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Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations

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Abstract

Exponential integrability properties of numerical approximations are a key tool towards establishing positive rates of strong and numerically weak convergence for a large class of nonlinear stochastic differential equations; cf. Cox et al. [3]. It turns out that well-known numerical approximation processes such as Euler-Maruyama approximations, linear-implicit Euler approximations and some tamed Euler approximations from the literature rarely preserve exponential integrability properties of the exact solution. The main contribution of this article is to identify a class of stopped increment-tamed Euler approximations which preserve exponential integrability properties of the exact solution under minor additional assumptions on the involved functions.

Contents

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1 Introduction

Let $T \in (0,\infty)$, $d, m \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions, let $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths and let $X\colon [0,T]\times\Omega \to \mathbb{R}^d$ be an adapted stochastic process with continuous sample paths satisfying the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{1}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$.

The goal of this paper is to identify numerical approximations $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, that converge in the strong sense to the exact solution of the SDE (1) and that *preserve exponential integrability properties* in the sense that $\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(U(Y_t^N)\big)\big] < \infty$ for all suffciently regular functions $U\colon \mathbb{R}^d \to [0,\infty)$ satisfying $\sup_{t\in[0,T]}\mathbb{E}\big[\exp(U(X_t))\big] < \infty$. Our main motivation for this is that such exponential integrability properties are a key tool towards establishing positive rates of strong and numerically weak convergence for a large class of nonlinear SDEs; cf. Cox et al. [3].

There are a number of SDEs in the literature that admit exponential integrability properties. We focus on Corollary 2.4 in Cox et al. [3] (see, for example, also Lemma 2.3 in Zhang [33]). Let $\rho \in [0,\infty)$, $U \in C^3(\mathbb{R}^d, [0,\infty))$ and $\bar{U} \in C(\mathbb{R}^d, \mathbb{R})$ satisfy $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, $\inf_{y\in\mathbb{R}^d}\bar{U}(y) > -\infty$ and

$$U'(x)\mu(x) + \tfrac{1}{2}\mathrm{tr}\Big(\sigma(x)\sigma(x)^*(\mathrm{Hess}\,U)(x)\Big) + \tfrac{1}{2}\left\|\sigma(x)^*(\nabla U)(x)\right\|^2 + \bar{U}(x) \le \rho\, U(x) \tag{2}$$

for all $x \in \mathbb{R}^d$. Then Corollary 2.4 in Cox et al. [3] yields that

$$\mathbb{E}\left[\exp\!\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\,ds\right)\right] \le \mathbb{E}\Big[e^{U(X_0)}\Big] \in (0,\infty) \tag{3}$$

for all $t \in [0,T]$. Section 4 lists a selection of SDEs from the literature which satisfy condition (2). Further instructive exponential integrability results for solutions of SDEs can be found, e.g., in [1, 6, 7, 9, 10, 20, 33]. In the light of inequality (3), the goal of this paper is, in particular, to identify numerical approximations that converge in the strong sense to the exact solution of the SDE (1) and that *preserve inequality* (3) in a suitable sense; see inequality (11) below.

It turns out that many well-known numerical methods for SDEs fail to preserve exponential integrability properties. For instance, in the special case $d = m = 1$, $\mu(x) = -x^3$ and $\sigma(x) = 1$ for all $x \in \mathbb{R}$, the SDE (1) reads as

$$X_t = X_0 - \int_0^t (X_s)^3\,ds + W_t \tag{4}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. In that case, inequality (2) holds with $\rho = 0$, $\varepsilon \in (0, \frac{1}{2}]$, $U(x) = \varepsilon|x|^4$ and $\bar{U}(x) = 4\varepsilon(1-2\varepsilon)x^6 - 6\varepsilon x^2$ for all $x \in \mathbb{R}$; see Subsection 5.1 below. Let $\varepsilon \in (0, \frac{1}{2}]$ be such that $\mathbb{E}\big[\exp(\varepsilon U(X_0))\big] < \infty$. Thus, Corollary 2.4 in Cox et al. [3] implies for all $\delta \in [0,\varepsilon]$ that

$$\sup_{t\in[0,T]}\mathbb{E}\left[\exp\!\left(\delta\,|X_t|^4 + \int_0^t 4\delta\,(1-2\delta)\,|X_s|^6 - 6\delta\,|X_s|^2\,ds\right)\right] \le \mathbb{E}\Big[e^{\delta|X_0|^4}\Big] < \infty \tag{5}$$

and, in particular, for all $\delta \in [0,\varepsilon] \cap [0, \frac{1}{2})$ that $\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big] < \infty$. If $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are the classical Euler-Maruyama approximations as defined in (134) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then moments are finite but unbounded in the sense that $\mathbb{E}\big[|Y_T^N|^p\big] < \infty$ for all $N \in \mathbb{N}$, $p \in (0,\infty)$ and $\lim_{N\to\infty}\mathbb{E}\big[|Y_T^N|^p\big] = \infty$ for all $p \in (0,\infty)$ (see Theorem 2.1 in [16] for the case $p \in [1,\infty)$ and Theorem 2.1 in [18]) whereas approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$ are infinite in the sense that $\inf_{t\in(0,T]}\mathbb{E}\big[\exp(p|Y_t^N|^q)\big] = \infty$ for all $N \in \mathbb{N}$, $p \in (0,\infty)$ and all $q \in (2,\infty)$; see Lemma 5.1 below. Next, if $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are the linear-implicit Euler approximations as defined in (136) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then strong convergence holds in the sense that $\lim_{N\to\infty}\mathbb{E}\big[|X_T - Y_T^N|^p\big] = 0$ for all $p \in (0,\infty)$ whereas approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$ are infinite in the sense that $\inf_{t\in(0,T]}\mathbb{E}\big[\exp(p|Y_t^N|^q)\big] = \infty$ for all $N \in \mathbb{N}$, $p \in (0,\infty)$ and all $q \in (2,\infty)$; see Lemma 5.2 below. Moreover, if $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, are tamed Euler approximations as defined in (144) or as in (149) below with $D_t = \mathbb{R}$, $t \in (0,T]$, then strong convergence holds in the sense that $\lim_{N\to\infty}\mathbb{E}\big[|X_T - Y_T^N|^p\big] = 0$ for all $p \in (0,\infty)$ whereas approximations of $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$ are finite but unbounded in the sense that $\sup_{t\in[0,T]}\mathbb{E}\big[\exp(\varepsilon|Y_t^N|^4)\big] < \infty$ for all $N \in \mathbb{N}$ and $\lim_{N\to\infty}\inf_{t\in(0,T]}\mathbb{E}\big[\exp(p|Y_t^N|^q)\big] = \infty$ for all $p \in (0,\infty)$ and all $q \in (3,\infty)$; see Corollary 5.4 and Corollary 5.5 below. Thus, Euler-Maruyama approximations, linear-implicit Euler approximations and tamed Euler approximations as defined in (144) or as in (149) are not suitable for numerically calculating $\mathbb{E}\big[\exp\big(\delta|X_t|^4\big)\big]$, $\delta \in (0,\varepsilon)$, $t \in (0,T]$. Lemma 5.3 below also indicates that further numerical one-step approximation methods whose one-step increment function grows sufficiently fast as the discretization step size decreases are not suitable for approximating expectations of exponential functionals in the generality of Theorem 1.1 below.

There are many results in the literature which prove uniform boundedness of polynomial moments of numerical approximations of certain nonlinear SDEs with superlinearly growing coefficients; see, e.g., [1, 2, 5, 8, 11, 12, 13, 14, 15, 17, 25, 22, 26, 27, 28, 30, 31, 32]. To the best of our knowledge, the only reference on exponential integrability properties of certain numerical approximations for nonlinear SDEs is Bou-Rabee & Hairer [1]. More precisely, Lemma 3.6 in Bou-Rabee & Hairer [1] implies that there exists $\theta \in (0,\beta)$ such that $\sup_{h\in(0,1]} \mathbb{E}\big[\exp(\theta U(\bar{X}^h_{\lfloor 1/h \rfloor}))\big] < \infty$ where $\bar{X}^h \colon \mathbb{N}_0 \to \mathbb{R}^d$, $h \in (0,1]$, is a 'patched' version of the Metropolis-Adjusted Langevin Algorithm (MALA) for the overdamped Langevin dynamics (see Subsection 4.9 below) where the potential energy function $U \in C^4(\mathbb{R}^d,\mathbb{R})$ satisfies certain assumptions; see [1] for the details. In addition, Proposition 5.2 in Bou-Rabee & Hairer [1] provides a one-step estimate for MALA which is a first step towards proving exponential integrability properties of a stopped version of MALA.

In this article, we propose the following method to approximate the solution of the SDE (1) and to preserve inequality (3) in a suitable sense. Let $Y^N \colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings satisfying $Y^N_0 = X_0$ and

$$Y^N_t = Y^N_{\frac{nT}{N}} + \mathbb{1}_{\left\{\|Y^N_{nT/N}\| \le \exp\left(|\ln(N/T)|^{1/2}\right)\right\}} \left[\frac{\mu(Y^N_{nT/N})(t-\frac{nT}{N}) + \sigma(Y^N_{nT/N})(W_t - W_{nT/N})}{1+\|\mu(Y^N_{nT/N})(t-\frac{nT}{N}) + \sigma(Y^N_{nT/N})(W_t - W_{nT/N})\|^2}\right] \tag{6}$$

for all $t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. This method differs from the classical Euler-Maruyama scheme in two aspects. First, the Euler-Maruyama increment is divided through by one plus the squared norm of the Euler-Maruyama increment. This ensures that the increments of the numerical method (6) are uniformly bounded. Second, the approximation paths with $N \in \mathbb{N}$ time discretizations are stopped after leaving the set $\{x \in \mathbb{R}^d \colon \|x\| \le \exp\big(|\ln(N/T)|^{1/2}\big)\}$ where we choose the stopping levels mainly such that $\lim_{N\to\infty} \exp\big(|\ln(N/T)|^{1/2}\big) N^{-p} = 0$ for all $p \in (0,1]$. These a priori bounds give us control on certain rare events. In addition, observe that the numerical approximations $\{0,1,\ldots,N\}\times\Omega \ni (n,\omega) \mapsto Y^N_{nT/N}(\omega) \in \mathbb{R}^d$, $N \in \mathbb{N}$, can be easily implemented recursively. In fact, this implementation requires only a few additional arithmetical operations in each recursion step compared to the classical Euler-Maruyama approximations. Theorem 1.1 below, shows that the numerical approximations (6) preserve inequality (3) in a suitable sense under slightly stronger assumptions on μ, σ, U and $\bar{U}$.

