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Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations

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Abstract

Recently, Hairer et. al [14] showed that there exist SDEs with infinitely often differentiable and globally bounded coefficient functions whose solutions fail to be locally Lipschitz continuous in the strong L^p-sense with respect to the initial value for every $p \in [1, \infty]$. In this article we provide sufficient conditions on the coefficient functions of the SDE and on $p \in (0, \infty]$ which ensure local Lipschitz continuity in the strong L^p-sense with respect to the initial value and we establish explicit estimates for the local Lipschitz continuity constants. In particular, we prove local Lipschitz continuity in the initial value for several nonlinear SDEs from the literature such as the stochastic van der Pol oscillator, Brownian dynamics, the Cox-Ingersoll-Ross processes and the Cahn-Hilliard-Cook equation. As an application of our estimates, we obtain strong completeness for several nonlinear SDEs.

Contents

1 Introduction

Let $d, m \in \mathbb{N}$, let $O \subseteq \mathbb{R}^d$ be an open set, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$ be continuous functions and let $X^x\colon [0,\infty)\times\Omega \to O$, $x \in O$, be adapted stochastic processes with continuous sample paths which solve the stochastic differential equation (SDE)

$$X_t^x = x + \int_0^t \mu(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dW_s \tag{1}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$ and all $x \in O$.

An essential question in stochastic analysis is regularity of solution processes of the SDE (1) in the initial value. In this article, our main objective are sufficient conditions on μ, σ and $t, p \in (0,\infty)$ which ensure that the function $O \ni x \mapsto X_t^x \in L^p(\Omega;\mathbb{R}^d)$ is *locally Lipschitz continuous*. Especially, for every $t, p \in (0,\infty)$, our goal is a continuous function $\varphi_{t,p}\colon O^2 \to [0,\infty)$ such that for all $x, y \in O$ it holds that

$$\|X_t^x - X_t^y\|_{L^p(\Omega;\mathbb{R}^d)} \leq \varphi_{t,p}(x,y)\,\|x-y\|\,. \tag{2}$$

In addition, we want the functions $\varphi_{t,p}$, $t,p \in (0,\infty)$, to be as small and as explicit as possible. A well-known sufficient condition for (2) with $p = 2$ is the *global monotonicity* assumption that there exists a $c \in \mathbb{R}$ such that for all $x, y \in O$ it holds that

$$\langle x-y, \mu(x)-\mu(y)\rangle + \tfrac{1}{2}\,\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,\|x-y\|^2\,. \tag{3}$$

In that case, inequality (2) is satisfied with $p = 2$ and $\varphi_{t,2}(x,y) = e^{ct}$ for all $t \in (0,\infty)$ and all $x, y \in O$ (see, e.g., Assumption (H2) and Proposition 4.2.10 in Prévôt & Röckner [37]). Thus the global monotonicity property (3) implies for all $t \in [0,\infty)$ global Lipschitz continuity of the function $O \ni x \mapsto X_t^x \in L^2(\Omega;\mathbb{R}^d)$. Unfortunately, the coefficient functions of the majority of nonlinear SDEs from applications do not satisfy the global monotonicity assumption (3) (see Section 4 for a selection of examples). It remained an open problem to find conditions on μ, σ and $p \in [2,\infty)$ which are satisfied by most of the nonlinear SDEs from applications and which ensure for all $t \in [0,\infty)$ local Lipschitz continuity of the function $O \ni x \mapsto X_t^x \in L^p(\Omega;\mathbb{R}^d)$. In this article, we partially solve this problem.

In the deterministic case $\sigma \equiv 0$, the solution of (1) is always locally Lipschitz continuous in the initial value if μ is locally Lipschitz continuous. The stochastic case is more subtle than the deterministic case. To emphasize the challenge of the stochastic case, we consider the following example SDE. In the special case $d = 2$, $m = 1$, $D = \mathbb{R}^d$ and $\mu(x_1,x_2) = (x_1x_2, -(x_1)^2)$ for all $(x_1,x_2) \in \mathbb{R}^2$ the SDE (1) reads as

$$dX_t^x = \begin{pmatrix} X_t^{x,1}X_t^{x,2} \\ -(X_t^{x,1})^2 \end{pmatrix} dt + \sigma(X_t^x)\,dW_t \tag{4}$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^2$. Hairer et al. [14] prove that if $\sigma(x) = x$ for all $x \in \mathbb{R}^2$ in (4), then for any $t, p \in (0, \infty)$ the mapping $\mathbb{R}^2 \ni x \mapsto X_t^x \in L^p(\Omega; \mathbb{R}^2)$ is well-defined but not locally Lipschitz continuous. In addition, Theorem 1.2 in Hairer et al. [14] implies that this *loss of regularity phenomenon* can happen even in the case of globally bounded and smooth coefficients. In contrast, Corollary 2.31 below implies for the SDE (4) that if σ in (4) is globally bounded and globally Lipschitz continuous, then for all $t, p \in (0, \infty)$ the mapping $\mathbb{R}^2 \ni x \mapsto X_t^x \in L^p(\Omega; \mathbb{R}^2)$ is locally Lipschitz continuous. More generally, Corollary 2.31 ensures for the SDE (1) that if μ is differentiable with $\lim_{x\to\infty} \|\mu'(x)\|/\|x\|^2 = 0$ and $\limsup_{x\to\infty} \langle x, \mu(x)\rangle/\|x\|^2 < \infty$ and if σ is globally bounded and globally Lipschitz continuous, then for all $t, p \in (0, \infty)$ the mapping $\mathbb{R}^d \ni x \mapsto X_t^x \in L^p(\Omega; \mathbb{R}^d)$ is locally Lipschitz continuous.

It turns out that for some SDEs such as Cox-Ingersoll-Ross processes (Subsection 4.9) or the Cahn-Hilliard-Cook equation with space-time white noise (Subsection 4.12.2), it is appropriate to measure distance with a general function $V \in C^2(O^2, [0, \infty))$ rather than with the squared Euclidean distance $O^2 \ni (x, y) \mapsto \|x - y\|^2 \in [0, \infty)$. Then Itô's formula implies that $dV(X_t^x, X_t^y) = (\overline{\mathcal{G}}_{\mu,\sigma} V)(X_t^x, X_t^y)\, dt + (\overline{G}_\sigma V)(X_t^x, X_t^y)\, dW_t$ for all $t \in [0, \infty)$, $x, y \in O$ where the linear operators $\overline{\mathcal{G}}_{\mu,\sigma} \colon C^2(O^2, \mathbb{R}) \to C(O^2, \mathbb{R})$ (see (1.1) in Maslowski [34] and Ichikawa [23]) and $\overline{G}_\sigma \colon C^2(O^2, \mathbb{R}) \to C(O^2, \mathbb{R}^{1\times m})$ are defined by

$$(\overline{\mathcal{G}}_{\mu,\sigma}\phi)(x, y) := \big(\tfrac{\partial}{\partial x}\phi\big)(x, y)\, \mu(x) + \big(\tfrac{\partial}{\partial y}\phi\big)(x, y)\, \mu(y) + \frac{1}{2}\sum_{i=1}^{m} \big(\tfrac{\partial^2}{\partial x^2}\phi\big)(x, y)\big(\sigma_i(x), \sigma_i(x)\big)$$
$$+ \sum_{i=1}^{m} \big(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\phi\big)(x, y)\big(\sigma_i(x), \sigma_i(y)\big) + \frac{1}{2}\sum_{i=1}^{m} \big(\tfrac{\partial^2}{\partial y^2}\phi\big)(x, y)\big(\sigma_i(y), \sigma_i(y)\big) \tag{5}$$
$$(\overline{G}_\sigma\phi)(x, y) := \big(\tfrac{\partial}{\partial x}\phi\big)(x, y)\, \sigma(x) + \big(\tfrac{\partial}{\partial y}\phi\big)(x, y)\, \sigma(y) \tag{6}$$

for all $x, y \in O$ and all $\phi \in C^2(O^2, \mathbb{R})$. In terms of these operators, we formulate the first main result of this article.

Theorem 1.1. *Assume the above setting and let $t \in (0, \infty)$, $\alpha_0, \alpha_1, \beta_0, \beta_1, c \in [0, \infty)$, $V \in C^2(O^2, [0, \infty))$, $U_0, U_1 \in C^2(O, [0, \infty))$, $\overline{U} \in C(O, [0, \infty))$, $r, p, q_0, q_1 \in (0, \infty]$ with $\frac{1}{r} + \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{p}$ and $(V^{-1})(0) \subseteq (\overline{\mathcal{G}}_{\mu,\sigma} V)^{-1}(0) \cap (\overline{G}_\sigma V)^{-1}(0)$. Moreover, assume*

$$\tfrac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y)}{V(x,y)} + \tfrac{(r-1)\,\|(\overline{G}_\sigma V)(x,y)\|^2}{2\,(V(x,y))^2} \le c + \tfrac{U_0(x)+U_0(y)}{2q_0 t e^{\alpha_0 t}} + \tfrac{\overline{U}(x)+\overline{U}(y)}{2q_1 e^{\alpha_1 t}}, \tag{7}$$
$$U_0'(x)\mu(x) + \tfrac{\operatorname{tr}(\sigma(x)\sigma(x)^*(\operatorname{Hess} U_0)(x))}{2} + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|^2 \le \alpha_0 U_0(x) + \beta_0, \tag{8}$$
$$U_1'(x)\mu(x) + \tfrac{\operatorname{tr}(\sigma(x)\sigma(x)^*(\operatorname{Hess} U_1)(x))}{2} + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_1)(x)\|^2 + \overline{U}(x) \le \alpha_1 U_1(x) + \beta_1 \tag{9}$$

for all $x, y \in O$ with $V(x, y) \neq 0$ and $\sup_{x,y\in K, V(x,y)\neq 0} \frac{\|(\overline{G}_\sigma V)(x,y)\|}{V(x,y)} < \infty$ for all compact sets $K \subseteq O$. Then

$$\|V(X_t^x, X_t^y)\|_{L^p(\Omega;\mathbb{R}^d)} \le V(x, y) \exp\Big(ct + \textstyle\sum_{i=0}^{1} \frac{2\beta_i t + U_i(x) + U_i(y)}{2q_i}\Big) \tag{10}$$

for all $x, y \in O$.

Theorem 1.1 follows immediately from the more general version in Theorem 2.23 in Subsection 2.3.2 below. In addition, the second main result of this article, Theorem 2.29 in Subsection 2.3.3 below, establishes a sufficient condition on μ, σ and $t, p \in (0, \infty)$ which ensures that the function $O \ni x \mapsto X^x|_{[0,t]} \in L^p(\Omega; C([0, t], \mathbb{R}^d))$ is locally Lipschitz continuous, that is, loosely speaking, an estimate such as (2) with supremum over time inside the L^p-norm. Theorem 1.1 implies local Lipschitz continuity in the initial value for all examples in Section 4. In many of these examples in Section 4, the function V in Theorem 1.1 is the squared Euclidean distance, that is, $V(x, y) = \|x - y\|^2$ for all $x, y \in O$. In that case, in the notation of Theorem 1.1, condition (7) reads as

$$\tfrac{2\langle x-y, \mu(x)-\mu(y)\rangle + \|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|x-y\|^2} + \tfrac{2(r-1)\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \le c + \tfrac{U_0(x)+U_0(y)}{2q_0 t e^{\alpha_0 t}} + \tfrac{\overline{U}(x)+\overline{U}(y)}{2q_1 e^{\alpha_1 t}} \tag{11}$$

for all $(x,y) \in O^2$ with $x \neq y$ (see Example 2.15 and Corollary 2.25). For Cox-Ingersoll-Ross processes in Subsection 4.9 and for Wright-Fisher diffusions in Subsection 4.10, we choose V such that $\overline{G}_\sigma V \equiv 0$. This considerably simplifies condition (7) with the cost that in a second step we need to derive a local Lipschitz estimate from inequality (10). Moreover, for the Cahn-Hilliard-Cook equation with space-time white noise in Subsection 4.12.2, we take V to be the squared distance with respect to a non-Euclidean inner product. Also the function $O^2 \ni (x,y) \mapsto \|x-y\|^2 (1 + \|x\|^q + \|y\|^q) \in [0,\infty)$ for some $q \in [2,\infty)$ can be a good choice. We note that in a number of our examples, we could not verify the conditions (8) and (11) with $\bar{U} \equiv 0$. The key to many of our examples is either inequality (11) together with condition (9) with $\bar{U} \not\equiv 0$ (see Subsections 4.1, 4.2, 4.5, 4.11, 4.12.1) or to find a suitable function V which is not the squared Euclidean distance (see Subsections 4.9, 4.10 and 4.12.2).

The method of proof of Theorem 1.1 is to show under suitable assumptions (see Proposition 2.12 for details) that

$$\begin{aligned} &V(X_t^x, X_t^y) \\ &= V(x,y) \exp\!\left(\int_0^t \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(X_s^x, X_s^y)}{V(X_s^x, X_s^y)} \, ds \right) \exp\!\left(\int_0^t \frac{(\overline{G}_\sigma V)(X_s^x, X_s^y)}{V(X_s^x, X_s^y)} \, dW_s - \int_0^t \frac{\|(\overline{G}_\sigma V)(X_s^x, X_s^y)\|^2}{2\,(V(X_s^x, X_s^y))^2} \, ds \right) \end{aligned} \tag{12}$$

$\mathbb{P}$-a.s. for all $t \in (0,\infty)$ and all $x, y \in O$ where $\frac{0}{0} := 0$ and then to estimate the L^p-norm of the right-hand side for each $p \in (0,\infty)$. Now due to condition (7), it suffices to estimate exponential moments. Exponential moments, in turn, are guaranteed by conditions (8) and (9). More precisely, Corollary 2.4 together with conditions (8) and (9) implies that

$$\left\| \exp\!\left(\int_0^t \frac{U_0(X_s^x)}{2 q_0 t e^{\alpha_0 s}} \, ds \right) \right\|_{L^{2q_0}(\Omega;\mathbb{R})} \leq \frac{1}{t} \int_0^t \left\| \exp\!\left(\frac{U_0(X_s^x)}{2 q_0 e^{\alpha_0 s}} \right) \right\|_{L^{2q_0}(\Omega;\mathbb{R})} ds \leq \exp\!\left(\frac{\beta_0 t + U_0(x)}{2 q_0} \right) \tag{13}$$

$$\left\| \exp\!\left(\frac{U_1(X_t^x)}{2 q_1 e^{\alpha_1 t}} + \int_0^t \frac{\bar{U}(X_s^x)}{2 q_1 e^{\alpha_1 s}} \, ds \right) \right\|_{L^{2q_1}(\Omega;\mathbb{R})} \leq \exp\!\left(\frac{\beta_1 t + U_1(x)}{2 q_1} \right) \tag{14}$$

for all $x \in O$ and all $t \in (0,\infty)$. Note that the last exponential factor on the right-hand of (12) is a local exponential martingale so that the global monotonicity assumption (3) immediately implies for all $t \in [0,\infty)$ and all $x, y \in O$ that $\|X_t^x - X_t^y\|_{L^2(\Omega;\mathbb{R})} \leq \|x-y\| e^{ct}$.

There are a number of related results in the literature. The idea of a general function for measuring distance was already used in Theorem 1.2 in Maslowski [34] (cf. also Ichikawa [23] and, e.g., also Leha & Ritter [28, 29]) for studying long-time stability properties of SDEs under the assumption $(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y) \leq 0$ for all $x, y \in O$. The relation (12) with V being the squared Euclidean distance appeared in (14) in Zhang [54] and on page 311 in Taniguchi [48] and in (14) in Li [30] in terms of the derivative in probability (see Definition 4.9 in Krylov [27]) of the mapping $O \ni x \mapsto X^x \in L^0(\Omega;\mathbb{R}^d)$. Building on Taniguchi's equation for the squared norm of the derivative process, Theorem 5.1 and Theorem 3.1 in Li [30] proves a conditional result which implies that if O is a complete connected Riemannian manifold, if there exists an $x \in O$ such that $\mathbb{P}\big[\,\forall\, t \in [0,\infty)\colon X_t^x \in O\,\big] = 1$, if μ and σ are twice continuously differentiable and if there exists a measurable function $f\colon O \to [0,\infty)$ and a $p \in (0,\infty)$ such that for all $x \in O$, $v \in \mathbb{R}^d \setminus \{0\}$ it holds that $2\langle (\nabla\mu)(x)v, v\rangle + \sum_{i=1}^m \|\sigma_i'(x)(v)\|^2 + (p-2)\sum_{i=1}^m \|v\|^{-2} |\langle \sigma_i'(x)(v), v\rangle|^2 \leq 6pf(x)\|v\|^2$ and $\sum_{i=1}^m \|\sigma_i'(x)\|^2_{L(\mathbb{R}^d,\mathbb{R}^d)} \leq f(x)$ and such that for all $t \in (0,\infty)$ and all compact sets $K \subset O$ it holds that $\sup_{x\in K} \mathbb{E}\big[\exp\big(6p^2 \int_0^t f(X_s^x)\, \mathbb{1}_{\cap_{r\in[0,s]}\{X_r^x \in O\}}\, ds\big)\big] < \infty$, then for all $t \in (0,\infty)$ the mapping $O \ni x \mapsto X^x \in L^p(\Omega; C([0,t],O))$ is locally Lipschitz continuous (cf. also Corollary 2.28 below). In addition, Lemma 6.1 in Li [30] derives inequality (13) from inequality (8) in the case $\alpha_0 = 0$ and $O = \mathbb{R}^d$. Moreover, Theorem 6.2 in Li [30] proves inequality (8) with $\alpha_0 = 0$ and with $U_0(x) = \ln(1 + \|x\|^2)$ for all $x \in \mathbb{R}^d$ under a global log-Lipschitz type condition (see Li [30] for details). In addition, Corollary 6.3 in Li [30] and Theorem 1.7 in Fang, Imkeller and Zhang [12] prove locally Lipschitz continuity results under appropriate at most quadratic growth assumptions on μ, μ', σ and σ' (see Corollary 2.31 below for details). Furthermore, Lemma 2.3

in Zhang [54] implies that if $O = \mathbb{R}^d$, if $c \in (0,\infty)$ is a real number and if $U_0 \in C^2(\mathbb{R}^d, [1,\infty))$ is a function such that for all $x, y \in \mathbb{R}^d$ it holds that $U_0'(x)\mu(x) + \frac{\operatorname{tr}(\sigma(x)(\sigma(x))^*(\operatorname{Hess} U_0)(x))}{2} \leq c\, U_0(x)$, $\|\sigma(x)^*(\nabla U_0)(x)\|^2 \leq c\, U_0(x)$, $\langle x - y, \mu(x) - \mu(y)\rangle \leq c\,(U_0(x) + U_0(y))\,\|x - y\|^2$ and $\|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(U_0(x) + U_0(y))\,\|x - y\|^2$, then for all $p \in [2,\infty)$ there exists a $t \in (0,\infty)$ such that the mapping $\mathbb{R}^d \ni x \mapsto X^x|_{[0,t]} \in L^p(\Omega; C([0,t];\mathbb{R}^d))$ is locally Lipschitz continuous. In particular, Lemma 2.3 in Zhang [54] yields local Lipschitz continuity in the initial value for sufficiently small positive time points for the stochastic van der Pol oscillator in the case of globally bounded noise (Subsection 4.1), for the stochastic Duffing-van der Pol oscillator in the case of globally bounded noise (see Subsection 4.2), for the stochastic Lorenz equation with additive noise (see Subsection 4.3), for the Langevin dynamics under certain assumptions (see Subsection 4.4), for a model from experimental psychology (see Subsection 4.7) and for the stochastic Brusselator under certain assumptions (see Subsection 4.8). Moreover, local Lipschitz continuity in the initial value in the L^p-norm for any $p \in (0,\infty)$ and any time point for the vorticity formulation of the two-dimensional stochastic Navier-Stokes equations follows from Lemma 4.10 in Hairer & Mattingly [15]. Lemma 4.10 in Hairer & Mattingly [15] also includes an inequality similar to (14) in the case where U_1 is the squared Hilbert space norm for the vorticity formulation of the two-dimensional stochastic Navier-Stokes equations. Further instructive results on exponential moments can be found, for example, in [6, 11, 12, 17]. Theorem 1.1 in this article implies local Lipschitz continuity in the initial value for all $t, p \in (0,\infty)$ for the stochastic van der Pol oscillator with unbounded noise (Subsection 4.1), for the stochastic Duffing-van der Pol oscillator with unbounded noise (Subsection 4.2), for the overdamped Langevin dynamics under certain assumptions (Subsection 4.5), for the stochastic SIR model (Subsection 4.6), for Cox-Ingersoll-Ross processes, for the Ait-Sahalia interest rate model, for Heston's 3/2-volatility, for constant elasticity of variance processes (Subsection 4.9), for Wright-Fisher diffusions (Subsection 4.10), for the stochastic Burgers equation with a globally bounded diffusion coefficient (Subsection 4.11) and for the Cahn-Hilliard-Cook equation (Subsection 4.12). Note that in the case of stochastic partial differential equations (SPDEs), we first apply Theorem 1.1 to spatial discretizations of the considered SPDE and then apply Fatou's lemma.

We sketch three applications of Theorem 1.1. First, Theorem 1.1 and its uniform counterpart in Theorem 2.29 respectively can be applied to establish *strong completeness* of the SDE (1). More precisely, we show in Lemma 3.1 in Subsection 3 that if there exist a $p \in (d,\infty)$ and an $\varepsilon \in (0,1)$ such that $\bar{O} \ni x \mapsto (X^x_t)_{t\in[0,\varepsilon]} \in L^p(\Omega; C([0,\varepsilon], \bar{O}))$ is *locally Lipschitz continuous*, then the SDE (1) is *strongly complete* (we assume here that X, μ and σ are continuously extended to $\bar{O}$ in an appropriate way; see Lemma 3.1 for the precise assumptions and, e.g., Subsection 4.6 for the application of Lemma 3.1 to an SDE on a domain which is not equal to $\mathbb{R}^d$) and, thus, there exists a mapping $Z\colon \Omega \to C([0,\infty) \times \bar{O}, \bar{O})$ such that Z^x, $x \in \bar{O}$, solve (1). Then combining Lemma 3.1 with Theorem 2.29 yields strong completeness for all finite-dimensional examples in Section 4; see Section 4 for the precise assumptions. We emphasize that strong completeness may fail to hold even in the case of smooth and globally bounded coefficient functions; see Li & Scheutzow [31].

Secondly, a local Lipschitz estimate such as (2) is an important tool for proving strong and weak convergence rates of *numerical approximations* to the solution processes of the SDE (1). In the literature strong and weak convergence rates for numerical approximation processes for multidimensional SDEs are (except of in the case of Dörsek's insightful work [9]; see Corollary 3.2 in [9]) in general only known under the global monotonicity condition (3); see, e.g., [20, 18, 22, 33, 25] and the references mentioned therein. Inequality (2) and Theorem 1.1 respectively now allow us to establish strong and weak convergence rates for numerical approximation processes of SDEs which fail to satify the global monotonicity condition (3). More formally, let $T \in (0,\infty)$ and let $\hat{X}^{s,x}\colon [s,T] \times \Omega \to O$, $s \in [0,T]$, $x \in O$, be solution processes of $d\hat{X}^{s,x}_t = \mu(\hat{X}^{s,x}_t)\,dt + \sigma(\hat{X}^{s,x}_t)\,dW_t$ and $\hat{X}^{s,x}_s = x$ for $t \in [s,T]$, $s \in [0,T]$, $x \in O$, and for every $h \in (0,T]$ let $Y^{x,h}\colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in O$, be a family of one-step numerical approximation stochastic processes for the SDE (1) with step size $h \in (0,T]$ (cf., e.g., Section 2.1.4 in [21]). Heuristically speaking, the exact solution is the best approximation process so that for estimating the quantity $\|X_T^x - Y_T^{x,h}\|_{L^2(\Omega;\mathbb{R}^d)}$ for $x \in O$ and small $h \in (0,T]$ we need to estimate at least the quantity $\big\|X_T^x - \hat{X}_T^{h,Y_h^{x,h}}\big\|_{L^2(\Omega;\mathbb{R}^d)} = \big\|\hat{X}_T^{h,X_h^x} - \hat{X}_T^{h,Y_h^{x,h}}\big\|_{L^2(\Omega;\mathbb{R}^d)}$ for $x \in O$ and small $h \in (0,T]$. But this can be done with a local Lipschitz estimate such as (2) together with estimates on the one-step approximation errors $\|X_h^x - Y_h^{h,x}\|_{L^p(\Omega;\mathbb{R}^d)}$, $x \in O$, $h \in (0,T]$, $p \in (2,\infty)$, and together with suitable a priori estimates on the approximation processes (see, e.g., Section 2 in [21]). The detailed analysis of strong and weak convergence rates for numerical approximation processes based on (2) and Theorem 1.1 respectively will be the subject of future research articles.

A third application of Theorem 1.1 are moment bounds on the derivative process. If the coefficient functions μ and σ are continuously differentiable, then Theorem 4.10 in Krylov [27] shows that there exist stochastic processes $D^x\colon [0,\infty)\times\Omega \to \mathbb{R}^{d\times d}$, $x \in O$, such that for all $y \in O$, $t \in [0,\infty)$ it holds that $\sup_{s\in[0,t]} \|X_s^x - X_s^y - D_s^y\,(x-y)\|/\|x-y\| + \sup_{s\in[0,t]} \|D_s^x - D_s^y\|_{L(\mathbb{R}^d)} \to 0$ in probability as $y \neq x \to y$. So, if μ and σ are continuously differentiable and if inequality (2) holds, then dividing by $\|x-y\| \in (0,\infty)$ in (2) and applying Fatou's lemma (cf., e.g., Lemma 3.10 in [21]) immediately implies that $\|D_t^y\|_{L^p(\Omega;\mathbb{R}^{d\times d})} \leq d\,\sup_{v\in R^d,\|v\|\leq 1}\|D_t^y v\|_{L^p(\Omega;\mathbb{R}^d)} \leq d\,\varphi_{t,p}(y,y)$ for all $y \in O$ and with $t,p \in (0,\infty)$ as in inequality (2).

1.1 Notation

Throughout this article we use the following notation. For $d,m \in \mathbb{N} := \{1,2,\dots\}$, $v = (v_1,\dots,v_d) \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times m}$ we denote by $\|v\| := \big[|v_1|^2 + \ldots + |v_d|^2\big]^{1/2}$ and $\|A\| := \|A\|_{L(\mathbb{R}^m,\mathbb{R}^d)} = \sup_{v\in\mathbb{R}^m\setminus\{0\}}\big[\|Av\|/\|v\|\big]$ the Euclidean norm of v and the Euclidean operator norm of A respectively and we denote by A^* the transposed matrix of A. For two sets A and B we denote by $\mathcal{M}(A,B)$ the set of all mappings from A to B. For two measurable spaces $(A,\mathcal{A})$ and $(B,\mathcal{B})$ we denote by $\mathcal{L}^0(A;B)$ the set of all $\mathcal{A}/\mathcal{B}$-measurable mappings from A to B. For two normed vector spaces $(V_1,\|\cdot\|_{V_1})$ and $(V_2,\|\cdot\|_{V_2})$ satisfying $V_1 \neq \{0\}$ and a function $f\colon V_1 \to V_2$ from V_1 to V_2, we define $\|f\|_{\mathrm{Lip}(V_1,V_2)} := \sup_{v,w\in V_1, v\neq w}\frac{\|f(v)-f(w)\|_{V_2}}{\|v-w\|_{V_1}}$. If $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, if $I \subseteq \mathbb{R}$ is a closed and non-empty interval and if $(\mathcal{F}_t)_{t\in I}$ is a normal filtration on $(\Omega,\mathcal{F},\mathbb{P})$, then we call the quadrupel $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in I})$ a *stochastic basis* (see, e.g., Appendix E in Prévôt & Röckner [37]). For $d,m \in \mathbb{N}$, an open set $O \subseteq \mathbb{R}^d$ and two functions $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$, we define linear operators $\mathcal{G}_{\mu,\sigma}\colon C^2(O,\mathbb{R}) \to \mathcal{M}(O,\mathbb{R})$, $G_\sigma\colon C^1(O,\mathbb{R}) \to \mathcal{M}(O,\mathbb{R}^{1\times m})$, $\overline{\mathcal{G}}_{\mu,\sigma}\colon C^2(O^2,\mathbb{R}) \to \mathcal{M}(O^2,\mathbb{R})$ and $\overline{G}_\sigma\colon C^1(O^2,\mathbb{R}) \to \mathcal{M}(O^2,\mathbb{R}^{1\times m})$ by

$$(\mathcal{G}_{\mu,\sigma}\phi)(x) := \phi'(x)\,\mu(x) + \tfrac{1}{2}\operatorname{tr}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}\phi)(x)\big), \tag{15}$$

$$(G_\sigma\psi)(x) := \psi'(x)\,\sigma(x), \tag{16}$$

$$\begin{aligned}(\overline{\mathcal{G}}_{\mu,\sigma}\Phi)(x,y) &:= \big(\tfrac{\partial}{\partial x}\Phi\big)(x,y)\,\mu(x) + \big(\tfrac{\partial}{\partial y}\Phi\big)(x,y)\,\mu(y) + \frac{1}{2}\sum_{i=1}^m \big(\tfrac{\partial^2}{\partial x^2}\Phi\big)(x,y)\big(\sigma_i(x),\sigma_i(x)\big) \\ &\quad + \sum_{i=1}^m \big(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\Phi\big)(x,y)\big(\sigma_i(x),\sigma_i(y)\big) + \frac{1}{2}\sum_{i=1}^m \big(\tfrac{\partial^2}{\partial y^2}\Phi\big)(x,y)\big(\sigma_i(y),\sigma_i(y)\big),\end{aligned} \tag{17}$$

$$(\overline{G}_\sigma\Psi)(x,y) := \big(\tfrac{\partial}{\partial x}\Psi\big)(x,y)\,\sigma(x) + \big(\tfrac{\partial}{\partial y}\Psi\big)(x,y)\,\sigma(y) \tag{18}$$

for all $x \in O$, $\phi \in C^2(O,\mathbb{R})$, $\psi \in C^1(O,\mathbb{R})$, $\Phi \in C^2(O^2,\mathbb{R})$, $\Psi \in C^1(O^2,\mathbb{R})$. We call the linear operator $\mathcal{G}_{\mu,\sigma}$ defined in (15) *generator*, we call the linear operator G_σ defined in (16) *noise operator*, we call the linear operator $\overline{\mathcal{G}}_{\mu,\sigma}$ defined in (17) *extended generator* and we call the linear operator $\overline{G}_\sigma$ defined in (18) *extended noise operator*. The extended generator has been exploited in Ichikawa [23] and Maslowski [34] (see, e.g., also Leha & Ritter [28, 29]) Whereas these references rely on the extended generator, in our analysis below both the extended generator and the extended diffusion operator play an essential role. Throughout this article we also often calculate and formulate expressions in the extended real numbers $[-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}$. In particular, we frequently use the conventions $\frac{0}{0} = 0 \cdot \infty = 0$, $0^0 = 1$, $\frac{a}{0} = \infty$, $\frac{-a}{0} = -\infty$, $0^{-a} = \frac{1}{0^a} = \infty$, $\frac{b}{\infty} = 0^a = 0$ for all $a \in (0,\infty)$ and all $b \in \mathbb{R}$ and $\sup(\emptyset) = -\infty$. Moreover, if $m \in \mathbb{N}$, $T \in (0,\infty)$, if $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ is a stochastic basis, if $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ is a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion and if $X\colon [0,T]\times\Omega \to [-\infty,\infty]^{d\times m}$ is an adapted and product measurable stochastic process with $\int_0^T \|X_s\|^2\,ds < \infty$ $\mathbb{P}$-a.s., then we define $\int_0^T X_s\,dW_s \in L^2(\Omega;\mathbb{R}^d)$ by $\int_0^T X_s\,dW_s := \int_0^T \mathbb{1}_{\{X_s\in\mathbb{R}^{d\times m}\}} X_s\,dW_s$ $\mathbb{P}$-a.s. Furthermore, we define $x\wedge y := \min(x,y)$ and $x \vee y := \max(x,y)$ for all $x,y \in \mathbb{R}$. Finally, for two sets A, B and a function $f\colon A \to B$ we denote by $\operatorname{im}(f) = \{y \in B\colon (\exists\, x \in A\colon f(x) = y)\}$ the image of f.

1.2 Setting

Throughout this article we will frequently use the following setting. Let $d, m \in \mathbb{N}$, $T \in (0,\infty)$, let $O \subseteq \mathbb{R}^d$ be an open set, let $\mu \in \mathcal{L}^0(O;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(O;\mathbb{R}^{d\times m})$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion.

2 Strong stability analysis for solutions of SDEs

The main results of this section are Theorem 2.23 and Theorem 2.29 below which establish marginal and uniform strong stability estimates respectively.

2.1 Exponential integrability bounds for solutions of SDEs

The main result of this subsection, Proposition 2.3 below, establishes certain exponential integrability properites for solutions of SDEs. Further instructive results on exponential moments can, for example, be found in [15, 6, 11, 12, 17]. For the proof of Proposition 2.3, we first present two well-known auxiliary lemmas.

Lemma 2.1. *Let $T \in [0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $Z\colon [0,T]\times\Omega \to \mathbb{R}$ be a product measurable stochastic process with $\int_0^T \max(Z_t,0)\,dt < \infty$ $\mathbb{P}$-a.s. and $Z_t \geq 0$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$. Then $\int_0^T Z_t\,dt \geq 0$ $\mathbb{P}$-a.s.*

Proof of Lemma 2.1. Note that

$$0 \leq \mathbb{E}\big[\textstyle\int_0^T \max(Z_t,0)\,dt - \int_0^T Z_t\,dt\big] = \mathbb{E}\big[\int_0^T \max(Z_t,0) - Z_t\,dt\big] = \int_0^T \mathbb{E}[\max(Z_t,0) - Z_t]\,dt = 0$$

and hence that $0 \leq \int_0^T \max(Z_t,0)\,dt = \int_0^T Z_t\,dt$ $\mathbb{P}$-a.s. This finishes the proof of Lemma 2.1. ☐

For convenience of the reader, we recall the following well-known Lyapunov estimate (cf., e.g., the proof of Lemma 2.2 in Gyöngy & Krylov [13]).

Lemma 2.2 (A Lyapunov estimate)**.** *Assume the setting in Section 1.2, let $V \in C^{1,2}([0,T]\times O,[0,\infty))$, $\alpha \in \mathcal{L}^0([0,T];[0,\infty))$ with $\int_0^T \alpha(t)\,dt < \infty$, let $\tau\colon \Omega \to [0,T]$ be a stopping time and let $X\colon [0,T]\times\Omega \to O$ be an adapted stochastic process with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s)\| + \|\sigma(X_s)\|^2\,ds < \infty$ $\mathbb{P}$-a.s.,*

$$\begin{aligned}&\big(\tfrac{\partial}{\partial t}V\big)(t\wedge\tau, X_{t\wedge\tau}) + \big(\tfrac{\partial}{\partial x}V\big)(t\wedge\tau, X_{t\wedge\tau})\,\mu(X_{t\wedge\tau})\\ &+ \tfrac{1}{2}\operatorname{tr}\big(\sigma(X_{t\wedge\tau})\sigma(X_{t\wedge\tau})^*(\operatorname{Hess}_x V)(t\wedge\tau, X_{t\wedge\tau})\big) \leq \alpha(t\wedge\tau)\,V(t\wedge\tau, X_{t\wedge\tau})\end{aligned} \tag{19}$$

$\mathbb{P}$-a.s. and $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau}\mu(X_s)\,ds + \int_0^{t\wedge\tau}\sigma(X_s)\,dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Then $\mathbb{E}\big[V(\tau,X_\tau)\big] \leq \exp\big(\int_0^T \alpha(s)\,ds\big)\,\mathbb{E}\big[V(0,X_0)\big] \in [0,\infty]$.

Proof of Lemma 2.2. First, we assume w.l.o.g. that $\mathbb{E}[V(0, X_0)] < \infty$. Next define stopping times $\rho_n : \Omega \to [0, T]$, $n \in \mathbb{N}$, by

$$\rho_n := \inf\big(\{\tau\} \cup \big\{t \in [0, T] \colon \sup_{s\in[0,t]} V(s, X_s) + \textstyle\int_0^t \|\sigma(X_s)^*(\nabla_x V)(s, X_s)\|^2 \, ds \geq n\big\}\big)$$

for all $n \in \mathbb{N}$. Then note that Itô's formula proves that

$$V(t \wedge \rho_n, X_{t\wedge\rho_n}) = V(0, X_0) + \int_0^{t\wedge\rho_n} \big(\tfrac{\partial}{\partial x}V\big)(s, X_s)\, \sigma(X_s)\, dW_s \tag{20}$$
$$+ \int_0^{t\wedge\rho_n} \big(\tfrac{\partial}{\partial s}V\big)(s, X_s) + \big(\tfrac{\partial}{\partial x}V\big)(s, X_s)\, \mu(X_s) + \tfrac{1}{2}\mathrm{tr}\big(\sigma(X_s)\sigma(X_s)^*(\mathrm{Hess}_x V)(s, X_s)\big)\, ds$$

$\mathbb{P}$-a.s. for all $(t, n) \in [0, T] \times \mathbb{N}$ and assumption (19) and Lemma 2.1 hence imply that

$$V(t \wedge \rho_n, X_{t\wedge\rho_n}) \leq V(0, X_0) + \int_0^{t\wedge\rho_n} \big(\tfrac{\partial}{\partial x}V\big)(s, X_s)\, \sigma(X_s)\, dW_s + \int_0^{t\wedge\rho_n} \alpha(s)\, V(s, X_s)\, ds \tag{21}$$

$\mathbb{P}$-a.s. for all $(t, n) \in [0, T] \times \mathbb{N}$. Taking expectations then shows that

$$\begin{aligned} \mathbb{E}\big[V(t \wedge \rho_n, X_{t\wedge\rho_n})\big] &\leq \mathbb{E}\big[V(0, X_0)\big] + \int_0^t \alpha(s)\, \mathbb{E}\big[\mathbb{1}_{\{s\leq\rho_n\}} V(s, X_s)\big]\, ds \\ &\leq \mathbb{E}\big[V(0, X_0)\big] + \int_0^t \alpha(s)\, \mathbb{E}\big[V(s \wedge \rho_n, X_{s\wedge\rho_n})\big]\, ds \end{aligned} \tag{22}$$

for all $(t, n) \in [0, T] \times \mathbb{N}$. The estimate $\mathbb{E}\big[V(t \wedge \rho_n, X_{t\wedge\rho_n})\big] \leq n + \mathbb{E}\big[V(0, X_0)\big] < \infty$ for all $(t, n) \in [0, T] \times \mathbb{N}$ and Gronwall's lemma therefore yield $\mathbb{E}\big[V(t \wedge \rho_n, X_{t\wedge\rho_n})\big] \leq e^{\int_0^t \alpha(s)\, ds}\, \mathbb{E}\big[V(0, X_0)\big]$ for all $(t, n) \in [0, T] \times \mathbb{N}$. This and Fatou's lemma complete the proof of Lemma 2.2. $\square$

The next proposition proves exponential integrability properites for solution processes of SDEs.

