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Abstract

The strong numerical approximation of semilinear stochastic partial differential equations (SPDEs) driven by infinite dimensional Wiener processes is investigated. There are a number of results in the literature that show that Euler-type approximation methods converge strongly, under suitable assumptions, to the exact solutions of such SPDEs with strong order $1/2$ or at least with strong order $1/2 - \varepsilon$ where $\varepsilon > 0$ is arbitrarily small. Recent results extend these results and show that Milstein-type approximation methods converge, under suitable assumptions, to the exact solutions of such SPDEs with strong order $1 - \varepsilon$. It has also been shown that splitting-up approximation methods converge, under suitable assumptions, with strong order 1 to the exact solutions of such SPDEs. In this article an exponential Wagner-Platen type numerical approximation method for such SPDEs is proposed and shown to converge, under suitable assumptions, with strong order $3/2 - \varepsilon$ to the exact solutions of such SPDEs.

1 Introduction

We investigate the strong numerical approximation of semilinear stochastic partial differential equations (SPDEs) driven by infinite dimensional Wiener processes. To illustrate the results, we concentrate in this introductory section on the following simple example SPDE. Let $H = L^2((0, 1); \mathbb{R})$ be the \mathbb{R} -Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0, 1)$ to \mathbb{R} , let $A: D(A) \subset H \rightarrow H$ be the Laplacian with Dirichlet boundary conditions, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a stochastic basis (see, e.g., Appendix E in Prévôt & Röckner [36]), let $r \in (1, \infty)$, $\xi \in D(A)$, let $W: [0, T] \times \Omega \rightarrow H$ be a standard $(\mathcal{F}_t)_{t \in [0, T]}$ -Wiener process with the covariance operator $Q := A^{-1} \in L(H)$ and let $X: [0, T] \times \Omega \rightarrow H$ be an adapted stochastic process with continuous sample paths satisfying the stochastic heat equation with linear multiplicative noise

$$dX_t(x) = \left[\frac{\partial^2}{\partial x^2} X_t(x) \right] dt + X_t(x) dW_t(x), \quad X_t(0) = X_t(1) = 0, \quad X_0(x) = \xi(x) \quad (1)$$

for $(t, x) \in [0, T] \times H$. The convergence result in Theorem 1 below can also be applied to a much more general class of SPDEs with more general covariance operators Q (see Section 2 for details) but for simplicity of presentation we restrict ourselves to the SPDE (1) in this introductory section. Our goal is then to compute a strong numerical approximation of the SPDE (1).

There are a good number of results in the literature that show that *Euler-type approximation methods for SPDEs* (such as the linear-implicit Euler method or the exponential Euler method; see, e.g., Section 3.3.1 in Da Prato et al. [6] for an overview on different Euler type approximations methods for SPDEs) converge to the solution process X of the SPDE (1) with strong order $1/2$ or at least with strong order $1/2 - \varepsilon$ where $\varepsilon \in (0, 1/2)$ is arbitrarily small (see, e.g., [5, 13, 14, 24, 29]). Further references on numerical approximations for SPDEs can also be found in the overview articles Gyöngy [11] and Jentzen & Kloeden [18].

Recent results extend the above mentioned results for Euler type approximation methods and prove that *Milstein-type approximation methods for SPDEs* converge with strong order $1 - \varepsilon$ or 1 to the solution process X of the SPDE (1) (see, e.g., [1, 2, 20, 25, 26, 30, 37]). An overview on Milstein-type approximation methods for SPDEs can also be found in Section 3.3.2 in Da Prato et al. [6]. Beside

Milstein-type approximation methods, it has also been established in the literature that *splitting-up approximation methods for SPDEs* converge with strong order 1 to the solution process X of the SPDE (1) (see, e.g., Gyöngy & Krylov [12]). Further references for splitting-up methods can also be found in the overview article Gyöngy [11]. Beside Milstein type methods and splitting-up methods, the choice of *suitable non-uniform time discretizations* is another approach for obtaining higher order strong convergence rates for SPDEs; see [32, 33, 34, 35] for details on such methods and their optimality. Moreover, in the case of SPDEs with additive noise, higher order strong convergence rates for SPDEs can also be obtained by using suitable *linear functionals of the noise process in the numerical scheme*; see, e.g., [4, 3, 8, 16, 17, 19, 21, 27, 28, 31, 38] and, e.g., (135)–(141) in Da Prato et al. [6] for an overview. For instance, in [19] it is proved in the case of additive noise, that the accelerated exponential Euler method converges, under suitable assumptions, with strong order $1 - \varepsilon$ to the exact solution of the SPDE under consideration. Furthermore, higher order strong temporal convergence rates of stochastic Taylor schemes for spectral Galerkin discretizations of SPDEs driven by one dimensional Brownian motions are established in Grecksch & Kloeden [10] and Kloeden & Shott [23].

Here we introduce an *exponential Wagner-Platen type numerical approximation method for SPDEs* (see (3) below) and in Theorem 1 below we prove that this method converges with strong order $3/2 - \varepsilon$ to the solution process X of the SPDE (1). Further details can be found in Section 3 below.

In Section 2 the abstract general setting used in this article is described. Section 3 introduces the above mentioned exponential Wagner-Platen method. In addition, in Section 3 we establish in Proposition 1 an a priori moment bound for the exponential Wagner-Platen method and present a convergence analysis theorem, Theorem 1, of the exponential Wagner-Platen method. Furthermore, Lemmas 1–3 in Section 3 illustrate how the exponential Wagner-Platen method in Section 3 can be simulated. The proofs of Proposition 1, Theorem 1, Lemma 2 and Lemma 3 are postponed to Section 4.

2 Setting

Throughout this article suppose that the following setting is fulfilled. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$, let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable \mathbb{R} -Hilbert spaces, let $Q \in L(U)$ be a trace class operator and let $(U_0, \langle \cdot, \cdot \rangle_{U_0}, \|\cdot\|_{U_0})$ be the \mathbb{R} -Hilbert space given by $U_0 = Q^{1/2}(U)$ and $\langle v, w \rangle_{U_0} = \langle Q^{-1/2}v, Q^{-1/2}w \rangle_U$ for all $v, w \in U_0$.

Assumption 1 (Linear operator A). *Let \mathcal{I} be a finite or countable set, let $(\lambda_i)_{i \in \mathcal{I}} \subset (0, \infty)$ be a family of real numbers with $\inf_{i \in \mathcal{I}} \lambda_i \in (0, \infty)$, let $(e_i)_{i \in \mathcal{I}}$ be an orthonormal basis of H and let $A: D(A) \subset H \rightarrow H$ be a linear operator with $D(A) = \{w \in H: \sum_{i \in \mathcal{I}} |\lambda_i|^2 |\langle e_i, w \rangle_H|^2 < \infty\}$ and $Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, v \rangle_H e_i$ for all $v \in D(A)$.*

In the following we denote by $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, the \mathbb{R} -Hilbert spaces given by $H_r = D((-A)^r)$ and $\langle v, w \rangle_{H_r} = \langle (-A)^r v, (-A)^r w \rangle_H$ for all $v, w \in H_r$ and all $r \in \mathbb{R}$.

Assumption 2 (Drift term F). *Let $\gamma \in [1, \frac{3}{2}]$, $\alpha \in (\gamma - 1, \gamma]$ and let $F \in C^2(H, H)$ be a globally Lipschitz continuous mapping with $F(H_\alpha) \subseteq H_\alpha$ and $\sup_{v \in H_\alpha} \frac{\|F(v)\|_{H_\alpha}}{1 + \|v\|_{H_\alpha}} < \infty$.*

Assumption 3 (Diffusion term B). *Let $\beta \in (\gamma - \frac{1}{2}, \gamma]$, $\delta \in (\gamma - 1, \beta]$ and let $B \in C^2(H, HS(U_0, H))$ be a globally Lipschitz continuous mapping with $B(H_\beta) \subseteq HS(U_0, H_\beta)$, $B'(v) \in L(H_\delta, HS(U_0, H_\delta))$ for all $v \in H_\gamma$ and $\sup_{v \in H_\beta} \frac{\|B(v)\|_{HS(U_0, H_\beta)}}{1 + \|v\|_{H_\beta}} + \sup_{v \in H_\gamma} \|B'(v)\|_{L(H_\delta, HS(U_0, H_\delta))} < \infty$.*

Assumption 4 (Initial value ξ). *Let $\xi: \Omega \rightarrow H_\gamma$ be an $\mathcal{F}/\mathcal{B}(H_\gamma)$ -measurable mapping.*

It is well known (see, e.g., Theorem 7.4 (i) in Da Prato & Zabczyk [7]) that the above assumptions ensure the existence of an up to modifications unique predictable stochastic process $X: [0, T] \times \Omega \rightarrow H$ satisfying $\int_0^T \|X_s\|_H^2 ds < \infty$ \mathbb{P} -a.s. and

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)}F(X_s) ds + \int_0^t e^{A(t-s)}B(X_s) dW_s \quad (2)$$

\mathbb{P} -a.s. for all $t \in [0, T]$.

3 An exponential Wagner-Platen type scheme for SPDEs

This section introduces and analyzes an exponential Wagner-Platen type approximation scheme for the SPDE (2). To formulate this scheme, let \mathcal{J} be a finite or countable set and let $(g_j)_{j \in \mathcal{J}} \subset U_0$ be an arbitrary orthonormal basis of U_0 . Then let $Y_m^M : \Omega \rightarrow H_\gamma$, $m \in \{0, 1, \dots, M\}$, $M \in \mathbb{N}$, be $\mathcal{F}/\mathcal{B}(H_\gamma)$ -measurable mappings satisfying $Y_0^M = \xi$ and

$$\begin{aligned}
Y_{m+1}^M &= e^{A \frac{T}{2M}} \left\{ e^{A \frac{T}{2M}} Y_m^M + \frac{T}{M} F(Y_m^M) + \frac{T^2}{2M^2} F'(Y_m^M) [AY_m^M + F(Y_m^M)] \right. \\
&\quad + F'(Y_m^M) \left(\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s B(Y_m^M) dW_u ds \right) + \sum_{j \in \mathcal{J}} \frac{T^2}{4M^2} F''(Y_m^M) (B(Y_m^M)g_j, B(Y_m^M)g_j) \\
&\quad + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B(Y_m^M) dW_s + A \left[\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s B(Y_m^M) dW_u ds - \frac{T}{2M} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B(Y_m^M) dW_s \right] \\
&\quad + \frac{T}{M} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^M) (AY_m^M + F(Y_m^M)) dW_s + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^M) \left(\int_{\frac{mT}{M}}^s B(Y_m^M) dW_u \right) dW_s \quad (3) \\
&\quad - \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s B'(Y_m^M) (AY_m^M + F(Y_m^M)) dW_u ds \\
&\quad + \frac{1}{2} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B''(Y_m^M) \left(\int_{\frac{mT}{M}}^s B(Y_m^M) dW_u, \int_{\frac{mT}{M}}^s B(Y_m^M) dW_u \right) dW_s \\
&\quad \left. + \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^M) \left[\int_{\frac{mT}{M}}^s B'(Y_m^M) \left(\int_{\frac{mT}{M}}^u B(Y_m^M) dW_v \right) dW_u \right] dW_s \right\}
\end{aligned}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M-1\}$ and all $M \in \mathbb{N}$. The setting in Section 2 ensures that the random variables $Y_m^M : \Omega \rightarrow H_\gamma$, $m \in \{0, 1, \dots, M\}$, $M \in \mathbb{N}$, do indeed exist. In addition, observe that the identity

$$\begin{aligned}
&\mathbb{E} \left[F''(Y_k^M) \left(\int_{\frac{kT}{M}}^t B(Y_k^M) dW_u, \int_{\frac{kT}{M}}^t B(Y_k^M) dW_u \right) \middle| \mathcal{F}_{\frac{kT}{M}} \right] \\
&= \sum_{j \in \mathcal{J}} \int_{\frac{kT}{M}}^t F''(Y_k^M) (B(Y_k^M)g_j, B(Y_k^M)g_j) du
\end{aligned} \quad (4)$$

\mathbb{P} -a.s. for all $t \in [\frac{kT}{M}, \frac{(k+1)T}{M}]$, all $k \in \{0, 1, \dots, M-1\}$ and all $M \in \mathbb{N}$ illustrates that (3) does not depend on the special choice of the orthonormal basis $(g_j)_{j \in \mathcal{J}}$ of U_0 . The following proposition establishes an a priori moment bound for the numerical approximations Y_m^M , $m \in \{0, 1, \dots, M\}$, $M \in \mathbb{N}$.

