Renormalized powers of Ornstein-Uhlenbeck processes and well-posedness of stochastic Ginzburg-Landau equations

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Abstract

This article analyzes well-definedness and regularity of renormalized powers of Ornstein-Uhlenbeck processes and uses this analysis to establish local existence, uniqueness and regularity of strong solutions of stochastic Ginzburg-Landau equations with polynomial nonlinearities in two space dimensions and with quadratic nonlinearities in three space dimensions.

1 Introduction

The first part of this article (see Section 2 below) investigates well-definedness and regularity of suitable renormalized powers of Ornstein-Uhlenbeck processes. More formally, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $d \in \mathbb{N} := \{1, 2, \dots\}$, $n \in \{2, 3, 4, \dots\}$ and let $(W_t)_{t\in\mathbb{R}}$ be a two-sided cylindrical I-Wiener process on the $\mathbb{R}$-Hilbert space $L^2([0, 2\pi]^d, \mathbb{R})$ of equivalence classes of Lebesgue square integrable functions from $[0, 2\pi]^d$ to $\mathbb{R}$. Moreover, let $C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$ be the space of periodic continuous functions from $[0, 2\pi]^d$ to $\mathbb{R}$, let $A\colon D(A) \subset C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R}) \to C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$ be the Laplacian with periodic boundary conditions on $C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$ minus the identity operator (see (5) below for details) and consider the stationary solution $V_t = \int_{-\infty}^t e^{A(t-s)}\, dW_s$, $t \in \mathbb{R}$, of the SPDE

$$dV_t = AV_t\, dt + dW_t \tag{1}$$

for $t \in \mathbb{R}$. Note that the process V_t, $t \in \mathbb{R}$, does in the case $d \geq 2$ $\mathbb{P}$-almost surely not take values in a function space anymore but in $D((-A)^{(2-d)/4-\varepsilon})$ (see, for instance, Da Prato & Zabczyk [5]). Nonetheless, powers of V are well defined in a suitable sense in the case $d = 2$. Indeed, n-th renormalized power of V, that is, the stochastic process $:(V_t)^n:$, $t \in \mathbb{R}$, is well defined and its regularity is analyzed in the case $d = 2$ in Lemma 3.2 in Da Prato & Debussche [3] (see, e.g., also [20, 8, 4] for further details on the definition of the n-th renormalized power). Proposition 14 in this article extends the regularity statement of this result and also establish well definedness of $:(V_t)^2:$, $t \in \mathbb{R}$, in the case $d = 3$. Moreover, if $d = 3$, $n \geq 3$ or if $d \geq 4$, then $:(V_t)^n:$, $t \in \mathbb{R}$, can not be defined anymore (see Section 7.1 in Da Prato & Tubaro [4] in the case $d = n = 3$ and Lemma 16 below in the general case). Although $:(V_t)^3:$, $t \in \mathbb{R}$, does not make sense in the case $d = 3$, we establish in Proposition 19 and Lemma 21 below that the processes $\int_{t_0}^t :(V_s)^n:\, ds$, $t \in [t_0, \infty)$, $t_0 \in \mathbb{R}$, (which we refer as *averaged Wick powers*) are well defined if and only if $\frac{n+1}{n-1} > \frac{d}{2}$ (i.e., if and only if $d \in \{1, 2\}$ or ($d = 3$ and $n \in \{2, 3, 4\}$) or ($d \in \{4, 5\}$ and $n = 2$)). The integral thus mollifies the renormalized power in a suitable sense and allows us to define $\int_{t_0}^t :(V_s)^3:\, ds$, $t \in [t_0, \infty)$, $t_0 \in \mathbb{R}$, even in the case $d = 3$. Another possibility to extend the definition of $:(V_t)^n:$, $t \in \mathbb{R}$, is to consider the process $\int_{-\infty}^t e^{A(t-s)} :(V_s)^n:\, ds$, $t \in \mathbb{R}$, which we refer as *convolutional Wick power*. Proposition 24 and Lemma 25 prove that $\int_{-\infty}^t e^{A(t-s)} :(V_s)^n:\, ds$, $t \in \mathbb{R}$, is (as in the case of averaged Wick powers) well defined if and only if $\frac{n+1}{n-1} > \frac{d}{2}$. Proposition 24 also proves that convolutional Wick powers enjoy more regularity properties than averaged Wick powers constructed in Proposition 19. Our analysis of convolutional Wick powers is inspired by a Walsh-expansion for the KPZ equation in the fundamental recent article Hairer [9]. For details on the results on Wick power, averaged Wick powers and convolutional Wick powers the reader is referred to the summary in Subsection 2.7 below.

The above outlined results on the well-definedness and regularity of renormalized powers of V are used in the second part of this article (see Section 3 below) to analyzes strong solutions of stochastic Ginzburg-Landau equations with polynomial nonlinearities. More formally, let $\eta, \kappa_0, \kappa_1, \ldots, \kappa_n \in \mathbb{R}$, let $x_0 \in D((-A)^\eta)$ and consider a solution process $(X_t)_{t\in[0,\infty)}$ of the SPDE

$$dX_t = \Big[AX_t + :\Big(\sum\nolimits_{i=0}^n \kappa_i \, (X_t)^i\Big): \Big] \, dt + dW_t \tag{2}$$

for $t \in [0, \infty)$ with the initial condition $X_0 = x_0$ and where the expression $:\big(\sum_{i=0}^n \kappa_i (X_t)^i\big):$ is a suitable renormalization of the term $\sum_{i=0}^n \kappa_i (X_t)^i$ for $t \in [0, \infty)$ (see Subsections 3.2 and 3.3 below for further details). The parameter $\eta \in \mathbb{R}$ thus measures the regularity of the initial value. SPDEs of the form (2) have a strong connection to models from quantum field theory; see [17]. Local and global existence, uniqueness and regularity of solutions of SPDEs of the form (2) (and suitable mollified versions of (2) respectively) have been intensively studied in the last two decades; see, e.g., the monograph [5] and the references mentioned therein for the one-dimensional case $d = 1$ and see [12, 2, 1, 5, 6, 7, 13, 15, 3] for the more subtle two-dimensional case $d = 2$. In this article we are mainly interested in strong solutions of (2) and we therefore review results for strong solutions of (2) in a bit more detail in the following.

In the case $d = 1$, global existence, uniqueness and regularity of strong solutions follows, e.g., from Section 7.2 in Da Prato & Zabczyk [5] if n is odd and if $\kappa_n < 0$. In the case $d = 1$ the expression $\sum_{i=0}^n \kappa_i (X_t)^i$ appearing in (2) is well defined and it is not necessary to replace it by its renormalization $:\big(\sum_{i=0}^n \kappa_i (X_t)^i\big):$ for $t \in [0, \infty)$. Moreover, note that the solution process $(X_t)_{t\in[0,\infty)}$ of the SPDE (2) satisfies $\mathbb{P}\big[X_t \in D((-A)^{1/4-\varepsilon}) \cup \{\infty\}\big] = 1$ for all $t, \varepsilon \in (0, \infty)$ in the case $d = 1$. The solution process thus takes $\mathbb{P}$-almost surely values in $D((-A)^{1/4-\varepsilon}) \cup \{\infty\}$ in the case $d = 1$ where $\varepsilon \in (0, \infty)$ is arbitrarily small. Here and below the solution process takes the value ∞ after its possible blow up (e.g., if $\kappa_n > 0$).

In the case $d = 2$ the renormalization is necessary and can not be avoided (see Walsh [21] and, e.g., Section 1 in Hairer et al. [10]). In the case $d = 2$ local existence, uniqueness and regularity of solutions of (2) have been established in Proposition 4.4 in Da Prato & Debussche [3] if the condition

$$\eta > \inf_{p\in(n,\infty)} \left(\max\!\left\{ \frac{-2}{p\,(2n+1)}, \frac{-1}{(n-1)} \left(1 - \frac{n}{p}\right) \right\} \right) = - \sup_{p\in(n,\infty)} \left(\min\!\left\{ \frac{2}{p\,(2n+1)}, \frac{1}{(n-1)} \left(1 - \frac{n}{p}\right) \right\} \right) \tag{3}$$

is fulfilled beside other assumptions (see also Theorem 4.2 in [3] for the corresponding global existence result). The first main result of this article, Theorem 31 in Subsection 3.2, extends Da Prato & Debussche's result by establishing local existence of strong solutions in the case $d = 2$ for a larger class of initial values, that is, if the condition

$$\eta > -\frac{2}{n} \tag{4}$$

is fulfilled instead of (3). Clearly, assumption (4) is less restrictive than assumption (3). In addition, under assumption (4), Theorem 31 establishes more regularity of the solution process of the SPDE (2).

The reader is referred to (186) in Subsection 3.2 for a detailed comparison of the regularity statement in Proposition 4.4 in Da Prato & Debussche [3] and of the regularity statement in Theorem 31 below. Under assumption (4), Theorem 31 also shows that the solution process $(X_t)_{t\in[0,\infty)}$ of the SPDE (2) satisfies $\mathbb{P}[X_t \in D((-A)^{-\varepsilon}) \cup \{\infty\}] = 1$ for all $t, \varepsilon \in (0,\infty)$ and all $r \in (-\infty, 0)$ in the case $d = 2$. The solution process thus takes $\mathbb{P}$-almost surely values in $D((-A)^{-\varepsilon}) \cup \{\infty\}$ in the case $d = 2$ where $\varepsilon \in (0,\infty)$ is arbitrarily small.

The next main result of this article is devoted to *the case* $d = 3$ *and* $n = 2$. More precisely, Theorem 32 in Subsection 3.3, proves local existence, uniqueness and regularity of strong solutions of (2) in the case $d = 3$ and $n = 2$ if the condition $\eta > -1$ is fulfilled. Under these assumptions, Theorem 32 proves that the solution process of the SPDE (2) satisfies $\mathbb{P}\big[X_t \in D((-A)^{-1/4-\varepsilon}) \cup \{\infty\}\big] = 1$ for all $t, \varepsilon \in (0,\infty)$. The solution process thus takes $\mathbb{P}$-almost surely values in $D((-A)^{-1/4-\varepsilon}) \cup \{\infty\}$ in the case $d = 3$ and $n = 2$ and $\eta > -1$ where $\varepsilon \in (0,\infty)$ is arbitrarily small. To the best of our knowledge, Theorem 32 is the first result in the literature that establish local existence of solutions of the SPDE (2) in the three dimensional case $d = 3$. The proof of Theorem 32 is based on a detailed analysis of mild solutions of determinisitic nonautonomous partial differential equations in Subsection 3.1 and on the analysis of $:(V_t)^2:$, $t \in \mathbb{R}$, in three dimensions $d = 3$ (see Section 2).

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1.1 Notation

Throughout this article the following conventions are used. If Ω is a set and $\mathcal{F} \subset \mathcal{P}(\Omega)$ is a subsets of the power set of Ω, then we denote by $\sigma_\Omega(\mathcal{F})$ the sigma-algebra on Ω which is generated by $\mathcal{F}$. If $(E, \mathcal{E})$ is a topological space, then we denote by $\mathcal{B}(E) := \sigma_E(\mathcal{E})$ the Borel sigma-algebra of $(E, \mathcal{E})$. Furthermore, if $d \in \mathbb{N} := \{1, 2, \dots\}$, then we denote by $C_\mathcal{P}([0, 2\pi]^d, \mathbb{R})$ the $\mathbb{R}$-Banach space of periodic continuous functions from $[0, 2\pi]^d$ to $\mathbb{R}$ and by $\mathcal{A}_d \colon D(\mathcal{A}_d) \subset C_\mathcal{P}([0, 2\pi]^d, \mathbb{R}) \to C_\mathcal{P}([0, 2\pi]^d, \mathbb{R})$ the generator of a strongly continuous analytic semigroup which satisfies

$$D(\mathcal{A}_d) \supset \Bigg\{ v \in C_P([0, 2\pi]^d, \mathbb{R}) \colon \Big(\exists\, w \in C^2(\mathbb{R}^d, \mathbb{R}) \colon \Big[\forall\, x \in \mathbb{R}^d \colon \forall\, j \in \{1, \dots, d\} \colon w(x) = w\big(x + 2\pi e_j^{(d)}\big) \Big] \wedge \Big[w|_{[0,2\pi]^d} = v \Big] \Big) \Bigg\} \tag{5}$$

and $\mathcal{A}_d v = \triangle v - v$ for all $v \in D(\mathcal{A}_d)$. The fact that such an operator exists and is unique can, e.g., be proved by considering the Laplacian on the whole $\mathbb{R}^d$. In addition, if $d \in \mathbb{N}$ and $r \in \mathbb{R}$, then we denote by

$$\big(\mathcal{C}^r_\mathcal{P}([0, 2\pi]^d, \mathbb{R}), \|\cdot\|_{\mathcal{C}^r_\mathcal{P}([0,2\pi]^d,\mathbb{R})}\big) := \Big(D\big((-\mathcal{A}_d)^{r/2} \big), \big\| (-\mathcal{A}_d)^{r/2}(\cdot) \big\|_{C([0,2\pi]^d,\mathbb{R})} \Big) \tag{6}$$

the $\mathbb{R}$-Banach space of the domain of the $\frac{r}{2}$-fractional power of $\mathcal{A}_d$. Finally, we observe that there exist real numbers $c^{(d)}_{\alpha,\beta,\gamma} \in [0,\infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$, $d \in \mathbb{N}$, such that for every $d \in \mathbb{N}$, every $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta > 0$ and $\gamma < \min(\alpha, \beta)$, every $v \in \mathcal{C}^\alpha_\mathcal{P}([0, 2\pi]^d, \mathbb{R})$ and every $w \in \mathcal{C}^\beta_\mathcal{P}([0, 2\pi]^d, \mathbb{R})$ it holds that $v \cdot w \in \mathcal{C}^\gamma_\mathcal{P}([0, 2\pi]^d, \mathbb{R})$ and that

$$\|v \cdot w\|_{\mathcal{C}^\gamma_\mathcal{P}([0,2\pi]^d,\mathbb{R})} \le c^{(d)}_{\alpha,\beta,\gamma} \, \|v\|_{\mathcal{C}^\alpha_\mathcal{P}([0,2\pi]^d,\mathbb{R})} \, \|w\|_{\mathcal{C}^\beta_\mathcal{P}([0,2\pi]^d,\mathbb{R})} \,. \tag{7}$$

More details on interpolation spaces and analytic semigroups can, e.g, be found in the excellent books Lunardi [14], Van Neerven [16] and Sell & You [19]. Finally, throughout this article, if $(V, \|\cdot\|_V)$ is an $\mathbb{R}$-Banach space, then we equip the set $V \cup \{\infty\}$ with the topology

$$\Bigg\{ A \subset \Big(V \cup \{\infty\}\Big) \colon \Big(\forall\, a \in A \backslash \{\infty\} \colon \Big[\exists\, \varepsilon \in (0,\infty) \colon \{ y \in V \colon \|y - v\|_V < \varepsilon \} \subset A \Big] \Big) \text{ and } \Big(\infty \in A \Rightarrow \Big[\exists\, R \in (0,\infty) \colon \{ y \in V \colon \|y\|_V > R \} \subset A \Big] \Big) \Bigg\} \tag{8}$$

and we observe that the pairing consisting of $V \cup \{\infty\}$ and (8) is a complete metrizable topological space.

2 Renormalized powers of Ornstein-Uhlenbeck processes

2.1 Setting and assumptions

Throughout Section 2 we will frequently assume that the following setting is fulfilled. Let $d \in \mathbb{N}$, let $\delta \colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ be a function defined through

$$\delta_{v,w} := \begin{cases} 1 & : v = w \\ 0 & : v \neq w \end{cases} \tag{9}$$

for all $v, w \in \mathbb{R}^d$ and let $g_v \colon [0, 2\pi]^d \to \mathbb{C}$, $v \in \mathbb{Z}^d$, be a family of functions defined through

$$g_v(x) := e^{\mathbf{i}\langle v,x\rangle_{\mathbb{R}^d}} = e^{\mathbf{i}(v_1x_1+\ldots+v_dx_d)} \tag{10}$$

for all $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ and all $x = (x_1, \ldots, x_d) \in [0, 2\pi]^d$. Next let $\big(H := L^2((0, 2\pi)^d; \mathbb{C}), \langle\cdot,\cdot\rangle_H \left\|\cdot\right\|_H\big)$ be the $\mathbb{C}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0, 2\pi)^d$ to $\mathbb{C}$ with $\langle v, w\rangle_H = \int_{(0,2\pi)^d} \overline{v(x)} \cdot w(x)\, dx$ for all $v, w \in H$. Observe that $(2\pi)^{-\frac{d}{2}}\, g_v$, $v \in \mathbb{Z}^d$, is an orthonormal basis of H and that $y = \sum_{v\in\mathbb{Z}^d} \frac{1}{(2\pi)^d} \langle g_v, y\rangle_H\, g_v$ for all $y \in H$. Moreover, let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, let $\mathcal{P}_m := \{(i, j) \in \{1, 2, \ldots, m\}^2 \colon i < j\}$, $m \in \mathbb{N}$, be sets and let $\Theta \colon \cup_{m=1}^{\infty} (\mathbb{N}_0)^{\mathcal{P}_m} \to \cup_{m=1}^{\infty} (\mathbb{N}_0)^m$ be a function defined through

$$\Theta(\alpha) := \left(\sum_{\substack{(i,j)\in\mathcal{P}_m \\ i=1 \text{ or } j=1}} \alpha_{(i,j)}\,, \quad \ldots \quad , \sum_{\substack{(i,j)\in\mathcal{P}_m \\ i=m \text{ or } j=m}} \alpha_{(i,j)} \right) \in (\mathbb{N}_0)^m \tag{11}$$

for all $\alpha \in (\mathbb{N}_0)^{\mathcal{P}_m}$ and all $m \in \mathbb{N}$. Furthermore, we denote by

$$\Phi := \big\{\varphi \colon \mathbb{Z}^d \to [0, \infty) \colon \big(\forall\, v \in \mathbb{Z}^d \colon \varphi_v = \varphi_{-v}\big)\big\} \tag{12}$$

the set of all functions from $\mathbb{Z}^d$ to $[0, \infty)$ that are symmetric with respect to the origin and equipp it with the Fréchet metric

$$d_\Phi(\varphi, \psi) := \sum_{k\in\mathbb{Z}^d} \frac{\min(1, \varphi_k - \psi_k)}{2^{(|k_1|+\ldots+|k_d|)}} \tag{13}$$

for all $\varphi, \psi \in \Phi$. Next define $\Phi_0 := \{\varphi \in \Phi \colon \varphi_k = 0 \text{ for almost all } k \in \mathbb{Z}^d\} \subset \Phi$ and $\Phi_{0,\leq 1} := \{\varphi \in \Phi_0 \colon (\forall\, k \in \mathbb{Z}^d \colon \varphi_k \in [0, 1])\} \subset \Phi$. In addition, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\beta^v \colon \mathbb{R} \times \Omega \to \mathbb{C}$, $v \in \mathbb{Z}^d$, be a family of jointly Gaussian complex valued stochastic processes with continuous sample paths and with

$$\overline{\beta^v_t} = \beta^{-v}_t \qquad \text{and} \qquad \mathbb{E}\Big[\overline{\beta^v_{t_1}}\beta^w_{t_2}\Big] = \begin{cases} \delta_{v,w} \min(|t_1|\,, |t_2|) & : t_1 \cdot t_2 \geq 0 \\ 0 & : t_1 \cdot t_2 < 0 \end{cases} \tag{14}$$

for all $t, t_1, t_2 \in \mathbb{R}$ and all $v, w \in \mathbb{Z}^d$. Observe that β^v, $v \in \mathbb{Z}^d$, are two-sided complex valued standard Brownian motions. Moreover, let $V^\varphi \colon \mathbb{R}\times\Omega \to \mathcal{C}_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$, $\varphi \in \Phi_0$, be a family of stochastic processes with continuous sample paths satisfying

$$V^\varphi_t = \sum_{v\in\mathbb{Z}^d} \sqrt{2}\, \varphi_v \left[\int_{-\infty}^t e^{-\lambda_v(t-s)}\, d\beta^v_s\right] g_v \tag{15}$$

$\mathbb{P}$-almost surely for all $t \in \mathbb{R}$ and all $\varphi = (\varphi_v)_{v\in\mathbb{Z}^d} \in \Phi_0$. Observe that

$$\frac{1}{(2\pi)^{2d}}\, \mathbb{E}\left[\overline{\Big\langle g_{v_1}, V^{\varphi^{(1)}}_{t_1}\Big\rangle_H} \Big\langle g_{v_2}, V^{\varphi^{(2)}}_{t_2}\Big\rangle_H\right] = \frac{\delta_{v_1,v_2}\, \varphi^{(1)}_{v_1} \varphi^{(2)}_{v_2}\, e^{-\lambda_{v_1}|t_2-t_1|}}{\lambda_{v_1}} \tag{16}$$

for all $t_1, t_2 \in \mathbb{R}$, $v_1, v_2 \in \mathbb{Z}^d$ and all $\varphi^{(1)} = (\varphi^{(1)}_v)_{v\in\mathbb{Z}^d}, \varphi^{(2)} = (\varphi^{(2)}_v)_{v\in\mathbb{Z}^d} \in \Phi_0$ and that

$$\begin{aligned} \mathbb{E}\left[\overline{V^{\varphi^{(1)}}_{t_1}(x_1)}\, V^{\varphi^{(2)}}_{t_2}(x_2)\right] &= \sum_{v\in\mathbb{Z}^d} \frac{1}{(2\pi)^{2d}}\, \mathbb{E}\left[\overline{\Big\langle g_v, V^{\varphi^{(1)}}_{t_1}\Big\rangle_H} \Big\langle g_v, V^{\varphi^{(2)}}_{t_2}\Big\rangle_H\right] g_v(x_2 - x_1) \\ &= \sum_{v\in\mathbb{Z}^d} \frac{\varphi^{(1)}_v \varphi^{(2)}_v\, e^{-\lambda_v|t_2-t_1|}\, g_v(x_1 - x_2)}{\lambda_v} \end{aligned} \tag{17}$$

for all $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$, $t_1, t_2 \in \mathbb{R}$ and all $x_1, x_2 \in [0, 2\pi]^d$. Moreover, if $n \in \mathbb{N}$, then we denote by $\mathcal{W}_n \subset L^2(\Omega; \mathbb{R})$ the closure in $L^2(\Omega; \mathbb{R})$ of the set

$$\bigcup_{k \in \mathbb{N}} \bigcup_{\substack{p\colon \mathbb{R}^k \to \mathbb{R} \text{ is a} \\ \text{polyn. of degree } n}} \bigcup_{\substack{v_1, \dots, v_k \\ \in \mathbb{Z}^d}} \bigcup_{\substack{t_1, \dots, t_k \\ \in \mathbb{R}}} \left\{ p(\beta^{v_1}_{t_1}, \dots, \beta^{v_k}_{t_k}) \right\}. \tag{18}$$

Note for every $n \in \mathbb{N}$ that the $\mathbb{R}$-Hilbert space $\mathcal{W}_n$ is the direct sum of the first n Wiener chaoses; see, e.g., Section 4 in Da Prato & Tubaro [4] and Section A.1 in Hairer [9]. Furthermore, let $H_n \colon \mathbb{R} \to \mathbb{R}$, $n \in \{0, 1, 2, \dots\}$, be the unique functions satisfying

$$e^{-\frac{t^2}{2} + tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_n(x) \tag{19}$$

for all $t, x \in \mathbb{R}$. The functions H_n, $n \in \{0, 1, 2, \dots\}$, are typically referred as *(probabilists') Hermite polynomials* in the literature. Note that $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, ... for all $x \in \mathbb{R}$. In addition, if $Z \colon \Omega \to \mathbb{R}$ is a centered real valued Gaussian random variable and if $n \in \mathbb{N}_0$, then we denote by $:Z^n: \colon \Omega \to \mathbb{R}$ the *n-th Wick power* of Z, that is, the random variable given by

$$:Z^n: = \begin{cases} \left(\mathbb{E}\left[Z^2\right]\right)^{\frac{n}{2}} H_n\!\left(\frac{Z}{\sqrt{\mathbb{E}[Z^2]}}\right) & : \mathbb{E}[Z^2] > 0 \\ Z^n & : \mathbb{E}[Z^2] = 0 \end{cases} \tag{20}$$

(see, e.g., page 9 in Simon [20]). Moreover, we denote by $:(V^\varphi)^n: \colon \mathbb{R} \times \Omega \to \mathcal{C}_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$, $\varphi \in \Phi_0$, $n \in \mathbb{N}_0$, the stochastic processes with continuous sample paths given by

$$\big(:(V^\varphi_t)^n: \big)(x) = :(V^\varphi_t(x))^n: \tag{21}$$

for all $t \in \mathbb{R}$, $x \in [0, 2\pi]^d$, $\varphi \in \Phi_0$ and all $n \in \mathbb{N}_0$. Note that $:(V^\varphi_t)^0: = 1$, $:(V^\varphi_t)^1: = V^\varphi_t$, $:(V^\varphi_t)^2: = (V^\varphi_t)^2 - \mathbb{E}\big[(V^\varphi_t)^2\big] = (V^\varphi_t)^2 - \sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}$, $:(V^\varphi_t)^3: = (V^\varphi_t)^3 - 3V^\varphi_t \mathbb{E}\big[(V^\varphi_t)^2\big] = (V^\varphi_t)^3 - 3V^\varphi_t \big(\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}\big)$, ... for all $t \in \mathbb{R}$ and all $\varphi \in \Phi_0$. In addition, we denote by $\circ\, \big(V^\varphi_{t_0,(\cdot)}\big)^n \circ \colon [t_0, \infty) \times \Omega \to C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$, $\varphi \in \Phi_0$, $n \in \mathbb{N}_0$, $t_0 \in \mathbb{R}$, the stochastic processes with continuous sample paths defined by

$$\circ\, (V^\varphi_{t_0,t})^n \circ := \int_{t_0}^{t} :(V^\varphi_s)^n: \, ds \tag{22}$$

for all $\varphi \in \Phi_0$, $n \in \mathbb{N}_0$ and all $t_0, t \in \mathbb{R}$ with $t_0 \leq t$ and we denote by $\bullet\, \big(V^\varphi\big)^n \bullet \colon \mathbb{R} \times \Omega \to C_{\mathcal{P}}([0, 2\pi]^d, \mathbb{R})$, $\varphi \in \Phi_0$, $n \in \mathbb{N}_0$, the stochastic processes with continuous sample paths defined by

$$\bullet\, (V^\varphi_t)^n \bullet := \int_{-\infty}^{t} e^{\mathcal{A}_d (t-s)} \big[:(V^\varphi_s)^n: \big] \, ds \tag{23}$$

for all $\varphi \in \Phi_0$, $n \in \mathbb{N}_0$ and all $t \in \mathbb{R}$. The readers who are familiar with quantum field theory should distinguish the concept of the "time-ordered product" in quantum field theory (see, for instance, Peskin & Schroeder [18]) from the averaged and the convolutional Wick power defined above. Finally, note that $\big(V^\varphi_t(x)\big)^n$, $:\!(V^\varphi_t(x))^n\!:$, $\circ \big(V^\varphi_{t_0,t}(x)\big)^n \circ$, $\bullet\, (V^\varphi_t(x))^n \bullet \in \mathcal{W}_n$ for all $n \in \mathbb{N}$, $x \in [0, 2\pi]^d$ and all $t_0, t \in \mathbb{R}$ with $t_0 \leq t$.

2.2 Hypercontractivity estimates

The following lemma allows us to calculate regularities of suitable stochastic processes by computing their correlations in Fourier space. It is quite similar to Proposition A.2 in Hairer [9].