Theorem 1.1. *Assume the above setting, let $p, c \in [1,\infty)$, let $\tau_N \colon \Omega \to [0,T]$, $N \in \mathbb{N}$, be mappings satisfying $\tau_N = \inf\big(\{t \in \{0, \frac{T}{N}, \frac{2T}{N}, \ldots, T\} \colon \|Y^N_t\| > \exp(|\ln(N/T)|^{1/2})\} \cup \{T\}\big)$ for all $N \in \mathbb{N}$ and assume that*

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \le c\,(1 + \|x\|^c)\,, \tag{7}$$

$$|\bar{U}(x) - \bar{U}(y)| \le c\,(1 + \|x\|^c + \|y\|^c)\,\|x-y\|, \tag{8}$$

$$\|x\|^{1/c} \le c\,(1 + U(x)) \tag{9}$$

$$\|U^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \le c\,(1+U(x))^{\max(1-i/p,0)} \tag{10}$$

for all $x, y \in \mathbb{R}^d$ and all $i \in \{1,2,3\}$. Then it holds for all $r \in (0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y^N_t\|^r\big]\big) = 0$, that

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\frac{U(Y^N_t)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \frac{\bar{U}(Y^N_s)}{e^{\rho s}}\,ds\right)\right] \le \mathbb{E}\Big[e^{U(X_0)}\Big] < \infty \tag{11}$$

and that $\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big(\frac{U(Y^N_t)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \frac{\bar{U}(Y^N_s)}{e^{\rho s}}\,ds\Big)\Big] < \infty$.

Theorem 1.1 is a special case of our main result, Corollary 3.8 below, in which the state space of the exact solution of the SDE under consideration is an open subset of $\mathbb{R}^d$. The proof of Theorem 1.1 is thus omitted. Corollary 3.8, in turn, follows from our general result on exponential integrability properties of stopped increment-tamed Euler-Maruyama schemes, Theorem 2.8 below, and from convergence in probability of stopped increment-tamed Euler-Maruyama schemes, Corollary 3.7 below. To the best of our knowledge, Theorem 1.1 and its generalization in Corollary 3.8 below respectively are the first results in the literature which imply exponential integrability properties for the stochastic Ginzburg-Landau equation in Subsection 4.2, for the stochastic Lorenz equation with additive noise in Subsection 4.3, for the stochastic van der Pol oscillator in Subsection 4.4, for the stochastic Duffing-van der Pol oscillator in Subsection 4.5, for the model from experimental psychology in Subsection 4.6, for the stochastic SIR model in Subsection 4.7, or – under additional assumptions on the model – for the Langevin dynamics in Subsection 4.8.

1.1 Notation

Throughout this article the following notation is used. For $d, m \in \mathbb{N}$ and a $d \times m$-matrix $A \in \mathbb{R}^{d\times m}$, we denote by $A^* \in \mathbb{R}^{m\times d}$ the transpose of the matrix A and by $\|A\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ the Hilbert-Schmidt norm of the matrix A. In addition, for $d, m \in \mathbb{N}$ and arbitrary functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$, we denote by $\mathcal{G}_{\mu,\sigma}\colon C^2(\mathbb{R}^d,\mathbb{R}) \to \{f\colon \mathbb{R}^d \to \mathbb{R}\}$ the formal generator associated to μ and σ defined by

$$(\mathcal{G}_{\mu,\sigma}\varphi)(x) := \langle \mu(x), (\nabla\varphi)(x)\rangle + \tfrac{1}{2}\operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}\varphi)(x)\big) \tag{12}$$

for all $x \in \mathbb{R}^d$ and all $\varphi \in C^2(\mathbb{R}^d,\mathbb{R})$. Moreover, for $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\lambda_A\colon \mathcal{B}(A) \to [0,\infty]$ the Lebesgue-Borel measure on $A \subseteq \mathbb{R}^d$. For two measurable spaces $(A,\mathcal{A})$ and $(B,\mathcal{B})$, we denote by $\mathcal{L}^0(A;B)$ the set of all $\mathcal{A}/\mathcal{B}$-measurable mappings from A to B. In addition, for $n, d \in \mathbb{N}$, $p, c \in [0,\infty]$, a set $B \subseteq \mathbb{R}$ and an open set $A \subset \mathbb{R}^d$, we denote by $C^n_{p,c}(A,B)$ (cf. (1.12) in [15]) the set

$$C^n_{p,c}(A,B) := \left\{ f \in C^{n-1}(A,B)\colon \begin{array}{c} f^{(n-1)} \text{ is locally Lipschitz continuous and for} \\ \lambda_{\mathbb{R}^d}\text{-almost all } x \in \mathbb{R}^d \text{ and all } i \in \{1,2,\dots,n\} \text{ it} \\ \text{holds } \|f^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c\,[1+|f(x)|]^{\max(1-i/p,0)} \end{array} \right\}. \tag{13}$$

Next, we define $x \vee y := \max(x,y)$ and $x \wedge y := \min(x,y)$ for all $x, y \in \mathbb{R}$. Finally, let $\lfloor\cdot\rfloor_h\colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be mappings given by $\lfloor t\rfloor_h = \max\{s \in \{0, h, -h, 2h, -2h, \dots\}\colon s \leq t\}$ for all $t \in \mathbb{R}$ and all $h \in (0,\infty)$.

2 Exponential moments for numerical approximation processes

2.1 From one-step estimates to exponential moments

The following proposition, Proposition 2.1, is an extended and generalized version of Corollary 2.2 in [15]. The proof of Proposition 2.1 is similar to the proof of Proposition 2.1 in [15].

Proposition 2.1. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(E,\mathcal{E})$ be a measurable space, let $V \in \mathcal{L}^0(E;[0,\infty])$, let $Z\colon \mathbb{N}_0 \times \Omega \to E$ be a stochastic process, let $\gamma_n \in [0,\infty)$, $n \in \mathbb{N}_0$, $\delta_n \in (0,\infty]$, $n \in \mathbb{N}_0$, and $\Omega_n \in \mathcal{F}$, $n \in \mathbb{N}_0$, be sequences satisfying $\Omega_0 = \Omega$, $\Omega_n \backslash \Omega_{n+1} \subseteq \{V(Z_n) > \delta_n\}$ and*

$$\mathbb{E}\Big[\mathbb{1}_{\Omega_{n+1}} V(Z_{n+1})\Big] \leq \gamma_n \cdot \mathbb{E}\Big[\mathbb{1}_{\Omega_n} V(Z_n)\Big] \tag{14}$$

for all $n \in \mathbb{N}_0$. Then it holds for all $n \in \mathbb{N}_0$, $p \in [1,\infty]$, $\bar{V} \in \mathcal{L}^0(E;[0,\infty])$ with $\bar{V} \leq V$ that

$$\mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Z_n)\big] \leq \left(\prod_{k=0}^{n-1}\gamma_k\right)\cdot \mathbb{E}\big[V(Z_0)\big], \quad \mathbb{P}\left[(\Omega_n)^c\right] \leq \left(\sum_{k=0}^{n-1}\frac{\prod_{l=0}^{k-1}\gamma_l}{\delta_k}\right)\mathbb{E}\big[V(Z_0)\big], \tag{15}$$

$$\mathbb{E}\big[\bar{V}(Z_n)\big] \leq \left(\prod_{k=0}^{n-1}\gamma_k\right)\cdot \mathbb{E}\big[V(Z_0)\big] + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})}\left[\bigg(\sum_{k=0}^{n-1}\frac{\prod_{l=0}^{k-1}\gamma_l}{\delta_k}\bigg)\mathbb{E}\big[V(Z_0)\big]\right]^{(1-\frac{1}{p})}. \tag{16}$$

Proof of Proposition 2.1. The first inequality in (15) is an easy consequence of (14). To arrive at the second estimate in (15), note first for all $n \in \mathbb{N}$ that

$$(\Omega_n)^c = \big(\Omega_{n-1}\backslash\Omega_n\big) \uplus \big((\Omega_{n-1})^c\backslash\Omega_n\big) \subseteq \big(\Omega_{n-1}\backslash\Omega_n\big) \cup \big((\Omega_{n-1})^c\big). \tag{17}$$

Iterating inclusion (17) and using $\Omega_0 = \Omega$ shows for all $n \in \mathbb{N}_0$ that

$$\begin{aligned} (\Omega_n)^c &\subseteq \left(\bigcup_{k=0}^{n-1}(\Omega_k\backslash\Omega_{k+1})\right) \bigcup \big((\Omega_0)^c\big) = \bigcup_{k=0}^{n-1}(\Omega_k\backslash\Omega_{k+1}) = \bigcup_{k=0}^{n-1}\Big(\Omega_k \cap \big(\Omega_k\backslash\Omega_{k+1}\big)\Big) \\ &\subseteq \bigcup_{k=0}^{n-1}\left(\Omega_k \cap \{V(Z_k) > \delta_k\}\right) = \bigcup_{k=0}^{n-1}\{\mathbb{1}_{\Omega_k}V(Z_k) > \delta_k\}. \end{aligned} \tag{18}$$