Proposition 1. *There exists a universal non-decreasing function $C : [0, \infty) \rightarrow [0, \infty)$ such that if the setting in Section 2 is fulfilled, if $p \in [2, \infty)$ and if*

$$\begin{aligned}
K &:= \|A^{-1}\|_{L(H)} + \frac{1}{(1 - \max(\gamma - \alpha, 2(\gamma - \beta), 2(\gamma - \delta - \frac{1}{2}), \gamma - \frac{1}{2}))} + T + p + \sup_{v \in H_\alpha} \frac{\|F(v)\|_{H_\alpha}}{1 + \|v\|_{H_\alpha}} \\
&\quad + \sup_{v \in H} \|F'(v)\|_{L(H)} + \sup_{v \in H_\beta} \frac{\|B(v)\|_{HS(U_0, H_\beta)}}{1 + \|v\|_{H_\beta}} + \sup_{v \in H} \|B'(v)\|_{L(H, HS(U_0, H))} + \sup_{v \in H} \|B''(v)\|_{L(H_\delta, HS(U_0, H_\delta))} \\
&\quad + \sup_{v \in H} \frac{(\|F''(v)\|_{L^{(2)}(H, H)} + \|B''(v)\|_{L^{(2)}(H, HS(U_0, H))}) \|B(v)\|_{HS(U_0, H)}^2}{1 + \|v\|_H} < \infty, \quad (5)
\end{aligned}$$

$$\text{then} \quad \sup_{M \in \mathbb{N}} \sup_{m \in \{0, 1, \dots, M\}} \|Y_m^M\|_{L^p(\Omega; H_\gamma)} \leq C(K) (1 + \|\xi\|_{L^p(\Omega; H_\gamma)}). \quad (6)$$

The proof of Proposition 1 is postponed to Subsection 4.1 below. The next theorem estimates the strong temporal approximation error of the numerical approximations Y_m^M , $m \in \{0, 1, \dots, M\}$, $M \in \mathbb{N}$.

Theorem 1. *There exists a universal non-decreasing function $C: [0, \infty) \rightarrow [0, \infty)$ such that if the setting in Section 2 is fulfilled and if*

$$\begin{aligned} K := & \sup_{m \in \{0, 1, \dots, M\}, M \in \mathbb{N}} \|Y_m^M\|_{L^6(\Omega; H_\gamma)} + \|A^{-1}\|_{L(H)} + T + \frac{1}{(\min(\alpha + 1, \beta + \frac{1}{2}, \delta + 1, \frac{3}{2}) - \gamma)} \\ & + \sup_{v \in H_\alpha} \frac{\|F(v)\|_{H_\alpha}}{1 + \|v\|_{H_\alpha}} + \sup_{v \in H_\beta} \frac{\|B(v)\|_{HS(U_0, H_\beta)}}{1 + \|v\|_{H_\beta}} + \sup_{v \in H_\gamma} \|B'(v)\|_{L(H_\delta, HS(U_0, H_\delta))} \\ & + \sum_{i=0}^2 \sup_{v, w \in H, v \neq w} \left[\frac{\|F^{(i)}(v) - F^{(i)}(w)\|_{L^{(i)}(H, H)} + \|B^{(i)}(v) - B^{(i)}(w)\|_{L^{(i)}(H, HS(U_0, H))}}{\|v - w\|_H} \right] < \infty, \end{aligned} \quad (7)$$

$$\text{then it holds for all } M \in \mathbb{N} \text{ that } \sup_{m \in \{0, 1, \dots, M\}} \|X_{mh} - Y_m^M\|_{L^2(\Omega; H)} \leq C(K) M^{-\gamma}. \quad (8)$$

The proof of Theorem 1 is given in Subsection 4.2 below. The following lemmas (Lemma 1, Lemma 2 and Lemma 3) show under suitable assumptions how the scheme (3) can be simulated. Lemma 1 is a slightly more general statement than display (83) in [20] (see also Remark 1 and Subsection 5.7 in [20]).

Lemma 1 (Commutative noise of the first kind for SPDEs). *Assume the setting in Section 2 and assume for all $v \in H$ that the bilinear Hilbert Schmidt operator $B'(v)(B(v)) \in H^{(2)}(U_0, H)$ is symmetric, i.e., assume that*

$$\left(B'(v)(B(v)u_1) \right)(u_2) = \left(B'(v)(B(v)u_2) \right)(u_1) \quad (9)$$

for all $v \in H$ and all $u_1, u_2 \in U_0$. If $U = U_0$ (which is equivalent to $\dim(U) < \infty$), then

$$\begin{aligned} & \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B(Z) dW_u \right) dW_s \\ & = \frac{1}{2} B'(Z) \left(B(Z) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) - \frac{(t - t_0)}{2} \sum_{i \in \mathcal{J}} B'(Z) \left(B(Z) g_i \right) g_i \end{aligned} \quad (10)$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$.

The proof of Lemma 1 is entirely analogous to the proof of (83) in [20] and therefore omitted. The next lemma treats the case of *commutative noise of second kind for SPDEs*. Assumption (11) is the abstract coordinate free analogue of (4.13) in Section 10.4 in Kloeden & Platen [22].

Lemma 2 (Commutative noise of the second kind for SPDEs). *Assume the setting in Section 2 and assume for all $v \in H$ that the trilinear Hilbert Schmidt operators $B'(v)(B'(v)(B(v))) \in H^{(3)}(U_0, H)$ and $B''(v)(B(v), B(v)) \in H^{(3)}(U_0, H)$ are symmetric, i.e., assume that*

$$\begin{aligned} & \left(B'(v) \left(B'(v)(B(v)u_1) \right) u_2 + B''(v)(B(v)u_1, B(v)u_2) \right) (u_3) \\ & = \left(B'(v) \left(B'(v)(B(v)u_{\pi(1)}) \right) u_{\pi(2)} + B''(v)(B(v)u_{\pi(1)}, B(v)u_{\pi(2)}) \right) (u_{\pi(3)}) \end{aligned} \quad (11)$$

for all $v \in H$, all $u_1, u_2, u_3 \in U_0$ and all $\pi \in S_3$. If $U = U_0$ (which is equivalent to $\dim(U) < \infty$), then

$$\begin{aligned} & \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) dW_v \right) dW_u \right) dW_s + \frac{1}{2} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) dW_u, \int_{t_0}^s B(Z) dW_u \right) dW_s \\ & = \frac{1}{6} B'(Z) \left(B'(Z) \left(B(Z) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \\ & + \frac{1}{6} B''(Z) \left(B(Z) (W_t - W_{t_0}), B(Z) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \\ & - \frac{(t - t_0)}{2} \sum_{i \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) (W_t - W_{t_0}) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) (W_t - W_{t_0}) \right] \end{aligned} \quad (12)$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$.

The proof of Lemma 2 is postponed to Subsection 4.3 below. Combining Lemma 1 and Lemma 2 shows that if $U = U_0$ and if (9) and (11) are fulfilled then the numerical scheme (3) satisfies

$$\begin{aligned}
Y_{m+1}^M &= e^{A\frac{T}{2M}} \left\{ e^{A\frac{T}{2M}} Y_m^M + \frac{T}{M} F(Y_m^M) + F'(Y_m^M) \frac{T^2}{2M^2} [AY_m^M + F(Y_m^M)] \right. \\
&\quad + F'(Y_m^M) \left(B(Y_m^M) \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} (W_s - W_{\frac{mT}{M}}) ds \right) + \sum_{j \in \mathcal{J}} \frac{T^2}{4M^2} F''(Y_m^M) (B(Y_m^M) g_j, B(Y_m^M) g_j) \\
&\quad + B(Y_m^M) \left(W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}} \right) + A \left[B(Y_m^M) \left(\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} (W_s - W_{\frac{mT}{M}}) ds - \frac{T}{2M} (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right) \right] \\
&\quad + \left(B'(Y_m^M) (AY_m^M + F(Y_m^M)) \right) \left(\frac{T}{M} (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) - \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} (W_s - W_{\frac{mT}{M}}) ds \right) \\
&\quad + \frac{1}{2} B'(Y_m^M) \left(B(Y_m^M) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \\
&\quad + \frac{1}{6} B''(Y_m^M) \left(B(Y_m^M) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}), B(Y_m^M) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \\
&\quad + \frac{1}{6} B'(Y_m^M) \left(B'(Y_m^M) \left(B(Y_m^M) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \\
&\quad - \frac{T}{2M} \sum_{j \in \mathcal{J}} B'(Y_m^M) (B(Y_m^M) g_j) g_j - \frac{T}{2M} \sum_{j \in \mathcal{J}} B''(Y_m^M) (B(Y_m^M) g_j, B(Y_m^M) g_j) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \\
&\quad \left. - \frac{T}{2M} \sum_{j \in \mathcal{J}} B'(Y_m^M) (B'(Y_m^M) (B(Y_m^M) g_j) g_j) (W_{\frac{(m+1)T}{M}} - W_{\frac{mT}{M}}) \right\} \tag{13}
\end{aligned}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M-1\}$ and all $M \in \mathbb{N}$. The next lemma, Lemma 3, illustrates for all $t_0, t \in [0, T]$ with $t_0 \leq t$ how the Gaussian distributed random variable $(W_t - W_{t_0}, \int_{t_0}^t (W_s - W_{t_0}) ds) \in U \times U$ can be simulated. Lemma 3 generalizes (4.2)–(4.3) in Section 10.4 in Kloeden & Platen [22] for finite dimensional SODEs to infinite dimensional Wiener processes. The proof of Lemma 3 is given in Subsection 4.4 below.

Lemma 3 (Covariance operator). *Assume the setting in Section 2. Then*

$$\left(\text{Cov} \left(\begin{array}{c} W_t - W_{t_0} \\ \int_{t_0}^t (W_s - W_{t_0}) ds \end{array} \right) \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (t - t_0) Qu_1 + \frac{1}{2} (t - t_0)^2 Qu_2 \\ \frac{1}{2} (t - t_0)^2 Qu_1 + \frac{1}{3} (t - t_0)^3 Qu_2 \end{pmatrix} \tag{14}$$

for all $u_1, u_2 \in U$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$.