Lemma 1. *Assume the setting of Subsection 2.1, let $n \in \mathbb{N}$ and let $a, b \in \mathbb{R}$ with $a < b$. Then there exist real numbers $\chi^{n,d,p,a,b}_{\alpha,\hat{\alpha},\beta,\hat{\beta}} \in [0, \infty)$, $p, \alpha, \hat{\alpha}, \beta, \hat{\beta} \in \mathbb{R}$, such that*

$$\begin{aligned} &\|X\|_{L^p(\Omega; C^\alpha([a,b], \mathcal{C}^{2\beta}_{\mathcal{P}}([0,2\pi]^d, \mathbb{R})))} \\ &\leq \chi^{n,d,p,a,b}_{\alpha,\hat{\alpha},\beta,\hat{\beta}} \left[\sup_{\substack{t_1, t_2 \\ \in [a,b], \\ t_1 \neq t_2}} \sum_{\substack{v_1, \\ v_2 \\ \in \mathbb{Z}^d}} \left[\frac{\left|\mathbb{E}\left[\langle g_{v_1}, X_{t_1} \rangle_H \langle g_{v_2}, X_{t_1} \rangle_H\right]\right|}{\left(\lambda_{v_1} \lambda_{v_2}\right)^{-\hat{\beta}}} + \frac{\left|\mathbb{E}\left[\langle g_{v_1}, X_{t_1} - X_{t_2} \rangle_H \langle g_{v_2}, X_{t_1} - X_{t_2} \rangle_H\right]\right|}{\left(\lambda_{v_1} \lambda_{v_2}\right)^{-\hat{\beta}} |t_1 - t_2|^{2\hat{\alpha}}} \right] \right]^{\frac{1}{2}} \end{aligned} \tag{24}$$

for all $p \in (0,\infty)$*,* $\hat{\alpha} \in (\alpha, 1)$*,* $\alpha \in (0,1)$*,* $\hat{\beta} \in (\beta, \infty)$*,* $\beta \in \mathbb{R}$ *and all stochastic processes* $X\colon [a,b] \times \Omega \to \cap_{r\in\mathbb{R}} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^d, \mathbb{R})$ *with continuous sample paths which satisfy for every* $t \in [a,b]$ *and every* $x \in [0,2\pi]^d$ *that* $X_t(x) \in \mathcal{W}_n$*.*

Proof of Lemma 1. Hypercontractivity (see, e.g., Lemma A.1 in Hairer [9]) ensures that there exist real numbers $\kappa_{k,p} \in [0,\infty)$, $k \in \mathbb{N}$, $p \in [2,\infty)$, such that

$$\mathbb{E}\big[\,|Y|^p\,\big] \le \kappa_{k,p} \left(\mathbb{E}\Big[|Y|^2\Big]\right)^{\frac{p}{2}} \tag{25}$$

for all $p \in [2,\infty)$, $Y \in \mathcal{W}_k$ and all $k \in \mathbb{N}$. Note that

$$\begin{aligned}
&\big\|(-A)^{\hat{\beta}} X\big\|_{C^{\hat{\alpha}}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))} = \sup_{t\in[a,b]} \big\|(-A)^{\hat{\beta}} X_t\big\|_{L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R}))} \\
&+ \sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}} \frac{\big\|(-A)^{\hat{\beta}}(X_{t_1}-X_{t_2})\big\|_{L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R}))}}{|t_1-t_2|^{\hat{\alpha}}} \\
&= \sup_{t\in[a,b]} \left\{ \int_{(0,2\pi)^d} \mathbb{E}\Big[\big|((-A)^{\hat{\beta}} X_t)(x)\big|^p\Big]\, dx \right\}^{\frac{1}{p}} \\
&+ \sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}} \frac{\left\{\int_{(0,2\pi)^d} \mathbb{E}\Big[\big|((-A)^{\hat{\beta}}(X_{t_1}-X_{t_2}))(x)\big|^p\Big]\, dx\right\}^{\frac{1}{p}}}{|t_1-t_2|^{\hat{\alpha}}} \\
&= \sup_{t\in[a,b]} \left\{ \int_{(0,2\pi)^d} \mathbb{E}\left[\Big|\sum_{v\in\mathbb{Z}^d} (\lambda_v)^{\hat{\beta}} \langle g_v, X_t\rangle_H\, g_v(x)\Big|^p\right] dx \right\}^{\frac{1}{p}} \\
&+ \sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}} \frac{\left\{\int_{(0,2\pi)^d} \mathbb{E}\Big[\big|\sum_{v\in\mathbb{Z}^d} (\lambda_v)^{\hat{\beta}} \langle g_v, X_{t_1}-X_{t_2}\rangle_H\, g_v(x)\big|^p\Big]\, dx\right\}^{\frac{1}{p}}}{|t_1-t_2|^{\hat{\alpha}}}
\end{aligned} \tag{26}$$

for all $p \in (0,\infty)$, $\hat{\alpha} \in (0,1)$, $\hat{\beta} \in \mathbb{R}$ and all stochastic processes $X\colon [a,b]\times\Omega \to \cap_{r\in\mathbb{R}} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})$. Estimate (25) hence implies that

$$\begin{aligned}
&\big\|(-A)^{\hat{\beta}} X\big\|_{C^{\hat{\alpha}}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))} \\
&\le \frac{\kappa_{n,p}}{(2\pi)^d} \left[\sup_{t\in[a,b]} \left\{ \int_{(0,2\pi)^d} \left(\mathbb{E}\left[\Big|\sum\nolimits_{v\in\mathbb{Z}^d} (\lambda_v)^{\hat{\beta}} \langle g_v, X_t\rangle_H\, g_v(x)\Big|^2\right]\right)^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} \right. \\
&\left. + \sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}} \frac{\left\{\int_{(0,2\pi)^d} \left(\mathbb{E}\Big[\big|\sum_{v\in\mathbb{Z}^d} (\lambda_v)^{\hat{\beta}} \langle g_v, X_{t_1}-X_{t_2}\rangle_H\, g_v(x)\big|^2\Big]\right)^{\frac{p}{2}} dx\right\}^{\frac{1}{p}}}{|t_1-t_2|^{\hat{\alpha}}} \right] \\
&= \frac{\kappa_{n,p}}{(2\pi)^d} \left[\sup_{t\in[a,b]} \left\{ \int_{(0,2\pi)^d} \left(\sum_{v_1,v_2\in\mathbb{Z}^d} \frac{\mathbb{E}\Big[\langle g_{v_1}, X_t\rangle_H \langle g_{v_2}, X_t\rangle_H\Big]\, g_{(v_2-v_1)}(x)}{(\lambda_{v_1}\lambda_{v_2})^{-\hat{\beta}}} \right)^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} \right. \\
&\left. + \sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}} \left\{ \int_{(0,2\pi)^d} \left(\sum_{v_1,v_2\in\mathbb{Z}^d} \frac{\mathbb{E}\Big[\langle g_{v_1}, X_{t_1}-X_{t_2}\rangle_H \langle g_{v_2}, X_{t_1}-X_{t_2}\rangle_H\Big]\, g_{(v_2-v_1)}(x)}{(\lambda_{v_1}\lambda_{v_2})^{-\hat{\beta}} |t_1-t_2|^{2\hat{\alpha}}} \right)^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} \right]
\end{aligned} \tag{27}$$

for all $p \in (0,\infty)$, $\hat{\alpha} \in (0,1)$, $\hat{\beta} \in \mathbb{R}$ and all stochastic processes $X\colon [a,b]\times\Omega \to \cap_{r\in\mathbb{R}} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})$ which satisfy for every $t \in [a,b]$ and every $x \in [0,2\pi]^d$ that $X_t(x) \in \mathcal{W}_n$. This implies

$$\begin{aligned}
&\big\|(-A)^{\hat\beta} X\big\|_{C^{\hat\alpha}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))} \\
&\leq \frac{\kappa_{n,p}}{(2\pi)^d}\left[\sup_{t\in[a,b]}\left\{\int_{(0,2\pi)^d}\left(\sum\nolimits_{v_1,v_2\in\mathbb{Z}^d}\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_t\rangle_H\langle g_{v_2},X_t\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}}\right)^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}\right. \\
&\left.+\sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}}\left\{\int_{(0,2\pi)^d}\left(\sum\nolimits_{v_1,v_2\in\mathbb{Z}^d}\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_{t_1}-X_{t_2}\rangle_H\langle g_{v_2},X_{t_1}-X_{t_2}\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}|t_1-t_2|^{2\hat\alpha}}\right)^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}\right] \\
&=\frac{\kappa_{n,p}}{(2\pi)^d}\left[\sup_{t\in[a,b]}\left\{\sum\nolimits_{v_1,v_2\in\mathbb{Z}^d}\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_t\rangle_H\langle g_{v_2},X_t\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}}\right\}^{\frac{1}{2}}\right. \\
&\left.+\sup_{\substack{t_1,t_2\in[a,b]\\ t_1\neq t_2}}\left\{\sum\nolimits_{v_1,v_2\in\mathbb{Z}^d}\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_{t_1}-X_{t_2}\rangle_H\langle g_{v_2},X_{t_1}-X_{t_2}\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}|t_1-t_2|^{2\hat\alpha}}\right\}^{\frac{1}{2}}\right]
\end{aligned} \tag{28}$$

and hence

$$\begin{aligned}
&\big\|(-A)^{\hat\beta} X\big\|_{C^{\hat\alpha}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))} \\
&\leq \kappa_{n,p}\left[\sup_{\substack{t_1,t_2\\ \in[a,b]\\ t_1\neq t_2}}\sum_{\substack{v_1,v_2\\ \in\mathbb{Z}^d}}\left\{\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_{t_1}\rangle_H\langle g_{v_2},X_{t_1}\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}}+\frac{\left|\mathbb{E}\left[\langle g_{v_1},X_{t_1}-X_{t_2}\rangle_H\langle g_{v_2},X_{t_1}-X_{t_2}\rangle_H\right]\right|}{(\lambda_{v_1}\lambda_{v_2})^{-\hat\beta}|t_1-t_2|^{2\hat\alpha}}\right\}\right]^{\frac{1}{2}}
\end{aligned} \tag{29}$$

for all $p \in (0,\infty)$, $\hat\alpha \in (0,1)$, $\hat\beta \in \mathbb{R}$ and all stochastic processes $X\colon [a,b]\times\Omega \to \cap_{r\in\mathbb{R}}\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})$ which satisfy for every $t \in [a,b]$ and every $x \in [0,2\pi]^d$ that $X_t(x) \in \mathcal{W}_n$. Moreover, the Sobolev embedding theorem ensures that there exist real numbers $\rho^{p,\alpha,\tilde\alpha}_{\beta,\hat\beta} \in [0,\infty)$, $p,\alpha,\tilde\alpha,\beta,\hat\beta \in \mathbb{R}$, and $\bar\rho^{p,\tilde\alpha,\hat\alpha} \in [0,\infty)$, $p,\tilde\alpha,\hat\alpha \in \mathbb{R}$, such that

$$\begin{aligned}
&\|X\|_{L^p(\Omega;C^\alpha([a,b],\mathcal{C}^{2\beta}_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})))} = \big\|(-A)^\beta X\big\|_{L^p(\Omega;C^\alpha([a,b],C_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})))} \\
&\leq \rho^{p,\alpha,\tilde\alpha}_{\beta,\hat\beta}\,\big\|(-A)^{\hat\beta}X\big\|_{L^p(\Omega;W^{\tilde\alpha,p}([a,b],L^p((0,2\pi)^d;\mathbb{R})))} \\
&= \rho^{p,\alpha,\tilde\alpha}_{\beta,\hat\beta}\,\big\|(-A)^{\hat\beta}X\big\|_{W^{\tilde\alpha,p}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))} \\
&\leq \rho^{p,\alpha,\tilde\alpha}_{\beta,\hat\beta}\,\bar\rho^{p,\tilde\alpha,\hat\alpha}\,\big\|(-A)^{\hat\beta}X\big\|_{C^{\hat\alpha}([a,b],L^p(\Omega;L^p((0,2\pi)^d;\mathbb{R})))}
\end{aligned} \tag{30}$$

for all stochastic processes $X\colon [a,b]\times\Omega \to \cap_{r\in\mathbb{R}}\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})$ with continuous sample paths and all $p \in (0,\infty)$, $\alpha,\tilde\alpha,\hat\alpha \in (0,1)$, $\beta,\hat\beta \in \mathbb{R}$ with $\hat\alpha > \tilde\alpha$, $\tilde\alpha-\alpha > \frac{1}{p}$ and $\hat\beta-\beta > \frac{d}{p}$. Combining (29) and (30) implies (24) and this completes the proof of Lemma 1. □

2.3 Estimates for discrete convolutions

We first state three well known lemmas that we will use below.

Lemma 2 (Finiteness of infinite sums). *Let $d \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and let $\lambda_x \in [1,\infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1+(x_1)^2+\ldots+(x_d)^2$ for all $x=(x_1,\ldots,x_d) \in \mathbb{R}^d$. Then $\sum_{k\in\mathbb{Z}^d}\frac{1}{(\lambda_k)^\alpha} < \infty$ if and only if $\alpha > \frac{d}{2}$.*

Lemma 3 (Growth rate of finite sums). *Let $d \in \mathbb{N}$, $\alpha \in [0,\frac{d}{2})$, $\beta \in \mathbb{R}$, $c \in (0,\infty)$ and let $\lambda_x \in [1,\infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1+(x_1)^2+\ldots+(x_d)^2$ for all $x=(x_1,\ldots,x_d) \in \mathbb{R}^d$. Then $\sup_{v\in\mathbb{Z}^d}\Big[\sum_{k\in\mathbb{Z}^d,\|k\|_{\mathbb{R}^d}\leq c\|v\|_{\mathbb{R}^d}}\frac{(\lambda_v)^\beta}{(\lambda_k)^\alpha}\Big] < \infty$ if and only if $\beta \leq \alpha-\frac{d}{2}$.*

Lemma 4 (Growth rate of infinite sums). *Let $d \in \mathbb{N}$, $\alpha \in (\frac{d}{2},\infty)$, $\beta \in \mathbb{R}$, $c \in (0,\infty)$ and let $\lambda_x \in [1,\infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1+(x_1)^2+\ldots+(x_d)^2$ for all $x=(x_1,\ldots,x_d) \in \mathbb{R}^d$. Then $\sup_{v\in\mathbb{Z}^d}\Big[\sum_{k\in\mathbb{Z}^d,\|k\|_{\mathbb{R}^d}>c\|v\|_{\mathbb{R}^d}}\frac{(\lambda_v)^\beta}{(\lambda_k)^\alpha}\Big] < \infty$ if and only if $\beta \leq \alpha-\frac{d}{2}$.*

Lemmas 2–4 can all be proved by estimating the sums through suitable Lebesgue integrals and then by using polar coordinates. The proofs of Lemmas 2–4 are straightforward and well known and therefore omitted.

Lemma 5 (Two-sided bounds for discrete convolutions)**.** *Let $d \in \mathbb{N}$ and let $\lambda_x \in [1, \infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1 + (x_1)^2 + \ldots + (x_d)^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then*

$$\frac{4^{-\beta}}{(\lambda_v)^\beta} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha} \right] \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha (\lambda_{v-k})^\beta} \leq \frac{4^{\beta}}{(\lambda_v)^\beta} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha} \right], \tag{31}$$

$$\frac{4^{-\alpha}}{(\lambda_v)^\alpha} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{3}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\beta} \right] \leq \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|k\|_{\mathbb{R}^d} \leq 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha (\lambda_{v-k})^\beta} \leq \frac{4^{\alpha}}{(\lambda_v)^\alpha} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ 3\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\beta} \right], \tag{32}$$

$$4^{-\beta} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} > \\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{(\alpha+\beta)}} \right] \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} > \\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha (\lambda_{v-k})^\beta} \leq 4^{\beta} \left[\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} > \\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{(\alpha+\beta)}} \right] \tag{33}$$

for all $v \in \mathbb{Z}^d$ and all $\alpha, \beta \in [0, \infty)$.

Proof of Lemma 5. First of all, observe that

$$\begin{aligned} &\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{3^{-\beta}}{(\lambda_k)^\alpha (\lambda_v)^\beta} \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha \left(\frac{9}{4}\lambda_v\right)^\beta} \\ &\leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha (\lambda_{v-k})^\beta} \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha \left(\frac{\lambda_v}{4}\right)^\beta} \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{4^\beta}{(\lambda_k)^\alpha (\lambda_v)^\beta} \end{aligned} \tag{34}$$

for all $v \in \mathbb{R}^d$. This proves (31). Furthermore, note that

$$\begin{aligned} &\sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} < \\ \frac{1}{2}\|v\|_{\mathbb{R}^d}}} \frac{4^{-\alpha}}{(\lambda_k)^\beta (\lambda_v)^\alpha} \leq \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|v-k\|_{\mathbb{R}^d} \leq \frac{3}{2}\|v\|_{\mathbb{R}^d}}} \frac{4^{-\alpha}}{(\lambda_k)^\beta (\lambda_v)^\alpha} = \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|k\|_{\mathbb{R}^d} \leq 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(4\lambda_v)^\alpha (\lambda_{v-k})^\beta} \\ &\leq \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|k\|_{\mathbb{R}^d} \leq 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^\alpha (\lambda_{v-k})^\beta} \leq \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|k\|_{\mathbb{R}^d} \leq 2\|v\|_{\mathbb{R}^d}}} \frac{1}{\left(\frac{\lambda_v}{4}\right)^\alpha (\lambda_{v-k})^\beta} \\ &= \sum_{\substack{k \in \mathbb{Z}^d, \frac{1}{2}\|v\|_{\mathbb{R}^d} < \\ \|v-k\|_{\mathbb{R}^d} \leq 2\|v\|_{\mathbb{R}^d}}} \frac{4^\alpha}{(\lambda_k)^\beta (\lambda_v)^\alpha} \leq \sum_{\substack{k \in \mathbb{Z}^d, \\ \|k\|_{\mathbb{R}^d} \leq \\ 3\|v\|_{\mathbb{R}^d}}} \frac{4^\alpha}{(\lambda_k)^\beta (\lambda_v)^\alpha} \end{aligned} \tag{35}$$

for all $v \in \mathbb{Z}^d$. This establishes (32). Finally, observe that

$$\begin{aligned}
&\sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{3^{-\beta}}{(\lambda_k)^{(\alpha+\beta)}} \leq \sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{\alpha}\left[1+\left[\,\|k\|_{\mathbb{R}^d}+\|v\|_{\mathbb{R}^d}\,\right]^2\right]^{\beta}} \\
&\leq \sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{\alpha}\,(\lambda_{v-k})^{\beta}} \leq \sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{\alpha}\left(1+\left[\|k\|_{\mathbb{R}^d}-\|v\|_{\mathbb{R}^d}\right]^2\right)^{\beta}} \\
&\leq \sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{1}{(\lambda_k)^{\alpha}\left(\frac{\lambda_k}{4}\right)^{\beta}} = \sum_{\substack{k\in\mathbb{Z}^d,\\ \|k\|_{\mathbb{R}^d}>\\ 2\|v\|_{\mathbb{R}^d}}} \frac{4^{\beta}}{(\lambda_k)^{(\alpha+\beta)}}
\end{aligned} \tag{36}$$

for all $v \in \mathbb{R}^d$. This shows (33). The proof of Lemma 5 is thus completed. □

The next elementary lemma, Lemma 6, is a direct consequence of Lemma 2 and of (33) in Lemma 5. The proof of Lemma 6 is clear and therefore omitted.

Lemma 6 (Finiteness of discrete convolutions)**.** *Let $d \in \mathbb{N}$, $\alpha, \beta \in [0,\infty)$, $v \in \mathbb{Z}^d$ and let $\lambda_x \in [1,\infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1 + (x_1)^2 + \ldots + (x_d)^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then $\sum_{k\in\mathbb{Z}^d} \frac{1}{(\lambda_k)^{\alpha}(\lambda_{v-k})^{\beta}} < \infty$ if and only if $\alpha + \beta > \frac{d}{2}$.*

The next lemma, Lemma 7, follows from Lemmas 3, 4 and 5.

Lemma 7 (Regularity of discrete convolutions)**.** *Let $d \in \mathbb{N}$, $\alpha, \beta, \gamma \in [0,\infty)$ be real numbers with $\alpha + \beta > \frac{d}{2} \neq \max(\alpha,\beta)$ and let $\lambda_x \in [1,\infty)$, $x \in \mathbb{R}^d$, be real numbers with $\lambda_x = 1 + (x_1)^2 + \ldots + (x_d)^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then*

$$\sup_{v\in\mathbb{Z}^d}\left[\sum_{k\in\mathbb{Z}^d} \frac{(\lambda_v)^{\gamma}}{(\lambda_k)^{\alpha}\,(\lambda_{v-k})^{\beta}}\right] < \infty \tag{37}$$

if and only if $\gamma \leq \min(\alpha, \beta, \alpha+\beta-\frac{d}{2})$.

Proof of Lemma 7. Note that $\sum_{k\in\mathbb{Z}^d} \frac{(\lambda_v)^{\gamma}}{(\lambda_k)^{\alpha}(\lambda_{v-k})^{\beta}} = \sum_{k\in\mathbb{Z}^d} \frac{(\lambda_v)^{\gamma}}{(\lambda_k)^{\beta}(\lambda_{v-k})^{\alpha}}$ for all $v \in \mathbb{Z}^d$. W.l.o.g. we assume that $\alpha \leq \beta$. This ensures that $\beta \neq \frac{d}{2}$. Moreover, Lemma 3 and (31) in Lemma 5 prove that

$$\begin{aligned}
&\left(\sup_{v\in\mathbb{Z}^d}\left[\sum_{k\in\mathbb{Z}^d,\|k\|_{\mathbb{R}^d}\leq\frac{1}{2}\|v\|_{\mathbb{R}^d}} \frac{(\lambda_v)^{\gamma}}{(\lambda_k)^{\alpha}\,(\lambda_{v-k})^{\beta}}\right] < \infty\right) \\
&\Leftrightarrow \left(\left[\left(\gamma \leq \alpha+\beta-\tfrac{d}{2}\right)\wedge\left(\alpha<\tfrac{d}{2}\right)\right]\vee\left[\left(\gamma<\beta\right)\wedge\left(\alpha=\tfrac{d}{2}\right)\right]\vee\left[\left(\gamma\leq\beta\right)\wedge\left(\alpha>\tfrac{d}{2}\right)\right]\right) \\
&\Leftarrow \left(\gamma \leq \min(\alpha,\beta,\alpha+\beta-\tfrac{d}{2})\right).
\end{aligned} \tag{38}$$

In addition, Lemma 3 and (32) in Lemma 5 show that

$$\begin{aligned}
&\left(\sup_{v\in\mathbb{Z}^d}\left[\sum_{k\in\mathbb{Z}^d,\frac{1}{2}\|v\|_{\mathbb{R}^d}<\|k\|_{\mathbb{R}^d}\leq 2\|v\|_{\mathbb{R}^d}} \frac{(\lambda_v)^{\gamma}}{(\lambda_k)^{\alpha}\,(\lambda_{v-k})^{\beta}}\right] < \infty\right) \\
&\Leftrightarrow \left(\left[\left(\gamma \leq \alpha+\beta-\tfrac{d}{2}\right)\wedge\left(\beta<\tfrac{d}{2}\right)\right]\vee\left[\left(\gamma\leq\alpha\right)\wedge\left(\beta>\tfrac{d}{2}\right)\right]\right) \\
&\Leftrightarrow \left(\left[\left(\gamma \leq \min(\alpha,\alpha+\beta-\tfrac{d}{2})\right)\wedge\left(\beta<\tfrac{d}{2}\right)\right]\vee\left[\left(\gamma\leq\min(\alpha,\alpha+\beta-\tfrac{d}{2})\right)\wedge\left(\beta>\tfrac{d}{2}\right)\right]\right) \\
&\Leftrightarrow \left(\gamma \leq \min(\alpha,\alpha+\beta-\tfrac{d}{2})\right) \Leftrightarrow \left(\gamma \leq \min(\alpha,\beta,\alpha+\beta-\tfrac{d}{2})\right).
\end{aligned} \tag{39}$$

Finally, Lemma 4 and (33) in Lemma 5 prove that

$$\left(\sup_{v\in\mathbb{Z}^d}\left[\sum_{k\in\mathbb{Z}^d,\|k\|_{\mathbb{R}^d}>2\|v\|_{\mathbb{R}^d}}\frac{(\lambda_v)^\gamma}{(\lambda_k)^\alpha\,(\lambda_{v-k})^\beta}\right]<\infty\right)\Leftrightarrow\left(\gamma\leq\alpha+\beta-\tfrac{d}{2}\right). \tag{40}$$

Combining (38)–(40) completes the proof of Lemma 7. □

Corollary 8 (Regularity of discrete convolutions)**.** *Let $d\in\mathbb{N}$, $\alpha,\beta\in[0,\infty)$ and let $\lambda_x\in[1,\infty)$, $x\in\mathbb{R}^d$, be real numbers with $\lambda_x=1+(x_1)^2+\ldots+(x_d)^2$ for all $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$. Then $\sup_{v\in\mathbb{Z}^d}\Big[\sum_{k\in\mathbb{Z}^d}\frac{(\lambda_v)^\gamma}{(\lambda_k)^\alpha(\lambda_{v-k})^\beta}\Big]<\infty$ for all $\gamma\in\big[0,\min(\alpha,\beta,\alpha+\beta-\frac{d}{2})\big)$.*

2.4 Wick powers of Ornstein-Uhlenbeck processes

The next elementary lemma is, e.g., similar to Lemma 2.4 in Da Prato & Tubaro [4] and Corollary 8.3.2 in Glimm & Jaffe [8].

Lemma 9 (Expectations of products of Wick powers of Gaussian random variables)**.** *Assume the setting of Subsection 2.1, let $m\in\mathbb{N}$ and let $Z=(Z_1,\ldots,Z_m)\colon\Omega\to\mathbb{R}^m$ be a centered jointly normally distributed random variable. Then*

$$\mathbb{E}\Big[(:(Z_1)^{n_1}:)\cdot(:(Z_2)^{n_2}:)\cdot\ldots\cdot(:(Z_m)^{n_m}:)\Big]=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left[\prod_{(i,j)\in\mathcal{P}_m}\big(\mathbb{E}[Z_iZ_j]\big)^{\alpha_{(i,j)}}\right] \tag{41}$$

for all $n=(n_1,\ldots,n_m)\in(\mathbb{N}_0)^m$.

Proof of Lemma 9. W.l.o.g. we assume that $\mathbb{E}\big[(Z_i)^2\big]>0$ for all $i\in\{1,2,\ldots,m\}$. Next throughout this proof let $\hat{Z}_i\colon\Omega\to\mathbb{R}$, $i\in\{1,2,\ldots,m\}$, be random variables defined through

$$\hat{Z}_i:=\frac{Z_i}{\big(\mathbb{E}\big[(Z_i)^2\big]\big)^{1/2}} \tag{42}$$

for all $i\in\{1,2,\ldots,m\}$. The definition of the Hermite polynomials H_n, $n\in\{0,1,2,\ldots\}$, then proves that

$$\begin{aligned}
&\sum_{n=(n_1,n_2,\ldots,n_m)\in\mathbb{N}_0}\frac{(s_1)^{n_1}\cdot(s_2)^{n_2}\cdot\ldots\cdot(s_m)^{n_m}\cdot\mathbb{E}\big[\prod_{i=1}^m H_{n_i}(\hat{Z}_i)\big]}{n_1!\,n_2!\,\ldots\,n_m!}\\
&=\mathbb{E}\left[\prod_{i=1}^m\left(\sum_{n_i=0}^\infty\frac{(s_i)^{n_i}}{n_i!}H_{n_i}(\hat{Z}_k)\right)\right]=\mathbb{E}\left[\prod_{i=1}^m\exp\left(-\frac{(s_i)^2}{2}+s_i\hat{Z}_i\right)\right]\\
&=\exp\left(\frac{-\sum_{i=1}^m(s_i)^2}{2}\right)\mathbb{E}\left[\exp\left(\sum_{i=1}^m s_i\hat{Z}_i\right)\right]\\
&=\exp\left(\frac{-\sum_{i=1}^m(s_i)^2}{2}+\frac{1}{2}\,\mathbb{E}\left[\left(\sum_{i=1}^m s_i\hat{Z}_i\right)^2\right]\right)\\
&=\exp\left(\tfrac{-\sum_{i=1}^m(s_i)^2+\sum_{i,j=1}^m s_is_j\mathbb{E}[\hat{Z}_i\hat{Z}_j]}{2}\right)=\prod_{\substack{i,j\in\{1,2,\ldots,m\}\\ i<j}}\exp\Big(s_is_j\mathbb{E}\Big[\hat{Z}_i\hat{Z}_j\Big]\Big)
\end{aligned} \tag{43}$$

and the identity $e^{s_1s_2c}=\sum_{n=0}^\infty\frac{(s_1s_2)^nc^n}{n!}$ for all $s_1,s_2,c\in\mathbb{R}$ therefore shows that

$$\begin{aligned}
&\sum_{n=(n_1,n_2,\ldots,n_m)\in\mathbb{N}_0}\frac{(s_1)^{n_1}\cdot(s_2)^{n_2}\cdot\ldots\cdot(s_m)^{n_m}\cdot\mathbb{E}\big[\prod_{i=1}^m H_{n_i}(\hat{Z}_i)\big]}{n!}\\
&=\prod_{(i,j)\in\mathcal{P}_m}\left(\sum_{\alpha_{(i,j)}=0}^\infty\frac{(s_is_j)^{\alpha_{(i,j)}}\Big\{\mathbb{E}\Big[\hat{Z}_i\hat{Z}_j\Big]\Big\}^{\alpha_{(i,j)}}}{\alpha_{(i,j)}!}\right)\\
&=\sum_{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}}\frac{1}{\alpha!}\left(\prod_{(i,j)\in\mathcal{P}_m}(s_is_j)^{\alpha_{(i,j)}}\Big\{\mathbb{E}\Big[\hat{Z}_i\hat{Z}_j\Big]\Big\}^{\alpha_{(i,j)}}\right).
\end{aligned} \tag{44}$$

This implies

$$\frac{1}{n!}\,\mathbb{E}\Bigg[\prod_{i=1}^{m} H_{n_i}(\hat{Z}_i)\Bigg] = \sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}} \frac{1}{\alpha!}\left[\prod_{(i,j)\in\mathcal{P}_m}\left\{\mathbb{E}\Big[\hat{Z}_i\hat{Z}_j\Big]\right\}^{\alpha_{(i,j)}}\right] \tag{45}$$

and hence

$$\begin{aligned}\mathbb{E}\Bigg[\prod_{i=1}^{m}\big(:(Z_i)^n:\big)\Bigg] &= \big\{\mathbb{E}\big[(Z_1)^2\big]\big\}^{\frac{n_1}{2}}\cdot\ldots\cdot\big\{\mathbb{E}\big[(Z_m)^2\big]\big\}^{\frac{n_m}{2}}\\ &\quad\cdot\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left[\prod_{(i,j)\in\mathcal{P}_m}\left\{\frac{\mathbb{E}[Z_iZ_j]}{\sqrt{\mathbb{E}[(Z_i)^2]\,\mathbb{E}[(Z_j)^2]}}\right\}^{\alpha_{(i,j)}}\right]\end{aligned} \tag{46}$$

for all $n=(n_1,\ldots,n_m)\in(\mathbb{N}_0)^m$. The definition of the function $\Theta\colon \cup_{m=1}^{\infty}(\mathbb{N}_0)^{\mathcal{P}_m}\to\cup_{m=1}^{\infty}(\mathbb{N}_0)^m$ therefore completes the proof of Lemma 9. □

Remark 10 (Wick's theorem)**.** *Assume the setting of Subsection 2.1, let $m\in\mathbb{N}$ and let $Z=(Z_1,\ldots,Z_m)\colon\Omega\to\mathbb{R}^m$ be a centered jointly normally distributed random variable. Then Lemma 9 implies that*

$$\mathbb{E}\big[Z_1\cdot Z_2\cdot\ldots\cdot Z_m\big] = \sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=(1,1,\ldots,1)}} n!\left[\prod_{(i,j)\in\mathcal{P}_m}\big(\mathbb{E}\big[Z_iZ_j\big]\big)^{\alpha_{(i,j)}}\right]. \tag{47}$$

Equation (47) *is often referred as* Wick's theorem *in the literature (see, e.g., Proposition 5.2 in Hairer [9]).*

The next lemma is a direct consequence of Lemma 9.