Proposition 2.3 (Exponential integrability properties)**.** *Assume the setting in Section 1.2, let $U \in C^{1,2}([0, T] \times O, \mathbb{R})$, $\overline{U} \in \mathcal{L}^0([0, T] \times O; \mathbb{R})$, let $\tau : \Omega \to [0, T]$ be a stopping time and let $X : [0, T] \times \Omega \to O$ be an adapted stochastic process with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s)\| + \|\sigma(X_s)\|^2 + |\overline{U}(s, X_s)|\, ds < \infty$ $\mathbb{P}$-a.s., $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \mu(X_s)\, ds + \int_0^{t\wedge\tau} \sigma(X_s)\, dW_s$ $\mathbb{P}$-a.s. for all $t \in [0, T]$ and*

$$(\tfrac{\partial}{\partial t}U)(t, x) + (\tfrac{\partial}{\partial x}U)(t, x)\, \mu(x) + \tfrac{\mathrm{tr}(\sigma(x)\,\sigma(x)^*(\mathrm{Hess}_x U)(t,x)) + \|\sigma(x)^*(\nabla_x U)(t,x)\|^2}{2} \leq -\overline{U}(t, x) \tag{23}$$

for all $(t, x) \in \cup_{\omega\in\Omega}\{(s, X_s(\omega)) \in [0, T] \times O \colon s \in [0, \tau(\omega)]\}$. Then

$$\mathbb{E}\Big[e^{r\left[U(\tau, X_\tau) + \int_0^\tau \overline{U}(s, X_s) + \frac{(1-r)}{2}\|\sigma(X_s)^*(\nabla_x U)(s, X_s)\|^2\, ds\right]}\Big] \leq \mathbb{E}\big[e^{r\, U(0, X_0)}\big] \in [0, \infty] \tag{24}$$

for all $r \in [0, \infty)$.

Proof of Proposition 2.3. First of all, let $r \in [0, \infty)$ and define $\bar{V} : [0, T] \times O \times \mathbb{R} \to \mathbb{R}$ by $\bar{V}(t, x, y) = \exp(r\, [U(t, x) + y])$ for all $(t, x, y) \in [0, T] \times O \times \mathbb{R}$. Then note that assumption (23) implies that

$$\begin{aligned} &(\tfrac{\partial}{\partial t}\bar{V})(t, x, y) + (\tfrac{\partial}{\partial x}\bar{V})(t, x, y)\, \mu(x) + \tfrac{1}{2}\,\mathrm{tr}(\sigma(x)\sigma(x)^*(\mathrm{Hess}_x \bar{V})(t, x, y)) \\ &+ (\tfrac{\partial}{\partial y}\bar{V})(t, x, y)\, \Big[\overline{U}(t, x) + \tfrac{(1-r)}{2}\|\sigma(x)^*(\nabla_x U)(t, x)\|^2\Big] \\ &= r\bar{V}(t, x, y)\Big[(\tfrac{\partial}{\partial t}U)(t, x) + (\tfrac{\partial}{\partial x}U)(t, x)\, \mu(x) + \overline{U}(t, x) + \tfrac{(1-r)}{2}\|\sigma(x)^*(\nabla_x U)(t, x)\|^2 \\ &\qquad + \tfrac{1}{2}\,\mathrm{tr}(\sigma(x)\,\sigma(x)^*(\mathrm{Hess}_x U)(t, x)) + \tfrac{r}{2}\|\sigma(x)^*(\nabla_x U)(t, x)\|^2\Big] \\ &= r\bar{V}(t, x, y)\Big[(\tfrac{\partial}{\partial t}U)(t, x) + (\tfrac{\partial}{\partial x}U)(t, x)\, \mu(x) + \overline{U}(t, x) \\ &\qquad + \tfrac{1}{2}\,\mathrm{tr}(\sigma(x)\,\sigma(x)^*(\mathrm{Hess}_x U)(t, x)) + \tfrac{1}{2}\|\sigma(x)^*(\nabla_x U)(t, x)\|^2\Big] \leq 0 \end{aligned} \tag{25}$$

for all $(t,x,y) \in \cup_{\omega\in\Omega}\{(s,X_s(\omega)) \in [0,T]\times O\colon s\in[0,\tau(\omega)]\}\times\mathbb{R}$. Next let $Y\colon [0,T]\times\Omega\to\mathbb{R}$ be an adapted stochastic process with continuous sample paths satisfying $Y_t = \int_0^{t\wedge\tau} \overline{U}(s,X_s) + \frac{(1-r)}{2}\|\sigma(X_s)^*(\nabla_x U)(s,X_s)\|^2\,ds$ $\mathbb{P}$-a.s. for all $t\in[0,T]$. Then we get from (25) that

$$\begin{aligned}&(\tfrac{\partial}{\partial t}\bar{V})(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau}) + (\tfrac{\partial}{\partial x}\bar{V})(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau})\,\mu(X_{t\wedge\tau})\\ &+(\tfrac{\partial}{\partial y}\bar{V})(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau})\left[\overline{U}(t\wedge\tau,X_{t\wedge\tau}) + \tfrac{(1-r)}{2}\|\sigma(X_{t\wedge\tau})^*(\nabla_x U)(t\wedge\tau,X_{t\wedge\tau})\|^2\right]\\ &+\tfrac{1}{2}\operatorname{tr}\big(\sigma(X_{t\wedge\tau})\,\sigma(X_{t\wedge\tau})^*\,(\operatorname{Hess}_x\bar{V})(t\wedge\tau,X_{t\wedge\tau},Y_{t\wedge\tau})\big)\le 0\end{aligned}\tag{26}$$

for all $t\in[0,T]$. An application of Lemma 2.2 hence completes the proof of Proposition 2.3. □

The next corollary, Corollary 2.4, specialises Proposition 2.3 to the case where $U(t,x) = e^{-\alpha t}\,U(0,x)$ and $\overline{U}(t,x) = e^{-\alpha t}\,\hat{U}(t,x)$ for all $(t,x)\in[0,T]\times O$ and some $\alpha\in\mathbb{R}$.

Corollary 2.4 (Exponential integrability properties (time-independent version))**.** *Assume the setting in Section 1.2, let $\alpha\in\mathbb{R}$, $U\in C^2(O,\mathbb{R})$, $\overline{U}\in\mathcal{L}^0([0,T]\times O;\mathbb{R})$, let $\tau\colon\Omega\to[0,T]$ be a stopping time and let $X\colon[0,T]\times\Omega\to O$ be an adapted stochastic process with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s)\| + \|\sigma(X_s)\|^2 + |\overline{U}(s,X_s)|\,ds<\infty$ $\mathbb{P}$-a.s., $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau}\mu(X_s)\,ds + \int_0^{t\wedge\tau}\sigma(X_s)\,dW_s$ $\mathbb{P}$-a.s. for all $t\in[0,T]$ and*

$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2e^{\alpha t}}\,\|\sigma(x)^*\,(\nabla U)(x)\|^2 + \overline{U}(t,x)\le\alpha U(x)\tag{27}$$

for all $(t,x)\in[0,T]\times\cup_{\omega\in\Omega}\{X_s(\omega)\in O\colon s\in[0,\tau(\omega)]\}$. Then

$$\mathbb{E}\!\left[\exp\!\left(\tfrac{U(X_\tau)}{e^{\alpha\tau}} + \int_0^\tau \tfrac{\overline{U}(s,X_s)}{e^{\alpha s}}\,ds\right)\right]\le\mathbb{E}\Big[\exp\big(U(X_0)\big)\Big]\in[0,\infty].\tag{28}$$

A slightly different formulation of Corollary 2.4 is presented in the following corollary.

Corollary 2.5. *Assume the setting in Section 1.2, let $\tau\colon\Omega\to[0,T]$ be a stopping time and let $X\colon[0,T]\times\Omega\to O$ be an adapted stochastic process with continuous sample paths satisfying $\int_0^\tau\|\mu(X_s)\| + \|\sigma(X_s)\|^2\,ds<\infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau}\mu(X_s)\,ds + \int_0^{t\wedge\tau}\sigma(X_s)\,dW_s$ $\mathbb{P}$-a.s. for all $t\in[0,T]$. Then*

$$\mathbb{E}\!\left[e^{e^{-\alpha\tau}U(X_\tau) + \int_0^\tau e^{-\alpha s}\left[\alpha U(X_s) - (\mathcal{G}_{\mu,\sigma}U)(X_s) - \frac{e^{-\alpha s}}{2}\|\sigma(X_s)^*(\nabla U)(X_s)\|^2\right]ds}\right]\le\mathbb{E}\big[e^{U(X_0)}\big]\in[0,\infty]\tag{29}$$

for all $\alpha\in\mathbb{R}$ and all $U\in C^2(O,\mathbb{R})$.

We illustrate Corollary 2.5 by three simple examples. First, observe that if $r\in\mathbb{R}$ and if U in Corollary 2.5 satisfies $U(x) = r\,\|x\|^2$ for all $x\in O$, then (29) shows for every $\alpha\in\mathbb{R}$ that

$$\mathbb{E}\!\left[e^{\frac{r}{e^{\alpha\tau}}\|X_\tau\|^2 + \int_0^\tau\frac{r}{e^{\alpha s}}\left[\alpha\|X_s\|^2 - 2\langle X_s,\mu(X_s)\rangle - \|\sigma(X_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} - \frac{2r}{e^{\alpha s}}\|\sigma(X_s)^*X_s\|^2\right]ds}\right]\le\mathbb{E}\!\left[e^{r\|X_0\|^2}\right].$$

Second, note that if $\varepsilon\in(0,\infty)$ and if μ and σ in Corollary 2.5 satisfy $\mu(x) = -(\nabla U)(x)$ and $\sigma(x) = \sqrt{\varepsilon}I$ for all $x\in O$ and some $U\in C^2(O,\mathbb{R})$, then (29) implies for every $\alpha,r\in\mathbb{R}$ that

$$\mathbb{E}\!\left[e^{\frac{r}{e^{\alpha\tau}}U(X_\tau) + \int_0^\tau\frac{2r}{e^{\alpha s}}\left[\alpha U(X_s) + \left[1-\frac{\varepsilon r}{2e^{\alpha s}}\right]\|(\nabla U)(X_s)\|^2 - \varepsilon(\triangle U)(X_s)\right]ds}\right]\le\mathbb{E}\!\left[e^{rU(X_0)}\right].\tag{30}$$

A result related to (30) can, e.g., be found in Lemma 2.5 in Bou-Rabee & Hairer [6]. Finally, observe that if $r\in\mathbb{R}$ and if U in Corollary 2.5 satisfies $U(x) = r\ln\!\big(1+\|x\|^2\big)$ for all $x\in O$, then (29) implies for every $\alpha\in\mathbb{R}$ that

$$\begin{aligned}&\mathbb{E}\!\left[\big(1+\|X_\tau\|^2\big)^{\frac{r}{e^{\alpha\tau}}}\,e^{\int_0^\tau\frac{r}{e^{\alpha s}}\left[\alpha\ln(1+\|X_s\|^2) - \frac{2\langle X_s,\mu(X_s)\rangle - \|\sigma(X_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{(1+\|X_s\|^2)} + \left[1-\frac{r}{e^{\alpha s}}\right]\frac{2\,\|\sigma(X_s)^*X_s\|^2}{(1+\|X_s\|^2)^2}\right]ds}\right]\\ &\le\mathbb{E}\Big[\big(1+\|X_0\|^2\big)^r\Big].\end{aligned}\tag{31}$$

The following corollary of (31) states a moment estimate for solutions of SDEs which is interesting on its own.

Corollary 2.6. *Assume the setting in Section 1.2, let $p, c \in \mathbb{R}$, $\alpha \in [0,\infty)$, let $\tau\colon \Omega \to [0,T]$ be a stopping time and let $X\colon [0,T]\times\Omega \to O$ be an adapted stochastic process with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s)\| + \|\sigma(X_s)\|^2\, ds < \infty$ $\mathbb{P}$-a.s., $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \mu(X_s)\, ds + \int_0^{t\wedge\tau} \sigma(X_s)\, dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and*

$$2\langle x, \mu(x)\rangle + \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \tfrac{2\,(p-1)\,\|\sigma(x)^* x\|^2}{(1+\|x\|^2)} \leq \big(c + \alpha \ln(1+\|x\|^2)\big)\,\big(1+\|x\|^2\big) \tag{32}$$

for all $x \in \mathrm{im}(X)$. Then $\mathbb{E}\big[e^{-c\tau}(1+\|X_\tau\|^2)^{pe^{-\alpha\tau}}\big] \leq \mathbb{E}\big[(1+\|X_0\|^2)^p\big]$.

Lemma 2.7 (Exponential martingales)**.** *Let $m \in \mathbb{N}$, $T \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion and let $A\colon [0,T]\times\Omega \to \mathbb{R}^m$ be an adapted and product measurable stochastic process satisfying $\int_0^T \|A_s\|^2\, ds < \infty$ $\mathbb{P}$-a.s. Then it holds for all $p \in (1,\infty]$ that*

$$\begin{aligned}
&\left\| \sup_{t\in[0,T]} \exp\!\left(\int_0^t \langle A_s, dW_s\rangle - \tfrac{1}{2}\int_0^t \|A_s\|^2\, ds \right) \right\|_{L^p(\Omega;\mathbb{R})} \\
&\quad\leq \frac{1}{(1-\frac{1}{p})} \inf_{q\in[p,\infty]} \left\| \exp\!\left(\tfrac{1}{2}\Big[\tfrac{1}{(\frac{1}{p}-\frac{1}{q})} - 1\Big] \int_0^T \|A_s\|^2\, ds \right) \right\|_{L^q(\Omega;\mathbb{R})} && (33)\\
&\quad\leq \frac{1}{(1-\frac{1}{p})} \left\| \exp\!\left(\big(p - \tfrac{1}{2}\big) \int_0^T \|A_s\|^2\, ds \right) \right\|_{L^{2p}(\Omega;\mathbb{R})} \in [0,\infty]. && (34)
\end{aligned}$$

Proof of Lemma 2.7. Inequality (34) follows from inequality (33) by taking $q = 2p$. It thus remains to prove inequality (33). If the right-hand side of (33) is infinite, then the proof is complete. So for the rest of the proof, we assume that the right-hand side of (33) is finite. If the infimum on the right-hand side of (33) is attained at $q = p$, then necessarily $\int_0^T \|A_s\|^2\, ds = 0$ $\mathbb{P}$-a.s. In that case, both sides of (33) are equal to 1 and this completes the proof in that case. So for the rest of the proof, we assume that $p \in (1,\infty)$ and that the infimum on the right-hand side of (33) is not attained at $q = p$. Let $Z^{(r)}\colon [0,T]\times\Omega \to \mathbb{R}$, $r \in \mathbb{R}$, be adapted stochastic processes with continuous sample paths satisfying

$$Z_t^{(r)} = \exp\!\left(r \int_0^t \langle A_s, dW_s\rangle - \tfrac{1}{2} r^2 \int_0^t \|A_s\|^2\, ds \right) \tag{35}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, $r \in \mathbb{R}$. It follows from, e.g., [24, Lemma 18.21] that for every $r \in \mathbb{R}$ the process $Z^{(r)}$ is a local martingale. For every $r \in \mathbb{R}$, let $\tau_{r,n}\colon \Omega \to [0,T]$, $n \in \mathbb{N}$, be a localizing sequence of stopping times for $Z^{(r)}$. Doob's martingale inequality and Hölder's inequality imply that for every $q, r \in (p,\infty)$, $n \in \mathbb{N}$ with $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$ it holds that

$$\begin{aligned}
\left\| \sup_{t\in[0,T\wedge\tau_{r,n}]} Z_t^{(1)} \right\|_{L^p(\Omega;\mathbb{R})} &\leq \frac{p}{(p-1)} \left\| Z^{(1)}_{T\wedge\tau_{r,n}} \right\|_{L^p(\Omega;\mathbb{R})} \\
&= \frac{p}{(p-1)} \left\| \Big(Z^{(r)}_{T\wedge\tau_{r,n}} \Big)^{\frac{1}{r}} \exp\!\left(\tfrac{1}{2}(r-1) \int_0^{T\wedge\tau_{r,n}} \|A_s\|^2\, ds \right) \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \frac{p}{(p-1)} \left(\mathbb{E}\Big[Z^{(r)}_{T\wedge\tau_{r,n}} \Big] \right)^{\frac{1}{r}} \left\| \exp\!\left(\tfrac{1}{2}(r-1) \int_0^T \|A_s\|^2\, ds \right) \right\|_{L^q(\Omega;\mathbb{R})} \\
&= \frac{p}{(p-1)} \left\| \exp\!\left(\tfrac{1}{2} \left(\tfrac{1}{(\frac{1}{p}-\frac{1}{q})} - 1 \right) \int_0^T \|A_s\|^2\, ds \right) \right\|_{L^q(\Omega;\mathbb{R})}.
\end{aligned} \tag{36}$$

Letting $n \to \infty$ and applying the monotone convergence theorem implies inequality (33). The proof of of Lemma 2.7 is thus completed. ☐

2.2 An identity for Lyapunov-type functions

In Lemma 2.10 below, a simple identity for suitable Lyapunov-type functions is proved. In the proof of Lemma 2.10 the following stochastic version of the Gronwall lemma is used. For completeness its proof is given below.

Lemma 2.8. *Let $m \in \mathbb{N}$, $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\tau \colon \Omega \to [0,T]$ be a stopping time, let $X \colon [0,T] \times \Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths and let $\hat{A}, A \colon [0,T] \times \Omega \to [-\infty,\infty]$ and $\hat{B} \colon [0,T] \times \Omega \to [-\infty,\infty]^{1\times m}$ be adapted and product measurable stochastic processes satisfying $\int_0^\tau |A_s| + |\hat{A}_s| + \|\hat{B}_s\|^2 \, ds < \infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} A_s \, ds + \int_0^{t\wedge\tau} \hat{B}_s X_s \, dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $\mathbb{1}_{\{t<\tau\}} A_t \leq \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$. Then*

$$X_\tau \leq \exp\left(\int_0^\tau \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 \quad \mathbb{P}\text{-a.s.} \tag{37}$$

If, in addition, $\mathbb{1}_{\{t<\tau\}} A_t = \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$, then (37) holds with equality.

Proof of Lemma 2.8. Let $Y \colon [0,T] \times \Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths satisfying

$$Y_t = X_{t\wedge\tau} \exp\left(-\int_0^{t\wedge\tau} \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds - \int_0^{t\wedge\tau} \hat{B}_s \, dW_s \right) \tag{38}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. Then Itô's formula proves that

$$\begin{aligned} Y_t = X_0 &+ \int_0^{t\wedge\tau} A_s \exp\left(-\int_0^{s\wedge\tau} \left[\hat{A}_u - \tfrac{1}{2}\|\hat{B}_u\|^2 \right] du - \int_0^{s\wedge\tau} \hat{B}_u \, dW_u \right) ds \\ &- \int_0^{t\wedge\tau} \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] Y_s \, ds + \int_0^{t\wedge\tau} Y_s \left[\hat{B}_s - \hat{B}_s \right] dW_s \\ &+ \tfrac{1}{2} \int_0^{t\wedge\tau} Y_s \, \|\hat{B}_s\|^2 \, ds - \int_0^{t\wedge\tau} Y_s \, \|\hat{B}_s\|^2 \, ds \\ = X_0 &+ \int_0^{t\wedge\tau} A_s \exp\left(-\int_0^{s\wedge\tau} \left[\hat{A}_u - \tfrac{1}{2}\|\hat{B}_u\|^2 \right] du - \int_0^{s\wedge\tau} \hat{B}_u \, dW_u \right) ds - \int_0^{t\wedge\tau} \hat{A}_s Y_s \, ds \end{aligned} \tag{39}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. The assumption $\mathbb{1}_{\{s<\tau\}}(\hat{A}_s X_s - A_s) \geq 0$ $\mathbb{P}$-a.s. for Lebesgue-almost all $s \in [0,T]$ together with Lemma 2.1 and $\int_0^\tau |A_s| + |\hat{A}_s X_s| \, ds < \infty$ $\mathbb{P}$-a.s. hence implies that $Y_t \leq X_0$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and, in particular, that $Y_T \leq X_0$ $\mathbb{P}$-a.s. In addition, observe that if $\mathbb{1}_{\{s<\tau\}} A_s = \mathbb{1}_{\{s<\tau\}} \hat{A}_s X_s$ $\mathbb{P}$-a.s. for Lebesgue-almost all $s \in [0,T]$, then Lemma 2.1 implies that $Y_T = X_0$ $\mathbb{P}$-a.s. This finishes the proof of Lemma 2.8. □

The following corollary shows that if X in Lemma 2.8 is nonnegative, then fewer integrability assumptions are needed.

Corollary 2.9. *Let $m \in \mathbb{N}$, $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\tau \colon \Omega \to [0,T]$ be a stopping time, let $X \colon [0,T] \times \Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths and let $\hat{A}, A \colon [0,T] \times \Omega \to [-\infty,\infty]$ and $\hat{B} \colon [0,T] \times \Omega \to [-\infty,\infty]^{1\times m}$ be adapted and product measurable stochastic processes satisfying $X_{t\wedge\tau} \geq 0$ $\mathbb{P}$-a.s., $\int_0^\tau |A_s| + \max(\hat{A}_s, 0) + \|\hat{B}_s\|^2 \, ds < \infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} A_s \, ds + \int_0^{t\wedge\tau} \hat{B}_s X_s \, dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $\mathbb{1}_{\{t<\tau\}} A_t \leq \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$. Then*

$$X_\tau \leq \exp\left(\int_0^\tau \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 \quad \mathbb{P}\text{-a.s.} \tag{40}$$

If, in addition, $\mathbb{1}_{\{t<\tau\}} A_t = \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$, then (40) holds with equality.

Proof of Corollary 2.9. As $\mathbb{1}_{\{t<\tau\}} A_t \leq \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t = \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_{t\wedge\tau} \leq \mathbb{1}_{\{t<\tau\}} \max(\hat{A}_t, -n) X_t$ $\mathbb{P}$-a.s. and $\int_0^\tau |\max(\hat{A}_s, -n)|\, ds < \infty$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$ and all $n \in \mathbb{N}$, Lemma 2.8 implies

$$X_\tau \leq \exp\!\left(\int_0^\tau \left[\max(\hat{A}_s, -n) - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 \quad \mathbb{P}\text{-a.s.} \tag{41}$$

for all $n \in \mathbb{N}$. Now the monotone convergence theorem shows

$$\begin{aligned} \lim_{n\to\infty} \int_0^\tau \max(\hat{A}_s, -n)\, ds &= \int_0^\tau \max(\hat{A}_s, 0)\, ds - \lim_{n\to\infty} \int_0^\tau \min(\max(-\hat{A}_s, 0), n)\, ds \\ &= \int_0^\tau \max(\hat{A}_s, 0)\, ds - \int_0^\tau \max(-\hat{A}_s, 0)\, ds = \int_0^\tau \hat{A}_s \, ds. \end{aligned} \tag{42}$$

So letting $n \to \infty$ on the right-hand side of (41) yields

$$X_\tau \leq \exp\!\left(\int_0^\tau \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 \quad \mathbb{P}\text{-a.s.} \tag{43}$$

This proves (40). For the rest of the proof, we assume that $\mathbb{1}_{\{t<\tau\}} A_t = \mathbb{1}_{\{t<\tau\}} \hat{A}_t X_t$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0,T]$. Define a sequence of stopping times $\rho_n \colon \Omega \to [0,T]$, $n \in \mathbb{N}$, by $\rho_n := \inf\big(\{\tau\} \cup \{t \in [0,T] \colon \int_0^t |\hat{A}_s|\, ds \geq n\}\big)$ for all $n \in \mathbb{N}$. Then left-continuity of the mapping $[0,T] \ni t \mapsto \int_0^t |\hat{A}_s|\, ds \in [0,\infty]$ implies that $\int_0^{\rho_n} |\hat{A}_s|\, ds \leq n$ for all $n \in \mathbb{N}$. Consequently, Lemma 2.8 shows

$$X_{\rho_n} = \exp\!\left(\int_0^{\rho_n} \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^{\rho_n} \hat{B}_s \, dW_s \right) X_0 \qquad \mathbb{P}\text{-a.s.} \tag{44}$$

for all $n \in \mathbb{N}$. Now the assumption that $X_\tau \geq 0$ $\mathbb{P}$-a.s., inequality (43) and $\int_0^\tau \|\hat{B}_s\|^2\, ds + |\int_0^\tau \hat{B}_s\, dW_s| < \infty$ $\mathbb{P}$-a.s. imply

$$0 \leq X_\tau \mathbb{1}_{\{\int_0^\tau \hat{A}_s\, ds = -\infty\}} \leq \mathbb{1}_{\{\int_0^\tau \hat{A}_s\, ds = -\infty\}} \exp\!\left(\int_0^\tau \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 = 0 \tag{45}$$

$\mathbb{P}$-a.s. Moreover, it follows from $\mathbb{1}_{\{\int_0^\tau \hat{A}_s\, ds > -\infty\}} = \mathbb{1}_{\{\int_0^\tau |\hat{A}_s|\, ds < \infty\}} = \mathbb{1}_{(\cup_{n\in\mathbb{N}}\{\rho_n = \tau\})} = \lim_{n\to\infty} \big(\mathbb{1}_{\{\rho_n=\tau\}}\big)$ $\mathbb{P}$-a.s. and from (44) that

$$\begin{aligned} X_\tau \mathbb{1}_{\{\int_0^\tau \hat{A}_s\, ds > -\infty\}} &= \lim_{n\to\infty} \big(X_\tau \mathbb{1}_{\{\rho_n=\tau\}} \big) = \lim_{n\to\infty} \big(X_{\rho_n} \mathbb{1}_{\{\rho_n=\tau\}} \big) \\ &= \lim_{n\to\infty} \left[\mathbb{1}_{\{\rho_n=\tau\}} \exp\!\left(\int_0^{\rho_n} \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^{\rho_n} \hat{B}_s \, dW_s \right) X_0 \right] \\ &= \mathbb{1}_{\{\int_0^\tau \hat{A}_s\, ds > -\infty\}} \exp\!\left(\int_0^\tau \left[\hat{A}_s - \tfrac{1}{2}\|\hat{B}_s\|^2 \right] ds + \int_0^\tau \hat{B}_s \, dW_s \right) X_0 \end{aligned} \tag{46}$$

$\mathbb{P}$-a.s. Combining (45) and (46) finishes the proof of Corollary 2.9. ☐

Lemma 2.10 (An identity for Lyapunov-type functions)**.** *Assume the setting in Section 1.2, let $V \in C^2(O, \mathbb{R})$, let $\tau \colon \Omega \to [0,T]$ be a stopping time, let $X \colon [0,T] \times \Omega \to O$ be an adapted stochastic process with continuous sample paths satisfying $V(X_{t\wedge\tau}) \geq 0$ $\mathbb{P}$-a.s., $\int_0^\tau \|\mu(X_s)\| + \|\sigma(X_s)\|^2 + \max\big(\frac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)}, 0\big) + \frac{\|(G_\sigma V)(X_s)\|^2}{(V(X_s))^2}\, ds < \infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \mu(X_s)\, ds + \int_0^{t\wedge\tau} \sigma(X_s)\, dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Then*

$$V(X_\tau) \leq V(X_0) \exp\!\left(\int_0^\tau \left[\frac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)} - \frac{\|(G_\sigma V)(X_s)\|^2}{2\,(V(X_s))^2} \right] ds + \int_0^\tau \frac{(G_\sigma V)(X_s)}{V(X_s)}\, dW_s \right) \quad \mathbb{P}\text{-a.s.} \tag{47}$$

If, in addition, $(V^{-1})(0) \subseteq (\mathcal{G}_{\mu,\sigma}V)^{-1}([0,\infty))$, then equality holds in (47).

Proof of Lemma 2.10. Path continuity together with $V \in C^2(O, \mathbb{R})$ implies that $\int_0^\tau |(\mathcal{G}_{\mu,\sigma}V)(X_s)| + \|(G_\sigma V)(X_s)\|^2 \, ds < \infty$ $\mathbb{P}$-a.s. Next the inequality $a < 0 = \frac{a}{0} \cdot 0$ for all $a \in (-\infty, 0)$ implies $\mathbb{1}_{\{t<\tau\}}\big((\mathcal{G}_{\mu,\sigma}V)(X_t) - \frac{(\mathcal{G}_{\mu,\sigma}V)(X_t)}{V(X_t)} V(X_t)\big)\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_t)<0\}} \leq 0$ $\mathbb{P}$-a.s. for all $t \in [0, T]$. Moreover, the assumption that $\int_0^\tau \max\big(\frac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)}, 0\big) + \frac{\|(G_\sigma V)(X_s)\|^2}{(V(X_s))^2} \, ds < \infty$ $\mathbb{P}$-a.s. yields that

$$\int_0^\tau \big((\mathcal{G}_{\mu,\sigma}V)(X_s)\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_s)\geq 0\}} + \|(G_\sigma V)(X_s)\|^2\big)\mathbb{1}_{\{V(X_s)=0\}} \, ds = 0 \tag{48}$$

$\mathbb{P}$-a.s. and, consequently, that

$$\int_0^\tau |(\mathcal{G}_{\mu,\sigma}V)(X_s) - \tfrac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)} V(X_s)|\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_s)\geq 0\}} + \|(G_\sigma V)(X_s) - \tfrac{(G_\sigma V)(X_s)}{V(X_s)} V(X_s)\|^2 \, ds = 0 \tag{49}$$

$\mathbb{P}$-a.s. This together with Itô's formula results in

$$\begin{aligned} V(X_{t\wedge\tau}) &= V(X_0) + \int_0^{t\wedge\tau} (\mathcal{G}_{\mu,\sigma}V)(X_s) \, ds + \int_0^{t\wedge\tau} (G_\sigma V)(X_s) \, dW_s \\ &= V(X_0) + \int_0^{t\wedge\tau} (\mathcal{G}_{\mu,\sigma}V)(X_s) \, ds + \int_0^{t\wedge\tau} \tfrac{(G_\sigma V)(X_s)}{V(X_s)} \cdot V(X_s) \, dW_s \end{aligned} \tag{50}$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Moreover, (49) and Fubini's theorem also imply that

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_0^\tau |(\mathcal{G}_{\mu,\sigma}V)(X_s) - \tfrac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)} V(X_s)|\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_s)\geq 0\}} \, ds\right] \\ &= \int_0^T \mathbb{E}\left[\mathbb{1}_{\{t<\tau\}}|(\mathcal{G}_{\mu,\sigma}V)(X_t) - \tfrac{(\mathcal{G}_{\mu,\sigma}V)(X_t)}{V(X_t)} V(X_t)|\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_t)\geq 0\}}\right] dt \end{aligned} \tag{51}$$

so that $\mathbb{1}_{\{t<\tau\}}\big((\mathcal{G}_{\mu,\sigma}V)(X_t) - \frac{(\mathcal{G}_{\mu,\sigma}V)(X_t)}{V(X_t)} V(X_t)\big)\mathbb{1}_{\{(\mathcal{G}_{\mu,\sigma}V)(X_t)\geq 0\}} = 0$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0, T]$. Combining these observations then shows that $\mathbb{1}_{\{t<\tau\}}\big((\mathcal{G}_{\mu,\sigma}V)(X_t) - \frac{(\mathcal{G}_{\mu,\sigma}V)(X_t)}{V(X_t)} V(X_t)\big) \leq 0$ $\mathbb{P}$-a.s. for Lebesgue-almost all $t \in [0, T]$. Applying Corollary 2.9 to the stochastic process $V(X_t)$, $t \in [0, T]$, and to the stopping time τ together with $\int_0^\tau |(\mathcal{G}_{\mu,\sigma}V)(X_s)| + \max\big(\frac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)}, 0\big) + \frac{\|(G_\sigma V)(X_s)\|^2}{(V(X_s))^2} \, ds < \infty$ $\mathbb{P}$-a.s. yields then that

$$V(X_\tau) \leq V(X_0) \exp\left(\int_0^\tau \left[\tfrac{(\mathcal{G}_{\mu,\sigma}V)(X_s)}{V(X_s)} - \tfrac{\|(G_\sigma V)(X_s)\|^2}{2\,(V(X_s))^2}\right] ds + \int_0^\tau \tfrac{(G_\sigma V)(X_s)}{V(X_s)} \, dW_s\right) \quad \mathbb{P}\text{-a.s.} \tag{52}$$

This proves inequality (47). The additional assumption $(V^{-1})(0) \subseteq (\mathcal{G}_{\mu,\sigma}V)^{-1}([0, \infty))$ implies $\mathbb{1}_{\{t<\tau\}}\big((\mathcal{G}_{\mu,\sigma}V)(X_t) - \frac{(\mathcal{G}_{\mu,\sigma}V)(X_t)}{V(X_t)} V(X_t)\big) = 0$ $\mathbb{P}$-a.s. for all $t \in [0, T]$. In that case, Corollary 2.9 yields equality in (47). This completes the proof of Lemma 2.10. □

2.3 Two solution approach

In this subsection, we apply Lemma 2.10 to the bivariate process of two solutions of the same SDE (see Proposition 2.12 below). From this, we then derive marginal and uniform estimates for this bivariate process (see Proposition 2.17 below and Proposition 2.26 below respectively).

The next remark illustrates the relation between the extended generator in (17) and the "standard" generator in (15) and between the extended noise operator in (18) and the "standard" noise operator in (16). The proof of Remark 2.11 is clear and therefore omitted.

Remark 2.11 (Relation between generator and extended generator and between noise operator and extended noise operator)**.** *Let $d, m \in \mathbb{N}$, let $O \subseteq \mathbb{R}^d$ be an open set, let $\mu\colon O \to \mathbb{R}^d$ and $\sigma\colon O \to \mathbb{R}^{d\times m}$ be functions and define functions $\overline{\mu}\colon O^2 \to \mathbb{R}^{2d}$ and $\overline{\sigma}\colon O^2 \to \mathbb{R}^{(2d)\times m}$ by $\overline{\mu}(x, y) = (\mu(x), \mu(y))$ and by $\overline{\sigma}(x, y)u = (\sigma(x)u, \sigma(y)u)$ for all $x, y \in O$ and all $u \in \mathbb{R}^m$. Then $\overline{\mathcal{G}}_{\mu,\sigma} = \mathcal{G}_{\overline{\mu},\overline{\sigma}}$ and $\overline{G}_\sigma = G_{\overline{\sigma}}$.*

Using both the extended generator and the extended noise operator, we now establish in Proposition 2.12 an elementary identity which is crucial for the results developed in this article. Proposition 2.12 follows immediately from Remark 2.11 and Lemma 2.10. Proposition 2.12 generalizes a relation on page 1935 in Zhang [54].

Proposition 2.12 (Two solution approach)**.** *Assume the setting in Section 1.2, let* $\tau\colon \Omega \to [0,T]$ *be a stopping time, let* $V \in C^2(O^2, \mathbb{R})$*, let* $X^i\colon [0,T]\times\Omega \to O$*,* $i \in \{1,2\}$*, be adapted stochastic processes with continuous sample paths satisfying* $\int_0^\tau \max\big(\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^1,X_s^2)}{V(X_s^1,X_s^2)},0\big)+\|\mu(X_s^i)\|+\|\sigma(X_s^i)\|^2+\frac{\|(\overline{G}_\sigma V)(X_s^1,X_s^2)\|^2}{(V(X_s^1,X_s^2))^2}\,ds < \infty$ *$\mathbb{P}$-a.s.,* $V(X^1_{t\wedge\tau}, X^2_{t\wedge\tau}) \geq 0$ *$\mathbb{P}$-a.s. and* $X^i_{t\wedge\tau} = X_0^i + \int_0^{t\wedge\tau}\mu(X_s^i)\,ds + \int_0^{t\wedge\tau}\sigma(X_s^i)\,dW_s$ *$\mathbb{P}$-a.s. for all* $(t,i) \in [0,T] \in \{1,2\}$*. Then*

$$\begin{aligned}
&V(X_\tau^1, X_\tau^2) \\
&\leq V(X_0^1, X_0^2)\exp\bigg(\int_0^\tau \Big[\tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^1,X_s^2)}{V(X_s^1,X_s^2)} - \tfrac{\|(\overline{G}_\sigma V)(X_s^1,X_s^2)\|^2}{2(V(X_s^1,X_s^2))^2}\Big]\,ds + \int_0^\tau \tfrac{(\overline{G}_\sigma V)(X_s^1,X_s^2)}{V(X_s^1,X_s^2)}\,dW_s\bigg) \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{53}$$

If, in addition, $(V^{-1})(0) \subseteq (\overline{\mathcal{G}}_{\mu,\sigma}V)^{-1}([0,\infty))$*, then equality holds in* (53)*.*

2.3.1 Calculations for the extended generator and the extended noise operator

In this subsection we calculate the extended generator and the extended noise operator applied to a suitable class of twice continuously differentiable functions (see Lemma 2.14 below). In these calculations the following well-known remark is used.