4 Proofs

Throughout the rest of this article we use the mappings $[\cdot]_M, \lceil \cdot \rceil_M, \langle \cdot \rangle_M : [0, T] \rightarrow [0, T]$, $M \in \mathbb{N}$, given by

$$[t]_M := \max \left\{ s \in \left\{ 0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T \right\} : s \leq t \right\}, \quad \lceil t \rceil_M := \min \left\{ s \in \left\{ 0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T \right\} : s \geq t \right\} \tag{15}$$

and $\langle t \rangle_M := \frac{1}{2} ([t]_M + \lceil t \rceil_M)$ for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Moreover, let $\hat{X}^M : [0, T] \times \Omega \rightarrow H$, $M \in \mathbb{N}$, be optional measurable stochastic processes satisfying $\sup_{s \in [0, T]} \mathbb{E} [\| \hat{X}_s^M \|^6_H] < \infty$ for all $M \in \mathbb{N}$ and

$$\hat{X}_t^M = e^{A(t - \lceil t \rceil_M)} Y_{(M/T)\lceil t \rceil_M}^M + \int_{\lceil t \rceil_M}^t e^{A(t-s)} F(\hat{X}_s^M) ds + \int_{\lceil t \rceil_M}^t e^{A(t-s)} B(\hat{X}_s^M) dW_s \tag{16}$$

\mathbb{P} -a.s. for all $t \in [0, T]$ and all $M \in \mathbb{N}$ (see, e.g., Theorem 7.4 (i) in Da Prato & Zabczyk [7]). In addition, let $\Phi^M : [0, T] \times \Omega \rightarrow H$, $M \in \mathbb{N}$, and $\Psi^M : [0, T] \times \Omega \rightarrow HS(U_0, H)$, $M \in \mathbb{N}$, be optional measurable

stochastic processes satisfying

$$\begin{aligned}\Phi_t^M &= F(\hat{X}_{[t]_M^M}^M) + F'(\hat{X}_{[t]_M^M}^M) \int_{[t]_M^M}^t \left(A\hat{X}_{[t]_M^M}^M + F(\hat{X}_{[t]_M^M}^M) \right) du + F'(\hat{X}_{[t]_M^M}^M) \int_{[t]_M^M}^t B(\hat{X}_{[t]_M^M}^M) dW_u \\ &\quad + \frac{1}{2} \sum_{j \in \mathcal{J}} \int_{[t]_M^M}^t F''(\hat{X}_{[t]_M^M}^M) \left(B(\hat{X}_{[t]_M^M}^M) g_j, B(Y_{[t]_M^M}^M) g_j \right) du\end{aligned}\tag{17}$$

\mathbb{P} -a.s. and

$$\begin{aligned}\Psi_t^M &= B(\hat{X}_{[t]_M^M}^M) + B'(\hat{X}_{[t]_M^M}^M) \int_{[t]_M^M}^t \left(B(\hat{X}_{[t]_M^M}^M) + B'(\hat{X}_{[t]_M^M}^M) \int_{[t]_M^M}^u B(\hat{X}_{[t]_M^M}^M) dW_v \right) dW_u \\ &\quad + B'(\hat{X}_{[t]_M^M}^M) \int_{[t]_M^M}^t \left(A\hat{X}_{[t]_M^M}^M + F(\hat{X}_{[t]_M^M}^M) \right) du + \frac{1}{2} B''(\hat{X}_{[t]_M^M}^M) \left(\int_{[t]_M^M}^t B(\hat{X}_{[t]_M^M}^M) dW_u, \int_{[t]_M^M}^t B(\hat{X}_{[t]_M^M}^M) dW_u \right)\end{aligned}\tag{18}$$

\mathbb{P} -a.s. for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Note that Itô's formula shows

$$\begin{aligned}& A \left[\int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s B(Y_m^M) dW_u ds - \frac{T}{2M} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B(Y_m^M) dW_s \right] \\ &= A \left[\frac{T}{2M} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B(\hat{X}_{[t]_M^M}^M) dW_s - \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} (s - \frac{mT}{M}) B(Y_m^M) dW_s \right] \\ &= \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M^M}^M) dW_s\end{aligned}\tag{19}$$

\mathbb{P} -a.s. and

$$\begin{aligned}& \frac{T}{M} \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} B'(Y_m^M) (AY_m^M + F(Y_m^M)) dW_s - \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{\frac{mT}{M}}^s B'(Y_m^M) (AY_m^M + F(Y_m^M)) dW_u ds \\ &= \int_{\frac{mT}{M}}^{\frac{(m+1)T}{M}} \int_{[s]_M^M}^s B'(\hat{X}_{[s]_M^M}^M) \left(A\hat{X}_{[s]_M^M}^M + F(\hat{X}_{[s]_M^M}^M) \right) du dW_s\end{aligned}\tag{20}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M-1\}$ and all $M \in \mathbb{N}$. Next we combine (3), (19) and (20) to obtain that

$$Y_m^M = e^{A\frac{mT}{M}} \xi + \int_0^{\frac{mT}{M}} e^{A(\frac{mT}{M} - \langle s \rangle_M)} \Phi_s^M ds + \int_0^{\frac{mT}{M}} e^{A(\frac{mT}{M} - \langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M^M}^M) \right) dW_s\tag{21}$$

\mathbb{P} -a.s. for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$.

4.1 Proof of Proposition 1

Throughout this proof $C: [0, \infty) \rightarrow [0, \infty)$ is a universal non-decreasing function which changes from line to line. Let $\theta \in [0, \infty)$ be defined by $\theta := \max(\gamma - \alpha, \gamma - \frac{1}{2}, 2(\gamma - \beta), 2(\gamma - \delta - \frac{1}{2}))$. Observe that

Assumption 2 and Assumption 3 ensure that $\theta < 1$. Next note that

$$\begin{aligned}
& \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \Phi_s^M \right\|_{L^p(\Omega; H_{(\gamma-\theta)})}^2 \\
& \leq C(K) \left\| (-A)^{(\gamma-\theta-\alpha)} \right\|_{L(H)}^2 \left\| F(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; H_\alpha)}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\theta)} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| F'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s (A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M)) du \right\|_{L^p(\Omega; H)}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\theta)} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| F'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s B(\hat{X}_{[s]_M}^M) dW_u \right\|_{L^p(\Omega; H)}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\theta)} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left[\int_{[s]_M}^s \left\| F''(\hat{X}_{[s]_M}^M) \right\|_{L^{(2)}(H, H)} \left\| B(\hat{X}_{[s]_M}^M) \right\|_{HS(U_0, H)}^2 \right]_{L^p(\Omega; \mathbb{R})}^2 du
\end{aligned} \tag{22}$$

and Lemma 7.7 in Da Prato and Zabczyk [7] hence shows

$$\begin{aligned}
& \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \Phi_s^M \right\|_{L^p(\Omega; H_{(\gamma-\theta)})}^2 \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\alpha)}^2 \right) + \frac{C(K) (s - [s]_M)}{(t - \langle s \rangle_M)^{2(\gamma-\theta)}} \int_{[s]_M}^s \left\| A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; H)}^2 du \\
& + \frac{C(K)}{(t - \langle s \rangle_M)^{2(\gamma-\theta)}} \int_{[s]_M}^s \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; H)}^2 du + \frac{C(K)}{(t - \langle s \rangle_M)^{2(\gamma-\theta)}} \left[\int_{[s]_M}^s \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H)} \right) du \right]^2 \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) + \frac{C(K) (s - [s]_M)}{(t - \langle s \rangle_M)^{2(\gamma-\theta)}} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right)
\end{aligned} \tag{23}$$

for all $s \in [0, t)$ and all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$. In the next step we combine the estimate

$$\begin{aligned}
& \int_0^t (t - \langle s \rangle_M)^{-\theta} ds = \left[\frac{T}{M} \right]^{-\theta} \sum_{l=0}^{\frac{tM}{M}-1} \int_{\frac{lT}{M}}^{\frac{(l+1)T}{M}} \left(\frac{tM}{T} - l - \frac{1}{2} \right)^{-\theta} ds \\
& = \left[\frac{T}{M} \right]^{(1-\theta)} \sum_{l=0}^{\frac{tM}{M}-1} \left(l + \frac{1}{2} \right)^{-\theta} \leq \left[\frac{T}{M} \right]^{(1-\theta)} \sum_{l=0}^{\frac{tM}{M}-1} \left(\frac{l+1}{2} \right)^{-\theta} \leq 2 \left[\frac{T}{M} \right]^{(1-\theta)} \sum_{l=1}^{\frac{tM}{M}} l^{-\theta} \\
& \leq \frac{C(K)}{M^{(1-\theta)}} \left(1 + \int_1^M s^{-\theta} ds \right) = \frac{C(K)}{M^{(1-\theta)}} \left(1 + \frac{(M^{1-\theta}) - 1}{(1-\theta)} \right) \leq C(K)
\end{aligned} \tag{24}$$

for all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$ with inequality (23) to obtain that

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-\langle s \rangle_M)} \Phi_s^M ds \right\|_{L^p(\Omega; H_\gamma)}^2 \\
& \leq \left[\int_0^t 2^\theta (t - \langle s \rangle_M)^{-\theta} \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \Phi_s^M \right\|_{L^p(\Omega; H_{(\gamma-\theta)})} ds \right]^2 \\
& \leq \left[\int_0^t 2^\theta (t - \langle s \rangle_M)^{-\theta} ds \right] \left[\int_0^t 2^\theta (t - \langle s \rangle_M)^{-\theta} \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \Phi_s^M \right\|_{L^p(\Omega; H_{(\gamma-\theta)})}^2 ds \right] \\
& \leq C(K) \int_0^t (t - \langle s \rangle_M)^{-\theta} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) ds
\end{aligned} \tag{25}$$

for all $s \in [0, t)$ and all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$. In addition, note that

$$\begin{aligned}
& \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) \right\|_{L^p(\Omega; HS(U_0, H_{(\gamma-\theta/2)}))}^2 \\
& \leq C(K) \left\| (-A)^{(\gamma-\frac{\theta}{2}-\beta)} \right\|_{L(H)}^2 \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H_\beta))}^2 \\
& + C(K) (\langle s \rangle_M - s)^2 \left\| (-A)^{(\gamma-\frac{\theta}{2}-\beta)} \right\|_{L(H)}^2 \left\| A e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H_\beta))}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\frac{\theta}{2})} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s \left(A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M) \right) du \right\|_{L^p(\Omega; HS(U_0, H))}^2 \quad (26) \\
& + C(K) \left\| (-A)^{(\gamma-\frac{\theta}{2}-\delta)} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s B(\hat{X}_{[s]_M}^M) dW_u \right\|_{L^p(\Omega; HS(U_0, H_\delta))}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\frac{\theta}{2})} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^u B(\hat{X}_{[s]_M}^M) dW_v dW_u \right\|_{L^p(\Omega; HS(U_0, H))}^2 \\
& + C(K) \left\| (-A)^{(\gamma-\frac{\theta}{2})} e^{\frac{A}{2}(t-\langle s \rangle_M)} \right\|_{L(H)}^2 \left\| B''(\hat{X}_{[s]_M}^M) \left(\int_{[s]_M}^s B(\hat{X}_{[s]_M}^M) dW_u, \int_{[s]_M}^s B(\hat{X}_{[s]_M}^M) dW_u \right) \right\|_{L^p(\Omega; HS(U_0, H))}^2
\end{aligned}$$