Corollary 11 (Products of Wick powers of V^{φ}, $\varphi\in\Phi_0$, in real space)**.** *Assume the setting of Subsection 2.1, let $m\in\mathbb{N}$ and let $n=(n_1,n_2,\ldots,n_m)\in(\mathbb{N}_0)^m\backslash\{0\}$. Then*

$$\begin{aligned}&\mathbb{E}\Big[\Big(:(V_{t_1}^{\varphi^{(1)}})^{n_1}:\Big)(x_1)\cdot\Big(:(V_{t_2}^{\varphi^{(2)}})^{n_2}:\Big)(x_2)\cdot\ldots\cdot\Big(:(V_{t_m}^{\varphi^{(m)}})^{n_m}:\Big)(x_m)\Big]\\ &=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left[\prod_{(i,j)\in\mathcal{P}_m}\left[\sum_{k\in\mathbb{Z}^d}\frac{\varphi_k^{(i)}\,\varphi_k^{(j)}\,g_k(x_i-x_j)\,e^{-\lambda_k|t_i-t_j|}}{\lambda_k}\right]^{\alpha_{(i,j)}}\right]\\ &=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\sum_{k\in(\mathbb{Z}^d)^{\left\{\substack{(A,l)\in\\ \mathcal{P}_m\times\mathbb{N}:\\ l\leq\alpha_A}\right\}}}\prod_{(i,j,r)\in\left\{\substack{(A,l)\in\\ \mathcal{P}_m\times\mathbb{N}:\\ l\leq\alpha_A}\right\}}\frac{\varphi^{(i)}_{k_{(i,j,r)}}\,\varphi^{(j)}_{k_{(i,j,r)}}\,e^{-\lambda_{k_{(i,j,r)}}|t_i-t_j|}\,g_{k_{(i,j,r)}}(x_i-x_j)}{\lambda_{k_{(i,j,r)}}}\end{aligned} \tag{48}$$

for all $t_1,t_2,\ldots,t_m\in\mathbb{R}$, $x_1,x_2,\ldots,x_m\in[0,2\pi]^d$ and all $\varphi^{(1)}=(\varphi_k^{(1)})_{k\in\mathbb{Z}^d}$, $\varphi^{(2)}(\varphi_k^{(2)})_{k\in\mathbb{Z}^d}$, $\ldots$, $\varphi^{(m)}=(\varphi_k^{(m)})_{k\in\mathbb{Z}^d}\in\Phi_0$.

Proof of Corollary 11. Combining Lemma 9 and equation (17) implies that

$$\begin{aligned}&\mathbb{E}\Big[\Big(:(V_{t_1}^{\varphi^{(1)}})^{n_1}:\Big)(x_1)\cdot\Big(:(V_{t_2}^{\varphi^{(2)}})^{n_2}:\Big)(x_2)\cdot\ldots\cdot\Big(:(V_{t_m}^{\varphi^{(m)}})^{n_m}:\Big)(x_m)\Big]\\ &=\mathbb{E}\Big[\Big(:\Big(V_{t_1}^{\varphi^{(1)}}(x_1)\Big)^{n_1}:\Big)\cdot\Big(:\Big(V_{t_2}^{\varphi^{(2)}}(x_2)\Big)^{n_2}:\Big)\cdot\ldots\cdot\Big(:\Big(V_{t_{n_m}}^{\varphi^{(n_m)}}(x_{n_m})\Big)^{n_m}:\Big)\Big]\\ &=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left[\prod_{(i,j)\in\mathcal{P}_m}\Big(\mathbb{E}\Big[V_{t_{n_i}}^{\varphi^{(i)}}(x_{n_i})\,V_{t_{n_j}}^{\varphi^{(j)}}(x_{n_j})\Big]\Big)^{\alpha_{(i,j)}}\right]\\ &=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left(\prod_{(i,j)\in\mathcal{P}_m}\left[\sum_{k\in\mathbb{Z}^d}\frac{\varphi_k^{(i)}\,\varphi_k^{(j)}\,e^{-\lambda_k|t_i-t_j|}\,g_k(x_i-x_j)}{\lambda_k}\right]^{\alpha_{(i,j)}}\right)\end{aligned} \tag{49}$$

and therefore

$$\begin{aligned}
&\mathbb{E}\Big[\Big(:\big(V_{t_1}^{\varphi^{(1)}}\big)^{n_1}:\Big)(x_1)\cdot\Big(:\big(V_{t_2}^{\varphi^{(2)}}\big)^{n_2}:\Big)(x_2)\cdot\ldots\cdot\Big(:\big(V_{t_m}^{\varphi^{(m)}}\big)^{n_m}:\Big)(x_m)\Big]\\
&=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\left(\prod_{(i,j)\in\mathcal{P}_m}\left[\sum_{\substack{k_1,k_2,\ldots,\\ k_{\alpha_{(i,j)}}\in\mathbb{Z}^d}}\prod_{l=1}^{\alpha_{(i,j)}}\left\{\frac{\varphi_{k_l}^{(i)}\,\varphi_{k_l}^{(j)}\,e^{-\lambda_{k_l}|t_i-t_j|}\,g_{k_l}(x_i-x_j)}{\lambda_{k_l}}\right\}\right]\right)\\
&=\sum_{\substack{\alpha\in(\mathbb{N}_0)^{\mathcal{P}_m}\\ \Theta(\alpha)=n}}\frac{n!}{\alpha!}\sum_{k\in(\mathbb{Z}^d)^{\left\{\substack{(A,l)\in\\ \mathcal{P}_m\times\mathbb{N}:\\ l\leq\alpha_A}\right\}}}\prod_{(i,j,r)\in\left\{\substack{(A,l)\in\\ \mathcal{P}_m\times\mathbb{N}:\\ l\leq\alpha_A}\right\}}\frac{\varphi_{k_{(i,j,r)}}^{(i)}\,\varphi_{k_{(i,j,r)}}^{(j)}\,e^{-\lambda_{k_{(i,j,r)}}|t_i-t_j|}\,g_{k_{(i,j,r)}}(x_i-x_j)}{\lambda_{k_{(i,j,r)}}}
\end{aligned}\tag{50}$$

for all $t_1,t_2,\ldots,t_m\in\mathbb{R}$, $x_1,x_2,\ldots,x_m\in[0,2\pi]^d$ and all $\varphi^{(1)},\varphi^{(2)},\ldots,\varphi^{(m)}\in\Phi_0$. The proof of Corollary 11 is thus completed. □

In the special case $m=2$, Corollary 11 reduces to the following result.

Corollary 12 (Correlation of Wick powers of V^φ, $\varphi\in\Phi_0$, in real space)**.** *Assume the setting of Subsection 2.1. Then*

$$\begin{aligned}
&\mathbb{E}\Big[\Big(:\big(V_{t_1}^{\varphi^{(1)}}\big)^{n_1}:\Big)(x_1)\cdot\Big(:\big(V_{t_2}^{\varphi^{(2)}}\big)^{n_2}:\Big)(x_2)\Big]\\
&=n_1!\,\delta_{n_1,n_2}\left[\sum_{k\in\mathbb{Z}^d}\frac{\varphi_k^{(1)}\,\varphi_k^{(2)}\,g_k(x_1-x_2)\,e^{-\lambda_k|t_1-t_2|}}{\lambda_k}\right]^{n_1}\\
&=\begin{cases} n_1!\,\delta_{n_1,n_2}\left[\sum_{k_1,\ldots,k_{n_1}\in\mathbb{Z}^d}\prod_{r=1}^{n_1}\frac{\varphi_{k_r}^{(1)}\,\varphi_{k_r}^{(2)}\,e^{-\lambda_{k_r}|t_1-t_2|}\,g_{k_r}(x_1-x_2)}{\lambda_{k_r}}\right] & : n_1\cdot n_2\neq 0\\ \delta_{n_1,n_2} & : n_1\cdot n_2=0\end{cases}
\end{aligned}\tag{51}$$

for all $t_1,t_2\in\mathbb{R}$, $x_1,x_2\in[0,2\pi]^d$, $\varphi^{(1)},\varphi^{(2)}\in\Phi_0$ and all $n_1,n_2\in\mathbb{N}_0$.

Corollary 12 investigates correlations of Wick powers of V^φ, $\varphi\in\Phi_0$, in real space. The next lemma studies correlations of Wick powers of V^φ, $\varphi\in\Phi_0$, in Fourier space. Its proof makes use of Corollary 12.

Lemma 13 (Correlation of Wick powers of V^φ, $\varphi\in\Phi_0$, in Fourier space)**.** *Assume the setting of Subsection 2.1. Then*

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}}\,\mathbb{E}\left[\overline{\Big\langle g_{k_1},:\big(V_{t_1}^{\varphi^{(1)}}\big)^{n_1}:\Big\rangle_H}\Big\langle g_{k_2},:\big(V_{t_2}^{\varphi^{(2)}}\big)^{n_2}:\Big\rangle_H\right]\\
&=\begin{cases} n_1!\,\delta_{n_1,n_2}\,\delta_{k_1,k_2}\left[\sum_{\substack{l_1,\ldots,l_{n_1}\in\mathbb{Z}^d\\ l_1+\ldots+l_{n_1}=k_1}}\left\{\prod_{i=1}^{n_1}\frac{\varphi_{l_i}^{(1)}\,\varphi_{l_i}^{(2)}\,e^{-\lambda_{l_i}|t_1-t_2|}}{\lambda_{l_i}}\right\}\right] & : n_1\cdot n_2\neq 0\\ \delta_{n_1,n_2}\,\delta_{k_1,k_2}\,\delta_{k_1,0} & : n_1\cdot n_2=0\end{cases}
\end{aligned}\tag{52}$$

for all $t_1,t_2\in\mathbb{R}$, $k_1,k_2\in\mathbb{Z}^d$, $\varphi^{(1)},\varphi^{(2)}\in\Phi_0$ and all $n_1,n_2\in\mathbb{N}_0$.

Proof of Lemma 13. First of all, observe that

$$\begin{aligned}
&\mathbb{E}\left[\overline{\Big\langle g_{k_1},:\big(V_{t_1}^{\varphi^{(1)}}\big)^{n_1}:\Big\rangle_H}\Big\langle g_{k_2},:\big(V_{t_2}^{\varphi^{(2)}}\big)^{n_2}:\Big\rangle_H\right]\\
&=\int_{(0,2\pi)^d}\int_{(0,2\pi)^d}\mathbb{E}\left[\overline{\Big(:\big(V_{t_1}^{\varphi^{(1)}}\big)^{n_1}:\Big)(x_1)}\cdot\Big(:\big(V_{t_2}^{\varphi^{(2)}}\big)^{n_2}:\Big)(x_2)\right]g_{-k_1}(x_1)\,g_{k_2}(x_2)\,dx_1\,dx_2
\end{aligned}\tag{53}$$

for all $t_1,t_2\in\mathbb{R}$, $k_1,k_2\in\mathbb{Z}^d$, $\varphi^{(1)},\varphi^{(2)}\in\Phi_0$ and all $n_1,n_2\in\mathbb{N}_0$. Equation (53) and Corollary 12 prove (52) in the case $(n_1,n_2)\in(\mathbb{N}_0)^2\backslash\{(k,k)\in\mathbb{N}^2\colon k\in\mathbb{N}\}$. Furthermore, equation (53), Corollary 12 and the integral transformation theorem imply that

$$
\begin{aligned}
&\mathbb{E}\Big[\overline{\big\langle g_{k_1}, :(V_{t_1}^{\varphi^{(1)}})^n: \big\rangle_H}\, \big\langle g_{k_2}, :(V_{t_2}^{\varphi^{(2)}})^n: \big\rangle_H\Big] \\
&= n! \int_{(0,2\pi)^d} \int_{(0,2\pi)^d} \left[\sum_{\substack{v_1, \\ \dots, \\ v_n \\ \in \mathbb{Z}^d}} \frac{\prod_{r=1}^n \big[\varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\, e^{-\lambda_{v_r}|t_1-t_2|}\, g_{v_r}(x_1-x_2)\big]}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}} \right] g_{-k_1}(x_1)\, g_{k_2}(x_2)\, dx_1\, dx_2 \\
&= n! \int_{(0,2\pi)^d} \int_{(0,2\pi)^d - x_2} \left[\sum_{\substack{v_1,\dots, \\ v_n \in \mathbb{Z}^d}} \frac{\prod_{r=1}^n \big[\varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\, e^{-\lambda_{v_r}|t_1-t_2|}\, g_{v_r}(y)\big]}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}} \right] g_{-k_1}(y+x_2)\, g_{k_2}(x_2)\, dy\, dx_2 \qquad (54)\\
&= n! \int_{(0,2\pi)^d} \left[\sum_{\substack{v_1,\dots, \\ v_n \in \mathbb{Z}^d}} \int_{(0,2\pi)^d - x_2} \frac{\prod_{r=1}^n \big[\varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\, e^{-\lambda_{v_r}|t_1-t_2|}\, g_{v_r}(y)\big]}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}}\, g_{-k_1}(y)\, dy \right] g_{(k_2-k_1)}(x_2)\, dx_2 \\
&= \delta_{k_1,k_2}\, n!\, (2\pi)^d \left[\sum_{v_1,\dots,v_n \in \mathbb{Z}^d} \int_{(0,2\pi)^d} \frac{\prod_{r=1}^n \big[\varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\, e^{-\lambda_{v_r}|t_1-t_2|}\, g_{v_r}(y)\big]}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}}\, g_{-k_1}(y)\, dy \right]
\end{aligned}
$$

for all $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}^d$, $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$ and all $n \in \mathbb{N}$. This shows that

$$
\begin{aligned}
&\mathbb{E}\Big[\overline{\big\langle g_{k_1}, :(V_{t_1}^{\varphi^{(1)}})^n: \big\rangle_H}\, \big\langle g_{k_2}, :(V_{t_2}^{\varphi^{(2)}})^n: \big\rangle_H\Big] \\
&= \delta_{k_1,k_2}\, n!\, (2\pi)^d \left[\sum_{v_1,\dots,v_n \in \mathbb{Z}^d} \int_{(0,2\pi)^d} \frac{\prod_{r=1}^n \big[\varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\, e^{-\lambda_{v_r}|t_1-t_2|}\, g_{v_r}(y)\big]}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}}\, g_{-k_1}(y)\, dy \right] \\
&= \delta_{k_1,k_2}\, n!\, (2\pi)^d \sum_{v_1,\dots,v_n \in \mathbb{Z}^d} \int_{(0,2\pi)^d} \frac{\big[\prod_{r=1}^n \varphi_{v_r}^{(1)}\, \varphi_{v_r}^{(2)}\big]\, g_{(-k_1+\sum_{r=1}^n v_r)}(y)\, e^{-[\sum_{r=1}^n \lambda_{v_r}]|t_1-t_2|}}{\lambda_{v_1}\cdot\ldots\cdot\lambda_{v_n}}\, dy \qquad (55)\\
&= \delta_{k_1,k_2}\, n!\, (2\pi)^{2d} \sum_{\substack{v_1,\dots,v_n \in \mathbb{Z}^d \\ v_1+\ldots+v_n = k_1}} \prod_{i=1}^n \left[\frac{\varphi_{v_i}^{(1)}\, \varphi_{v_i}^{(2)}\, e^{-\lambda_{v_i}|t_1-t_2|}}{\lambda_{v_i}} \right]
\end{aligned}
$$

for all $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}^d$, $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$ and all $n \in \mathbb{N}$. The proof of Lemma 13 is thus completed. □

The next result, Proposition 14, proves convergence of Wick powers in the case $(n,d) \in (\{2,3,\dots\} \times \{2\}) \cup \{(2,3)\}$. The proof of Proposition 14 makes use of Lemma 13.

Proposition 14 (Convergence of Wick powers)**.** *Assume the setting of Subsection 2.1 and let* $(n,d) \in (\{2,3,\dots\} \times \{2\}) \cup \{(2,3)\}$. *Then there exists an up to indistinguishability unique stochastic process* $:(V)^n: \colon \mathbb{R} \times \Omega \to \cap_{\beta \in (-\infty, 2-d)} \mathcal{C}_{\mathcal{P}}^{\beta}([0,2\pi]^d, \mathbb{R})$ *with continuous sample paths which satisfies for every* $T, p \in (0,\infty)$, $\alpha \in (0, \frac{4-d}{4})$, $\beta \in \mathbb{R}$ *with* $2\alpha + \beta < 2 - d$ *that*

$$
\|:(V^{\varphi})^n: - :(V)^n:\|_{L^p(\Omega; C^{\alpha}([-T,T], \mathcal{C}_{\mathcal{P}}^{\beta}([0,2\pi]^d, \mathbb{R})))} \to 0 \qquad \textit{as} \qquad \Phi_{0,\le 1} \ni \varphi \to 1. \qquad (56)
$$

Proof of Proposition 14. We apply Lemma 13 four times to obtain that

$$
\begin{aligned}
&\mathbb{E}\Big[\overline{\big\langle g_{k_1}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n: \big\rangle_H}\, \big\langle g_{k_2}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n: \big\rangle_H\Big] \\
&= n!\, (2\pi)^{2d}\, \delta_{k_1,k_2} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \left\{ \prod_{i=1}^n \frac{[\varphi_{l_i}]^2}{\lambda_{l_i}} - 2 \prod_{i=1}^n \frac{\varphi_{l_i}\psi_{l_i}}{\lambda_{l_i}} + \prod_{i=1}^n \frac{[\psi_{l_i}]^2}{\lambda_{l_i}} \right\} \right] \qquad (57)\\
&= n!\, (2\pi)^{2d}\, \delta_{k_1,k_2} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2}{\big(\prod_{i=1}^n \lambda_{l_i}\big)} \right]
\end{aligned}
$$

for all $t \in \mathbb{R}$, $\varphi, \psi \in \Phi_0$ and all $k_1, k_2 \in \mathbb{Z}^d$. Next observe that

$$\begin{aligned}
\left|\prod_{i=1}^{n} \varphi_{l_i} - \prod_{i=1}^{n} \psi_{l_i}\right| &= \left|\sum_{i=1}^{n} \left[\prod_{j=1}^{i} \varphi_{l_i}\right] \left[\prod_{j=i+1}^{n} \psi_{l_i}\right] - \left[\prod_{j=1}^{i-1} \varphi_{l_i}\right] \left[\prod_{j=i}^{n} \psi_{l_i}\right]\right| \\
&= \sum_{i=1}^{n} \underbrace{\left[\prod_{j=1}^{i-1} \varphi_{l_i}\right]}_{\leq 1} \underbrace{\left[\prod_{j=i+1}^{n} \psi_{l_i}\right]}_{\leq 1} |\varphi_{l_i} - \psi_{l_i}| \leq \underbrace{\sum_{i=1}^{n} |\varphi_{l_i} - \psi_{l_i}|}_{\leq n}
\end{aligned} \tag{58}$$

for all $t \in \mathbb{R}$, $l_1, \ldots, l_n \in \mathbb{Z}^d$ and all $\varphi, \psi \in \Phi_{0,\leq 1}$. Combining (57) and (58) implies that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \sum_{k_1,k_2\in\mathbb{Z}^d} \frac{\left|\mathbb{E}\left[\left\langle g_{k_1}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n:\right\rangle_H \left\langle g_{k_2}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n:\right\rangle_H\right]\right|}{(\lambda_{k_1}\lambda_{k_2})^{-\beta}} \\
&\leq n!\,(2\pi)^{2d} \left[\sum_{k\in\mathbb{Z}^d} \sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k}} \frac{(\lambda_k)^{2\beta} \left(\sum_{i=1}^n |\varphi_{l_i} - \psi_{l_i}|\right)^2}{\left(\prod_{i=1}^n \lambda_{l_i}\right)}\right]
\end{aligned} \tag{59}$$

for all $\beta \in \mathbb{R}$ and all $\varphi, \psi \in \Phi_{0,\leq 1}$. Morever, combining the identity

$$\sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k}} \frac{1}{\left(\prod_{i=1}^n \lambda_{l_i}\right)} = \sum_{l_1\in\mathbb{Z}^d} \frac{1}{\lambda_{l_1}} \left[\sum_{l_2\in\mathbb{Z}^d} \frac{1}{\lambda_{l_2}} \left[\cdots \left[\sum_{l_{n-1}\in\mathbb{Z}^d} \frac{1}{\lambda_{l_{n-1}} \cdot \lambda_{(k-l_1-\ldots-l_{n-1})}}\right]\right]\right] \tag{60}$$

for all $k \in \mathbb{Z}^d$ with Corollary 8 and with the assumption $(n, d) \in (\{2, 3, \ldots\} \times \{2\}) \cup \{(2, 3)\}$ proves that $\sup_{k\in\mathbb{Z}^d} \left[\sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k}} \frac{(\lambda_k)^{\beta}}{\left(\prod_{i=1}^n \lambda_{l_i}\right)}\right] < \infty$ for all $\beta \in (-\infty, 2 - \frac{d}{2})$. This implies that

$$\sum_{k\in\mathbb{Z}^d} \sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k}} \frac{(\lambda_k)^{2\beta}}{\left(\prod_{i=1}^n \lambda_{l_i}\right)} < \infty \tag{61}$$

for all $\beta \in (-\infty, \frac{2-d}{2})$. Dominated convergence and (59) therefore show for every $\beta \in (-\infty, \frac{2-d}{2})$ that

$$\sup_{t\in\mathbb{R}} \sum_{k_1,k_2\in\mathbb{Z}^d} \frac{\left|\mathbb{E}\left[\left\langle g_{k_1}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n:\right\rangle_H \left\langle g_{k_2}, :(V_t^{\varphi})^n: - :(V_t^{\psi})^n:\right\rangle_H\right]\right|}{(\lambda_{k_1}\lambda_{k_2})^{-\beta}} \to 0 \quad \text{as} \quad (\Phi_{0,\leq 1})^2 \ni (\varphi, \psi) \to (1, 1). \tag{62}$$

Next observe that Lemma 13 shows that

$$\begin{aligned}
&\mathbb{E}\left[\begin{array}{l} \left\langle g_{-k_1}, \left[:(V_{t_1}^{\varphi})^n: - :(V_{t_1}^{\psi})^n:\right] - \left[:(V_{t_2}^{\varphi})^n: - :(V_{t_2}^{\psi})^n:\right]\right\rangle_H \\ \cdot \left\langle g_{k_2}, \left[:(V_{t_1}^{\varphi})^n: - :(V_{t_1}^{\psi})^n:\right] - \left[:(V_{t_2}^{\varphi})^n: - :(V_{t_2}^{\psi})^n:\right]\right\rangle_H \end{array}\right] \\
&= n!\,(2\pi)^{2d}\,\delta_{k_1,k_2} \sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k_1}} \frac{\left(2\prod_{i=1}^n (\varphi_{l_i})^2 - 4\prod_{i=1}^n \varphi_{l_i}\psi_{l_i} + 2\prod_{i=1}^n (\psi_{l_i})^2\right)\left(1 - e^{-\sum_{i=1}^n \lambda_{l_i}|t_1-t_2|}\right)}{\left(\prod_{i=1}^n \lambda_{l_i}\right)} \\
&= 2\,n!\,(2\pi)^{2d}\,\delta_{k_1,k_2} \left[\sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k_1}} \frac{\left(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\right)^2 \left(1 - e^{-\sum_{i=1}^n \lambda_{l_i}|t_1-t_2|}\right)}{\left(\prod_{i=1}^n \lambda_{l_i}\right)}\right] \\
&\leq n!\,(2\pi)^{4d}\,\delta_{k_1,k_2} \left[\sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k_1}} \frac{\left(\sum_{i=1}^n |\varphi_{l_i} - \psi_{l_i}|\right)^2 \left(\sum_{i=1}^n \lambda_{l_i}\right)^{2\alpha}}{\left(\prod_{i=1}^n \lambda_{l_i}\right)}\right] |t_1 - t_2|^{2\alpha} \\
&\leq n!\,(2\pi)^{4d}\,\delta_{k_1,k_2} \left[\sum_{\substack{l_1,\ldots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=k_1}} \frac{\left(\sum_{i=1}^n |\varphi_{l_i} - \psi_{l_i}|\right)^2 \left(\sum_{i=1}^n (\lambda_{l_i})^{2\alpha}\right)}{\left(\prod_{i=1}^n \lambda_{l_i}\right)}\right] |t_1 - t_2|^{2\alpha}
\end{aligned} \tag{63}$$

for all $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}^d$, $\alpha \in (0, \frac{1}{2}]$ and all $\varphi, \psi \in \Phi_{0,\leq 1}$ where we used $1 - e^{-x} \leq x^{2\alpha}$ for all $\alpha \in [0, \frac{1}{2}]$ and all $x \in [0, \infty)$ and $\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i} \leq \sum_{i=1}^n |\varphi_{l_i} - \psi_{l_i}|$ for all $l_1, \ldots, l_n \in \mathbb{Z}^d$ and all $\varphi, \psi \in \Phi_{0,\leq 1}$ (cf. (58)) in the last but one line of (63) and where we used $\left(\sum_{i=1}^n \lambda_{l_i}\right)^{2\alpha} \leq \sum_{i=1}^n \left(\lambda_{l_i}\right)^{2\alpha}$ for all $\alpha \in [0, \frac{1}{2}]$ in the last line of (63). Moreover, Corollary 8 proves that

$$\sum_{k \in \mathbb{Z}^d} \left[\sum_{\substack{l_1, \ldots, l_n \in \mathbb{Z}^d \\ l_1 + \ldots + l_n = k_1}} \frac{\left(\lambda_{l_n}\right)^{2\alpha} \left(\lambda_k\right)^{2\beta}}{\left(\prod_{i=1}^n \lambda_{l_i}\right)} \right] < \infty \tag{64}$$

for all $\alpha \in (0, \frac{4-d}{4})$, $\beta \in \mathbb{R}$ with $\alpha + \beta < \frac{2-d}{2}$. Dominated convergence and (63) therefore show for every $\alpha \in (0, \frac{4-d}{4})$, $\beta \in \mathbb{R}$ with $\alpha + \beta < \frac{2-d}{2}$ that

$$\sup_{\substack{t_1, t_2 \in \mathbb{R} \\ t_1 \neq t_2}} \left[\sum_{k_1, k_2 \in \mathbb{Z}^d} \frac{\left| \mathbb{E}\left[\begin{array}{l} \left\langle g_{-k_1}, \left[:(V_{t_1}^{\varphi})^n: - :(V_{t_1}^{\psi})^n: \right] - \left[:(V_{t_2}^{\varphi})^n: - :(V_{t_2}^{\psi})^n: \right] \right\rangle_H \\ \cdot \left\langle g_{k_2}, \left[:(V_{t_1}^{\varphi})^n: - :(V_{t_1}^{\psi})^n: \right] - \left[:(V_{t_2}^{\varphi})^n: - :(V_{t_2}^{\psi})^n: \right] \right\rangle_H \end{array} \right] \right|}{\left(\lambda_{k_1} \lambda_{k_2}\right)^{-\beta} |t_1 - t_2|^{2\alpha}} \right] \to 0 \ \text{ as } \ \left(\Phi_{0,\leq 1}\right)^2 \ni (\varphi, \psi) \to (1, 1). \tag{65}$$

Combining (62) and (65) with Lemma 1 completes the proof of Proposition 14. ☐

The next proposition is well known in the literature (see, for instance, Da Prato & Zabczyk [5] for related results and references) and its proof is therefore omitted.

Proposition 15 (Ornstein-Uhlenbeck processes). *Assume the setting of Subsection 2.1 and let $d \in \mathbb{N}$. Then there exists an up to indistinguishability unique stochastic process $V \colon \mathbb{R} \times \Omega \to \cap_{\beta \in (-\infty, \frac{2-d}{2})} \mathcal{C}_{\mathcal{P}}^{\beta}([0, 2\pi]^d, \mathbb{R})$ with continuous sample paths which satisfies for every $T, p \in (0, \infty)$, $\alpha \in (0, \frac{1}{2})$, $\beta \in \mathbb{R}$ with $2\alpha + \beta < \frac{2-d}{2}$ that*

$$\|V^{\varphi} - V\|_{L^p(\Omega; C^{\alpha}([-T,T], \mathcal{C}_{\mathcal{P}}^{\beta}([0,2\pi]^d, \mathbb{R})))} \to 0 \qquad \text{as} \qquad \Phi_{0,\leq 1} \ni \varphi \to 1. \tag{66}$$

Proposition 14 shows convergence of Wick powers in the case $(n, d) \in (\{2, 3, \ldots\} \times \{2\}) \cup \{(2, 3)\}$. In the case $(n, d) \in (\{3, 4, \ldots\} \times \{3\}) \cup (\{2, 3, \ldots\} \times \{4, 5, \ldots\})$, Wick powers do not converge anymore. This is the subject of the next lemma. In the case $d = n = 3$, a statement similar to the next lemma has been formulated in Section 7 in Da Prato & Tubaro [4].

