows analogously to the previous case, only this time we use the fact that $\mathrm{Ker}\bar{\mathsf{W}} = \mathrm{span}\{1\}$ and $v_0 = 1$ as a test function in (A.16a). Then, since $g \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$ we get

$$\Lambda\left\langle \omega^{-1}\,,\,1\right\rangle = 0,$$

and therefore $\Lambda \equiv 0$. From this, and using similar arguments as before, we derive the augmented variational problem. The uniqueness and existence for any $g \in \widetilde{H}^{-1/2}_{\langle 0\rangle}(\Gamma)$ follows from the ellipticity as well. In particular, when testing with $v = 1$, we deduce

$$\beta\left\langle w\,,\,\omega^{-1}\right\rangle\left\langle 1\,,\,\omega^{-1}\right\rangle = 0, \qquad \left\langle 1\,,\,\omega^{-1}\right\rangle > 0, \tag{A.17}$$

from where w must lie in $H_*^{1/2}(\Gamma)$,thus showing the equivalence between the augmented and original problem (2.9).
□

Appendix B. Stability results for $X_h = \mathcal{S}_0^{0,1}(\Gamma_h) \subset X = \widetilde{H}^{1/2}$, $Y_h = \bar{\mathcal{S}}^{-1,0}(\hat{\Gamma}_h) \subset Y = H^{-1/2}$ (case B).

As a tool we rely on the "tent functions" $b_k \in \mathcal{S}^{0,1}(\Gamma_h)$, $k = 1, \ldots, N-2$, defined by $b_k(x_i) = \delta_{(k+1)i}$ (Kronecker symbol), and we write $\omega_k := \mathrm{supp}(b_k)$, which consists of two adjacent mesh intervals for $k = 1, \ldots, N-2$. Refer to Figure 3.2 for an illustration (drawn in blue). Moreover, we employ the piecewise constant functions $q_j \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h)$, $j = 1, \ldots, N-2$ which are equal to 1 on (η_{j-1}, η_j) and vanish outside this interval of the dual mesh, see Figure 3.2 (red/green).

B.1. Proof of Lemma 4.4. We use our trial bases to write $w_h := \sum_{j=1}^{N-2} w_i b_i \in \mathcal{S}^{0,1}(\Gamma_h)$, and $\psi_h := \sum_{j=1}^{N-2} \psi_j q_j \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h)$. Hence, we can write the dual product operator in its matricial form

$$|\langle \psi_h\,,\,w_h\rangle| := \left|\boldsymbol{\psi}^T\,\mathsf{T}_B\,\mathbf{w}\right|, \tag{B.1}$$

where $\mathsf{T}_B[l,k] := \langle b_l\,,\,q_k\rangle$. Furthermore, $\mathsf{T}_B \in \mathbb{R}^{N-2,N-2}$ is a tridiagonal matrix given by the following expression

$$\mathsf{T}_B = \frac{1}{8}\begin{pmatrix} h_1+3c_1 & h_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \vdots & & \ddots & & \ddots & & \vdots & 0 \\ 0 & 0 & \cdots & h_i & 3c_i & h_{i+1} & \cdots & 0 & 0 \\ 0 & \vdots & & \ddots & & \ddots & & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & h_{N-2} & 3c_{N-2}+h_{N-1} \end{pmatrix},$$

where $h_i = x_{i+1} - x_i > 0$ and $c_i := h_i + h_{i+1} > 0$. Observe T_B is a strict diagonally dominant matrix, futhermore it is symmetric. Then, by Gershgorin we know its eigenvalues $\lambda_1, ..., \lambda_k, ...\lambda_{N-2} \in \mathbb{R}$ satisfy

$$\lambda_k \in (3c_1, 5h_1+3h_2) \cup \bigcup_{i=2}^{N-3}(2c_i, 4c_i) \cup (3c_{N-2}, 5h_{N-2}+3h_{N-1}),\ \forall\, k = 1,\dots,N-2.$$