and Lemma 7.7 in Da Prato & Zabczyk [7] hence implies

$$\begin{aligned}
& \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) \right\|_{L^p(\Omega; HS(U_0, H_{(\gamma-\theta/2)}))}^2 \\
& \leq C(K) \left(1 + \frac{(\langle s \rangle_M - s)^2}{(t - \langle s \rangle_M)^2} \right) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\beta)}^2 \right) \\
& + C(K) \frac{([s]_M - s)}{(t - \langle s \rangle_M)^{2(\gamma-\frac{\theta}{2})}} \int_{[s]_M}^s \left\| A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; H)}^2 du \\
& + C(K) (t - \langle s \rangle_M)^{-2(\gamma-\frac{\theta}{2}-\delta)} \int_{[s]_M}^s \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H_\delta))}^2 du \\
& + C(K) (t - \langle s \rangle_M)^{-2(\gamma-\frac{\theta}{2})} \int_{[s]_M}^s \int_{[s]_M}^u \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H))}^2 dv du \\
& + C(K) (t - \langle s \rangle_M)^{-2(\gamma-\frac{\theta}{2})} \left\| \left\| B''(\hat{X}_{[s]_M}^M) \right\|_{L^{(2)}(H, HS(U_0, H))} \right\| \left\| \int_{[s]_M}^s B(\hat{X}_{[s]_M}^M) dW_u \right\|_H^2 \left\| \right\|_{L^p(\Omega; \mathbb{R})}^2 \quad (27)
\end{aligned}$$

for all $s \in [0, t)$ and all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$. This and again Lemma 7.7 in Da Prato & Zabczyk [7] imply

$$\begin{aligned}
& \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) \right\|_{L^p(\Omega; HS(U_0, H_{(\gamma-\theta/2)}))}^2 \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) \\
& + C(K) (t - \langle s \rangle_M)^{-2(\gamma-\frac{\theta}{2})} \left\| \int_{[s]_M}^s \left\| B''(\hat{X}_{[s]_M}^M) \right\|_{L^{(2)}(H, HS(U_0, H))}^{1/2} B(\hat{X}_{[s]_M}^M) dW_u \right\|_{L^{2p}(\Omega; H)}^4 \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) \\
& + C(K) (t - \langle s \rangle_M)^{-2(\gamma-\frac{\theta}{2})} \left[\int_{[s]_M}^s \left\| \left\| B''(\hat{X}_{[s]_M}^M) \right\|_{L^{(2)}(H, HS(U_0, H))} \right\|_{HS(U_0, H)}^2 \right\|_{L^p(\Omega; \mathbb{R})} du \right]^2 \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) + \frac{C(K) (\lfloor s \rfloor_M - s)^2}{(t - \langle s \rangle_M)^{2(\gamma-\frac{\theta}{2})}} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H)}^2 \right) \\
& \leq C(K) \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right)
\end{aligned} \tag{28}$$

for all $s \in [0, t)$ and all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$. Again Lemma 7.7 in Da Prato and Zabczyk [7] hence shows

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) dW_s \right\|_{L^p(\Omega; H_\gamma)}^2 \\
& \leq C(K) \int_0^t (t - \langle s \rangle_M)^{-\theta} \left\| e^{\frac{A}{2}(t-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) \right\|_{L^p(\Omega; HS(U_0, H_{(\gamma-\theta/2)}))}^2 ds \\
& \leq C(K) \int_0^t (t - \langle s \rangle_M)^{-\theta} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) ds
\end{aligned} \tag{29}$$

for all $s \in [0, t)$ and all $t \in \{0, \frac{T}{M}, \dots, \frac{(M-1)T}{M}, T\}$. Next we combine (21), (25), (29) and (24) to obtain that

$$\begin{aligned}
\|Y_m^M\|_{L^p(\Omega; H_\gamma)}^2 & \leq C(K) \left\| e^{A\frac{mT}{M}} \xi \right\|_{L^p(\Omega; H_\gamma)}^2 + C(K) \left\| \int_0^{\frac{mT}{M}} e^{A(\frac{mT}{M}-\langle s \rangle_M)} \Phi_s^M ds \right\|_{L^p(\Omega; H_\gamma)}^2 \\
& + C(K) \left\| \int_0^{\frac{mT}{M}} e^{A(\frac{mT}{M}-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) dW_s \right\|_{L^p(\Omega; H_\gamma)}^2 \\
& \leq C(K) \|\xi\|_{L^p(\Omega; H_\gamma)}^2 + C(K) \int_0^{\frac{mT}{M}} \left(\frac{mT}{M} - \langle s \rangle_M \right)^{-\theta} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 \right) ds \\
& \leq C(K) \left(1 + \|\xi\|_{L^p(\Omega; H_\gamma)}^2 \right) + C(K) \int_0^{\frac{mT}{M}} \frac{2^\theta}{(2[\frac{mT}{M} - \langle s \rangle_M])^\theta} \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\gamma)}^2 ds \\
& = C(K) \left(1 + \|\xi\|_{L^p(\Omega; H_\gamma)}^2 \right) + C(K) \left[\frac{T}{M} \right]^{(1-\theta)} \sum_{l=0}^{m-1} \frac{2^\theta}{(m-l + [m-l-1])^\theta} \|Y_l^M\|_{L^p(\Omega; H_\gamma)}^2 \\
& \leq C(K) \left(1 + \|\xi\|_{L^p(\Omega; H_\gamma)}^2 \right) + C(K) M^{-(1-\theta)} \sum_{l=0}^{m-1} (m-l)^{-\theta} \|Y_l^M\|_{L^p(\Omega; H_\gamma)}^2
\end{aligned} \tag{30}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. In the next step we use the mappings $E_\varepsilon : [0, \infty) \rightarrow [0, \infty)$, $\varepsilon \in (0, \infty)$, defined by $E_\varepsilon(t) := \sum_{n=0}^{\infty} \frac{t^{n\varepsilon}}{\Gamma(n\varepsilon+1)}$ for all $\varepsilon \in (0, \infty)$ and all $t \in [0, \infty)$ (see Section 7 in

Henry [15]) and apply a generalized version of the discrete Gronwall lemma (see Theorem 6.1 in Dixon & McKee [9]) to (30) to obtain

$$\begin{aligned}
\|Y_m^M\|_{L^p(\Omega; H_\gamma)}^2 &\leq C(K) \left(1 + \|\xi\|_{L^p(\Omega; H_\gamma)}^2\right) E_{(1-\theta)} \left(2M \left(C(K)M^{-(1-\theta)}\Gamma(1-\theta)\right)^{\frac{1}{(1-\theta)}}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. In addition, note that

$$\begin{aligned}
E_{(1-\theta)}(C(K)) &= \sum_{n=0}^{\infty} \frac{C(K)^{n(1-\theta)}}{\Gamma(n(1-\theta)+1)} = \sum_{n=0}^{\lceil \frac{1}{K} \rceil} \frac{C(K)^{n(1-\theta)}}{\Gamma(n(1-\theta)+1)} + \sum_{n=\lceil \frac{1}{K} \rceil+1}^{\infty} \frac{C(K)^{n(1-\theta)}}{\Gamma(n(1-\theta)+1)} \\
&\leq \left(\left\lceil \frac{1}{K} \right\rceil + 1\right) C(K)^{(2+\frac{1}{K})} + \sum_{n=\lceil \frac{1}{K} \rceil+1}^{\infty} \frac{C(K)^{\frac{n}{K}}}{\Gamma\left(\frac{n}{K}+1\right)} \leq C(K) + \sum_{n=0}^{\infty} \frac{C(K)^{\frac{n}{K}}}{\Gamma\left(\frac{n}{K}+1\right)} \\
&= C(K) + E_{\frac{1}{K}}(C(K)) \leq C(K).
\end{aligned} \tag{32}$$

Combining (31) and (32) then gives

$$\|Y_m^M\|_{L^p(\Omega; H_\gamma)}^2 \leq C(K) \left(1 + \|\xi\|_{L^p(\Omega; H_\gamma)}^2\right) \tag{33}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. This completes the proof of Proposition 1.

4.2 Proof of Theorem 1

Throughout this proof $C: [0, \infty) \rightarrow [0, \infty)$ is a universal non-decreasing function which changes from line to line. Note that Jensen's inequality implies

$$\|\hat{X}_t^M\|_{L^p(\Omega; H)}^2 \leq \|\hat{X}_t^M\|_{L^6(\Omega; H)}^2 \leq C(K) \left(1 + \|\hat{X}_{[t]_M}^M\|_{L^6(\Omega; H)}^2\right) \leq C(K) \left(1 + \|\hat{X}_{[t]_M}^M\|_{L^6(\Omega; H_\gamma)}^2\right) \leq C(K) \tag{34}$$

and

$$\|\hat{X}_t^M - \hat{X}_{[t]_M}^M\|_{L^p(\Omega; H)} \leq \|\hat{X}_t^M - \hat{X}_{[t]_M}^M\|_{L^6(\Omega; H)} \leq C(K) (t - [t]_M)^{1/2} \tag{35}$$

for all $p \in (0, 6]$, $t \in [0, T]$ and all $M \in \mathbb{N}$. Moreover, let $Z^M: [0, T] \times \Omega \rightarrow H$, $M \in \mathbb{N}$, be stochastic processes satisfying

$$Z_t^M = e^{At} X_0 + \int_0^t e^{A(t-s)} F(\hat{X}_s^M) ds + \int_0^t e^{A(t-s)} B(\hat{X}_s^M) dW_s \tag{36}$$

\mathbb{P} -a.s. for all $t \in [0, T]$ and all $M \in \mathbb{N}$ and let $\hat{Z}^M: \{0, 1, \dots, M\} \times \Omega \rightarrow H$, $M \in \mathbb{N}$, be stochastic processes satisfying

$$\hat{Z}_t^M = e^{At} X_0 + \int_0^t e^{A(t-s)} \Phi_s^M ds + \int_0^t e^{A(t-s)} \Psi_s^M dW_s \tag{37}$$