Lemma 16 (Divergence of Wick powers). *Assume the setting of Subsection 2.1, let $d \in \{3, 4, \ldots\}$, $n \in \{2, 3, \ldots\}$ be natural numbers with $d + n \geq 6$ and let $C_0, C_1, \ldots, C_{n-1} \colon \Phi_0 \to \mathbb{R}$ be arbitrary functions. Then it holds for every $v \in \mathbb{Z}^d$ and every $t \in \mathbb{R}$ that*

$$\mathbb{E}\left[\left| \left\langle g_v, (V_t^{\varphi})^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot (V_t^{\varphi})^k \right\rangle_H \right|^2 \right] \to \infty \qquad \text{as} \qquad \Phi_0 \ni \varphi \to 1. \tag{67}$$

Proof of Lemma 16. Throughout this proof let $\hat{C}_0, \hat{C}_1, \ldots \hat{C}_n \colon \Phi_0 \to \mathbb{R}$ be the unique functions satisfying $\hat{C}_0(0) = -C_0(0)$, $\hat{C}_1(0) = -C_1(0)$, $\ldots$, $\hat{C}_{n-1}(0) = -C_{n-1}(0)$, $\hat{C}_n(0) = 1$ and

$$x^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot x^k = \sum_{k=0}^{n} \hat{C}_k(\varphi) \cdot \left[\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v} \right]^{\frac{k}{2}} \cdot H_k\left(\frac{x}{\sqrt{\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}}} \right) \tag{68}$$

for all $x \in \mathbb{R}$, $\varphi = (\varphi_v)_{v \in \mathbb{Z}^d} \in \Phi_0 \backslash \{0\}$ and all $t \in \mathbb{R}$. This ensures that $\hat{C}_n(\varphi) = 1$ and

$$(V_t^{\varphi})^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot (V_t^{\varphi})^k = \sum_{k=0}^{n} \hat{C}_k(\varphi) \cdot \left(:(V_t^{\varphi})^k: \right) \tag{69}$$

for all $\varphi \in \Phi_0$ and all $t \in \mathbb{R}$. Lemma 13 hence implies that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left\langle g_v, (V_t^\varphi)^n - \sum_{k=0}^{n-1} C_k(\varphi)\cdot (V_t^\varphi)^k\right\rangle_H\right|^2\right] = \mathbb{E}\left[\left|\sum_{k=0}^{n} \left\langle g_v, \hat{C}_k(\varphi)\left(:(V_t^\varphi)^k:\right)\right\rangle_H\right|^2\right] \\
&= \sum_{k,l=0}^{n} \hat{C}_k(\varphi)\cdot \hat{C}_l(\varphi)\cdot \mathbb{E}\Big[\overline{\langle g_v, :(V_t^\varphi)^k:\rangle_H}\,\langle g_v, :(V_t^\varphi)^l:\rangle_H\Big] \\
&= \sum_{k=0}^{n} \left|\hat{C}_k(\varphi)\right|^2 \mathbb{E}\Big[\left|\langle g_v, :(V_t^\varphi)^k:\rangle_H\right|^2\Big] \geq \left|\hat{C}_n(\varphi)\right|^2 \mathbb{E}\Big[\left|\langle g_v, :(V_t^\varphi)^n:\rangle_H\right|^2\Big] \\
&= n!\,(2\pi)^{2d}\left[\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=v}} \left\{\prod_{i=1}^{n}\frac{(\varphi_{l_i})^2}{\lambda_{l_i}}\right\}\right] \geq \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=v}} \frac{(\varphi_{l_1})^2\cdot\ldots\cdot(\varphi_{l_n})^2}{\lambda_{l_1}\cdot\ldots\cdot\lambda_{l_n}}
\end{aligned}
\tag{70}
$$

for all $v \in \mathbb{Z}^d$ and all $\varphi \in \Phi_0$. Next note that the estimate

$$
\sum_{l_1,l_2\in\mathbb{Z}^3}\frac{1}{\lambda_{l_1}\lambda_{l_2}\lambda_{(v-l_1-l_2)}} \geq \sum_{l_1,l_2\in\mathbb{Z}^3}\frac{1}{3\big(1+\|l_1\|_{\mathbb{R}^3}^2\big)\big(1+\|l_2\|_{\mathbb{R}^3}^2\big)\big(1+\|l_1\|_{\mathbb{R}^3}^2+\|l_2\|_{\mathbb{R}^3}^2+\|v\|_{\mathbb{R}^3}^2\big)} = \infty \tag{71}
$$

for all $v \in \mathbb{Z}^d$ together with the assumptions $d \geq 3$, $n \geq 2$ and $d + n \geq 6$ and Lemma 6 implies that

$$
\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\ldots+l_n=v}}\frac{1}{\lambda_{l_1}\cdot\ldots\cdot\lambda_{l_n}} = \sum_{l_1,\dots,l_{n-1}\in\mathbb{Z}^d}\frac{1}{\lambda_{l_1}\cdot\ldots\cdot\lambda_{l_{n-1}}\cdot\lambda_{(v-l_1-\ldots-l_{n-1})}} = \infty \tag{72}
$$

for all $v \in \mathbb{Z}^d$. Combining this with (70) completes the proof of Lemma 16. □

2.5 Averaged Wick powers of Ornstein-Uhlenbeck processes

In the previous subsection it has been proved in the case $d = 3$ that for every $t \in \mathbb{R}$ the family $:(V_t^\varphi)^3:$, $\varphi \in \Phi_{0,\leq 1}$, *does not converge* as $\Phi_{0,\leq 1} \ni \varphi \to 1 \in \Phi_{0,\leq 1}$ (see Lemma 16). In this subsection we prove in the case $d = 3$ that for every $(t_0, t) \in \{(s_0, s) \in \mathbb{R}^2 \colon s_0 \leq s\}$ the family $\circ(V_{t_0,t}^\varphi)^3\circ = \int_{t_0}^{t} :(V_s^\varphi)^3: ds$, $\varphi \in \Phi_{0,\leq 1}$, *does converge* as $\Phi_{0,\leq 1} \ni \varphi \to 1 \in \Phi_{0,\leq 1}$ (see Proposition 19).

Lemma 17 (Correlation of averaged Wick powers of V^φ, $\varphi \in \Phi_0$, in Fourier space)**.** *Assume the setting of Subsection 2.1. Then*

$$
\frac{1}{(2\pi)^{2d}}\,\mathbb{E}\left[\overline{\left\langle g_{k_1}, \circ\big(V_{t_0,t_1}^{\varphi^{(1)}}\big)^{n_1}\circ\right\rangle_H}\left\langle g_{k_2}, \circ\big(V_{t_0,t_2}^{\varphi^{(2)}}\big)^{n_2}\circ\right\rangle_H\right]
= \begin{cases} n_1!\,\delta_{n_1,n_2}\,\delta_{k_1,k_2}\displaystyle\sum_{\substack{l_1,\dots,l_{n_1}\in\mathbb{Z}^d\\ l_1+\ldots+l_{n_1}=k_1}} \frac{\prod_{i=1}^{n_1}\varphi_{l_i}^{(1)}\,\varphi_{l_i}^{(2)}}{\prod_{i=1}^{n_1}\lambda_{l_i}}\int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-(\sum_{i=1}^{n}\lambda_{l_i})|s_1-s_2|}\,ds_2\,ds_1 & : n_1 n_2 \neq 0 \\ \delta_{n_1,n_2}\,\delta_{k_1,k_2} & : n_1 n_2 = 0 \end{cases} \tag{73}
$$

for all $k_1, k_2 \in \mathbb{Z}^d$, $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$, $n_1, n_2 \in \mathbb{N}_0$ and all $t_0, t_1, t_2 \in \mathbb{R}$ with $t_0 \leq \min(t_1, t_2)$.

Lemma 17 is an immediate consequence of Lemma 13 and the proof of Lemma 17 is therefore omitted.

Lemma 18 (Time integrals for averaged Wick powers)**.** *Assume the setting of Subsection 2.1. Then*

$$
\int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-c|s_1-s_2|}\,ds_2\,ds_1 = \frac{2(\min(t_1,t_2)-t_0)}{c} + \frac{\Big(\Big[\sum_{j=1}^{2} e^{-c(t_j-t_0)}\Big]-1-e^{-c(t_2-t_1)}\Big)}{c^2} \tag{74}
$$

and

$$
\frac{(t_1-t_0)\Big(1-e^{-\frac{c(t_1-t_0)}{2}}\Big)}{c} \leq \int_{t_0}^{t_1}\int_{t_0}^{t_1} e^{-c|s_1-s_2|}\,ds_2\,ds_1 = \frac{2}{c^2}\Big(e^{-c(t_1-t_0)} - [1-(t_1-t_0)\,c]\Big) \leq \frac{2\,(t_1-t_0)^\theta}{c^{(2-\theta)}} \tag{75}
$$

for all $c \in (0, \infty)$, $\theta \in [1, 2]$ and all $t_0, t_1, t_2 \in \mathbb{R}$ with $t_0 \leq \min(t_1, t_2)$.

Proof of Lemma 18. Note that

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{t_0}^{t_2} \mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_2\le u_1\}}(s_1,s_2)\cdot e^{-c|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{t_0}^{t_1}\int_{t_0}^{s_1} e^{-c(s_1-s_2)}\,ds_2\,ds_1=\int_{t_0}^{t_1}\frac{\left(1-e^{-c(s_1-t_0)}\right)}{c}\,ds_1=\frac{(t_1-t_0)}{c}+\frac{\left(e^{-c(t_1-t_0)}-1\right)}{c^2}\end{aligned}\tag{76}$$

and hence

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{t_0}^{t_2} \mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_1\le u_2\le t_1\}}(s_1,s_2)\cdot e^{-c|s_1-s_2|}\,ds_1\,ds_2\\&=\int_{t_0}^{t_1}\int_{t_0}^{s_2} e^{-c(s_2-s_1)}\,ds_1\,ds_2=\frac{(t_1-t_0)}{c}+\frac{\left(e^{-c(t_1-t_0)}-1\right)}{c^2}\end{aligned}\tag{77}$$

for all $c\in(0,\infty)$ and all $t_0,t_1,t_2\in\mathbb{R}$ with $t_0\le t_1\le t_2$. Furthermore, observe that

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{t_0}^{t_2} \mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_1\le t_1\le u_2\}}(s_1,s_2)\cdot e^{-c|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{t_1}^{t_2}\int_{t_0}^{t_1} e^{-c(s_2-s_1)}\,ds_1\,ds_2=\int_{t_1}^{t_2}\frac{\left(e^{-c(s_2-t_1)}-e^{-c(s_2-t_0)}\right)}{c}\,ds_2\\&=\frac{-\left(e^{-c(t_2-t_1)}-1\right)+\left(e^{-c(t_2-t_0)}-e^{-c(t_1-t_0)}\right)}{c^2}=\frac{1+e^{-c(t_2-t_0)}-e^{-c(t_2-t_1)}-e^{-c(t_1-t_0)}}{c^2}\end{aligned}\tag{78}$$

for all $c\in(0,\infty)$ and all $t_0,t_1,t_2\in\mathbb{R}$ with $t_0\le t_1\le t_2$. Combining (76)–(78) results in

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-c|s_1-s_2|}\,ds_2\,ds_1\\&=\frac{2\,(t_1-t_0)}{c}+\frac{2\left(e^{-c(t_1-t_0)}-1\right)}{c^2}+\frac{\left(1+e^{-c(t_2-t_0)}-e^{-c(t_2-t_1)}-e^{-c(t_1-t_0)}\right)}{c^2}\\&=\frac{2\,(t_1-t_0)}{c}+\frac{\left(e^{-c(t_1-t_0)}+e^{-c(t_2-t_0)}-1-e^{-c(t_2-t_1)}\right)}{c^2}\end{aligned}\tag{79}$$

for all $c\in(0,\infty)$ and all $t_0,t_1,t_2\in\mathbb{R}$ with $t_0\le t_1\le t_2$. In addition, observe that

$$\begin{aligned}&\frac{|y|\left(1-e^{\frac{y}{2}}\right)}{2}=\int_y^{\frac{y}{2}}1-e^{\frac{y}{2}}\,ds\le\int_y^{\frac{y}{2}}1-e^{s}\,ds\le\int_y^0 1-e^s\,ds\\&=e^y-(1+y)=\int_y^0 1-e^s\,ds\le\int_y^0[1-e^s]^\theta\,ds\le\int_y^0\left[\int_s^0 e^u\,du\right]^\theta ds\\&\le\int_y^0|s|^\theta\,ds=\int_0^{-y}s^\theta\,ds=\frac{|y|^{(1+\theta)}}{(1+\theta)}\le|y|^{(1+\theta)}\end{aligned}\tag{80}$$

for all $y\in(-\infty,0]$ and all $\theta\in[0,1]$. Combining this with (79) completes the proof of Lemma 18. ☐

The next result, Proposition 19, establishes convergence of averaged Wick powers under the assumption that $n,d\in\{2,3,\dots\}$ with $\frac{n+1}{n-1}>\frac{d}{2}$. The proof of Proposition 19 exploits Lemma 1, Lemma 17 and Lemma 18.

Proposition 19 (Convergence of averaged Wick powers)**.** *Assume the setting of Subsection 2.1, let $t_0\in\mathbb{R}$ and let $n,d\in\{2,3,\dots\}$ with $\frac{n+1}{n-1}>\frac{d}{2}$. Then there exists an up to indistinguishability unique stochastic process*

$$\circ\left(V_{t_0,(\cdot)}\right)^n\circ\colon[t_0,\infty)\times\Omega\to\cap_{\beta\in(-\infty,1+\frac{1}{n}-\frac{d}{2}+(n-1)\min(1+\frac{1}{n}-\frac{d}{2},0))}\mathcal{C}^\beta_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})\tag{81}$$

with continuous sample paths which satisfies for every $T\in(t_0,\infty)$, $p\in(0,\infty)$, $\alpha\in(0,1)$ and every $\beta\in(-\infty,1+\frac{1}{n}-\frac{d}{2}+(n-1)\min(1+\frac{1}{n}-\frac{d}{2},0))$ that

$$\|\circ(V^\varphi_{t_0,(\cdot)})^n\circ-\circ(V_{t_0,(\cdot)})^n\circ\|_{L^p(\Omega;C^\alpha([t_0,T],\mathcal{C}^\beta_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})))}\to 0\qquad as\qquad \Phi_{0,\le 1}\ni\varphi\to 1.\tag{82}$$

Proof of Proposition 19. Lemma 17 and Lemma 18 imply

$$
\begin{aligned}
&\frac{1}{(2\pi)^{2d}}\,\mathbb{E}\Big[\overline{\Big\langle g_{k_1}, \circ (V^{\varphi}_{\hat t,t})^n \circ - \circ (V^{\psi}_{\hat t,t})^n \circ\Big\rangle_H}\,\Big\langle g_{k_2}, \circ (V^{\varphi}_{\hat t,t})^n \circ - \circ (V^{\psi}_{\hat t,t})^n \circ\Big\rangle_H\Big]\\
&= n!\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \left[\begin{array}{c}\Big\{\prod_{i=1}^n \frac{[\varphi_{l_i}]^2}{\lambda_{l_i}} - 2\prod_{i=1}^n \frac{\varphi_{l_i}\psi_{l_i}}{\lambda_{l_i}} + \prod_{i=1}^n \frac{[\psi_{l_i}]^2}{\lambda_{l_i}}\Big\}\\ \cdot \int_{\hat t}^t \int_{\hat t}^t e^{-[\sum_{i=1}^n \lambda_{l_i}]|s_2-s_1|}\,ds_2\,ds_1\end{array}\right]\\
&= n!\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2}{\big(\prod_{i=1}^n \lambda_{l_i}\big)} \int_{\hat t}^t \int_{\hat t}^t e^{-[\sum_{i=1}^n \lambda_{l_i}]|s_2-s_1|}\,ds_2\,ds_1\\
&\le n!\,2\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2 (t-\hat t)}{\big(\prod_{i=1}^n \lambda_{l_i}\big)\big(\sum_{i=1}^n \lambda_{l_i}\big)}\\
&\le n!\,2\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2 (t-\hat t)}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+1/n)}\Big)}
\end{aligned}
\tag{83}
$$

for all $k_1, k_2 \in \mathbb{Z}^d$ and all $\hat t, t \in \mathbb{R}$ with $\hat t \le t$. Next note that Corollary 8 ensures that

$$
\sup_{k_1\in\mathbb{Z}^d}\left[\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \frac{(\lambda_{k_1})^{\gamma}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+1/n)}\Big)}\right] < \infty \tag{84}
$$

for all $\gamma \in \big(0, 1+\frac{1}{n} + (n-1)\min\big(1+\frac{1}{n}-\frac{d}{2}, 0\big)\big)$ and therefore, we obtain that

$$
\sum_{k_1\in\mathbb{Z}^d} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=k_1}} \frac{(\lambda_{k_1})^{2\beta}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+1/n)}\Big)} < \infty \tag{85}
$$

for all $\beta \in \big(-\infty, \frac{1}{2}+\frac{1}{2n} + (n-1)\min\big(\frac{1}{2}+\frac{1}{2n}-\frac{d}{4}, 0\big) - \frac{d}{4}\big)$. Combining this, (83) and dominated convergence implies for every $\beta \in \big(-\infty, \frac{1}{2}+\frac{1}{2n} - \frac{d}{4} + (n-1)\min\big(\frac{1}{2}+\frac{1}{2n}-\frac{d}{4}, 0\big)\big)$ that

$$
\sup_{\substack{\hat t, t\in\mathbb{R},\\ \hat t<t}} \sum_{k_1,k_2\in\mathbb{Z}^d} \frac{\Big|\mathbb{E}\Big[\overline{\Big\langle g_{k_1}, \circ\big(V^{\varphi}_{\hat t,t}\big)^n \circ - \circ\big(V^{\psi}_{\hat t,t}\big)^n \circ\Big\rangle_H}\Big\langle g_{k_2}, \circ\big(V^{\varphi}_{\hat t,t}\big)^n \circ - \circ\big(V^{\psi}_{\hat t,t}\big)^n \circ\Big\rangle_H\Big]\Big|}{(t-\hat t)\cdot(\lambda_{k_1}\lambda_{k_2})^{-\beta}} \to 0 \tag{86}
$$

as $(\Phi_{0,\le 1})^2 \ni (\varphi,\psi) \to (1,1)$. In the next step observe that Definition (22) implies that

$$
\begin{aligned}
&\mathbb{E}\left[\begin{array}{l}\Big\langle g_{-k_1}, \Big[\circ(V^{\varphi}_{\hat t,t_1})^n \circ - \circ (V^{\psi}_{\hat t,t_1})^n\circ\Big] - \Big[\circ(V^{\varphi}_{\hat t,t_2})^n \circ - \circ (V^{\psi}_{\hat t,t_2})^n\circ\Big]\Big\rangle_H\\ \cdot\Big\langle g_{k_2}, \Big[\circ(V^{\varphi}_{\hat t,t_1})^n \circ - \circ (V^{\psi}_{\hat t,t_1})^n\circ\Big] - \Big[\circ(V^{\varphi}_{\hat t,t_2})^n \circ - \circ (V^{\psi}_{\hat t,t_2})^n\circ\Big]\Big\rangle_H\end{array}\right]\\
&= \mathbb{E}\Big[\Big\langle g_{-k_1}, \circ(V^{\varphi}_{t_1,t_2})^n \circ - \circ (V^{\psi}_{t_1,t_2})^n\circ\Big\rangle_H \Big\langle g_{k_2}, \circ(V^{\varphi}_{t_1,t_2})^n \circ - \circ (V^{\psi}_{t_1,t_2})^n\circ\Big\rangle_H\Big]
\end{aligned}
\tag{87}
$$

for all $k_1, k_2 \in \mathbb{Z}^d$, $\varphi, \psi \in \Phi_{0,\le 1}$ and all $t_1, t_2 \in \mathbb{R}$ with $\hat t \le t_1 \le t_2$. Combining this with (86) shows for every $\beta \in \big(-\infty, \frac{1}{2}+\frac{1}{2n} - \frac{d}{4} + (n-1)\min\big(\frac{1}{2}+\frac{1}{2n}-\frac{d}{4}, 0\big)\big)$ that

$$
\sup_{\substack{\hat t\in\mathbb{R},\\ t_1,t_2\in[\hat t,\infty),\\ t_1\neq t_2}} \left[\sum_{k_1,k_2\in\mathbb{Z}^d} \frac{\left|\mathbb{E}\left[\begin{array}{l}\Big\langle g_{-k_1}, \Big[\circ(V^{\varphi}_{\hat t,t_1})^n \circ - \circ (V^{\psi}_{\hat t,t_1})^n\circ\Big] - \Big[\circ(V^{\varphi}_{\hat t,t_2})^n \circ - \circ (V^{\psi}_{\hat t,t_2})^n\circ\Big]\Big\rangle_H\\ \cdot\Big\langle g_{k_2}, \Big[\circ(V^{\varphi}_{\hat t,t_1})^n \circ - \circ (V^{\psi}_{\hat t,t_1})^n\circ\Big] - \Big[\circ(V^{\varphi}_{\hat t,t_2})^n \circ - \circ (V^{\psi}_{\hat t,t_2})^n\circ\Big]\Big\rangle_H\end{array}\right]\right|}{(\lambda_{k_1}\lambda_{k_2})^{-\beta}|t_1-t_2|}\right] \to 0 \tag{88}
$$

as $(\Phi_{0,\le 1})^2 \ni (\varphi,\psi) \to (1,1)$. Combining (86) and (88) with Lemma 1 completes the proof of Proposition 19. ◻

Proposition 19 shows convergence of averaged Wick powers under the assumption that $n, d \in \{2, 3, \dots\}$ with $\frac{n+1}{n-1} > \frac{d}{2}$. Lemma 21 below, in particular, proves that averaged Wick powers fail to converge if $n, d \in \{2, 3, \dots\}$ with $\frac{n+1}{n-1} \le \frac{d}{2}$. In the proof of Lemma 21 the following lemma is used.

Lemma 20. *Assume the setting of Subsection 2.1 and let $n, d \in \{2, 3, \dots\}$ with $\frac{(n+1)}{(n-1)} \leq \frac{d}{2}$. Then $\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{(\prod_{i=1}^n \lambda_{l_i})(\lambda_v + \sum_{i=1}^n \lambda_{l_i})} = \infty$ for all $v \in \mathbb{Z}^d$.*

Proof of Lemma 20. Note that

$$
\begin{aligned}
&\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{\left(\prod_{i=1}^n \lambda_{l_i}\right)\left(\lambda_v + \sum_{i=1}^n \lambda_{l_i}\right)} \\
&\geq \sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{\left(\lambda_v + \sum_{i=1}^n \lambda_{l_i}\right)^{(n+1)}} \geq \sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{\left(\lambda_v + \lambda_{(v-l_1-\dots-l_n)} + \sum_{i=1}^{n-1} \lambda_{l_i}\right)^{(n+1)}} \\
&\geq \frac{1}{(n+1)^{(n+1)}} \sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{\left(\lambda_v + \sum_{i=1}^{n-1} \lambda_{l_i}\right)^{(n+1)}} \\
&\geq \frac{1}{\left(2\lambda_v\,(n+1)\right)^{(n+1)}} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\dots+l_n=v}} \frac{1}{\left(\sum_{i=1}^{n-1} \lambda_{l_i}\right)^{(n+1)}}\right] \\
&= \frac{1}{\left(2\lambda_v\,(n+1)\right)^{(n+1)}} \left[\sum_{k \in \mathbb{Z}^{d(n-1)}} \frac{1}{\left(1 + \|k\|^2_{\mathbb{R}^{d(n-1)}}\right)^{(n+1)}}\right] = \infty
\end{aligned}
\tag{89}
$$

for all $v \in \mathbb{Z}^d$. The proof of Lemma 20 is thus completed. ☐

Lemma 21 (Divergence of averaged Wick powers)**.** *Assume the setting of Subsection 2.1, let $n, d \in \{2, 3, \dots\}$ with $\frac{(n+1)}{(n-1)} \leq \frac{d}{2}$ and let $C_0, C_1, \dots, C_{n-1} \colon \Phi_0 \to \mathbb{R}$ be arbitrary functions. Then it holds for every $v \in \mathbb{Z}^d$ and every $t_0, t \in \mathbb{R}$ with $t_0 < t$ that*

$$
\mathbb{E}\left[\left|\left\langle g_v, \int_{t_0}^t \left((V_s^\varphi)^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot (V_s^\varphi)^k \right) ds \right\rangle_H\right|^2\right] \to \infty \qquad as \qquad \Phi_0 \ni \varphi \to 1. \tag{90}
$$

Proof of Lemma 21. Throughout this proof let $\hat{C}_0, \hat{C}_1, \dots \hat{C}_n \colon \Phi_0 \to \mathbb{R}$ be the unique functions satisfying $\hat{C}_0(0) = -C_0(0)$, $\hat{C}_1(0) = -C_1(0)$, $\dots$, $\hat{C}_{n-1}(0) = -C_{n-1}(0)$, $\hat{C}_n(0) = 1$ and

$$
x^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot x^k = \sum_{k=0}^{n} \hat{C}_k(\varphi) \cdot \left[\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}\right]^{\frac{k}{2}} \cdot H_k\left(\frac{x}{\sqrt{\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}}}\right) \tag{91}
$$

for all $x \in \mathbb{R}$, $\varphi \in \Phi_0 \backslash \{0\}$ and all $t \in \mathbb{R}$ (cf. (68)). Then Lemma 17 and Lemma 18 imply that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left\langle g_v, \int_{t_0}^t \left((V_s^\varphi)^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot (V_s^\varphi)^k \right) ds \right\rangle_H \right|^2\right] = \mathbb{E}\left[\left| \sum_{k=0}^n \left\langle g_v, \int_{t_0}^t \hat{C}_k(\varphi) \left(:(V_s^\varphi)^k: \right) ds \right\rangle_H \right|^2\right] \\
&= \sum_{k,l=0}^n \hat{C}_k(\varphi) \cdot \hat{C}_l(\varphi) \cdot \mathbb{E}\Big[\overline{\langle g_v, \circ(V_{t_0,t}^\varphi)^k \circ \rangle_H} \langle g_v, \circ(V_{t_0,t}^\varphi)^l \circ \rangle_H \Big] \\
&= \sum_{k=0}^n \left| \hat{C}_k(\varphi) \right|^2 \mathbb{E}\Big[\left| \langle g_v, \circ(V_{t_0,t}^\varphi)^k \circ \rangle_H \right|^2\Big] \geq \left| \hat{C}_n(\varphi) \right|^2 \mathbb{E}\Big[\left| \langle g_v, \circ(V_{t_0,t}^\varphi)^n \circ \rangle_H \right|^2\Big] \\
&= n!\, (2\pi)^{2d} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n=v}} \left\{ \prod_{i=1}^n \frac{(\varphi_{l_i})^2}{\lambda_{l_i}} \right\} \left\{ \int_{t_0}^t \int_{t_0}^t e^{-(\sum_{i=1}^n \lambda_{l_i})|s_1-s_2|}\, ds_2\, ds_1 \right\} \right] \\
&\geq n!\, (2\pi)^{2d} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n=v}} \left(\prod_{i=1}^n \frac{(\varphi_{l_i})^2}{\lambda_{l_i}} \right) \frac{(t-t_0)\left(1-e^{-\frac{(t-t_0)}{2}}\right)}{(\sum_{i=1}^n \lambda_{l_i})} \right] \\
&\geq n!\, (2\pi)^{2d} \left[\sum_{\substack{l_1,\dots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n=v}} \frac{\left(\prod_{i=1}^n (\varphi_{l_i})^2\right)(t-t_0)\left(1-e^{-\frac{(t-t_0)}{2}}\right)}{(\prod_{i=1}^n \lambda_{l_i})(\lambda_v + \sum_{i=1}^n \lambda_{l_i})} \right]
\end{aligned}
\tag{92}
$$

for all $v \in \mathbb{Z}^d$, $t_0, t \in \mathbb{R}$ with $t_0 \leq t$ and all $\varphi \in \Phi_0$. Combining this with Lemma 20 completes the proof of Lemma 21. ☐

2.6 Convolutional Wick powers of Ornstein-Uhlenbeck processes

Lemma 22 (Correlation of convolutional Wick powers of V^φ, $\varphi \in \Phi_0$, in Fourier space)**.** *Assume the setting of Subsection 2.1. Then*

$$
\begin{aligned}
&\frac{1}{(2\pi)^{2d}} \mathbb{E}\Big[\overline{\langle g_{k_1}, \bullet(V_{t_1}^{\varphi_1})^{n_1} \bullet \rangle_H} \langle g_{k_2}, \bullet(V_{t_2}^{\varphi_2})^{n_2} \bullet \rangle_H \Big] \\
&= \begin{cases} n_1!\, \delta_{n_1,n_2}\, \delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_{n_1} \in \mathbb{Z}^d \\ l_1+\ldots+l_{n_1}=k_1}} \Bigg[\frac{\left(\prod_{i=1}^{n_1} \varphi_{l_i}^{(1)}\, \varphi_{l_i}^{(2)}\right)}{(\prod_{i=1}^{n_1} \lambda_{l_i})} & \\ \quad \cdot \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-\lambda_{k_1}(t_1-s_1+t_2-s_2)-(\sum_{i=1}^{n_1} \lambda_{l_i})|s_1-s_2|}\, ds_2\, ds_1 \Bigg] & : n_1 n_2 \neq 0 \\ \delta_{n_1,n_2}\, \delta_{k_1,k_2}\, \delta_{k_1,0} & : n_1 n_2 = 0 \end{cases}
\end{aligned}
\tag{93}
$$

for all $t_1, t_2 \in \mathbb{R}$, $k_1, k_2 \in \mathbb{Z}^d$, $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$ and all $n_1, n_2 \in \mathbb{N}_0$.

Proof of Lemma 22. Combining the identity

$$
\begin{aligned}
&\frac{1}{(2\pi)^{2d}} \mathbb{E}\left[\overline{\left\langle g_{k_1}, \bullet\left(V_{t_1}^{\varphi^{(1)}}\right)^{n_1} \bullet \right\rangle_H} \left\langle g_{k_2}, \bullet\left(V_{t_2}^{\varphi^{(2)}}\right)^{n_2} \bullet \right\rangle_H \right] \\
&= \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-\lambda_{k_1}(t_1-s_1)}\, e^{-\lambda_{k_2}(t_2-s_2)}\, \frac{\mathbb{E}\left[\overline{\left\langle g_{k_1}, :\left(V_{s_1}^{\varphi^{(1)}}\right)^{n_1}: \right\rangle_H} \left\langle g_{k_2}, :\left(V_{s_2}^{\varphi^{(2)}}\right)^{n_2}: \right\rangle_H \right]}{(2\pi)^{2d}}\, ds_1\, ds_2
\end{aligned}
\tag{94}
$$

for all $\varphi^{(1)}, \varphi^{(2)} \in \Phi_0$, $k_1, k_2 \in \mathbb{Z}^d$, $n_1, n_2 \in \mathbb{N}_0$ and all $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$ with Lemma 13 completes the proof of Lemma 22. ☐

Lemma 23 (Time integrals for convolutional Wick powers)**.** *Assume the setting of Subsection 2.1. Then*

$$
\int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\, ds_2\, ds_1 = \frac{e^{-a(t_2-t_1)}}{a\,(a+b)} + \begin{cases} \frac{\left(e^{-b(t_2-t_1)} - e^{-a(t_2-t_1)}\right)}{(a-b)(a+b)} & : a \neq b \\ \frac{(t_2-t_1)e^{-a(t_2-t_1)}}{(a+b)} & : a = b \end{cases}
\tag{95}
$$

and

$$
\int_{-\infty}^t \int_{-\infty}^t e^{-a(2t-s_1-s_2)-b|s_1-s_2|}\, ds_2\, ds_1 = \frac{1}{a\,(a+b)}
\tag{96}
$$

for all $a,b\in(0,\infty)$ and all $t,t_1,t_2\in\mathbb{R}$ with $t_1\le t_2$.