Consequently, $\lambda_{min} > 0$ and T_B is invertible. Recall $w_h := \sum_{i=1}^{N-2} w_i b_i$. Now let $\psi_h^* := \sum_{i=1}^{N-2} w_i q_i$. Then, by setting $w_0 = w_{N-1} = 0$, we have

$$\begin{aligned}\langle \psi_h^*\,,\,w_h\rangle = \mathbf{w}^T\,\mathsf{T}_B\,\mathbf{w} = \frac{1}{8}\Big\{&w_1((4h_1+3h_2)w_1 + h_2w_2)\\ &+\sum_{i=2}^{N-3} w_i(h_iw_{i-1} + 3(h_i+h_{i+1})w_i + h_{i+1}w_{i+1})\\ &+ w_{N-2}(h_{N-2}w_{N-3} + (3h_{N-2}+4h_{N-1})w_{N-2})\Big\}\\ = \frac{1}{8}\Big\{&\sum_{i=1}^{N-2}(h_i+h_{i+1})w_i^2 + h_1w_1^2 + h_{N-1}w_{N-2}^2\Big\}\\ + \frac{1}{8}\Big\{&w_1(2(h_1+h_2)w_1 + h_2w_2)\\ &+\sum_{i=2}^{N-3} w_i(h_iw_{i-1} + 2(h_i+h_{i+1})w_i + h_{i+1}w_{i+1})\\ &+ w_{N-2}(h_{N-2}w_{N-3} + 2(h_{N-2}+h_{N-1})w_{N-2})\Big\},\end{aligned} \quad \text{(B.2)}$$

where the last expression on the right hand side can be seen as $\mathbf{w}^T\tilde{\mathsf{T}}\mathbf{w}$, with $\tilde{\mathsf{T}}$ denoting a strict diagonally dominant matrix. Hence, that last term is positive and we can bound (B.2) by

$$\langle \psi_h^*\,,\,w_h\rangle \geq \frac{1}{8}\Big\{\sum_{i=1}^{N-2}(h_i+h_{i+1})w_i^2 + h_1w_1^2 + h_{N-1}w_{N-2}^2\Big\} \geq \frac{1}{3}(D_B\mathbf{w},\mathbf{w}), \quad \text{(B.3)}$$

where $D_B = \mathrm{diag}(\mathsf{T}_B)$. On the other hand,

$$\|w_h\|^2_{L^2(\Gamma)} = \langle w_h\,,\,w_h\rangle = \frac{1}{3}\mathbf{w}^T\mathrm{diag}(h_i+h_{i+1})\mathbf{w} \leq \frac{8}{9}\mathbf{w}^T\mathrm{diag}(\mathsf{T}_B)\mathbf{w} = \frac{8}{9}(D_B\mathbf{w},\mathbf{w}), \quad \text{(B.4)}$$

and

$$\begin{aligned}\|\psi_h^*\|^2_{L^2(\Gamma)} &= \langle \psi_h^* \,,\, \psi_h^* \rangle \\ &= \frac{1}{2}\mathbf{w}^T \begin{pmatrix} (2h_1+h_2) & 0 & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots & 0 \\ \cdots & 0 & (h_i+h_{i+1}) & 0 & \cdots \\ 0 & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 0 & 0 & (h_{N-2}+2h_{N-1}) \end{pmatrix} \mathbf{w} \\ &\leq \frac{1}{2}\mathbf{w}^T \mathrm{diag}(\mathsf{T}_B)\mathbf{w} = \frac{1}{2}(D_B\mathbf{w},\mathbf{w}). \end{aligned} \tag{B.5}$$