Remark 2.13 (Derivatives of powers of the norm function)**.** *Let* $p \in [2,\infty)$ *and let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be an $\mathbb{R}$-Hilbert space. Then the function* $F\colon H \to \mathbb{R}$ *given by* $F(x) = \|x\|^p$ *for all* $x \in H$ *is twice continuously differentiable and fulfills*

$$\begin{aligned}
F'(x)(v) &= p\,\|x\|^{(p-2)}\,\langle x, v\rangle\,, \\
F''(x)(v,w) &= p\,\|x\|^{(p-2)}\,\langle v,w\rangle + p\,(p-2)\,\|x\|^{(p-4)}\,\langle x,v\rangle\,\langle x,w\rangle
\end{aligned} \tag{54}$$

for all $x,v,w \in H$*. In particular, the function* $G\colon H^2 \to \mathbb{R}$ *given by* $G(x,y) = \|x-y\|^p$ *for all* $x,y \in H$ *is twice continuously differentiable and fulfills*

$$\begin{aligned}
\Big(\big(\tfrac{\partial}{\partial x}G\big)(x,y)\Big)(v) &= -\Big(\big(\tfrac{\partial}{\partial y}G\big)(x,y)\Big)(v) = p\,\|x-y\|^{(p-2)}\,\langle x-y, v\rangle\,, \\
\Big(\big(\tfrac{\partial^2}{\partial x^2}G\big)(x,y)\Big)(v,w) &= \Big(\big(\tfrac{\partial^2}{\partial y^2}G\big)(x,y)\Big)(v,w) = -\Big(\big(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}G\big)(x,y)\Big)(v,w) \\
&= p\,\|x-y\|^{(p-2)}\,\langle v,w\rangle + p\,(p-2)\,\|x-y\|^{(p-4)}\,\langle x-y,v\rangle\,\langle x-y,w\rangle
\end{aligned} \tag{55}$$

for all $x,y,v,w \in H$*.*

The next lemma, Lemma 2.14, is the main result of this subsection. It calculates the extended generator and of the extended noise operator when applied to a suitable class of twice continuously differentiable functions.

Lemma 2.14. *Let* $d,m,k \in \mathbb{N}$*,* $p \in [2,\infty)$*, let* $O \subseteq \mathbb{R}^d$ *be an open set, let* $\mu \in \mathcal{L}^0(O;\mathbb{R}^d)$*,* $\sigma \in \mathcal{L}^0(O;\mathbb{R}^{d\times m})$*,* $\Phi \in C^2(O,\mathbb{R}^k)$ *and let* $V\colon O^2 \to \mathbb{R}$ *be given by* $V(x,y) = \|\Phi(x)-\Phi(y)\|^p$ *for all* $x,y \in O$*. Then* $V \in C^2(O^2,\mathbb{R})$ *and*

$$\frac{\big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{|V(x,y)|^2} = \frac{p^2\sum_{i=1}^m |\langle \Phi(x)-\Phi(y), \Phi'(x)\,\sigma_i(x) - \Phi'(y)\,\sigma_i(y)\rangle|^2}{\|\Phi(x)-\Phi(y)\|^4} \qquad \text{and} \tag{56}$$

$$\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(x,y)}{V(x,y)} = \frac{p\,\langle \Phi(x)-\Phi(y), \Phi'(x)\,\mu(x)-\Phi'(y)\,\mu(y)\rangle}{\|\Phi(x)-\Phi(y)\|^2} + \frac{(p-2)\,\big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{2\,p\,|V(x,y)|^2}$$
$$+ \frac{p\sum_{i=1}^m \big\langle \Phi(x)-\Phi(y), \Phi''(x)\big(\sigma_i(x),\sigma_i(x)\big)-\Phi''(y)\big(\sigma_i(y),\sigma_i(y)\big)\big\rangle}{2\,\|\Phi(x)-\Phi(y)\|^2} \tag{57}$$
$$+ \frac{p\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\,\|\Phi'(x)\,\sigma(x)-\Phi'(y)\,\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^k)}}{2\,\|\Phi(x)-\Phi(y)\|^p}$$

for all $x,y\in O$.

Proof of Lemma 2.14. First, note that the chain rule together with Remark 2.13 shows that $V\in C^2(O^2,\mathbb{R})$. Next observe that Remark 2.13 implies that

$$\begin{aligned}
&(\overline{\mathcal{G}}_{\mu,\sigma}V)(x,y) = p\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\,\langle \Phi(x)-\Phi(y), \Phi'(x)\,\mu(x)-\Phi'(y)\,\mu(y)\rangle \\
&+ \frac{p}{2}\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-2)}\,\big\langle \Phi(x)-\Phi(y), \Phi''(x)\big(\sigma_i(x),\sigma_i(x)\big)-\Phi''(y)\big(\sigma_i(y),\sigma_i(y)\big)\big\rangle \\
&+ \sum_{i=1}^m p\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\left[\tfrac{\|\Phi'(x)\,\sigma_i(x)\|^2+\|\Phi'(y)\,\sigma_i(y)\|^2}{2} - \langle \Phi'(x)\,\sigma_i(x), \Phi'(y)\,\sigma_i(y)\rangle\right] \\
&+ \frac{p\,(p-2)}{2}\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-4)}\,|\langle \Phi(x)-\Phi(y), \Phi'(x)\,\sigma_i(x)\rangle|^2 \\
&+ \frac{p\,(p-2)}{2}\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-4)}\,|\langle \Phi(x)-\Phi(y), \Phi'(y)\,\sigma_i(y)\rangle|^2 \\
&- p\,(p-2)\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-4)}\,\langle \Phi(x)-\Phi(y), \Phi'(x)\,\sigma_i(x)\rangle\,\langle \Phi(x)-\Phi(y), \Phi'(y)\,\sigma_i(y)\rangle
\end{aligned} \tag{58}$$

and

$$(\overline{G}_\sigma V)(x,y) = p\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\,(\Phi(x)-\Phi(y))^*\,(\Phi'(x)\,\sigma(x)-\Phi'(y)\,\sigma(y)) \tag{59}$$

for all $x,y\in O$. This shows that

$$\begin{aligned}
&(\overline{\mathcal{G}}_{\mu,\sigma}V)(x,y) = p\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\,\langle \Phi(x)-\Phi(y), \Phi'(x)\,\mu(x)-\Phi'(y)\,\mu(y)\rangle \\
&+ \frac{p}{2}\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-2)}\,\big\langle \Phi(x)-\Phi(y), \Phi''(x)\big(\sigma_i(x),\sigma_i(x)\big)-\Phi''(y)\big(\sigma_i(y),\sigma_i(y)\big)\big\rangle \\
&+ \frac{p}{2}\,\|\Phi(x)-\Phi(y)\|^{(p-2)}\,\|\Phi'(x)\,\sigma(x)-\Phi'(y)\,\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^k)} \\
&+ \frac{p\,(p-2)}{2}\sum_{i=1}^m \|\Phi(x)-\Phi(y)\|^{(p-4)}\,|\langle \Phi(x)-\Phi(y), \Phi'(x)\,\sigma_i(x)-\Phi'(y)\,\sigma_i(y)\rangle|^2
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\big\|(\overline{G}_\sigma V)(x,y)\big\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R})} \\
&= p^2\,\|\Phi(x)-\Phi(y)\|^{(2p-4)}\left[\sum_{i=1}^m |\langle \Phi(x)-\Phi(y), \Phi'(x)\,\sigma_i(x)-\Phi'(y)\,\sigma_i(y)\rangle|^2\right]
\end{aligned} \tag{61}$$

for all $x, y \in O$. Hence, we obtain that

$$
\begin{aligned}
\frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y)}{V(x,y)} &= \frac{p \, \langle \Phi(x) - \Phi(y), \Phi'(x) \, \mu(x) - \Phi'(y) \, \mu(y) \rangle}{\|\Phi(x) - \Phi(y)\|^2} \\
&+ \frac{p \sum_{i=1}^m \big\langle \Phi(x) - \Phi(y), \Phi''(x)\big(\sigma_i(x), \sigma_i(x)\big) - \Phi''(y)\big(\sigma_i(y), \sigma_i(y)\big) \big\rangle}{2 \, \|\Phi(x) - \Phi(y)\|^2} \\
&+ \frac{p \, \|\Phi(x) - \Phi(y)\|^{(p-2)} \, \|\Phi'(x) \, \sigma(x) - \Phi'(y) \, \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k)}}{2 \, \|\Phi(x) - \Phi(y)\|^p} \\
&+ \frac{p \, (p-2) \sum_{i=1}^m \|\Phi(x) - \Phi(y)\|^{(p-4)} \, |\langle \Phi(x) - \Phi(y), \Phi'(x) \, \sigma_i(x) - \Phi'(y) \, \sigma_i(y) \rangle|^2}{2 \, \|\Phi(x) - \Phi(y)\|^p}
\end{aligned}
\tag{62}
$$

and

$$
\begin{aligned}
\frac{\big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{|V(x,y)|^2} &= \frac{\big\|(\overline{G}_\sigma V)(x,y)\big\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R})}}{|V(x,y)|^2} \\
&= \frac{p^2 \, \|\Phi(x) - \Phi(y)\|^{(2p-4)} \, \big[\sum_{i=1}^m |\langle \Phi(x) - \Phi(y), \Phi'(x) \, \sigma_i(x) - \Phi'(y) \, \sigma_i(y) \rangle|^2\big]}{\|\Phi(x) - \Phi(y)\|^{2p}} \\
&= \frac{p^2 \sum_{i=1}^m |\langle \Phi(x) - \Phi(y), \Phi'(x) \, \sigma_i(x) - \Phi'(y) \, \sigma_i(y) \rangle|^2}{\|\Phi(x) - \Phi(y)\|^4}
\end{aligned}
\tag{63}
$$

and therefore

$$
\begin{aligned}
\frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y)}{V(x,y)} &= \frac{p \, \langle \Phi(x) - \Phi(y), \Phi'(x) \, \mu(x) - \Phi'(y) \, \mu(y) \rangle}{\|\Phi(x) - \Phi(y)\|^2} \\
&+ \frac{p \sum_{i=1}^m \big\langle \Phi(x) - \Phi(y), \Phi''(x)\big(\sigma_i(x), \sigma_i(x)\big) - \Phi''(y)\big(\sigma_i(y), \sigma_i(y)\big) \big\rangle}{2 \, \|\Phi(x) - \Phi(y)\|^2} \\
&+ \frac{p \, \|\Phi(x) - \Phi(y)\|^{(p-2)} \, \|\Phi'(x) \, \sigma(x) - \Phi'(y) \, \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^k)}}{2 \, \|\Phi(x) - \Phi(y)\|^p} + \frac{(p-2) \, \big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{2 \, p \, |V(x,y)|^2}
\end{aligned}
\tag{64}
$$

for all $x, y \in O$. This completes the proof of Lemma 2.14. ☐

Next we illustrate (56) and (57) in Lemma 2.14 by two simple corollaries, Example 2.15 and Example 2.16. Example 2.15 is the special case of Lemma 2.14 where $\Phi(x) = x$ for all $x \in O$.

Example 2.15. *Let $d, m \in \mathbb{N}$, $p \in [2, \infty)$, let $O \subseteq \mathbb{R}^d$ be an open set, let $\mu \in \mathcal{L}^0(O; \mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(O; \mathbb{R}^{d \times m})$ and let $V \colon O^2 \to \mathbb{R}$ be given by $V(x,y) = \|x - y\|^p$ for all $x, y \in O$. Then $V \in C^2(O^2, \mathbb{R})$ and*

$$
\frac{\big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{|V(x,y)|^2} = \frac{p^2 \, \|(\sigma(x) - \sigma(y))^*(x - y)\|^2}{\|x - y\|^4} \qquad \text{and} \tag{65}
$$

$$
\begin{aligned}
\frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y)}{V(x,y)} &= p \left[\frac{\langle x - y, \mu(x) - \mu(y) \rangle + \frac{1}{2} \|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)}}{\|x - y\|^2} \right] + \frac{(p-2) \, \big\|(\overline{G}_\sigma V)(x,y)\big\|^2}{2 \, p \, |V(x,y)|^2} \\
&\leq p \left[\frac{\langle x - y, \mu(x) - \mu(y) \rangle + \frac{(p-1)}{2} \|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m, \mathbb{R}^d)}}{\|x - y\|^2} \right]
\end{aligned}
\tag{66}
$$

for all $x, y \in O$.

Example 2.16 is the special case of Lemma 2.14 where $O = (0, \infty) \subset \mathbb{R}$ and where $\Phi(x) = x^q$ for all $x \in O$ and some $q \in \mathbb{R}$.

Example 2.16. *Let $p \in [2,\infty)$, $q \in \mathbb{R}$ and let $V\colon (0,\infty)^2 \to \mathbb{R}$ be given by $V(x,y) = |x^q - y^q|^p$ for all $x, y \in (0,\infty)$. Then $V \in C^2((0,\infty)^2, \mathbb{R})$ and*

$$\frac{\left\|(\overline{G}_\sigma V)(x,y)\right\|^2}{|V(x,y)|^2} = \frac{p^2 q^2 \left(x^{(q-1)}\,\sigma(x) - y^{(q-1)}\,\sigma(y)\right)^2}{\left(x^q - y^q\right)^2} \qquad \text{and} \tag{67}$$

$$\begin{aligned} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(x,y)}{V(x,y)} &= \frac{pq\left(x^{(q-1)}\,\mu(x) - y^{(q-1)}\,\mu(y) + \frac{(q-1)}{2}\left[x^{(q-2)}(\sigma(x))^2 - y^{(q-2)}(\sigma(y))^2\right]\right)}{(x^q - y^q)} \\ &+ \frac{p\,(p-1)\,q^2\left(x^{(q-1)}\,\sigma(x) - y^{(q-1)}\,\sigma(y)\right)^2}{2\left(x^q - y^q\right)^2} \end{aligned} \tag{68}$$

for all $x, y \in (0,\infty)$.

2.3.2 Marginal strong stability analysis for solutions of SDEs

Proposition 2.17 (Marginal strong stability analysis)**.** *Assume the setting in Section 1.2, let $x, y \in O$, $V \in C^2(O^2, [0,\infty))$, let $\tau\colon \Omega \to [0,T]$ be a stopping time and let $X^z\colon [0,T]\times\Omega \to O$, $z \in \{x,y\}$, be adapted stochastic processes with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s^z)\| + \|\sigma(X_s^z)\|^2 + \max\!\big(\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}, 0\big) + \frac{\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{(V(X_s^x,X_s^y))^2}\,ds < \infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau}^z = z + \int_0^{t\wedge\tau}\mu(X_s^z)\,ds + \int_0^{t\wedge\tau}\sigma(X_s^z)\,dW_s$ $\mathbb{P}$-a.s. for all $(t,z) \in [0,T]\times\{x,y\}$. Then it holds for all $p, q, r \in (0,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ that*

$$\|V(X_\tau^x, X_\tau^y)\|_{L^r(\Omega;\mathbb{R})} \leq V(x,y)\left\| e^{\int_0^\tau \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)} + \frac{(p-1)\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds}\right\|_{L^q(\Omega;\mathbb{R})}. \tag{69}$$

In addition, if $\int_0^\tau \|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2\,ds = 0$ $\mathbb{P}$-a.s. and $(V^{-1})(0) \subseteq (\overline{\mathcal{G}}_{\mu,\sigma}V)^{-1}([0,\infty))$, then it holds for all $r \in (0,\infty]$ that

$$\|V(X_\tau^x, X_\tau^y)\|_{L^r(\Omega;\mathbb{R})} = V(x,y)\left\| e^{\int_0^\tau \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,ds}\right\|_{L^r(\Omega;\mathbb{R})}. \tag{70}$$

Proof of Proposition 2.17. Proposition 2.12 together with Hölder's inequality implies that

$$\begin{aligned} &\left\|V(X_\tau^x, X_\tau^y)\right\|_{L^r(\Omega;\mathbb{R})} \\ &\leq V(x,y)\left\|\exp\!\left(\int_0^\tau \left[\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)} - \frac{\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2(V(X_s^x,X_s^y))^2}\right]ds + \int_0^\tau \frac{(\overline{G}_\sigma V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,dW_s\right)\right\|_{L^r(\Omega;\mathbb{R})} \\ &\leq V(x,y)\left\|\exp\!\left(\int_0^\tau \left[\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)} + \frac{(p-1)\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\right]ds\right)\right\|_{L^q(\Omega;\mathbb{R})} \\ &\quad\cdot\left\|\exp\!\left(\int_0^\tau \frac{(\overline{G}_\sigma V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,dW_s - \int_0^\tau \frac{p\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds\right)\right\|_{L^p(\Omega;\mathbb{R})} \\ &\leq V(x,y)\left\|\exp\!\left(\int_0^\tau \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)} + \frac{(p-1)\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds\right)\right\|_{L^q(\Omega;\mathbb{R})} \end{aligned} \tag{71}$$

for all $p, q, r \in (0,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This proves (69). In the next step we observe that if $\int_0^\tau \|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2\,ds = 0$ $\mathbb{P}$-a.s. and if $(V^{-1})(0) \subseteq (\overline{\mathcal{G}}_{\mu,\sigma}V)^{-1}([0,\infty))$, then Proposition 2.12 proves that

$$V(X_\tau^x, X_\tau^y) = \exp\!\left(\int_0^\tau \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,ds\right)V(x,y) \tag{72}$$

$\mathbb{P}$-a.s. and hence

$$\|V(X_\tau^x, X_\tau^y)\|_{L^r(\Omega;\mathbb{R})} = V(x,y)\left\|\exp\!\left(\int_0^\tau \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,ds\right)\right\|_{L^r(\Omega;\mathbb{R})} \tag{73}$$

for all $r \in (0,\infty]$. The proof of Proposition 2.17 is thus completed. □

Remark 2.18. *Note in the setting of Proposition 2.17 that if* $\hat{V} \in C^2(O,[0,\infty))$, $\hat{g} \in C(O,\mathbb{R} \cup \{\infty,-\infty\})$, $x = y$ *and if* $V(v,w) = \hat{V}(v)$ *and* $g(v,w) = \hat{g}(v)$ *for all* $v,w \in O$, *then* (69) *in Proposition 2.17 reduces to an estimate for* $\|\hat{V}(X_\tau^x)\|_{L^r(\Omega;\mathbb{R})}$.

The next corollary follows immediately from Proposition 2.17 and Example 2.15. It establishes an estimate for the L^r-norm of the difference of two solutions of the same SDE starting in different initial values for $r \in (0,\infty]$.

Corollary 2.19. *Assume the setting in Section 1.2, let* $x,y \in O$, *let* $\tau\colon \Omega \to [0,T]$ *be a stopping time, let* $X^z\colon [0,T]\times\Omega \to O$, $z \in \{x,y\}$, *be adapted stochastic processes with continuous sample paths satisfying* $\int_0^\tau \|\mu(X_s^z)\| + \|\sigma(X_s^z)\|^2 + \frac{\max(\langle X_s^x - X_s^y, \mu(X_s^x)-\mu(X_s^y)\rangle, 0) + \|\sigma(X_s^x)-\sigma(X_s^y)\|^2}{\|X_s^x - X_s^y\|^2}\,ds < \infty$ $\mathbb{P}$*-a.s. and* $X_{t\wedge\tau}^z = z + \int_0^{t\wedge\tau}\mu(X_s^z)\,ds + \int_0^{t\wedge\tau}\sigma(X_s^z)\,dW_s$ $\mathbb{P}$*-a.s. for all* $(t,z) \in [0,T]\times\{x,y\}$. *Then*

$$\begin{aligned}
&\|X_\tau^x - X_\tau^y\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \|x-y\| \left\| \exp\!\left(\int_0^\tau \left[\begin{array}{r} \frac{\langle X_s^x - X_s^y, \mu(X_s^x)-\mu(X_s^y)\rangle + \frac{1}{2}\|\sigma(X_s^x)-\sigma(X_s^y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|X_s^x - X_s^y\|^2} \\ + \frac{(\frac{p}{2}-1)\,\|(\sigma(X_s^x)-\sigma(X_s^y))^*(X_s^x-X_s^y)\|^2}{\|X_s^x-X_s^y\|^4} \end{array} \right] ds \right) \right\|_{L^q(\Omega;\mathbb{R})}
\end{aligned} \tag{74}$$

for all $p,q,r \in (0,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.

Corollary 2.19 simplifies in the case of one-dimensional SDEs. More precisely, (74) reads as

$$\begin{aligned}
&\|X_t^x - X_t^y\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \|x-y\| \left\| \exp\!\left(\int_0^t \frac{(X_s^x - X_s^y)\,(\mu(X_s^x)-\mu(X_s^y)) + \frac{p-1}{2}\|\sigma(X_s^x)-\sigma(X_s^y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R})}}{|X_s^x-X_s^y|^2}\,ds \right) \right\|_{L^q(\Omega;\mathbb{R})}
\end{aligned} \tag{75}$$

for all $p,q,r \in (0,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ in the case $d=1$. To analyze and to estimate the term appearing in the exponent in (74) in Corollary 2.19, the following elementary remark can be useful (see, e.g., Subsection 4.4 below).

Remark 2.20. *Let* $D \subseteq \mathbb{R}^d$ *be an open set and let* $F\colon D \to \mathbb{R}^d$ *be a continuously differentiable function. Then*

$$\forall\, x \in D\colon \quad \sup_{v \in \mathbb{R}^d\setminus\{0\}} \left[\frac{\langle v, F'(x)v\rangle}{\|v\|^2}\right] \leq \sup_{y \in D\setminus\{x\}} \left[\frac{\langle x-y, F(x)-F(y)\rangle}{\|x-y\|^2}\right], \tag{76}$$

$$\forall x,y \in D \text{ with } x \neq y: \quad \frac{\langle x-y, F(x)-F(y)\rangle}{\|x-y\|^2} \leq \sup_{r\in[0,1]} \sup_{v \in \mathbb{R}^d\setminus\{0\}} \left[\frac{\langle v, F'(rx+(1-r)y)v\rangle}{\|v\|^2}\right], \tag{77}$$

$$\sup_{x\in D}\sup_{v \in \mathbb{R}^d\setminus\{0\}} \left[\frac{\langle v, F'(x)v\rangle}{\|v\|^2}\right] = \sup_{\substack{x,y\in D\\ x\neq y}} \left[\frac{\langle x-y, F(x)-F(y)\rangle}{\|x-y\|^2}\right]. \tag{78}$$

Lemma 2.22 below uses Corollary 2.4 above to estimate expectations of certain exponential integrals. Besides Corollary 2.4, the proof of Lemma 2.22 also uses the following well-known consequence of Jensen's inequality in the next lemma (see, e.g., inequality (19) in Li [30]).

Lemma 2.21. *Let* $T \in (0,\infty)$, *let* $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space and let* $A\colon [0,T]\times\Omega \to \mathbb{R}$ *be a product measurable stochastic process with* $\int_0^T \max(0,A_s)\,ds < \infty$ $\mathbb{P}$*-a.s. or with* $\int_0^T \max(0,-A_s)\,ds < \infty$ $\mathbb{P}$*-a.s. Then it holds for all* $p \in [1,\infty]$ *that*

$$\left\|\exp\!\left(\int_0^T A_t\,dt\right)\right\|_{L^p(\Omega;\mathbb{R})} \leq \frac{1}{T}\int_0^T \|\exp(TA_t)\|_{L^p(\Omega;\mathbb{R})}\,dt \leq \sup_{t\in[0,T]} \|\exp(TA_t)\|_{L^p(\Omega;\mathbb{R})}. \tag{79}$$

Proof of Lemma 2.21. Jensen's inequality and Minkowski's integral inequality imply that

$$\begin{aligned}
&\left\|\exp\!\left(\int_0^T A_t\,dt\right)\right\|_{L^p(\Omega;\mathbb{R})} = \left\|\exp\!\left(\frac{1}{T}\int_0^T TA_t\,dt\right)\right\|_{L^p(\Omega;\mathbb{R})}\\
&\le \left\|\frac{1}{T}\int_0^T \exp(TA_t)\,dt\right\|_{L^p(\Omega;\mathbb{R})} = \frac{1}{T}\left\|\int_0^T \exp(TA_t)\,dt\right\|_{L^p(\Omega;\mathbb{R})}\\
&\le \frac{1}{T}\int_0^T \|\exp(TA_t)\|_{L^p(\Omega;\mathbb{R})}\,dt \le \sup_{t\in[0,T]}\|\exp(TA_t)\|_{L^p(\Omega;\mathbb{R})}
\end{aligned}\tag{80}$$

for all $p\in[1,\infty]$. This completes the proof of Lemma 2.21. □

Using Lemma 2.21 and Corollary 2.4, we are now ready to prove Lemma 2.22. Lemma 2.22 is crucial for Theorems 2.23 and 2.29 below.

Lemma 2.22. *Assume the setting in Section 1.2, let $k\in\mathbb{N}$, $x,y\in O$, $\alpha_0,\alpha_1,\beta_0,\beta_1\in\mathbb{R}^k$, $p\in(0,\infty]$, $q_0,q_1\in(0,\infty]^k$ with $\sum_{i=0}^{1}\sum_{l=1}^{k}\frac{1}{q_{i,l}}=\frac{1}{p}$, $U_0=(U_{0,l})_{l\in\{1,\ldots,k\}}$, $U_1=(U_{1,l})_{l\in\{1,\ldots,k\}}\in C^2(O,\mathbb{R}^k)$, $\overline{U}=(\overline{U}_l)_{l\in\{1,\ldots,k\}}\in C(O,\mathbb{R}^k)$, $V\in\mathcal{L}^0(O^2;\mathbb{R})$, $c\in\mathcal{L}^0([0,T];\mathbb{R})$, let $\tau\colon\Omega\to[0,T]$ be a stopping time and let $X^z\colon[0,T]\times\Omega\to O$, $z\in\{x,y\}$, be adapted stochastic processes with continuous sample paths satisfying $\int_0^\tau \max(c(s),0)+\|\mu(X_s^z)\|+\|\sigma(X_s^z)\|^2+\max(V(X_s^x,X_s^y),0)\,ds<\infty$ $\mathbb{P}$-a.s. and $X^z_{t\wedge\tau}=z+\int_0^{t\wedge\tau}\mu(X_s^z)\,ds+\int_0^{t\wedge\tau}\sigma(X_s^z)\,dW_s$ $\mathbb{P}$-a.s. for all $(t,z)\in[0,T]\times\{x,y\}$ and*

$$V(v,w)\le c(t)+\textstyle\sum_{n=1}^{k}\left[\frac{U_{0,n}(v)+U_{0,n}(w)}{2q_{0,n}Te^{\alpha_{0,n}t}}+\frac{\overline{U}_n(v)+\overline{U}_n(w)}{2q_{1,n}e^{\alpha_{1,n}t}}\right]\qquad\text{and}\tag{81}$$

$$(\mathcal{G}_{\mu,\sigma}U_{i,l})(u)+\tfrac{1}{2e^{\alpha_{i,l}t}}\|\sigma(u)^*(\nabla U_{i,l})(u)\|^2+\mathbb{1}_{\{1\}}(i)\cdot\overline{U}_l(u)\le\alpha_{i,l}U_{i,l}(u)+\beta_{i,l}\tag{82}$$

for all $i\in\{0,1\}$, $l\in\{1,\ldots,k\}$, $u\in\{v,w\}$, $(v,w)\in\operatorname{im}(X_t^x)\times\operatorname{im}(X_t^y)$, $t\in[0,T]$. Then

$$\begin{aligned}
&\left\|e^{\int_0^\tau V(X_s^x,X_s^y)\,ds}\right\|_{L^p(\Omega;\mathbb{R})}\le\exp\!\left(\sum_{l=1}^{k}\left[\int_0^T\frac{\beta_{0,l}\,(1-\frac{s}{T})}{q_{0,l}e^{\alpha_{0,l}s}}\,ds+\sum_{i=0}^{1}\frac{U_{i,l}(x)+U_{i,l}(y)}{2q_{i,l}}\right]\right)\\
&\cdot\left\|\exp\!\left(\int_0^\tau c(s)\,ds+\sum_{l=1}^{k}\left[\int_0^\tau\frac{\beta_{1,l}}{q_{1,l}e^{\alpha_{1,l}s}}\,ds-\frac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l}e^{\alpha_{1,l}\tau}}-\int_\tau^T\frac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l}Te^{\alpha_{0,l}s}}\,ds\right]\right)\right\|_{L^\infty(\Omega;\mathbb{R})}.
\end{aligned}$$

Proof of Lemma 2.22. First of all, observe that Hölder's inequality proves that

$$\begin{aligned}
&\left\|e^{\int_0^\tau V(X_s^x,X_s^y)\,ds}\right\|_{L^p(\Omega;\mathbb{R})}\\
&\le\left\|\exp\!\left(\int_0^\tau c(s)\,ds-\sum_{l=1}^{k}\frac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l}e^{\alpha_{1,l}\tau}}-\sum_{l=1}^{k}\int_\tau^T\frac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l}Te^{\alpha_{0,l}s}}\,ds+\sum_{l=1}^{k}\int_0^\tau\frac{\beta_{1,l}}{q_{1,l}e^{\alpha_{1,l}s}}\,ds\right)\right\|_{L^\infty(\Omega;\mathbb{R})}\\
&\cdot\left\|\exp\!\left(\int_0^\tau V(X_s^x,X_s^y)-c(s)-\sum_{l=1}^{k}\frac{\overline{U}_{1,l}(X_s^x)+\overline{U}_{1,l}(X_s^y)}{2q_{1,l}e^{\alpha_{1,l}s}}\,ds+\sum_{l=1}^{k}\int_\tau^T\frac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l}Te^{\alpha_{0,l}s}}\,ds\right)\right\|_{L^{[\sum_{l=1}^{k}\frac{1}{q_{0,l}}]^{-1}}(\Omega;\mathbb{R})}\\
&\cdot\left\|\exp\!\left(\sum_{l=1}^{k}\left[\frac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l}e^{\alpha_{1,l}\tau}}+\int_0^\tau\frac{\overline{U}_{1,l}(X_s^x)+\overline{U}_{1,l}(X_s^y)}{2q_{1,l}e^{\alpha_{1,l}s}}\,ds-\int_0^\tau\frac{\beta_{1,l}}{q_{1,l}e^{\alpha_{1,l}s}}\,ds\right]\right)\right\|_{L^{[\sum_{l=1}^{k}1/q_{1,l}]^{-1}}(\Omega;\mathbb{R})}.
\end{aligned}$$

Hence, (81) implies that

$$\begin{aligned}
&\left\|e^{\int_0^\tau V(X_s^x,X_s^y)\,ds}\right\|_{L^p(\Omega;\mathbb{R})}\le\left\|\exp\!\left(\sum_{l=1}^{k}\int_0^T\frac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l}Te^{\alpha_{0,l}s}}\,ds\right)\right\|_{L^{[\sum_{l=1}^{k}1/q_{0,l}]^{-1}}(\Omega;\mathbb{R})}\\
&\cdot\left\|\exp\!\left(\sum_{l=1}^{k}\left[\frac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l}e^{\alpha_{1,l}\tau}}+\int_0^\tau\frac{\overline{U}_{1,l}(X_s^x)+\overline{U}_{1,l}(X_s^y)-2\beta_{1,l}}{2q_{1,l}e^{\alpha_{1,l}s}}\,ds\right]\right)\right\|_{L^{[\sum_{l=1}^{k}1/q_{1,l}]^{-1}}(\Omega;\mathbb{R})}\\
&\cdot\left\|\exp\!\left(\int_0^\tau c(s)\,ds+\sum_{l=1}^{k}\int_0^\tau\frac{\beta_{1,l}}{q_{1,l}e^{\alpha_{1,l}s}}\,ds-\sum_{l=1}^{k}\frac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l}e^{\alpha_{1,l}\tau}}-\sum_{l=1}^{k}\int_\tau^T\frac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l}Te^{\alpha_{0,l}s}}\,ds\right)\right\|_{L^\infty(\Omega;\mathbb{R})}
\end{aligned}$$

and again Hölder's inequality therefore proves that

$$
\begin{aligned}
&\left\| e^{\int_0^\tau V(X_s^x, X_s^y)\,ds} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \prod_{l=1}^{k} \left\| \exp\!\left(\int_0^T \frac{U_{0,l}(X_s^x)}{2q_{0,l} T e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{2q_{0,l} T e^{\alpha_{0,l} u}}\,du\,ds \right) \right\|_{L^{2q_{0,l}}(\Omega;\mathbb{R})} \\
&\cdot \prod_{l=1}^{k} \left\| \exp\!\left(\int_0^T \frac{U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{2q_{0,l} T e^{\alpha_{0,l} u}}\,du\,ds \right) \right\|_{L^{2q_{0,l}}(\Omega;\mathbb{R})} \\
&\cdot \prod_{l=1}^{k} \left\| \exp\!\left(\frac{U_{1,l}(X_\tau^x)}{2q_{1,l} e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^x) - \beta_{1,l}}{2q_{1,l} e^{\alpha_{1,l} s}}\,ds \right) \right\|_{L^{2q_{1,l}}(\Omega;\mathbb{R})} \\
&\cdot \prod_{l=1}^{k} \left\| \exp\!\left(\frac{U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^y) - \beta_{1,l}}{2q_{1,l} e^{\alpha_{1,l} s}}\,ds \right) \right\|_{L^{2q_{1,l}}(\Omega;\mathbb{R})} \cdot \exp\!\left(\sum_{l=1}^{k} \int_0^T \int_0^s \frac{\beta_{0,l}}{q_{0,l} T e^{\alpha_{0,l} u}}\,du\,ds \right) \\
&\cdot \left\| \exp\!\left(\int_0^\tau c(s)\,ds + \sum_{l=1}^{k} \int_0^\tau \frac{\beta_{1,l}}{q_{1,l} e^{\alpha_{1,l} s}}\,ds - \sum_{l=1}^{k} \frac{U_{1,l}(X_\tau^x) + U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} - \sum_{l=1}^{k} \int_\tau^T \frac{U_{0,l}(X_s^x) + U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}}\,ds \right) \right\|_{L^\infty(\Omega;\mathbb{R})}.
\end{aligned}
$$

Lemma 2.21 hence implies that

$$
\begin{aligned}
&\left\| e^{\int_0^\tau V(X_s^x, X_s^y)\,ds} \right\|_{L^p(\Omega;\mathbb{R})} \leq \exp\!\left(\sum_{l=1}^{k} \int_0^T \frac{\beta_{0,l}(T-s)}{q_{0,l} T e^{\alpha_{0,l} s}}\,ds \right) \\
&\cdot \prod_{l=1}^{k} \left[\sup_{s\in[0,T]} \left\| \exp\!\left(\frac{U_{0,l}(X_s^x)}{2q_{0,l} e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{2q_{0,l} e^{\alpha_{0,l} u}}\,du \right) \right\|_{L^{2q_{0,l}}(\Omega;\mathbb{R})} \right] \\
&\cdot \prod_{l=1}^{k} \left[\sup_{s\in[0,T]} \left\| \exp\!\left(\frac{U_{0,l}(X_s^y)}{2q_{0,l} e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{2q_{0,l} e^{\alpha_{0,l} u}}\,du \right) \right\|_{L^{2q_{0,l}}(\Omega;\mathbb{R})} \right] \\
&\cdot \prod_{l=1}^{k} \left| \mathbb{E}\!\left[\exp\!\left(\frac{U_{1,l}(X_\tau^x)}{e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^x) - \beta_{1,l}}{e^{\alpha_{1,l} s}}\,ds \right) \right] \mathbb{E}\!\left[\exp\!\left(\frac{U_{1,l}(X_\tau^y)}{e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^y) - \beta_{1,l}}{e^{\alpha_{1,l} s}}\,ds \right) \right] \right|^{\frac{1}{2q_{1,l}}} \\
&\cdot \left\| \exp\!\left(\int_0^\tau c(s)\,ds + \sum_{l=1}^{k} \int_0^\tau \frac{\beta_{1,l}}{q_{1,l} e^{\alpha_{1,l} s}}\,ds - \sum_{l=1}^{k} \frac{U_{1,l}(X_\tau^x) + U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} - \sum_{l=1}^{k} \int_\tau^T \frac{U_{0,l}(X_s^x) + U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}}\,ds \right) \right\|_{L^\infty(\Omega;\mathbb{R})}
\end{aligned}
$$

and this shows that

$$
\begin{aligned}
&\left\| e^{\int_0^\tau V(X_s^x, X_s^y)\,ds} \right\|_{L^p(\Omega;\mathbb{R})} \leq \exp\!\left(\sum_{l=1}^{k} \int_0^T \frac{\beta_{0,l}(T-s)}{q_{0,l} T e^{\alpha_{0,l} s}}\,ds \right) \\
&\cdot \prod_{l=1}^{k} \left| \sup_{s\in[0,T]} \mathbb{E}\!\left[\exp\!\left(\frac{U_{0,l}(X_s^x)}{e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{e^{\alpha_{0,l} u}}\,du \right) \right] \cdot \sup_{s\in[0,T]} \mathbb{E}\!\left[\exp\!\left(\frac{U_{0,l}(X_s^y)}{e^{\alpha_{0,l} s}} - \int_0^s \frac{\beta_{0,l}}{e^{\alpha_{0,l} u}}\,du \right) \right] \right|^{\frac{1}{2q_{0,l}}} \\
&\cdot \prod_{l=1}^{k} \left| \mathbb{E}\!\left[\exp\!\left(\frac{U_{1,l}(X_\tau^x)}{e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^x) - \beta_{1,l}}{e^{\alpha_{1,l} s}}\,ds \right) \right] \mathbb{E}\!\left[\exp\!\left(\frac{U_{1,l}(X_\tau^y)}{e^{\alpha_{1,l}\tau}} + \int_0^\tau \frac{\overline{U}_{1,l}(X_s^y) - \beta_{1,l}}{e^{\alpha_{1,l} s}}\,ds \right) \right] \right|^{\frac{1}{2q_{1,l}}} \\
&\cdot \left\| \exp\!\left(\int_0^\tau c(s)\,ds + \sum_{l=1}^{k} \int_0^\tau \frac{\beta_{1,l}}{q_{1,l} e^{\alpha_{1,l} s}}\,ds - \sum_{l=1}^{k} \frac{U_{1,l}(X_\tau^x) + U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} - \sum_{l=1}^{k} \int_\tau^T \frac{U_{0,l}(X_s^x) + U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}}\,ds \right) \right\|_{L^\infty(\Omega;\mathbb{R})}.
\end{aligned} \tag{83}
$$

Corollary 2.4 therefore proves that

$$
\begin{aligned}
&\left\| e^{\int_0^\tau V(X_s^x, X_s^y)\, ds} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \exp\!\left(\sum_{l=1}^{k} \int_0^T \tfrac{\beta_{0,l}(1-\frac{s}{T})}{q_{0,l} e^{\alpha_{0,l} s}}\, ds \right) \cdot \left[\prod_{l=1}^{k} \exp\!\left(\tfrac{U_{0,l}(x)+U_{0,l}(y)}{2q_{0,l}} \right) \right] \cdot \left[\prod_{l=1}^{k} \exp\!\left(\tfrac{U_{1,l}(x)+U_{1,l}(x)}{2q_{1,l}} \right) \right] \\
&\cdot \left\| \exp\!\left(\int_0^\tau c(s)\, ds + \sum_{l=1}^{k} \int_0^\tau \tfrac{\beta_{1,l}}{q_{1,l} e^{\alpha_{1,l} s}}\, ds - \sum_{l=1}^{k} \tfrac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} - \sum_{l=1}^{k} \int_\tau^T \tfrac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}}\, ds \right) \right\|_{L^\infty(\Omega;\mathbb{R})}
\end{aligned}
\tag{84}
$$

and this implies that

$$
\begin{aligned}
&\left\| e^{\int_0^\tau V(X_s^x, X_s^y)\, ds} \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \exp\!\left(\sum_{l=1}^{k} \left[\int_0^T \tfrac{\beta_{0,l}(1-\frac{s}{T})}{q_{0,l} e^{\alpha_{0,l} s}}\, ds + \tfrac{U_{0,l}(x)+U_{0,l}(y)}{2q_{0,l}} + \tfrac{U_{1,l}(x)+U_{1,l}(x)}{2q_{1,l}} \right] \right) \\
&\cdot \left\| \exp\!\left(\int_0^\tau c(s)\, ds + \sum_{l=1}^{k} \int_0^\tau \tfrac{\beta_{1,l}}{q_{1,l} e^{\alpha_{1,l} s}}\, ds - \sum_{l=1}^{k} \tfrac{U_{1,l}(X_\tau^x)+U_{1,l}(X_\tau^y)}{2q_{1,l} e^{\alpha_{1,l}\tau}} - \sum_{l=1}^{k} \int_\tau^T \tfrac{U_{0,l}(X_s^x)+U_{0,l}(X_s^y)}{2q_{0,l} T e^{\alpha_{0,l} s}}\, ds \right) \right\|_{L^\infty(\Omega;\mathbb{R})} .
\end{aligned}
$$

The proof of Lemma 2.22 is thus completed. □

In the next step we present Theorem 2.23 which is the main result of this subsection. It is an immediate consequence of Proposition 2.17 and Lemma 2.22. Theorem 2.23 appeared in the special case $c = 0$ and $U_0 \equiv 0 \equiv U_1$ in Theorem 2.1 in Maslowski [34] (cf. Ichikawa [23] and, e.g., Leha & Ritter [28, 29])). In Section 4 below various examples of SDEs are presented that fulfill the assumptions of Theorem 2.23.