Additivity of the probability measure $\mathbb{P}$, Markov's inequality and the first inequality in (15) therefore imply for all $n \in \mathbb{N}_0$ that

$$\begin{aligned} \mathbb{P}\big[(\Omega_n)^c\big] &\leq \sum_{k=0}^{n-1}\mathbb{P}\Big[\mathbb{1}_{\Omega_k}V(Z_k) > \delta_k\Big] \leq \sum_{k=0}^{n-1}\left[\frac{\mathbb{E}\big[\mathbb{1}_{\Omega_k}V(Z_k)\big]}{\delta_k}\right] \\ &\leq \sum_{k=0}^{n-1}\left[\frac{\Big(\prod_{l=0}^{k-1}\gamma_l\Big)\cdot\mathbb{E}[V(Z_0)]}{\delta_k}\right] \end{aligned} \tag{19}$$

This is the second inequality in (15) and the proof of (15) is thus completed. Next observe that Hölder's inequality implies for all $\tilde{\Omega} \in \mathcal{F}$, $p \in [1,\infty]$ and all $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable mappings $X\colon \Omega \to [0,\infty]$ that

$$\mathbb{E}[X] \leq \mathbb{E}[\mathbb{1}_{\tilde{\Omega}}X] + \big(\mathbb{P}\big[(\tilde{\Omega})^c\big]\big)^{(1-1/p)}\|X\|_{L^p(\Omega;\mathbb{R})}. \tag{20}$$

Combining (15) and (20) finally results in

$$
\begin{aligned}
\mathbb{E}\big[\bar{V}(Z_n)\big] &\leq \mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Z_n)\big] + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})} \left(\mathbb{P}\big[(\Omega_n)^c\big]\right)^{(1-1/p)} \\
&\leq \left(\prod_{k=0}^{n-1} \gamma_k\right) \cdot \mathbb{E}\big[V(Z_0)\big] + \|\bar{V}(Z_n)\|_{L^p(\Omega;\mathbb{R})} \left[\left(\sum_{k=0}^{n-1} \frac{\prod_{l=0}^{k-1} \gamma_l}{\delta_k}\right) \mathbb{E}\big[V(Z_0)\big]\right]^{(1-\frac{1}{p})}
\end{aligned} \tag{21}
$$

for all $n \in \mathbb{N}_0$, $p \in [1, \infty]$ and all $\mathcal{E}/\mathcal{B}([0,\infty])$-measurable functions $\bar{V} \colon E \to [0, \infty]$ with $\bar{V}(x) \leq V(x)$ for all $x \in E$. The proof of Proposition 2.1 is thus completed. □

The next elementary lemma (Lemma 2.2) establishes an a prior bound based on a specific class of path dependent Lyapunov-type functions (see (22) below for details and cf., e.g., also Section 3.1 in Schurz [29]). For completeness the proof of Lemma 2.2 is given below.

Lemma 2.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $\rho, T \in [0, \infty)$, $h \in (0, T)$, $c \in \mathbb{R}$, $U, \bar{U} \in \mathcal{L}^0(E; \mathbb{R})$, $A \in \mathcal{E}$ and let $Y : [0,T] \times \Omega \to E$ be a product measurable stochastic process satisfying $\int_0^T \mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, |\bar{U}(Y_r)| \, dr < \infty$ and*

$$
\mathbb{E}\!\left[\exp\!\left(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right) \Big| \, (Y_r)_{r \in [0, nh]}\right] \leq \exp\!\left(-cnh + \tfrac{U(Y_{nh})}{e^{\rho nh}} + \int_0^{nh} \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right) \tag{22}
$$

$\mathbb{P}$-a.s. for all $t \in (nh, (n+1)h] \cap [0,T]$ and all $n \in \mathbb{N}_0$. Then it holds for all $t \in [0,T]$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right)\right] \leq e^{ct} \, \mathbb{E}\!\left[e^{U(Y_0)}\right]. \tag{23}
$$

Proof of Lemma 2.2. Assumption (22) implies for all $t \in [0,T]$ that

$$
\begin{aligned}
&\mathbb{E}\!\left[\exp\!\left(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right)\right] \\
&= \mathbb{E}\!\left[\mathbb{E}\!\left[\exp\!\left(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right)\right] \Big| \, (Y_s)_{s \in [0, \lfloor t \rfloor_h]}\right] \\
&\leq \mathbb{E}\!\left[\exp\!\left(-c\lfloor t \rfloor_h + \tfrac{U(Y_{\lfloor t \rfloor_h})}{e^{\rho \lfloor t \rfloor_h}} + \int_0^{\lfloor t \rfloor_h} \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right)\right] \\
&\leq \ldots \leq \mathbb{E}\!\left[\exp\!\left(\tfrac{U(Y_0)}{e^{\rho \cdot 0}}\right)\right] = \mathbb{E}\!\left[e^{U(Y_0)}\right].
\end{aligned} \tag{24}
$$

This completes the proof of Lemma 2.2. □

The next corollary, Corollary 2.3, specialises Lemma 2.2 to the case where the product measurable stochastic process appearing in (22) and (23) is an appropriate one-step approximation process for an SDE driven by a standard Brownian motion; see (25) below for details.

Corollary 2.3. *Let $T \in (0, \infty)$, $h \in (0, T]$, $d, m \in \mathbb{N}$, $\rho, c \in [0, \infty)$, $U \in \mathcal{L}^0(\mathbb{R}^d; [0, \infty))$, $\bar{U} \in \mathcal{L}^0(\mathbb{R}^d; \mathbb{R})$, $\Phi \in \mathcal{L}^0(\mathbb{R}^d \times [0,T] \times \mathbb{R}^m; \mathbb{R}^d)$, $A \in \mathcal{B}(\mathbb{R}^d)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$ be a filtrated probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion with continuous sample paths, let $Y \colon [0,T] \times \Omega \to \mathbb{R}^d$ be an adapted stochastic process satisfying*

$$
Y_t = \mathbb{1}_{A^c}(Y_{nh}) \cdot Y_{nh} + \mathbb{1}_A(Y_{nh}) \cdot \Phi(Y_{nh}, t - nh, W_t - W_{nh}) \tag{25}
$$

for all $t \in (nh, (n+1)h] \cap [0,T]$ and all $n \in \mathbb{N}_0$ and assume $\int_0^T \mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, |\bar{U}(Y_r)| \, dr < \infty$ and

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{\mathbb{1}_A(x) \, U(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(x) \, \bar{U}(\Phi(x,s,W_s))}{e^{\rho s}} \, ds\right)\right] \leq e^{ct + U(x)} \tag{26}
$$

for all $(t,x) \in (0,h] \times \mathbb{R}^d$. Then it holds for all $t \in [0,T]$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{U(Y_t)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(Y_{\lfloor r \rfloor_h}) \, \bar{U}(Y_r)}{e^{\rho r}} \, dr\right)\right] \leq e^{ct} \, \mathbb{E}\!\left[e^{U(Y_0)}\right]. \tag{27}
$$

Proof of Corollary 2.3. We prove Corollary 2.3 through an application of Lemma 2.2. For this we observe that assumption (26) implies for all $(t,x) \in (0,h] \times \mathbb{R}^d$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{\mathbb{1}_A(x) \, U(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_A(x) \, \bar{U}(\Phi(x,s,W_s))}{e^{\rho s}} \, ds\right)\right] \leq e^{ct + \mathbb{1}_A(x) \, U(x)}. \tag{28}
$$

Next note that equation (25), Jensen's inequality and inequality (28) imply that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(Y_t)}{e^{\rho t}} + \int\limits_{nh}^{t} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(Y_s)}{e^{\rho s}}\,ds\Bigg)\,\Big|\,(Y_s)_{s\in[0,nh]}\Bigg] \\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(\Phi(Y_{nh},t-nh,W_t-W_{nh}))}{e^{\rho t}} + \int\limits_{nh}^{t} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(\Phi(Y_{nh},s-nh,W_s-W_{nh}))}{e^{\rho s}}\,ds\Bigg)\,\Big|\,(Y_s)_{s\in[0,nh]}\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(\Phi(Y_{nh},t-nh,W_t-W_{nh}))}{e^{\rho(t-nh)}} + \int\limits_{0}^{t-nh} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(\Phi(Y_{nh},s,W_{nh+s}-W_{nh}))}{e^{\rho s}}\,ds\Bigg)\Bigg]^{\exp(-\rho nh)}\,\Big|\,(Y_s)_{s\in[0,nh]}\Bigg] \\
&\leq \Bigg|\mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(\Phi(Y_{nh},t-nh,W_t-W_{nh}))}{e^{\rho(t-nh)}} + \int\limits_{0}^{t-nh} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(\Phi(Y_{nh},s,W_{nh+s}-W_{nh}))}{e^{\rho s}}\,ds\Bigg)\,\Big|\,(Y_s)_{s\in[0,nh]}\Bigg]\Bigg|^{\exp(-\rho nh)} \\
&\leq \Big|e^{c(t-nh)+\mathbb{1}_A(Y_{nh})\,U(Y_{nh})}\Big|^{\exp(-\rho nh)} \leq \exp\Big(c\,(t-nh) + \tfrac{\mathbb{1}_A(Y_{nh})\,U(Y_{nh})}{e^{\rho nh}}\Big)
\end{aligned}
$$

$\mathbb{P}$-a.s. for all $t \in (nh,(n+1)h] \cap [0,T]$ and all $n \in \mathbb{N}_0$. Combining this with (25) shows that

$$
\begin{aligned}
&\mathbb{E}\Bigg[\exp\Bigg(-ct + \tfrac{U(Y_t)}{e^{\rho t}} + \int\limits_{0}^{t} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Big|\,(Y_r)_{r\in[0,nh]}\Bigg] \\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{A^c}(Y_{nh})\,U(Y_t)}{e^{\rho t}} + \int\limits_{nh}^{t} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Big|\,(Y_r)_{r\in[0,nh]}\Bigg] \\
&\quad \cdot \exp\Bigg(-ct + \int\limits_{0}^{nh} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg) \\
&= \mathbb{E}\Bigg[\exp\Bigg(\tfrac{\mathbb{1}_A(Y_{nh})\,U(Y_t)}{e^{\rho t}} + \tfrac{\mathbb{1}_{A^c}(Y_{nh})\,U(Y_{nh})}{e^{\rho t}} + \int\limits_{nh}^{t} \tfrac{\mathbb{1}_A(Y_{nh})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg)\,\Big|\,(Y_r)_{r\in[0,nh]}\Bigg] \\
&\quad \cdot \exp\Bigg(-ct + \int\limits_{0}^{nh} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg) \\
&\leq \exp\Bigg(c\,(t-nh) + \tfrac{\mathbb{1}_A(Y_{nh})\,U(Y_{nh})}{e^{\rho nh}} - ct + \tfrac{\mathbb{1}_{A^c}(Y_{nh})\,U(Y_{nh})}{e^{\rho t}} + \int\limits_{0}^{nh} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg) \\
&\leq \exp\Bigg(-cnh + \tfrac{U(Y_{nh})}{e^{\rho nh}} + \int\limits_{0}^{nh} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\,\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg)
\end{aligned}
\tag{29}
$$