Proof of Lemma 23. First of all, note that

$$\begin{aligned}&\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_2\le u_1\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{-\infty}^{t_1}\int_{-\infty}^{t_1}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_2\le u_1\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{-\infty}^{t_1}\int_{-\infty}^{s_1}e^{-a(t_1-s_1+t_2-s_2)-b(s_1-s_2)}\,ds_2\,ds_1\\&=\frac{\int_{-\infty}^{t_1}e^{-a(t_1+t_2-2s_1)}\,ds_1}{(a+b)}=\frac{e^{-a(t_2-t_1)}-e^{-a(t_1+t_2-2t_0)}}{2a\,(a+b)},\end{aligned}\tag{97}$$

that

$$\begin{aligned}&\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_1\le u_2\le t_1\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{-\infty}^{t_1}\int_{-\infty}^{t_1}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_1\le u_2\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{-\infty}^{t_1}\int_{-\infty}^{t_1}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_2\le u_1\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\frac{e^{-a(t_2-t_1)}-e^{-a(t_1+t_2-2t_0)}}{2a\,(a+b)}\end{aligned}\tag{98}$$

and that

$$\begin{aligned}&\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\mathbb{1}_{\{(u_1,u_2)\in\mathbb{R}^2\colon\, u_1\le t_1\le u_2\}}(s_1,s_2)\cdot e^{-a(t_1-s_1+t_2-s_2)-b|s_1-s_2|}\,ds_2\,ds_1\\&=\int_{t_1}^{t_2}\int_{-\infty}^{t_1}e^{-a(t_1-s_1+t_2-s_2)-b(s_2-s_1)}\,ds_1\,ds_2\\&=\frac{1}{(a+b)}\int_{t_1}^{t_2}\left(e^{-a(t_2-s_2)-b(s_2-t_1)}\right)ds_2=\begin{cases}\frac{e^{-b(t_2-t_1)}-e^{-a(t_2-t_1)}}{(a-b)(a+b)} & : a\neq b\\ \frac{(t_2-t_1)e^{-a(t_2-t_1)}}{(a+b)} & : a=b\end{cases}\end{aligned}\tag{99}$$

for all $t_1,t_2\in\mathbb{R}$ with $t_1\le t_2$. Combining (97)–(99) proves that

$$\int_{-\infty}^{t_1}\int_{-\infty}^{t_2}e^{-a(t_1-s_1+t_2-s_2)-b|s_2-s_1|}\,ds_2\,ds_1=\frac{e^{-a(t_2-t_1)}}{a\,(a+b)}+\begin{cases}\frac{e^{-a(t_2-t_1)}-e^{-b(t_2-t_1)}}{(a-b)(a+b)} & : a\neq b\\ \frac{(t_2-t_1)e^{-a(t_2-t_1)}}{(a+b)} & : a=b\end{cases}\tag{100}$$

for all $t_1,t_2\in\mathbb{R}$ with $t_1\le t_2$. The proof of Lemma 23 is thus completed. ☐

The next proposition proves convergence of convolutional Wick powers under the assumption that $n,d\in\{2,3,\dots\}$ with $\frac{n+1}{n-1}>\frac{d}{2}$. Its proof uses Lemma 22, Lemma 23 and Lemma 1.

Proposition 24 (Convergence of convolutional Wick powers)**.** *Assume the setting of Subsection 2.1 and let $n,d\in\{2,3,\dots\}$ with $\frac{n+1}{n-1}>\frac{d}{2}$. Then there exists an up to indistinguishability unique stochastic process*

$$\bullet\,(V)^n\bullet\colon\mathbb{R}\times\Omega\to\cap_{\beta\in(-\infty,2+\frac{n(2-d)}{2})}\mathcal{C}^{\beta}_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})\tag{101}$$

with continuous sample paths which satisfies for every $T,p\in(0,\infty)$ and every $\alpha\in(0,\frac{1}{2})$, $\beta\in\mathbb{R}$ with $2\alpha+\beta<2+\frac{n(2-d)}{2}$ that

$$\|\bullet(V^{\varphi})^n\bullet-\bullet\,(V)^n\bullet\|_{L^p(\Omega;\mathcal{C}^{\alpha}([-T,T],\mathcal{C}^{\beta}_{\mathcal{P}}([0,2\pi]^d,\mathbb{R})))}\to 0\qquad as\qquad \Phi_{0,\le 1}\ni\varphi\to 1.\tag{102}$$

Proof of Proposition 24. Lemma 22 and Lemma 23 imply

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}}\,\mathbb{E}\Big[\overline{\big\langle g_{k_1}, \bullet(V_t^{\varphi})^n\bullet - \bullet(V_t^{\psi})^n\bullet\big\rangle_H}\big\langle g_{k_2}, \bullet(V_t^{\varphi})^n\bullet - \bullet(V_t^{\psi})^n\bullet\big\rangle_H\Big] \\
&= n!\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d \\ l_1+\ldots+l_n=k_1}} \left[\begin{array}{c} \Big\{\prod_{i=1}^n \frac{[\varphi_{l_i}]^2}{\lambda_{l_i}} - 2\prod_{i=1}^n \frac{\varphi_{l_i}\psi_{l_i}}{\lambda_{l_i}} + \prod_{i=1}^n \frac{[\psi_{l_i}]^2}{\lambda_{l_i}}\Big\} \\ \cdot \int_{-\infty}^t \int_{-\infty}^t e^{-\lambda_{k_1}(2t-s_1-s_2)-(\sum_{i=1}^n \lambda_{l_i})|s_2-s_1|}\,ds_2\,ds_1 \end{array}\right] \\
&= n!\,\delta_{k_1,k_2} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d \\ l_1+\ldots+l_n=k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2}{\big(\prod_{i=1}^n \lambda_{l_i}\big)\,\lambda_{k_1}\,\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)} \\
&\leq \frac{n!\,\delta_{k_1,k_2}}{(\lambda_{k_1})^{\max(4-d,1)}} \left[\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d \\ l_1+\ldots+l_n=k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)} \right]
\end{aligned} \tag{103}$$

for all $k_1, k_2 \in \mathbb{Z}^d$, $t \in \mathbb{R}$ and all $\varphi, \psi \in \Phi_0$. Next note that Corollary 8 ensures that

$$\sup_{k_1\in\mathbb{Z}^d} \left[\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d \\ l_1+\ldots+l_n=k_1}} \frac{(\lambda_{k_1})^{\gamma}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)} \right] < \infty \tag{104}$$

for all $\gamma \in \big(-\infty, \frac{d}{2} + \min(d-2,1) + n\,(1-\frac{d}{2})\big)$. Therefore, we obtain that

$$\sum_{k_1\in\mathbb{Z}^d} \sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d \\ l_1+\ldots+l_n=k_1}} \frac{(\lambda_{k_1})^{(2\beta-\max(4-d,1))}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)} < \infty \tag{105}$$

for all $\beta \in \big(-\infty, \frac{4-(d-2)n}{4}\big)$. Combining this with (103) and dominated convergence shows for every $\beta \in \big(-\infty, 1-\frac{(d-2)n}{4}\big)$ that

$$\sup_{t\in\mathbb{R}} \sum_{k_1,k_2\in\mathbb{Z}^d} \frac{\Big|\mathbb{E}\Big[\overline{\big\langle g_{k_1}, \bullet(V_t^{\varphi})^n\bullet - \bullet(V_t^{\psi})^n\bullet\big\rangle_H}\big\langle g_{k_2}, \bullet(V_t^{\varphi})^n\bullet - \bullet(V_t^{\psi})^n\bullet\big\rangle_H\Big]\Big|}{(\lambda_{k_1}\lambda_{k_2})^{-\beta}} \to 0 \quad \text{as} \quad (\Phi_{0,\leq 1})^2 \ni (\varphi,\psi) \to (1,1). \tag{106}$$

In the next step let $h_{k_1,l_1,\dots,l_n}\colon \mathbb{R}^2 \to \mathbb{R}$, $k_1, l_1, \dots, l_n \in \mathbb{Z}^d$, be functions defined through

$$h_{k_1,l_1,\dots,l_n}(t_1,t_2) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} e^{-\lambda_{k_1}(t_1-s_1+t_2-s_2)-(\sum_{i=1}^n \lambda_{l_i})|s_1-s_2|}\,ds_2\,ds_1 \tag{107}$$

for all $t_1, t_2 \in \mathbb{R}$ and all $k_1, l_1, \dots, l_n \in \mathbb{Z}^d$. Then observe that Lemma 23 implies that

$$\begin{aligned}
&h_{k_1,l_1,\dots,l_n}(t_1,t_1) - 2h_{k_1,l_1,\dots,l_n}(t_1,t_2) + h_{k_1,l_1,\dots,l_n}(t_2,t_2) \\
&= \frac{2\big(1-e^{-\lambda_{k_1}(t_2-t_1)}\big)}{\lambda_{k_1}\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)} - 2\cdot \begin{cases} \frac{e^{-(\sum_{i=1}^n \lambda_{l_i})(t_2-t_1)} - e^{-\lambda_{k_1}(t_2-t_1)}}{\big(\lambda_{k_1}-\sum_{i=1}^n \lambda_{l_i}\big)\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)} & : \lambda_{k_1} \neq \sum_{i=1}^n \lambda_{l_i} \\ \frac{(t_2-t_1)e^{-\lambda_{k_1}(t_2-t_1)}}{\big(\lambda_{k_1}+[\sum_{i=1}^n \lambda_{l_i}]\big)} & : \lambda_{k_1} = \sum_{i=1}^n \lambda_{l_i} \end{cases} \\
&\leq \frac{2\big(1-e^{-\lambda_{k_1}(t_2-t_1)}\big)}{\lambda_{k_1}\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)} \leq \frac{2\,(\lambda_{k_1})^{2\alpha}\,(t_2-t_1)^{2\alpha}}{\lambda_{k_1}\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)} = \frac{2\,(\lambda_{k_1})^{(2\alpha-1)}\,(t_2-t_1)^{2\alpha}}{\big(\lambda_{k_1}+\sum_{i=1}^n \lambda_{l_i}\big)}
\end{aligned} \tag{108}$$

for all $k_1, l_1, \ldots, l_n \in \mathbb{Z}^d$, $\alpha \in [0, \frac{1}{2}]$ and all $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$. Lemma 22 hence shows that

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}}\,\mathbb{E}\left[\begin{array}{l}\Big\langle g_{-k_1}, \Big[\bullet (V^{\varphi}_{t_1})^n \bullet - \bullet (V^{\psi}_{t_1})^n \bullet\Big] - \Big[\bullet (V^{\varphi}_{t_2})^n \bullet - \bullet (V^{\psi}_{t_2})^n \bullet\Big]\Big\rangle_H \\ \cdot \Big\langle g_{k_2}, \Big[\bullet (V^{\varphi}_{t_1})^n \bullet - \bullet (V^{\psi}_{t_1})^n \bullet\Big] - \Big[\bullet (V^{\varphi}_{t_2})^n \bullet - \bullet (V^{\psi}_{t_2})^n \bullet\Big]\Big\rangle_H\end{array}\right] \\
&= n!\, \delta_{k_1,k_2} \sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{\big(\prod_{i=1}^n (\varphi_{l_i})^2 - 2\prod_{i=1}^n \varphi_{l_i}\psi_{l_i} + \prod_{i=1}^n (\psi_{l_i})^2\big)\big(h_{k_1,l_1,\ldots,l_n}(t_1,t_1) - 2h_{k_1,l_1,\ldots,l_n}(t_1,t_2) + h_{k_1,l_1,\ldots,l_n}(t_2,t_2)\big)}{\big(\prod_{i=1}^n \lambda_{l_i}\big)} \\
&= n!\, \delta_{k_1,k_2} \sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2\big(h_{k_1,l_1,\ldots,l_n}(t_1,t_1) - 2h_{k_1,l_1,\ldots,l_n}(t_1,t_2) + h_{k_1,l_1,\ldots,l_n}(t_2,t_2)\big)}{\big(\prod_{i=1}^n \lambda_{l_i}\big)} \\
&\leq (n+1)!\, \delta_{k_1,k_2} \left[\sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2 (\lambda_{k_1})^{(2\alpha-1)}}{\big(\prod_{i=1}^n \lambda_{l_i}\big)\big(\lambda_{k_1} + \sum_{i=1}^n \lambda_{l_i}\big)}\right] (t_2 - t_1)^{2\alpha} \\
&\leq \frac{(n+1)!\, \delta_{k_1,k_2}}{(\lambda_{k_1})^{(\max(4-d,1)-2\alpha)}} \left[\sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{\big(\prod_{i=1}^n \varphi_{l_i} - \prod_{i=1}^n \psi_{l_i}\big)^2}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)}\right] (t_2 - t_1)^{2\alpha}
\end{aligned} \tag{109}$$

for all $k_1, k_2 \in \mathbb{Z}^d$, $\varphi, \psi \in \Phi_{0,\leq 1}$, $\alpha \in [0, \frac{1}{2}]$ and all $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$. In addition, Corollary 8 ensures that

$$\sup_{k_1 \in \mathbb{Z}^d} \left[\sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{(\lambda_{k_1})^{\gamma}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)}\right] < \infty \tag{110}$$

for all $\gamma \in \big(-\infty, \frac{d}{2} + \min(d-2,1) + n\,\big(1 - \frac{d}{2}\big)\big)$. Therefore, we obtain that

$$\sum_{k_1 \in \mathbb{Z}^d} \sum_{\substack{l_1,\ldots,l_n \in \mathbb{Z}^d \\ l_1+\ldots+l_n = k_1}} \frac{(\lambda_{k_1})^{(2\alpha+2\beta-\max(4-d,1))}}{\Big(\prod_{i=1}^n (\lambda_{l_i})^{(1+\frac{\min(d-2,1)}{n})}\Big)} < \infty \tag{111}$$

for all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta < 1 - \frac{(d-2)n}{4}$. Combining this with (109) and dominated convergence implies for every $\alpha \in [0, \frac{1}{2}]$, $\beta \in \mathbb{R}$ with $\alpha + \beta < 1 - \frac{(d-2)n}{4}$ that

$$\sup_{\substack{t_1,t_2 \in \mathbb{R} \\ t_1 \neq t_2}} \left[\sum_{k_1,k_2 \in \mathbb{Z}^d} \frac{\left|\mathbb{E}\left[\begin{array}{l}\Big\langle g_{-k_1}, \Big[\bullet (V^{\varphi}_{t_1})^n \bullet - \bullet (V^{\psi}_{t_1})^n \bullet\Big] - \Big[\bullet (V^{\varphi}_{t_2})^n \bullet - \bullet (V^{\psi}_{t_2})^n \bullet\Big]\Big\rangle_H \\ \cdot \Big\langle g_{k_2}, \Big[\bullet (V^{\varphi}_{t_1})^n \bullet - \bullet (V^{\psi}_{t_1})^n \bullet\Big] - \Big[\bullet (V^{\varphi}_{t_2})^n \bullet - \bullet (V^{\psi}_{t_2})^n \bullet\Big]\Big\rangle_H\end{array}\right]\right|}{\big(\lambda_{k_1}\lambda_{k_2}\big)^{-\beta} |t_1 - t_2|^{2\alpha}}\right] \to 0 \;\text{ as }\; (\Phi_{0,\leq 1})^2 \ni (\varphi,\psi) \to (1,1). \tag{112}$$

Combining (106) and (112) with Lemma 1 completes the proof of Proposition 24. □

Proposition 24 shows convergence of convolutional Wick powers under the assumption that $n, d \in \{2, 3, \ldots\}$ with $\frac{n+1}{n-1} > \frac{d}{2}$. In the case $n, d \in \{2, 3, \ldots\}$ with $\frac{n+1}{n-1} \leq \frac{d}{2}$, convolutional Wick powers fail to converge. This is the subject of the next lemma.

Lemma 25 (Divergence of convolutional Wick powers)**.** *Assume the setting of Subsection 2.1, let $n, d \in \{2, 3, \ldots\}$ with $\frac{n+1}{n-1} \leq \frac{d}{2}$ and let $C_0, C_1, \ldots, C_{n-1} \colon \Phi_0 \to \mathbb{R}$ be arbitrary functions. Then it holds for every $v \in \mathbb{Z}^d$ and every $t \in \mathbb{R}$ that*

$$\mathbb{E}\left[\left|\left\langle g_v, \int_{-\infty}^{t} e^{A(t-s)} \left((V^{\varphi}_s)^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot (V^{\varphi}_s)^k\right) ds\right\rangle_H\right|^2\right] \to \infty \qquad \textit{as} \qquad \Phi_0 \ni \varphi \to 1. \tag{113}$$

Proof of Lemma 25. Throughout this proof let $\hat{C}_0, \hat{C}_1, \ldots \hat{C}_n \colon \Phi_0 \to \mathbb{R}$ be the unique functions satisfying $\hat{C}_0(0) = -C_0(0)$, $\hat{C}_1(0) = -C_1(0)$, $\ldots$, $\hat{C}_{n-1}(0) = -C_{n-1}(0)$, $\hat{C}_n(0) = 1$ and

$$x^n - \sum_{k=0}^{n-1} C_k(\varphi) \cdot x^k = \sum_{k=0}^{n} \hat{C}_k(\varphi) \cdot \left[\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}\right]^{\frac{k}{2}} \cdot H_k\left(\frac{x}{\sqrt{\sum_{v \in \mathbb{Z}^d} \frac{(\varphi_v)^2}{\lambda_v}}}\right) \tag{114}$$

for all $x \in \mathbb{R}$, $\varphi \in \Phi_0\backslash\{0\}$ and all $t \in \mathbb{R}$ (cf. (68)). Then Lemma 22 implies that

$$\begin{aligned}
&\mathbb{E}\left[\left|\left\langle g_v, \int_{-\infty}^t e^{A(t-s)} \left((V_s^\varphi)^n - \sum_{k=0}^{n-1} C_k(\varphi)\cdot (V_s^\varphi)^k\right) ds\right\rangle_H\right|^2\right] \\
&= \mathbb{E}\left[\left|\sum_{k=0}^n \left\langle g_v, \int_{-\infty}^t e^{A(t-s)} \left[\hat{C}_k(\varphi)\left(:(V_t^\varphi)^k:\right)\right] ds\right\rangle_H\right|^2\right] \\
&= \sum_{k,l=0}^n \hat{C}_k(\varphi)\cdot \hat{C}_l(\varphi)\cdot \mathbb{E}\Big[\overline{\langle g_v, \bullet(V_t^\varphi)^k\bullet\rangle_H}\,\langle g_v, \bullet(V_t^\varphi)^l\bullet\rangle_H\Big] \\
&= \sum_{k=0}^n \left|\hat{C}_k(\varphi)\right|^2 \mathbb{E}\Big[\left|\langle g_v, \bullet(V_t^\varphi)^k\bullet\rangle_H\right|^2\Big] \geq \left|\hat{C}_n(\varphi)\right|^2 \mathbb{E}\Big[\left|\langle g_v, \bullet(V_t^\varphi)^n\bullet\rangle_H\right|^2\Big] \\
&= \frac{n!\,(2\pi)^{2d}}{\lambda_v}\left[\sum_{\substack{l_1,\dots,l_n\in\mathbb{Z}^d\\ l_1+\dots+l_n=v}} \frac{\left(\prod_{i=1}^n (\varphi_{l_i})^2\right)}{\left(\prod_{i=1}^n \lambda_{l_i}\right)\left(\lambda_v + \sum_{i=1}^n \lambda_{l_i}\right)}\right]
\end{aligned} \tag{115}$$

for all $t \in \mathbb{R}$, $v \in \mathbb{Z}^d$ and all $\varphi \in \Phi_0$. Combining this with Lemma 20 completes the proof of Lemma 25. □

2.7 Summary

The following table briefly summarizes the results of Proposition 14, Proposition 19 and Proposition 24 and of Lemma 16, Lemma 21 and Lemma 25. Recall that the main arguments for the results from Propositions 14, 19 and 24 presented in the table are certain summability properities; see (57) and (61) in the case of Wick powers, (83) and (85) in the case of averaged Wick powers and (103) and (105) in the case of convolutional Wick powers. In the table $\varepsilon \in (0,\infty)$ is an arbitrarily small positive real number, $\mathcal{C}_\mathcal{P}^\alpha$ is an abbreviation for $\mathcal{C}_\mathcal{P}^\alpha([0,2\pi]^d,\mathbb{R})$ where $\alpha \in \mathbb{R}$ and $d \in \mathbb{N}$ and the expressions *WP*, *AWP* and *CWP* are abbreviations for *Wick powers*, *averaged Wick powers* and *convolutional Wick powers* respectively.

⋮	⋮	⋮	⋮	⋮	⋮	
n = 5	**WP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$ **AWP:** $\mathcal{C}_{\mathcal{P}}^{1/5-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{2-\varepsilon}$	No WP No AWP No CWP	No WP No AWP No CWP	No WP No AWP No CWP	No WP No AWP No CWP	…
n = 4	**WP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$ **AWP:** $\mathcal{C}_{\mathcal{P}}^{1/4-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{2-\varepsilon}$	No WP **AWP:** $\mathcal{C}_{\mathcal{P}}^{-1-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$	No WP No AWP No CWP	No WP No AWP No CWP	No WP No AWP No CWP	…
n = 3	**WP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$ **AWP:** $\mathcal{C}_{\mathcal{P}}^{1/3-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{2-\varepsilon}$	No WP **AWP:** $\mathcal{C}_{\mathcal{P}}^{-1/2-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{1/2-\varepsilon}$	No WP No AWP No CWP	No WP No AWP No CWP	No WP No AWP No CWP	…
n = 2	**WP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$ **AWP:** $\mathcal{C}_{\mathcal{P}}^{1/2-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{2-\varepsilon}$	**WP:** $\mathcal{C}_{\mathcal{P}}^{-1-\varepsilon}$ **AWP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{1-\varepsilon}$	No WP **AWP:** $\mathcal{C}_{\mathcal{P}}^{-1-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{-\varepsilon}$	No WP **AWP:** $\mathcal{C}_{\mathcal{P}}^{-2-\varepsilon}$ **CWP:** $\mathcal{C}_{\mathcal{P}}^{-1-\varepsilon}$	No WP No AWP No CWP	…
	d = 2	d = 3	d = 4	d = 5	d = 6	…

3 Stochastic partial differential equations (SPDEs)

3.1 Local existence and uniqueness of mild solutions of deterministic nonautonomous partial differential equations

This subsection investigates local existence and uniqueness questions for mild solutions of deterministic nonautonomous evolution equations of the form

$$\frac{\partial}{\partial t}x(t) = Ax(t) + \sum_{i=1}^{n} F_i(t, x(t)) \tag{116}$$

on a real Banach space $(U, \|\cdot\|_U)$ for $t \in [t_0, T]$ where $t_0, T \in \mathbb{R}$ are real numbers with $t_0 < T$, where $A\colon D(A) \subset U \to U$ is a negative generator of a strongly continuous analytic semigroup, where $n \in \mathbb{N}$ is a natural number and where $F_1, \ldots, F_n$ are suitable functions that are locally Lipschitz continuous on appropriate spaces.

To investigate these questions, we impose the following setting. Throughout this subsection, let $(U, \|\cdot\|_U)$ be a real Banach space, let $A\colon D(A) \subset U \to U$ be a negative generator of a strongly continuous analytic semigroup on U and let $\big(U_r, \|\cdot\|_{U_r}\big) := (D((-A)^r), \|(-A)^r(\cdot)\|_U)$ for all $r \in \mathbb{R}$. Next define

$$\|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} := \sup_{t\in[t_0,T]} \sum_{i=1}^{n} \left[\|F_i(t,0)\|_{U_{\alpha_i}} + \sup_{\substack{x,y\in U_{\max(\beta_i,\gamma_i)}\\ x\neq y}} \frac{\|F_i(t,x) - F_i(t,y)\|_{U_{\alpha_i}}}{\big(1 + \|x\|^{\delta_i}_{U_{\beta_i}} + \|y\|^{\delta_i}_{U_{\beta_i}}\big)\, \|x-y\|_{U_{\gamma_i}}} \right] \in [0,\infty] \tag{117}$$

for all $F = (F_1, \dots, F_n) \in C([t_0,T] \times U_{\max(\beta_1,\gamma_1)}, U_{\alpha_1}) \times \dots \times C([t_0,T] \times U_{\max(\beta_n,\gamma_n)}, U_{\alpha_n})$, $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $t_0 \in (-\infty, T)$, $T \in \mathbb{R}$ and all $n \in \mathbb{N}$. Furthermore, define

$$\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T]) := \Big\{ F \in \Big(C([t_0,T] \times U_{\max(\beta_1,\gamma_1)}, U_{\alpha_1}) \times \dots \times C([t_0,T] \times U_{\max(\beta_n,\gamma_n)}, U_{\alpha_n}) \Big) \colon \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} < \infty \Big\} \tag{118}$$

for all $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $t_0 \in (-\infty, T)$, $T \in \mathbb{R}$ and all $n \in \mathbb{N}$. Observe that the pairs $\big(\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T]), \|\cdot\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\big)$ for $\alpha, \beta, \gamma \in \mathbb{R}^n$, $\delta \in [0,\infty)^n$, $t_0 \in (-\infty, T)$, $T \in \mathbb{R}$ and $n \in \mathbb{N}$ are normed real vector spaces. In the next step define

$$\begin{aligned} \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) := \Big\{ &(F_1, \dots, F_n) \in \Big(C([t_0,\infty) \times U_{\max(\beta_1,\gamma_1)}, U_{\alpha_1}) \times \dots \\ &\times C([t_0,\infty) \times U_{\max(\beta_n,\gamma_n)}, U_{\alpha_n}) \Big) \colon \Big(\forall\, T \in (t_0,\infty) \colon \\ &\|(F_1|_{[t_0,T] \times U_{\max(\beta_1,\gamma_1)}}, \dots, F_n|_{[t_0,T] \times U_{\max(\beta_n,\gamma_n)}})\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} < \infty \Big) \Big\} \end{aligned} \tag{119}$$

for all $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $t_0 \in \mathbb{R}$ and all $n \in \mathbb{N}$. Moreover, we equip $\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$ with the metric $d_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))} \colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \to [0,\infty)$ defined through

$$\begin{aligned} d_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))}(F,G) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, \big\|((F_1 - G_1)|_{[t_0,t_0+k] \times U_{\max(\beta_1,\gamma_1)}}, \dots, \\ (F_n - G_n)|_{[t_0,t_0+k] \times U_{\max(\beta_n,\gamma_n)}})\big\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,t_0+k])}\Big) \end{aligned} \tag{120}$$

for all $F = (F_1, \dots, F_n)$, $G = (G_1, \dots, G_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $t_0 \in \mathbb{R}$ and all $n \in \mathbb{N}$. Finally, note that the triangle inequality and the definition of $\|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}$ imply that

$$\begin{aligned} &\|F_i(t,x)\|_{U_{\alpha_i}} \le \|F_i(t,x) - F_i(t,0)\|_{U_{\alpha_i}} + \|F_i(t,0)\|_{U_{\alpha_i}} \\ &\le \left[\sup_{y \in U_{\max(\beta_i,\gamma_i)} \setminus \{0\}} \frac{\|F_i(t,y) - F_i(t,0)\|_{U_{\alpha_i}}}{\big(1 + \|y\|^{\delta_i}_{U_{\beta_i}}\big)\|y\|_{U_{\gamma_i}}} + \|F_i(t,0)\|_{U_{\alpha_i}} \right] \big(1 + \|x\|^{\delta_i}_{U_{\beta_i}}\big)\big(1 + \|x\|_{U_{\gamma_i}}\big) \\ &\le \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \big(1 + \|x\|^{\delta_i}_{U_{\beta_i}}\big) \big(1 + \|x\|_{U_{\gamma_i}}\big) \end{aligned} \tag{121}$$

for all $t \in [t_0,T]$, $x \in U_{\max(\beta_i,\gamma_i)}$, $i \in \{1, 2, \dots, n\}$, $F = (F_1, \dots, F_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$, $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $t_0 \in (-\infty, T)$, $T \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Lemma 26 (Local existence and uniqueness of mild solutions)**.** *Assume the setting in the beginning of Subsection 3.1, let $r_0, t_0 \in \mathbb{R}$, $T \in (t_0,\infty)$, $v \in U_{r_0}$, $n \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in [r_0,\infty)^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0,\infty)^n$, $r_1 \in [\max(\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n), 1 + \min(\alpha_1, \dots, \alpha_n))$ with $\max_{i \in \{1,\dots,n\}}[\gamma_i - \min(\alpha_i, r_0) + \delta_i(\beta_i - r_0)] < 1$ and let $F = (F_1, \dots, F_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$. Then there exist a real number $\tau \in (t_0,T]$ such that there exists a unique continuous function $x \colon [t_0,\tau] \to U_{r_0}$ satisfying $x|_{(t_0,\tau]} \in C((t_0,\tau], U_{r_1})$, $\sup_{s \in (t_0,\tau]} (s - t_0)^{(r_1 - r_0)} \|x(s)\|_{U_{r_1}} < \infty$ and $x(t) = e^{A(t-t_0)} v + \sum_{i=1}^{n} \int_{t_0}^{t} e^{A(t-s)} F_i(s, x(s))\, ds$ for all $t \in [t_0,\tau]$.*

Observe that all integrals appearing in Lemma 26 are well-defined. Indeed, under the assumptions of Lemma 26 it holds that if $\tau \in (t_0,T]$ and if $x \colon [t_0,\tau] \to U_{r_0}$ is a continuous function which satisfies $x|_{(t_0,\tau]} \in C((t_0,\tau], U_{r_1})$ and $\sup_{s \in (t_0,\tau]} (s - t_0)^{(r_1 - r_0)} \|x(s)\|_{U_{r_1}} < \infty$, then (121) and interpolation (see, e.g., Theorem 37.6 in Sell & You [19]) imply that

$$\begin{aligned}
&\int_{t_0}^t \left\| e^{A(t-s)} F_i(s,x(s)) \right\|_{U_{r_1}} ds \leq \int_{t_0}^t \| e^{A(t-s)} \|_{L(U_{\alpha_i},U_{r_1})} \| F_i(s,x(s)) \|_{U_{\alpha_i}} ds \\
&\leq \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[\sup_{s\in(0,T-t_0]} \frac{\|e^{As}\|_{L(U_{\alpha_i},U_{r_1})}}{s^{\min(\alpha_i-r_1,0)}} \right] \int_{t_0}^t \frac{\left(1+\|x(s)\|^{\delta_i}_{U_{\beta_i}}\right)\left(1+\|x(s)\|_{U_{\gamma_i}}\right)}{(t-s)^{\max(r_1-\alpha_i,0)}} ds \\
&\leq \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[\sup_{s\in(0,T-t_0]} \frac{\|e^{As}\|_{L(U_{\alpha_i},U_{r_1})}}{s^{\min(\alpha_i-r_1,0)}} \right] \left[\sup_{s\in(t_0,\tau]} \frac{(1+\|x(s)\|_{U_{\gamma_i}})}{(s-t_0)^{(r_0-\gamma_i)}} \right] \\
&\cdot \left[\sup_{s\in(t_0,\tau]} \frac{(1+\|x(s)\|^{\delta_i}_{U_{\beta_i}})}{(s-t_0)^{\delta_i(r_0-\beta_i)}} \right] \int_{t_0}^t \frac{1}{(t-s)^{\max(r_1-\alpha_i,0)}\,(s-t_0)^{(\gamma_i-r_0+\delta_i(\beta_i-r_0))}} ds < \infty
\end{aligned} \tag{122}$$

for all $t\in[t_0,\tau]$ and all $i\in\{1,2,\ldots,n\}$ where we used $r_1 < 1+\min_{j\in\{1,\ldots,n\}} \alpha_j \leq 1+\alpha_i$ and $\gamma_i - r_0 + \delta_i(\beta_i-r_0) < 1$ for all $i\in\{1,2,\ldots,n\}$ in the last line of (122). We now present the proof of Lemma 26.