Theorem 2.23. *Assume the setting in Section 1.2, let $x, y \in O$, $k \in \mathbb{N}$, $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}^k$, $c \in \mathcal{L}^0([0,T];\mathbb{R})$, $V \in C^2(O^2,[0,\infty))$, $U_0 \in C^2(O,\mathbb{R}^k)$, $U_1 \in C^2(O,[0,\infty)^k)$, $\overline{U} \in C(O,\mathbb{R}^k)$, $r, p \in (0,\infty]$, $q_0, q_1 \in (0,\infty]^k$ with $\frac{1}{p} + \sum_{i=0}^{1} \sum_{l=1}^{k} \frac{1}{q_{i,l}} = \frac{1}{r}$ and let $X^z \colon [0,T] \times \Omega \to O$, $z \in \{x,y\}$, be adapted stochastic processes with continuous sample paths satisfying*

$$
\tfrac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(v,w)}{V(v,w)} + \tfrac{(p-1)\, \|(\overline{G}_\sigma V)(v,w)\|^2}{2\,(V(v,w))^2} \leq c(t) + \textstyle\sum_{n=1}^{k} \left[\tfrac{U_{0,n}(v)+U_{0,n}(w)}{2q_{0,n} T e^{\alpha_{0,n} t}} + \tfrac{\overline{U}_n(v)+\overline{U}_n(w)}{2q_{1,n} e^{\alpha_{1,n} t}} \right] \tag{85}
$$

$$
\text{and} \qquad (\mathcal{G}_{\mu,\sigma} U_{i,l})(u) + \tfrac{1}{2 e^{\alpha_{i,l} t}} \|\sigma(u)^* (\nabla U_{i,l})(u)\|^2 + \mathbb{1}_{\{1\}}(i) \cdot \overline{U}_l(u) \leq \alpha_{i,l} U_{i,l}(u) + \beta_{i,l} \tag{86}
$$

for all $i \in \{0,1\}$, $l \in \{1,\ldots,k\}$, $u \in \{v,w\}$, $(v,w) \in \operatorname{im}(X_t^x) \times \operatorname{im}(X_t^y)$, $t \in [0,T]$ and $\int_0^T |c(s)| + \|\mu(X_s^z)\| + \|\sigma(X_s^z)\|^2 + \frac{\|(\overline{G}_\sigma V)(X_s^x, X_s^y)\|^2}{(V(X_s^x, X_s^y))^2}\, ds < \infty$ $\mathbb{P}$-a.s. and $X_t^z = z + \int_0^t \mu(X_s^z)\, ds + \int_0^t \sigma(X_s^z)\, dW_s$ $\mathbb{P}$-a.s. for all $(t,z) \in [0,T] \times \{x,y\}$. Then

$$
\|V(X_T^x, X_T^y)\|_{L^r(\Omega;\mathbb{R})} \leq \exp\!\left(\int_0^T c(s)\, ds + \sum_{i=0}^{1} \sum_{l=1}^{k} \left[\int_0^T \tfrac{\beta_{i,l}\, (1-\frac{s}{T})^{(1-i)}}{q_{i,l}\, e^{\alpha_{i,l} s}}\, ds + \tfrac{U_{i,l}(x)+U_{i,l}(y)}{2q_{i,l}} \right] \right) V(x,y) .
$$

Corollary 2.25 below specialises Theorem 2.23 to the case where μ and σ are locally Lipschitz continuous functions, where $k = 1$ and where $V \in C^2(O^2,[0,\infty))$ satisfies $V(x,y) = \|x-y\|^2$ for all $x, y \in O$. For this the following result from the literature (see, e.g., Theorem 2.1 in Yamada & Ogura [53]) is needed.

Lemma 2.24. *Assume the setting in Section 1.2, let $\mu \colon O \to \mathbb{R}^d$ and $\sigma \colon O \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous, let $\tau \colon \Omega \to [0,T]$ be a stopping time and let $X^i \colon [0,T] \times \Omega \to O$, $i \in \{1,2\}$, be adapted stochastic processes with continuous sample paths satisfying $X_{t\wedge\tau}^i = X_0^i + \int_0^{t\wedge\tau} \mu(X_s^i)\, ds + \int_0^{t\wedge\tau} \sigma(X_s^i)\, dW_s$ $\mathbb{P}$-a.s. for all $(t,i) \in [0,T] \times \{1,2\}$. Then $\mathbb{P}[\{\exists\, t \in [0,\tau] \colon X_t^1 = X_t^2\} \cap \{X_0 \neq Y_0\}] = 0$.*

We are now ready to present the promised Corollary 2.25. It follows immediately from Theorem 2.23 and Example 2.15.

Corollary 2.25. *Assume the setting in Section 1.2, let* $\mu\colon O \to \mathbb{R}^d$ *and* $\sigma\colon O \to \mathbb{R}^{d\times m}$ *be locally Lipschitz continuous, let* $\alpha_0, \alpha_1, \beta_0, \beta_1, c \in \mathbb{R}$, $r, p, q_0, q_1 \in (0, \infty]$ *with* $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{r}$, $U_0 \in C^2(O, \mathbb{R})$, $U_1 \in C^2(O, [0, \infty))$, $\overline{U} \in C(O, \mathbb{R})$ *and let* $X^x\colon [0, T] \times \Omega \to O$, $x \in O$, *be adapted stochastic processes with continuous sample paths satisfying* $X^x_t = x + \int_0^t \mu(X^x_s)\, ds + \int_0^t \sigma(X^x_s)\, dW_s$ $\mathbb{P}$*-a.s.,* $(\mathcal{G}_{\mu,\sigma} U_i)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U_i)(x)\|^2 + \mathbb{1}_{\{1\}}(i) \cdot \overline{U}(x) \le \alpha_i U_i(x) + \beta_i$ *and*

$$\frac{\langle x-y, \mu(x)-\mu(y)\rangle + \frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|x-y\|^2} + \frac{(\frac{p}{2}-1)\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \le c + \frac{U_0(x)+U_0(y)}{2q_0 T e^{\alpha_0 t}} + \frac{\overline{U}(x)+\overline{U}(y)}{2q_1 e^{\alpha_1 t}}$$

for all $i \in \{0, 1\}$, $(t, x, y) \in [0, T] \times O^2$ *with* $x \neq y$*. Then it holds for all* $x, y \in O$ *that*

$$\|X^x_T - X^y_T\|_{L^r(\Omega;\mathbb{R}^d)} \le \exp\Big(cT + \textstyle\sum_{i=0}^1 \big[\int_0^T \frac{\beta_i\,(1-\frac{s}{T})^{(1-i)}}{q_i e^{\alpha_i s}}\, ds + \frac{U_i(x)+U_i(y)}{2q_i}\big]\Big)\, \|x - y\|. \tag{87}$$

2.3.3 Uniform strong stability analysis for solutions of SDEs

Proposition 2.26. *Assume the setting in Section 1.2, let* $x, y \in O$, $V \in C^2(O^2, [0, \infty))$*, let* $\tau\colon \Omega \to [0, T]$ *be a stopping time and let* $X^z\colon [0, T] \times \Omega \to O$, $z \in \{x, y\}$*, be adapted stochastic processes with continuous sample paths satisfying* $\int_0^\tau \|\mu(X^z_s)\| + \|\sigma(X^z_s)\|^2 + \max\big(\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)}, 0\big) + \frac{\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{(V(X^x_s,X^y_s))^2}\, ds < \infty$ $\mathbb{P}$*-a.s. and* $X^z_{t\wedge\tau} = z + \int_0^{t\wedge\tau} \mu(X^z_s)\, ds + \int_0^{t\wedge\tau} \sigma(X^z_s)\, dW_s$ $\mathbb{P}$*-a.s. for all* $(t, z) \in [0, T] \times \{x, y\}$*. Then*

$$\begin{aligned}
\Big\|\sup_{t\in[0,\tau]} V(X^x_t, X^y_t)\Big\|_{L^r(\Omega;\mathbb{R})} &\le \frac{V(x,y)}{\big[1-\frac{\theta}{p}\big]^{1/\theta}} \Big\| \exp\Big(\big[\tfrac{1}{(\frac{1}{p}-\frac{1}{v})} - \theta\big] \int_0^\tau \tfrac{\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|_{L^v(\Omega;\mathbb{R})} \\
&\quad \cdot \Big\|\exp\Big(\sup_{t\in[0,\tau]} \int_0^t \tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)} + \tfrac{(\theta-1)\,\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|_{L^q(\Omega;\mathbb{R})}
\end{aligned}$$

for all $v \in [p, \infty]$, $\theta \in (0, p)$ *and all* $p, q, r \in (0, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof of Proposition 2.26. Let $p, q, r \in (0, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\theta \in (0, p)$. Then Hölder's inequality proves that

$$\begin{aligned}
&\Big\|\sup_{t\in[0,\tau]} \exp\Big(\int_0^t \tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)} - \tfrac{\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds + \int_0^t \tfrac{(\overline{G}_\sigma V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)}\, dW_s\Big)\Big\|_{L^r(\Omega;\mathbb{R})} \\
&\le \Big\|\sup_{t\in[0,\tau]} \exp\Big(\int_0^t \tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)} + \tfrac{(\theta-1)\,\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|_{L^q(\Omega;\mathbb{R})} \\
&\quad \cdot \Big\|\sup_{t\in[0,\tau]} \exp\Big(\int_0^t \tfrac{(\overline{G}_\sigma V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)}\, dW_s - \int_0^t \tfrac{\theta\,\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|_{L^p(\Omega;\mathbb{R})} \\
&= \Big\|\exp\Big(\sup_{t\in[0,\tau]} \int_0^t \tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)} + \tfrac{(\theta-1)\,\|(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|_{L^q(\Omega;\mathbb{R})} \\
&\quad \cdot \Big\|\sup_{t\in[0,T]} \exp\Big(\int_0^t \tfrac{\mathbb{1}_{\{s<\tau\}}\,\theta\,(\overline{G}_\sigma V)(X^x_s,X^y_s)}{V(X^x_s,X^y_s)}\, dW_s - \int_0^t \tfrac{\mathbb{1}_{\{s<\tau\}}\|\theta(\overline{G}_\sigma V)(X^x_s,X^y_s)\|^2}{2\,(V(X^x_s,X^y_s))^2}\, ds\Big)\Big\|^{1/\theta}_{L^{p/\theta}(\Omega;\mathbb{R})}.
\end{aligned} \tag{88}$$

In addition, Lemma 2.7 gives

$$
\begin{aligned}
&\left\| \sup_{t\in[0,T]} \exp\!\left(\int_0^t \tfrac{\mathbb{1}_{\{s<\tau\}}\,\theta\,(\overline{G}_\sigma V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)}\,dW_s - \int_0^t \tfrac{\mathbb{1}_{\{s<\tau\}}\|\theta(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^{p/\theta}(\Omega;\mathbb{R})}^{1/\theta} \\
&\le \inf_{v\in[\frac{p}{\theta},\infty]} \left[\frac{1}{(1-\frac{\theta}{p})} \left\| \exp\!\left(\tfrac{1}{2}\Big[\tfrac{1}{(\frac{\theta}{p}-\frac{1}{v})}-1\Big] \int_0^\tau \tfrac{\theta^2\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^v(\Omega;\mathbb{R})} \right]^{1/\theta} \\
&= \frac{1}{\big[1-\frac{\theta}{p}\big]^{1/\theta}} \inf_{v\in[\frac{p}{\theta},\infty]} \left\| \exp\!\left(\Big[\tfrac{1}{(\frac{\theta}{p}-\frac{1}{v})}-1\Big] \int_0^\tau \tfrac{\theta\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^{v\theta}(\Omega;\mathbb{R})} \\
&= \frac{1}{\big[1-\frac{\theta}{p}\big]^{1/\theta}} \inf_{v\in[p,\infty]} \left\| \exp\!\left(\Big[\tfrac{1}{(\frac{1}{p}-\frac{1}{v})}-\theta\Big] \int_0^\tau \tfrac{\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^{v}(\Omega;\mathbb{R})}.
\end{aligned}
\tag{89}
$$

Combining (88), (89) and Proposition 2.12 proves that

$$
\begin{aligned}
&\left\| \sup_{t\in[0,\tau]} |V(X_t^x,X_t^y)| \right\|_{L^r(\Omega;\mathbb{R})} \\
&\le \frac{|V(x,y)|}{\big[1-\frac{\theta}{p}\big]^{1/\theta}} \left\| \exp\!\left(\Big[\tfrac{1}{(\frac{1}{p}-\frac{1}{v})}-\theta\Big] \int_0^\tau \tfrac{\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^{v}(\Omega;\mathbb{R})} \\
&\quad\cdot \left\| \exp\!\left(\sup_{t\in[0,\tau]} \int_0^t \tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^x,X_s^y)}{V(X_s^x,X_s^y)} + \tfrac{(\theta-1)\,\|(\overline{G}_\sigma V)(X_s^x,X_s^y)\|^2}{2\,(V(X_s^x,X_s^y))^2}\,ds \right) \right\|_{L^{q}(\Omega;\mathbb{R})}
\end{aligned}
\tag{90}
$$

for all $v \in [p,\infty]$. This completes the proof of Proposition 2.26. □

The next remark to Proposition 2.26 is the uniform counterpart to Remark 2.18.

Remark 2.27. *Note in the setting of Proposition 2.26 that if $\hat{V} \in C^2(O,[0,\infty))$, if $x = y$ and if $V(v,w) = \hat{V}(v)$ for all $v,w \in O$, then Proposition 2.26 reduces to an estimate for $\|\sup_{t\in[0,\tau]} \hat{V}(X_t^x)\|_{L^r(\Omega;\mathbb{R})}$ for $p,q,r \in (0,\infty]$, $v \in [p,\infty]$, $\theta \in (0,p)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.*

The next corollary, Corollary 2.28, specialises Proposition 2.26 to the case where the function $V \in C^2(O^2,[0,\infty))$ satisfies $V(x,y) = \|x-y\|^2$ for all $x,y \in O$ (cf. Theorem 5.1 in Li [30]). Corollary 2.28 follows immediately from Proposition 2.26 and Example 2.15.

Corollary 2.28. *Assume the setting in Section 1.2, let $x,y \in O$, let $\tau\colon \Omega \to [0,T]$ be a stopping time and let $X^z\colon [0,T]\times\Omega \to O$, $z \in \{x,y\}$, be adapted stochastic processes with continuous sample paths satisfying $\int_0^\tau \|\mu(X_s^z)\| + \|\sigma(X_s^z)\|^2 + \frac{\max(\langle X_s^x - X_s^y, \mu(X_s^x)-\mu(X_s^y)\rangle,0)+\|\sigma(X_s^x)-\sigma(X_s^y)\|^2}{\|X_s^x-X_s^y\|^2}\,ds < \infty$ $\mathbb{P}$-a.s. and $X_{t\wedge\tau}^z = z + \int_0^{t\wedge\tau} \mu(X_s^z)\,ds + \int_0^{t\wedge\tau} \sigma(X_s^z)\,dW_s$ $\mathbb{P}$-a.s. for all $(t,z) \in [0,T]\times\{x,y\}$. Then*

$$
\begin{aligned}
&\left\| \sup_{t\in[0,\tau]} \|X_t^x - X_t^y\| \right\|_{L^r(\Omega;\mathbb{R})} \\
&\le \frac{\|x-y\|}{\big[1-\frac{\theta}{p}\big]^{1/\theta}} \left\| \exp\!\left(\Big[\tfrac{1}{(\frac{1}{p}-\frac{1}{v})}-\theta\Big] \int_0^\tau \tfrac{\|(\sigma(X_s^x)-\sigma(X_s^y))^*(X_s^x-X_s^y)\|^2}{2\,\|X_s^x-X_s^y\|^4}\,ds \right) \right\|_{L^{v}(\Omega;\mathbb{R})} \\
&\quad\cdot \left\| \exp\!\left(\sup_{t\in[0,\tau]} \int_0^t \left[\begin{array}{r} \frac{\langle X_s^x-X_s^y,\mu(X_s^x)-\mu(X_s^y)\rangle+\frac{1}{2}\|\sigma(X_s^x)-\sigma(X_s^y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|X_s^x-X_s^y\|^2} \\ +\frac{(\theta-1)\,\|(\sigma(X_s^x)-\sigma(X_s^y))^*(X_s^x-X_s^y)\|^2}{\|X_s^x-X_s^y\|^4} \end{array} \right] ds \right) \right\|_{L^{q}(\Omega;\mathbb{R})}
\end{aligned}
\tag{91}
$$

for all $v \in [p,\infty]$, $\theta \in (0,p)$ and all $p,q,r \in (0,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$.

The next theorem is the uniform counterpart to Theorem 2.23. It follows directly from Proposition 2.26 and from Lemma 2.22.

Theorem 2.29. *Assume the setting in Section 1.2, let $x, y \in O$, $k \in \mathbb{N}$, $r, p \in (0, \infty]$, $\theta \in (0, p)$, $\rho \in [p, \infty]$, $(\alpha_{i,j,l})_{i,j\in\{0,1\},l\in\{1,\ldots,k\}}, (\beta_{i,j,l})_{i,j\in\{0,1\},l\in\{1,\ldots,k\}} \subset \mathbb{R}$, $(q_{i,j,l})_{i,j\in\{0,1\},l\in\{1,\ldots,k\}} \subset [r, \infty]$ with $\sum_{i=0}^{1}\sum_{l=1}^{k} \frac{1}{q_{0,i,l}} = \frac{1}{\rho}$ and $\frac{1}{p} + \sum_{i=0}^{1}\sum_{l=1}^{k} \frac{1}{q_{1,i,l}} = \frac{1}{r}$, $c_0, c_1 \in \mathcal{L}^0([0,T]; \mathbb{R})$, $V \in C^2(O^2, [0,\infty))$, $U_{0,0}, U_{1,0} \in C^2(O, \mathbb{R}^k)$, $U_{0,1}, U_{1,1} \in C^2(O, [0,\infty)^k)$, $\overline{U}_{0,1}, \overline{U}_{1,1} \in C(O, \mathbb{R}^k)$ and let $X^z \colon [0,T] \times \Omega \to O$, $z \in \{x, y\}$, be adapted stochastic processes with continuous sample paths satisfying $\int_0^T c_0(s) + c_1(s) + \|\mu(X_s^z)\| + \|\sigma(X_s^z)\|^2 \, ds < \infty$ $\mathbb{P}$-a.s. and $X_t^z = z + \int_0^t \mu(X_s^z)\, ds + \int_0^t \sigma(X_s^z)\, dW_s$ $\mathbb{P}$-a.s. for all $(t, z) \in [0,T] \times \{x, y\}$ and*

$$(\mathcal{G}_{\mu,\sigma} U_{i,j,l})(u) + \tfrac{1}{2e^{\alpha_{i,j,l}t}} \|\sigma(u)^*(\nabla U_{i,j,l})(u)\|^2 + \mathbb{1}_{\{1\}}(j) \cdot \overline{U}_{i,j,l}(u) \le \alpha_{i,j,l} U_{i,j,l}(u) + \beta_{i,j,l}, \tag{92}$$

$$\left[\tfrac{1}{(2/p-2/\rho)} - \tfrac{\theta}{2}\right] \tfrac{\|(\overline{G}_\sigma V)(v,w)\|^2}{(V(v,w))^2} \le c_0(t) + \textstyle\sum_{n=1}^{k} \left[\tfrac{U_{0,0,n}(v)+U_{0,0,n}(w)}{2q_{0,0,n}Te^{\alpha_{0,0,n}t}} + \tfrac{\overline{U}_{0,1,n}(v)+\overline{U}_{0,1,n}(w)}{2q_{0,1,n}e^{\alpha_{0,1,n}t}}\right] \quad \textit{and} \tag{93}$$

$$0 \vee \left[\tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(v,w)}{V(v,w)} + \tfrac{(\theta-1)\,\|(\overline{G}_\sigma V)(v,w)\|^2}{2\,(V(v,w))^2}\right] \le c_1(t) + \textstyle\sum_{n=1}^{k} \left[\tfrac{U_{1,0,n}(v)+U_{1,0,n}(w)}{2q_{1,0,n}Te^{\alpha_{1,0,n}t}} + \tfrac{\overline{U}_{1,1,n}(v)+\overline{U}_{1,1,n}(w)}{2q_{1,1,n}e^{\alpha_{1,1,n}t}}\right] \tag{94}$$

for all $i \in \{0,1\}$, $l \in \{1, \ldots, k\}$, $u \in \{v, w\}$, $(v,w) \in \operatorname{im}(X_t^x) \times \operatorname{im}(X_t^y)$, $t \in [0,T]$. Then

$$\begin{aligned} &\left\| \sup_{t\in[0,T]} |V(X_t^x, X_t^y)| \right\|_{L^r(\Omega;\mathbb{R})} \\ &\le \exp\!\left(\int_0^T c_0(s) + c_1(s)\, ds + \sum_{i,j=0}^{1} \sum_{l=1}^{k} \left[\int_0^T \tfrac{\beta_{i,j,l}\,(1-\frac{s}{T})^{(1-j)}}{q_{i,j,l}\, e^{\alpha_{i,j,l}s}}\, ds + \tfrac{U_{i,j,l}(x)+U_{i,j,l}(y)}{2q_{i,j,l}} \right] \right) \frac{V(x,y)}{\left[1 - \frac{\theta}{p}\right]^{1/\theta}}. \end{aligned} \tag{95}$$

The next corollary specialises Theorem 2.29 to the case where $k = 1$, where V satisfies $V(x,y) = \|x - y\|^2$ for all $x, y \in O$ and where μ and σ are continuous (cf. Lemma 2.3 in Zhang [54]). It follows directly from Theorem 2.29 and from Example 2.15.

Corollary 2.30. *Assume the setting in Section 1.2, let $x, y \in O$, $r, p \in (0, \infty]$, $\theta \in (0, p)$, $\rho \in [p, \infty]$, $(\alpha_{i,j})_{i,j\in\{0,1\}}, (\beta_{i,j})_{i,j\in\{0,1\}} \subset \mathbb{R}$, $(q_{i,j})_{i,j\in\{0,1\}} \subset [r, \infty]$ with $\sum_{i=0}^{1} \frac{1}{q_{0,i}} = \frac{1}{\rho}$ and $\frac{1}{p} + \sum_{i=0}^{1} \frac{1}{q_{1,i}} = \frac{1}{r}$, $c_0, c_1 \in C([0,T], \mathbb{R})$, $U_{0,0}, U_{1,0} \in C^2(O, \mathbb{R})$, $U_{0,1}, U_{1,1} \in C^2(O, [0,\infty))$, $\overline{U}_{0,1}, \overline{U}_{1,1} \in C(O, \mathbb{R})$, $\mu \in C(O, \mathbb{R}^d)$, $\sigma \in C(O, \mathbb{R}^{d\times m})$ and let $X^z \colon [0,T] \times \Omega \to O$, $z \in \{x, y\}$, be adapted stochastic processes with continuous sample paths satisfying $X_t^z = z + \int_0^t \mu(X_s^z)\, ds + \int_0^t \sigma(X_s^z)\, dW_s$ $\mathbb{P}$-a.s. for all $(t, z) \in [0,T] \times \{x, y\}$ and*

$$(\mathcal{G}_{\mu,\sigma} U_{i,j})(u) + \tfrac{1}{2e^{\alpha_{i,j}t}} \|\sigma(u)^*(\nabla U_{i,j})(u)\|^2 + \mathbb{1}_{\{1\}}(j) \cdot \overline{U}_{i,j}(u) \le \alpha_{i,j} U_{i,j}(u) + \beta_{i,j}, \tag{96}$$

$$2?\left[\tfrac{1}{(2/p-2/\rho)} - \tfrac{\theta}{2}\right] \tfrac{\|(v-w)^*(\sigma(v)-\sigma(w))\|^2}{\|v-w\|^4} \le c_0(t) + \tfrac{U_{0,0}(v)+U_{0,0}(w)}{2q_{0,0}Te^{\alpha_{0,0}t}} + \tfrac{\overline{U}_{0,1}(v)+\overline{U}_{0,1}(w)}{2q_{0,1}e^{\alpha_{0,1}t}} \quad \textit{and} \tag{97}$$

$$\begin{aligned} &\max\!\left\{0, \tfrac{\langle v-w, \mu(v)-\mu(w)\rangle + \frac{1}{2}\|\sigma(v)-\sigma(w)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|v-w\|^2} + \tfrac{(\theta/2-1)\,\|(v-w)^*(\sigma(v)-\sigma(w))\|^2}{\|v-w\|^4}\right\} \\ &\le c_1(t) + \tfrac{U_{1,0}(v)+U_{1,0}(w)}{2q_{1,0}Te^{\alpha_{1,0}t}} + \tfrac{\overline{U}_{1,1}(v)+\overline{U}_{1,1}(w)}{2q_{1,1}e^{\alpha_{1,1}t}} \end{aligned} \tag{98}$$

for all $i, j \in \{0,1\}$, $u \in \{v, w\}$, $(v,w) \in \operatorname{im}(X_t^x) \times \operatorname{im}(X_t^y)$, $t \in [0,T]$. Then

$$\left\| \sup_{t\in[0,T]} \frac{\|X_t^x - X_t^y\|}{\|x-y\|} \right\|_{L^r(\Omega;\mathbb{R})} \le \frac{e^{\int_0^T c_0(s)+c_1(s)\,ds}}{[1-\frac{\theta}{p}]^{1/\theta}} \exp\!\left(\sum_{i,j=0}^{1} \left[\int_0^T \tfrac{\beta_{i,j}\,(1-\frac{s}{T})^{(1-j)}}{q_{i,j}\, e^{\alpha_{i,j}s}}\, ds + \tfrac{U_{i,j}(x)+U_{i,j}(y)}{2q_{i,j}} \right] \right).$$

The next corollary (Corollary 2.31) is the special case of Corollary 2.30 where $U_{0,0}(x) = U_{1,0}(x) = c\,(1 + \|x\|^2)^\varepsilon$ and $U_{0,1}(x) = U_{1,1}(x) = \bar{U}_{0,1}(x) = \bar{U}_{1,1}(x) = 0$ for all $x \in \mathbb{R}^d$ and some $c \in (0, \infty)$, $\varepsilon \in (0, 1]$ and where $q_{0,1} = \infty = q_{1,1}$, $q_{1,0} = 2r = p$, $q_{0,0} = 4r = \rho$, $\theta = 2$, $\beta_{0,0} = \beta_{1,0} = \beta$, $\beta_{0,1} = \beta_{1,1} = 0$, $r \in (1, \infty]$. Corollary 2.31 is related to Theorem 1.7 in Fang, Imkeller & Zhang [12] and to Corollary 6.3 in Li [30].

Corollary 2.31. *Assume the setting in Section 1.2, let $x, y \in O$, $\mu \in C(O, \mathbb{R}^d)$, $\alpha, \beta, c_0, c_1 \in [0, \infty)$, $c \in (0, \infty)$, $r \in (1, \infty]$ and let $X^z\colon [0,T] \times \Omega \to O$, $z \in \{x, y\}$, be adapted stochastic processes with continuous sample paths satisfying $X_t^z = z + \int_0^t \mu(X_s^z)\, ds + \int_0^t \sigma(X_s^z)\, dW_s$ $\mathbb{P}$-a.s. for all $(t, z) \in [0, T] \times \{x, y\}$ and*

$$\begin{aligned}
\tfrac{\langle v-w, \mu(v)-\mu(w)\rangle + \frac{1}{2}\|\sigma(v)-\sigma(w)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|v-w\|^2} + \tfrac{\|(\sigma(v)-\sigma(w))^*(v-w)\|^2}{2\,\|v-w\|^4} &\le c_1 + \tfrac{c\,(1+\|v\|^2)^\varepsilon + c\,(1+\|w\|^2)^\varepsilon}{4rTe^{\alpha T}}, \\
\tfrac{(4r-2)\,\|(\sigma(v)-\sigma(w))^*(v-w)\|^2}{2\,\|v-w\|^4} &\le c_0 + \tfrac{c\,(1+\|v\|^2)^\varepsilon + c\,(1+\|w\|^2)^\varepsilon}{8rTe^{\alpha T}}, \\
2\,\langle u, \mu(u)\rangle + \|\sigma(u)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \tfrac{2c\varepsilon\|\sigma(u)^* u\|^2}{(1+\|u\|^2)^{(1-\varepsilon)}} &\le \tfrac{\alpha}{\varepsilon}\left(1+\|u\|^2\right) + \tfrac{\beta}{c\varepsilon}\left(1+\|u\|^2\right)^{(1-\varepsilon)}
\end{aligned} \tag{99}$$

for all $u \in \{v, w\}$, $(v, w) \in \operatorname{im}(X^x) \times \operatorname{im}(X^y)$. Then

$$\left\| \sup_{t\in[0,T]} \frac{\|X_t^x - X_t^y\|}{\|x-y\|} \right\|_{L^r(\Omega;\mathbb{R})} \le \left[1 - \tfrac{1}{r}\right]^{-\frac{1}{2}} \exp\!\Big((c_0 + c_1)T + \tfrac{\beta T + c(1+\|x\|^2)^\varepsilon + c(1+\|y\|^2)^\varepsilon}{2r} \Big). \tag{100}$$

3 Strong completeness of SDEs

The theory developed in Subsection 2.3 can be used to establish *strong completeness* of SDEs with non-globally Lipschitz continuous nonlinearities by combining it with a suitable Kolmogorov argument; see Lemma 3.1 and Lemma 3.2 below. As, e.g., in the proof of Theorem 2.4 in Zhang [54] we exploit here that local Lipschitz continuity estimates on a time interval of positive length are sufficient to establish strong completeness on the whole time interval $[0, \infty)$.

First, however, we recall some notation: for $\alpha \in [0, 1]$, $d \in \mathbb{N}$, a set $D \subseteq \mathbb{R}^d$ and an $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$, we define the real Banach space of globally bounded and α-Hölder continuous functions from D to E by

$$C_b^\alpha(D, E) := \left\{ f \in C(D, E)\colon \sup_{x\in D} \|f(x)\|_E + \sup_{x,y\in D, x\neq y} \tfrac{\|f(x)-f(y)\|_E}{\|x-y\|^\alpha} < \infty \right\}. \tag{101}$$

In addition, for $\alpha \in [0, 1]$, $d \in \mathbb{N}$, a set $D \subseteq \mathbb{R}^d$ and an $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$, we say that a mapping $f\colon D \to E$ is *locally α-Hölder continuous* if for every $x \in D$ there exists a relatively open set $U \subseteq D$ containing x such that $f|_U \in C_b^\alpha(U, E)$.

Lemma 3.1 (Strong completeness based on uniform strong stability estimates)**.** *Let $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be a closed set, let $\mu \in \mathcal{L}^0(D; \mathbb{R}^d)$ and $\sigma \in \mathcal{L}^0(D; \mathbb{R}^{d\times m})$ be locally Lipschitz continuous, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0, \infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion and assume that for every $x \in D$ there exists an adapted stochastic process $X^x\colon [0, \infty) \times \Omega \to D$ with continuous sample paths satisfying*

$$X_t^x = x + \int_0^t \mu(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{102}$$

$\mathbb{P}$-a.s. for every $t \in [0, \infty)$. In addition, assume that there exist $\varepsilon \in (0, \infty)$, $p \in (d, \infty)$, $\alpha \in (\frac{d}{p}, 1]$ such that for every $x \in D$ it holds that $(X_t^x)_{t\in[0,\varepsilon]} \in L^p(\Omega; C([0, \varepsilon], \mathbb{R}^d))$ and such that the mapping $D \ni x \mapsto (X_t^x)_{t\in[0,\varepsilon]} \in L^p(\Omega; C([0, \varepsilon], \mathbb{R}^d))$ is locally α-Hölder continuous. Then there exists a $Y \in \mathcal{L}^0([0, \infty) \times D \times \Omega; D)$ such that for every $\omega \in \Omega$ it holds that $Y(\cdot, \cdot, \omega) \in C([0, \infty) \times D, D)$ and such that for every $x \in D$ it holds that $(Y(t, x))_{t\in[0,\infty)} = (X_t^x)_{t\in[0,\infty)}$ $\mathbb{P}$-a.s.

Lemma 3.2 (Strong completeness based on marginal strong stability estimates)**.** *Let $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be a closed set, let $\mu \in \mathcal{L}^0(D; \mathbb{R}^d)$ and $\sigma \in \mathcal{L}^0(D; \mathbb{R}^{d\times m})$ be locally Lipschitz continuous,* *let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ *be a stochastic basis, let* $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ *be a standard* $(\mathcal{F}_t)_{t\in[0,\infty)}$*-Brownian motion and assume that for every* $x \in D$ *there exists an adapted stochastic process* $X^x\colon [0,\infty)\times\Omega \to D$ *with continuous sample paths satisfying*

$$X_t^x = x + \int_0^t \mu(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{103}$$

$\mathbb{P}$*-a.s. for every* $t \in [0,\infty)$*. In addition, assume that there exist* $\varepsilon \in (0,\infty)$*,* $p \in (d+1,\infty)$*,* $\alpha \in (\frac{d+1}{p}, 1]$ *such that for every* $(t,x) \in [0,\varepsilon]\times D$ *it holds that* $X_t^x \in L^p(\Omega;\mathbb{R}^d)$ *and such that the mapping* $[0,\varepsilon]\times D \ni (t,x) \mapsto X_t^x \in L^p(\Omega;\mathbb{R}^d)$ *is locally* α*-Hölder continuous. Then there exists a* $Y \in \mathcal{L}^0([0,\infty)\times D\times\Omega; D)$ *such that for every* $\omega\in\Omega$ *it holds that* $Y(\cdot,\cdot,\omega) \in C([0,\infty)\times D, D)$ *and such that for every* $x \in D$ *it holds that* $(Y(t,x))_{t\in[0,\infty)} = (X_t^x)_{t\in[0,\infty)}$ $\mathbb{P}$*-a.s.*

3.1 Theorems of Yamada-Watanabe and of Kolmogorov-Chentsov type

We shall use a Yamada-Watanabe type theorem (see Theorem 3.4 below) and the Kolmogorov-Chentsov theorem (see Theorem 3.5 below) to prove Lemmas 3.1 and 3.2. But first of all, we recall the following well-known notion from the literature.