Finally, by combining (B.3), (B.5) and (B.4), we obtain

$$\begin{aligned}\sup_{\psi_h \in \mathcal{S}^{-1,0}(\Gamma_h)} \frac{|\langle \psi_h \,,\, w_h \rangle|}{\|\psi_h\|_{L^2(\Gamma)}} &\geq \frac{\langle \psi_h^* \,,\, w_h \rangle}{\|\psi_h^*\|_{L^2(\Gamma)}} \\ &\geq \frac{\sqrt{2}}{3}(D_B\alpha,\alpha)^{1/2} \\ &\geq \sqrt{\frac{9}{8}}\frac{\sqrt{2}}{3}\,\|w_h\|_{L^2(\Gamma)} = \frac{1}{2}\,\|w_h\|_{L^2(\Gamma)}\,. \end{aligned} \tag{B.6}$$

□

B.2. Proof of Proposition 4.5 . We aim to prove the H^1-stability for $\tilde{Q}_h$. With this purpose in mind, we will introduce a quasi interpolation operator as in [26, Section 1.5].

First recall $\omega_k = \mathrm{supp}\{b_k\}$, then define the related space locally by $X_h(\omega_k) := \{b_j|_{\omega_k} \;:\; b_j \in \mathcal{S}_0^{0,1}(\Gamma_h)\}$. Let Q_h^k denote the Galerkin L^2-Projection onto the local trial space $X_h(\omega_k)$, such that for $u \in L^2(\omega_k)$

$$\left\langle Q_h^k u \,,\, v_h \right\rangle_{L^2(\omega_k)} = \langle u \,,\, v_h \rangle_{L_2(\omega_k)}\,, \qquad \forall\, v_h \,\in\, X_h(\omega_k),\; h \,\in\, \mathbb{H}. \tag{B.7}$$

Due to Assumption 4.1, we have the stability estimate as well as the quasi optimal error estimate

$$\left\|Q_h^k u\right\|_{L^2(\omega_k)} \leq \|u\|_{L^2(\omega_k)}\,, \text{ for all } u \in L^2(\omega_k), \tag{B.8}$$

$$\left\|(\mathsf{Id} - Q_h^k)u\right\|_{L^2(\omega_k)} \leq c_{st}^{loc}\hat{h}_k\,|u|_{H^1(\omega_k)}\,, \text{ for all } u \in H^1(\omega_k). \tag{B.9}$$

Futhermore, local quasi-uniformity gives us the following stability estimate

$$\left\|Q_h^k u\right\|_{H^1(\omega_k)} \leq \tilde{c}_{st}^{loc}\hat{h}_k\,\|u\|_{H^1(\omega_k)} \text{ for all } u \in H^1(\omega_k). \tag{B.10}$$

Then, it is possible to define a quasi interpolation operator by

$$(P_h u)(x) = \sum_{k=1}^{N-2}(Q_h^k u)(x_k)\cdot b_k(x), \tag{B.11}$$

which is also a projection onto $\mathcal{S}_0^{0,1}(\Gamma_h)$. Moreover, P_h have properties which will be key pieces for the proof of Proposition 4.5. We introduce these results in the following two lemmas.

LEMMA B.1 (Extension of Lemma 1.9 [26]). *Let* $u \in H_0^1(\Gamma)$. *Then, there exists a positive constant* c_{p1} *independent of* h *such that*

$$\|(\mathsf{Id} - P_h)u\|_{L^2(\Gamma)} \leq c_{p1} \sum_{k\in J(l)} \hat{h}_k \, |u|_{H^1(\omega_k)}\,, \qquad l = 1, \dots, N-1. \tag{B.12}$$

Moreover,

$$\|P_h u\|_{H^1(\Gamma)} \leq c_{p1} \, \|u\|_{H^1(\Gamma)}\,, \quad \textit{for all } \; u \in H_0^1(\Gamma), \tag{B.13}$$

and

$$\sum_{k=1}^{N-2} \hat{h}_k^{-2} \, \|(\mathsf{Id} - P_h)u\|^2_{L^2(\omega_k)} \leq c_{p1} \, \|u\|_{H^1(\Gamma)}\,, \quad \textit{for all} \, u \in H_0^1(\Gamma). \tag{B.14}$$

Since the only difference with the original Lemma is due to the endpoints, where the arguments involved also hold, proof follows from [26, Lemma 1.9].