$\mathbb{P}$-a.s. for all $t \in (nh,(n+1)h] \cap [0,T]$ and all $n \in \mathbb{N}_0$. Combining this with Lemma 2.2 yields for all $t \in [0,T]$ that

$$
\mathbb{E}\Bigg[\exp\Bigg(\tfrac{U(Y_t)}{e^{\rho t}} + \int\limits_{0}^{t} \tfrac{\mathbb{1}_A(Y_{\lfloor r\rfloor_h})\bar{U}(Y_r)}{e^{\rho r}}\,dr\Bigg)\Bigg] \leq e^{ct}\,\mathbb{E}\Big[e^{U(Y_0)}\Big].
$$

This finishes the proof of Corollary 2.3. ☐

2.2 A one-step estimate for exponential moments

In Lemma 2.7 below a one-step estimate for exponential moments (see (26) in Corollary 2.3 above) is proved for a general class of stopped one-step numerical approximation schemes. The proof of Lemma 2.7 uses the elementary estimate in Lemma 2.5 below. Moreover, the proof of Lemma 2.5 exploits the following well-known lemma, Lemma 2.4. For completeness the proof of Lemma 2.4 is given below.

Lemma 2.4. *It holds for all $x \in \mathbb{R}$ that*

$$
e^x = 2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) - \frac{1}{e^x} \leq 2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right). \tag{30}
$$

Proof of Lemma 2.4. Note for all $x \in \mathbb{R}$ that

$$
e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) - \left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right). \tag{31}
$$

This implies for all $x \in \mathbb{R}$ that

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + \left(\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right] - e^{-x}\right) \\
&= 2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right) - e^{-x} \leq 2\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right).
\end{aligned}
\tag{32}
$$

The proof of Lemma 2.4 is thus completed. ☐

Lemma 2.5. *Let $T \in [0,\infty)$, $d,m \in \mathbb{N}$, $A \in \mathbb{R}^{d\times m}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion. Then it holds for all $t \in [0,T]$ that $\mathbb{E}\big[e^{\|AW_t\|}\big] \leq 2\exp\big(\frac{t}{2}\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\big)$.*

Proof of Lemma 2.5. First of all, define functions $f_n\colon \mathbb{R}^m \to [0,\infty)$, $n \in \mathbb{N}_0$, by $f_n(x) := \|Ax\|^{2n}$ for all $x \in \mathbb{R}^m$ and all $n \in \mathbb{N}_0$. Then note for all $x \in \mathbb{R}^m$ and all $n \in \mathbb{N}$ that

$$\begin{aligned}
\operatorname{trace}((\operatorname{Hess} f_n)(x)) &= \operatorname{trace}\Big(2n\,\|Ax\|^{(2n-2)}A^*A + \mathbb{1}_{\{x\neq 0\}}\,2n\,(2n-2)\,\|Ax\|^{(2n-4)}\,(A^*Ax)\,(A^*Ax)^*\Big)\\
&= 2n\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \mathbb{1}_{\{x\neq 0\}}\,2n\,(2n-2)\,\|Ax\|^{(2n-4)}\,\|A^*Ax\|^2\\
&\leq 2n\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + 2n\,(2n-2)\,\|Ax\|^{(2n-2)}\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\\
&= 2n\,(2n-1)\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\;f_{n-1}(x)\,.
\end{aligned} \tag{33}$$

Itô's formula hence shows for all $s_0 \in [0,T]$ and all $n \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{E}\Big[\|AW_{s_0}\|^{2n}\Big] &= \mathbb{E}\big[f_n(W_{s_0})\big] = \frac{1}{2}\int_0^{s_0}\mathbb{E}\big[\operatorname{trace}((\operatorname{Hess} f_n)(W_{s_1}))\big]\,ds_1\\
&\leq \frac{1}{2}\int_0^{s_0} 2n\,(2n-1)\,\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\,\mathbb{E}\big[f_{n-1}(W_{s_1})\big]\,ds_1\\
&\leq \ldots \leq \frac{(2n)!}{2^n}\,\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\int_0^{s_0}\int_0^{s_1}\cdots\int_0^{s_{n-1}}\mathbb{E}\big[f_0(W_{s_n})\big]\,ds_n\cdots ds_2\,ds_1\\
&= \frac{(2n)!}{2^n n!}\,\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\,(s_0)^n\,.
\end{aligned} \tag{34}$$

Combining this with Lemma 2.4 implies for all $t \in [0,T]$ that

$$\mathbb{E}\Big[e^{\|AW_t\|}\Big] \leq 2\left(\sum_{n=0}^{\infty}\frac{\mathbb{E}\big[\|AW_t\|^{2n}\big]}{(2n)!}\right) \leq 2\left(\sum_{n=0}^{\infty}\frac{t^n\|A\|^{2n}_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{2^n n!}\right) = 2e^{\frac{t}{2}\|A\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}. \tag{35}$$

This finishes the proof of Lemma 2.5. □

Beside Lemma 2.5, the proof of Lemma 2.7 also uses the following lemma (Lemma 2.6). Lemma 2.6 is an immediately consequence of (2.63) in Lemma 2.12 in [15].

Lemma 2.6. *Let $c,p \in [1,\infty)$, $V \in C^1_{p,c}(\mathbb{R}^d,[0,\infty))$. Then it holds for all $x,y \in \mathbb{R}^d$ that $1 + V(x+y) \leq c^p\,2^{(p-1)}\,(1 + V(x) + \|y\|^p)$.*

Lemma 2.7. *Let $\alpha,h \in (0,\infty)$, $d,m \in \mathbb{N}$, $c,p \in [1,\infty)$, $\gamma_0,\gamma_1,\ldots,\gamma_6,\gamma_7,\rho \in [0,\infty)$, $\mu \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $\bar{U} \in C(\mathbb{R}^d,\mathbb{R})$, $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, $\Phi \in C^{0,1,2}(\mathbb{R}^d\times[0,h]\times\mathbb{R}^m,\mathbb{R}^d)$, let $D_t \subseteq \{x \in \mathbb{R}^d\colon U(x) \leq \frac{c}{t^\alpha}\}$, $t \in (0,h]$, be a non-increasing family of sets, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,h]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths, assume*

$$\|\mu(x)\| \leq c\,(1 + |U(x)|^{\gamma_0})\,, \qquad \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(1 + |U(x)|^{\gamma_1})\,, \tag{36}$$

$$\Phi(x,0,0) = x, \qquad (\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \rho\cdot U(x) \tag{37}$$

for all $x \in \mathbb{R}^d$, assume $\Phi(x,t,y) = x$ for all $x \in \mathbb{R}^d\backslash D_t$, $t \in (0,h]$, $y \in \mathbb{R}^m$, assume

$$\big\|\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) - \mu(x)\big\|_{L^4(\Omega;\mathbb{R}^d)} \leq cs^{\gamma_2}, \tag{38}$$

$$\big\|\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) - \sigma(x)\big\|_{L^8(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} \leq cs^{\gamma_3}, \tag{39}$$

$$\big\|(\triangle_y\Phi)(x,s,W_s)\big\|_{L^4(\Omega;\mathbb{R}^d)} \leq cs^{\gamma_4}, \tag{40}$$

$$\|\Phi(x,s,W_s) - x\|_{L^r(\Omega;\mathbb{R}^d)} \leq c\min\!\big(r, 1 + |U(x)|^{\gamma_5}, (1 + |U(x)|^{\gamma_5})\,\|\mu(x)s + \sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)}\big) \tag{41}$$

for all $r \in [1,\infty)$, $x \in D_s, s \in (0,h], y \in \mathbb{R}^m$ and assume $|\bar{U}(x)| \leq c\,(1 + |U(x)|^{\gamma_6})$ and $|\bar{U}(x) - \bar{U}(y)| \leq c\,(1 + |U(x)|^{\gamma_7} + |U(y)|^{\gamma_7})\,\|x-y\|$ for all $x,y \in \mathbb{R}^d$. Then it holds for all $(t,x) \in (0,h]\times\mathbb{R}^d$ that

$$\begin{aligned}
&\mathbb{E}\left[\exp\!\left(\tfrac{U(\Phi(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_t}(x)\,\bar{U}(\Phi(x,s,W_s))}{e^{\rho s}}\,ds\right)\right]\\
&\leq e^{U(x)}\left[1 + \int_0^t \exp\!\left(\tfrac{s\,[2c]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right)\tfrac{\max(\rho,1)\,[2pc\max(s,1)]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\ldots,\gamma_5)}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\,ds\right].
\end{aligned} \tag{42}$$

Proof of Lemma 2.7. Let $Y^x\colon [0,h]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, be stochastic processes given by $Y^x_s=\Phi(x,s,W_s)$ for all $s\in[0,h]$, $x\in\mathbb{R}^d$. The assumption that $\Phi(x,s,y)=x$ for all $x\in\mathbb{R}^d\backslash D_s$, $s\in(0,h]$, $y\in\mathbb{R}^d$ implies that $\Phi(x,s,W_s)=x$ for all $x\in\mathbb{R}^d\backslash D_s$, $s\in(0,h]$. This shows for all $t\in(0,h]$, $x\in\mathbb{R}^d\backslash D_t$ that

$$\mathbb{E}\bigg[\exp\bigg(\tfrac{U(\Phi(x,t,W_t))}{e^{\rho t}}+\int_0^t \tfrac{\mathbb{1}_{D_t}(x)\,\bar U(\Phi(x,r,W_r))}{e^{\rho s}}\,dr\bigg)\bigg]=\exp\big(e^{-\rho t}U(x)\big)\le e^{U(x)}. \tag{43}$$