Definition 3.3. *Let* $d, m \in \mathbb{N}$*, let* $D \subseteq \mathbb{R}^d$ *be a closed set, let* $\mu \in \mathcal{L}^0(D;\mathbb{R}^d)$*,* $\sigma \in \mathcal{L}^0(D;\mathbb{R}^{d\times m})$*, and let* $\nu\colon \mathcal{B}(D) \to [0,1]$ *be a probability measure. We say that* solutions to the SDE with coefficients (μ,σ) and initial distribution ν are pathwise unique *if for every stochastic basis* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$*, every standard* $(\mathcal{F}_t)_{t\in[0,\infty)}$*-Brownian motion* $W\colon [0,\infty)\times\Omega\to\mathbb{R}^m$ *and all adapted stochastic processes* $X^1, X^2\colon [0,\infty)\times\Omega \to D$ *with continuous sample paths and with* $\mathbb{P}\big[X_0^1 = X_0^2\big] = 1$*,* $\mathbb{P}_{X_0^1} = \nu$ *and*

$$X_t^i = X_0^i + \int_0^t \mu(X_s^i)\, ds + \int_0^t \sigma(X_s^i)\, dW_s \tag{104}$$

$\mathbb{P}$*-a.s. for every* $(i,t) \in \{1,2\}\times[0,\infty)$*, it holds that* $\mathbb{P}\big[X^1 = X^2\big] = 1$*.*

The following theorem is a slightly modified version of Theorem 21.14 in [24]. For the definition of *universally adapted* we refer to [24, Page 423].

Theorem 3.4. *Let* $d, m \in \mathbb{N}$*, let* $D \subseteq \mathbb{R}^d$ *be a non-empty closed set, let* $\mu \in \mathcal{L}^0(D;\mathbb{R}^d)$*,* $\sigma \in \mathcal{L}^0(D;\mathbb{R}^{d\times m})$*, let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ *be a stochastic basis, let* $W\colon [0,\infty)\times\Omega\to\mathbb{R}^m$ *be a standard* $(\mathcal{F}_t)_{t\in[0,\infty)}$*-Brownian motion and assume that for every* $x\in D$ *there exists an adapted stochastic process* $X^x\colon [0,\infty)\times\Omega\to D$ *with continuous sample paths satisfying*

$$X_t^x = x + \int_0^t \mu(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{105}$$

$\mathbb{P}$*-a.s. for every* $t\in[0,\infty)$*. Moreover, assume that for every* $x \in D$ *solutions to the SDE with coefficients* (μ,σ) *and initial distribution* δ_x *are pathwise unique. Then there exists a Borel measurable and universally adapted function*

$$F\colon D\times C([0,\infty),\mathbb{R}^m) \to C([0,\infty),\mathbb{R}^d) \tag{106}$$

such that for every stochastic basis $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t)_{t\in[0,\infty)})$*, every standard* $(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)}$*-Brownian motion* $\tilde W\colon [0,\infty)\times\tilde\Omega\to\mathbb{R}^m$ *and every* $\tilde X_0 \in \mathcal{L}^0(\tilde\Omega, \tilde{\mathcal{F}}_0; D)$ *it holds that the process* $\tilde X\colon [0,\infty)\times\tilde\Omega \to \mathbb{R}^d$ *defined by* $\tilde X := F(\tilde X_0, \tilde W)$ *is the up to indistinguishability unique* $(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)}$*-adapted stochastic process with continuous sample paths satisfying* $\tilde X \in C([0,\infty), D)$ $\tilde{\mathbb{P}}$*-a.s. and*

$$\tilde X_t = \tilde X_0 + \int_0^t \mu(\tilde X_s)\, ds + \int_0^t \sigma(\tilde X_s)\, d\tilde W_s \tag{107}$$

$\tilde{\mathbb{P}}$*-a.s. for every* $t\in[0,\infty)$*.*

Proof of Theorem 3.4. Let $\hat{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\hat{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be mappings given by $\hat{\mu}(x) = 0$, $\hat{\sigma}(x) = 0$ for every $x \in \mathbb{R}^d \backslash D$ and $\hat{\mu}(x) = \mu(x)$, $\hat{\sigma}(x) = \sigma(x)$ for every $x \in D$. Observe that $\hat{\mu} \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$ and $\hat{\sigma} \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Moreover, let $\hat{X}^x\colon [0,\infty)\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be given by $\hat{X}^x_t = x$ for every $x \in \mathbb{R}^d\backslash D$, $t \in [0,\infty)$ and by $\hat{X}^x_t = X^x_t$ for every $x \in D$, $t \in [0,\infty)$. Note for every $x \in \mathbb{R}^d$ that $\hat{X}^x$ is an adapted stochastic process with continuous sample paths and observe that

$$\hat{X}^x_t = x + \int_0^t \hat{\mu}(\hat{X}^x_s)\,ds + \int_0^t \hat{\sigma}(\hat{X}^x_s)\,dW_s \tag{108}$$

$\mathbb{P}$-a.s. for every $(t,x) \in [0,\infty)\times\mathbb{R}^d$. It follows from the fact that $\mathbb{R}^d\backslash D$ is open, that for every $x \in \mathbb{R}^d$ the solutions to the SDE with coefficients $(\hat{\mu},\hat{\sigma})$ and initial distribution δ_x are pathwise unique. In conclusion, the conditions of [24, Theorem 21.14] are satisfied, which provides the existence measurable and universally adapted mapping $F\colon D\times C([0,\infty),\mathbb{R}^m) \to C([0,\infty),\mathbb{R}^d)$ such that for every stochastic basis $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)})$, every standard $(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)}$-Brownian motion $\tilde{W}\colon [0,\infty)\times\tilde{\Omega} \to \mathbb{R}^m$ and every $\tilde{X}_0 \in \mathcal{L}^0(\tilde{\Omega};D)$ it holds that $\tilde{X} := F(\tilde{X}_0,\tilde{W})$ is the up to indistinguishability unique $(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)}$-adapted stochastic process with continuous sample paths satisfying

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \hat{\mu}(\tilde{X}_s)\,ds + \int_0^t \hat{\sigma}(\tilde{X}_s)\,d\tilde{W}_s \tag{109}$$

$\tilde{\mathbb{P}}$-a.s. for every $t \in [0,\infty)$. All that remains to be proven is that if $\tilde{X}_0 \in \mathcal{L}^0(\tilde{\Omega},\tilde{\mathcal{F}}_0;D)$, then $F(\tilde{X}_0,\tilde{W}) \in C([0,\infty),D)$ $\tilde{\mathbb{P}}$-a.s. This follows from the fact that by construction it holds for all $x \in D$ that $F(x,W) = X^x$ $\mathbb{P}$-a.s. and that $\tilde{W}$ is independent of $\tilde{X}_0$. □

The next theorem provides a suitable extension to Theorem 2.1 in [35].

Theorem 3.5. *Let $d \in \mathbb{N}$, $p \in (d,\infty)$, $\alpha \in (\frac{d}{p},1]$, let $(E,\|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space, let $F \subseteq E$ be a non-empty closed set, let $D \subseteq \mathbb{R}^d$ be a non-empty set and let $X \in C^\alpha_b(D,L^p(\Omega;F))$. Then there exists a $Y \in \cap_{\beta\in(0,\alpha-\frac{d}{p})} L^p(\Omega;C^\beta_b(\overline{D},F))$ such that for every $x \in D$ it holds that $Y(x) = X(x)$ $\mathbb{P}$-a.s.*

Proof of Theorem 3.5. In the case that $F = E$, the proof of this theorem is provided in Theorem 2.1 in [35]. In the case that $F \neq E$, we first observe that by the above there exists a $\hat{Y} \in \cap_{\beta\in(0,\alpha-\frac{d}{p})} L^p(\Omega;C^\beta_b(\overline{D},E))$ such that for every $x \in D$ it holds that $\hat{Y}(x) = X(x)$ $\mathbb{P}$-a.s. As $D \subseteq \mathbb{R}^d$ is separable, there exists a sequence $x_n \in D$, $n \in \mathbb{N}$, such that $\{x_n \in D\colon n \in \mathbb{N}\}$ is dense in D. This in turn implies that $\overline{\{x_n \in D\colon n \in \mathbb{N}\}} = \overline{D}$. Next we define $\Omega_0 \in \mathcal{F}$ by

$$\Omega_0 := \cap_{n\in\mathbb{N}}\{\hat{Y}(x_n) = X(x_n)\} = \cap_{n\in\mathbb{N}}\{\omega \in \Omega\colon \hat{Y}(x_n,\omega) = X(x_n,\omega)\} \tag{110}$$

and we observe that $\mathbb{P}\big[\Omega_0\big] = 1$. Moreover, note for every $\omega \in \Omega_0$, $y \in \overline{D}$ that there exists an increasing sequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} x_{n_k} = y$ and the continuity of $\hat{Y}(\cdot,\omega)$ and the closedness of F hence show that

$$\hat{Y}(y,\omega) = \hat{Y}\Big(\lim_{k\to\infty} x_{n_k},\omega\Big) = \lim_{k\to\infty}\hat{Y}(x_{n_k},\omega) = \lim_{k\to\infty} X(x_{n_k},\omega) \in F. \tag{111}$$

This proves for every $(\omega,y) \in \Omega_0\times\overline{D}$ that $Y(\omega,y) \in F$. In the next step let $\xi \in F$ be arbitrary and let $Y\colon \Omega \to C^\beta_b(\overline{D},F)$ be given by $Y(x,\omega) = \hat{Y}(x,\omega)$ for every $x \in \overline{D}$, $\omega \in \Omega_0$ and by $Y(x,\omega) = \xi$ for every $x \in \overline{D}$, $\omega \in \Omega\backslash\Omega_0$. By construction it holds that $\mathbb{P}\big[Y = \hat{Y}\big] = 1$ and for every $\omega \in \Omega$, $x \in \overline{D}$ that $Y(\omega,x) \in F$. The proof of Theorem 3.5 is thus completed. □

Remark 3.6. *The Kolmogorov-Chentsov theorem as provided in [35, Theorem 2.1] can be obtained from an extension result that can be found in the book of Stein [47].*

More specifically, it is proven in [47, Section VI.2.2.1] that for every $d \in \mathbb{N}$ and every $\alpha \in (0,1]$ there exists a real number $C \in [0,\infty)$ such that for every $D \subseteq \mathbb{R}^d$ there exists a mapping $\mathcal{E}_0\colon C_b^\alpha(D,\mathbb{R}^d) \to C_b^\alpha(\mathbb{R}^d,\mathbb{R}^d)$ such that for every $f \in C_b^\alpha(D,\mathbb{R}^d)$ it holds that $\mathcal{E}_0(f)|_D = f$ and

$$\|\mathcal{E}_0(f)\|_{C_b^\alpha(\mathbb{R}^d,\mathbb{R}^d)} \leq C\,\|f\|_{C_b^\alpha(D,\mathbb{R}^d)}. \tag{112}$$

The proof carries over mutatis mutandis to the space $C_b^\alpha(D,E)$, where $(E,\|\cdot\|_E)$ is a real Banach space. This extension operator in combination with the Kolmogorov-Chentsov theorem on rectangles provided, e.g., in [39, Theorem I.2.1] proves the Kolmogorov-Chentsov theorem as provided in [35, Theorem 2.1].

Note that this version of the proof of the Kolmogorov-Chentsov theorem does not rely on Sobolev embeddings. So, no assumptions are required regarding the smoothness of the domain D.

3.2 Proofs of the strong completeness results

Proof of Lemma 3.1. As μ and σ are locally Lipschitz continuous, using a standard localization argument one can prove for every $x \in D$ that solutions to the SDE with coefficients (μ,σ) and initial distribution δ_x are pathwise unique. Thus the conditions of Theorem 3.4 are satisfied. Let $F\colon D \times C([0,\infty),\mathbb{R}^m) \to C([0,\infty),\mathbb{R}^d)$ be the measurable and universally adapted function provided by that theorem. In particular, for every $x \in D$ we have

$$(X_t^x)_{t\in[0,\infty)} = F(x,W) \tag{113}$$

$\mathbb{P}$-a.s.

Throughout this proof we use the closed balls $B_R(0) \subseteq \mathbb{R}^d$, $R \in (0,\infty)$, given by $B_R(0) = \{x \in \mathbb{R}^d\colon \|x\| \leq R\}$ for every $R \in (0,\infty)$ and the stochastic processes $W^{*\theta_s}\colon [0,\infty)\times\Omega \to \mathbb{R}^m$, $s \in [0,\infty)$, given by $W_t^{*\theta_s} = W_{t+s} - W_s$ for every $s,t \in [0,\infty)$. Observe that $W^{*\theta_s}$ is a standard $(\mathcal{F}_{s+t})_{t\in[0,\infty)}$-Brownian motion for every $s \in [0,\infty)$. Clearly, it holds for every $s \in [0,\infty)$ that $F(x,W)$ and $F(x,W^{*\theta_s})$ are equal in distribution. It thus follows from (113) and the fact that $D \ni x \mapsto (X_t^x)_{t\in[0,\varepsilon]} \in L^p(\Omega; C([0,\varepsilon],\mathbb{R}^d))$ is locally α-Hölder continuous, that for every $n \in \mathbb{N}_0$ one has that $D \ni x \mapsto F(x,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \in L^p(\Omega; C([0,\varepsilon],\mathbb{R}^d))$ is locally α-Hölder continuous. As D is closed, it follows for every $n,R \in \mathbb{N}_0$ that $D \cap B_R(0)$ is a compact set and, consequently, that

$$\sup_{\substack{x,y\in D\cap B_R(0)\\ x\neq y}} \left[\frac{\big\|F(x,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} - F(y,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]}\big\|_{L^p(\Omega;C([0,\varepsilon],\mathbb{R}^d))}}{\|x-y\|^\alpha}\right] < \infty. \tag{114}$$

By applying Theorem 3.5 to the mappings $D\cap B_R(0) \ni x \mapsto F(x,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \in L^p(\Omega; C([0,\varepsilon],\mathbb{R}^d))$, and recalling that $F(x,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \in C([0,\varepsilon],D)$ $\mathbb{P}$-a.s., $(n,R) \in \mathbb{N}_0\times\mathbb{N}$, it follows that for every $n \in \mathbb{N}_0$ there exists a mapping $Y^{(n)} \in \mathcal{L}^0(D\times\Omega; C([0,\varepsilon],D))$ such that for every $x \in D$ it holds that $Y^{(n)}(x) = F(x,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]}$ $\mathbb{P}$-a.s. and such that for every $\omega \in \Omega$ it holds that $Y^{(n)}(\cdot,\omega) \in C(D, C([0,\varepsilon],D))$.

Claim. For every $n \in \mathbb{N}_0$ and every $\xi,\tilde{\xi} \in \mathcal{L}^0(\Omega,\mathcal{F}_{n\varepsilon}; D)$ satisfying $\xi = \tilde{\xi}$ $\mathbb{P}$-a.s. it holds that

$$Y^{(n)}(\tilde{\xi}) = F(\xi,W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \qquad \mathbb{P}\text{-a.s.} \tag{115}$$

Proof of the claim. Clearly, if $\xi,\tilde{\xi} \in \mathcal{L}^0(\Omega,\mathcal{F}_{n\varepsilon}; D)$ satisfy $\xi = \tilde{\xi}$ $\mathbb{P}$-a.s., then for every $n \in \mathbb{N}_0$ it holds that

$$Y^{(n)}(\xi) = Y^{(n)}(\tilde{\xi}) \tag{116}$$

$\mathbb{P}$-a.s. Fix $n \in \mathbb{N}_0$. If $\xi \in \mathcal{L}^0(\Omega,\mathcal{F}_{n\varepsilon}; D)$ is an $\mathcal{F}_{n\varepsilon}$-simple function, then $Y^{(n)}(\xi)$ satisfies

$$Y^{(n)}(\xi)(t) = \xi + \int_0^t \mu(Y^{(n)}(\xi)(s))\,ds + \int_0^t \sigma(Y^{(n)}(\xi)(s))\,dW_s^{*\theta_{n\varepsilon}} \tag{117}$$

$\mathbb{P}$-a.s. for every $t \in [0, \varepsilon]$. By the uniqueness statement in Theorem 3.4 it follows that

$$Y^{(n)}(\xi) = F(\xi, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \tag{118}$$

$\mathbb{P}$-a.s. Now let $\xi \in \mathcal{L}^0(\Omega, \mathcal{F}_{n\varepsilon}; D)$ be arbitrary and let $\xi_m \in \mathcal{L}^0(\Omega, \mathcal{F}_{n\varepsilon}; D)$, $m \in \mathbb{N}$, be $\mathcal{F}_{n\varepsilon}$-simple functions such that $\lim_{m\to\infty} \xi_m \to \xi$ $\mathbb{P}$-a.s. It follows from the fact that for every $\omega \in \Omega$ it holds that $Y^{(n)}(\cdot, \omega) \in C(D, C([0, \varepsilon], D))$ that for $\mathbb{P}$-almost every $\omega \in \Omega$ it holds that $\lim_{m\to\infty} Y^{(n)}(\xi_m(\omega), \omega) = Y^{(n)}(\xi(\omega), \omega)$. On the other hand, by [37, Proposition 3.2.1] it holds that

$$\left\| F(\xi_m, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} - F(\xi, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]} \right\|_{C([0,\varepsilon],\mathbb{R}^d)} \to 0 \tag{119}$$

in probability as $m \to \infty$. As $Y^{(n)}(\xi_m) = F(\xi_m, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]}$ $\mathbb{P}$-a.s., it follows that $Y^{(n)}(\xi) = F(\xi, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]}$ $\mathbb{P}$-a.s. □

Claim. For every $\xi \in \mathcal{L}^0(\Omega, \mathcal{F}_0; D)$ and every $s \in [0, \infty)$ it holds that

$$\big(F(\xi, W)(t)\big)_{t\in[s,\infty)} = \Big(F\big(F(\xi, W)(s), W^{*\theta_s}\big)(t - s)\Big)_{t\in[s,\infty)} \qquad \mathbb{P}\text{-a.s.} \tag{120}$$

Proof of the claim. Let $s \in [0, \infty)$ and $\xi \in \mathcal{L}^0(\Omega, \mathcal{F}_0; D)$ be arbitrary and define $\Omega_0 \subseteq \Omega$ by

$$\Omega_0 = \left\{ \omega \in \Omega \colon \begin{array}{c} F(\xi(\omega), W(\omega)) \in C([0, \infty), D) \text{ and} \\ F\big(F(\xi(\omega), W(\omega))(s), W^{*\theta_s}(\omega)\big) \in C([0, \infty), D) \end{array} \right\}. \tag{121}$$

Observe that $\mathbb{P}\big[\Omega_0\big] = 1$. Next let $Z\colon [0, \infty) \times \Omega \to D$ be given by $Z_t(\omega) = \xi(\omega)$ for all $(t, \omega) \in [0, \infty) \times \Omega \setminus \Omega_0$ and by

$$Z_t(\omega) = \begin{cases} F(\xi(\omega), W(\omega))(t) & : t \in [0, s] \\ F\big(F(\xi(\omega), W(\omega))(s), W^{*\theta_s}(\omega)\big)(t - s) & : t \in (s, \infty) \end{cases} \tag{122}$$

for every $(t, \omega) \in [0, \infty) \times \Omega_0$. One may easily verify that Z is an adapted stochastic process with continuous sample paths which satisfies

$$Z_t = \xi + \int_0^t \mu(Z_u)\, du + \int_0^t \sigma(Z_u)\, dW_u \tag{123}$$

$\mathbb{P}$-a.s. for every $t \in [0, \infty)$. Pathwise uniqueness of solutions to the SDE with coefficients (μ, σ) and initial distribution given by the law of ξ hence shows that $Z = F(\xi, W)$ $\mathbb{P}$-a.s. The proof of the claim is thus completed. □

In the next step let $Y\colon D \times [0, \infty) \times \Omega \to D$ be a mapping defined recursively by $Y(x, \omega, t) := Y^{(0)}(x, \omega, t)$ for every $x \in D$, $\omega \in \Omega$, $t \in [0, \varepsilon]$ and by

$$Y(x, t, \omega) = Y^{(n)}\big(Y(x, n\varepsilon, \omega), \omega, t - n\varepsilon\big) \tag{124}$$

for every $x \in D$, $\omega \in \Omega$, $t \in (n\varepsilon, (n + 1)\varepsilon]$, $n \in \mathbb{N}$. Note that Y is an adapted stochastic process with continuous sample paths. Moreover, observe that by construction it holds that $Y(n\varepsilon, x) = F(x, W)(n\varepsilon)$ $\mathbb{P}$-a.s. for every $x \in D$ and every $n \in \mathbb{N}_0$. This and the two claims above show that

$$\big(Y(t, x)\big)_{t\in[n\varepsilon,(n+1)\varepsilon]} = \Big(F\big(F(x, W)(n\varepsilon), W^{*\theta_{n\varepsilon}}\big)(t - n\varepsilon)\Big)_{t\in[n\varepsilon,(n+1)\varepsilon]} = F(x, W)|_{[n\varepsilon,(n+1)\varepsilon]} \tag{125}$$

$\mathbb{P}$-a.s. for every $x \in D$ and every $n \in \mathbb{N}_0$. Hence, we obtain that

$$(Y(t, x))_{t\in[0,\infty)} = F(x, W) = (X_t^x)_{t\in[0,\infty)} \tag{126}$$

$\mathbb{P}$-a.s. for every $x \in D$. Moreover, as $Y^{(n)}(\cdot, \omega) \in C(D, C([0, \varepsilon], D)) = C([0, \varepsilon] \times D, D)$ for every $\omega \in \Omega$, it immediately follows from our construction that $Y(\cdot, \cdot, \omega) \in C([0, \infty) \times D, D)$ for every $\omega \in \Omega$. This finishes the proof of Lemma 3.2. □

Proof of Lemma 3.2. As before, for every $x \in D$ the solutions to the SDE with coefficients (μ, σ) and initial distribution δ_x are pathwise unique. Let $F\colon D \times C([0,\infty), \mathbb{R}^m) \to C([0,\infty), \mathbb{R}^d)$ be the measurable and universally adapted function provided by Theorem 3.4. As before, it follows from the fact that the mapping $[0,\varepsilon] \times D \ni (t,x) \to X_t^x \in L^p(\Omega;\mathbb{R}^d)$ is locally α-Hölder continuous and Theorem 3.4 that for every $n, R \in \mathbb{N}_0$ it holds that

$$\sup_{\substack{(s,x),(t,y)\in[0,\varepsilon]\times(D\cap B_R(0))\\(s,x)\neq(t,y)}} \frac{\left\|F(x, W^{*\theta_{n\varepsilon}})(s) - F(y, W^{*\theta_{n\varepsilon}})(t)\right\|_{L^p(\Omega;\mathbb{R}^d)}}{(\|x-y\| + |s-t|)^\alpha} < \infty. \tag{127}$$

By applying Theorem 3.5 to the mappings $[0,\varepsilon] \times (D \cap B_R(0)) \ni (t,x) \mapsto F(x, W^{*\theta_{n\varepsilon}})(t) \in L^p(\Omega; D)$, $(n,R) \in \mathbb{N}_0 \times \mathbb{N}$, it follows for every $n \in \mathbb{N}_0$ that there exists a mapping $Y^{(n)} \in \mathcal{L}^0([0,\varepsilon] \times D \times \Omega; D)$ such that for every $x \in D$ and every $t \in [0,\varepsilon]$ it holds that $Y^{(n)}(t,x) = F(x, W^{*\theta_{n\varepsilon}})(t)$ $\mathbb{P}$-a.s. and such that for every $\omega \in \Omega$ it holds that $Y^{(n)}(\cdot,\cdot,\omega) \in C([0,\varepsilon] \times D, D)$. Path continuity implies for every $n \in \mathbb{N}_0$ and every $x \in D$ that $(Y^{(n)}(t,x))_{t\in[0,\varepsilon]} = F(x, W^{*\theta_{n\varepsilon}})|_{[0,\varepsilon]}$ $\mathbb{P}$-a.s. The rest of the proof is entirely analogous to the proof of Lemma 3.1. This finishes the proof of Lemma 3.2. □

3.3 Strong completeness for SDEs with additive noise

Theorems 2.23 and 2.29 together with Lemmas 3.2 and 3.1 can be used to prove strong completeness for SDEs. In the case of additive noise, another well-known possiblity for proving strong completeness is to subtract the driving noise process from the SDE and then to try to solve the resulting random ordinary differential equation (RODE) globally for every continuous trajectory of the driving noise process. This approach works, for instance, if the drift coefficient grows at most linearly. However, if the drift coefficient grows superlinearly then it might happen that the resulting RODE can not be solved globally for every continuous trajectory of the driving noise process. This is illustrated in the following example.

Let $\Omega = \{f \in C([0,\infty), \mathbb{R}^2)\colon f(0) = 0\}$, let $\mathcal{F} = \mathcal{B}(\Omega)$, let $P\colon \mathcal{F} \to [0,1]$ be the Wiener measure on $(\Omega, \mathcal{F})$, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}^2$ be given by $W_t(\omega) = \omega(t)$ for every $t \in [0,T]$, $\omega \in \Omega$, let $R = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ and let $X^x\colon [0,\infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \int_0^t \|X_s^x\|^2 R X_s^x \, ds + W_t \tag{128}$$

$\mathbb{P}$-a.s. for every $(t,x) \in [0,\infty) \times \mathbb{R}^2$. Then observe for every $\rho \in [0,\infty)$ and every $x \in \mathbb{R}^2$ that

$$2\rho \left\langle x, \mu(x) \right\rangle + \tfrac{1}{2} \operatorname{tr}\big(2\rho I\big) + \tfrac{1}{2} \left\|2\rho x\right\|^2 = 2\rho + 2\rho^2 \left\|x\right\|^2 = 2\rho + 2\rho \left[\rho\|x\|^2\right]. \tag{129}$$

Corollary 2.4 (with $U(x) = 1 + \rho\|x\|^2$ and $\bar{U}(x) \equiv 0$ in the setting of that corollary) hence implies for every $\rho, t \in [0,\infty)$ and every $x \in \mathbb{R}^2$ that

$$\mathbb{E}\left[\exp\left(\frac{\rho\|X_t^x\|^2}{e^{2\rho t}}\right)\right] \leq \exp\big(1 - e^{-2\rho t} + \rho \left\|x\right\|^2\big) \leq \exp\big(1 + \rho \left\|x\right\|^2\big). \tag{130}$$

In addition, note for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ with $x \neq y$ that

$$\begin{aligned} \frac{\langle x - y, \|x\|^2 Rx - \|y\|^2 Ry \rangle}{\|x-y\|^2} &= \frac{(\|x\|^2 - \|y\|^2) \left\langle x - y, R(x+y) \right\rangle}{2 \left\|x-y\right\|^2} \\ &= \frac{(\|x\| + \|y\|) \left\langle x - y, R(x+y) \right\rangle}{2 \left\|x-y\right\|} \leq \frac{(\|x\| + \|y\|) \left\|R(x+y)\right\|}{2} \leq \|x\|^2 + \|y\|^2. \end{aligned} \tag{131}$$

Corollary 2.30 (with $U_{0,0} = U_{i,1} = \bar{U}_{i,1} \equiv 0$, $U_{1,0} = 1 + \rho\|x\|^2$, $q_{1,0} = r$, $p = \infty$ and $\beta_{i,j} = 0$ for $i, j \in \{0, 1\}$ in the setting of that corollary) hence implies for every $x, y \in \mathbb{R}^2$, $\rho, T \in (0, \infty)$ and every $r \in \big(0, \frac{\rho\, e^{-2\rho T}}{2T}\big)$ that

$$\left\| \sup_{t\in[0,T]} \frac{\|X_t^x - X_t^y\|}{\|x - y\|} \right\|_{L^r(\Omega;\mathbb{R})} \leq \exp\!\left(\frac{2 + \rho\,\|x\|^2 + \rho\,\|y\|^2}{2r} \right). \tag{132}$$

Combining this and the fact that for every $\rho \in (0, \infty)$ it holds that $\lim_{T\searrow 0} \frac{\rho\, e^{-2\rho T}}{2T} = \infty$ with Lemma 3.1 shows that the SDE (128) is *strongly complete*. Next let $x_0 \in \mathbb{R}^2 \backslash \{0\}$ be arbitrary and let $\tau \in (0, \infty]$ be the unique maximal extended real number such that there exists a unique continuously differentiable function $z\colon [0, \tau) \to \mathbb{R}^2$ satisfying

$$z(0) = x_0 \qquad \text{and} \qquad \forall\, t \in [0, \tau)\colon \quad z'(t) = \left\| z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right\|^2 R \left(z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right). \tag{133}$$

Note that this definition ensures for every $t \in [0, \tau)$ that

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\|z(t)\|^2 = 2\,\langle z(t), z'(t)\rangle = 2 \left\| z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right\|^2 \left\langle z(t), R\left(z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right) \right\rangle \\
&= -2 \tfrac{t}{\|z(t)\|^{3/2}} \left\| z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right\|^2 \left\langle z(t), R^2 z(t) \right\rangle \\
&= 2t\|z(t)\|^{1/2} \left\| z(t) - \tfrac{tRz(t)}{\|z(t)\|^{3/2}} \right\|^2 \\
&= 2t\|z(t)\|^{1/2} \left[\|z(t)\|^2 + t^2\|z(t)\| \right] \geq 2t\|z(t)\|^{5/2} = 2t \left[\|z(t)\|^2 \right]^{5/4}.
\end{aligned} \tag{134}$$

This implies that τ is a real number, i.e., that $\tau < \infty$. Next let $\omega_0 \in \Omega$ be given by

$$\omega_0(t) = \begin{cases} -\frac{tRz(t)}{\|z(t)\|^{3/2}} & : t < \tau \\ 0 & : t \geq \tau \end{cases} \tag{135}$$

for every $t \in [0, \infty)$ and note that (133) implies that

$$z(0) = x_0 \quad \text{and} \quad \forall\, t \in [0, \tau)\colon \quad z'(t) = \|z(t) + W_t(\omega_0)\|^2\, R\,(z(t) + W_t(\omega_0)). \tag{136}$$

This proves that there do not exist stochastic processes $Y^x\colon [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, with continuous sample paths which satisfy for every $(t, x, \omega) \in [0, \infty) \times \mathbb{R}^2 \times \Omega$ that

$$Y_t^x(\omega) = x + \int_0^t \|Y_s^x(\omega) + W_s(\omega)\|^2\, R\,(Y_s^x(\omega) + W_s(\omega))\, ds. \tag{137}$$

In conclusion, the RODE (137) associated to the SODE (128) with additive noise can thus not be solved globally for every continuous trajectory of the driving noise process.

4 Examples of SDEs

In this section we apply Theorem 2.23 and Theorem 2.29 to several example SDEs from the literature.

4.1 Stochastic van der Pol oscillator

The van der Pol oscillator was proposed to describe stable oscillation; see van der Pol [52]. Timmer et. al [49] considered a stochastic version with additive noise acting on the velocity. Here we consider a more general version hereof. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $\alpha \in$

$(0,\infty)$, $\gamma,\delta,\eta_0,\eta_1\in[0,\infty)$, $m\in\mathbb{N}$, let $W\colon[0,\infty)\times\Omega\to\mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $g\colon\mathbb{R}\to\mathbb{R}^{1\times m}$ be a globally Lipschtiz continuous function with $\|g(y)\|^2\leq\eta_0+\eta_1 y^2$ for all $y\in\mathbb{R}$, let $\mu\colon\mathbb{R}^2\to\mathbb{R}^2$ and $\sigma\colon\mathbb{R}^2\to\mathbb{R}^{2\times m}$ be given by $\mu(x)=(x_2,(\gamma-\alpha(x_1)^2)\,x_2-\delta x_1)$ and $\sigma(x)u=(0,g(x_1)u)$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, $u\in\mathbb{R}^m$ and let $X^x\colon[0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x=x+\textstyle\int_0^t\mu(X_s)\,ds+\int_0^t\sigma(X_s^x)\,dW_s\tag{138}$$

$\mathbb{P}$-a.s. for all $(t,x)\in[0,\infty)\times\mathbb{R}^2$. Next we define a function $\vartheta\colon(0,\infty)\to[0,\infty)$ by $\vartheta(\rho):=\min_{r\in(0,\infty)}\big(\big[\frac{|\delta-1|}{r}+\eta_1\big]\vee\big[r\,|\delta-1|+2\gamma+4\eta_0\rho\big]\big)$ for all $\rho\in(0,\infty)$. If $\rho\in[0,\infty)$ and if $U,\overline{U}\colon\mathbb{R}^2\to\mathbb{R}$ are given by $U(x)=\rho\,\|x\|^2$ and $\overline{U}(x)=2\rho\,[\alpha-\rho\eta_1]\,(x_1x_2)^2$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, then it holds for every $x=(x_1,x_2)\in\mathbb{R}^2$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U)(x)+\tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2+\overline{U}(x)\\
&=2\rho\left[(1-\delta)\,x_1x_2+\gamma(x_2)^2-\alpha(x_1x_2)^2+\tfrac{1}{2}\|g(x_1)^*\|^2\right]+2(\rho x_2)^2\|g(x_1)^*\|^2+\overline{U}(x)\\
&\leq\rho\eta_0+2\rho\left[(1-\delta)\,x_1x_2+\tfrac{\eta_1}{2}(x_1)^2+[\gamma+2\eta_0\rho]\,(x_2)^2\right]+2\rho\,[\rho\eta_1-\alpha]\,(x_1x_2)^2+\overline{U}(x)\\
&\leq\rho\eta_0+2\rho\inf_{r\in(0,\infty)}\left[\left[\tfrac{|\delta-1|}{2r}+\tfrac{\eta_1}{2}\right](x_1)^2+\left[\tfrac{r\,|\delta-1|}{2}+\gamma+2\eta_0\rho\right](x_2)^2\right]\leq\rho\eta_0+\vartheta(\rho)\,U(x).
\end{aligned}\tag{139}$$

Corollary 2.4 hence proves for every $x\in\mathbb{R}^2$, $t\in[0,\infty)$, $\rho\in[0,\frac{\alpha}{\eta_1}]\cap\mathbb{R}$ that

$$\mathbb{E}\left[\exp\left(\tfrac{\rho}{e^{\vartheta(\rho)t}}\|X_t^x\|^2+\int_0^t\tfrac{2\rho(\alpha-\rho\eta_1)}{e^{\vartheta(\rho)s}}\left|X_s^{1,x}X_s^{2,x}\right|^2ds\right)\right]\leq e^{\int_0^t\frac{\rho\eta_0}{e^{\vartheta(\rho)s}}\,ds+\rho\|x\|^2}\leq e^{\frac{1}{4}+\rho\|x\|^2}.\tag{140}$$

In the next step we observe for every $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$ with $x_2\cdot y_2<0$ that $(x_2-y_2)\cdot((x_1)^2x_2-(y_1)^2y_2)\geq0$. Consequently, we get for every $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$ with $x\neq y$ that

$$\begin{aligned}
&-\frac{\alpha\,(x_2-y_2)\,[(x_1)^2x_2-(y_1)^2y_2]}{\|x-y\|^2}\leq-\mathbb{1}_{[0,\infty)}(x_2y_2)\cdot\frac{\alpha\,(x_2-y_2)\,[(x_1)^2x_2-(y_1)^2y_2]}{\|x-y\|^2}\\
&=-\mathbb{1}_{[0,\infty)}(x_2y_2)\cdot\frac{\alpha\,(|x_2|-|y_2|)\,[(x_1)^2|x_2|-(y_1)^2|y_2|]}{\|x-y\|^2}\\
&\leq-\mathbb{1}_{[0,\infty)}(x_2y_2)\cdot\frac{\alpha\,(|x_2|-|y_2|)\,((x_1)^2-(y_1)^2)\min(|x_2|,|y_2|)}{\|x-y\|^2}\\
&\leq\frac{\alpha\,|x_2-y_2|\,|(x_1)^2-(y_1)^2|\min(|x_2|,|y_2|)}{\|x-y\|^2}\cdot\mathbb{1}_{(-\infty,0]}\big((|x_2|-|y_2|)((x_1)^2-(y_1)^2)\big)\\
&\leq\tfrac{\alpha}{2}\,(|x_1|+|y_1|)\min(|x_2|,|y_2|)\cdot\mathbb{1}_{(-\infty,0]}((|x_2|-|y_2|)(|x_1|-|y_1|))\\
&\leq\tfrac{\alpha}{2}\,(|x_1|+|y_1|)\min(|x_2|,|y_2|)\leq\tfrac{\alpha}{2}\,[|x_1x_2|+|y_1y_2|]\,.
\end{aligned}\tag{141}$$

This implies for every $t,q,\theta\in(0,\infty)$, $\rho\in(0,\frac{\alpha}{\eta_1})$, $x=(x_1,x_2),y=(y_1,y_2)\in\mathbb{R}^2$ with $x\neq y$ that

$$\begin{aligned}
&\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle+\frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^2)}}{\|x-y\|^2}+\tfrac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4}\\
&=\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle+\frac{1}{2}\|g(x_1)^*-g(y_1)^*\|^2}{\|x-y\|^2}+\tfrac{(\frac{\theta}{2}-1)\,(x_2-y_2)^2\,\|g(x_1)^*-g(y_1)^*\|^2}{\|x-y\|^4}\\
&\leq\tfrac{\gamma+\sqrt{\gamma^2+(\delta-1)^2}}{2}+\tfrac{\alpha}{2}\,(|x_1||x_2|+|y_1||y_2|)+\tfrac{\frac{1}{2}\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}|x_1-y_1|^2}{\|x-y\|^2}\\
&\quad+\tfrac{|\frac{\theta}{2}-1|\,(x_1-y_1)^2\,(x_2-y_2)^2\,\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}}{\|x-y\|^4}\\
&\leq\tfrac{\gamma+\sqrt{\gamma^2+(\delta-1)^2}}{2}+\left[\tfrac{1}{2}+\tfrac{1}{4}\max(\tfrac{\theta}{2}-1,0)\right]\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}+\frac{q\alpha^2e^{\vartheta(\rho)t}}{8\rho[\alpha-\rho\eta_1]}\\
&\quad+\frac{2\rho\,[\alpha-\rho\eta_1]\,[(x_1x_2)^2+(y_1y_2)^2]}{2qe^{\vartheta(\rho)t}}.
\end{aligned}\tag{142}$$

Combining this and (139) with Corollary 2.30 proves for every $T \in (0, \infty)$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $\rho \in (0, \frac{\alpha}{\eta_1})$, $r, p, q \in (0, \infty]$, $\theta \in (0, p)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ that

$$\begin{aligned} &\left[1 - \tfrac{\theta}{p}\right]^{\frac{1}{\theta}} \left\| \sup_{t \in [0,T]} \frac{\|X_t^x - X_t^y\|}{\|x - y\|} \right\|_{L^r(\Omega;\mathbb{R})} \\ &\leq \exp\!\left(\tfrac{\left[\gamma + \sqrt{\gamma^2 + (\delta-1)^2}\right] T}{2} + \tfrac{[p + 2 \vee (4-\theta)]\, T\, \|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}}{8} + \tfrac{\int_0^T q \alpha^2 e^{\vartheta(\rho) s}\, ds}{8\rho[\alpha - \rho\eta_1]} + \tfrac{\frac{1}{2} + \rho\|x\|^2 + \rho\|y\|^2}{2q} \right). \end{aligned} \tag{143}$$

In particular, in the case of additive noise, i.e., $g(y) = g(0)$ for all $y \in \mathbb{R}$, this shows for every $T \in (0, \infty)$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $\rho \in (0, \frac{\alpha}{\eta_1})$, $r \in (0, \infty]$, $\theta \in (0, \infty)$ that

$$\left\| \sup_{t \in [0,T]} \frac{\|X_t^x - X_t^y\|}{\|x - y\|} \right\|_{L^r(\Omega;\mathbb{R})} \leq \exp\!\left(\tfrac{\left[\gamma + \sqrt{\gamma^2 + (\delta-1)^2}\right] T}{2} + \tfrac{\int_0^T r \alpha^2 e^{\vartheta(\rho) s}\, ds}{8\rho[\alpha - \rho\eta_1]} + \tfrac{\frac{1}{2} + \rho\|x\|^2 + \rho\|y\|^2}{2r} \right). \tag{144}$$

In addition, combining (143) with Lemma 3.1 proves that the stochastic Van der Pol oscillator (138) is strongly complete. Strong completeness for the SDE (138) follows from earlier results in the literature, namely from Theorem 2.4 (applied with $\mathcal{W}(x) = \|x\|^2$ for all $x \in \mathbb{R}^2$ and $\alpha = 1$) in Zhang [54] in the case of a globally bounded and globally Lipschitz continuous g and it follows with the method of Theorem 3.5 in Schenk-Hoppé [42] in the case where g is twice continuously differentiable with a globally bounded first derivative by showing for every $x \in \mathbb{R}^2$ that $[0, \infty) \ni t \mapsto \big(X_t^{x,1}, X_t^{x,2} - g(X_t^{x,1}) W_t\big) \in \mathbb{R}^2$ is the solution of an appropriate random ordinary differential equation (RODE).