\mathbb{P} -a.s. for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Next observe that the triangle inequality implies that

$$\left\|X_{\frac{mT}{M}} - Y_m^M\right\|_{L^2(\Omega; H)}^2 \leq 2 \left\|X_{\frac{mT}{M}} - Z_{\frac{mT}{M}}^M\right\|_{L^2(\Omega; H)}^2 + 2 \left[\left\|Z_{\frac{mT}{M}}^M - \hat{Z}_{\frac{mT}{M}}^M\right\|_{L^2(\Omega; H)} + \left\|\hat{Z}_{\frac{mT}{M}}^M - Y_m^M\right\|_{L^2(\Omega; H)} \right]^2 \tag{38}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. Moreover, note that

$$\begin{aligned}
\|X_t - Z_t^M\|_{L^2(\Omega; H)}^2 &\leq C(K) \left\| \int_0^t e^{A(t-s)} \left(F(X_s) - F(\hat{X}_s^M) \right) ds \right\|_{L^2(\Omega; H)}^2 \\
&\quad + C(K) \left\| \int_0^t e^{A(t-s)} \left(B(X_s) - B(\hat{X}_s^M) \right) dW_s \right\|_{L^2(\Omega; H)}^2 \\
&\leq C(K) \int_0^t \left(\|F(X_s) - F(\hat{X}_s^M)\|_{L^2(\Omega; H)}^2 + \|B(X_s) - B(\hat{X}_s^M)\|_{L^2(\Omega; HS(U_0, H))}^2 \right) ds \\
&\leq C(K) \int_0^t \|X_s - \hat{X}_s^M\|_{L^2(\Omega; H)}^2 ds
\end{aligned} \tag{39}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. In addition, (36), (37), (63) and (69) imply

$$\left\| Z_{\frac{mT}{M}}^M - \hat{Z}_{\frac{mT}{M}}^M \right\|_{L^2(\Omega; H)} \leq \left\| \int_0^{\frac{mT}{M}} e^{A(\frac{mT}{M}-s)} \left(F(\hat{X}_s^M) - \Phi_s^M \right) ds \right\|_{L^2(\Omega; H)} \tag{40}$$

$$+ \left[\int_0^{\frac{mT}{M}} \|B(\hat{X}_s^M) - \Psi_s^M\|_{L^2(\Omega; HS(U_0, H))}^2 ds \right]^{\frac{1}{2}} \tag{41}$$

$$\leq C(K)M^{-\gamma} + \left[\int_0^{\frac{mT}{M}} C(K)M^{-2\gamma} ds \right]^{\frac{1}{2}} \leq C(K)M^{-\gamma} \tag{42}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. Combining (38)–(42) yields

$$\left\| X_{\frac{mT}{M}} - Y_m^M \right\|_{L^2(\Omega; H)}^2 \leq C(K) \int_0^{\frac{mT}{M}} \|X_s - \hat{X}_s^M\|_{L^2(\Omega; H)}^2 ds + 2 \left[C(K)M^{-\gamma} + \left\| \hat{Z}_{\frac{mT}{M}}^M - Y_m^M \right\|_{L^2(\Omega; H)} \right]^2 \tag{43}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. Next (21), (37), (76) and (80) show

$$\left\| \hat{Z}_{\frac{mT}{M}}^M - Y_m^M \right\|_{L^2(\Omega; H)} \tag{44}$$

$$\leq \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} - e^{A(\frac{mT}{M}-\langle s \rangle_M)} \right) \Phi_s^M ds \right\|_{L^2(\Omega; H)} \tag{45}$$

$$+ \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} \Psi_s^M - e^{A(\frac{mT}{M}-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{\langle s \rangle_M}^M) \right) \right) dW_s \right\|_{L^2(\Omega; H)}^2 \tag{46}$$

$$\leq C(K)M^{-\min(\alpha+1, \frac{3}{2})} (1 + \log(M)) + C(K)M^{-\min(\beta+\frac{1}{2}, \delta+1, \frac{3}{2})} (1 + \log(M)) \leq C(K)M^{-\gamma} \tag{47}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. Combining this and (43) then yields

$$\begin{aligned}
\left\| X_{\frac{mT}{M}} - Y_m^M \right\|_{L^2(\Omega; H)}^2 &\leq C(K) \int_0^{\frac{mT}{M}} \|X_s - \hat{X}_s^M\|_{L^2(\Omega; H)}^2 ds + C(K)M^{-2\gamma} \\
&= C(K) \sum_{l=0}^{m-1} \int_{\frac{lT}{M}}^{\frac{(l+1)T}{M}} \|X_s - \hat{X}_s^M\|_{L^2(\Omega; H)}^2 ds + C(K)M^{-2\gamma} \\
&\leq C(K) \sum_{l=0}^{m-1} \int_{\frac{lT}{M}}^{\frac{(l+1)T}{M}} \|X_{\frac{lT}{M}} - Y_l^M\|_{L^2(\Omega; H)}^2 ds + C(K)M^{-2\gamma}
\end{aligned} \tag{48}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$. To finish the proof of Theorem 1 we apply the discrete Gronwall lemma to (48) and take square root to obtain

$$\left\| X_{\frac{mT}{M}} - Y_m^M \right\|_{L^2(\Omega; H)} \leq C(K)M^{-\gamma} \tag{49}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$.

4.2.1 Estimates for $\|Z_m^M - \hat{Z}_m^M\|_{L^2(\Omega; H)}$ for $m \in \{0, 1, \dots, M\}$ and $M \in \mathbb{N}$

The following well known lemma will be used frequently below.

Lemma 4. *Let the setting in Section 2 be fulfilled. Then*

$$\left\| (-tA)^{-\kappa} (e^{At} - I) \right\|_{L(H)} \leq 1 \quad (50)$$

for all $t \in (0, \infty)$ and all $\kappa \in [0, 1]$ and

$$\left\| (-tA)^{-\kappa} (e^{At} - I - tA) \right\|_{L(H)} \leq 1 \quad (51)$$

for all $t \in (0, \infty)$ and all $\kappa \in [1, 2]$.

With the help of Lemma 4 we first establish some estimates that we exploit in the estimation of (40) and (41). More formally, observe that Lemma 4 implies

$$\begin{aligned} & \left\| \int_{[t]_M}^t \left(e^{A(t-s)} F(\hat{X}_s^M) - F(\hat{X}_{[s]_M}^M) \right) ds \right\|_{L^2(\Omega; H)} \\ & \leq C(K) \int_{[t]_M}^t \left(\left\| e^{A(t-s)} \left(F(\hat{X}_s^M) - F(\hat{X}_{[s]_M}^M) \right) \right\|_{L^2(\Omega; H)} + \left\| \left(e^{A(t-s)} - I \right) F(\hat{X}_{[s]_M}^M) \right\|_{L^2(\Omega; H)} \right) ds \\ & \leq C(K) \int_{[t]_M}^t \left(\left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M \right\|_{L^2(\Omega; H)} + \left\| (-A)^{-\alpha} \left(e^{A(t-s)} - I \right) \right\|_{L(H)} \left\| F(\hat{X}_{[s]_M}^M) \right\|_{L^2(\Omega; H_\alpha)} \right) ds \\ & \leq C(K) \int_{[t]_M}^t \left(M^{-\frac{1}{2}} + M^{-\min(\alpha, 1)} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^2(\Omega; H_\alpha)} \right) \right) ds \leq C(K) M^{-\min(\alpha+1, \frac{3}{2})} \leq C(K) M^{-\gamma} \end{aligned} \quad (52)$$

and

$$\begin{aligned} & \left\| \int_{[t]_M}^t \left(e^{A(t-s)} B(\hat{X}_s^M) - B(\hat{X}_{[s]_M}^M) \right) dW_s \right\|_{L^p(\Omega; H)}^2 \\ & \leq C(K) \int_{[t]_M}^t \left\| e^{A(t-s)} \left(B(\hat{X}_s^M) - B(\hat{X}_{[s]_M}^M) \right) + \left(e^{A(t-s)} - I \right) B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H))}^2 ds \\ & \leq C(K) \int_{[t]_M}^t \left(\left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H)}^2 + \left\| (-A)^{-\beta} \left(e^{A(t-s)} - I \right) \right\|_{L(H)}^2 \left\| B(\hat{X}_{[s]_M}^M) \right\|_{L^p(\Omega; HS(U_0, H_\beta))}^2 \right) ds \\ & \leq C(K) \int_{[t]_M}^t \left(M^{-1} + M^{-\min(2\beta, 2)} \left(1 + \left\| \hat{X}_{[s]_M}^M \right\|_{L^p(\Omega; H_\beta)}^2 \right) \right) ds \leq C(K) M^{-\min(2\beta+1, 2)} \leq C(K) M^{-2} \end{aligned} \quad (53)$$

for all $t \in [0, T]$, all $M \in \mathbb{N}$ and all $p \in [2, 6]$. Additionally, Lemma 4, (52) and (35) show

$$\begin{aligned}
& \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t \left(A \hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M) \right) ds - \int_{[t]_M}^t B(\hat{X}_s^M) dW_s \right\|_{L^2(\Omega; H)} \\
& \leq \left\| \left(e^{A(t-[t]_M)} - I - (t - [t]_M) A \right) \hat{X}_{[t]_M}^M \right\|_{L^2(\Omega; H)} + \left\| \int_{[t]_M}^t \left(e^{A(t-s)} F(\hat{X}_s^M) - F(\hat{X}_{[s]_M}^M) \right) ds \right\|_{L^2(\Omega; H)} \\
& + \left\| \int_{[t]_M}^t \left(e^{A(t-s)} - I \right) B(\hat{X}_s^M) dW_s \right\|_{L^2(\Omega; H)} \\
& \leq \left\| (-A)^{-\gamma} \left(e^{A(t-[t]_M)} - I - (t - [t]_M) A \right) \right\|_{L(H)} \left\| \hat{X}_{[t]_M}^M \right\|_{L^2(\Omega; H_\gamma)} + C(K) M^{-\gamma} \\
& + \left[\int_{[t]_M}^t \left\| (-A)^{-\beta} \left(e^{A(t-s)} - I \right) \right\|_{L(H)}^2 \left\| B(\hat{X}_s^M) \right\|_{L^2(\Omega; HS(U_0, H_\beta))}^2 ds \right]^{\frac{1}{2}} \\
& \leq C(K) M^{-\gamma} + C(K) M^{-(\beta+\frac{1}{2})} \leq C(K) M^{-\gamma}
\end{aligned} \tag{54}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Moreover, Lemma 4 and (53) imply

$$\begin{aligned}
& \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right\|_{L^p(\Omega; H)} \\
& \leq \left\| (-A)^{-1} \left(e^{A(t-[t]_M)} - I \right) \right\|_{L(H)} \left\| \hat{X}_{[t]_M}^M \right\|_{L^p(\Omega; H_1)} + \int_{[t]_M}^t \left\| e^{A(t-s)} F(\hat{X}_s^M) \right\|_{L^p(\Omega; H)} ds \\
& + \left\| \int_{[t]_M}^t \left(e^{A(t-s)} B(\hat{X}_s^M) - B(\hat{X}_{[s]_M}^M) \right) dW_s \right\|_{L^p(\Omega; H)} \leq C(K) M^{-1}
\end{aligned} \tag{55}$$

for all $t \in [0, T]$, all $M \in \mathbb{N}$ and all $p \in [2, 6]$.