Moreover, note that Itô's formula implies that

$$\begin{aligned}
&\exp\bigg(e^{-\rho t}U(Y^x_t)+\int_0^t e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)-e^{U(x)}\\
&=\int_0^t \exp\bigg(e^{-\rho s}U(Y^x_s)+\int_0^s e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\,e^{-\rho s}U'(Y^x_s)\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\,dW_s\\
&\quad+\int_0^t \exp\bigg(e^{-\rho s}U(Y^x_s)+\int_0^s e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\,e^{-\rho s}\cdot\bigg(\bar U(Y^x_s)-\rho U(Y^x_s)+U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s)\\
&\qquad+\tfrac12\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big)\\
&\qquad+\tfrac12 e^{-\rho s}\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2+\tfrac12\textstyle\sum_{i=1}^m U'(Y^x_s)\big(\tfrac{\partial^2}{\partial y_i^2}\Phi\big)(x,s,W_s)\bigg)\,ds
\end{aligned} \tag{44}$$

$\mathbb{P}$-a.s. for all $x\in D_t$ and all $t\in[0,h]$. Next define a sequence $\tau_n\colon\Omega\to[0,h]$, $n\in\mathbb{N}$, of stopping times by $\tau_n=\inf(\{s\in[0,h]\colon \|W_s\|>n\}\cup\{h\})$ for all $n\in\mathbb{N}$. As for all $n\in\mathbb{N}$ and all $x\in\mathbb{R}^d$ the continuous function

$$[0,h]\times\mathbb{R}^m\ni(s,y)\mapsto\exp\bigg(\tfrac{U(\Phi(x,s,y))}{e^{\rho s}}+\int_0^s\tfrac{\bar U(\Phi(x,r,y))}{e^{\rho r}}\,dr\bigg)\,\tfrac{U'(\Phi(x,s,y))\,(\frac{\partial}{\partial y}\Phi)(x,s,y)}{e^{\rho s}}\in \operatorname{HS}(\mathbb{R}^m,\mathbb{R}^d)$$

is bounded on the compact set $[0,h]\times\{y\in\mathbb{R}^d\colon\|y\|\le n\}$, we see that the sequence τ_n, $n\in\mathbb{N}$, is a localizing sequence for the local martingale on the right-hand side of (44). Taking expectations on both sides of (44) hence shows for all $t\in[0,h]$, $x\in D_t$ and all $n\in\mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}U(Y^x_{t\wedge\tau_n})+\int_0^{t\wedge\tau_n}e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\bigg]-e^{U(x)}\\
&=\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\exp\bigg(e^{-\rho s}U(Y^x_s)+\int_0^s e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\,e^{-\rho s}\cdot\bigg(\bar U(Y^x_s)-\rho U(Y^x_s)\\
&\qquad+U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s)+\tfrac12\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big)\\
&\qquad+\tfrac12 e^{-\rho s}\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2+\tfrac12 U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)\bigg)\,ds\bigg].
\end{aligned} \tag{45}$$

Next assumption (37) yields for all $t\in[0,h]$, $x\in D_t$ and all $n\in\mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}U(Y^x_{t\wedge\tau_n})+\int_0^{t\wedge\tau_n}e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\bigg]-e^{U(x)}\\
&=\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\exp\bigg(e^{-\rho s}U(Y^x_s)+\int_0^s e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\,e^{-\rho s}\\
&\quad\cdot\bigg(-\rho U(x)+(\mathcal{G}_{\mu,\sigma}U)(x)+\tfrac12 e^{-\rho s}\,\|\sigma(x)^*(\nabla U)(x)\|^2+\bar U(x)\\
&\quad+\bar U(Y^x_s)-\bar U(x)-\rho(U(Y^x_s)-U(x))+U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s)-U'(x)\mu(x)\\
&\quad+\tfrac12\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big)-\tfrac12\operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\big)\\
&\quad+\tfrac12 e^{-\rho s}\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2-\tfrac12 e^{-\rho s}\,\|\sigma(x)^*(\nabla U)(x)\|^2+\tfrac{U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)}{2}\bigg)\,ds\bigg]\\
&\le\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\exp\bigg(e^{-\rho s}U(Y^x_s)+\int_0^s e^{-\rho r}\bar U(Y^x_r)\,dr\bigg)\\
&\quad\cdot\bigg(\big|\bar U(Y^x_s)-\bar U(x)\big|+\rho\,|U(Y^x_s)-U(x)|+\big|U'(Y^x_s)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s)-U'(x)\mu(x)\big|\\
&\quad+\tfrac12\Big|\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y^x_s)\Big)-\operatorname{trace}\big(\sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\big)\Big|\\
&\quad+\tfrac12 e^{-\rho s}\bigg|\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y^x_s)\Big\|^2-\|\sigma(x)^*(\nabla U)(x)\|^2\bigg|+\tfrac{|U'(Y^x_s)\,(\triangle_y\Phi)(x,s,W_s)|}{2}\bigg)\,ds\bigg].
\end{aligned} \tag{46}$$

Hence, Fatou's lemma, Fubini's theorem and Hölder's inequality imply for all $t \in [0,h]$, $x \in D_t$ that

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(e^{-\rho t}U(Y_t^x) + \int_0^t e^{-\rho r}\bar{U}(Y_r^x)\,dr\bigg)\bigg] - e^{U(x)} \\
&\leq \liminf_{n\to\infty} \mathbb{E}\bigg[\exp\bigg(e^{-\rho(t\wedge\tau_n)}U(Y_{t\wedge\tau_n}^x) + \int_0^{t\wedge\tau_n} e^{-\rho r}\bar{U}(Y_r^x)\,dr\bigg)\bigg] - e^{U(x)} \\
&\leq \mathbb{E}\bigg[\int_0^t \exp\bigg(U(Y_s^x) + \int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr\bigg) \\
&\quad \cdot \bigg(\big|\bar{U}(Y_s^x) - \bar{U}(x)\big| + \rho|U(Y_s^x) - U(x)| + \big|U'(Y_s^x)\left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - U'(x)\mu(x)\big| \\
&\quad + \tfrac{1}{2}\Big|\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y_s^x)\Big) - \operatorname{trace}\left(\sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\right)\Big| \\
&\quad + \tfrac{1}{2}e^{-\rho s}\bigg|\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y_s^x)\Big\|^2 - \|\sigma(x)^*(\nabla U)(x)\|^2\bigg| + \tfrac{|U'(Y_s^x)\,(\triangle_y\Phi)(x,s,W_s)|}{2}\bigg)\,ds\bigg] \\
&\leq \int_0^t \bigg\|\exp\bigg(U(Y_s^x) + \int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr\bigg)\bigg\|_{L^2(\Omega;\mathbb{R})} \\
&\quad \cdot \bigg[\rho\|U(Y_s^x) - U(x)\|_{L^2(\Omega;\mathbb{R})} + \big\|U'(Y_s^x)\big(\tfrac{\partial}{\partial s}\Phi\big)(x,s,W_s) - U'(x)\mu(x)\big\|_{L^2(\Omega;\mathbb{R})} \\
&\quad + \tfrac{1}{2}\Big\|\operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\operatorname{Hess}U)(Y_s^x) - \sigma(x)\sigma(x)^*(\operatorname{Hess}U)(x)\Big)\Big\|_{L^2(\Omega;\mathbb{R})} \\
&\quad + \tfrac{1}{2}e^{-\rho s}\bigg\|\Big\|\big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s)\big]^*(\nabla U)(Y_s^x)\Big\|^2 - \|\sigma(x)^*(\nabla U)(x)\|^2\bigg\|_{L^2(\Omega;\mathbb{R})} \\
&\quad + \tfrac{\|U'(Y_s^x)\,(\triangle_y\Phi)(x,s,W_s)\|_{L^2(\Omega;\mathbb{R})}}{2} + \big\|\bar{U}(Y_s^x) - \bar{U}(x)\big\|_{L^2(\Omega;\mathbb{R})}\bigg]\,ds\,.
\end{aligned} \tag{47}
$$

Next we estimate the L^2-norms on the right-hand side separately. Combining the assumption that $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$ with Lemma 2.6 implies for all $x,y \in \mathbb{R}^d$, $i \in \{0,1,2\}$ that

$$
\begin{aligned}
\big\|U^{(i)}(y) - U^{(i)}(x)\big\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} &\leq \int_0^1 \big\|U^{(i+1)}\big(x + r(y-x)\big)\big\|_{L^{(i+1)}(\mathbb{R}^d,\mathbb{R})}\,\|y-x\|\,dr \\
&\leq c\left[\int_0^1 \big[1 + U\big(x + r(y-x)\big)\big]^{\frac{\max(p-i-1,0)}{p}}\,dr\right]\|y-x\| \\
&\leq \frac{c\,(2c)^{\max(p-i-1,0)}}{2^{\max(p-i-1,0)/p}}\left([1+U(x)]^{\frac{\max(p-i-1,0)}{p}} + \|y-x\|^{\max(p-i-1,0)}\right)\|y-x\| \\
&\leq \tfrac{(2c)^p}{2}\left(1 + |U(x)|^{\frac{\max(p-i-1,0)}{p}} + \|y-x\|^{\max(p-i-1,0)}\right)\|y-x\|\,.
\end{aligned} \tag{48}
$$

This, in particular, shows for all $x,y \in \mathbb{R}^d$ that

$$
|U(y) - U(x)| \leq \tfrac{(2c)^p}{2}\left(1 + |U(x)|^{\frac{p-1}{p}} + \|y-x\|^{(p-1)}\right)\|y-x\|\,. \tag{49}
$$