4.2 Stochastic Duffing-van der Pol oscillator

The Duffing-van der Pol equation unifies both the Duffing equation and the van der Pol equation and has, for example, been used in certain flow-induced structural vibration problems (see Holmes & Rand [19]) and the references therein. Schenk-Hoppé [41] studied a stochastic version with affine-linear noise acting on the velocity (see also the references in [41]). Here we consider a more general version hereof. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,\infty)})$ be a stochastic basis, let $\eta_0, \eta_1, \alpha_1 \in [0, \infty)$, $\alpha_2, \alpha_3 \in (0, \infty)$, $m \in \mathbb{N}$, let $W \colon [0, \infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,\infty)}$-Brownian motion, let $g \colon \mathbb{R} \to \mathbb{R}^{1 \times m}$ be a globally Lipschitz continuous function with $\|g(y)\|^2 \leq \eta_0 + \eta_1 y^2$ for all $y \in \mathbb{R}$, let $\mu \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma \colon \mathbb{R}^2 \to \mathbb{R}^{2 \times m}$ be given by $\mu(x) = (x_2, \alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3)$ and $\sigma(x) u = (0, g(x_1) u)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$ and let $X^x = (X^{x,1}, X^{x,2}) \colon [0, \infty) \times \Omega \to \mathbb{R}^2$, $x \in \mathbb{R}^2$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \textstyle\int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{145}$$

$\mathbb{P}$-a.s. for all $(t, x) \in [0, \infty) \times \mathbb{R}^2$. If $\rho \in (0, \infty)$ and if $U \colon \mathbb{R}^2 \to \mathbb{R}$ is given by $U(x_1, x_2) = \rho\big[\frac{(x_1)^4}{2} + \alpha_1 (x_1)^2 + (x_2)^2\big]$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ (cf., e.g., (8) in Holmes & Rand [19]), then it holds for every $x = (x_1, x_2) \in \mathbb{R}^2$ that

$$\begin{aligned} &(\mathcal{G}_{\mu,\sigma} U)(x) + \tfrac{1}{2} \|\sigma(x)^* (\nabla U)(x)\|^2 \\ &= 2\rho\alpha_1 x_1 x_2 + 2\rho x_2 \left[\alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 x_2 (x_1)^2\right] + \rho \|g(x_1)\|^2 + 2(\rho x_2)^2 \|g(x_1)\|^2 \\ &\leq \rho\eta_0 + \rho \left[\eta_1 (x_1)^2 + 2 \left[\rho\eta_0 + \alpha_2\right] (x_2)^2\right] + 2\rho \left[\rho\eta_1 - \alpha_3\right] (x_1 x_2)^2 \\ &\leq \rho\eta_0 + \rho \left[\eta_1 - 2\alpha_1(\rho\eta_0 + \alpha_2)\right] (x_1)^2 + 2\rho \left(\rho\eta_0 + \alpha_2\right) \left[\alpha_1 (x_1)^2 + (x_2)^2\right] \\ &\quad + 2\rho \left[\rho\eta_1 - \alpha_3\right] (x_1 x_2)^2 \\ &\leq \rho\eta_0 + \tfrac{\rho |0 \vee (\eta_1 - 2\alpha_1(\rho\eta_0 + \alpha_2))|^2}{4(\rho\eta_0 + \alpha_2)} + 2 \left(\rho\eta_0 + \alpha_2\right) U(x) + 2\rho \left[\rho\eta_1 - \alpha_3\right] (x_1 x_2)^2. \end{aligned} \tag{146}$$

Corollary 2.4 hence proves for every $t \in (0,\infty)$, $\rho \in [0, \frac{\alpha_3}{\eta_1}] \cap \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$ that

$$\begin{aligned}
&\mathbb{E}\Bigg[\exp\Bigg(\frac{\rho\big(\frac{1}{2}[X_t^{x,1}]^4+\alpha_1[X_t^{x,1}]^2+[X_t^{x,2}]^2\big)}{\exp(2t[\rho\eta_0+\alpha_2])}+\int_0^t \frac{2\rho[\alpha_3-\rho\eta_1]\big[X_s^{x,1}X_s^{x,2}\big]^2}{\exp(2s[\rho\eta_0+\alpha_2])}\,ds\Bigg)\Bigg]\\
&\le \exp\Bigg(\int_0^t \frac{\rho\eta_0+\frac{\rho|0\vee(\eta_1-2\alpha_1[\rho\eta_0+\alpha_2])|^2}{4[\rho\eta_0+\alpha_2]}}{\exp(2s[\rho\eta_0+\alpha_2])}\,ds+\rho\Big[\frac{(x_1)^4}{2}+\alpha_1(x_1)^2+(x_2)^2\Big]\Bigg)\\
&\le e^{\big[\frac{1+\rho(\alpha_1)^2}{2}+\frac{t\rho(\eta_1)^2}{4(\rho\eta_0+\alpha_2)}+\rho(x_1)^4+\rho(x_2)^2\big]}.
\end{aligned} \tag{147}$$

In the next step we observe for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ with $x \neq y$ that

$$-\frac{\big[(x_1)^3-(y_1)^3\big][x_2-y_2]}{\|x-y\|^2} = -\frac{\big[(x_1)^2+x_1y_1+(y_1)^2\big][x_1-y_1][x_2-y_2]}{[x_1-y_1]^2+[x_2-y_2]^2} \le \frac{(x_1)^2+x_1y_1+(y_1)^2}{2}. \tag{148}$$

Combining (141) and (148) implies for every $\varepsilon, \theta \in (0,\infty)$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ with $x \neq y$ that

$$\begin{aligned}
&\frac{\langle x-y,\mu(x)-\mu(y)\rangle+\frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^2)}}{\|x-y\|^2}+\frac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4}\\
&\le \frac{\alpha_2+\sqrt{(\alpha_1-1)^2+(\alpha_2)^2}}{2}-\frac{\alpha_3\,[x_2(x_1)^2-y_2(y_1)^2]\,[x_2-y_2]+[(x_1)^3-(y_1)^3]\,[x_2-y_2]}{\|x-y\|^2}\\
&\quad+\frac{\|g(x_1)-g(y_1)\|^2}{2\,\|x-y\|^2}+\frac{(\frac{\theta}{2}-1)\,|x_2-y_2|^2\,\|g(x_1)-g(y_1)\|^2}{\|x-y\|^4}\\
&\le \frac{\alpha_2+\sqrt{(\alpha_1-1)^2+(\alpha_2)^2}}{2}+\frac{\max(\theta+2,4)}{8}\,\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}+\frac{\alpha_3}{2}\,[|x_1x_2|+|y_1y_2|]+\frac{3[(x_1)^2+(y_1)^2]}{4}.
\end{aligned} \tag{149}$$

Corollary 2.30 hence shows for every $T \in (0,\infty)$, $\rho_0 \in (0, \frac{\alpha_3}{\eta_1}] \cap \mathbb{R}$, $\rho_1 \in (0, \frac{\alpha_3}{\eta_1})$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $r, p, q_0, q_1 \in (0,\infty]$, $\theta \in (0,p)$ with $\frac{1}{p}+\frac{1}{q_0}+\frac{1}{q_1}=\frac{1}{r}$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^x-X_t^y\|\big\|_{L^r(\Omega;\mathbb{R}^2)}\\
&\le \exp\Bigg(\frac{\big[\alpha_2+\sqrt{(\alpha_1-1)^2+(\alpha_2)^2}\big]T}{2}+\frac{[p+2\vee(4-\theta)]T\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}}{8}+\int_0^T\frac{9q_0Te^{2s[\rho_0\eta_0+\alpha_2]}}{16\rho_0}\,ds\Bigg)\\
&\cdot\exp\Bigg(\int_0^T\frac{\Big[\rho_0\eta_0+\frac{|0\vee(\eta_1-2\alpha_1(\rho_0\eta_0+\alpha_2))|^2}{4(\rho_0\eta_0+\alpha_2)}\Big](1-\frac{s}{T})}{q_0e^{2s[\rho_0\eta_0+\alpha_2]}}+\frac{\Big[\rho_1\eta_0+\frac{|0\vee(\eta_1-2\alpha_1(\rho_1\eta_0+\alpha_2))|^2}{4(\rho_1\eta_0+\alpha_2)}\Big]}{q_1e^{2s[\rho_1\eta_0+\alpha_2]}}\,ds\Bigg)\\
&\cdot\exp\Bigg(\int_0^T\frac{q_1(\alpha_3)^2e^{2s[\rho_1\eta_0+\alpha_2]}}{8\rho_1[\rho_1\eta_1-\alpha_3]}\,ds+\Big[\frac{\rho_0}{q_0}+\frac{\rho_1}{q_1}\Big]\Big[\frac{(x_1)^4+(y_1)^4}{4}+\frac{\alpha_1[(x_1)^2+(y_1)^2]}{2}+\frac{(x_2)^2+(y_2)^2}{2}\Big]\Bigg)\frac{\|x-y\|}{[1-\theta/p]^{1/\theta}}.
\end{aligned}$$

This implies for every $T \in (0,\infty)$, $\rho_0, \rho_1 \in (0, \frac{\alpha_3}{\eta_1})$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $r, q_0, q_1 \in (0,\infty]$, $p \in (2,\infty]$ with $\frac{1}{p}+\frac{1}{q_0}+\frac{1}{q_1}=\frac{1}{r}$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^x-X_t^y\|\big\|_{L^r(\Omega;\mathbb{R}^2)}\\
&\le \exp\Bigg(\frac{1}{2q_0}+\frac{1}{2q_1}+\alpha_2T+\frac{(\alpha_1+1)T}{2}+\frac{(p+2)T\|g^*\|^2_{\mathrm{Lip}(\mathbb{R},\mathbb{R}^m)}}{8}+\sum_{i=0}^{1}\frac{2^i(\eta_1)^2T}{8q_i(\rho_i\eta_0+\alpha_2)}+\frac{q_0T^2e^{2T[\rho_0\eta_0+\alpha_2]}}{\rho_0}\Bigg)\\
&\cdot\exp\Bigg(\frac{q_1(\alpha_3)^2e^{2T[\rho_1\eta_0+\alpha_2]}}{8\rho_1[\alpha_3-\rho_1\eta_1]}+\Big[\frac{\rho_0}{q_0}+\frac{\rho_1}{q_1}\Big]\Big[\frac{(x_1)^4+(y_1)^4}{4}+\frac{\alpha_1[(x_1)^2+(y_1)^2]}{2}+\frac{(x_2)^2+(y_2)^2}{2}\Big]\Bigg)\frac{\|x-y\|}{\sqrt{1-2/p}}.
\end{aligned}$$

Combining this with Lemma 3.1 proves that the stochastic Duffing-van der Pol oscillator (145) is strongly complete. Strong completeness for the SDE (145) follows from earlier results in the literature, namely from Theorem 2.4 (applied with $\mathcal{W}(x) = \|x\|^2$ for all $x \in \mathbb{R}^2$ and $\alpha = 1$) in Zhang [54] for all globally bounded and globally Lipschitz continuous functions g and follows with the method of Theorem 3.5 in Schenk-Hoppé [42] for twice continuously differentiable functions g with globally bounded first derivative by showing that $[0,\infty) \ni t \mapsto (X_t^{x,1}, X_t^{x,2}-g(X_t^{x,1})W_t) \in \mathbb{R}^2$ is the solution of a random ordinary differential equation for every $x \in \mathbb{R}^2$.

4.3 Stochastic Lorenz equation with additive noise

Lorenz [32] suggested a three-dimensional ordinary differential equation as a simplified model of convection rolls in the atmosphere. As, for instance, in Zhou & E [55], we consider a stochastic version hereof with additive noise. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W = (W^1, W^2, W^3)\colon [0,\infty)\times\Omega \to \mathbb{R}^3$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\alpha_1, \alpha_2, \alpha_3, \beta \in [0,\infty)$ and let $A \in \mathbb{R}^{3\times 3}$, $\mu\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $\sigma\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{pmatrix} -\alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -1 & 0 \\ 0 & 0 & -\alpha_3 \end{pmatrix}, \qquad \mu\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ax + \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix} \tag{150}$$

and $\sigma(x) = \sqrt{\beta} I_{\mathbb{R}^3}$ for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Moreover, let $X^x\colon [0,\infty)\times\Omega \to \mathbb{R}^3$, $x \in \mathbb{R}^3$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \smallint_0^t \mu(X_s)\, ds + \smallint_0^t \sigma(X_s^x)\, dW_s \tag{151}$$

$\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times\mathbb{R}^3$. The processes X^x, $x \in \mathbb{R}^d$, are thus thus solution processes of the stochastic Lorenz equation in Zhou and E [55]. In the next step we define a real number $\vartheta \in [0,\infty)$ by

$$\vartheta := \min_{r\in(0,\infty)} \Big[\big[\tfrac{(\alpha_1+\alpha_2)^2}{r} - 2\alpha_1\big] \vee [r-1] \vee 0 \Big]. \tag{152}$$

If $\rho \in [0,\infty)$ and if $U\colon \mathbb{R}^3 \to \mathbb{R}$ is given by $U(x) = \rho\,\|x\|^2$ for all $x \in \mathbb{R}^3$, then it holds for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma} U)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 \\
&= 2\rho\,\langle x, \mu(x)\rangle + 3\rho\beta + 2\rho^2\beta\|x\|^2 \\
&= 2\rho\alpha_1 x_1(x_2 - x_1) + 2\rho x_2(\alpha_2 x_1 - x_2) - 2\rho\alpha_3(x_3)^2 + 3\rho\beta + 2\rho\beta U(x) \\
&= 2\rho(\alpha_1+\alpha_2)x_1x_2 - 2\rho\left[\alpha_1(x_1)^2 + (x_2)^2 + \alpha_3(x_3)^2\right] + 3\rho\beta + 2\rho\beta U(x) \\
&\leq \rho \cdot \inf_{r\in(0,\infty)} \Big[\big[\tfrac{(\alpha_1+\alpha_2)^2}{r} - 2\alpha_1\big](x_1)^2 + (r-1)\,(x_2)^2 - 2\alpha_3(x_3)^2 \Big] + 3\rho\beta + 2\rho\beta U(x) \\
&\leq 3\rho\beta + \left[2\rho\beta + \inf_{r\in(0,\infty)} \Big[\big[\tfrac{(\alpha_1+\alpha_2)^2}{r} - 2\alpha_1\big] \vee [r-1] \vee [-2\alpha_3] \Big] \right] U(x) \\
&\leq 3\rho\beta + [2\rho\beta + \vartheta]\, U(x).
\end{aligned} \tag{153}$$

Hence, Corollary 2.4 implies for every $x \in \mathbb{R}^3$ and every $t, \rho \in [0,\infty)$ that

$$\mathbb{E}\big[\exp\big(\tfrac{\rho}{e^{(2\rho\beta+\vartheta)t}}\|X_t^x\|^2\big)\big] \leq \exp\big(\smallint_0^t \tfrac{3\rho\beta}{e^{(2\rho\beta+\vartheta)s}}\, ds + \rho\,\|x\|^2\big) \leq \exp\big(\tfrac{3}{2} + \rho\,\|x\|^2\big). \tag{154}$$

Next we apply Corollary 2.30. For this observe for every $\theta \in (0,\infty]$ and every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ with $x \neq y$ that

$$\begin{aligned}
&\frac{\langle x-y, \mu(x)-\mu(y)\rangle + \frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^3)}}{\|x-y\|^2} + \frac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \\
&= \frac{\langle x-y, \mu(x)-\mu(y)\rangle}{\|x-y\|^2} \\
&= \frac{\max(\mathrm{spectrum}(A+A^*))}{2} + \frac{(x_3-y_3)\,(x_1x_2-y_1y_2) - (x_2-y_2)\,(x_1x_3-y_1y_3)}{\|x-y\|^2} \\
&= \frac{\max(\mathrm{spectrum}(A+A^*))}{2} + \frac{(x_1-y_1)\,(y_2x_3-x_2y_3)}{\|x-y\|^2} \\
&= \frac{\max(\mathrm{spectrum}(A+A^*))}{2} + \frac{(x_1-y_1)\,[(y_2+x_2)\,(x_3-y_3) - (x_2-y_2)\,(y_3+x_3)]}{2\,\|x-y\|^2} \\
&\leq \frac{\max(\mathrm{spectrum}(A+A^*))}{2} + \frac{|x_2|+|x_3|+|y_2|+|y_3|}{4}.
\end{aligned} \tag{155}$$

The estimate $a \leq \frac{\delta}{4} + \frac{a^2}{\delta}$ for all $a \in \mathbb{R}$, $\delta \in (0, \infty)$ hence proves for every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$ with $x \neq y$ and every $r, t, T, \rho \in (0, \infty)$, $\theta \in (0, \infty]$ that

$$\begin{aligned} &\frac{\langle x - y, \mu(x) - \mu(y)\rangle + \frac{1}{2}\|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^3)}}{\|x-y\|^2} + \frac{(\frac{\theta}{2} - 1)\,\|(\sigma(x) - \sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \\ &\leq \frac{\max(\text{spectrum}(A + A^*))}{2} + \frac{1}{4} \cdot \frac{rTe^{(2\rho\beta+\vartheta)t}}{8\rho} + \frac{8\rho}{rTe^{(2\rho\beta+\vartheta)t}} \cdot \frac{|x_2|^2 + |x_3|^2 + |y_2|^2 + |y_3|^2}{16} \\ &\leq \frac{\max(\text{spectrum}(A + A^*))}{2} + \frac{rTe^{(2\rho\beta+\vartheta)t}}{32\rho} + \frac{\rho\,[\|x\|^2 + \|y\|^2]}{2rTe^{(2\rho\beta+\vartheta)t}}. \end{aligned} \tag{156}$$

Corollary 2.30 hence implies for every $T, r, \rho \in (0, \infty)$ and every $x, y \in \mathbb{R}^3$ that

$$\left\| \sup_{t \in [0,T]} \frac{\|X_t^x - X_t^y\|}{\|x - y\|} \right\|_{L^r(\Omega;\mathbb{R}^3)} \leq \exp\!\left(\tfrac{\max(\text{spectrum}(A+A^*))T}{2} + \tfrac{rT^2 e^{(2\rho\beta+\vartheta)T}}{32\rho} + \tfrac{3 + \rho\,\|x\|^2 + \rho\,\|y\|^2}{2r} \right). \tag{157}$$

Combining this with Lemma 3.1 ensures that the stochastic Lorenz equation (151) is strongly complete. In the same way as above strong stability estimates of the form (157) and strong completeness can be proved if the diffusion coefficient is not necessarily constant as in (151) but globally bounded and globally Lipschitz continuous. Strong completeness for the SDE (151) follows also from inequality (156) and Theorem 2.4 in Zhang [54]. If the diffusion coefficient is linear and if $m = 1$, then strong completeness follows in the case $\alpha_2 = \alpha_1$ from Theorem 4.1 in Schmalfuß [44]. If the diffusion coefficient is merely globally Lipschitz continuous but not globally bounded, then it is still an *open question* whether strong stability estimates of the form (157) do hold (see also Section 2 in Hairer et al. [14] for a counterexample with a related drift coefficient function and a linear diffusion coefficient function) and also whether strong completeness does hold if σ is non-linear or if $m > 1$. Another way for establishing strong completeness for the stochastic Lorenz equation (151) in the case of additive noise is to substract the noise process and then to solve the resulting random orindary differential equations for every continuous trajectory of the driving noise process (cf. the remarks in Subsection 3.3).

4.4 Langevin dynamics

The *Langevin dynamics* is a well-known model for the dynamics of a molecular system. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $m \in \mathbb{N}$, $\gamma, \varepsilon \in (0, \infty)$, $U \in C^2(\mathbb{R}^m, \mathbb{R})$, let $W\colon [0, \infty) \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\mu\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ and $\sigma\colon \mathbb{R}^{2m} \to \mathbb{R}^{(2m)\times m}$ be given by $\mu(x) = (x_2, -(\nabla U)(x_1) - \gamma x_2)$ and $\sigma(x)u = (0, \sqrt{\varepsilon} u)$ for all $x = (x_1, x_2) \in \mathbb{R}^{2m}$, $u \in \mathbb{R}^m$ and let $X^x\colon [0, \infty) \times \Omega \to \mathbb{R}^{2m}$, $x \in \mathbb{R}^{2m}$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \textstyle\int_0^t \mu(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s \tag{158}$$

$\mathbb{P}$-a.s. for all $(t, x) \in [0, \infty) \times \mathbb{R}^{2m}$. Next observe that if $\rho \in [0, \infty)$ and if $U_0\colon \mathbb{R}^{2m} \to \mathbb{R}$ is given by $U_0(x) = \rho\, U(x_1) + \frac{\rho}{2}\,\|x_2\|^2$ for all $x = (x_1, x_2) \in \mathbb{R}^{2m}$, then it holds for every $x = (x_1, x_2) \in \mathbb{R}^{2m}$ that

$$(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|^2 = \tfrac{\rho\varepsilon m}{2} + \rho\left[\tfrac{\rho\varepsilon}{2} - \gamma\right]\|x_2\|^2. \tag{159}$$

Corollary 2.4 hence implies for every $x = (x_1, x_2) \in \mathbb{R}^{2m}$, $t, \rho \in [0, \infty)$ that

$$\mathbb{E}\!\left[\exp\!\left(\rho\, U(X_t^{x,1}) + \tfrac{\rho\,\|X_t^{x,2}\|^2}{2} + \int_0^t \rho\left[\gamma - \tfrac{\rho\varepsilon}{2}\right]\left\|X_s^{x,2}\right\|^2 ds\right)\right] = e^{[\frac{\rho\varepsilon m t}{2} + \rho\, U(x_1) + \frac{\rho}{2}\|x_2\|^2]}. \tag{160}$$

Combining (159) and Corollary 2.30 shows that if

$$\exists\, \rho \in [0, \tfrac{2\gamma}{\varepsilon}], r, T \in (0,\infty),\, c \in \mathbb{R}\colon\ \forall_{\substack{x,y\in\mathbb{R}^m\\ \text{with } x\neq y}}\colon\ \tfrac{\langle x-y, (\nabla U)(x)-(\nabla U)(y)\rangle}{\|x-y\|^2} \leq c + \tfrac{\rho U(x)+\rho U(y)}{2rT}, \tag{161}$$

then it holds for every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^{2m}$ that

$$\left\|\sup\nolimits_{t\in[0,T]} \|X_t^x - X_t^y\|\right\|_{L^r(\Omega;\mathbb{R})} \leq \exp\Big(\big[c + \tfrac{1}{2} + \tfrac{\rho\varepsilon m}{4r}\big]\, T + \tfrac{\rho U(x_1)+\rho U(y_1)}{2r}\Big)\ \|x-y\|. \tag{162}$$

This and Lemma 3.1 imply that if

$$\exists\, c \in [0,\infty)\colon \qquad \sup\nolimits_{x,y\in\mathbb{R}^m} \big[\tfrac{\langle x-y, (\nabla U)(y)-(\nabla U)(x)\rangle}{\|x-y\|^2} - cU(x) - cU(y)\big] < \infty, \tag{163}$$

then the SDE (158) is strongly complete. Strong completeness for the SDE (158) follows also from inequality (159) and Theorem 2.4 in Zhang [54]. Let us point out that even in the case of SDEs with additive noise such as (158) strong completeness is not clear in general; see Subsection 3.3 above for details.

4.5 Brownian dynamics (Overdamped Langevin dynamics)

Brownian dynamics is a simplified version of Langevin dynamics in the limit of no average acceleration and models the positions of molecules in a potential (see, for instance, Section 2.1 in Beskos & Stuart [4]). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $d \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, $\eta_0 \in [0,\infty)$, $\eta_1 \in \mathbb{R}$, $\eta_2 \in [0, \frac{2}{\varepsilon}]$, $U \in C^2(\mathbb{R}^d, [0,\infty))$ satisfy

$$\forall\, x \in \mathbb{R}^d\colon \qquad (\triangle U)(x) \leq \eta_0 + 2\eta_1 U(x) + \eta_2 \|(\nabla U)(x)\|^2, \tag{164}$$

let $W\colon [0,\infty)\times\Omega \to \mathbb{R}^d$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion and let $X^x\colon [0,\infty)\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x - \textstyle\int_0^t (\nabla U)(X_s^x)\, ds + \sqrt{\varepsilon} W_t \tag{165}$$

$\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times\mathbb{R}^d$. If $\rho \in [0,\infty)$ and $U_1, \overline{U}\colon \mathbb{R}^d \to \mathbb{R}$ satisfy for every $x \in \mathbb{R}^d$ that

$$U_1(x) = \rho\, U(x) \quad \text{and} \quad \overline{U}(x) = \rho\left(1 - \tfrac{\varepsilon}{2}(\eta_2+\rho)\right)\|(\nabla U)(x)\|^2, \tag{166}$$

then it holds for every $x \in \mathbb{R}^d$ that

$$\begin{aligned}
&(\mathcal{G}_{-\nabla U, \sqrt{\varepsilon} I} U_1)(x) + \tfrac{1}{2}\left\|\sqrt{\varepsilon}(\nabla U_1)(x)\right\|^2 + \overline{U}(x) \\
&= -\rho\,\|(\nabla U)(x)\|^2 + \tfrac{\varepsilon\rho}{2}\operatorname{tr}((\operatorname{Hess} U)(x)) + \tfrac{\varepsilon\rho^2}{2}\|(\nabla U)(x)\|^2 + \overline{U}(x) \\
&\leq \tfrac{\varepsilon\rho 2\eta_1}{2} U(x) + \rho\left[\tfrac{\varepsilon(\eta_2+\rho)}{2} - 1\right]\|(\nabla U)(x)\|^2 + \overline{U}(x) + \tfrac{\varepsilon\rho\eta_0}{2} = \tfrac{\varepsilon\rho\eta_0}{2} + \varepsilon\eta_1 U_1(x)\,.
\end{aligned} \tag{167}$$

Hence, Corollary 2.4 implies for every $t \in [0,\infty)$, $\rho \in [0, \frac{2}{\varepsilon} - \eta_2]$ and every $x \in \mathbb{R}^d$ that

$$\mathbb{E}\left[\exp\left(\tfrac{\rho U(X_t^x)}{e^{\varepsilon\eta_1 t}} + \int_0^t \tfrac{\rho(1-\frac{\varepsilon}{2}(\eta_2+\rho))}{e^{\varepsilon\eta_1 s}}\|(\nabla U)(X_s^x)\|^2 - \tfrac{\varepsilon\rho\eta_0}{2e^{\varepsilon\eta_1 s}} ds\right)\right] \leq e^{\rho U(x)}. \tag{168}$$

This exponential moment estimate generalizes Lemma 2.5 in Bou-Rabee & Hairer [6] in the case where the function Θ appearing in Lemma 2.5 in [6] satisfies $\Theta(u) = \exp(\rho u)$ for all $u \in \mathbb{R}$ and some $\rho \in [0, \frac{2}{\varepsilon} - \eta_2]$. If $T \in (0,\infty)$, $c \in \mathbb{R}$, $\rho_0, \rho_1 \in [0, \frac{2}{\varepsilon} - \eta_2]$, $r \in (0,\infty]$, $q_0, q_1 \in [r,\infty]$ with $\frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{r}$ satisfy for every $x, y \in \mathbb{R}^d$ that

$$\tfrac{\langle x-y, (\nabla U)(y)-(\nabla U)(x)\rangle}{\|x-y\|^2} \leq c + \tfrac{\rho_0}{2q_0 T e^{\varepsilon\eta_1 T}}\left[U(x) + U(y)\right] + \tfrac{\rho_1(1-\frac{\varepsilon}{2}(\eta_2+\rho_1))}{2q_1 e^{\varepsilon\eta_1 T}}\left[\|(\nabla U)(x)\|^2 + \|(\nabla U)(y)\|^2\right], \tag{169}$$

then Corollary 2.30 (applied with $U_{0,0} \equiv U_{0,1} \equiv \overline{U}_{0,1} \equiv 0$, $U_{1,0} = \rho_0 U$, $U_{1,1} = \rho_1 U_1$ and $c_0 \equiv 0$) together with inequalities (167) and (169) shows for every $x, y \in \mathbb{R}^d$ that

$$\left\|\sup_{t\in[0,T]} \|X_t^x - X_t^y\|\right\|_{L^r(\Omega;\mathbb{R})} \leq \exp\!\Big(cT + \big(\tfrac{\rho_0}{q_0} + \tfrac{\rho_1}{q_1}\big)\tfrac{\varepsilon\eta_0 T}{2} + \tfrac{\rho_0(U(x)+U(y))}{2q_0} + \tfrac{\rho_1(U(x)+U(y))}{2q_1}\Big)\,\|x-y\|\,. \tag{170}$$

Hence, if there exist $\overline{\rho}_0 \in (0,\infty)$ and $\overline{\rho}_1 \in \big(0, \frac{(1-\frac{\varepsilon}{2}\eta_2)^2}{4\varepsilon d}\big)$ such that

$$\sup_{x,y\in\mathbb{R}^d}\Big[\tfrac{\langle x-y,(\nabla U)(y)-(\nabla U)(x)\rangle}{\|x-y\|^2} - \overline{\rho}_0\,[U(x)+U(y)] - \overline{\rho}_1\,\big[\|(\nabla U)(x)\|^2 + \|(\nabla U)(y)\|^2\big]\Big] < \infty, \tag{171}$$

then there exist $T, c \in (0,\infty)$, $\rho_0, \rho_1 \in [0, \frac{2}{\varepsilon} - \eta_2]$, $r \in (d,\infty)$, $q_0, q_1 \in (r,\infty)$ with $\frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{r}$ such that inequality (170) holds and then Lemma 3.1 proves that the SDE (165) is strongly complete. Strong completeness for the SDE (165) follows from Theorem 2.4 in Zhang [54] under the stronger assumption that condition (164) holds with $\eta_2 = 0$ and that condition (171) holds with $\overline{\rho}_1 = 0$.