LEMMA B.2 (Extension of Lemma 2.3 [26]). *Let condition* (4.4) *be satisfied and* $q_k \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h)$, $k = 1, \dots, N-2$. *Then*

$$\sum_{l=1}^{N-1} h_l^{-2} \, \|v_h\|^2_{L^2(\tau_l)} \leq c_{p2} \sum_{k=1}^{N-2} \left[\frac{\langle v_h \, , \, q_k \rangle_{L^2(\Gamma)}}{\hat{h}_k \, \|q_k\|_{L^2(\Gamma)}} \right]^2, \tag{B.15}$$

for all $v_h \in \mathcal{S}_0^{0,1}(\Gamma_h)$ *with a positive constant* c_{p2}.

Proof. This proof can be derived by adapting Steinbach's original proof (similarly to what it was shown in [3]). Therefore, we introduce his notation for the following two index sets: $I(k) := \{l \in \{1, \dots, N-1\} : \tau_l \cap \omega_k \neq \emptyset\}$ (indices of elements τ_l where b_k is not zero) and $J(l) := \{k \in \{1, \dots, N-2\} : \omega_k \cap \tau_l \neq \emptyset)\}$ (indices of hat functions that do not vanish on τ_l). Since $v_h = \sum_{k=1}^{N-2} v_k b_k \in \mathcal{S}_0^{0,1}(\Gamma_h)$ we can write

$$\begin{aligned}\sum_{l=1}^{N-1} h_l^{-2} \, \|v_h\|_{L^2(\tau_l)} &\leq c_p \sum_{l=1}^{N-1} h_l^{-2} \sum_{k\in J(l)} v_k^2 \, \|b_k\|^2_{L^2(\tau_l)} \\ &\leq c_p \sum_{k=1}^{N-2} v_k^2 \sum_{l\in I(k)} h_l^{-2} \, \|b_k\|^2_{L^2(\tau_l)} = c_p \sum_{k=1}^{N-2} v_k^2 \gamma_k^2,\end{aligned}$$

where $\gamma_k := \sqrt{\sum_{l\in I(k)} h_l^{-2} \, \|b_k\|^2_{L^2(\tau_l)}}$. Setting $x_k := v_k \gamma_k$ this gives

$$\sum_{l=0}^{N-1} h_l^{-2} \, \|v_h\|^2_{L^2(\tau_l)} \leq c_p \, \|\mathbf{x}\|_2^2 \, .$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{N-2} \left[\frac{\langle v_h \, , \, q_k \rangle_{L^2(\Gamma)}}{\hat{h}_k \, \|q_k\|_{L^2(\Gamma)}} \right]^2 &= \sum_{k=1}^{N-2} \left[\sum_{j=1}^{N-2} v_j \frac{\langle b_j \, , \, q_k \rangle_{L^2(\Gamma)}}{\hat{h}_k \, \|q_k\|_{L^2(\Gamma)}} \right]^2 \\ &= \sum_{k=1}^{N-2} \left[\sum_{j=1}^{N-2} x_j \frac{\langle b_j \, , \, q_k \rangle_{L^2(\Gamma)}}{\gamma_j \hat{h}_k \, \|q_k\|_{L^2(\Gamma)}} \right]^2 = \|A\mathbf{x}\|_2^2 \, ,\end{aligned}$$

where A is a matrix given by

$$A := D_q^{-1}\tilde{G}_h D_\gamma^{-1}, \qquad D_q := \operatorname{diag}(\hat{h}_k \, \|q_k\|_{L^2(\omega_k)}), \qquad D_\gamma := \operatorname{diag}(\gamma_k).$$