Combining this with the inequality $|\bar{U}(x)| \leq c\,(1 + |U(x)|^{\gamma_6})$ for all $x \in \mathbb{R}^d$ and Hölder's inequality yields for all $s \in (0,h]$, $x \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(2U(Y_s^x) + 2\int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr - 2U(x)\bigg)\bigg] \leq \mathbb{E}\bigg[\exp\bigg(2\,|U(Y_s^x) - U(x)| + 2\int_0^s \big|\bar{U}(Y_r^x)\big|\,dr\bigg)\bigg] \\
&\leq \mathbb{E}\bigg[\exp\bigg((2c)^p\big[1 + |U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big]\,\|Y_s^x - x\| + 2c\int_0^s (1 + |U(Y_r^x)|^{\gamma_6})\,dr\bigg)\bigg] \\
&\leq \Big\|\exp\Big((2c)^p\big[1 + |U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big]\,\|Y_s^x - x\|\Big)\Big\|_{L^1(\Omega;\mathbb{R})}\bigg\|\exp\bigg(2c\int_0^s (1 + |U(Y_r^x)|^{\gamma_6})\,dr\bigg)\bigg\|_{L^\infty(\Omega;\mathbb{R})} \\
&\leq \mathbb{E}\Big[\exp\Big((2c)^p\big[1 + |U(x)|^{\frac{p-1}{p}} + \|Y_s^x - x\|^{(p-1)}\big]\,\|Y_s^x - x\|\Big)\Big]\exp\bigg(2c\int_0^s \big(1 + \|U(Y_r^x)\|^{\gamma_6}_{L^\infty(\Omega;\mathbb{R})}\big)\,dr\bigg)\,.
\end{aligned} \tag{50}
$$

Next we estimate the two factors on the right-hand side of (50) separately. Using Hölder's inequality and assumption (41) shows for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big((2c)^p\big[1+|U(x)|^{\frac{p-1}{p}}+\|Y_s^x-x\|^{(p-1)}\big]\,\|Y_s^x-x\|\Big)\Big]\\
&=\mathbb{E}\Bigg[\sum_{n=0}^{\infty}\tfrac{(2c)^{pn}}{n!}\big[1+|U(x)|^{\frac{p-1}{p}}+\|\Phi(x,s,W_s)-x\|^{(p-1)}\big]^n\,\|\Phi(x,s,W_s)-x\|^n\Bigg]\\
&=\sum_{n=0}^{\infty}\Big\|\tfrac{(2c)^{pn}}{n!}\big[1+|U(x)|^{\frac{p-1}{p}}+\|\Phi(x,s,W_s)-x\|^{(p-1)}\big]^n\,\|\Phi(x,s,W_s)-x\|^n\Big\|_{L^1(\Omega;\mathbb{R})}\\
&\le\sum_{n=0}^{\infty}\tfrac{(2c)^{pn}}{n!}\Big[1+|U(x)|^{\frac{p-1}{p}}+\|\Phi(x,s,W_s)-x\|^{(p-1)}_{L^\infty(\Omega;\mathbb{R}^d)}\Big]^n\,\|\Phi(x,s,W_s)-x\|^n_{L^n(\Omega;\mathbb{R}^d)}\\
&\le\sum_{n=0}^{\infty}\tfrac{(2c)^{pn}}{n!}\Big[1+|U(x)|^{\frac{p-1}{p}}+c^{(p-1)}(1+|U(x)|^{\gamma_5})^{(p-1)}\Big]^n\Big[c\,(1+|U(x)|^{\gamma_5})\,\|\mu(x)s+\sigma(x)W_s\|_{L^n(\Omega;\mathbb{R}^d)}\Big]^n\\
&=\sum_{n=0}^{\infty}\tfrac{(2c)^{pn}}{n!}\Big[c\big(1+|U(x)|^{\frac{p-1}{p}}\big)\big(1+|U(x)|^{\gamma_5}\big)+c^p(1+|U(x)|^{\gamma_5})^p\Big]^n\,\mathbb{E}\big[\,\|\mu(x)s+\sigma(x)W_s\|^n\,\big]\\
&=\mathbb{E}\Big[\exp\Big([2c]^p\Big[c\big(1+|U(x)|^{\frac{p-1}{p}}\big)\big(1+|U(x)|^{\gamma_5}\big)+c^p(1+|U(x)|^{\gamma_5})^p\Big]\,\|\mu(x)s+\sigma(x)W_s\|\Big)\Big].
\end{aligned}
\tag{51}
$$

Hence, assumption (36) and Lemma 2.5 yield for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big([2c]^p\,\big[1+|U(x)|^{\frac{p-1}{p}}+\|Y_s^x-x\|^{(p-1)}\big]\,\|Y_s^x-x\|\Big)\Big]\\
&\le\mathbb{E}\Big[\exp\Big(c\,[2c]^p\,\Big[1+|U(x)|^{\frac{p-1}{p}}+|U(x)|^{\gamma_5}+|U(x)|^{(\gamma_5+\frac{p-1}{p})}+[2c]^{(p-1)}\,(1+|U(x)|^{p\gamma_5})\Big]\,\|\mu(x)s+\sigma(x)W_s\|\Big)\Big]\\
&\le\mathbb{E}\Big[\exp\Big(2^pc^{(p+1)}\,\Big[1+\frac{c}{s^{\alpha(p-1)/p}}+\frac{c^{\gamma_5}}{s^{\alpha\gamma_5}}+\frac{c^{(\gamma_5+1)}}{s^{\alpha(\gamma_5+(p-1)/p)}}+[2c]^{(p-1)}\,\Big(1+\frac{c^{p\gamma_5}}{s^{\alpha p\gamma_5}}\Big)\Big]\,\|\mu(x)s+\sigma(x)W_s\|\Big)\Big]\\
&\le\mathbb{E}\Big[\exp\Big(2^pc^{(p+1)}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)}\,\Big[4c^{(\gamma_5+1)}+2^pc^{(p+p\gamma_5-1)}\Big]\,\|\mu(x)s+\sigma(x)W_s\|\Big)\Big]\\
&\le\mathbb{E}\Big[\exp\Big(2^{(2p+2)}c^{p(\gamma_5+3)}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)}\,(\|\mu(x)\|s+\|\sigma(x)W_s\|)\Big)\Big]\\
&=\exp\Big(2^{(2p+2)}c^{p(\gamma_5+3)}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)}\,\|\mu(x)\|s\Big)\,\mathbb{E}\Big[\exp\Big(2^{(2p+2)}c^{p(\gamma_5+3)}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)}\,\|\sigma(x)W_s\|\Big)\Big]\\
&\le\exp\Big(2^{(2p+2)}c^{p(\gamma_5+3)}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)}\,\|\mu(x)\|s\Big)\,2\exp\Big(\tfrac{s2^{(4p+4)}c^{2p(\gamma_5+3)}\,\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{2\,[\min(s,1)]^{2\alpha(p\gamma_5+1)}}\Big)\\
&\le 2\exp\Big(s\,2^{(4p+3)}c^{2p(\gamma_5+3)}\,[\min(s,1)]^{-2\alpha(p\gamma_5+1)}\,\Big[\|\mu(x)\|+\|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\Big]\Big)\\
&\le 2\exp\Big(s\,2^{(4p+3)}\,c^{2p(\gamma_5+3)}\,[\min(s,1)]^{-2\alpha(p\gamma_5+1)}\,\Big[c\,(1+|U(x)|^{\gamma_0})+c^2\,(1+|U(x)|^{\gamma_1})^2\Big]\Big)\\
&\le 2\exp\Big(s\,2^{(4p+3)}\,c^{2p(\gamma_5+3)}\,[\min(s,1)]^{-2\alpha(p\gamma_5+1)}\,\Big[c+c^{(1+\gamma_0)}s^{-\alpha\gamma_0}+2c^2+2c^{2(1+\gamma_1)}s^{-2\alpha\gamma_1}\Big]\Big)\\
&\le 2\exp\Big(s\,2^{(4p+3)}\,c^{2p(\gamma_5+3)}\,[\min(s,1)]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)}\,\Big[2c^{(1+\gamma_0)}+4c^{2(1+\gamma_1)}\Big]\Big)\\
&\le 2\exp\Big(s\,2^{(4p+6)}\,c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)}\,[\min(s,1)]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)}\Big).
\end{aligned}
\tag{52}
$$

Next we combine (49) with assumption (41) to obtain for all $r \in (0,h]$, $x \in D_r$ that

$$
\begin{aligned}
&\|U(Y_r^x)\|_{L^\infty(\Omega;\mathbb{R})}\le U(x)+\|U(Y_r^x)-U(x)\|_{L^\infty(\Omega;\mathbb{R})}\\
&\le U(x)+\tfrac{(2c)^p}{2}\,\Big\|\Big(1+|U(x)|^{(p-1)/p}+\|Y_r^x-x\|^{(p-1)}\Big)\,\|Y_r^x-x\|\Big\|_{L^\infty(\Omega;\mathbb{R})}\\
&\le U(x)+\tfrac{(2c)^p}{2}\,\Big[c\big(1+|U(x)|^{(p-1)/p}\big)\,(1+|U(x)|^{\gamma_5})+c^p\,(1+|U(x)|^{\gamma_5})^p\Big]\\
&\le U(x)+\tfrac{(2c)^p}{2}\,\big[2c\max(1,U(x))\cdot 2\max(1,|U(x)|^{\gamma_5})+[2c]^p\max(1,|U(x)|^{p\gamma_5})\big]\\
&\le U(x)+\tfrac{(2c)^p}{2}\,\big[4c\max(1,|U(x)|^{(\gamma_5+1)})+[2c]^p\max(1,|U(x)|^{p\gamma_5})\big]\\
&\le U(x)+\tfrac{(2c)^p}{2}\,\max(1,|U(x)|^{(p\gamma_5+1)})\,[4c+(2c)^p]\\
&\le U(x)+\tfrac{3}{2}\,[2c]^{2p}\max(1,|U(x)|^{(p\gamma_5+1)})\\
&\le 2^{(2p+1)}c^{2p}\max(1,|U(x)|^{(p\gamma_5+1)}).
\end{aligned}
\tag{53}
$$