4.6 Stochastic SIR model

The SIR model from epidemiology for the total number of *s*usceptible, *i*nfected and *r*ecovered individuals has been introduced by Anderson & May [3]. This section establishes strong stability estimates for the stochastic SIR model studied in Tornatore, Buccellato & Vetro [50]. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\alpha,\beta,\gamma,\delta \in (0,\infty)$, let $\mu\colon [0,\infty)^3 \to \mathbb{R}^3$ and $\sigma\colon [0,\infty)^3 \to \mathbb{R}^3$ be given by

$$\mu\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha x_1 x_2 - \delta x_1 + \delta \\ \alpha x_1 x_2 - (\gamma+\delta)x_2 \\ \gamma x_2 - \delta x_3 \end{pmatrix}, \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\beta x_1 x_2 \\ \beta x_1 x_2 \\ 0 \end{pmatrix} \tag{172}$$

for all $x = (x_1,x_2,x_3) \in (0,\infty)^3$ and let $X^x = (X^{x,1}, X^{x,2}, X^{x,3})\colon [0,\infty)\times\Omega \to [0,\infty)^3$, $x \in [0,\infty)^3$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \textstyle\int_0^t \mu(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dW_s \tag{173}$$

$\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times[0,\infty)^3$. For the stochastic SIR model it is well known that the sum of the first two coordinates serves as a Lyapunov-type function (cf., e.g., Tornatore, Buccellato & Vetro [50]). We use this to construct an exponential Lyapunov-type function in the sense of Corollary 2.4. More formally, if $\rho, \kappa \in [0,\infty)$ and if $U\colon \mathbb{R}^3 \to \mathbb{R}$ is given by $U(x) = \rho\,(x_1 + x_2 - \kappa)$ for all $x = (x_1,x_2,x_3) \in \mathbb{R}^3$, then it holds for every $x = (x_1,x_2,x_3) \in [0,\infty)^3$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2e^{-\delta t}}\|\sigma(x)^*(\nabla U)(x)\|^2 = \rho\,[-\alpha x_1x_2 - \delta x_1 + \delta + \alpha x_1 x_2 - (\gamma+\delta)x_2] \\
&= \rho\,[-\delta x_1 + \delta - \delta x_2] - \rho\gamma x_2 = -\delta\rho\,[x_1 + x_2 - 1] - \rho\gamma x_2 \\
&= -\delta U(x) - \delta\rho\,[\kappa - 1] - \rho\gamma x_2 \leq -\delta U(x) - \delta\rho\,[\kappa-1] \leq \delta\rho.
\end{aligned} \tag{174}$$

Corollary 2.4 hence implies for every $x = (x_1,x_2,x_3) \in [0,\infty)^3$, $\rho, t \in [0,\infty)$ that

$$\mathbb{E}\Big[e^{\rho\, e^{\delta t}(X_t^{x,1} + X_t^{x,2} - 1)}\Big] \leq e^{\rho(x_1+x_2-1)}, \quad \mathbb{E}\Big[e^{\rho\,(X_t^{x,1}+X_t^{x,2}) + \delta\rho\int_0^t (X_s^{x,1}+X_s^{x,2})\,ds}\Big] \leq e^{\rho(\delta t + x_1 + x_2)}.$$

Moreover, if $\rho \in [0,\infty)$ and if $U\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $U(x) = \rho\,(x_1+x_2-1)^2$ for all $x = (x_1,x_2,x_3) \in \mathbb{R}^3$, then it holds for every $x = (x_1,x_2,x_3) \in [0,\infty)^3$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{e^{2\delta t}}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 \\
&= 2\rho\,(x_1+x_2-1)\,[-\alpha x_1x_2 - \delta x_1 + \delta + \alpha x_1x_2 - (\gamma+\delta)x_2] \\
&= 2\rho\,(x_1+x_2-1)\,[-\delta x_1 + \delta - \delta x_2] - 2\rho\gamma\,(x_1+x_2-1)\,x_2 \\
&= -2\delta U(x) - 2\rho\gamma\,(x_1+x_2-1)\,x_2 \leq -2\delta U(x) - 2\rho\gamma\,(x_2-1)\,x_2 \\
&= -2\delta U(x) - 2\rho\gamma\,\big((x_2)^2 - x_2 + \tfrac{1}{4}\big) + \tfrac{\rho\gamma}{2} = -2\delta U(x) - 2\rho\gamma\,\big(x_2 - \tfrac{1}{2}\big)^2 + \tfrac{\rho\gamma}{2} \\
&\leq \tfrac{\rho\gamma}{2} - 2\delta U(x) \leq \tfrac{\rho\gamma}{2}\,.
\end{aligned} \tag{175}$$

Corollary 2.4 hence ensures for every $x = (x_1, x_2, x_3) \in [0,\infty)^3$, $\rho, t \in [0,\infty)$ that

$$\mathbb{E}\Big[\exp\Big(\rho\, e^{2\delta t}\, \big[X_t^{x,1} + X_t^{x,2} - 1\big]^2\Big)\Big] \leq \exp\big(\tfrac{\gamma}{4}\,\big[e^{2\delta t} - 1\big] + \rho\,[x_1 + x_2 - 1]^2\big) \tag{176}$$

and $\mathbb{E}\big[e^{\rho[X_t^{x,1}+X_t^{x,2}-1]^2 + 2\rho\delta\int_0^t [X_s^{x,1}+X_s^{x,2}-1]^2\,ds}\big] \leq e^{\frac{\rho\gamma t}{2} + \rho[x_1+x_2-1]^2}$. In the next step note for every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in [0,\infty)^3$ with $x \neq y$ that

$$\begin{aligned}
&\frac{\|(\sigma(x) - \sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \leq \frac{\|\sigma(x) - \sigma(y)\|^2}{\|x-y\|^2} = \frac{\beta^2\,(x_1x_2 - y_1y_2)^2}{\|x-y\|^2} \\
&= \frac{\beta^2\,[(x_1-y_1)(x_2+y_2) + (x_1+y_1)(x_2-y_2)]^2}{4\,\|x-y\|^2} \\
&\leq \tfrac{\beta^2}{4}\max([x_1+y_1]^2, [x_2+y_2]^2) + \tfrac{\beta^2}{8}[x_1+y_1][x_2+y_2] \leq \tfrac{\beta^2}{4}\,[x_1+y_1+x_2+y_2]^2 \\
&\leq \beta^2\,[x_1+x_2-1]^2 + \beta^2\,[y_1+y_2-1]^2 + 2\beta^2
\end{aligned} \tag{177}$$

and that

$$\begin{aligned}
&\frac{\langle x-y, \mu(x) - \mu(y)\rangle}{\|x-y\|^2} = \frac{\alpha\,[(x_2-y_2) - (x_1-y_1)]\,(x_1x_2 - y_1y_2)}{\|x-y\|^2} \\
&= \frac{\alpha\,(x_2 - y_2 - (x_1-y_1))\,[(x_1-y_1)(x_2+y_2) + (x_1+y_1)(x_2-y_2)]}{2\,\|x-y\|^2} \\
&\leq \tfrac{3\alpha}{4}\,(x_1+y_1) + \tfrac{\alpha}{4}\,(x_2+y_2) \leq \tfrac{3\alpha}{2} + \tfrac{3\alpha}{4}\,(x_1+x_2-1) + \tfrac{3\alpha}{4}\,(y_1+y_2-1)
\end{aligned} \tag{178}$$

This implies for every $\theta \in [0,\infty)$, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in [0,\infty)^3$ with $x \neq y$ that

$$\begin{aligned}
&\frac{\langle x-y, \mu(x)-\mu(y)\rangle + \frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R},\mathbb{R}^3)}}{\|x-y\|^2} + \frac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \\
&\leq \tfrac{(\sqrt{2}-1)\,\gamma}{2} - \delta + \tfrac{3\alpha}{4}\,(x_1+x_2) + \tfrac{3\alpha}{4}\,(y_1+y_2) + \tfrac{[1+|\theta-2|]\,\|\sigma(x)-\sigma(y)\|^2}{2\,\|x-y\|^2} \\
&\leq \gamma + \tfrac{3\alpha}{4}\,(x_1+x_2) + \tfrac{3\alpha}{4}\,(y_1+y_2) \\
&\quad + \beta^2\big[\tfrac{1}{2} + |\tfrac{\theta}{2} - 1|\big]\,\big([x_1+x_2-1]^2 + [y_1+y_2-1]^2 + 2\big)\,.
\end{aligned} \tag{179}$$

Combining (174), (175), (177) and (179) with Corollary 2.30 then shows for every $T \in [0,\infty)$, $r, p, q_{1,0,1}, q_{1,0,2} \in [2,\infty)$, $\eta \in [p,\infty]$, $\theta \in (0,p)$ with $\frac{1}{p} + \frac{1}{q_{1,0,1}} + \frac{1}{q_{1,0,2}} = \frac{1}{r}$ and every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in [0,\infty)^3$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^x - X_t^y\|\big\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \frac{\|x-y\|}{[1-\frac{\theta}{p}]^{1/\theta}} \cdot \exp\left(\begin{array}{l}
\Big(2\beta^2\left[\frac{1}{(2/p-2/\eta)} - \frac{\theta}{2}\right] + \gamma\Big)\,T + \beta^2\gamma T^2\left[\frac{1}{(2/p-2/\eta)} - \frac{\theta}{2}\right] \\
+\beta^2 T\left[\frac{1}{(2/p-2/\eta)} - \frac{\theta}{2}\right]\big[(x_1+x_2-1)^2 + (y_1+y_2-1)^2\big] \\
+\frac{3\alpha\delta T^2}{2} + \frac{3\alpha T}{4}(x_1+x_2+y_1+y_2) + \beta^2\gamma T^2\left[\frac{1}{2} + |\frac{\theta}{2}-1|\right] \\
+\beta^2 T\left[\frac{1}{2} + |\frac{\theta}{2}-1|\right]\big[(x_1+x_2-1)^2 + (y_1+y_2-1)^2\big]
\end{array}\right).
\end{aligned} \tag{180}$$

This implies for every $T \in [0,\infty)$, $r, p \in (2,\infty)$, $\eta \in [p,\infty]$, $\theta \in [2,p)$ with $\frac{1}{p} < \frac{1}{r}$ and every $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in [0,\infty)^3$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^x - X_t^y\|\big\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \frac{\|x-y\|}{[1-\frac{\theta}{p}]^{1/\theta}} \cdot \exp\left(\begin{array}{l}
\beta^2 T\left[\frac{1}{(1/p-1/\eta)} - \theta\right] + \gamma T + \frac{3\alpha T}{4}\big(2\delta T + x_1+x_2+y_1+y_2\big) \\
+\frac{\beta^2 T}{2}\left[\frac{1}{(1/p-1/\eta)} - 1\right]\big[\gamma T + (x_1+x_2-1)^2 + (y_1+y_2-1)^2\big]
\end{array}\right).
\end{aligned}$$

This finally shows for every $T \in [0,\infty)$, $r \in (2,\infty)$, $\theta \in [2,r)$ and every $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in [0,\infty)^3$ that

$$\begin{aligned} &\left\|\sup\nolimits_{t\in[0,T]}\|X_t^x - X_t^y\|\right\|_{L^r(\Omega;\mathbb{R})} \\ &\leq \frac{\|x-y\|}{\left[1-\frac{\theta}{r}\right]^{1/\theta}} \cdot \exp\left(\begin{array}{c} \beta^2 T(r-\theta) + \gamma T + \frac{3\alpha T}{4}\big(2\delta T + x_1 + x_2 + y_1 + y_2\big) \\ + \frac{\beta^2 T(r-1)}{2}\big[\gamma T + (x_1+x_2-1)^2 + (y_1+y_2-1)^2\big] \end{array} \right). \end{aligned} \tag{181}$$

Combining this with Lemma 3.1 proves that the stochastic SIR model (173) is strongly complete.

4.7 Experimental psychology model

Let $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty)\times\Omega\to\mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\alpha,\delta\in(0,\infty)$, $\beta\in\mathbb{R}$, let $\mu\colon\mathbb{R}^2\to\mathbb{R}^2$ and $\sigma\colon\mathbb{R}^2\to\mathbb{R}^2$ be given by

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (x_2)^2\,(\delta+4\alpha x_1) - \frac{\beta^2 x_1}{2} \\ -x_1 x_2\,(\delta+4\alpha x_1) - \frac{\beta^2 x_2}{2} \end{pmatrix}, \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\beta x_2 \\ \beta x_1 \end{pmatrix} \tag{182}$$

for all $x = (x_1,x_2)\in\mathbb{R}^2$ and let $X^x\colon[0,\infty)\times\Omega\to\mathbb{R}^2$, $x\in\mathbb{R}^2$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \textstyle\int_0^t \mu(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dW_s \tag{183}$$

$\mathbb{P}$-a.s. for all $(t,x)\in[0,\infty)\times\mathbb{R}^2$. The SDE (183) is a suitable transformed version of a model proposed in Haken, Kelso & Bunz [16] in the deterministic case and in Schöner, Haken & Kelso [45] in the stochastic case (see Section 7.2 in Kloeden & Platen [26] for details). If $p\in[1,\infty)$, $\rho\in(0,\infty)$ and if $U\colon\mathbb{R}^2\to\mathbb{R}$ satisfies $U(x) = \rho\|x\|^{2p}$ for all $x\in\mathbb{R}^2$, then it holds for every $x\in\mathbb{R}^2$ that

$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\|\sigma(x)^*\,(\nabla U)(x)\|^2 = 2p\rho\,\|x\|^{(2p-2)}\left[\langle x,\mu(x)\rangle + \tfrac{1}{2}\|\sigma(x)\|^2\right] = 0. \tag{184}$$

In addition, we get from Itô's formula that for every $x\in\mathbb{R}^2$, $t\in[0,\infty)$ it holds that $\|X_t^x\| = \|x\|$ $\mathbb{P}$-a.s. Next note for every $\theta\in(0,\infty)$, $x,y\in\mathbb{R}^2$ with $x\neq y$ that

$$\begin{aligned} &\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle + \frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R},\mathbb{R}^2)}}{\|x-y\|^2} + \tfrac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \\ &= \frac{\langle x-y,\mu(x)-\mu(y)\rangle + \frac{1}{2}\|\sigma(x)-\sigma(y)\|^2}{\|x-y\|^2} \\ &= \frac{(x_1-y_1)\left[(x_2)^2(\delta+4\alpha x_1) - \frac{\beta^2 x_1}{2} - (y_2)^2(\delta+4\alpha y_1) + \frac{\beta^2 y_1}{2}\right]}{\|x-y\|^2} \\ &+ \frac{(x_2-y_2)\left[-x_1x_2(\delta+4\alpha x_1) - \frac{\beta^2 x_2}{2} + y_1y_2(\delta+4\alpha y_1) + \frac{\beta^2 y_2}{2}\right]}{\|x-y\|^2} \\ &+ \frac{\frac{1}{2}\beta^2\left[(x_1-y_1)^2 + (x_2-y_2)^2\right]}{\|x-y\|^2}. \end{aligned} \tag{185}$$

This implies for every $\theta, \varepsilon \in (0,\infty)$, $x, y \in \mathbb{R}^2$ with $x \neq y$ that

$$
\begin{aligned}
&\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle+\frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R},\mathbb{R}^2)}}{\|x-y\|^2} + \tfrac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4}\\
&= \tfrac{(x_1-y_1)\left[(x_2)^2(\delta+4\alpha x_1)-(y_2)^2(\delta+4\alpha y_1)\right]}{\|x-y\|^2} - \tfrac{(x_2-y_2)[x_1x_2(\delta+4\alpha x_1)-y_1y_2(\delta+4\alpha y_1)]}{\|x-y\|^2}\\
&\leq \tfrac{\delta(x_1-y_1)(x_2-y_2)(x_2+y_2)}{\|x-y\|^2} - \tfrac{\delta(x_2-y_2)[(x_2-y_2)(x_1+y_1)+(x_1-y_1)(x_2+y_2)]}{2\,\|x-y\|^2}\\
&\quad + \tfrac{4\alpha(x_1-y_1)\left[(x_2)^2x_1-(y_2)^2y_1\right]}{\|x-y\|^2} - \tfrac{4\alpha(x_2-y_2)\left[(x_1)^2x_2-(y_1)^2y_2\right]}{\|x-y\|^2}\\
&\leq \tfrac{\delta(x_1-y_1)(x_2-y_2)(x_2+y_2)}{2\,\|x-y\|^2} - \tfrac{\delta(x_2-y_2)^2(x_1+y_1)}{2\,\|x-y\|^2}\\
&\quad + \tfrac{2\alpha(x_1-y_1)\left[(x_2-y_2)(x_2+y_2)(x_1+y_1)+((x_2)^2+(y_2)^2)(x_1-y_1)\right]}{\|x-y\|^2}\\
&\quad - \tfrac{2\alpha(x_2-y_2)\left[(x_1-y_1)(x_1+y_1)(x_2+y_2)+((y_1)^2+(x_1)^2)(x_2-y_2)\right]}{\|x-y\|^2}\\
&\leq \tfrac{\delta}{4}|x_2+y_2| + \tfrac{\delta}{2}|x_1+y_1| + \tfrac{2\alpha(x_1-y_1)^2((x_2)^2+(y_2)^2)-2\alpha(x_2-y_2)^2((x_1)^2+(y_1)^2)}{\|x-y\|^2}\\
&\leq \tfrac{\delta}{4}|x_2+y_2| + \tfrac{\delta}{2}|x_1+y_1| + 2\alpha\left[(x_2)^2+(y_2)^2\right]\\
&\leq \tfrac{\delta^2}{32\varepsilon} + \tfrac{\delta^2}{4(2\alpha+\varepsilon)} + [2\alpha+\varepsilon]\left[\|x\|^2+\|y\|^2\right].
\end{aligned}
\tag{186}
$$

Combining (184) and (186) with Corollary 2.30 proves for every $\varepsilon, r, T \in (0,\infty)$, $x, y \in \mathbb{R}^2$ with $x \neq y$ that

$$
\left\|\sup\nolimits_{t\in[0,T]}\|X^x_t - X^y_t\|\right\|_{L^r(\Omega;\mathbb{R})} \leq e^{\left[\frac{\delta^2 T}{32\varepsilon}+\frac{\delta^2 T}{4(2\alpha+\varepsilon)}+[2\alpha+\varepsilon]T\left(\|x\|^2+\|y\|^2\right)\right]}\|x-y\|. \tag{187}
$$

Combining this with Lemma 3.1 proves that the SDE (183) is strongly complete. Strong completeness for the SDE (183) follows also from the inequalities (184) and (186) together with Theorem 2.4 in Zhang [54].

4.8 Stochastic Brusselator in the well-stirred case

The Brusselator is a model for a trimolecular chemical reaction and has been studied in Prigogine & Lefever [38] and by other scientists from Brussels (cf. Tyson [51]). A stochastic version hereof has been proposed by Dawson [7] (see also Scheutzow [43]). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}^2$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion, let $\alpha, \delta \in (0,\infty)$, $m \in \mathbb{N}$, let $\sigma = (\sigma_{i,j})_{(i,j)\in\{1,2\}\times\{1,\dots,m\}}\colon [0,\infty)^2 \to \mathbb{R}^{2\times m}$ be a globally Lipschitz continuous function with $\forall\, y \in (0,\infty)\colon \sigma(0,y) = \sigma(y,0) = 0$ (cf. the last sentence in Section 1 in Scheutzow [43]), let $\mu = (\mu_1, \mu_2)\colon [0,\infty)^2 \to \mathbb{R}^2$ be given by

$$
\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \delta - (\alpha+1)\,x_1 + x_2\cdot(x_1)^2 \\ \alpha x_1 - x_2\cdot(x_1)^2 \end{pmatrix} \tag{188}
$$

for all $x = (x_1, x_2) \in [0,\infty)^2$ and let $X^x\colon [0,\infty)\times\Omega \to [0,\infty)^2$, $x \in [0,\infty)^2$, be adapted stochastic processes with continuous sample paths satisfying the stochastic Brusselator equation

$$
X^x_t = x + \textstyle\int_0^t \mu(X^x_s)\,ds + \int_0^t \sigma(X^x_s)\,dW_s \tag{189}
$$

$\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times[0,\infty)^2$. We first apply Proposition 2.17 to (189). More formally, if $V\colon [0,\infty)^2 \to \mathbb{R}$ is given by $V(x,y) = |\langle x-y,(1,1)\rangle|^2 = [(x_1+x_2)-(y_1+y_2)]^2$ for all $x = (x_1,x_2), y = (y_1,y_2) \in [0,\infty)^2$, then it holds for every $x = (x_1,x_2), y = (y_1,y_2) \in [0,\infty)^2$ with $x \neq y$ that

$$
\begin{aligned}
(\overline{\mathcal{G}}_{\overline{\mu},\overline{\sigma}}V)(x,y) &= 2\left[(x_1+x_2)-(y_1+y_2)\right]\left[(\mu_1(x)+\mu_2(x))-(\mu_1(y)+\mu_2(y))\right]\\
&\quad + \operatorname{tr}\left((\sigma(x)-\sigma(y))\,(\sigma(x)-\sigma(y))^*\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)\\
&= \sum_{i=1}^m \left[(\sigma_{1,i}(x)+\sigma_{2,i}(x))-(\sigma_{1,i}(y)+\sigma_{2,i}(y))\right]^2 = \left\|(\sigma(x)-\sigma(y))^*(1,1)\right\|^2.
\end{aligned}
\tag{190}
$$

Proposition 2.17 hence implies that if

$$\forall\, r \in (0,\infty)\colon \quad \sup_{x,y\in[0,\infty)^2, \|x\|,\|y\|\le r} \left(\tfrac{\|(\sigma(x)-\sigma(y))^*(1,1)\|}{|\langle x-y,(1,1)\rangle|} \right) < \infty, \tag{191}$$

then it holds for every $x, y \in [0,\infty)^2$, $T \in [0,\infty)$, $r,p,q \in (0,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ that

$$\begin{aligned} &\big\|(X_T^{x,1}-X_T^{y,1})+(X_T^{x,2}-X_T^{y,2})\big\|_{L^r(\Omega;\mathbb{R})} \\ &\le |(x_1-y_1)+(x_2-y_2)| \left\|\exp\!\left(\textstyle\int_0^T \tfrac{(p-1)\,\|(\sigma(X_s^x)-\sigma(X_s^y))^*(1,1)\|^2}{2\,|\langle X_s^x-X_s^y,(1,1)\rangle|^2}\,ds\right)\right\|_{L^q(\Omega;\mathbb{R})}. \end{aligned} \tag{192}$$

For instance, if $m=1$, $\beta\in\mathbb{R}$, $v\in\mathbb{R}^2$ and if $\sigma(x)=v+\beta x$ for all $x\in[0,\infty)^2$, then (192) shows for every $x,y\in[0,\infty)^2$, $T,r\in(0,\infty)$ that

$$\big\|(X_T^{x,1}-X_T^{y,1})+(X_T^{x,2}-X_T^{y,2})\big\|_{L^r(\Omega;\mathbb{R})} \le e^{\frac{(r-1)\beta^2 T}{2}}\,|(x_1-y_1)+(x_2-y_2)|\,. \tag{193}$$

In the following we additionally assume that $\eta := \sup_{y\in[0,\infty)^2}\|\sigma(y)^*(1,1)\| \in [0,\infty)$ (cf. the last sentence in Section 1 in Scheutzow [43]) and further investigate the stochastic Brusselator equation (189) under this additional assumption. In that case it holds that if $\rho\in(0,\infty)$ and if $U\colon\mathbb{R}^2\to\mathbb{R}$ is given by $U(x)=\rho\,(x_1+x_2)^2$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, then it holds for every $x=(x_1,x_2)\in[0,\infty)^2$, $\varepsilon\in(0,\infty)$ that

$$\begin{aligned} &(\mathcal{G}_{\mu,\sigma}U)(x)+\tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 \\ &= 2\rho\,(x_1+x_2)\,(\delta-\alpha x_1)+\rho\,\mathrm{tr}\!\left(\sigma(x)\sigma(x)^*\begin{pmatrix}1&1\\1&1\end{pmatrix}\right)+2\rho U(x)\|\sigma(x)^*(1,1)\|^2 \\ &\le 2\rho\delta\,(x_1+x_2)+\rho\eta^2+2\rho\eta^2 U(x) \le \tfrac{\delta^2}{2\varepsilon}+\rho\eta^2+2\rho\left[\eta^2+\varepsilon\right]U(x)\,. \end{aligned} \tag{194}$$

In addition, note for every $x=(x_1,x_2), y=(y_1,y_2)\in[0,\infty)^2$, $\theta\in(0,\infty)$ that

$$\begin{aligned} &\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle+\frac{1}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^2)}}{\|x-y\|^2}+\frac{(\frac{\theta}{2}-1)\,\|(\sigma(x)-\sigma(y))^*(x-y)\|^2}{\|x-y\|^4} \\ &\le \frac{(x_1)^2+(y_1)^2}{4}+\frac{3(x_1+y_1)(x_2+y_2)}{4}+\left[\tfrac{1}{2}+|\tfrac{\theta}{2}-1|\right]\|\sigma\|^2_{\mathrm{Lip}(\mathbb{R}^2,\mathrm{HS}(\mathbb{R}^2))} \\ &= \frac{(x_1)^2+(y_1)^2}{4}+\frac{3(x_1x_2+x_1y_2+y_1x_2+y_1y_2)}{4}+\left[\tfrac{1}{2}+|\tfrac{\theta}{2}-1|\right]\|\sigma\|^2_{\mathrm{Lip}(\mathbb{R}^2,\mathrm{HS}(\mathbb{R}^2))} \\ &\le (x_1+x_2)^2+(y_1+y_2)^2+\left[\tfrac{1}{2}+|\tfrac{\theta}{2}-1|\right]\|\sigma\|^2_{\mathrm{Lip}(\mathbb{R}^2,\mathrm{HS}(\mathbb{R}^2))}. \end{aligned} \tag{195}$$

Combining (194) and (195) with Corollary 2.30 hence implies that if $\eta=\sup_{y\in[0,\infty)^2}\|\sigma(y)^*(1,1)\|\in[0,\infty)$, then it holds for every $x=(x_1,x_2), y=(y_1,y_2)\in[0,\infty)^2$, $r,p,q\in(2,\infty)$, $T,\varepsilon,\rho\in(0,\infty)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $\exp\!\big(2\rho T[\eta^2+\varepsilon]\big)\le\frac{\rho}{2qT}$ that

$$\begin{aligned} &\big\|\sup\nolimits_{t\in[0,T]}\|X_t^x-X_t^y\|\big\|_{L^r(\Omega;\mathbb{R})} \\ &\le \frac{\|x-y\|}{\sqrt{1-2/p}}\exp\!\left(\tfrac{(p-1)T}{2}\,\|\sigma\|^2_{\mathrm{Lip}(\mathbb{R}^2,\mathrm{HS}(\mathbb{R}^2))}+\tfrac{\delta^2T/\varepsilon+1+\rho(x_1+x_2)^2+\rho(y_1+y_2)^2}{2q}\right). \end{aligned} \tag{196}$$

This together with Lemma 3.1 implies that the SDE (189) is strongly complete provided that σ is globally Lipschitz continuous and that $\sup_{y\in[0,\infty)^2}\|\sigma(y)^*(1,1)\|^2<\infty$.

4.9 Stochastic volatility processes and interest rate models (CIR, Ait-Sahalia, 3/2-model, CEV)

There are a number of models in the finance literature which generalize the Black-Scholes model by the use of a stochastic process for the squared volatility. The following SDE includes a number of these models for the squared volatility. Let $a \in (1,\infty)$, $b \in [\frac{1}{2},\infty)$, $c,\beta \in (0,\infty)$, $\alpha,\kappa \in [0,\infty)$, $\gamma,\delta \in \mathbb{R}$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion with continuous sample paths and let $X^x\colon [0,\infty)\times\Omega \to [0,\infty)$, $x \in (0,\infty)$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \int_0^t \left[\frac{\kappa}{(X_s^x)^c} + \delta + \gamma X_s^x - \alpha\,(X_s^x)^a\right] ds + \int_0^t \beta\,(X_s^x)^b\, dW_s \tag{197}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$ and all $x \in (0,\infty)$. The class (197) of processes includes Cox-Ingersoll-Ross processes ($b = 0.5$, $\gamma < 0 < \delta$, $\alpha = \kappa = 0$), Ait-Sahalia interest rate models, the volatility processes in Heston's 3/2-models ($b = 1.5$, $a = 2$, $\delta = \kappa = 0 \le \gamma$, $\alpha,\beta > 0$) and constant elasticity of variance processes ($b \in [0.5,1]$, $\alpha = \delta = \kappa = 0 \le \gamma$, $\beta > 0$). Let $\tau_y^x\colon \Omega \to [0,\infty]$, $x \in (0,\infty)$, $y \in [0,\infty)$, be given by $\tau_y^x = \inf\bigl(\{t \in [0,\infty)\colon X_t^x = y\} \cup \{\infty\}\bigr)$ for all $x \in (0,\infty)$ and all $y \in [0,\infty)$. For the rest of this subsection, we assume that the boundary point 0 is inaccessible, that is, that $\mathbb{P}\bigl[\tau_0^x = \infty\bigr] = 1$ for all $x \in (0,\infty)$. According to Feller's boundary classification (see, e.g., Theorem V.51.2 in [40]), the boundary point 0 is inaccessible if and only if *(i)* $\kappa > 0$ or *(ii)* $b = \frac{1}{2}$ and $2\delta \ge \beta^2$ or *(iii)* $b > \frac{1}{2}$ and $\delta > 0$ or *(iv)* $b \ge 1$ and $\delta \ge 0$.

In the case $b \neq 1$, the processes $(X^x)^{(1-b)}$, $x \in (0,\infty)$, satisfy an SDE with constant diffusion function (see, e.g., Alfonsi [2]) and globally one-sided Lipschitz continuous drift function (see, e.g., Dereich, Neuenkirch & Szpruch [8], Neuenkirch & Szpruch [36]). In the following calculation we exploit this observation together with the results in Section 2 to derive an estimate for the Lyapunov-type function $V\colon (0,\infty)^2 \to [0,\infty)$ given by $V(x,y) = |x^{(1-b)} - y^{(1-b)}|^2$ for all $x,y \in (0,\infty)$. For this, let $\mu,\sigma\colon (0,\infty) \to \mathbb{R}$ be given by $\mu(x) = \kappa x^{-c} + \delta + \gamma x - \alpha x^a$ and $\sigma(x) = \beta x^b$ for all $x \in (0,\infty)$. Then Example 2.16 implies that for all $x,y \in (0,\infty)$ it holds that

$$\tfrac{\|(\overline{G}_\sigma V)(x,y)\|^2}{|V(x,y)|^2} = \tfrac{4(1-b)^2\left[x^{-b}\beta x^b - y^{-b}\beta y^b\right]^2}{\left[x^{(1-b)} - y^{(1-b)}\right]^2} = 0 \tag{198}$$

and in the case $b \neq 1$ that for all $x,y \in (0,\infty)$ with $x \neq y$ it holds that

$$\begin{aligned}
&\tfrac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(x,y)}{V(x,y)} \\
&= \tfrac{2(1-b)\left(x^{-b}\left[\kappa x^{-c}+\delta+\gamma x-\alpha x^a\right]-y^{-b}\left[\kappa y^{-c}+\delta+\gamma y-\alpha y^a\right]-\frac{b}{2}\left[x^{1-b-2}\beta^2x^{2b}-y^{1-b-2}\beta^2y^{2b}\right]\right)}{x^{1-b}-y^{1-b}} \\
&= 2\,(1-b)\,\gamma + \tfrac{2(1-b)\int_y^x\left[-\delta b z^{(-b-1)}-\alpha(a-b)z^{(a-b-1)}-\kappa(c+b)z^{(-c-b-1)}-\frac{b(b-1)}{2}\beta^2 z^{(b-2)}\right]dz}{x^{1-b}-y^{1-b}} \\
&\le 2\,(1-b)\,\gamma + \tfrac{2(1-b)\left[\int_y^x z^{-b}\,dz\right]\left[\sup_{u\in(0,\infty)}\left(-\delta b u^{-1}-\alpha(a-b)u^{(a-1)}-\kappa(c+b)u^{(-c-1)}-\frac{b(b-1)}{2}\beta^2u^{(2b-2)}\right)\right]}{x^{(1-b)}-y^{(1-b)}} \\
&= 2\left[\sup_{u\in(0,\infty)}\left((1-b)\gamma-\tfrac{\delta b}{u}-\alpha(a-b)u^{(a-1)}-\tfrac{\kappa(c+b)}{u^{(c+1)}}+\tfrac{b(1-b)\beta^2}{2}u^{(2b-2)}\right)\right].
\end{aligned} \tag{199}$$

Proposition 2.17 with $r = \infty = p = q$ together with Lemma 2.24 hence yields for all $t \in [0,\infty)$ and all $x,y \in (0,\infty)$ that

$$\begin{aligned}
&\left\|(X_t^x)^{(1-b)} - (X_t^y)^{(1-b)}\right\|_{L^\infty(\Omega;\mathbb{R})} \le \left|x^{(1-b)} - y^{(1-b)}\right| e^{t\left[\sup_{u,v\in(0,\infty),u\neq v}\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(u,v)}{V(u,v)}\right]} \\
&\le \left|x^{(1-b)} - y^{(1-b)}\right| e^{t\sup_{u\in(0,\infty)}\left((1-b)\gamma-\frac{\delta b}{u}-\alpha(a-b)u^{(a-1)}-\frac{\kappa(b+c)}{u^{(c+1)}}+\frac{b(1-b)}{2}\beta^2u^{(2b-2)}\right)}.
\end{aligned} \tag{200}$$

Moreover, Proposition 2.26 with $v = \infty = p = q = r$ together with Lemma 2.24 yields for all $t \in [0,\infty)$ and all $x,y \in (0,\infty)$ that

$$\begin{aligned}
&\left\|\sup_{s\in[0,t]}\left|(X_s^x)^{(1-b)} - (X_s^y)^{(1-b)}\right|\right\|_{L^\infty(\Omega;\mathbb{R})} \le \left|x^{(1-b)} - y^{(1-b)}\right| e^{t\max\left\{0,\sup_{u,v\in(0,\infty)}\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(u,v)}{V(u,v)}\right\}} \\
&\le \left|x^{(1-b)} - y^{(1-b)}\right| e^{t\max\left\{0,\sup_{u\in(0,\infty)}\left((1-b)\gamma-\frac{\delta b}{u}-\alpha(a-b)u^{(a-1)}-\frac{\kappa(b+c)}{u^{(c+1)}}+\frac{b(1-b)}{2}\beta^2u^{(2b-2)}\right)\right\}}.
\end{aligned} \tag{201}$$

The right-hand sides of (199), of (200) and of (201) are finite if (i) $\kappa > 0$ or (ii) $b = \frac{1}{2}$ and $2\delta \geq \beta^2$ or (iii) $b > \frac{1}{2}$ and $\delta > 0$ or (iv) $b \geq 1$ and $\delta \geq 0$.

Next, for the convenience of the reader, we derive well-known moment estimates. For this let $\eta \in (0,\infty)$ be a fixed real number and let $U_p \colon (0,\infty) \to (0,\infty)$, $p \in \mathbb{R}$, be given by $U_p(x) = \eta + x^p$ for all $x \in (0,\infty)$, $p \in \mathbb{R}$. Then Lemma 2.2 implies for all $t \in [0,\infty)$, $x \in (0,\infty)$, $p \in \mathbb{R}$ that

$$\begin{aligned}&\mathbb{E}\big[\eta + (X_t^x)^p\big] \leq (\eta + x^p) \exp\Big(\sup_{u\in(0,\infty)} \tfrac{t\,(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\Big) \\ &= (\eta + x^p) \exp\left(\sup_{u\in(0,\infty)} \tfrac{t\,p\left[\kappa u^{(p-1-c)} + \delta u^{(p-1)} + \gamma u^p - \alpha u^{(p+a-1)} + \frac{1}{2}(p-1)\beta^2 u^{(p+2b-2)}\right]}{[\eta + u^p]}\right) \in [0,\infty].\end{aligned} \tag{202}$$

Next observe for all $p \in [1,\infty) \cap \big[(c+1)\mathbb{1}_{(0,\infty)}(\kappa), \infty\big)$ that if (i) $b \leq 1$ or (ii) $b < \frac{a+1}{2}$ and $\alpha > 0$ or (iii) $b = \frac{a+1}{2}$ and $2\alpha \geq (p-1)\beta^2$ or (iv) $p = 1$, then

$$\sup_{u\in(0,\infty)} \left[\tfrac{(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\right] = p \cdot \sup_{u\in(0,\infty)} \left[\tfrac{\kappa u^{(p-1-c)} + \delta u^{(p-1)} + \gamma u^p - \alpha u^{(p+a-1)} + \frac{1}{2}(p-1)\beta^2 u^{(p+2b-2)}}{\eta + u^p}\right] < \infty. \tag{203}$$

Moreover, note that Itô's formula shows that

$$\begin{aligned}U_p(X_t^x) &= U_p(x) + \int_0^t (\mathcal{G}_{\mu,\sigma}U_p)(X_r^x)\,dr + \int_0^t (G_\sigma U_p)(X_r^x)\,dW_r \\ &\leq U_p(x) + \left[\int_0^t U_p(X_r^x)\,dr\right]\left[\sup_{u\in(0,\infty)} \tfrac{(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\right] + \int_0^t (G_\sigma U_p)(X_r^x)\,dW_r\end{aligned} \tag{204}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$, $x \in (0,\infty)$, $p \in \mathbb{R}$. In addition, Jensen's inequality and Doob's martingale inequality yield for all $t \in [0,\infty)$, $x \in (0,\infty)$, $p \in \mathbb{R}$ that

$$\begin{aligned}&\left\| \sup_{s\in[0,t]} \left| \int_0^s (G_\sigma U_p)(X_r^x)\,dW_r \right| \right\|_{L^1(\Omega;\mathbb{R})} \leq \left\| \sup_{s\in[0,t]} \left| \int_0^s (G_\sigma U_p)(X_r^x)\,dW_r \right| \right\|_{L^2(\Omega;\mathbb{R})} \\ &\leq 2 \left\| \int_0^t |(G_\sigma U_p)(X_r^x)|^2\,dr \right\|_{L^1(\Omega;\mathbb{R})}^{1/2} = 2|p|\beta \left[\int_0^t \mathbb{E}\big[(X_r^x)^{(2p-2+2b)}\big]\,dr\right]^{1/2}.\end{aligned} \tag{205}$$

Then taking supremum over the time interval $[0, t\wedge\tau_n^x]$ for $t \in [0,\infty)$ in inequality (204), taking expectation and applying inequality (205) shows for all $t \in [0,\infty)$, $x \in (0,\infty)$, $p \in \mathbb{R}$, $n \in \mathbb{N}$ that

$$\begin{aligned}&\left\| \sup_{s\in[0,t\wedge\tau_n^x]} U_p(X_s^x) \right\|_{L^1(\Omega;\mathbb{R})} \leq U_p(x) + \left\| \int_0^{t\wedge\tau_n^x} U_p(X_r^x)\,dr \right\|_{L^1(\Omega;\mathbb{R})} \max\Big\{0, \sup_{u\in(0,\infty)} \tfrac{(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\Big\} \\ &+ \left\| \sup_{s\in[0,t]} \left| \int_0^s (G_\sigma U_p)(X_r^x)\,dW_r \right| \right\|_{L^1(\Omega;\mathbb{R})} \\ &\leq U_p(x) + 2|p|\beta \left[\int_0^t \mathbb{E}\big[(X_r^x)^{(2p-2+2b)}\big]\,dr\right]^{1/2} \\ &+ \left[\int_0^t \big\|\sup_{r\in[0,s\wedge\tau_n^x]} U_p(X_r^x)\big\|_{L^1(\Omega;\mathbb{R})}\,ds\right] \max\Big\{0, \sup_{u\in(0,\infty)} \tfrac{(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\Big\}.\end{aligned} \tag{206}$$

Now the monotone convergence theorem, Gronwall's inequality, inequality (206) and inequality (202) yield for all $t \in [0,\infty)$, $x \in (0,\infty)$, $p \in \mathbb{R}$ that

$$\begin{aligned}&\big\|\sup_{s\in[0,t]} U_p(X_s^x)\big\|_{L^1(\Omega;\mathbb{R})} = \lim_{n\to\infty} \big\|\sup_{s\in[0,t\wedge\tau_n^x]} U_p(X_s^x)\big\|_{L^1(\Omega;\mathbb{R})} \\ &\leq \left[U_p(x) + 2|p|\beta \left| \int_0^t \mathbb{E}\big[(X_r^x)^{(2p-2+2b)}\big]\,dr \right|^{1/2}\right] \exp\Big(t \max\Big\{0, \sup_{u\in(0,\infty)} \tfrac{(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\Big\}\Big) \\ &\leq \left[U_p(x) + 2|p|\beta \left| U_{2p-2+2b}(x) \int_0^t e^{\left[\sup_{u\in(0,\infty)} \frac{r\,(\mathcal{G}_{\mu,\sigma}U_{2p-2+2b})(u)}{U_{2p-2+2b}(u)}\right]}\,dr \right|^{1/2}\right] e^{\max\left\{0, \sup_{u\in(0,\infty)} \frac{t\,(\mathcal{G}_{\mu,\sigma}U_p)(u)}{U_p(u)}\right\}}.\end{aligned} \tag{207}$$

In the next step we observe that in the case $b \neq 1$ it holds for all $x, y \in (0,\infty)$ that

$$\begin{aligned}
|x-y| &= \left| \left[x^{1-b}\right]^{\frac{1}{(1-b)}} - \left[y^{1-b}\right]^{\frac{1}{(1-b)}} \right| = \left| \int_{y^{1-b}}^{x^{1-b}} \tfrac{1}{(1-b)}\, z^{\left[\frac{1}{1-b}-1\right]}\, dz \right| \leq \tfrac{\max(x^b,y^b)}{|1-b|} \left| x^{1-b} - y^{1-b} \right|, \\
\left| x^{1-b} - y^{1-b} \right| &= \left| (1-b) \int_y^x z^{-b}\, dz \right| \leq \tfrac{|1-b|}{\min(x^b,y^b)} |x-y|.
\end{aligned} \tag{208}$$