Proof of Lemma 26. Lemma 26 follows from an application of the Banach fixed point theorem. For this several preparations are needed. First, let $\kappa\in[0,\infty)$ be a real number defined through

$$\begin{aligned}
\kappa :=& \left[2 + r_1 - r_0 + T + \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \sum_{i=1}^n \delta_i \right]^{\left(4+|r_0|+|r_1|+\max_{i\in\{1,\ldots,n\}}|\alpha_i|\right)} \\
&+ \sum_{j=0}^1 \sum_{i=1}^n \left[\frac{1}{\min(1+\alpha_i-r_j,1)} + B_{(1+\min(\alpha_i-r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))} \right] \\
&+ \max_{j\in\{0,1\}} \max_{\theta\in\{r_0,r_1,\alpha_1,\ldots,\alpha_n\}} \sup_{t\in(t_0,T]} \left[(t-t_0)^{\max(r_j-\theta,0)} \left\| e^{A(t-t_0)} \right\|_{L(U_\theta,U_{r_j})} \right] \\
&+ \max_{\substack{\theta\in\{\beta_1,\ldots,\beta_n\} \\ \cup\{\gamma_1,\ldots,\gamma_n\}}} \sup_{\substack{v\in U_{r_1} \\ v\neq 0}} \left[1 + \frac{\|v\|_{U_\theta}}{\|v\|_{U_{r_1}}^{\frac{(\theta-r_0)}{(r_1-r_0)}} \|v\|_{U_{r_0}}^{\frac{(r_1-\theta)}{(r_1-r_0)}}} \right]^{(1+\sum_{i=1}^n \delta_i)} < \infty
\end{aligned} \tag{123}$$

where $B\colon (0,\infty)^2 \to (0,\infty)$ is the Beta function defined through $B_{(x,y)} := \int_0^1 (1-s)^{(x-1)}\, s^{(y-1)}\, ds$ for all $x,y\in(0,\infty)$. Observe that the quantity κ is indeed finite; see, e.g., Theorems 37.5 and 37.6 in Sell & You [19]. Next define real vector spaces $\mathcal{E}_{[t_0,\tau]} \subset C([t_0,\tau],U_{r_0})$, $\tau\in(t_0,T]$, through

$$\mathcal{E}_{[t_0,\tau]} := \left\{ x\in C([t_0,\tau],U_{r_0})\colon \begin{pmatrix} x|_{(t_0,\tau]} \in C((t_0,\tau],U_{r_1}) \text{ and} \\ \sup_{t\in(t_0,\tau]} (t-t_0)^{(r_1-r_0)} \|x(t)\|_{U_{r_1}} < \infty \end{pmatrix} \right\} \tag{124}$$

for all $\tau\in(t_0,T]$, define norms $\|\cdot\|_{\mathcal{E}_{[t_0,\tau]}}\colon \mathcal{E}_{[t_0,\tau]} \to [0,\infty)$, $\tau\in(t_0,T]$, through

$$\|x\|_{\mathcal{E}_{[t_0,\tau]}} := \sum_{j=0}^1 \left[\sup_{t\in(t_0,\tau]} \left[(t-t_0)^{(r_j-r_0)} \|x(t)\|_{U_{r_j}} \right] \right] \tag{125}$$

for all $\tau\in(t_0,T]$, define sets $\mathcal{E}_{[t_0,\tau],v} \subset \mathcal{E}_{[t_0,\tau]}$, $\tau\in(t_0,T]$, $v\in U_{r_0}$, through

$$\mathcal{E}_{[t_0,\tau],v} := \left\{ x\in\mathcal{E}_{[t_0,\tau]}\colon \|x\|_{\mathcal{E}_{[t_0,\tau]}} \leq \kappa^7 \left(1+\|v\|_{U_{r_0}}\right) \right\} \tag{126}$$

for all $\tau\in(t_0,T]$ and all $v\in U_{r_0}$ and define mappings $\Phi_{[t_0,\tau],v}\colon \mathcal{E}_{[t_0,\tau]} \to \mathcal{E}_{[t_0,\tau]}$, $\tau\in(t_0,T]$, $v\in U_{r_0}$, through

$$(\Phi_{[t_0,\tau],v}x)(t) := e^{A(t-t_0)}\, v + \sum_{i=1}^n \int_{t_0}^t e^{A(t-s)}\, F_i(s,x(s))\, ds \tag{127}$$

for all $t\in[t_0,\tau]$, $x\in\mathcal{E}_{[t_0,\tau]}$, $\tau\in(t_0,T]$ and all $v\in U_{r_0}$. Note that (122) ensures that the mappings $\Phi_{[t_0,\tau],v}$, $\tau\in(t_0,T]$, $v\in U_{r_0}$, are well-defined. We now establish a few estimates for the mappings $\Phi_{[t_0,\tau],v}$,

$\tau \in (t_0, T]$, $v \in U_{r_0}$. First, observe that

$$\begin{aligned}
&\left\|(\Phi_{[t_0,\tau],v}0)(t)\right\|_{U_{r_j}} \leq \left\|e^{A(t-t_0)}v\right\|_{U_{r_j}} + \sum_{i=1}^{n} \int_{t_0}^{t} \left\|e^{A(t-s)}\right\|_{L(U_{\alpha_i},U_{r_j})} \left\|F_i(s,0)\right\|_{U_{\alpha_i}} ds \\
&\leq \left\|e^{A(t-t_0)}\right\|_{L(U_{r_0},U_{r_j})} \|v\|_{U_{r_0}} + \kappa^2 \sum_{i=1}^{n} \int_{t_0}^{t} (t-s)^{\min(\alpha_i - r_j, 0)}\, ds \\
&\leq \kappa\, (t-t_0)^{(r_0-r_j)} \|v\|_{U_{r_0}} + \kappa^2 \sum_{i=1}^{n} \frac{(t-t_0)^{\min(1+\alpha_i-r_j,1)}}{\min(1+\alpha_i-r_j,1)} \\
&\leq \kappa^5\, (t-t_0)^{(r_0-r_j)} \left(1+\|v\|_{U_{r_0}}\right)
\end{aligned} \tag{128}$$

for all $j \in \{0,1\}$, $t \in (t_0, \tau]$, $\tau \in (t_0, T]$, $v \in U_{r_0}$ and hence

$$\left\|\Phi_{[t_0,\tau],v}(0)\right\|_{\mathcal{E}_{[t_0,\tau]}} \leq \kappa^6 \left(1+\|v\|_{U_{r_0}}\right) \tag{129}$$

for all $\tau \in (t_0, T]$, $v \in U_{r_0}$. In the next step observe that

$$\begin{aligned}
&\left\|(\Phi_{[t_0,\tau],v}x)(t) - (\Phi_{[t_0,\tau],v}y)(t)\right\|_{U_{r_j}} \leq \sum_{i=1}^{n} \int_{t_0}^{t} \left\|e^{A(t-s)}\left[F_i(s,x(s)) - F_i(s,y(s))\right]\right\|_{U_{r_j}} ds \\
&\leq \sum_{i=1}^{n} \int_{t_0}^{t} \left\|e^{A(t-s)}\right\|_{L(U_{\alpha_i},U_{r_j})} \left\|F_i(s,x(s)) - F_i(s,y(s))\right\|_{U_{\alpha_i}} ds \\
&\leq \kappa \sum_{i=1}^{n} \int_{t_0}^{t} (t-s)^{\min(\alpha_i-r_j,0)} \left\|F_i(s,x(s)) - F_i(s,y(s))\right\|_{U_{\alpha_i}} ds \\
&\leq \kappa^2 \sum_{i=1}^{n} \int_{t_0}^{t} (t-s)^{\min(\alpha_i-r_j,0)} \left[1 + \|x(s)\|_{U_{\beta_i}}^{\delta_i} + \|y(s)\|_{U_{\beta_i}}^{\delta_i}\right] \|x(s)-y(s)\|_{U_{\gamma_i}} ds \\
&\leq \kappa^3 \sum_{i=1}^{n} \int_{t_0}^{t} (t-s)^{\min(\alpha_i-r_j,0)} \|x(s)-y(s)\|_{U_{r_1}}^{\frac{(\gamma_i-r_0)}{(r_1-r_0)}} \|x(s)-y(s)\|_{U_{r_0}}^{\frac{(r_1-\gamma_i)}{(r_1-r_0)}} \\
&\quad \cdot \left[1 + \|x(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}} \|x(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}} + \|y(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}} \|y(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\right] ds
\end{aligned} \tag{130}$$

and therefore

$$\begin{aligned}
&(t-t_0)^{(r_j-r_0)} \left\|(\Phi_{[t_0,\tau],v}x)(t) - (\Phi_{[t_0,\tau],v}y)(t)\right\|_{U_{r_j}} \\
&\leq \kappa^3\, (t-t_0)^{(r_j-r_0)} \|x-y\|_{\mathcal{E}_{[t_0,t]}} \sum_{i=1}^{n} \int_{t_0}^{t} (t-s)^{\min(\alpha_i-r_j,0)} \\
&\cdot \left((s-t_0)^{(r_0-\gamma_i)} + (s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))} \left[\|x\|_{\mathcal{E}_{[t_0,t]}}^{\delta_i} + \|y\|_{\mathcal{E}_{[t_0,t]}}^{\delta_i}\right]\right) ds \\
&\leq \kappa^5\, (t-t_0)^{(r_j-r_0)} \left[1 + \|x\|_{\mathcal{E}_{[t_0,t]}} + \|y\|_{\mathcal{E}_{[t_0,t]}}\right]^{\left(1+\sum_{i=1}^{n}\delta_i\right)} \|x-y\|_{\mathcal{E}_{[t_0,t]}} \\
&\cdot \sum_{i=1}^{n} \int_{0}^{(t-t_0)} (t-t_0-s)^{\min(\alpha_i-r_j,0)}\, s^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\, ds \\
&= \kappa^5 \left[1 + \|x\|_{\mathcal{E}_{[t_0,t]}} + \|y\|_{\mathcal{E}_{[t_0,t]}}\right]^{\left(1+\sum_{i=1}^{n}\delta_i\right)} \|x-y\|_{\mathcal{E}_{[t_0,t]}} \\
&\cdot \sum_{i=1}^{n} (t-t_0)^{(1+\min(\alpha_i,r_j)-\gamma_i+\delta_i(r_0-\beta_i))}\, B_{(1+\min(\alpha_i-r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))}
\end{aligned} \tag{131}$$

and hence

$$
\begin{aligned}
&(t-t_0)^{(r_j-r_0)}\left\|(\Phi_{[t_0,\tau],v}x)(t)-(\Phi_{[t_0,\tau],v}y)(t)\right\|_{U_{r_j}}\\
&\leq \kappa^6\,(t-t_0)^{\min_{i\in\{1,\ldots,n\}}[1-(\gamma_i-\min(\alpha_i,r_j)+\delta_i(\beta_i-r_0))]}\left[1+\|x\|_{\mathcal{E}_{[t_0,t]}}+\|y\|_{\mathcal{E}_{[t_0,t]}}\right]^{\left(1+\sum_{i=1}^n\delta_i\right)}\\
&\quad\cdot\|x-y\|_{\mathcal{E}_{[t_0,t]}}\left[\sum_{i=1}^n B_{(1+\min(\alpha_i-r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))}\right]\\
&\leq \kappa^7\,(t-t_0)^{\left[1-\max_{i\in\{1,\ldots,n\}}(\gamma_i-\min(\alpha_i,r_j)+\delta_i(\beta_i-r_0))\right]}\left[1+\|x\|_{\mathcal{E}_{[t_0,t]}}+\|y\|_{\mathcal{E}_{[t_0,t]}}\right]^{\left(1+\sum_{i=1}^n\delta_i\right)}\\
&\quad\cdot\|x-y\|_{\mathcal{E}_{[t_0,t]}}
\end{aligned}
\tag{132}
$$

for all $j\in\{0,1\}$, $t\in(t_0,\tau]$, $x,y\in\mathcal{E}_{[t_0,\tau]}$, $\tau\in(t_0,T]$, $v\in U_{r_0}$. Hence, we get

$$
\begin{aligned}
&\left\|\Phi_{[t_0,\tau],v}(x)-\Phi_{[t_0,\tau],v}(y)\right\|_{\mathcal{E}_{[t_0,\tau]}}\\
&\leq \kappa^8\,(\tau-t_0)^{\left[1-\max_{i\in\{1,\ldots,n\}}(\gamma_i-\min(\alpha_i,r_0)+\delta_i(\beta_i-r_0))\right]}\\
&\quad\cdot\left[1+\|x\|_{\mathcal{E}_{[t_0,\tau]}}+\|y\|_{\mathcal{E}_{[t_0,\tau]}}\right]^{\kappa}\|x-y\|_{\mathcal{E}_{[t_0,\tau]}}
\end{aligned}
\tag{133}
$$

for all $x,y\in\mathcal{E}_{[t_0,\tau]}$, $v\in U_{r_0}$, $\tau\in(t_0,T]$. Combining (129) and (133) results in

$$
\begin{aligned}
&\left\|\Phi_{[t_0,\tau],v}(x)\right\|_{\mathcal{E}_{[t_0,\tau]}}\leq\left\|\Phi_{[t_0,\tau],v}(x)-\Phi_{[t_0,\tau],v}(0)\right\|_{\mathcal{E}_{[t_0,\tau]}}+\left\|\Phi_{[t_0,\tau],v}(0)\right\|_{\mathcal{E}_{[t_0,\tau]}}\\
&\leq\kappa^8\,(\tau-t_0)^{\left[1-\max_{i\in\{1,\ldots,n\}}(\gamma_i-\min(\alpha_i,r_0)+\delta_i(\beta_i-r_0))\right]}\left[1+\|x\|_{\mathcal{E}_{[t_0,\tau]}}\right]^{\kappa}\|x\|_{\mathcal{E}_{[t_0,\tau]}}\\
&\quad+\kappa^6\left(1+\|v\|_{U_{r_0}}\right)
\end{aligned}
\tag{134}
$$

for all $x\in\mathcal{E}_{[t_0,\tau]}$, $v\in U_{r_0}$, $\tau\in(t_0,T]$. The assumption

$$
\max_{i\in\{1,\ldots,n\}}\left[\gamma_i-\min(\alpha_i,r_0)+\delta_i\,(\beta_i-r_0)\right]<1
\tag{135}
$$

together with inequalities (133) and (134) implies that there exists a mapping $\rho\colon U_{r_0}\to(t_0,T]$ such that

$$
\begin{aligned}
\left\|\Phi_{[t_0,\rho(v)],v}(x)\right\|_{\mathcal{E}_{[t_0,\rho(v)]}}&\leq 1+\kappa^6\left(1+\|v\|_{U_{r_0}}\right),\\
\left\|\Phi_{[t_0,\rho(v)],v}(x)-\Phi_{[t_0,\rho(v)],v}(y)\right\|_{\mathcal{E}_{[t_0,\rho(v)]}}&\leq\frac{1}{2}\,\|x-y\|_{\mathcal{E}_{[t_0,\rho(v)]}}
\end{aligned}
\tag{136}
$$

for all $x,y\in\mathcal{E}_{[t_0,\rho(v)],v}$, $v\in U_{r_0}$. This ensures that $\Phi_{[t_0,\rho(v)],v}\big(\mathcal{E}_{[t_0,\rho(v)],v}\big)\subset\mathcal{E}_{[t_0,\rho(v)],v}$ for all $v\in U_{r_0}$. The Banach fixed point theorem hence proves that there exist unique functions $x_v\in\mathcal{E}_{[t_0,\rho(v)],v}$, $v\in U_{r_0}$, such that $\Phi_{[t_0,\rho(v)],v}(x_v)=x_v$ for all $v\in U_{r_0}$. This completes the proof of Lemma 26. □

Lemma 26 shows, under suitable assumptions, that there exists a unique local mild solution of (116). This solution can be extended to a maximal interval of definition. This is the subject of the next corollary. It follows directly from Lemma 26 and a standard argument from the ordinary differential equations literature and its proof is therefore omitted.

Corollary 27 (Maximal mild solutions)**.** *Assume the setting in the beginning of Subsection 3.1, let $r_0,t_0\in\mathbb{R}$, $T\in(t_0,\infty)$, $v\in U_{r_0}$, $n\in\mathbb{N}$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$, $\beta=(\beta_1,\ldots,\beta_n),\gamma=(\gamma_1,\ldots,\gamma_n)\in[r_0,\infty)^n$, $\delta=(\delta_1,\ldots,\delta_n)\in[0,\infty)^n$, $r_1\in[\max(\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n),1+\min(\alpha_1,\ldots,\alpha_n))$ with $\max_{i\in\{1,\ldots,n\}}[\gamma_i-\min(\alpha_i,r_0)+\delta_i(\beta_i-r_0)]<1$ and let $F=(F_1,\ldots,F_n)\in\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$. Then there exist a unique real number $\tau\in(t_0,T]$ and a unique continuous function $x\colon[t_0,\tau)\to U_{r_0}$ satisfying $x|_{(t_0,\tau)}\in C((t_0,\tau),U_{r_1})$, $\sup_{s\in(t_0,t]}(s-t_0)^{(r_1-r_0)}\|x(s)\|_{U_{r_1}}<\infty$, $\lim_{s\nearrow\tau}\left[\frac{1}{(T-s)}+\|x(s)\|_{U_{r_1}}\right]=\infty$ and $x(t)=e^{A(t-t_0)}v+\sum_{i=1}^n\int_{t_0}^t e^{A(t-s)}F_i(s,x(s))\,ds$ for all $t\in(t_0,\tau)$.*

The next result shows, under suitable assumptions, that the unique maximal mild solution of (116) enjoys a bit more regularity than the regularity asserted in Corollary 27.

Corollary 28 (More regularity for maximal mild solutions)**.** *Assume the setting in the beginning of Subsection 3.1, let $r_0,t_0\in\mathbb{R}$, $T\in(t_0,\infty)$, $v\in U_{r_0}$, $n\in\mathbb{N}$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$, $\beta=(\beta_1,\ldots,\beta_n),\gamma=(\gamma_1,\ldots,\gamma_n)\in[r_0,\infty)^n$, $\delta=(\delta_1,\ldots,\delta_n)\in[0,\infty)^n$ with $\max(\beta_1,\ldots,\beta_n,\gamma_1,\ldots,\gamma_n)<1+\min(\alpha_1,\ldots,\alpha_n)$ and $\max_{i\in\{1,\ldots,n\}}[\gamma_i-\min(\alpha_i,r_0)+\delta_i(\beta_i-r_0)]<1$ and let $F=(F_1,\ldots,F_n)\in\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$. Then* *there exist a unique real number $\tau \in (t_0, T]$ and a unique continuous function $x\colon [t_0, \tau) \to U_{r_0}$ satisfying $x|_{(t_0,\tau)} \in C((t_0, \tau), U_{r_1})$, $\sup_{s\in(t_0,t]} (s-t_0)^{(r_1-r_0)} \|x(s)\|_{U_{r_1}} < \infty$, $\lim_{s\nearrow\tau} \big[\|x(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}} + \frac{1}{(T-s)}\big] = \infty$ and $x(t) = e^{A(t-t_0)}\, v + \sum_{i=1}^n \int_{t_0}^t e^{A(t-s)}\, F_i(s, x(s))\, ds$ for all $t \in (t_0, \tau)$ and all $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$.*

Proof of Corollary 28. First of all, Corollary 27 implies that there exists a unique real number $\tau \in (t_0, T]$ and a unique continuous function $x\colon [t_0, \tau) \to U_{r_0}$ satisfying $x|_{(t_0,\tau)} \in C((t_0, \tau), U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)})$, $\lim_{s\nearrow\tau} \big[\frac{1}{(T-s)} + \|x(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}}\big] = \infty$ and

$$\sup_{s\in(t_0,t]} (s-t_0)^{(\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)-r_0)} \|x(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}} < \infty \tag{137}$$

and

$$x(t) = e^{A(t-t_0)}\, v + \sum_{i=1}^n \int_{t_0}^t e^{A(t-s)}\, F_i(s, x(s))\, ds \tag{138}$$

for all $t \in (t_0, \tau)$. Next we observe similar as in (122) that (121) and interpolation (see, e.g., Theorem 37.6 in Sell & You [19]) imply that

$$\begin{aligned}
&\int_{t_0}^t \big\|e^{A(t-s)}\, F_i(s, x(s))\big\|_{U_{r_1}} ds \leq \int_{t_0}^t \|e^{A(t-s)}\|_{L(U_{\alpha_i},U_{r_1})} \|F_i(s, x(s))\|_{U_{\alpha_i}} ds \\
&\leq \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[\sup_{s\in(0,T-t_0]} \tfrac{\|e^{As}\|_{L(U_{\alpha_i},U_{r_1})}}{s^{\min(\alpha_i-r_1,0)}}\right] \int_{t_0}^t \frac{\big(1 + \|x(s)\|^{\delta_i}_{U_{\beta_i}}\big)\big(1 + \|x(s)\|_{U_{\gamma_i}}\big)}{(t-s)^{\max(r_1-\alpha_i,0)}}\, ds \\
&\leq \|F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[\sup_{s\in(0,T-t_0]} \tfrac{\|e^{As}\|_{L(U_{\alpha_i},U_{r_1})}}{s^{\min(\alpha_i-r_1,0)}}\right] \left[\sup_{s\in(t_0,t]} \frac{(1 + \|x(s)\|_{U_{\gamma_i}})}{(s-t_0)^{(r_0-\gamma_i)}}\right] \\
&\cdot \left[\sup_{s\in(t_0,t]} \frac{(1 + \|x(s)\|^{\delta_i}_{U_{\beta_i}})}{(s-t_0)^{\delta_i(r_0-\beta_i)}}\right] \int_{t_0}^t \frac{1}{(t-s)^{\max(r_1-\alpha_i,0)}\,(s-t_0)^{(\gamma_i-r_0+\delta_i(\beta_i-r_0))}}\, ds < \infty
\end{aligned} \tag{139}$$

for all $t \in [t_0, \tau)$, $i \in \{1, 2, \dots, n\}$ and all $r_1 \in (-\infty, 1 + \min(\alpha_1, \dots, \alpha_n))$ where we used $\gamma_i - r_0 + \delta_i(\beta_i - r_0) < 1$ for all $i \in \{1, 2, \dots, n\}$ in the last line of (139). This proves that $x(t) \in U_{r_1}$ for all $t \in (t_0, \tau)$ and all $r_1 \in (-\infty, 1 + \min(\alpha_1, \dots, \alpha_n))$ and that

$$\sup_{s\in(t_0,t]} (s-t_0)^{(r_1-r_0)} \|x(s)\|_{U_{r_1}} < \infty \tag{140}$$

for all $t \in (t_0, \tau)$ and all $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$. Applying Lemma 26 then proves that $x|_{(t_0,\tau)} \in C((t_0, \tau), U_{r_1})$ for all $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$. This completes the proof of Lemma 28. ☐

We now present and prove the main result of this subsection. It shows, under suitable assumptions, that the unique local mild solutions of (116) depend continuously in an appropriate sense on the possibly nonlinear vector fields in (116).

Theorem 29 (Continuous dependence on the data on bounded time intervals)**.** *Assume the setting in the beginning of Subsection 3.1 and let $r_0 \in \mathbb{R}$, $n \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \dots, \beta_n), \gamma = (\gamma_1, \dots, \gamma_n) \in [r_0, \infty)^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0, \infty)^n$ with $\max(\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n) < 1 + \min(\alpha_1, \dots, \alpha_n)$ and $\max_{i\in\{1,\dots,n\}} \big[\gamma_i - \min(\alpha_i, r_0) + (\beta_i - r_0)\delta_i\big] < 1$. Then there exist unique lower semicontinuous functions $\tau^{t_0,T}\colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0} \to (t_0, T]$, $t_0, T \in \mathbb{R}$ with $t_0 < T$, and unique functions $x^{t_0,T}\colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0} \to \cup_{s\in(t_0,T]} C([t_0, s), U_{r_0})$, $t_0, T \in \mathbb{R}$ with $t_0 < T$, which satisfy $x^{t_0,T}_{F,v} \in C([t_0, \tau^{t_0,T}_{F,v}), U_{r_0})$, $x^{t_0,T}_{F,v}|_{(t_0,\tau^{t_0,T}_{F,v})} \in C((t_0, \tau^{t_0,T}_{F,v}), U_{r_1})$, $\sup_{s\in(t_0,t]}(s-t_0)^{(r_1-r_0)} \|x^{t_0,T}_{F,v}(s)\|_{U_{r_1}} < \infty$ and*

$$\lim_{s\nearrow\tau^{t_0,T}_{F,v}} \Big[\tfrac{1}{(T-s)} + \|x^{t_0,T}_{F,v}(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}}\Big] = \infty \tag{141}$$

and

$$x^{t_0,T}_{F,v}(t) = e^{A(t-t_0)}\, v + \sum_{i=1}^n \int_{t_0}^t e^{A(t-s)}\, F_i(s, x^{t_0,T}_{F,v}(s))\, ds \tag{142}$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v})$, $v \in U_{r_0}$, $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$, $F = (F_1, \dots, F_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. In addition, it holds for every $t_0, T \in \mathbb{R}$ with $t_0 < T$, every $t \in (t_0, T]$ and every $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$ that the function

$$\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0} \ni (F, v) \mapsto \left\{ \begin{array}{ll} x^{t_0,T}_{F,v}(t) & : t < \tau_{F,v} \\ \infty & : t \geq \tau_{F,v} \end{array} \right\} \in U_{r_1} \cup \{\infty\} \tag{143}$$

is Borel measurable. Moreover, it holds that

$$\lim_{N\to\infty} \sup_{s\in(t_0,t]} \left[\begin{array}{c} (s-t_0)^{(r_1-r_0)} \, \|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\|_{U_{r_1}} \\ + \|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\|_{U_{r_0}} \end{array} \right] = 0 \tag{144}$$

for all $t \in (t_0, \tau_{F,v})$, $r_1 \in [r_0, 1+\min(\alpha_1,\dots,\alpha_n))$, $(v_N)_{N\in\mathbb{N}} \subset U_{r_0}$, $(F_N)_{N\in\mathbb{N}} \subset \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ *with* $\lim_{N\to\infty} \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} = \lim_{N\to\infty} \|v_1 - v_N\|_{U_{r_0}} = 0$ *and all* $t_0, T \in \mathbb{R}$ *with* $t_0 < T$.

Proof of Theorem 29. First of all, observe that Corollary 28 ensures that there exist unique functions $\tau^{t_0,T} \colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T]) \times U_{r_0} \to (t_0,T]$, $t_0,T \in \mathbb{R}$ with $t_0 < T$, and $x^{t_0,T} \colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T]) \times U_{r_0} \to \cup_{s\in(t_0,T]} C([t_0,s),U_{r_0})$, $t_0,T \in \mathbb{R}$ with $t_0 < T$, satisfying $x^{t_0,T}_{F,v} \in C([t_0,\tau^{t_0,T}_{F,v}),U_{r_0})$, $x^{t_0,T}_{F,v}|_{(t_0,\tau^{t_0,T}_{F,v})} \in C((t_0,\tau^{t_0,T}_{F,v}),U_{r_1})$, $\sup_{s\in(t_0,t]} (s-t_0)^{(r_1-r_0)} \|x^{t_0,T}_{F,v}(s)\|_{U_{r_1}} < \infty$ and

$$\lim_{s\nearrow\tau_{F,v}} \left[\frac{1}{(T-s)} + \|x^{t_0,T}_{F,v}(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}} \right] = \infty \tag{145}$$

and

$$x^{t_0,T}_{F,v}(t) = e^{A(t-t_0)}\, v + \sum_{i=1}^{n} \int_{t_0}^{t} e^{A(t-s)}\, F_i(s, x^{t_0,T}_{F,v}(s))\, ds \tag{146}$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v})$, $v \in U_{r_0}$, $F = (F_1,\dots,F_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$, $t_0,T \in \mathbb{R}$ with $t_0 < T$ and all $r_1 \in [r_0, 1+\min(\alpha_1,\dots,\alpha_n))$. It thus remains to prove that $\tau^{t_0,T}$, $t_0,T \in \mathbb{R}$ with $t_0 < T$, are lower semicontinuous and that (143) and (144) are fulfilled.