Let $\bar{G}_h = H^{-1}\tilde{G}_h H$. Define for any $\mathbf{y} \in \mathbb{R}^{N-2}$

$$u_h := \sum_{k=1}^{N-2} h_k y_k b_k \; \in \mathcal{S}_0^{0,1}(\Gamma_h), \qquad \phi_h := \sum_{k=1}^{N-2} h_k^{-1} y_k q_k \; \in \mathcal{S}^{-1,0}(\hat{\Gamma}_h).$$

Then, using

$$(H_l^{-1}\tilde{G}_l H_l \mathbf{x}_l, \mathbf{x}_l) \geq c_0 (D_l \mathbf{x}_l, \mathbf{x}_l) \quad \text{for all } \mathbf{x}_l \in \mathbb{R}^{M_l}, \qquad l = 1...N-1, \tag{B.16}$$

which is transposed to (4.4), we derive the following bound:

$$\begin{aligned}(\bar{G}_h \mathbf{y}, \mathbf{y}) = (H^{-1}\tilde{G}_h H \mathbf{y}, \mathbf{y}) = (\tilde{G}_h H \mathbf{y}, H^{-1}\mathbf{y}) = \langle u_h \, , \, \phi_h \rangle_{L^2(\Gamma)} = \sum_{l=1}^{N-1} \langle u_h \, , \, \phi_h \rangle_{L^2(\tau_l)} \\ = \sum_{l=1}^{N-1} (H_l^{-1}\tilde{G}_l H_l \mathbf{y}_l, \mathbf{y}_l) \geq c_0 \sum_{l=1}^{N-1} (D_l \mathbf{y}_l, \mathbf{y}_l) = c_0 (D\mathbf{y}, \mathbf{y}).\end{aligned}$$

Now, set $D_h^{1/2} := \operatorname{diag}(\|b_k\|_{L^2(\omega_k)})$. From

$$\begin{aligned}c_0 \left\| D_h^{1/2}\mathbf{y} \right\|_2^2 = c_0 (D\mathbf{y}, \mathbf{y}) \leq (\bar{G}_h \mathbf{y}, \mathbf{y}) = (D_h^{-1/2}\bar{G}_h \mathbf{y}, D_h^{1/2}\mathbf{y}) \\ \leq \left\| D_h^{-1/2}\bar{G}_h \mathbf{y} \right\|_2 \left\| D_h^{1/2}\mathbf{y} \right\|_2,\end{aligned}$$

we conclude that

$$c_0 \left\| D_h^{1/2}\mathbf{y} \right\|_2 \leq \left\| D_h^{-1/2}\bar{G}_h \mathbf{y} \right\|_2.$$

Taking $\mathbf{z} := D_\gamma \mathbf{y}$, this is equivalent to

$$c_0 \left\| D_h^{1/2} D_\gamma^{-1}\mathbf{z} \right\|_2 \leq \left\| D_h^{-1/2} D_q D_q^{-1}\bar{G}_h D_\gamma^{-1}\mathbf{z} \right\|_2.$$

From (B.4) and (B.5), the definition of $\hat{h}_k$, and local quasi-uniformity, the ratio of the diagonal entries satisfies

$$\frac{D_h^{1/2}[k,k]}{D_\gamma[k,k]} = \frac{\|b_k\|_{L^2(\Gamma)}}{\sqrt{\sum_{l \in I(k)} h_l^{-2} \, \|b_k\|_{L^2(\tau_l)}^2}} \geq c\hat{h}_k,$$

due to

$$\begin{aligned}\frac{D_h^{1/2}[k,k]}{D_\gamma[k,k]} = \frac{\sqrt{\frac{h_{k-1}+h_k}{3}}}{\sqrt{h_{k-1}^{-2}\frac{h_{k-1}}{3} + h_k^{-2}\frac{h_k}{3}}} = \frac{\sqrt{\frac{h_{k-1}+h_k}{3}}}{\sqrt{\frac{h_{k-1}+h_k}{3h_k h_{k-1}}}} = \sqrt{h_k h_{k-1}} \\ \geq \sqrt{\frac{1}{c_L}h_k^2} = \sqrt{\frac{1}{c_L}}h_k \geq c_Q\sqrt{\frac{1}{c_L}}\hat{h}_k, \qquad \text{for } k = 1..N-2,\end{aligned}$$

and

$$\frac{D_q[k,k]}{D_h^{1/2}[k,k]} = \frac{\hat{h_k}\,\|q_k\|_{L^2(\Gamma)}}{\|b_k\|_{L^2(\Gamma)}} \leq c\hat{h}_k.$$