4.2.1.1 Estimation of (40) Note that

$$\begin{aligned}
F(\hat{X}_t^M) &= F(\hat{X}_{[t]_M}^M) + F'(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M \right) + \frac{1}{2} F''(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M, \hat{X}_t^M - \hat{X}_{[t]_M}^M \right) \\
& + \int_0^1 \left(F'' \left(\hat{X}_{[t]_M}^M + r(\hat{X}_t^M - \hat{X}_{[t]_M}^M) \right) - F''(\hat{X}_{[t]_M}^M) \right) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M, \hat{X}_t^M - \hat{X}_{[t]_M}^M \right) (1-r) dr
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
& \left\| \int_0^1 \left(F'' \left(\hat{X}_{[t]_M}^M + r(\hat{X}_t^M - \hat{X}_{[t]_M}^M) \right) - F''(\hat{X}_{[t]_M}^M) \right) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M, \hat{X}_t^M - \hat{X}_{[t]_M}^M \right) (1-r) dr \right\|_{L^2(\Omega; H)} \\
& \leq \int_0^1 \left\| \left\| F'' \left(\hat{X}_{[t]_M}^M + r(\hat{X}_t^M - \hat{X}_{[t]_M}^M) \right) - F''(\hat{X}_{[t]_M}^M) \right\|_{L^{(2)}(H, H)} \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M \right\|_H^2 \right\|_{L^2(\Omega; \mathbb{R})} dr \\
& \leq C(K) \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M \right\|_{L^6(\Omega; H)}^3 \leq C(K) M^{-\frac{3}{2}}
\end{aligned} \tag{57}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. The remainder terms in (57) and (64) are here estimated similarly as in Kruse [25]. Combining (56) and (57) then shows

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-s)} \left(F(\hat{X}_s^M) - \Phi_s^M \right) ds \right\|_{L^2(\Omega; H)} \\
& \leq \left\| \int_0^t e^{A(t-s)} F'(\hat{X}_{[s]_M}^M) \left(\hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s \left(A\hat{X}_{[u]_M}^M + F(\hat{X}_{[u]_M}^M) \right) du - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) ds \right\|_{L^2(\Omega; H)} \\
& + \left\| \int_0^t e^{A(t-s)} F''(\hat{X}_{[s]_M}^M) \left(\hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \hat{X}_s^M - \hat{X}_{[s]_M}^M + \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) ds \right\|_{L^2(\Omega; H)} \\
& + \left\| \int_0^t e^{A(t-s)} \left(F''(\hat{X}_{[s]_M}^M) \left(\int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right. \right. \\
& \quad \left. \left. - \sum_{j \in \mathcal{J}} \int_{[s]_M}^s F''(\hat{X}_{[s]_M}^M) \left(B(\hat{X}_{[u]_M}^M) g_j, B(\hat{X}_{[u]_M}^M) g_j \right) du \right) ds \right\|_{L^2(\Omega; H)} + C(K) M^{-\frac{3}{2}}
\end{aligned} \tag{58}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Moreover, (54) and (35) imply

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-s)} F'(\hat{X}_{[s]_M}^M) \left(\hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s \left(A\hat{X}_{[u]_M}^M + F(\hat{X}_{[u]_M}^M) \right) du - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) ds \right\|_{L^2(\Omega; H)} \\
& \leq C(K) \int_0^t \left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s \left(A\hat{X}_{[u]_M}^M + F(\hat{X}_{[u]_M}^M) \right) du - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^2(\Omega; H)} ds \\
& + C(K) \left[\frac{T}{M} \sum_{l=0}^{M\lfloor t \rfloor_M / T} \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} \left\| \int_{\frac{lT}{M}}^s \left(B(\hat{X}_u^M) - B(\hat{X}_{[u]_M}^M) \right) dW_u \right\|_{L^2(\Omega; H)}^2 ds \right]^{\frac{1}{2}} \\
& \leq C(K) \int_0^t M^{-\gamma} ds + C(K) \left[M^{-1} \sum_{l=0}^{M\lfloor t \rfloor_M / T} \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} \int_{\frac{lT}{M}}^s \left\| \hat{X}_u^M - \hat{X}_{[u]_M}^M \right\|_{L^2(\Omega; H)}^2 du ds \right]^{\frac{1}{2}} \\
& \leq C(K) M^{-\gamma} + C(K) \left[\sum_{l=0}^{M\lfloor t \rfloor_M / T} M^{-4} \right]^{\frac{1}{2}} \leq C(K) M^{-\gamma}
\end{aligned} \tag{59}$$

and Lemma 7.7 in Da Prato and Zabczyk [7], (55) and (35) show

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-s)} F''(\hat{X}_{[s]_M}^M) \left(\hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \hat{X}_s^M - \hat{X}_{[s]_M}^M + \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) ds \right\|_{L^2(\Omega; H)} \\
& \leq C(K) \int_0^t \left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^4(\Omega; H)} \left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M + \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^4(\Omega; H)} ds \\
& \leq C(K) M^{-1} \int_0^t \left(\left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M \right\|_{L^4(\Omega; H)} + \left\| \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^4(\Omega; H)} \right) ds \\
& \leq C(K) M^{-1} \int_0^t \left(M^{-\frac{1}{2}} + \left[\int_{[s]_M}^s \left\| B(\hat{X}_{[u]_M}^M) \right\|_{L^4(\Omega; HS(U_0, H))}^2 du \right]^{\frac{1}{2}} \right) ds \leq C(K) M^{-\frac{3}{2}}
\end{aligned} \tag{60}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Furthermore, observe that

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-s)} \left[F''(\hat{X}_{[s]_M}^M) \left(\int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right. \right. \\
& \quad \left. \left. - \sum_{j \in \mathcal{J}} \int_{[s]_M}^s F''(\hat{X}_{[s]_M}^M) \left(B(\hat{X}_{[u]_M}^M) g_j, B(\hat{X}_{[u]_M}^M) g_j \right) du \right] ds \right\|_{L^2(\Omega; H)}^2 \\
&= \sum_{l=0}^{M[t]_M/T} \left\| \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} e^{A(t-s)} \left[F''(\hat{X}_{[s]_M}^M) \left(\int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right. \right. \\
& \quad \left. \left. - \sum_{j \in \mathcal{J}} \int_{[s]_M}^s F''(\hat{X}_{[s]_M}^M) \left(B(\hat{X}_{[u]_M}^M) g_j, B(\hat{X}_{[u]_M}^M) g_j \right) du \right] ds \right\|_{L^2(\Omega; H)}^2 \\
&\leq \frac{C(K)T}{M} \sum_{l=0}^{M[t]_M/T} \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} \left[\left\| F''(\hat{X}_{\frac{lT}{M}}^M) \left(\int_{\frac{lT}{M}}^s B(\hat{X}_{[u]_M}^M) dW_u, \int_{\frac{lT}{M}}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right\|_{L^2(\Omega; H)}^2 \right. \\
& \quad \left. + \left\| \sum_{j \in \mathcal{J}} \int_{\frac{lT}{M}}^s F''(\hat{X}_{\frac{lT}{M}}^M) \left(B(\hat{X}_{[u]_M}^M) g_j, B(\hat{X}_{[u]_M}^M) g_j \right) du \right\|_{L^2(\Omega; H)}^2 \right] ds
\end{aligned} \tag{61}$$

and Lemma 7.7 in Da Prato and Zabczyk [7] hence implies

$$\begin{aligned}
& \left\| \int_0^t e^{A(t-s)} \left[F''(\hat{X}_{[s]_M}^M) \left(\int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u, \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right. \right. \\
& \quad \left. \left. - \sum_{j \in \mathcal{J}} \int_{[s]_M}^s F''(\hat{X}_{[s]_M}^M) \left(B(\hat{X}_{[u]_M}^M) g_j, B(\hat{X}_{[u]_M}^M) g_j \right) du \right] ds \right\|_{L^2(\Omega; H)}^2 \\
&\leq \frac{C(K)}{M} \sum_{l=0}^{M[t]_M/T} \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} \left[\left\| \int_{\frac{lT}{M}}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^4(\Omega; H)}^4 + \frac{T}{M} \int_{\frac{lT}{M}}^s \left\| B(\hat{X}_{[u]_M}^M) \right\|_{L^4(\Omega; HS(U_0, H))}^4 du \right] ds \\
&\leq \frac{C(K)}{M} \sum_{l=0}^{M[t]_M/T} \int_{\frac{lT}{M}}^{\min(\frac{(l+1)T}{M}, t)} M^{-2} ds \leq C(K)M^{-3}
\end{aligned} \tag{62}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Finally, combining (58)–(62) yields

$$\left\| \int_0^t e^{A(t-s)} \left(F(\hat{X}_s^M) - \Phi_s^M \right) ds \right\|_{L^2(\Omega; H)} \leq C(K)M^{-\gamma} \tag{63}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$.

4.2.1.2 Estimation of (41) Similar as in the previous subsection a Taylor expansion of $B: H \rightarrow HS(U_0, H)$ and the estimate

$$\begin{aligned}
& \left\| \int_0^1 \left(B'' \left(\hat{X}_{[t]_M}^M + r(\hat{X}_t^M - \hat{X}_{[t]_M}^M) \right) - B''(\hat{X}_{[t]_M}^M) \right) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M, \hat{X}_t^M - \hat{X}_{[t]_M}^M \right) (1-r) dr \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq C(K)M^{-\frac{3}{2}} \tag{64}
\end{aligned}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$ give

$$\begin{aligned}
& \left\| B(\hat{X}_t^M) - \Psi_t^M \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq \left\| B'(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t (A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M)) ds - \int_{[t]_M}^t B(\hat{X}_s^M) dW_s \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + \left\| B'(\hat{X}_{[t]_M}^M) \int_{[t]_M}^t \left(B(\hat{X}_s^M) - B(\hat{X}_{[s]_M}^M) - B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) dW_s \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + \left\| B''(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s, \hat{X}_t^M - \hat{X}_{[t]_M}^M + \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + C(K)M^{-\frac{3}{2}}
\end{aligned} \tag{65}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Moreover, (54) implies

$$\begin{aligned}
& \left\| B'(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t (A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M)) ds - \int_{[t]_M}^t B(\hat{X}_s^M) dW_s \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq C(K) \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t (A\hat{X}_{[s]_M}^M + F(\hat{X}_{[s]_M}^M)) ds - \int_{[t]_M}^t B(\hat{X}_s^M) dW_s \right\|_{L^2(\Omega; H)} \\
& \leq C(K)M^{-\gamma}
\end{aligned} \tag{66}$$

and a further Taylor expansion of $B: H \rightarrow HS(U_0, H)$, (55) and (35) show

$$\begin{aligned}
& \left\| B'(\hat{X}_{[t]_M}^M) \int_{[t]_M}^t \left(B(\hat{X}_s^M) - B(\hat{X}_{[s]_M}^M) - B'(\hat{X}_{[s]_M}^M) \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) dW_s \right\|_{L^2(\Omega; HS(U_0, H))}^2 \\
& \leq C(K) \int_{[t]_M}^t \left\| B'(\hat{X}_{[s]_M}^M) \left(\hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right) \right\|_{L^2(\Omega; HS(U_0, H))}^2 ds \\
& + C(K) \int_{[t]_M}^t \left\| \int_0^1 B''(\hat{X}_{[s]_M}^M + r(\hat{X}_s^M - \hat{X}_{[s]_M}^M)) (\hat{X}_s^M - \hat{X}_{[s]_M}^M, \hat{X}_s^M - \hat{X}_{[s]_M}^M) (1-r) dr \right\|_{L^2(\Omega; HS(U_0, H))}^2 ds \\
& \leq C(K) \int_{[t]_M}^t \left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M - \int_{[s]_M}^s B(\hat{X}_{[u]_M}^M) dW_u \right\|_{L^2(\Omega; H)}^2 ds + C(K) \int_{[t]_M}^t \left\| \hat{X}_s^M - \hat{X}_{[s]_M}^M \right\|_{L^4(\Omega; H)}^4 ds \\
& \leq C(K)M^{-3}
\end{aligned} \tag{67}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$. Furthermore, Lemma 7.7 in Da Prato and Zabczyk [7], (55) and (35) imply

$$\begin{aligned}
& \left\| B''(\hat{X}_{[t]_M}^M) \left(\hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s, \hat{X}_t^M - \hat{X}_{[t]_M}^M + \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M - \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right\|_{L^4(\Omega; H)} \left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M + \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right\|_{L^4(\Omega; H)} \\
& \leq C(K)M^{-1} \left[\left\| \hat{X}_t^M - \hat{X}_{[t]_M}^M \right\|_{L^4(\Omega; H)} + \left\| \int_{[t]_M}^t B(\hat{X}_{[s]_M}^M) dW_s \right\|_{L^4(\Omega; H)} \right] \leq C(K)M^{-\frac{3}{2}}
\end{aligned} \tag{68}$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$ due to (55). Combining (65)–(68) then finally yields

$$\left\| B(\hat{X}_t^M) - \Psi_t^M \right\|_{L^2(\Omega; HS(U_0, H))} \leq C(K)M^{-\gamma} \quad (69)$$

for all $t \in [0, T]$ and all $M \in \mathbb{N}$.