For this let $r_1 \in [\max(\beta_1,\dots,\beta_n,\delta_1,\dots,\delta_n), 1+\min(\alpha_1,\dots,\alpha_n))$ be an arbitrary real number and let $\kappa_{[t_0,T]} \in [0,\infty)$, $t_0,T \in \mathbb{R}$ with $t_0 < T$, be real numbers defined through

$$\begin{aligned} \kappa_{[t_0,T]} &:= \sum_{j=0}^{1} \sum_{i=1}^{n} \left[\frac{1}{(1+\alpha_i - r_j)} + B_{(1+\min(\alpha_i - r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))} \right] \\ &+ \left[2 + n + r_1 - r_0 + |T-t_0| + \sum_{i=1}^{n} \delta_i \right]^{\left(4+|r_0|+|r_1|+\max_{i\in\{1,\dots,n\}}|\alpha_i|\right)} \\ &+ \max_{j\in\{0,1\}} \max_{\theta\in\{r_0,r_1,\alpha_1,\dots,\alpha_n\}} \sup_{t\in(t_0,T]} \left[(t-t_0)^{\max(r_j-\theta,0)} \, \|e^{A(t-t_0)}\|_{L(U_\theta,U_{r_j})} \right] \\ &+ \max_{\substack{\theta\in\{\beta_1,\dots,\beta_n\} \\ \cup\{\gamma_1,\dots,\gamma_n\}}} \sup_{\substack{v\in U_{r_1} \\ v\neq 0}} \left[1 + \frac{\|v\|_{U_\theta}}{\|v\|_{U_{r_1}}^{\frac{(\theta-r_0)}{(r_1-r_0)}} \, \|v\|_{U_{r_0}}^{\frac{(r_1-\theta)}{(r_1-r_0)}}} + \frac{\|v\|_{U_{r_0}}}{\|v\|_{U_{r_1}}} \right]^{(1+\sum_{i=1}^n \delta_i)} < \infty \end{aligned} \tag{147}$$

for all $t_0,T \in \mathbb{R}$ with $t_0 < T$ where $B \colon (0,\infty)^2 \to (0,\infty)$ is the Beta function defined through $B_{(x,y)} := \int_0^1 (1-s)^{(x-1)}\, s^{(y-1)}\, ds$ for all $x,y \in (0,\infty)$. Then observe that

$$\begin{aligned} &\big\|x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde F,\tilde v}(t)\big\|_{U_{r_j}} \leq \big\|e^{A(t-t_0)}\big\|_{L(U_{r_k},U_{r_j})} \, \|v - \tilde v\|_{U_{r_k}} \\ &+ \sum_{i=1}^{n} \int_{t_0}^{t} \big\|e^{A(t-s)}\big\|_{L(U_{\alpha_i},U_{r_j})} \big\|F_i(s, x^{t_0,T}_{F,v}(s)) - \tilde F_i(s, x^{t_0,T}_{\tilde F,\tilde v}(s))\big\|_{U_{\alpha_i}} ds \\ &\leq \kappa_{[t_0,T]} \, (t-t_0)^{\min(r_k - r_j,0)} \, \|v - \tilde v\|_{U_{r_k}} \\ &+ \sum_{i=1}^{n} \int_{t_0}^{t} \kappa_{[t_0,T]} \, (t-t_0)^{\min(\alpha_i - r_j,0)} \, \|F_i(s, x^{t_0,T}_{F,v}(s)) - \tilde F_i(s, x^{t_0,T}_{F,v}(s))\|_{U_{\alpha_i}} \, ds \\ &+ \sum_{i=1}^{n} \int_{t_0}^{t} \kappa_{[t_0,T]} \, (t-t_0)^{\min(\alpha_i - r_j,0)} \, \|\tilde F_i(s, x^{t_0,T}_{F,v}(s)) - \tilde F_i(s, x^{t_0,T}_{\tilde F,\tilde v}(s))\|_{U_{\alpha_i}} \, ds \end{aligned} \tag{148}$$

and inequality (121) therefore implies that

$$
\begin{aligned}
&\big\|x_{F,v}^{t_0,T}(t) - x_{\tilde F,\tilde v}^{t_0,T}(t)\big\|_{U_{r_j}} \leq \kappa_{[t_0,T]}\,(t-t_0)^{\min(r_k-r_j,0)}\,\|v-\tilde v\|_{U_{r_k}} \\
&+ \kappa_{[t_0,T]}\,\|F-\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \\
&\cdot \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\left(1+\|x_{F,v}^{t_0,T}(s)\|_{\beta_i}^{\delta_i}\right)\left(1+\|x_{F,v}^{t_0,T}(s)\|_{U_{\gamma_i}}\right) ds \\
&+ \kappa_{[t_0,T]}\,\|\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \\
&\cdot \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\left(1+\|x_{F,v}^{t_0,T}(s)\|_{U_{\beta_i}}^{\delta_i}+\|x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{\beta_i}}^{\delta_i}\right) \\
&\cdot \|x_{F,v}^{t_0,T}(s) - x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{\gamma_i}}\,ds
\end{aligned}
\tag{149}
$$

and the definition of $\kappa_{[t_0,T]}$ hence shows that

$$
\begin{aligned}
&\big\|x_{F,v}^{t_0,T}(t) - x_{\tilde F,\tilde v}^{t_0,T}(t)\big\|_{U_{r_j}} \leq \kappa_{[t_0,T]}\,(t-t_0)^{\min(r_k-r_j,0)}\,\|v-\tilde v\|_{U_{r_k}} \\
&+ \left[\kappa_{[t_0,T]}\right]^3 \|F-\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)} \\
&\cdot \left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}}\right]\left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_0}}^{\frac{(r_1-\gamma_i)}{(r_1-r_0)}}\,\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}^{\frac{(\gamma_i-r_0)}{(r_1-r_0)}}\right] ds \\
&+ \left[\kappa_{[t_0,T]}\right]^3 \|\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)} \\
&\cdot \left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}} + \|x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,\|x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}}\right] \\
&\cdot \|x_{F,v}^{t_0,T}(s) - x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_0}}^{\frac{(r_1-\gamma_i)}{(r_1-r_0)}}\,\|x_{F,v}^{t_0,T}(s) - x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}^{\frac{(\gamma_i-r_0)}{(r_1-r_0)}}\,ds
\end{aligned}
\tag{150}
$$

for all $j,k \in \{0,1\}$, $t \in (t_0, \tau_{F,v}^{t_0,T}) \cap (t_0, \tau_{\tilde F,\tilde v}^{t_0,T})$, $v,\tilde v \in U_{r_0}$, $F,\tilde F \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. This, in particular, implies that

$$
\begin{aligned}
&\big\|x_{F,v}^{t_0,T}(t) - x_{\tilde F,\tilde v}^{t_0,T}(t)\big\|_{U_{r_1}} \leq \kappa_{[t_0,T]}\,\|v-\tilde v\|_{U_{r_1}} \\
&+ \left[\kappa_{[t_0,T]}\right]^5 \|F-\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_1,0)} \\
&\cdot \left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}^{\delta_i}\right]\left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}\right] ds \\
&+ \left[\kappa_{[t_0,T]}\right]^5 \|\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_1,0)} \\
&\cdot \left[1+\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}^{\delta_i}+\|x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}^{\delta_i}\right]\|x_{F,v}^{t_0,T}(s) - x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}\,ds
\end{aligned}
\tag{151}
$$

and the estimates $\left(1+|x|^{\delta_i}\right)\left(1+|x|\right) \leq \kappa\,(1+|x|)^{(2+\sum_{j=1}^n \delta_j)}$ and $\left(1+|x|^{\delta_i}+|y|^{\delta_i}\right) \leq \kappa\,(1+|x|+|y|)^{(1+\sum_{j=1}^n \delta_j)}$ for all $x,y \in \mathbb{R}$ and all $i \in \{1,\dots,n\}$ hence give

$$
\begin{aligned}
&\big\|x_{F,v}^{t_0,T}(t) - x_{\tilde F,\tilde v}^{t_0,T}(t)\big\|_{U_{r_1}} \leq \kappa_{[t_0,T]}\,\|v-\tilde v\|_{U_{r_1}} \\
&+ \left[\kappa_{[t_0,T]}\right]^6 \|F-\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\left[1+\sup_{s\in[t_0,t]}\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}\right]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_1,0)}\,ds \\
&+ \left[\kappa_{[t_0,T]}\right]^6 \|\tilde F\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\left[1+\sup_{s\in[t_0,t]}\|x_{F,v}^{t_0,T}(s)\|_{U_{r_1}}+\sup_{s\in[t_0,t]}\|x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}\right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_1,0)}\,\|x_{F,v}^{t_0,T}(s) - x_{\tilde F,\tilde v}^{t_0,T}(s)\|_{U_{r_1}}\,ds
\end{aligned}
\tag{152}
$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_1}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. Therefore, we obtain that

$$\begin{aligned}
&\left\| x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t) \right\|_{U_{r_1}} \leq \kappa_{[t_0,T]} \left\| v - \tilde{v} \right\|_{U_{r_1}} \\
&+ \left[\kappa_{[t_0,T]} \right]^8 \| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} \right]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot \int_{t_0}^t (t-s)^{\min(\alpha_1 - r_1, \ldots, \alpha_n - r_1, 0)} \, ds \\
&+ \left[\kappa_{[t_0,T]} \right]^8 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \|_{U_{r_1}} \right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \int_{t_0}^t (t-s)^{\min(\alpha_1 - r_1, \ldots, \alpha_n - r_1, 0)} \, \| x^{t_0,T}_{F,v}(s) - x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \|_{U_{r_1}} \, ds
\end{aligned} \tag{153}$$

and hence

$$\begin{aligned}
&\left\| x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t) \right\|_{U_{r_1}} \\
&\leq \left[\kappa_{[t_0,T]} \right]^{10} \left[\left\| v - \tilde{v} \right\|_{U_{r_1}} + \| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \right] \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} \right]^{(2+\sum_{i=1}^n \delta_i)} \\
&+ \left[\kappa_{[t_0,T]} \right]^8 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \|_{U_{r_1}} \right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \int_{t_0}^t (t-s)^{\min(\alpha_1 - r_1, \ldots, \alpha_n - r_1, 0)} \, \| x^{t_0,T}_{F,v}(s) - x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \|_{U_{r_1}} \, ds
\end{aligned} \tag{154}$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_1}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. A generalization of Gronwall's lemma (see Lemma 7.1.1 in Henry [11]) therefore implies

$$\begin{aligned}
&\sup_{s \in [t_0,t]} \left\| x^{t_0,T}_{F,v}(s) - x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \right\|_{U_{r_1}} \leq E_{\min(\alpha_1 - r_1, \ldots, \alpha_n - r_1, 0)} \Bigg[\left[\kappa_{[t_0,T]} \right]^9 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \\
&\cdot \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{\tilde{F},\tilde{v}}(s) \|_{U_{r_1}} \right]^{(1+\sum_{i=1}^n \delta_i)} \Bigg] \left[\kappa_{[t_0,T]} \right]^{10} \\
&\cdot \left[\left\| v - \tilde{v} \right\|_{U_{r_1}} + \| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \right] \left[1 + \sup_{s \in [t_0,t]} \| x^{t_0,T}_{F,v}(s) \|_{U_{r_1}} \right]^{(2+\sum_{i=1}^n \delta_i)}
\end{aligned} \tag{155}$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_1}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$ where $E_r \colon [0,\infty) \to [0,\infty)$, $r \in (-1, 0]$, is a family of functions defined through $E_r(x) := \sum_{n=0}^\infty \frac{(x \cdot \Gamma(r+1))^n}{\Gamma(n(r+1)+1)}$ for all $x \in [0,\infty)$ and all $r \in (-1,0]$. As in (126) and (127), we now define sets $\mathcal{E}_{[t_0,T]}$, $t_0, T \in \mathbb{R}$ with $t_0 < T$, and functions $\|\cdot\|_{\mathcal{E}_{[t_0,T]}} : \mathcal{E}_{[t_0,T]} \to [0,\infty)$, $t_0, T \in \mathbb{R}$ with $t_0 < T$, by

$$\mathcal{E}_{[t_0,T]} := \left\{ y \in C([t_0,T], U_{r_0}) \colon \begin{pmatrix} y|_{(t_0,T]} \in C((t_0,T], U_{r_1}) \text{ and} \\ \sup_{t \in (t_0,T]} (t-t_0)^{(r_1 - r_0)} \|y(t)\|_{U_{r_1}} < \infty \end{pmatrix} \right\} \tag{156}$$

for all $t_0, T \in \mathbb{R}$ with $t_0 < T$ and by $\|y\|_{\mathcal{E}_{[t_0,T]}} := \sum_{j=0}^1 \sup_{t \in (t_0,\tau]} (t-t_0)^{(r_j - r_0)} \|y(t)\|_{U_{r_j}}$ for all $y \in \mathcal{E}_{[t_0,T]}$,

$t_0, T \in \mathbb{R}$ with $t_0 < T$. Then we get from (150) that

$$
\begin{aligned}
&\left\|x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t)\right\|_{U_{r_j}} \leq \kappa_{[t_0,T]}\,(t-t_0)^{(r_0-r_j)}\,\|v-\tilde{v}\|_{U_{r_0}} \\
&+ \left[\kappa_{[t_0,T]}\right]^4 \|F-\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))} \\
&\cdot \left[1 + \|x^{t_0,T}_{F,v}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,(s-t_0)^{(\beta_i-r_0)\delta_i}\,\|x^{t_0,T}_{F,v}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}}\right] \\
&\cdot \left[1 + \|x^{t_0,T}_{F,v}(s)\|_{U_{r_0}}^{\frac{(r_1-\gamma_i)}{(r_1-r_0)}}\,(s-t_0)^{(\gamma_i-r_0)}\,\|x^{t_0,T}_{F,v}(s)\|_{U_{r_1}}^{\frac{(\gamma_i-r_0)}{(r_1-r_0)}}\right] ds \\
&+ \left[\kappa_{[t_0,T]}\right]^4 \|\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))} \\
&\cdot \left[1 + \|x^{t_0,T}_{F,v}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,(s-t_0)^{(\beta_i-r_0)\delta_i}\,\|x^{t_0,T}_{F,v}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}}\right. \\
&\left. + \|x^{t_0,T}_{\tilde{F},\tilde{v}}(s)\|_{U_{r_0}}^{\frac{(r_1-\beta_i)\delta_i}{(r_1-r_0)}}\,(s-t_0)^{(\beta_i-r_0)\delta_i}\,\|x^{t_0,T}_{\tilde{F},\tilde{v}}(s)\|_{U_{r_1}}^{\frac{(\beta_i-r_0)\delta_i}{(r_1-r_0)}}\right] \\
&\cdot \|x^{t_0,T}_{F,v}(s) - x^{t_0,T}_{\tilde{F},\tilde{v}}(s)\|_{U_{r_0}}^{\frac{(r_1-\gamma_i)}{(r_1-r_0)}}\,(s-t_0)^{(\gamma_i-r_0)}\,\|x^{t_0,T}_{F,v}(s) - x^{t_0,T}_{\tilde{F},\tilde{v}}(s)\|_{U_{r_1}}^{\frac{(\gamma_i-r_0)}{(r_1-r_0)}}\,ds
\end{aligned}
\tag{157}
$$

and therefore

$$
\begin{aligned}
&\left\|x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t)\right\|_{U_{r_j}} \leq \kappa_{[t_0,T]}\,(t-t_0)^{(r_0-r_j)}\,\|v-\tilde{v}\|_{U_{r_0}} \\
&+ \left[\kappa_{[t_0,T]}\right]^4 \|F-\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,ds \\
&\cdot \left[1 + \|x^{t_0,T}_{F,v}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}^{\delta_i}\right]\left[1 + \|x^{t_0,T}_{F,v}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}\right] \\
&+ \left[\kappa_{[t_0,T]}\right]^4 \|\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,ds \\
&\cdot \left[1 + \|x^{t_0,T}_{F,v}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}^{\delta_i} + \|x^{t_0,T}_{\tilde{F},\tilde{v}}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}^{\delta_i}\right] \|(x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}
\end{aligned}
\tag{158}
$$

and the estimates $\left(1+|x|^{\delta_i}\right)\left(1+|x|\right) \leq \kappa\,(1+|x|)^{(2+\sum_{j=1}^n \delta_j)}$ and $\left(1+|x|^{\delta_i}+|y|^{\delta_i}\right) \leq \kappa\,(1+|x|+|y|)^{(1+\sum_{j=1}^n \delta_j)}$ for all $x, y \in \mathbb{R}$ and all $i \in \{1,\ldots,n\}$ hence show that

$$
\begin{aligned}
&\left\|x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t)\right\|_{U_{r_j}} \leq \kappa_{[t_0,T]}\,(t-t_0)^{(r_0-r_j)}\,\|v-\tilde{v}\|_{U_{r_0}} \\
&+ \left[\kappa_{[t_0,T]}\right]^5 \|F-\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \|x^{t_0,T}_{F,v}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}\right]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,ds \\
&+ \left[\kappa_{[t_0,T]}\right]^5 \|\tilde{F}\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \|x^{t_0,T}_{F,v}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}} + \|x^{t_0,T}_{\tilde{F},\tilde{v}}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}\right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \|(x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}} \sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,ds
\end{aligned}
\tag{159}
$$

for all $j \in \{0,1\}$, $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_0}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. The estimate

$$
\begin{aligned}
&\sum_{i=1}^n \int_{t_0}^t (t-s)^{\min(\alpha_i-r_j,0)}\,(s-t_0)^{(r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,ds \\
&= \sum_{i=1}^n (t-t_0)^{(1+\min(\alpha_i-r_j,0)+r_0-\gamma_i+\delta_i(r_0-\beta_i))}\,B_{(1+\min(\alpha_i-r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))} \\
&\leq \kappa_{[t_0,T]}\,(t-t_0)^{\left[r_0-r_j+\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))\right]} \\
&\quad \cdot \sum_{i=1}^n B_{(1+\min(\alpha_i-r_j,0),1+r_0-\gamma_i+\delta_i(r_0-\beta_i))} \\
&\leq \left[\kappa_{[t_0,T]}\right]^2 (t-t_0)^{\left[r_0-r_j+\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))\right]}
\end{aligned}
\tag{160}
$$

for all $j \in \{0,1\}$, $t \in (t_0, T]$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$ therefore proves that

$$
\begin{aligned}
&(t-t_0)^{(r_j - r_0)} \left\| x^{t_0,T}_{F,v}(t) - x^{t_0,T}_{\tilde{F},\tilde{v}}(t) \right\|_{U_{r_j}} \leq \kappa_{[t_0,T]} \left\| v - \tilde{v} \right\|_{U_{r_0}} \\
&+ \left[\kappa_{[t_0,T]}\right]^7 \| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot (t-t_0)^{\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))} \\
&+ \left[\kappa_{[t_0,T]}\right]^7 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| x^{t_0,T}_{\tilde{F},\tilde{v}}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \| (x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} (t-t_0)^{\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))}
\end{aligned}
\tag{161}
$$

for all $j \in \{0,1\}$, $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_0}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. Hence, we obtain

$$
\begin{aligned}
&\| (x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \leq \left[\kappa_{[t_0,T]}\right]^2 \| v - \tilde{v} \|_{U_{r_0}} \\
&+ \left[\kappa_{[t_0,T]}\right]^8 \| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot (t-t_0)^{\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))} \\
&+ \left[\kappa_{[t_0,T]}\right]^8 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| x^{t_0,T}_{\tilde{F},\tilde{v}}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(1+\sum_{i=1}^n \delta_i)} \\
&\cdot \| (x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} (t-t_0)^{\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))}
\end{aligned}
\tag{162}
$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_0}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$. Rearranging finally results in

$$
\begin{aligned}
&\| (x^{t_0,T}_{F,v} - x^{t_0,T}_{\tilde{F},\tilde{v}})|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \Bigg[1 - (t-t_0)^{\min_{i\in\{1,\ldots,n\}}(1+\min(\alpha_i,r_0)-\gamma_i+\delta_i(r_0-\beta_i))} \\
&\cdot \left[\kappa_{[t_0,T]}\right]^8 \| \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| x^{t_0,T}_{\tilde{F},\tilde{v}}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(1+\sum_{i=1}^n \delta_i)} \Bigg] \\
&\leq \left[\kappa_{[t_0,T]}\right]^9 \left[\| F - \tilde{F} \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \| v - \tilde{v} \|_{U_{r_0}} \right] \left[1 + \| x^{t_0,T}_{F,v}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(2+\sum_{i=1}^n \delta_i)}
\end{aligned}
\tag{163}
$$

for all $t \in (t_0, \tau^{t_0,T}_{F,v}) \cap (t_0, \tau^{t_0,T}_{\tilde{F},\tilde{v}})$, $v, \tilde{v} \in U_{r_0}$, $F, \tilde{F} \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $t_0, T \in \mathbb{R}$ with $t_0 < T$.

We now use (155) and (163) to prove (144). For this let $t_0, T \in \mathbb{R}$ be real numbers with $t_0 < T$, let $\varepsilon \in (0,1]$ be a real number defined through $\varepsilon := \min_{i\in\{1,\ldots,n\}} (1 + \min(\alpha_i, r_0) - \gamma_i + \delta_i(r_0 - \beta_i))$ and let $(v_N)_{N\in\mathbb{N}} \subset U_{r_0}$ and $F_N = (F_{N,1}, \ldots, F_{N,n}) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])$, $N \in \mathbb{N}$, be sequences with $\lim_{N\to\infty} \|v_1 - v_N\|_{U_{r_0}} = \lim_{N\to\infty} \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} = 0$ and $\|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \leq 1$ for all $N \in \mathbb{N}$. Then observe that (163) ensures that

$$
\begin{aligned}
&\left\| (x^{t_0,T}_{F_1,v_1} - x^{t_0,T}_{F_N,v_N})|_{[t_0,t]} \right\|_{\mathcal{E}_{[t_0,t]}} \Bigg[1 - \left[\kappa_{[t_0,T]}\right]^8 (t-t_0)^\varepsilon \Big[2 + \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \\
&+ \| x^{t_0,T}_{F_N,v_N}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| F_1 \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \Big]^{(2+\sum_{i=1}^n \delta_i)} \Bigg] \leq \left[\kappa_{[t_0,T]}\right]^9 \\
&\cdot \left[\| F_1 - F_N \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \| v_1 - v_N \|_{U_{r_0}} \right] \left[1 + \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(2+\sum_{i=1}^n \delta_i)}
\end{aligned}
\tag{164}
$$

for all $t \in (t_0, \tau^{t_0,T}_{F_1,v_1}) \cap (t_0, \tau^{t_0,T}_{F_N,v_N})$ and all $N \in \mathbb{N}$. This implies that

$$
\begin{aligned}
&\| (x^{t_0,T}_{F_1,v_1} - x^{t_0,T}_{F_N,v_N})|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \\
&\cdot \left[1 - \left[\kappa_{[t_0,T]}\right]^8 (t-t_0)^\varepsilon \left[4 + 2 \, \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| F_1 \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \right]^{(2+\sum_{i=1}^n \delta_i)} \right] \\
&\leq \left[\kappa_{[t_0,T]}\right]^9 \left[\| F_1 - F_N \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \| v_1 - v_N \|_{U_{r_0}} \right] \left[1 + \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} \right]^{(2+\sum_{i=1}^n \delta_i)}
\end{aligned}
\tag{165}
$$

for all $t \in \left\{ s \in (t_0, \tau^{t_0,T}_{F_1,v_1}) \cap (t_0, \tau^{t_0,T}_{F_N,v_N}) \colon \| x^{t_0,T}_{F_N,v_N}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,s]}} \leq 2 + \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,s]}} \right\}$ and all $N \in \mathbb{N}$. In the next step let $\hat{t} \in (t_0, \tau^{t_0,T}_{F_1,v_1})$ and $\hat{N} \in \mathbb{N}$ be real numbers with the property that

$$
\left[\kappa_{[t_0,T]}\right]^8 (t-t_0)^\varepsilon \left[4 + 2 \, \| x^{t_0,T}_{F_1,v_1}|_{[t_0,t]} \|_{\mathcal{E}_{[t_0,t]}} + \| F_1 \|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} \right]^{(2+\sum_{i=1}^n \delta_i)} \leq \tfrac{1}{2}
\tag{166}
$$

for all $t \in (t_0, \hat{t}]$ and with the property that

$$\left[\kappa_{[t_0,T]}\right]^9 \left[\|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \|v_1 - v_N\|_{U_{r_0}}\right] \cdot \left[1 + \|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}\right]^{(2+\sum_{i=1}^n \delta_i)} \leq \tfrac{1}{2} \tag{167}$$

for all $N \in \{\hat{N}, \hat{N}+1, \dots\} =: \hat{\mathbb{N}}$. Then we obtain from (165) that

$$\|(x^{t_0,T}_{F_1,v_1} - x^{t_0,T}_{F_N,v_N})|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}} \leq 2\left[\kappa_{[t_0,T]}\right]^9 \cdot \left[\|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \|v_1 - v_N\|_{U_{r_0}}\right]\left[1 + \|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}\right]^{(2+\sum_{i=1}^n \delta_i)} \leq 1 \tag{168}$$

for all $t \in \left\{s \in (t_0,\hat{t}] \cap (t_0, \tau^{t_0,T}_{F_N,v_N}) \colon \|x^{t_0,T}_{F_N,v_N}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,s]}} \leq 2 + \|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,s]}}\right\}$ and all $N \in \hat{\mathbb{N}}$. This implies that

$$\|(x^{t_0,T}_{F_1,v_1} - x^{t_0,T}_{F_N,v_N})|_{[t_0,\hat{t}]}\|_{\mathcal{E}_{[t_0,\hat{t}]}} \leq 2\left[\kappa_{[t_0,T]}\right]^9 \cdot \left[\|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])} + \|v_1 - v_N\|_{U_{r_0}}\right]\left[1 + \|x^{t_0,T}_{F_1,v_1}|_{[t_0,\hat{t}]}\|_{\mathcal{E}_{[t_0,\hat{t}]}}\right]^{(2+\sum_{i=1}^n \delta_i)} \tag{169}$$

for all $N \in \hat{\mathbb{N}}$ and we hence get

$$\lim_{N\to\infty} \|(x^{t_0,T}_{F_1,v_1} - x^{t_0,T}_{F_N,v_N})|_{[t_0,\hat{t}]}\|_{\mathcal{E}_{[t_0,\hat{t}]}} = 0. \tag{170}$$