We derive this last result from

$$\frac{D_q[k,k]}{D_h^{1/2}[k,k]} = \frac{\hat{h_k}\,\|q_k\|_{L^2(\Gamma)}}{\|b_k\|_{L^2(\Gamma)}} = \sqrt{\frac{1}{6}}(h_{k-1}+h_k) = 3\sqrt{\frac{1}{6}}\hat{h_k}, \qquad \text{for } k = 2,..,N-2,$$

$$\frac{D_q[k,k]}{D_h^{1/2}[k,k]} = \frac{\hat{h_k}\,\|q_k\|_{L^2(\Gamma)}}{\|b_k\|_{L^2(\Gamma)}} \leq \sqrt{\frac{1}{3}}(h_{k-1}+h_k) = 3\sqrt{\frac{1}{3}}\hat{h_k}, \qquad \text{for } k = 1, N-2.$$

Thus, by taking $\mathbf{x} = H\mathbf{z}$

$$c_p\,\|\mathbf{x}\|_2 = c_p\,\|H\mathbf{z}\|_2 \leq \left\|HD_q^{-1}\bar{G}_h D_\gamma^{-1}\mathbf{z}\right\|_2 = \left\|HD_q^{-1}H^{-1}\tilde{G}_h H D_\gamma^{-1}H^{-1}\mathbf{x}\right\|_2 = \|A\mathbf{x}\|_2\,.$$

∎

Proof of Proposition (4.5). With the above, we can finally show

$$\begin{aligned}
\|\tilde{Q}_h u\|^2_{H^1(\Gamma)} &\leq 2\left\{\|P_h u\|^2_{H^1(\Gamma)} + \|(\tilde{Q}_h - P_h)u\|^2_{H^1(\Gamma)}\right\} \\
\text{(B.13)} \quad &\leq 2\left\{c_{p1}\|u\|^2_{H^1(\Gamma)} + \|(\tilde{Q}_h - P_h)u\|^2_{H^1(\Gamma)}\right\} \\
&\leq 2\left\{c_{p1}\|u\|^2_{H^1(\Gamma)} + \sum_{l=1}^{N-1} h_l^{-2}\|(\tilde{Q}_h - P_h)u\|^2_{L^2(\tau_l)}\right\} \\
\text{(Lemma } B.1) \quad &\leq 2\left\{c_{p1}\|u\|^2_{H^1(\Gamma)} + c_{p2}\sum_{k=1}^{N-2}\left[\frac{\left\langle(\tilde{Q}_h - P_h)u\,,\,q_k\right\rangle_{L^2(\Gamma)}}{\hat{h}_k\|q_k\|_{L^2(\Gamma)}}\right]^2\right\} \\
&\leq 2\left\{c_{p1}\|u\|^2_{H^1(\Gamma)} + c_{p2}\sum_{k=1}^{N-2}\left[\frac{\langle(\mathsf{Id} - P_h)u\,,\,q_k\rangle_{L^2(\omega_k)}}{\hat{h}_k\|q_k\|_{L^2(\Gamma)}}\right]^2\right\} \\
&\leq 2\left\{c_{p1}\|u\|^2_{H^1(\Gamma)} + c_{p2}\sum_{k=1}^{N-2}\hat{h}_k^{-2}\|(\mathsf{Id} - P_h)u\|^2_{L^2(\omega_k)}\right\} \\
\text{(B.14)} \quad &\leq \tilde{c}_{st}\|u\|^2_{H^1(\Gamma)}.
\end{aligned} \tag{B.17}$$

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