Therefore, it holds for all $s \in (0,h]$, $x \in D_s$ that

$$\begin{aligned}
2c\int_0^s \big(1 + \|U(Y_r^x)\|_{L^\infty(\Omega;\mathbb{R})}^{\gamma_6}\big)\,dr &\leq 2sc + 2sc\left(2^{(2p+1)}c^{2p}\max\big(1,|U(x)|^{(p\gamma_5+1)}\big)\right)^{\gamma_6} \\
&\leq 2sc + 2sc\,2^{(2p+1)\gamma_6}\,c^{2p\gamma_6}\max\big(1,|U(x)|^{(p\gamma_5+1)\gamma_6}\big) \\
&\leq 2sc + s\,2^{[(2p+1)\gamma_6+1]}\,c^{(2p\gamma_6+1+(p\gamma_5+1)\gamma_6)}\max\big(1,s^{-\alpha(p\gamma_5+1)\gamma_6}\big) \\
&\leq s\,2^{[(2p+1)\gamma_6+2]}\,c^{[(p(\gamma_5+2)+1)\gamma_6+1]}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)\gamma_6}\,.
\end{aligned} \tag{54}$$

Inserting (52) and (54) into (50) then shows for all $s \in (0,h]$, $x \in D_s$ that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(2U(Y_s^x) + 2\int_0^s e^{-\rho r}\bar{U}(Y_r^x)\,dr - 2U(x)\right)\right] \\
&\leq 2\exp\Big(s\,2^{(4p+6)}\,c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)}\,[\min(s,1)]^{-\alpha(2p\gamma_5+2+\gamma_0+2\gamma_1)}\Big) \\
&\quad\cdot\exp\Big(s\,2^{[(2p+1)\gamma_6+2]}\,c^{[(p(\gamma_5+2)+1)\gamma_6+1]}\,[\min(s,1)]^{-\alpha(p\gamma_5+1)\gamma_6}\Big) \\
&\leq 2\exp\left(\left[2^{(4p+6)}\,c^{2p(\max(\gamma_0/2,\gamma_1)+\gamma_5+4)} + 2^{[(2p+1)\gamma_6+2]}\,c^{[p\gamma_6(\gamma_5+3)+1]}\right]\tfrac{s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \\
&\leq 2\exp\Big(2^{[1+(2p+3)(\gamma_6+2)]}\,c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\,[\min(s,1)]^{-\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}\,s\Big)\,.
\end{aligned} \tag{55}$$

Therefore, we obtain for all $s \in (0,h]$, $x \in D_s$ that

$$\left\|\exp\left(U(Y_s^x) + \int_0^s \tfrac{\bar{U}(Y_r^x)}{e^{\rho r}}\,dr\right)\right\|_{L^2(\Omega;\mathbb{R})} \leq \sqrt{2}\exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\,c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\,s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right)e^{U(x)}\,. \tag{56}$$

Moreover, assumption (41) and the inequality $\|\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^m)} \leq \sqrt{sr(r-1)/2}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ for all $r \in [2,\infty)$, $s \in [0,h]$ (see, e.g., Lemma 7.7 in Da Prato & Zabczyk [4]) imply for all $r \in [2,\infty)$, $s \in (0,h]$, $x \in D_s$ that

$$\begin{aligned}
&\|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} = \|\Phi(x,s,W_s) - x\|_{L^r(\Omega;\mathbb{R}^d)} \leq c\,(1+|U(x)|^{\gamma_5})\,\|\mu(x)s + \sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\leq c\left(1 + c^{\gamma_5}s^{-\alpha\gamma_5}\right)\left(\|\mu(x)\|\,s + \sqrt{sr(r-1)/2}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\right) \\
&\leq c\left(1 + c^{\gamma_5}s^{-\alpha\gamma_5}\right)\left(cs\left(1 + c^{\gamma_0}s^{-\alpha\gamma_0}\right) + c\sqrt{sr(r-1)/2}\left(1 + c^{\gamma_1}s^{-\alpha\gamma_1}\right)\right) \\
&\leq 2c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[\min(s,1)]^{-\alpha\gamma_5}\left(1 + \sqrt{r(r-1)/2}\right)\sqrt{s}\max\big(\sqrt{s}\left(1 + s^{-\alpha\gamma_0}\right), 1 + s^{-\alpha\gamma_1}\big) \\
&\leq 2c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[\min(s,1)]^{-\alpha\gamma_5}\,r\sqrt{s}\,[\max(s,1)]^{1/2}\max\big(1 + s^{-\alpha\gamma_0}, 1 + s^{-\alpha\gamma_1}\big) \\
&\leq 4rc^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}s^{1/2}\,[\max(s,1)]^{1/2}\,[\min(s,1)]^{-\alpha(\gamma_0+\gamma_1+\gamma_5)} \\
&= 4rc^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}\,.
\end{aligned} \tag{57}$$

Combining (41) and (57) with Hölder's inequality and inequality (48) yields for all $r \in [2,\infty)$, $i \in \{0,1,2\}$, $s \in (0,h]$, $x \in D_s$ that

$$\begin{aligned}
&\big\|U^{(i)}(Y_s^x) - U^{(i)}(x)\big\|_{L^r(\Omega;L^{(i)}(\mathbb{R}^d,\mathbb{R}))} \\
&\leq \left\|\tfrac{(2c)^p}{2}\left(1 + |U(x)|^{\frac{\max(p-i-1,0)}{p}} + \|Y_s^x - x\|^{\max(p-i-1,0)}\right)\|Y_s^x - x\|\right\|_{L^r(\Omega;\mathbb{R})} \\
&\leq \tfrac{(2c)^p}{2}\left(\|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} + |U(x)|^{\frac{\max(p-i-1,0)}{p}}\|Y_s^x - x\|_{L^r(\Omega;\mathbb{R}^d)} + \|Y_s^x - x\|^{\max(p-i,1)}_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)}\right) \\
&\leq \tfrac{(2c)^p}{2}\left(1 + \tfrac{c}{s^{\alpha\max(p-i-1,0)/p}} + \|Y_s^x - x\|^{\max(p-i-1,0)}_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)}\right)\|Y_s^x - x\|_{L^{r\cdot\max(p-i,1)}(\Omega;\mathbb{R}^d)} \\
&\leq \tfrac{(2c)^p}{2}\left[\tfrac{2c}{[\min(s,1)]^{\alpha}} + [crp]^{\max(p-i-1,0)}\right]4rp\,c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)}\max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]} \\
&\leq 2^{(p+1)}\,c^{(p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[2crp + [crp]^p]\max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\leq 6\,c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\,[2rp]^p\max(s,1)\,[\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]}\,.
\end{aligned} \tag{58}$$

This together with the assumption that $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, Hölder's inequality and assumptions (36) and

(38) shows for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\left\| U'(Y_s^x) \left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - U'(x)\,\mu(x) \right\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \left\| \|U'(Y_s^x)\|_{L(\mathbb{R}^d,\mathbb{R})} \left\| \left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x) \right\| \right\|_{L^2(\Omega;\mathbb{R})} + \|U'(Y_s^x) - U'(x)\|_{L^2(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \|\mu(x)\| \\
&\leq \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})} \left\| \left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x) \right\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \left[\|\mu(x)\| + \left\| \left(\tfrac{\partial}{\partial s}\Phi\right)(x,s,W_s) - \mu(x) \right\|_{L^4(\Omega;\mathbb{R}^d)} \right] \\
&\leq c\,[1+U(x)]^{\frac{(p-1)}{p}}\, c s^{\gamma_2} \\
&\quad + 6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [8p]^p \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \left[c\left(1 + c^{\gamma_0} s^{-\alpha\gamma_0}\right) + c s^{\gamma_2} \right] \qquad (59) \\
&\leq c^2 s^{\gamma_2} \left[1 + c s^{-\alpha}\right]^{(p-1)/p} \\
&+ 6\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\, [8p]^p \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]}\, [2 + s^{\gamma_2}] \\
&\leq 2c^3 s^{\gamma_2}\, [\min(s,1)]^{-\alpha} + 18\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\, [8p]^p\, [\max(s,1)]^{(1+\gamma_2)}\, [\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&= 2c^3\, [\max(s,1)]^{\gamma_2}\, [\min(s,1)]^{(\gamma_2-\alpha)} \\
&+ 18\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\, [8p]^p\, [\max(s,1)]^{(1+\gamma_2)}\, [\min(s,1)]^{[1/2-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\leq 20\, [8p]^p\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)}\, [\max(s,1)]^{(1+\gamma_2)}\, [\min(s,1)]^{[\min(\gamma_2,1/2)-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]}.
\end{aligned}
$$

Analogously, (58), the assumption that $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, Hölder's inequality and assumptions (36) and (39) show for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\left\| U'(Y_s^x) \left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - U'(x)\,\sigma(x) \right\|_{L^4(\Omega;L(\mathbb{R}^m,\mathbb{R}))} \\
&\leq \left\| \|U'(Y_s^x)\|_{L(\mathbb{R}^d,\mathbb{R})} \left\| \left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x) \right\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \right\|_{L^4(\Omega;\mathbb{R})} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})} \left\| \left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x) \right\|_{L^4(\Omega;L(\mathbb{R}^m,\mathbb{R}^d))} \\
&\quad + \|U'(Y_s^x) - U'(x)\|_{L^8(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \left[\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} + \left\| \left(\tfrac{\partial}{\partial y}\Phi\right)(x,s,W_s) - \sigma(x) \right\|_{L^8(\Omega;L(\mathbb{R}^m,\mathbb{R}^d))} \right] \\
&\leq c\,[1+U(x)]^{(p-1)/p}\, c s^{\gamma_3} \\
&\quad + 6\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)}\, [16p]^p \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \left[c\left(1 + c^{\gamma_1} s^{-\alpha\gamma_1}\right) + c s^{\gamma_3} \right] \\
&\leq c^2 s^{\gamma_3} \left[1 + c s^{-\alpha}\right]^{(p-1)/p} \\
&\quad + 6\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [16p]^p \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}\, [2 + s^{\gamma_3}] \\
&\leq 2c^3 s^{\gamma_3}\, [\min(s,1)]^{-\alpha} + 18\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [16p]^p\, [\max(s,1)]^{(1+\gamma_3)}\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\leq 20\, [16p]^p\, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)}\, [\max(s,1)]^{(1+\gamma_3)}\, [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}.
\end{aligned}
\tag{60}
$$