The estimate (201) together with the inequalities (208) and monotonicity of solutions of (197) with respect to the initial values implies in the case $b \neq 1$ that for all $x, y, t, p \in (0,\infty)$ it holds that

$$\begin{aligned}
&\left\| \sup_{s\in[0,t]} |X_s^x - X_s^y| \right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \frac{1}{|1-b|} \left\| \sup_{s\in[0,t]} \max\!\left\{ (X_s^x)^b, (X_s^y)^b \right\} \right\|_{L^p(\Omega;\mathbb{R})} \left\| \sup_{s\in[0,t]} \left| (X_s^x)^{1-b} - (X_s^y)^{1-b} \right| \right\|_{L^\infty(\Omega;\mathbb{R})} \\
&\leq \frac{1}{|1-b|} \left[\max_{z\in\{x,y\}} \left\| \sup_{s\in[0,t]} (X_s^z)^b \right\|_{L^p(\Omega;\mathbb{R})} \right] \left\| \sup_{s\in[0,t]} \left| (X_s^x)^{1-b} - (X_s^y)^{1-b} \right| \right\|_{L^\infty(\Omega;\mathbb{R})} \\
&\leq \frac{\left| x^{1-b} - y^{1-b} \right|}{|1-b|} \left[\max_{z\in\{x,y\}} \left\| \sup_{s\in[0,t]} X_s^z \right\|_{L^{pb}(\Omega;\mathbb{R})}^b \right] e^{t \max\left\{0, \sup_{u,v\in(0,\infty)} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(u,v)}{V(u,v)}\right\}} \\
&\leq \frac{|x-y|}{\min(x^b,y^b)} \left[\max_{z\in\{x,y\}} \left\| \sup_{s\in[0,t]} X_s^z \right\|_{L^{pb}(\Omega;\mathbb{R})}^b \right] e^{t \max\left\{0, \sup_{u,v\in(0,\infty)} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(u,v)}{V(u,v)}\right\}}
\end{aligned} \tag{209}$$

and inserting (207) hence shows in the case $b \neq 1$ that for all $x, y, t, p \in (0,\infty)$ it holds that

$$\begin{aligned}
\left\| \sup_{s\in[0,t]} |X_s^x - X_s^y| \right\|_{L^p(\Omega;\mathbb{R})} &\leq \frac{|x-y|}{\min(x^b,y^b)}\, e^{t\left[\max\left\{0, \sup_{u,v\in(0,\infty)} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(u,v)}{V(u,v)}\right\} + \max\left\{0, \sup_{u\in(0,\infty)} \frac{(\mathcal{G}_{\mu,\sigma} U_p)(u)}{p\, U_p(u)}\right\}\right]} \\
&\cdot \max_{z\in\{x,y\}} \left[U_{pb}(z) + 2pb\beta \left[U_{2(pb+b-1)}(z) \int_0^t e^{\left[\sup_{u\in(0,\infty)} \frac{r\,(\mathcal{G}_{\mu,\sigma} U_{2pb-2+2b})(u)}{U_{2pb-2+2b}(u)}\right]} dr \right]^{1/2} \right]^{1/p}.
\end{aligned} \tag{210}$$

This implies in the case $b \neq 1$, $\eta \geq 1$ that for all $x, y, t \in (0,\infty)$, $p \in [1,\infty)$ it holds that

$$\begin{aligned}
&\left\| \sup_{s\in[0,t]} |X_s^x - X_s^y| \right\|_{L^p(\Omega;\mathbb{R})} \leq |x-y|\, \frac{\left[1 + \eta + \max(x^b,y^b)\right]}{\min(x^b,y^b)} \left[1 + \left[2pb\beta t^2\right]^{\frac{1}{p}} \right] \\
&e^{t\left[\max\left\{0, \sup_{u,v\in(0,\infty)} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(u,v)}{V(u,v)}\right\} + \max\left\{0, \sup_{u\in(0,\infty)} \frac{(\mathcal{G}_{\mu,\sigma} U_p)(u)}{p\, U_p(u)}\right\} + \max\left\{0, \sup_{u\in(0,\infty)} \frac{t\,(\mathcal{G}_{\mu,\sigma} U_{2pb-2+2b})(u)}{2p\, U_{2pb-2+2b}(u)}\right\}\right]}.
\end{aligned} \tag{211}$$

Combining the statement below (201) and (203) shows that the right-hand sides of (210) and of (211) are finite for all $x, y, t \in (0,\infty)$, $p \in \left[\max\left\{1, (c+1)\mathbb{1}_{(0,\infty)}(\kappa), (\frac{c+3}{2b} - 1)\mathbb{1}_{(0,\infty)}(\kappa)\right\}, \infty\right)$ if (i) $\left[b = \frac{1}{2} \text{ and } (2\delta \geq \beta^2 \text{ or } \kappa > 0)\right]$ or if (ii) $\left[\frac{1}{2} < b \leq 1 \text{ and } \kappa + \max(0,\delta) > 0\right]$ or if (iii) $\left[1 \leq b < \frac{a+1}{2},\ \alpha > 0 \text{ and } (\delta \geq 0 \text{ or } \kappa > 0)\right]$ or if (iv) $\left[b = \frac{a+1}{2},\ \max\{\frac{p-1}{2}, pb + b - \frac{3}{2}\} \leq \frac{\alpha}{\beta^2} \text{ and } (\delta \geq 0 \text{ or } \kappa > 0)\right]$. Analogously, the inequalities (200), (202) and (208) show in the case $b \neq 1$ that for all $x, y, t, p \in (0,\infty)$ it holds that

$$\|X_t^x - X_t^y\|_{L^p(\Omega;\mathbb{R})} \leq \frac{|x-y|\left[\eta + \max\{x^{pb}, y^{pb}\}\right]^{\frac{1}{p}}}{\min(x^b,y^b)}\, e^{t\left[\sup_{u,v\in(0,\infty), u\neq v} \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(u,v)}{V(u,v)} + \sup_{u\in(0,\infty)} \frac{(\mathcal{G}_{\mu,\sigma} U_p)(u)}{p\, U_p(u)}\right]}. \tag{212}$$

Combining the statement below (201) and (203) shows that the right-hand sides of (210), of (211) and of (212) are finite for all $x, y, t \in (0,\infty)$, $p \in [1,\infty) \cap \big[(c+1)\mathbb{1}_{(0,\infty)}(\kappa), \infty\big)$ if (i) $\big[b = \frac{1}{2}$ and $(2\delta \geq \beta^2$ or $\kappa > 0)\big]$ or if (ii) $\big[\frac{1}{2} < b \leq 1$ and $\kappa + \max(0,\delta) > 0\big]$ or if (iii) $\big[1 \leq b < \frac{a+1}{2}$, $\alpha > 0$ and $(\delta \geq 0$ or $\kappa > 0)\big]$ or if (iv) $\big[b = \frac{a+1}{2}$, $(p-1)\beta^2 \leq 2\alpha$ and $(\delta \geq 0$ or $\kappa > 0)\big]$ or if (v) $\big[b \geq 1$, $\delta \geq 0$ and $p = 1\big]$.

4.10 Wright-Fisher diffusion

In biology, the Wright-Fisher diffusion is a model for the relative frequence of type 'A' in a panmictic population of constant population size with two types 'A' and 'a'; see, e.g., Chapter 7 in Durrett [10]. Let $s \in \mathbb{R}$ denote the relative fitness advantage of type 'A', let $\rho_0 \in [0,\infty)$ be the mutation rate from type 'a' to type 'A', let $\rho_1 \in [0,\infty)$ be the mutation rate from type 'A' to type 'a' and let $\beta \in (0,\infty)$ be the inverse of the (effective) number of haploid individuals. Moreover, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $W\colon [0,\infty) \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,\infty)}$-Brownian motion with continuous sample paths and let $X^x\colon [0,\infty) \times \Omega \to [0,1]$, $x \in (0,1)$, be adapted stochastic processes with continuous sample paths satisfying

$$X_t^x = x + \int_0^t \rho_0(1 - X_r^x) - \rho_1 X_r^x + s X_r^x(1 - X_r^x)\,dr + \int_0^t \sqrt{\beta X_r^x(1 - X_r^x)}\,dW_r \tag{213}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$ and all $x \in (0,1)$. In addition, define stopping times $\tau_h^x\colon \Omega \to [0,\infty]$, $x \in (0,1)$, $h \in [0,1]$, by $\tau_h^x := \inf(\{t \in [0,\infty)\colon X_t^x = h\} \cup \{\infty\})$ for all $x \in (0,1)$ and all $h \in [0,1]$. Feller's boundary classification (e.g., Theorem V.51.2 in [40]) implies that $\mathbb{P}[\tau_0^x = \infty] = 1$ for all $x \in (0,1)$ if and only if $2\rho_0 \geq \beta$ and that $\mathbb{P}[\tau_1^x = \infty] = 1$ for all $x \in (0,1)$ if and only if $2\rho_1 \geq \beta$.

In the case $\rho_0, \eta_1 \in [\frac{\beta}{4}, \infty)$ the processes $\arcsin(\sqrt{X^x})$, $x \in (0,\infty)$, satisfy an SDE with constant diffusion function and non-increasing drift function; see, e.g., Neuenkirch & Szpruch [36]. In the following calculation, we exploit this observation together with the results in Section 2 to derive an estimate for the Lyapunov-type function $V\colon (0,1)^2 \to \mathbb{R}$ given by $V(x,y) = |\arcsin(\sqrt{x}) - \arcsin(\sqrt{y})|^2$ for all $x, y \in (0,1)$. Let $\mu, \sigma\colon (0,1) \to \mathbb{R}$ be given by $\mu(x) = \rho_0(1-x) - \rho_1 x + sx(1-x)$ and $\sigma(x) = \sqrt{\beta x(1-x)}$ for all $x \in (0,1)$ and let $f\colon (0,\pi/2) \to \mathbb{R}$ be given by

$$\begin{aligned} f(x) &= \left[\tfrac{\rho_0(1-y) - \rho_1 y + sy(1-y) - (1-2y)\frac{\beta}{4}}{\sqrt{y(1-y)}}\right]_{y=(\sin(x))^2} \\ &= \tfrac{\rho_0(\cos(x))^2 - \rho_1(\sin(x))^2 + s(\sin(x))^2(\cos(x))^2 - \left((\cos(x))^2 - (\sin(x))^2\right)\frac{\beta}{4}}{\sin(x)\cos(x)} \\ &= \left(\rho_0 - \tfrac{\beta}{4}\right)\tfrac{1}{\tan(x)} - \left(\rho_1 - \tfrac{\beta}{4}\right)\tan(x) + \tfrac{s}{2}\sin(2x) \end{aligned} \tag{214}$$

for all $x \in (0,\pi/2)$. Next we infer from $(0,\pi/2) \ni x \mapsto \tan(x) \in (0,\infty)$ being an increasing function that

$$\tfrac{f(x)-f(y)}{x-y} \leq \tfrac{s}{2}\tfrac{\sin(2x)-\sin(2y)}{x-y} \leq |s| \tag{215}$$

for all $x, y \in (0,\pi/2)$ in the case $\rho_0, \rho_1 \in [\frac{\beta}{4}, \infty)$. Now we apply Lemma 2.14 and inequality (215) to obtain that for all $x, y \in (0,1)$ it holds that

$$\tfrac{\|(\overline{G}_\sigma V)(x,y)\|^2}{|V(x,y)|^2} = \frac{4\left[\frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{1}{2\sqrt{x}}\sqrt{\beta x(1-x)} - \frac{1}{\sqrt{1-(\sqrt{y})^2}}\frac{1}{2\sqrt{y}}\sqrt{\beta y(1-y)}\right]^2}{\left[\arcsin(\sqrt{x}) - \arcsin(\sqrt{y})\right]^2} = 0 \tag{216}$$

and

$$
\begin{aligned}
\frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(x,y)}{V(x,y)} &= 2\cdot\frac{\frac{1}{2\sqrt{x(1-x)}}(\rho_0(1-x)-\rho_1x+sx(1-x))-\frac{1}{2\sqrt{y(1-y)}}(\rho_0(1-y)-\rho_1y+sy(1-y))}{\big[\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\big]} \\
&\quad+\frac{\Big[\frac{(2x-1)}{4[x(1-x)]^{3/2}}\beta x(1-x)-\frac{(2y-1)}{4[y(1-y)]^{3/2}}\beta y(1-y)\Big]}{\big[\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\big]} \\
&=\frac{\frac{1}{\sqrt{x(1-x)}}\big(\rho_0(1-x)-\rho_1x+sx(1-x)-(1-2x)\frac{\beta}{4}\big)-\frac{1}{\sqrt{y(1-y)}}\big(\rho_0(1-y)-\rho_1y+sy(1-y)-(1-2y)\frac{\beta}{4}\big)}{\big[\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\big]} \\
&=\frac{f\big(\arcsin(\sqrt{x})\big)-f\big(\arcsin(\sqrt{y})\big)}{\arcsin(\sqrt{x})-\arcsin(\sqrt{y})}\le|s|.
\end{aligned}
\tag{217}
$$

Hence, Proposition 2.26 with $O=(0,1)$ and with $v=\infty=p=q=r$ shows that in the case $\rho_0,\rho_1\in[\frac{\beta}{2},\infty)$ it holds for all $t\in[0,\infty)$ and all $x,y\in(0,1)$ that

$$
\left\|\sup_{r\in[0,t]}\left|\arcsin\big(\sqrt{X_r^x}\big)-\arcsin\big(\sqrt{X_r^y}\big)\right|^2\right\|_{L^\infty(\Omega;\mathbb{R})}\le\left|\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\right|^2e^{t|s|}. \tag{218}
$$

Clearly, this implies that if $\rho_0,\rho_1\in[\frac{\beta}{2},\infty)$, then it holds for all $t\in[0,\infty)$, $x,y\in(0,1)$ that

$$
\left\|\sup_{r\in[0,t]}\left|\arcsin\big(\sqrt{X_r^x}\big)-\arcsin\big(\sqrt{X_r^y}\big)\right|\right\|_{L^\infty(\Omega;\mathbb{R})}\le e^{\frac{t|s|}{2}}\left|\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\right|. \tag{219}
$$

This together with the estimates

$$
\begin{aligned}
|\arcsin(\sqrt{x})-\arcsin(\sqrt{y})| &= \left|\int_y^x\frac{1}{\sqrt{4z(1-z)}}\,dz\right|\le|x-y|\left[\max_{z\in\{x,y\}}\frac{1}{\sqrt{4z(1-z)}}\right], \\
|x-y| &= \left|\big[\sin(\arcsin(\sqrt{x}))\big]^2-\big[\sin(\arcsin(\sqrt{y}))\big]^2\right| \\
&= \left|\int_{\arcsin(\sqrt{y})}^{\arcsin(\sqrt{x})}2\sin(z)\cos(z)\,dz\right|\le\left|\arcsin(\sqrt{x})-\arcsin(\sqrt{y})\right|
\end{aligned}
\tag{220}
$$

for all $x,y\in(0,1)$ implies that if $\rho_0,\rho_1\in[\frac{\beta}{2},\infty)$, then it holds for all $t\in[0,\infty)$, $x,y\in(0,1)$ that

$$
\left\|\sup_{r\in[0,t]}|X_r^x-X_r^y|\right\|_{L^\infty(\Omega;\mathbb{R})}\le\left[\max_{z\in\{x,y\}}\frac{\exp\big(\frac{t|s|}{2}\big)}{\sqrt{4z(1-z)}}\right]|x-y|. \tag{221}
$$

4.11 Stochastic Burgers equation with a globally bounded diffusion coefficient and trace class noise

Let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)=(L^2((0,1);\mathbb{R}),\langle\cdot,\cdot\rangle_{L^2((0,1);\mathbb{R})},\|\cdot\|_{L^2((0,1);\mathbb{R})})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $c\in\mathbb{R}\backslash\{0\}$, let $F\colon H\to W^{-1,1}((0,1);\mathbb{R})$ be given by $F(v)=\frac{c}{2}(v^2)'$ for all $v\in H$, let $B\colon H\to\mathrm{HS}(H)$ be a globally Lipschitz continuous function with $\eta:=\sup_{x\in H}\|B(x)\|^2_{\mathrm{HS}(H)}\in(0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $(W_t)_{t\in[0,\infty)}$ be a cylindrical I-Wiener process on H with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $e_k\in H$, $k\in\mathbb{N}$, be given by $e_k(y)=\sqrt{2}\sin(k\pi y)$ for all $y\in(0,1)$, $k\in\mathbb{N}$, let $P_n\in L(H)$, $n\in\mathbb{N}$, be given by $P_n(v)=\sum_{k=1}^n\langle e_k,v\rangle_H e_k$ for all $v\in H$, $n\in\mathbb{N}$ and let $A\colon D(A)\subset H\to H$ be the Laplacian with Dirichlet boundary conditions, that is, $D(A)=\{v\in H^2((0,1),\mathbb{R})\colon v(0)=v(1)=0\}=\{v\in H\colon\sum_{k=1}^\infty k^4|\langle e_k,v\rangle_H|^2<\infty\}$. Moreover, let $X^x\colon[0,\infty)\times\Omega\to H$, $x\in H$, be adapted stochastic processes with continuous sample paths satisfying $X_t^x=e^{At}x+\int_0^te^{A(t-s)}F(X_s^x)\,ds+\int_0^te^{A(t-s)}B(X_s^x)\,dW_s$ $\mathbb{P}$-a.s. for all $(t,x)\in[0,\infty)\times H$, let $\mu_n\colon P_n(H)\to P_n(H)$ and $\sigma_n\colon P_n(H)\to\mathrm{HS}(H,P_n(H))$ be given by $\mu_n(v)=Av+P_n(F(v))$ and $\sigma_n(v)u=P_n\big(B(v)P_n(u)\big)$ for all $v\in P_n(H)$, $u\in H$, $n\in\mathbb{N}$ and let $X^{x,n}\colon[0,\infty)\times\Omega\to P_n(H)$,

$x \in P_n(H)$, $n \in \mathbb{N}$, be adapted stochastic processes with continuous sample paths satisfying $X_t^{x,n} = x + \int_0^t \mu_n(X_s^{x,n})\,ds + \int_0^t \sigma_n(X_s^{x,n})\,dW_s$ $\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty) \times P_n(H)$ and all $n \in \mathbb{N}$. Then note that if $n \in \mathbb{N}$, if $\rho \in [0,\infty)$ and if $U\colon P_n(H) \to \mathbb{R}$ is given by $U(x) = \rho\,\|x\|_H^2$ for all $x \in P_n(H)$, then it holds for every $x \in P_n(H)$ that

$$\begin{aligned}
&U'(x)\,\mu_n(x) + \tfrac{1}{2}\operatorname{tr}\big(\sigma_n(x)\,\sigma_n(x)^*\,(\operatorname{Hess} U)(x)\big) + \tfrac{1}{2}\,\|\sigma_n(x)^*(\nabla U)(x)\|_H^2 \\
&= 2\rho\,\langle x, Ax + F(x)\rangle_H + \rho\,\|\sigma_n(x)\|^2_{\mathrm{HS}(H,P_n(H))} + 2\rho^2\|\sigma_n(x)^* x\|_H^2 \\
&\leq -2\rho\,\|(-A)^{1/2}x\|_H^2 + \rho\,\|B(x)\|^2_{\mathrm{HS}(H)} + 2\rho^2\,\|B(x)^* x\|_H^2 \\
&\leq \rho\eta - 2\rho\,\|(-A)^{1/2}x\|_H^2 + 2\rho^2\eta\|x\|_H^2 \leq \rho\eta + 2\rho\left[\tfrac{\rho\eta}{\pi} - 1\right]\|(-A)^{1/2}x\|_H^2.
\end{aligned} \tag{222}$$

As for every $n \in \mathbb{N}$ it holds that $(P_n(H), \langle\cdot,\cdot\rangle_{P_n(H)}, \|\cdot\|_{P_n(H)})$ is isometric isomorph to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, \|\cdot\|)$, it follows from (222) with $\rho = \frac{\pi}{2\eta}$ and Corollary 2.4 that for every $n \in \mathbb{N}$, $t \in [0,\infty)$, $x \in P_n(H)$ it holds that

$$\mathbb{E}\!\left[\exp\!\left(\tfrac{\pi}{2\eta}\|X_t^{x,n}\|_H^2 + \int_0^t \tfrac{\pi}{2\eta}\,\big\|(-A)^{1/2}X_s^{x,n}\big\|_H^2\,ds\right)\right] \leq e^{\frac{\pi t}{2} + \frac{\pi}{2\eta}\|x\|_H^2}. \tag{223}$$

In the next step we note for every $n \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, $p \in (0,\infty]$, $x, y \in P_n(H)$ with $x \neq y$ that

$$\left[\tfrac{1}{(2/p)} - 1\right]\tfrac{\|(\sigma_n(x)-\sigma_n(y))^*(x-y)\|_H^2}{\|x-y\|_H^4} \leq \tfrac{(p-2)}{2}\,\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))} \qquad \text{and} \tag{224}$$

$$\begin{aligned}
&\max\!\left\{0, \tfrac{\langle x-y, \mu_n(x)-\mu_n(y)\rangle_H + \frac{1}{2}\|\sigma_n(x)-\sigma_n(y)\|^2_{\mathrm{HS}(P_n(H))}}{\|x-y\|_H^2}\right\} \\
&\leq \max\!\left\{0, \tfrac{-\frac{c}{4}\langle (x-y)^2, (x+y)'\rangle_H - \|(-A)^{1/2}(x-y)\|_H^2}{\|x-y\|_H^2} + \tfrac{1}{2}\,\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}\right\} \\
&\leq \max\!\left\{0, \tfrac{|c|\|x'+y'\|_H\|x-y\|_H\|x-y\|_{L^\infty((0,1);\mathbb{R})} - \|(-A)^{1/2}(x-y)\|_H^2}{4\,\|x-y\|_H^2} + \tfrac{1}{2}\,\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}\right\} \\
&\leq \max\!\left\{0, \tfrac{\varepsilon\left\|(-A)^{1/2}(x+y)\right\|_H^2}{4} + \tfrac{\frac{c^2}{16\varepsilon}\|x-y\|^2_{L^\infty((0,1);\mathbb{R})} - \|(-A)^{1/2}(x-y)\|_H^2}{\|x-y\|_H^2} + \tfrac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}\right\}.
\end{aligned} \tag{225}$$

Using the fact that there exist a function $\kappa\colon (0,\infty) \to (0,\infty)$ such that for every $r \in (0,\infty)$, $u \in D((-A)^{1/2})$ it holds that $\|u\|^2_{L^\infty((0,1);\mathbb{R})} \leq \kappa(r)\,\|u\|_H^2 + r\,\big\|(-A)^{1/2}u\big\|_H^2$ (see, e.g., Theorem 37.5 in Sell & You [46]) hence shows for every $n \in \mathbb{N}$, $q \in (0,\infty)$, $x, y \in P_n(H)$ with $x \neq y$ that

$$\begin{aligned}
&\max\!\left\{0, \tfrac{\langle x-y, \mu_n(x)-\mu_n(y)\rangle_H + \frac{1}{2}\|\sigma_n(x)-\sigma_n(y)\|^2_{\mathrm{HS}(P_n(H))}}{\|v-w\|_H^2}\right\} \\
&\leq \tfrac{\frac{\pi}{2\eta}\left\|(-A)^{1/2}x\right\|_H^2 + \frac{\pi}{2\eta}\left\|(-A)^{1/2}y\right\|_H^2}{2q} + \tfrac{c^2\eta q\kappa(8\pi/(c^2\eta q))}{8\pi} + \tfrac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}.
\end{aligned} \tag{226}$$

Combining (222), (224) and (226) with Corollary 2.30 implies for every $n \in \mathbb{N}$, $T \in (0,\infty)$, $x, y \in P_n(H)$, $r, q \in (2,\infty)$, $p \in (r,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^{x,n} - X_t^{y,n}\|_H\big\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \frac{\|x-y\|}{\sqrt{1-2/p}}\exp\!\Big(\tfrac{c^2\eta q T\kappa(8\pi/(c^2\eta q))}{8\pi} + \tfrac{(p-1)T\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2} + \tfrac{\pi T}{2q} + \tfrac{\pi\|x\|_H^2 + \pi\|y\|_H^2}{4\eta q}\Big).
\end{aligned} \tag{227}$$

Fatou's lemma applied to (227) (cf., e.g., Alabert & Gyöngy [1] and Section 4.3 in Blömker & Jentzen [5]) then shows for every $x, y \in H$, $T \in (0,\infty)$, $r, q \in (2,\infty)$, $p \in (r,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ that

$$\begin{aligned}
&\big\|\sup\nolimits_{t\in[0,T]}\|X_t^{x} - X_t^{y}\|_H\big\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \frac{\|x-y\|}{\sqrt{1-2/p}}\exp\!\Big(\tfrac{c^2\eta q T\kappa(8\pi/(c^2\eta q))}{8\pi} + \tfrac{(p-1)T\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2} + \tfrac{\pi T}{2q} + \tfrac{\pi\|x\|_H^2 + \pi\|y\|_H^2}{4\eta q}\Big).
\end{aligned} \tag{228}$$

4.12 Cahn-Hilliard-Cook equation

Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2((0,1);\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2((0,1);\mathbb{R})}, \|\cdot\|_{L^2((0,1);\mathbb{R})})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $c \in (0,\infty)$, $\beta \in [0,\frac{1}{2})$, let $F\colon L^6((0,1);\mathbb{R}) \to W^{-2,2}((0,1);\mathbb{R})$ be given by $F(v) = c\,\triangle(v^3 - v) = c\,(v^3 - v)''$ for all $v \in L^6((0,1);\mathbb{R})$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ be a stochastic basis, let $(W_t)_{t\in[0,\infty)}$ be a cylindrical I-Wiener process on H with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$, let $e_k \in H$, $k \in \mathbb{N}$, be given by $e_k(y) = \sqrt{2}\sin(k\pi y)$ for all $y \in (0,1)$, $k \in \mathbb{N}$, let $P_n \in L(H)$, $n \in \mathbb{N}$, be given by $P_n(v) = \sum_{k=1}^n \langle e_k, v\rangle_H\, e_k$ for all $v \in H$, $n \in \mathbb{N}$, let $A\colon D(A) \subset H \to H$ be given by $D(A) = \{v \in H\colon \sum_{k=1}^\infty k^8\, |\langle e_k, v\rangle_H|^2 < \infty\}$ and $Av = -v''''$ for all $v \in D(A)$ and let $B\colon H \to \operatorname{HS}(H, D((-A)^{-\beta}))$ be a globally Lipschitz continuous function. Moreover, let $X^x\colon [0,\infty)\times\Omega \to H$, $x \in H$, be adapted stochastic processes with continuous sample paths satisfying $X_t^x = e^{At}x + \int_0^t e^{A(t-s)}F(X_s^x)\,ds + \int_0^t e^{A(t-s)}B(X_s^x)\,dW_s$ $\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times H$, let $\mu_n\colon P_n(H) \to P_n(H)$ and $\sigma_n\colon P_n(H) \to \operatorname{HS}(H, P_n(H))$ be given by $\mu_n(v) = Av + P_n(F(v))$ and $\sigma_n(v)u = P_n\big(B(v)P_n(u)\big)$ for all $v \in P_n(H)$, $u \in H$, $n \in \mathbb{N}$ and let $X^{x,n}\colon [0,\infty)\times\Omega \to P_n(H)$, $x \in P_n(H)$, $n \in \mathbb{N}$, be adapted stochastic processes with continuous sample paths satisfying $X_t^{x,n} = x + \int_0^t \mu_n(X_s^{x,n})\,ds + \int_0^t \sigma_n(X_s^{x,n})\,dW_s$ $\mathbb{P}$-a.s. for all $(t,x) \in [0,\infty)\times P_n(H)$ and all $n \in \mathbb{N}$.

4.12.1 Globally bounded diffusion coefficient and trace class noise

In this subsection we assume in addition to the assumptions in the beginning of Subsection 4.12 that $B(H) \subset \operatorname{HS}(H)$ and that $\eta \in [0,\infty)$ is a real number which satisfies $\sup_{x\in H}\|B(x)\|_{\operatorname{HS}(H)}^2 \leq \eta$. Next note for every $v \in D(A)$ that $\|v'\|_H^2 \leq \|v\|_H\,\|v''\|_H$ (c.f., e.g., Theorem 37.5 in Sell & You [46]). This shows that if $n \in \mathbb{N}$, if $\rho \in [0,\infty)$ and if $U\colon P_n(H) \to \mathbb{R}$ is given by $U(x) = \rho\,\|x\|_H^2$ for all $x \in P_n(H)$, then it holds for every $\varepsilon, \delta \in (0,\infty)$, $x \in P_n(H)$ that

$$\begin{aligned}
&U'(x)\,\mu_n(x) + \tfrac{1}{2}\operatorname{tr}\!\big(\sigma_n(x)\,\sigma_n(x)^*\,(\operatorname{Hess}U)(x)\big) + \tfrac{1}{2}\,\|\sigma_n(x)^*(\nabla U)(x)\|_H^2 \\
&= 2\rho\,\langle x, Ax + F(x)\rangle_H + \rho\,\|\sigma_n(x)\|_{\operatorname{HS}(H,P_n(H))}^2 + 2\rho^2\|\sigma_n(x)^*x\|_H^2 \\
&\leq 2\rho\left[c\|x'\|_H^2 - \|x''\|_H^2 - 3c\left\langle x^2, (x')^2\right\rangle_H\right] + \rho\,\|B(x)\|_{\operatorname{HS}(H)}^2 + 2\rho^2\,\|B(x)^*x\|_H^2 \\
&\leq 2\rho\left[\rho\eta\|x\|_H^2 + c\|x\|_H\|x''\|_H - \|x''\|_H^2 - 3c\left\langle x^2, (x')^2\right\rangle_H\right] + \rho\eta \\
&\leq 2\rho\left[\left(\rho\eta + \tfrac{c^2}{4\varepsilon}\right)\int_0^1 [x(y)]^2\cdot\left[\tfrac{\partial}{\partial y}y\right]dy + (\varepsilon - 1)\,\|x''\|_H^2 - 3c\left\langle x^2, (x')^2\right\rangle_H\right] + \rho\eta \\
&\leq 2\rho\left[\int_0^1 \left|\sqrt{2\delta}\,x'(y)\,x(y)\right|\cdot\left|\frac{\sqrt{2}y\left[\rho\eta + \frac{c^2}{4\varepsilon}\right]}{\sqrt{\delta}}\right|dy + (\varepsilon - 1)\,\|x''\|_H^2 - 3c\left\langle x^2, (x')^2\right\rangle_H\right] + \rho\eta \\
&\leq 2\rho\left[\int_0^1 \left[\frac{y^2\left[\rho\eta + c^2/(4\varepsilon)\right]^2}{\delta}\right]dy + (\varepsilon - 1)\,\|x''\|_H^2 + (\delta - 3c)\left\langle x^2, (x')^2\right\rangle_H\right] + \rho\eta \\
&= \frac{2\rho\left[\rho\eta + c^2/(4\varepsilon)\right]^2}{3\delta} + \rho\eta + 2\rho\,(\varepsilon - 1)\,\|x''\|_H^2 + 2\rho\,(\delta - 3c)\,\|x'x\|_H^2\,.
\end{aligned} \tag{229}$$

As for every $n \in \mathbb{N}$ it holds that $(P_n(H), \langle\cdot,\cdot\rangle_{P_n(H)}, \|\cdot\|_{P_n(H)})$ is isometric isomorph to $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, \|\cdot\|)$, it follows from (229) and Corollary 2.4 that for every $n \in \mathbb{N}$, $t \in [0,\infty)$, $\varepsilon, \delta, \rho \in (0,\infty)$, $x \in P_n(H)$ it holds that

$$\mathbb{E}\Big[e^{\rho\,\|X_t^{x,n}\|_H^2 + 2\rho\int_0^t (1-\varepsilon)\|(X_s^{x,n})''\|_H^2 + (3c-\delta)\|(X_s^{x,n})'X_s^{x,n}\|_H^2\,ds}\Big] \leq e^{\frac{2\rho t}{3\delta}\left[\rho\eta + c^2/(4\varepsilon)\right]^2 + \rho\eta t + \rho\|x\|_H^2}. \tag{230}$$

In the next step we note for every $n \in \mathbb{N}$, $p \in (0,\infty]$, $x, y \in P_n(H)$ with $x \neq y$ that

$$\left[\tfrac{1}{(2/p)} - 1\right]\frac{\|(\sigma_n(x) - \sigma_n(y))^*(x-y)\|_H^2}{\|x-y\|_H^4} \leq \tfrac{(p-2)}{2}\,\|B\|_{\operatorname{Lip}(H,\operatorname{HS}(H))} \qquad \text{and} \tag{231}$$

$$
\begin{aligned}
&0 \vee \left[\frac{\langle x-y,\mu_n(x)-\mu_n(y)\rangle_H+\frac{1}{2}\|\sigma_n(x)-\sigma_n(y)\|^2_{\mathrm{HS}(P_n(H))}}{\|x-y\|^2_H}\right] \qquad (232)\\
&\leq 0 \vee \left[\frac{c\|(x-y)'\|^2_H-\|(x-y)''\|^2_H-c\big\langle (x-y)',\big[(x-y)(x^2+xy+y^2)\big]'\big\rangle_H}{\|x-y\|^2_H}+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}\right]\\
&\leq 0 \vee \left[\frac{-c\big\langle [(x-y)']^2,x^2+xy+y^2\big\rangle_H-c\langle (x-y)'(x-y),2x'x+x'y+xy'+2y'y\rangle_H}{\|x-y\|^2_H}+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}\right]+\frac{c^2}{4}\\
&\leq 0 \vee \left[\frac{-\frac{c}{2}\big\langle [(x-y)']^2,x^2+y^2\big\rangle_H+c\langle (x-y)'(x-y),(|x|+|y|)(|x'|+|y'|)\rangle_H}{\|x-y\|^2_H}+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}\right]+\frac{c^2}{4}\\
&\leq \frac{4c\big\langle (x-y)^2,(|x'|+|y'|)^2\big\rangle_H}{\|x-y\|^2_H}+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}+2c^2}{2}\leq 4c\,\||x'|+|y'|\|^2_{L^\infty((0,1);\mathbb{R})}+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}+2c^2}{2}\\
&\leq \tfrac{8c}{3}\|x''\|^2_H+\tfrac{8c}{3}\|y''\|^2_H+\frac{\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}}{2}+\frac{c^2}{4}
\end{aligned}
$$

where we used the estimates $c\|v'\|^2_H \leq \frac{c^2}{4}\|v\|^2_H + \|v''\|^2_H$ and $\|v'\|_{L^\infty((0,1);\mathbb{R})} \leq \frac{1}{\sqrt{3}}\|v''\|_H$ for all $v \in D(A)$ in the second and last inequality respectively. Combining (229), (231) and (232) with Corollary 2.30 implies for every $n \in \mathbb{N}$, $T \in (0,\infty)$, $x,y \in P_n(H)$, $\varepsilon \in (0,1)$, $r,q \in (2,\infty)$, $p \in (r,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ that

$$
\begin{aligned}
&\sqrt{1-\tfrac{2}{p}}\left\|\sup_{t\in[0,T]}\frac{\|X^{x,n}_t-X^{y,n}_t\|_H}{\|x-y\|_H}\right\|_{L^r(\Omega;\mathbb{R})} \qquad (233)\\
&\leq \exp\!\left(\tfrac{(p-1)T}{2}\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}+\tfrac{c^2T}{4}+\tfrac{16T}{27(1-\varepsilon)}\left[\tfrac{8cq\eta}{3(1-\varepsilon)}+\tfrac{c^2}{4\varepsilon}\right]^2+\tfrac{8c\eta T}{3(1-\varepsilon)}+\tfrac{4c\|x\|^2_H+4c\|y\|^2_H}{3(1-\varepsilon)}\right).
\end{aligned}
$$

Fatou's lemma applied to (233) then shows for every $T \in (0,\infty)$, $x,y \in H$, $\varepsilon \in (0,1)$, $r,q \in (2,\infty)$, $p \in (r,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ that

$$
\begin{aligned}
&\sqrt{1-\tfrac{2}{p}}\left\|\sup_{t\in[0,T]}\frac{\|X^{x}_t-X^{y}_t\|_H}{\|x-y\|_H}\right\|_{L^r(\Omega;\mathbb{R})} \qquad (234)\\
&\leq \exp\!\left(\tfrac{(p-1)T}{2}\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(H))}+\tfrac{c^2T}{4}+\tfrac{16T}{27(1-\varepsilon)}\left[\tfrac{8cq\eta}{3(1-\varepsilon)}+\tfrac{c^2}{4\varepsilon}\right]^2+\tfrac{8c\eta T}{3(1-\varepsilon)}+\tfrac{4c\|x\|^2_H+4c\|y\|^2_H}{3(1-\varepsilon)}\right).
\end{aligned}
$$

4.12.2 Space-time white noise

In this subsection we assume in addition to the assumptions in the beginning of Subsection 4.12 that there exists a real number $\eta \in [0,\infty)$ such that for every $v,w \in H$ it holds that $\|(-A)^{-1/4}(B(v)-B(w))\|^2_{\mathrm{HS}(H)} \leq \eta\|(-A)^{-1/4}(v-w)\|^2_H$. Then note for every $r \in (0,\infty]$, $\theta \in (0,r)$, $n \in \mathbb{N}$, $x,y \in P_N(H)$ with $x \neq y$ that

$$
\left[\tfrac{r}{2}-\tfrac{\theta}{2}\right]\frac{\big\|((-A)^{-1/4}(x-y))^*((-A)^{-1/4}[\sigma_n(x)-\sigma_n(y)])\big\|^2_H}{\|(-A)^{-1/4}(x-y)\|^4_H} \leq \left[\tfrac{r}{2}-\tfrac{\theta}{2}\right]\eta \qquad \text{and} \qquad (235)
$$

$$
\begin{aligned}
&0 \vee \left[\frac{\big\langle (-A)^{-1/4}(x-y),(-A)^{-1/4}(\mu(x)-\mu(y))\big\rangle_H+\frac{1}{2}\big\|(-A)^{-1/4}(\sigma_n(x)-\sigma_n(y))\big\|^2_{\mathrm{HS}(H)}}{\big\|(-A)^{-1/4}(x-y)\big\|^2_H}\right] \qquad (236)\\
&\leq 0 \vee \left[\frac{\|x-y\|^2_H-\|(x-y)'\|^2_H}{\|(x-y)'\|^2_H}\right]+\frac{\eta}{2}=\frac{\eta}{2}.
\end{aligned}
$$

Combining this Theorem 2.29 and Fatou's lemma implies for every $T \in (0,\infty)$, $r \in (0,\infty]$, $\theta \in (0,r)$, $x,y \in H$ that

$$
\left\|\sup_{t\in[0,T]}\big\|(-A)^{-1/4}(X^x_t-X^y_t)\big\|_H\right\|_{L^r(\Omega;\mathbb{R})} \leq \frac{\exp\!\Big(\frac{[r-\min(1,\theta-1)]\eta T}{2}\Big)\big\|(-A)^{-1/4}(x-y)\big\|_H}{\big[1-\frac{\theta}{r}\big]^{1/\theta}}.
$$

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