4.2.2 Estimates for $\|\hat{Z}_m^M - Y_m^M\|_{L^2(\Omega; H)}$ for $m \in \{0, 1, \dots, M\}$ and $M \in \mathbb{N}$

The following well known lemma is used in this subsection.

Lemma 5. *Let the setting in Section 2 be fulfilled. Then*

$$\left\| (-tA)^{-\kappa} \left(e^{As} - e^{A\frac{t}{2}} \right) \right\|_{L(H)} \leq 1 \quad (70)$$

for all $s \in [0, t]$, all $t \in (0, \infty)$ and all $\kappa \in [0, 1]$ and

$$\left\| \int_0^t (-tA)^{-\kappa} \left(e^{Au} - e^{A\frac{t}{2}} \right) du \right\|_{L(H)} \leq t, \quad (71)$$

$$\left\| (-tA)^{-\kappa} \left(e^{As} - e^{A\frac{t}{2}} \left(I + \left(s - \frac{t}{2} \right) A \right) \right) \right\|_{L(H)} \leq 2 \quad (72)$$

for all $s \in [0, t]$, all $t \in (0, \infty)$ and all $\kappa \in [0, 2]$.

4.2.2.1 Estimation of (45) First of all, note that

$$\begin{aligned} & \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} - e^{A(\frac{mT}{M}-\langle s \rangle_M)} \right) \Phi_s^M ds \right\|_{L^2(\Omega; H)} \\ & \leq \left\| \int_{\frac{(m-1)T}{M}}^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} - e^{A\frac{T}{2M}} \right) \Phi_s^M ds \right\|_{L^2(\Omega; H)} \\ & + \sum_{l=0}^{m-2} \left\| (-A) e^{A(\frac{m-l-1)T}{M}} \right\|_{L(H)} \left\| \int_{\frac{lT}{M}}^{\frac{(l+1)T}{M}} (-A)^{-1} \left(e^{A(\frac{(l+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \Phi_s^M ds \right\|_{L^2(\Omega; H)} \end{aligned} \quad (73)$$

for all $m \in \{1, 2, \dots, M\}$ and $M \in \mathbb{N}$. Moreover, observe that

$$\begin{aligned} & \left\| \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \Phi_s^M ds \right\|_{L^2(\Omega; H)} \\ & \leq \left\| \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} (-A)^{-(\kappa+\alpha)} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) ds \right\|_{L(H)} \|F(Y_k^M)\|_{L^2(\Omega; H_\alpha)} \\ & + \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| F'(Y_k^M) \int_{\frac{kT}{M}}^s (AY_k^M + F(Y_k^M)) du \right\|_{L^2(\Omega; H)} ds \\ & + \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| F'(Y_k^M) \int_{\frac{kT}{M}}^s B(Y_k^M) dW_u \right\|_{L^2(\Omega; H)} ds \\ & + \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \int_{\frac{kT}{M}}^s \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \sum_{j \in \mathcal{J}} F''(Y_k^M) (B(Y_k^M)g_j, B(Y_k^M)g_j) \right\|_{L^2(\Omega; H)} du ds \end{aligned} \quad (74)$$

and Lemma 5 hence gives

$$\begin{aligned}
& \left\| \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \Phi_s^M ds \right\|_{L^2(\Omega;H)} \\
& \leq C(K)M^{-(1+\min(\kappa+\alpha,2))} + C(K)M^{-\kappa} \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \int_{\frac{kT}{M}}^s \|AY_k^M + F(Y_k^M)\|_{L^2(\Omega;H)} du ds \\
& + C(K)M^{-\kappa} \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \left[\int_{\frac{kT}{M}}^s \|B(Y_k^M)\|_{L^2(\Omega;HS(U_0,H))}^2 du \right]^{\frac{1}{2}} ds \\
& + C(K)M^{-(\kappa+1)} \int_{\frac{kT}{M}}^{\frac{(k+1)T}{M}} \left\| \sum_{j \in \mathcal{J}} F''(Y_k^M) (B(Y_k^M)g_j, B(Y_k^M)g_j) \right\|_{L^2(\Omega;H)} ds \\
& \leq C(K)M^{-(\kappa+1+\min(\alpha, \frac{1}{2}))}
\end{aligned} \tag{75}$$

for all $k \in \{0, 1, \dots, M-1\}$, all $M \in \mathbb{N}$ and all $\kappa \in [0, 1]$. Combining (73) and (75) implies

$$\begin{aligned}
& \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} - e^{A(\frac{mT}{M}-\langle s \rangle_M)} \right) \Phi_s^M ds \right\|_{L^2(\Omega;H)} \\
& \leq C(K)M^{-(1+\min(\alpha, \frac{1}{2}))} + C(K)M^{-(2+\min(\alpha, \frac{1}{2}))} \sum_{l=1}^{m-1} \left\| (-A) e^{A\frac{lT}{M}} \right\|_{L(H)} \\
& \leq C(K)M^{-(1+\min(\alpha, \frac{1}{2}))} + C(K)M^{-(1+\min(\alpha, \frac{1}{2}))} \left[\sum_{l=1}^{m-1} \frac{1}{l} \right] \leq C(K)M^{-\min(\alpha+1, \frac{3}{2})} (1 + \log(M))
\end{aligned} \tag{76}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$.

4.2.2.2 Estimation of (46) First of all, observe that

$$\begin{aligned}
& \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} \Psi_s^M - e^{A(\frac{mT}{M}-\langle s \rangle_M)} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M) \right) \right) dW_s \right\|_{L^2(\Omega;H)}^2 \\
& \leq \int_{\frac{(m-1)T}{M}}^{\frac{mT}{M}} \left\| e^{A(\frac{mT}{M}-s)} \Psi_s^M - e^{A\frac{T}{2M}} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(Y_{(m-1)}^M) \right) \right\|_{L^2(\Omega;HS(U_0,H))}^2 ds \\
& + \sum_{l=0}^{m-2} \int_{\frac{lT}{M}}^{\frac{(l+1)T}{M}} \left\| (-A)^{\frac{1}{2}} e^{A\frac{(m-l-1)T}{M}} \right\|^2 \\
& \quad \cdot \left\| (-A)^{-\frac{1}{2}} \left(e^{A(\frac{(l+1)T}{M}-s)} \Psi_s^M - e^{A\frac{T}{2M}} \left(\Psi_s^M + (\langle s \rangle_M - s) AB(Y_l^M) \right) \right) \right\|_{L^2(\Omega;HS(U_0,H))}^2 ds
\end{aligned} \tag{77}$$

for all $m \in \{1, 2, \dots, M\}$ and all $M \in \mathbb{N}$. In addition, note that

$$\begin{aligned}
& \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} \Psi_s^M - e^{A\frac{T}{2M}} (\Psi_s^M + (\langle s \rangle_M - s) AB(Y_k^M)) \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq \left\| (-A)^{-(\kappa+\beta)} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} (I + (\langle s \rangle_M - s) A) \right) \right\|_{L(H)} \left\| (-A)^\beta B(Y_k^M) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| B'(Y_k^M) \int_{\frac{kT}{M}}^s (AY_k^M + F(Y_k^M)) du \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + \left\| (-A)^{-(\kappa+\delta)} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| (-A)^\delta B'(Y_k^M) \int_{\frac{kT}{M}}^s B(Y_k^M) dW_u \right\|_{L^2(\Omega; HS(U_0, H))} \\
& + C(K) \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| \int_{\frac{kT}{M}}^s B'(Y_k^M) \left(\int_{\frac{kT}{M}}^u B(Y_k^M) dW_v \right) dW_u \right\|_{L^2(\Omega; H)} \\
& + C(K) \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} - e^{A\frac{T}{2M}} \right) \right\|_{L(H)} \left\| \int_{\frac{kT}{M}}^s B(Y_k^M) dW_u \right\|_{L^4(\Omega; H)}^2
\end{aligned} \tag{78}$$

and Lemma 7.7 in Da Prato and Zabczyk [7] and Lemma 5 hence imply

$$\begin{aligned}
& \left\| (-A)^{-\kappa} \left(e^{A(\frac{(k+1)T}{M}-s)} \Psi_s^M - e^{A\frac{T}{2M}} (\Psi_s^M + (\langle s \rangle_M - s) AB(Y_k^M)) \right) \right\|_{L^2(\Omega; HS(U_0, H))} \\
& \leq C(K) M^{-(\kappa+\beta)} + C(K) M^{-\kappa} \int_{\frac{kT}{M}}^s \left(\|AY_k^M\|_{L^2(\Omega; H)} + \|F(Y_k^M)\|_{L^2(\Omega; H)} \right) du \\
& + C(K) M^{-(\kappa+\delta)} \left(\int_{\frac{kT}{M}}^s \left\| (-A)^\delta B(Y_k^M) \right\|_{L^2(\Omega; HS(U_0, H))}^2 du \right)^{\frac{1}{2}} \\
& + C(K) M^{-\kappa} \left(\int_{\frac{kT}{M}}^s \int_{\frac{kT}{M}}^u \|B(Y_k^M)\|_{L^2(\Omega; HS(U_0, H))}^2 dv du \right)^{\frac{1}{2}} + C(K) M^{-\kappa} \int_{\frac{kT}{M}}^s \|B(Y_k^M)\|_{L^4(\Omega; HS(U_0, H))}^2 du \\
& \leq C(K) M^{-(\kappa+\beta)} + C(K) M^{-(\kappa+1)} + C(K) M^{-(\kappa+\delta+\frac{1}{2})} \leq C(K) M^{-(\kappa+\min(\beta, \delta+\frac{1}{2}, 1))}
\end{aligned} \tag{79}$$

for all $s \in [\frac{kT}{M}, \frac{(k+1)T}{M}]$, all $k \in \{0, 1, \dots, M-1\}$, all $M \in \mathbb{N}$ and all $\kappa \in [0, \frac{1}{2}]$. Combining (77) and (79) then shows

$$\begin{aligned}
& \left\| \int_0^{\frac{mT}{M}} \left(e^{A(\frac{mT}{M}-s)} \Psi_s^M - e^{A(\frac{mT}{M}-\langle s \rangle_M)} (\Psi_s^M + (\langle s \rangle_M - s) AB(\hat{X}_{[s]_M}^M)) \right) dW_s \right\|_{L^2(\Omega; H)}^2 \\
& \leq C(K) M^{-2 \min(\beta+\frac{1}{2}, \delta+1, \frac{3}{2})} + C(K) M^{-2 \min(\beta+\frac{1}{2}, \delta+1, \frac{3}{2})} \left[\sum_{l=1}^{m-1} \frac{1}{l} \right] \\
& \leq C(K) M^{-2 \min(\beta+\frac{1}{2}, \delta+1, \frac{3}{2})} (1 + \log(M))
\end{aligned} \tag{80}$$

for all $m \in \{0, 1, \dots, M\}$ and all $M \in \mathbb{N}$.