In the next step we define $\hat{v}_N \in U_{r_1}$, $N \in \hat{\mathbb{N}} \cup \{1\}$, through $\hat{v}_N := x^{t_0,T}_{F_N,v_N}(\hat{t})$ for all $N \in \hat{\mathbb{N}} \cup \{1\}$ and we define $\hat{F}_N \in C^n_{\alpha,\beta,\gamma,\delta}([\hat{t},T])$, $N \in \hat{\mathbb{N}} \cup \{1\}$, through $\hat{F}_N := (F_{N,1}|_{[\hat{t},T]}, \dots, F_{N,n}|_{[\hat{t},T]})$ for all $N \in \hat{\mathbb{N}} \cup \{1\}$. Note that $(\hat{v}_N)_{N\in\hat{\mathbb{N}}\cup\{1\}}$ is well-defined since $\hat{t} < \tau^{t_0,T}_{F_N,v_N}$ for all $N \in \hat{\mathbb{N}} \cup \{1\}$. Furthermore, we obtain from (155) that

$$\begin{aligned}\sup_{s\in[\hat{t},t]} \left\|x^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}(s) - x^{\hat{t},T}_{\hat{F}_N,\hat{v}_N}(s)\right\|_{U_{r_1}} &\leq E_{\min(\alpha_1-r_1,\dots,\alpha_n-r_1,0)}\Bigg[\left[\kappa_{[\hat{t},T]}\right]^9 \|\hat{F}_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([\hat{t},T])} \\ &\cdot \left[1 + \sup_{s\in[\hat{t},t]} \|x^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}(s)\|_{U_{r_1}} + \sup_{s\in[\hat{t},t]} \|x^{\hat{t},T}_{\hat{F}_N,\hat{v}_N}(s)\|_{U_{r_1}}\right]^{(1+\sum_{i=1}^n \delta_i)}\Bigg]\left[\kappa_{[\hat{t},T]}\right]^{10} \\ &\cdot \left[\|\hat{v}_1 - \hat{v}_N\|_{U_{r_1}} + \|\hat{F}_1 - \hat{F}_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([\hat{t},T])}\right]\left[1 + \sup_{s\in[\hat{t},t]} \|x^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}(s)\|_{U_{r_1}}\right]^{(2+\sum_{i=1}^n \delta_i)}\end{aligned} \tag{171}$$

for all $t \in (\hat{t}, \tau^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}) \cap (\hat{t}, \tau^{\hat{t},T}_{\hat{F}_N,\hat{v}_N})$ and all $N \in \hat{\mathbb{N}}$. This implies that

$$\begin{aligned}\sup_{s\in[\hat{t},t]} \left\|x^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}(s) - x^{\hat{t},T}_{\hat{F}_N,\hat{v}_N}(s)\right\|_{U_{r_1}} &\leq \left[\kappa_{[t_0,T]}\right]^{10} \cdot E_{\min(\alpha_1-r_1,\dots,\alpha_n-r_1,0)}\Bigg[\left[\kappa_{[t_0,T]}\right]^9 \\ &\cdot \left[2 + \sup_{s\in[\hat{t},t]} \|x^{\hat{t},T}_{\hat{F}_1,\hat{v}_1}(s)\|_{U_{r_1}} + \sup_{s\in[\hat{t},t]} \|x^{\hat{t},T}_{\hat{F}_N,\hat{v}_N}(s)\|_{U_{r_1}} + \|F_1\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\right]^{(2+\sum_{i=1}^n \delta_i)}\Bigg] \\ &\cdot \left[\|\hat{v}_1 - \hat{v}_N\|_{U_{r_1}} + \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\right]\left[1 + \frac{\|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}}{(\hat{t}-t_0)^{(r_1-r_0)}}\right]^{(2+\sum_{i=1}^n \delta_i)}\end{aligned} \tag{172}$$

and therefore

$$\begin{aligned}\sup_{s\in[\hat{t},t]} \left\|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\right\|_{U_{r_1}} &\leq \left[\kappa_{[t_0,T]}\right]^{10} \cdot E_{\min(\alpha_1-r_1,\dots,\alpha_n-r_1,0)}\Bigg[\left[\kappa_{[t_0,T]}\right]^9 \\ &\cdot \left[4 + 2\sup_{s\in[\hat{t},t]} \|x^{t_0,T}_{F_1,v_1}(s)\|_{U_{r_1}} + \|F_1\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\right]^{(2+\sum_{i=1}^n \delta_i)}\Bigg] \\ &\cdot \left[\|\hat{v}_1 - \hat{v}_N\|_{U_{r_1}} + \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\right]\left[1 + \frac{\|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}}{(\hat{t}-t_0)^{(r_1-r_0)}}\right]^{(2+\sum_{i=1}^n \delta_i)}\end{aligned} \tag{173}$$

for all $t \in \{s \in (\hat{t}, \tau^{t_0,T}_{F_1,v_1}) \cap (\hat{t}, \tau^{t_0,T}_{F_N,v_N})\colon \sup_{u\in[\hat{t},s]} \|x^{t_0,T}_{F_N,v_N}(u)\|_{U_{r_1}} \leq 2 + \sup_{u\in[\hat{t},s]} \|x^{t_0,T}_{F_1,v_1}(u)\|_{U_{r_1}}\}$ and all $N \in \hat{\mathbb{N}}$. In the next step we observe that (170) proves that there exists a non-decreasing family $N_t \in \hat{\mathbb{N}}$, $t \in (\hat{t}, \tau^{t_0,T}_{F_1,v_1})$, of natural numbers such that

$$
\begin{aligned}
&E_{\min(\alpha_1-r_1,\dots,\alpha_n-r_1,0)}\Bigg[\Big[4 + 2\sup_{s\in[\hat{t},t]} \|x^{t_0,T}_{F_1,v_1}(s)\|_{U_{r_1}} + \|F_1\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\Big]^{(2+\sum_{i=1}^n \delta_i)} \\
&\cdot [\kappa_{[t_0,T]}]^9\Bigg] [\kappa_{[t_0,T]}]^{10} \Big[\|\hat{v}_1 - \hat{v}_N\|_{U_{r_1}} + \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\Big] \\
&\cdot \left[1 + \frac{\|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}}{(\hat{t}-t_0)^{(r_1-r_0)}}\right]^{(2+\sum_{i=1}^n \delta_i)} \leq 1
\end{aligned}
\tag{174}
$$

for all $N \in \{N_t, N_t + 1, \dots\}$ and all $t \in (\hat{t}, \tau^{t_0,T}_{F_1,v_1})$. Combining this with (173) results in

$$
\begin{aligned}
&\sup_{s\in[\hat{t},t]} \|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\|_{U_{r_1}} \leq E_{\min(\alpha_1-r_1,\dots,\alpha_n-r_1,0)}\Bigg[[\kappa_{[t_0,T]}]^9 \\
&\cdot \Big[4 + 2\sup_{s\in[\hat{t},t]} \|x^{t_0,T}_{F_1,v_1}(s)\|_{U_{r_1}} + \|F_1\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\Big]^{(2+\sum_{i=1}^n \delta_i)}\Bigg] [\kappa_{[t_0,T]}]^{10} \\
&\cdot \Big[\|\hat{v}_1 - \hat{v}_N\|_{U_{r_1}} + \|F_1 - F_N\|_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,T])}\Big] \left[1 + \frac{\|x^{t_0,T}_{F_1,v_1}|_{[t_0,t]}\|_{\mathcal{E}_{[t_0,t]}}}{(\hat{t}-t_0)^{(r_1-r_0)}}\right]^{(2+\sum_{i=1}^n \delta_i)}
\end{aligned}
\tag{175}
$$

for all $N \in \{N_t, N_t + 1, \dots\}$ and all $t \in (\hat{t}, \tau^{t_0,T}_{F_1,v_1})$. Inequality (175) implies that $\tau^{t_0,T}$ is lower semicontinuous and combining (175) with (169) proves that

$$
\lim_{N\to\infty} \sup_{s\in(t_0,t]} \left[\begin{array}{c} (s-t_0)^{(r_1-r_0)} \|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\|_{U_{r_1}} \\ + \|x^{t_0,T}_{F_1,v_1}(s) - x^{t_0,T}_{F_N,v_N}(s)\|_{U_{r_0}} \end{array}\right] = 0 \tag{176}
$$

for all $t \in (t_0, \tau_{F_1,v_1})$. Interpolation (see, e.g., Theorem 37.6 in Sell & You [19]) hence implies that (144) is fulfilled. Since every lower semicontinuous function is Borel measurable, we obtain that $\tau^{t_0,T}$ is Borel measurable. Therefore, we get for every $t \in [t_0, T]$ that the sets $\{(F, v) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0}\colon \tau^{t_0,T}_{F,v} > t\}$ and $\{(F, v) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0}\colon \tau^{t_0,T}_{F,v} \leq t\}$ are Borel measurable subsets of $\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0}$ and (144) implies for every $t \in (t_0, T]$ and every $r \in [r_0, r_1]$ that the mapping $\{(F, v) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0, T]) \times U_{r_0}\colon \tau^{t_0,T}_{F,v} > t\} \ni (F, v) \mapsto x_{F,v}(t) \in U_r$ is continuous and, in particular, Borel measurable. These two facts imply (143) and this completes the proof of Theorem 29. □

Theorem 29 investigates solutions of (116) on a bounded time interval. The next corollary extends this result to unbounded time intervals.

Corollary 30 (Continuous dependence on the data on unbounded time intervals)**.** *Assume the setting in the beginning of Subsection 3.1 and let* $t_0, r_0 \in \mathbb{R}$, $n \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \dots, \beta_n)$, $\gamma = (\gamma_1, \dots, \gamma_n) \in [r_0, \infty)^n$, $\delta = (\delta_1, \dots, \delta_n) \in [0, \infty)^n$ *with* $\max(\beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n) < 1 + \min(\alpha_1, \dots, \alpha_n)$ *and*

$$
\max_{i\in\{1,\dots,n\}} \big[\gamma_i - \min(\alpha_i, r_0) + \delta_i(\beta_i - r_0)\big] < 1. \tag{177}
$$

Then there exist a unique lower semicontinuous function $\tau\colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \to (t_0, \infty]$ *and a unique function* $x\colon \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \to \cup_{s\in(t_0,\infty]} C([t_0, s), U_{r_0})$ *satisfying* $x_{F,v} \in C([t_0, \tau_{F,v}), U_{r_0})$, $x_{F,v}|_{(t_0,\tau_{F,v})} \in C((t_0, \tau_{F,v}), U_{r_1})$, $\sup_{s\in(t_0,t]}(s - t_0)^{(r_1-r_0)} \|x_{F,v}(s)\|_{U_{r_1}} < \infty$ *and*

$$
\lim_{s\nearrow\tau_{F,v}} \big[\tau_{F,v} + \|x_{F,v}(s)\|_{U_{\max(\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n)}}\big] = \infty \tag{178}
$$

and

$$
x_{F,v}(t) = e^{A(t-t_0)}\, v + \sum_{i=1}^n \int_{t_0}^t e^{A(t-s)}\, F_i(s, x_{F,v}(s))\, ds \tag{179}
$$

for all $t \in (t_0, \tau_{F,v})$, $v \in U_{r_0}$, $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$ *and all* $F = (F_1, \dots, F_n) \in \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$. *In addition, it holds for every* $t \in (t_0, \infty)$ *and every* $r_1 \in [r_0, 1 + \min(\alpha_1, \dots, \alpha_n))$ *that the function*

$$
\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \ni (F, v) \mapsto \left\{\begin{array}{ll} x_{F,v}(t) & : t < \tau_{F,v} \\ \infty & : t \geq \tau_{F,v} \end{array}\right\} \in U_{r_1} \cup \{\infty\} \qquad \text{(Measurability property)}
$$

is Borel measurable. Moreover, it holds that

$$\lim_{N\to\infty} \sup_{s\in(t_0,t]} \left[\begin{array}{c} (s-t_0)^{(r_1-r_0)} \|x_{F_1,v_1}(s) - x_{F_N,v_N}(s)\|_{U_{r_1}} \\ + \|x_{F_1,v_1}(s) - x_{F_N,v_N}(s)\|_{U_{r_0}} \end{array} \right] = 0 \qquad \text{(Continuity property)}$$

for all $t \in (t_0, \tau_{F_1,v_1})$, $r_1 \in [r_0, 1+\min(\alpha_1,\dots,\alpha_n))$, $(v_N)_{N\in\mathbb{N}} \subset U_{r_0}$, $(F_N)_{N\in\mathbb{N}} \subset \mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$ *with* $\lim_{N\to\infty} d_{\mathcal{C}^n_{\alpha,\beta,\gamma,\delta}([t_0,\infty))}(F_1,F_N) = \lim_{N\to\infty} \|v_1 - v_N\|_{U_{r_0}} = 0$.

Corollary 30 follows immediately from Theorem 29 and its proof is therefore omitted.

3.2 SPDEs with space-time white noise and polynomial nonlinearities in two space dimensions

The aim of this subsection is to prove local existence and uniqueness of mild solutions of SPDEs in two space dimensions with polynomial nonlinearities of the form

$$dX_t = \left[\triangle X_t + \kappa_n(t) : (X_t)^n : + \ldots + \kappa_2(t) : (X_t)^2 : + \kappa_1(t) X_t + \kappa_0(t)\right] dt + dW_t \tag{180}$$

for $t \in [0,\infty)$ with periodic boundary conditions on $(0,2\pi)^2$ where $n \in \mathbb{N}$ is an arbitrary natural number, where $\kappa_0, \kappa_1, \dots, \kappa_n \in C([0,\infty),\mathbb{R})$ are arbitrary continuous functions, where $(W_t)_{t\geq 0}$ is a cylindrical I-Wiener process and where $:(X_t)^2:$, ..., $:(X_t)^n:$ are suitable renormalizations of $(X_t)^2$, ..., $(X_t)^n$ for $t \in [0,\infty)$. The precise result is formulated in the following theorem.

Theorem 31 (Polynomial nonlinearities in two space dimensions)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $n \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $\kappa_0, \kappa_1, \dots, \kappa_n \in C([t_0,\infty),\mathbb{R})$, $\eta \in (-\frac{2}{n}, 0)$, *let* $V = :(V)^1:, :(V)^2:, \dots, :(V)^n: : [t_0,\infty) \times \Omega \to \cap_{r\in(-\infty,0)} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})$ *be stochastic processes with continuous sample paths given by Propositions 14 and 15 and let* $\xi\colon \Omega \to \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})$ *be a random variable. Then there exists a unique random variable* $\tau\colon \Omega \to (t_0,\infty]$ *and a unique stochastic process* $X\colon [t_0,\infty)\times\Omega \to \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^2,\mathbb{R}) \cup \{\infty\}$ *such that for every* $\omega \in \Omega$ *it holds that* $X_t(\omega) = \infty$ *for all* $t \in [\tau(\omega),\infty)$, *that*

$$(X_s(\omega))_{s\in[t_0,\tau(\omega))} \in C\big([t_0,\tau(\omega)), \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big), \tag{181}$$

$$(X_s(\omega) - V_s(\omega))_{s\in(t_0,\infty)} \in C\big((t_0,\infty), \cap_{\nu\in(0,2)} [\mathcal{C}^\nu_{\mathcal{P}}([0,2\pi]^2,\mathbb{R}) \cup \{\infty\}]\big), \tag{182}$$

$$\sup_{s\in(t_0,t]} (s-t_0)^{\frac{(r-\eta)}{2}} \|X_s(\omega) - V_s(\omega)\|_{\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})} < \infty \tag{183}$$

for all $r \in [\eta, 2)$ *and all* $t \in (t_0, \tau(\omega))$ *and that*

$$\begin{aligned} X_t(\omega) = e^{\mathcal{A}_2(t-t_0)}\,\xi(\omega) + V_t(\omega) - e^{\mathcal{A}_2(t-t_0)}\,V_{t_0}(\omega) + \int_{t_0}^t e^{\mathcal{A}_2(t-s)} \Bigg(\kappa_0(t) + (\kappa_1(t)+1)\,X_t(\omega) \\ + \sum_{w=2}^n \kappa_w(t) \Bigg[(X_t(\omega) - V_t(\omega))^w + \sum_{k=0}^{w-1} \binom{w}{k} (X_t(\omega) - V_t(\omega))^k \Big(:(V_t)^{(w-k)}: \Big)(\omega) \Bigg] \Bigg)\, ds \end{aligned} \tag{184}$$

for all $t \in [t_0, \tau(\omega))$*. In that sense, the stochastic process* X *is a local mild solution of the SPDE* (180).

Let us briefly compare Proposition 4.4 in Da Prato & Debussche [3] with Theorem 31 above. In the setting of Theorem 31 we note that

$$\begin{aligned} &\int_{t_0}^t \|X_s(\omega) - V_s(\omega)\|^p_{\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})}\, ds \\ &\leq \left[\sup_{s\in(t_0,t]} (s-t_0)^{\frac{p(r-\eta)}{2}} \|X_s(\omega) - V_s(\omega)\|^p_{\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})} \right] \int_{t_0}^t (s-t_0)^{\frac{p(\eta-r)}{2}}\, ds \\ &= \left[\sup_{s\in(t_0,t]} (s-t_0)^{\frac{p(r-\eta)}{2}} \|X_s(\omega) - V_s(\omega)\|^p_{\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})} \right] \frac{(t-t_0)^{\left(1+\frac{p(\eta-r)}{2}\right)}}{\left(1+\frac{p(\eta-r)}{2}\right)} < \infty \end{aligned} \tag{185}$$

and hence

$$(X_s(\omega) - V_s(\omega))_{s\in[t_0,t]} \in C\big([t_0,t]; \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big) \cap L^p\big([t_0,t]; \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big) \tag{186}$$

for all $t \in (t_0, \tau(\omega))$, $\omega \in \Omega$, $r \in [\eta, \frac{2}{p} + \eta)$ and all $p \in (0,\infty)$. Equation (186) implies the regularity statement in Proposition 4.4 in Da Prato & Debussche [3] and this demonstrates that Theorem 31 above implies Proposition 4.4 in Da Prato & Debussche [3].

Proof of Theorem 31. We show Theorem 31 through an application of Corollary 30. For this application define $(U, \|\cdot\|_U) := \big(\mathcal{C}^0_{\mathcal{P}}([0,2\pi]^2,\mathbb{R}), \|\cdot\|_{\mathcal{C}^0_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})}\big)$ and $\big(U_r, \|\cdot\|_{U_r}\big) := (D((-\mathcal{A}_2)^r), \|(-\mathcal{A}_2)^r(\cdot)\|_U)$ for all $r \in \mathbb{R}$. Moreover, define $r_0 := \frac{\eta}{2} \in (-\frac{1}{n}, 0)$ and let $\varepsilon \in (0, \min(\frac{1}{2}, \frac{1}{n} + r_0))$ be a real number. Observe that this ensures that $n\,(\varepsilon - r_0) < 1$. Next define $\alpha := -\varepsilon$, $\beta := \varepsilon$, $\gamma := \varepsilon$, $\delta := n-1$ and let $F^\omega \colon [t_0,\infty) \times U_{\max(\beta,\gamma)} \to U_\alpha$, $\omega \in \Omega$, be functions defined through

$$F^\omega(t,y) = \kappa_0(t) + (\kappa_1(t)+1)\,(y + V_t(\omega)) + \sum_{w=2}^{n} \kappa_w(t) \left[y^w + \sum_{k=0}^{w-1} \binom{w}{k} y^k \Big(:(V_t)^{(w-k)}: \Big)(\omega) \right] \tag{187}$$

for all $y \in U_{\max(\beta,\gamma)}$, $t \in [t_0,\infty)$, $\omega \in \Omega$. Then note for every $\omega \in \Omega$ that $F^\omega \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$; see (119) for the definition of $\mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$. Next observe that

$$\big[\max(\beta,\gamma), 1+\alpha\big) = \big[\varepsilon, 1-\varepsilon\big) \neq \emptyset \tag{188}$$

and

$$\begin{aligned} \gamma - \min(\alpha, r_0) + (\beta - r_0)\delta &= \varepsilon - \min(-\varepsilon, r_0) + (\varepsilon - r_0)\,(n-1) \\ &= \varepsilon - r_0 + (\varepsilon - r_0)\,(n-1) = n\,(\varepsilon - r_0) < 1. \end{aligned} \tag{189}$$

We can thus apply Corollary 30 to obtain the existence of a unique lower semicontinuous function $\rho \colon \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,T]) \times U_{r_0} \to (t_0,\infty]$ and to obtain the existence of a unique function $y \colon \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \to \cup_{s\in(t_0,\infty]} C([t_0,s),U_{r_0})$ which satisfy $y_{G,v} \in C([t_0,\rho_{G,v}),U_{r_0})$, $y_{G,v}|_{(t_0,\rho_{G,v})} \in C((t_0,\rho_{G,v}),U_{r_1})$ and

$$\sup_{s\in(t_0,t]} (s-t_0)^{(r_1-r_0)}\,\|y_{G,v}(s)\|_{U_{r_1}} < \infty = \lim_{s\nearrow\rho_{G,v}} \big[\rho_{G,V} + \|y_{G,v}(s)\|_{U_\varepsilon}\big] \tag{190}$$

and

$$y_{G,v}(t) = e^{\mathcal{A}_2(t-t_0)}\,v + \int_{t_0}^{t} e^{\mathcal{A}_2(t-s)}\,G(s, y_{G,v}(s))\,ds \tag{191}$$

for all $t \in (t_0,\rho_{G,v})$, $v \in U_{r_0}$, $r_1 \in \big[\frac{\eta}{2}, 1-\varepsilon\big)$, $G \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,T])$ and all $T \in (0,\infty)$. Next we define functions $\tau \colon \Omega \to (t_0,\infty]$ and $X \colon [t_0,\infty) \times \Omega \to U_{r_0} \cup \{\infty\}$ through $\tau(\omega) := \rho_{F^\omega,\,\xi(\omega)-V_{t_0}(\omega)}$ for all $\omega \in \Omega$ and through

$$X_t(\omega) := \begin{cases} y_{F^\omega,\,\xi(\omega)-V_{t_0}(\omega)}(t) + V_t(\omega) & : t < \tau(\omega) \\ \infty & : t \geq \tau(\omega) \end{cases} \tag{192}$$

for all $t \in [t_0,\infty)$ and all $\omega \in \Omega$. This definition together with (191) ensures that

$$X_t(\omega) - V_t(\omega) = e^{\mathcal{A}_2(t-t_0)}\big(\xi(\omega) - V_{t_0}(\omega)\big) + \int_{t_0}^{t} e^{\mathcal{A}_2(t-s)}\,F^\omega\big(s, X_t(\omega) - V_t(\omega)\big)\,ds \tag{193}$$

for all $t \in (t_0,\tau(\omega)$ and all $\omega \in \Omega$. Combining this with (187) proves that X fulfills (184). In the next step we note that

$$\begin{aligned} &\mathcal{B}\Big(C\big([t_0,\infty), \big[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big]^{\times n}\big)\Big) \\ &= \sigma_{C([t_0,\infty),[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})]^{\times n})}\Big(C\big([t_0,\infty), \big[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big]^{\times n}\big) \\ &\qquad \ni f \mapsto f(t) \in \big[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big]^{\times n} : t \in [t_0,\infty)\Big). \end{aligned} \tag{194}$$

This implies that the mapping

$$\Omega \ni \omega \mapsto \big(V_t(\omega), (:(V_t)^2:)(\omega), \ldots, (:(V_t)^n:)(\omega)\big)_{t\in[t_0,\infty)} \in C\big([t_0,\infty), \big[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big]^{\times n}\big) \tag{195}$$

is $\mathcal{F}/\mathcal{B}\big(C([t_0,\infty), \big[\mathcal{C}^{-\varepsilon/2}_{\mathcal{P}}([0,2\pi]^2,\mathbb{R})\big]^{\times n})\big)$-measurable. This ensures that the mapping

$$\Omega \ni \omega \mapsto \big(F^\omega, \xi(\omega)\big) \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \tag{196}$$

is $\mathcal{F}/\mathcal{B}\big(\mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0}\big)$-measurable. Combining this with Corollary 30 proves that τ is a random variable and that X is a stochastic process (see (Measurability property) in Corollary 30 for details). Since $\varepsilon \in (0, \min(\frac{1}{2}, \frac{1}{n} + r_0))$ was arbitrary, the proof of Theorem 31 is completed. □

3.3 SPDEs with space-time white noise and quadratic nonlinearities in three space dimensions

The aim of this subsection is to prove local existence and uniqueness of mild solutions of SPDEs in three space dimensions with quadratic nonlinearities of the form

$$dX_t = \big[\triangle X_t + \kappa_2(t) : (X_t)^2 : + \kappa_1(t) X_t + \kappa_0(t)\big]\, dt + dW_t \tag{197}$$

for $t \in [0,\infty)$ with periodic boundary conditions on $(0,2\pi)^3$ where $\kappa_0, \kappa_1, \kappa_2 \in C([0,\infty),\mathbb{R})$ are arbitrary continuous functions, where $(W_t)_{t\geq 0}$ is a cylindrical I-Wiener process and where $:(X_t)^2:$ is a suitable renormalization of $(X_t)^2$ for $t \in [0,\infty)$. The precise result is formulated in the following theorem.

Theorem 32 (Quadratic nonlinearities in three space dimensions)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $t_0 \in \mathbb{R}$, $\kappa_0, \kappa_1, \kappa_2 \in C([t_0,\infty),\mathbb{R})$, $\eta \in (-1, -\frac{1}{2})$, let $V\colon [t_0,\infty) \times \Omega \to \cap_{r \in (-\infty,-1/2)} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})$ and $:(V)^2:\colon [t_0,\infty) \times \Omega \to \cap_{r\in(-\infty,-1)} \mathcal{C}^r_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})$ be stochastic processes with continuous sample paths given by Propositions 14 and 15 and let $\xi\colon \Omega \to \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})$ be a random variable. Then there exists a unique random variable $\tau\colon \Omega \to (t_0,\infty]$ and a unique stochastic process $X\colon [t_0,\infty)\times\Omega \to \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^3,\mathbb{R}) \cup \{\infty\}$ such that for every $\omega \in \Omega$ it holds that $X_t(\omega) = \infty$ for all $t \in [\tau(\omega),\infty)$, that*

$$(X_s(\omega))_{s\in[t_0,\tau(\omega))} \in C([t_0,\tau(\omega)), \mathcal{C}^\eta_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})), \tag{198}$$

$$(X_s(\omega) - V_s(\omega))_{s\in(t_0,\infty)} \in C\big((t_0,\infty), \cap_{\nu\in(\frac{1}{2},1)} [\mathcal{C}^\nu_{\mathcal{P}}([0,2\pi]^3,\mathbb{R}) \cup \{\infty\}]\big), \tag{199}$$

$$\sup_{s\in(t_0,t]} (s-t_0)^{\frac{(r-\eta)}{2}} \|X_s(\omega) - V_s(\omega)\|_{\mathcal{C}^r_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})} < \infty \tag{200}$$

for all $r \in [\eta,1)$ and all $t \in (t_0,\tau(\omega))$ and that

$$\begin{aligned} X_t(\omega) = e^{\mathcal{A}_3(t-t_0)}\, \xi(\omega) + V_t(\omega) - e^{\mathcal{A}_3(t-t_0)}\, V_0(\omega) + \int_{t_0}^t e^{\mathcal{A}_3(t-s)} \Big[\kappa_2(t)\Big((X_t(\omega) - V_t(\omega))^2 \\ + 2\,(X_t(\omega) - V_t(\omega))\, V_t(\omega) + \big(:(V_t)^2:\big)(\omega)\Big) + (\kappa_1(t)+1)\, X_t(\omega) + \kappa_0(t)\Big] ds \end{aligned} \tag{201}$$

for all $t \in [t_0,\tau(\omega))$. In that sense, the stochastic process X is a local mild solution of the SPDE (197).

Proof of Theorem 32. We show Theorem 32 through an application of Corollary 30. For this application define $(U, \|\cdot\|_U) := \big(\mathcal{C}^0_{\mathcal{P}}([0,2\pi]^3,\mathbb{R}), \|\cdot\|_{\mathcal{C}^0_{\mathcal{P}}([0,2\pi]^3,\mathbb{R})}\big)$ and $\big(U_r, \|\cdot\|_{U_r}\big) := (D((-\mathcal{A}_3)^r), \|(-\mathcal{A}_3)^r(\cdot)\|_U)$ for all $r \in \mathbb{R}$. Moreover, define $r_0 := \frac{\eta}{2} \in (-\frac{1}{2}, -\frac{1}{4})$ and let $\varepsilon \in (0, \frac{1}{4} + \frac{r_0}{2})$ be a real number. Observe that this ensures that $2\varepsilon - r_0 < \frac{1}{2}$ and that $\varepsilon < \frac{1}{8}$. Next define $\alpha := -\frac{1}{2} - \varepsilon$, $\beta := -\frac{1}{4} - \frac{\varepsilon}{2}$, $\gamma := \frac{1}{4} + \varepsilon$ and $\delta := 1$ and let $F^\omega\colon [t_0,\infty) \times U_{\max(\beta,\gamma)} \to U_{\alpha_i}$, $\omega \in \Omega$, be functions defined through

$$F^\omega(t,y) := \kappa_2(t)\,\big(y^2 + 2\,V_t(\omega)\,y + \big(:(V_t)^2:\big)(\omega)\big) + (\kappa_1(t)+1)\,(y + V_t(\omega)) + \kappa_0(t) \tag{202}$$

for all $y \in U_{\max(\beta,\gamma)}$, $t \in [t_0,\infty)$, $\omega \in \Omega$. Then note for every $\omega \in \Omega$ that $F^\omega \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$; see (119) for the definition of $\mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$. Next observe that

$$\big[\max(\beta,\gamma), 1+\alpha\big) = \big[\tfrac{1}{4}+\varepsilon, \tfrac{1}{2}-\varepsilon\big) \neq \emptyset \tag{203}$$

and

$$\begin{aligned} &\gamma - \min(\alpha, r_0) + (\beta - r_0)\delta = \gamma + \beta - \alpha - r_0 \\ &= \tfrac{\varepsilon}{2} + \tfrac{1}{2} + \varepsilon - r_0 \leq \tfrac{1}{2} + 2\varepsilon - r_0 < 1. \end{aligned} \tag{204}$$

We can thus apply Theorem 29 to obtain the existence of a unique lower semicontinuous function $\rho\colon \mathcal{C}^1_{\alpha,\beta,\gamma}([t_0,\infty)) \times U_{r_0} \to (t_0,\infty]$ and to obtain the existence of a unique function $y\colon \mathcal{C}^1_{\alpha,\beta,\gamma}([t_0,\infty)) \times U_{r_0} \to \cup_{s\in(t_0,\infty]} C([t_0,s), U_{r_0})$ which satisfy $y_{G,v} \in C([t_0,\rho_{G,v}), U_{r_0})$, $y_{G,v}|_{(t_0,\rho_{G,v})} \in C((t_0,\rho_{G,v}), U_{r_1})$ and

$$\sup_{s\in(t_0,t]} (s-t_0)^{(r_1-r_0)} \|y_{G,v}(s)\|_{U_{r_1}} < \infty = \lim_{s\nearrow\rho_{G,v}} \Big[\rho_{G,v} + \|y_{G,v}(s)\|_{U_{\frac{1}{4}+\varepsilon}}\Big] \tag{205}$$

and

$$y_{G,v}(t) = e^{\mathcal{A}_3(t-t_0)}\, v + \int_{t_0}^t e^{\mathcal{A}_3(t-s)}\, G(s, y_{G,v}(s))\, ds \tag{206}$$

for all $t \in (t_0, \rho_{G,v})$, $v \in U_{r_0}$, $r_1 \in \left[\frac{\eta}{2}, \frac{1}{2} - \varepsilon\right)$ and all $G \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty))$. Next we define functions $\tau\colon \Omega \to (t_0, \infty]$ and $X\colon [t_0,\infty) \times \Omega \to U_{r_0} \cup \{\infty\}$ through $\tau(\omega) := \rho_{F^\omega,\, \xi(\omega) - V_{t_0}(\omega)}$ for all $\omega \in \Omega$ and through

$$X_t(\omega) := \begin{cases} y_{F^\omega,\, \xi(\omega) - V_{t_0}(\omega)}(t) + V_t(\omega) & : t < \tau(\omega) \\ \infty & : t \geq \tau(\omega) \end{cases} \tag{207}$$

for all $t \in [t_0, \infty)$ and all $\omega \in \Omega$. This definition together with (206) ensures that

$$X_t(\omega) - V_t(\omega) = e^{\mathcal{A}_3(t-t_0)}\big(\xi(\omega) - V_{t_0}(\omega)\big) + \int_{t_0}^t e^{\mathcal{A}_3(t-s)}\, F^\omega(s, X_t(\omega) - V_t(\omega))\, ds \tag{208}$$

for all $t \in (t_0, \tau(\omega))$ and all $\omega \in \Omega$. Combining this with (202) proves that X fulfills (201). In the next step we note that

$$\begin{aligned} &\mathcal{B}\Big(C\big([t_0,\infty), \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R})\big)\Big) \\ &= \sigma_{C([t_0,\infty), \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}))}\Big(\\ &C\big([t_0,\infty), \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R})\big) \\ &\ni f \mapsto f(t) \in \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R})\colon t \in [t_0,\infty)\Big). \end{aligned} \tag{209}$$

This implies that the mapping

$$\Omega \ni \omega \mapsto \big(V_t(\omega), (:(V_t)^2:)(\omega)\big)_{t \in [t_0,\infty)} \in C\big([t_0,\infty), \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R})\big) \tag{210}$$

is $\mathcal{F}/\mathcal{B}\big(C([t_0,\infty), \mathcal{C}_{\mathcal{P}}^{-(1+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}) \times \mathcal{C}_{\mathcal{P}}^{-(2+\varepsilon)/2}([0,2\pi]^3,\mathbb{R}))\big)$-measurable and this shows that the mapping

$$\Omega \ni \omega \mapsto \big(F^\omega, \xi(\omega) - V_{t_0}(\omega)\big) \in \mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0} \tag{211}$$

is $\mathcal{F}/\mathcal{B}\big(\mathcal{C}^1_{\alpha,\beta,\gamma,\delta}([t_0,\infty)) \times U_{r_0}\big)$-measurable. Combining this with Corollary 30 proves that τ is a random variable and that X is a stochastic process (see (Measurability property) in Corollary 30 for details). Since $\varepsilon \in (0, \frac{1}{4} + \frac{r_0}{2})$ was arbitrary, the proof of Theorem 32 is completed. □

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