4.3 Proof of Lemma 2

Throughout this proof let $I_{(i),t_0,t}, I_{(i,j),t_0,t}, I_{(i,j,k),t_0,t}: \Omega \rightarrow \mathbb{R}, i, j, k \in \mathcal{J}, t_0, t \in [0, T], t_0 \leq t$, be random variables satisfying

$$\begin{aligned} I_{(i),t_0,t} &= \int_{t_0}^t \langle g_i, dW_s \rangle_{U_0} \\ I_{(i,j),t_0,t} &= \int_{t_0}^t \int_{t_0}^s \langle g_i, dW_u \rangle_{U_0} \langle g_j, dW_s \rangle_{U_0} \\ I_{(i,j,k),t_0,t} &= \int_{t_0}^t \int_{t_0}^s \int_{t_0}^u \langle g_i, dW_v \rangle_{U_0} \langle g_j, dW_u \rangle_{U_0} \langle g_k, dW_s \rangle_{U_0} \end{aligned} \quad (81)$$

\mathbb{P} -a.s. for all $i, j, k \in \mathcal{J}$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. In addition, we use the following well known identities for stochastic integrals (see, e.g., (10.3.15) and (10.4.14) in [22])

$$I_{(i),t_0,t} I_{(j),t_0,t} = I_{(i,j),t_0,t} + I_{(j,i),t_0,t}, \quad I_{(i,i),t_0,t} = \frac{1}{6} \left(I_{(i),t_0,t}^2 - 3(t - t_0) \right) I_{(i),t_0,t} \quad (82)$$

\mathbb{P} -a.s. for all $i, j \in \mathcal{J}$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$ and

$$\begin{aligned} I_{(i,j,k),t_0,t} + I_{(j,i,k),t_0,t} + I_{(j,k,i),t_0,t} + I_{(i,k,j),t_0,t} + I_{(k,j,i),t_0,t} + I_{(k,i,j),t_0,t} &= I_{(i),t_0,t} I_{(j),t_0,t} I_{(k),t_0,t} \\ I_{(i,j,j),t_0,t} + I_{(j,i,j),t_0,t} + I_{(j,j,i),t_0,t} &= \frac{1}{2} I_{(i),t_0,t} \left(I_{(j),t_0,t}^2 - (t - t_0) \right) \end{aligned} \quad (83)$$

\mathbb{P} -a.s. for all $i, j, k \in \mathcal{J}, i \neq j, i \neq k, j \neq k$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. Next note that

$$\begin{aligned} & \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) dW_v \right) dW_u \right) dW_s \\ &= \sum_{i,j,k \in \mathcal{J}} \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) g_i \langle g_i, dW_v \rangle_{U_0} \right) g_j \langle g_j, dW_u \rangle_{U_0} \right) g_k \langle g_k, dW_s \rangle_{U_0} \\ &= \sum_{i,j,k \in \mathcal{J}} B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_j \right) g_k I_{(i,j,k),t_0,t} \end{aligned} \quad (84)$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. In addition, (82) shows

$$\begin{aligned} & \frac{1}{2} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) dW_u, \int_{t_0}^s B(Z) dW_u \right) dW_s \\ &= \frac{1}{2} \sum_{i,j,k \in \mathcal{J}} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) g_i \langle g_i, dW_u \rangle_{U_0}, \int_{t_0}^s B(Z) g_j \langle g_j, dW_u \rangle_{U_0} \right) g_k \langle g_k, dW_s \rangle_{U_0} \\ &= \frac{1}{2} \sum_{i,j,k \in \mathcal{J}} B''(Z) \left(B(Z) g_i, B(Z) g_j \right) g_k \int_{t_0}^t I_{(i),t_0,s} I_{(j),t_0,s} d\langle g_k, W_s \rangle_{U_0} \\ &= \frac{1}{2} \sum_{i,j,k \in \mathcal{J}} B''(Z) \left(B(Z) g_i, B(Z) g_j \right) g_k \left(I_{(i,j,k),t_0,t} + I_{(j,i,k),t_0,t} \right) \\ &= \sum_{i,j,k \in \mathcal{J}} B''(Z) \left(B(Z) g_i, B(Z) g_j \right) g_k I_{(i,j,k),t_0,t} \end{aligned} \quad (85)$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. Combining (84) and (85) then gives

$$\begin{aligned}
& \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) dW_v \right) dW_u \right) dW_s + \frac{1}{2} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) dW_u, \int_{t_0}^s B(Z) dW_u \right) dW_s \\
&= \sum_{i,j,k \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_j \right) + B''(Z) \left(B(Z) g_i, B(Z) g_j \right) \right] g_k I_{(i,j,k),t_0,t} \\
&= \frac{1}{6} \sum_{\substack{i,j,k \in \mathcal{J} \\ i \neq j, i \neq k, j \neq k}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_j \right) + B''(Z) \left(B(Z) g_i, B(Z) g_j \right) \right] g_k \left[I_{(i,j,k),t_0,t} + I_{(j,i,k),t_0,t} \right. \\
&\quad \left. + I_{(j,k,i),t_0,t} + I_{(i,k,j),t_0,t} + I_{(k,j,i),t_0,t} + I_{(k,i,j),t_0,t} \right] \quad (86) \\
&+ \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) \right] g_j \left[I_{(i,j,i),t_0,t} + I_{(j,i,i),t_0,t} + I_{(i,i,j),t_0,t} \right] \\
&+ \sum_{i \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) \right] g_i I_{(i,i,i),t_0,t}
\end{aligned}$$

and (82)–(83) hence show

$$\begin{aligned}
& \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) dW_v \right) dW_u \right) dW_s + \frac{1}{2} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) dW_u, \int_{t_0}^s B(Z) dW_u \right) dW_s \\
&= \frac{1}{6} \sum_{\substack{i,j,k \in \mathcal{J} \\ i \neq j, i \neq k, j \neq k}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_j \right) + B''(Z) \left(B(Z) g_i, B(Z) g_j \right) \right] g_k I_{(i),t_0,t} I_{(j),t_0,t} I_{(k),t_0,t} \\
&+ \frac{1}{2} \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) \right] g_j I_{(j),t_0,t} \left(I_{(i),t_0,t}^2 - (t - t_0) \right) \quad (87) \\
&+ \frac{1}{6} \sum_{i \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) \right] g_i \left(I_{(i),t_0,t}^2 - 3(t - t_0) \right) I_{(i),t_0,t}
\end{aligned}$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. This finally yields

$$\begin{aligned}
& \int_{t_0}^t B'(Z) \left(\int_{t_0}^s B'(Z) \left(\int_{t_0}^u B(Z) dW_v \right) dW_u \right) dW_s + \frac{1}{2} \int_{t_0}^t B''(Z) \left(\int_{t_0}^s B(Z) dW_u, \int_{t_0}^s B(Z) dW_u \right) dW_s \\
&= \frac{1}{6} \sum_{i,j,k \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_j \right) + B''(Z) \left(B(Z) g_i, B(Z) g_j \right) \right] g_k I_{(i),t_0,t} I_{(j),t_0,t} I_{(k),t_0,t} \\
&- \frac{(t - t_0)}{2} \sum_{i,j \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) \right] g_j I_{(j),t_0,t} \\
&= \frac{1}{6} B'(Z) \left(B'(Z) \left(B(Z) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \quad (88) \\
&+ \frac{1}{6} B''(Z) \left(B(Z) (W_t - W_{t_0}), B(Z) (W_t - W_{t_0}) \right) (W_t - W_{t_0}) \\
&- \frac{(t - t_0)}{2} \sum_{i \in \mathcal{J}} \left[B'(Z) \left(B'(Z) \left(B(Z) g_i \right) g_i \right) (W_t - W_{t_0}) + B''(Z) \left(B(Z) g_i, B(Z) g_i \right) (W_t - W_{t_0}) \right]
\end{aligned}$$

\mathbb{P} -a.s. for all $\mathcal{F}_{t_0}/\mathcal{B}(H)$ -measurable mappings $Z: \Omega \rightarrow H$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$.

4.4 Proof of Lemma 3

First of all, note that

$$\begin{aligned}
& \mathbb{E} \left[\left\langle v, \int_{t_0}^t (W_{s_1} - W_{t_0}) ds_1 \right\rangle_U \left\langle w, \int_{t_0}^t (W_{s_2} - W_{t_0}) ds_2 \right\rangle_U \right] \\
&= \int_{t_0}^t \int_{t_0}^t \mathbb{E} \left[\left\langle v, (W_{s_1} - W_{t_0}) \right\rangle_U \left\langle w, (W_{s_2} - W_{t_0}) \right\rangle_U \right] ds_1 ds_2 \\
&= \int_{t_0}^t \int_{t_0}^t \left\langle v, (\min(s_1, s_2) - t_0) Qw \right\rangle_U ds_1 ds_2 \\
&= \int_{t_0}^t \int_{t_0}^{s_2} \left\langle v, s_1 Qw \right\rangle_U ds_1 ds_2 + \int_{t_0}^t \int_{s_2}^t \left\langle v, s_2 Qw \right\rangle_U ds_1 ds_2 - \int_{t_0}^t \int_{t_0}^t \left\langle v, t_0 Qw \right\rangle_U ds_1 ds_2 \\
&= \left\langle v, \frac{1}{3}(t - t_0)^3 Qw \right\rangle_U
\end{aligned} \tag{89}$$

for all $v, w \in U$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. Moreover, observe that

$$\begin{aligned}
& \mathbb{E} \left[\left\langle v, (W_t - W_{t_0}) \right\rangle_U \left\langle w, \int_{t_0}^t (W_s - W_{t_0}) ds \right\rangle_U \right] \\
&= \int_{t_0}^t \mathbb{E} \left[\left\langle v, (W_t - W_{t_0}) \right\rangle_U \left\langle w, (W_s - W_{t_0}) \right\rangle_U \right] ds = \int_{t_0}^t \left\langle v, (s - t_0) Qw \right\rangle_U ds \\
&= \left\langle v, \frac{1}{2}(t - t_0)^2 Qw \right\rangle_U
\end{aligned} \tag{90}$$

for all $v, w \in U$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$. Combining (89) and (90) then yields

$$\begin{aligned}
& \mathbb{E} \left[\begin{pmatrix} \left\langle v, (W_t - W_{t_0}) \right\rangle_U \\ \left\langle v, \int_{t_0}^t (W_s - W_{t_0}) ds \right\rangle_U \end{pmatrix} \begin{pmatrix} \left\langle w, (W_t - W_{t_0}) \right\rangle_U \\ \left\langle w, \int_{t_0}^t (W_s - W_{t_0}) ds \right\rangle_U \end{pmatrix}^* \right] \\
&= \begin{pmatrix} \left\langle v, (t - t_0) Qw \right\rangle_U & \left\langle v, \frac{1}{2}(t - t_0)^2 Qw \right\rangle_U \\ \left\langle v, \frac{1}{2}(t - t_0)^2 Qw \right\rangle_U & \left\langle v, \frac{1}{3}(t - t_0)^3 Qw \right\rangle_U \end{pmatrix}
\end{aligned} \tag{91}$$

for all $v, w \in U$ and all $t_0, t \in [0, T]$ with $t_0 \leq t$.

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