In the next step we note for all $A_1, A_2 \in \mathbb{R}^{d\times m}$, $B_1, B_2 \in \mathbb{R}^{d\times d}$ that

$$
\begin{aligned}
&\operatorname{trace}(A_1 A_1^* B_1 - A_2 A_2^* B_2) = \operatorname{trace}(A_1^* B_1 A_1 - A_2^* B_2 A_2) = \langle A_1, B_1 A_1 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} - \langle A_2, B_2 A_2 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&= \langle A_2, (B_1 - B_2) A_2 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \langle A_1 - A_2, B_1 A_1 \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \langle A_2, B_1 (A_1 - A_2) \rangle_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|(B_1 - B_2) A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|B_1 A_1\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad + \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|B_1 (A_1 - A_2)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq \|B_1 - B_2\|_{L(\mathbb{R}^d)} \|A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|B_1\|_{L(\mathbb{R}^d)} \left[\|A_1\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \right] \qquad (61) \\
&\leq \|B_1 - B_2\|_{L(\mathbb{R}^d)} \|A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad + \left[\|A_1 - A_2\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + 2\,\|A_1 - A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|A_2\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \right] \left[\|B_1 - B_2\|_{L(\mathbb{R}^d)} + \|B_2\|_{L(\mathbb{R}^d)} \right].
\end{aligned}
$$

We apply this inequality with $A_1 = (\frac{\partial}{\partial y}\Phi)(x,s,W_s)$, $A_2 = \sigma(x)$, $B_1 = (\operatorname{Hess} U)(Y_s^x)$ and $B_2 = (\operatorname{Hess} U)(x)$ for $s \in [0,h]$, take expectations, apply Hölder's inequality, the assumption that $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, and apply inequalities (36), (39) and (58) to obtain for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\Big\| \operatorname{trace}\Big(\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big]^* (\operatorname{Hess} U)(Y_s^x) - \sigma(x)\,\sigma(x)^*\,(\operatorname{Hess} U)(x) \Big) \Big\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \|(\operatorname{Hess} U)(Y_s^x) - (\operatorname{Hess} U)(x)\|_{L^2(\Omega;L(\mathbb{R}^d))} \, \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\quad + \Big\| \big\| \big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) - \sigma(x) \big\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + 2 \, \big\| \big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) - \sigma(x) \big\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \Big\|_{L^4(\Omega;\mathbb{R})} \\
&\quad \cdot \Big[\|(\operatorname{Hess} U)(Y_s^x) - (\operatorname{Hess} U)(x)\|_{L^4(\Omega;L(\mathbb{R}^d))} + \|(\operatorname{Hess} U)(x)\|_{L(\mathbb{R}^d)} \Big] \\
&\leq 6 \, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} \, [4p]^p \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \, 2c^2 \left(1 + c^{2\gamma_1} s^{-2\alpha\gamma_1}\right) \\
&\quad + \left[c^2 s^{2\gamma_3} + 2cs^{\gamma_3} c \left(1 + c^{\gamma_1} s^{-\alpha\gamma_1}\right) \right] \\
&\quad \cdot \Big[6 \, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} \, [8p]^p \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + c \, [1+U(x)]^{\max(p-2,0)/p} \Big] \\
&\leq 24 \, [4p]^p \, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + c^{(2+\gamma_1)} \, s^{\gamma_3} \left[s^{\gamma_3} + 2 + 2s^{-\alpha\gamma_1} \right] \\
&\quad \cdot \Big[6 \, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} \, [8p]^p \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + 2c^2 \, [\min(s,1)]^{-\alpha} \Big] \\
&\leq 24 \, [4p]^p \, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + 5 \, c^{(2p+4+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \, [\max(s,1)]^{(1+2\gamma_3)} \, [\min(s,1)]^{[\gamma_3-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \, [6 \, [8p]^p + 2] \\
&\leq 24 \, [4p]^p \, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\quad + 35 \, [8p]^p \, c^{(2p+4+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \, [\max(s,1)]^{(1+2\gamma_3)} \, [\min(s,1)]^{[\gamma_3-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\leq 47 \, [8p]^p \, c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} \, [\max(s,1)]^{(2\gamma_3+1)} \, [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \, .
\end{aligned} \tag{62}
$$

Next we apply the inequality $\big| \|a\|^2 - \|b\|^2 \big| \leq \|a-b\| \, (\|a-b\| + 2\,\|b\|)$ for all $a,b \in \mathbb{R}^m$, Hölder's inequality and inequalities (60) and (36) to obtain for all $s \in (0,h]$ satisfying $x \in D_s$ that

$$
\begin{aligned}
&\Big\| \big\| \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big]^* (\nabla U)(Y_s^x) \big\|^2 - \|\sigma(x)^* (\nabla U)(x)\|^2 \Big\|_{L^2(\Omega;\mathbb{R})} \\
&\leq \big\| \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big]^* (\nabla U)(Y_s^x) - \sigma(x)^* (\nabla U)(x) \big\|_{L^4(\Omega;\mathbb{R}^m)} \\
&\quad \cdot \Big\| \big\| \big[\big(\tfrac{\partial}{\partial y}\Phi\big)(x,s,W_s) \big]^* (\nabla U)(Y_s^x) - \sigma(x)^* (\nabla U)(x) \big\| + 2 \, \|\sigma(x)^*\|_{L(\mathbb{R}^d,\mathbb{R}^m)} \, \|(\nabla U)(x)\| \Big\|_{L^4(\Omega;\mathbb{R})} \\
&\leq 20 \, [16p]^p \, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \, [\max(s,1)]^{(1+\gamma_3)} \, [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\quad \cdot \left[\tfrac{20[16p]^p c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} [\max(s,1)]^{(1+\gamma_3)}}{[\min(s,1)]^{-[\min(\gamma_3,1/2)-\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]}} + 2c \left(1 + c^{\gamma_1} s^{-\alpha\gamma_1}\right) c \left[1 + cs^{-\alpha}\right]^{(p-1)/p} \right] \\
&\leq 20 \, [16p]^p \, c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \, [\max(s,1)]^{(1+\gamma_3)} \, [\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\quad \cdot \Big[20 \, [16p]^p \, [\max(s,1)]^{(1+\gamma_3)} + 8 \Big] \\
&\leq \left[2^9 p \right]^{2p} c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} \, [\max(s,1)]^{(2+2\gamma_3)} \, [\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \, .
\end{aligned} \tag{63}
$$

In addition, we note that Hölder's inequality, the assumption that $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$ and the inequalities (40) and (58) imply for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\|U'(Y_s^x) \, (\triangle_y \Phi)(x,s,W_s)\|_{L^2(\Omega;\mathbb{R})} \leq \|U'(Y_s^x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))} \, \|(\triangle_y \Phi)(x,s,W_s)\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\leq \Big(\|U'(Y_s^x) - U'(x)\|_{L^4(\Omega;L(\mathbb{R}^d,\mathbb{R}))} + \|U'(x)\|_{L(\mathbb{R}^d,\mathbb{R})} \Big) \, \|(\triangle_y \Phi)(x,s,W_s)\|_{L^4(\Omega;\mathbb{R}^d)} \\
&\leq \Big(6 \, [8p]^p \, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + c \, [1+U(x)]^{(p-1)/p} \Big) \, cs^{\gamma_4} \\
&\leq \Big(6 \, [8p]^p \, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} \max(s,1) \, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} + 2c^2 \, [\min(s,1)]^{-\alpha} \Big) \, cs^{\gamma_4} \\
&\leq (6 \, [8p]^p + 2) \, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)} \, [\max(s,1)]^{(\gamma_4+1)} \, [\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\leq 7 \, [8p]^p \, c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)} \, [\max(s,1)]^{(\gamma_4+1)} \, [\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \, .
\end{aligned} \tag{64}
$$

Moreover, using the inequality $\big|\bar{U}(x) - \bar{U}(y)\big| \leq c \, (1 + |U(x)|^{\gamma_7} + |U(y)|^{\gamma_7}) \, \|x-y\|$ for all $x,y \in \mathbb{R}^d$ and inequalities (53) and (57) shows for all $s \in (0,h]$, $x \in D_s$ that

$$
\begin{aligned}
&\left\|\bar{U}(Y_s^x) - \bar{U}(x)\right\|_{L^2(\Omega;\mathbb{R})} \leq \left\|c\left(1 + |U(x)|^{\gamma_7} + |U(Y_s^x)|^{\gamma_7}\right) \|Y_s^x - x\|\right\|_{L^2(\Omega;\mathbb{R})} \\
&\leq c\left[1 + |U(x)|^{\gamma_7} + \|U(Y_s^x)\|_{L^\infty(\Omega;\mathbb{R})}^{\gamma_7}\right] \|Y_s^x - x\|_{L^2(\Omega;\mathbb{R}^d)} \\
&\leq c\left[1 + |U(x)|^{\gamma_7} + \left[2^{(2p+1)} c^{2p} \max(1, |U(x)|^{(p\gamma_5+1)})\right]^{\gamma_7}\right] \tfrac{8\, c^{(2+\max(\gamma_0,\gamma_1)+\gamma_5)} \max(s,1)}{[\min(s,1)]^{-[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}} \\
&\leq \left[1 + c^{\gamma_7} s^{-\alpha\gamma_7} + 2^{(2p+1)\gamma_7} c^{(2p+p\gamma_5+1)\gamma_7} [\min(s,1)]^{-\alpha\gamma_7(p\gamma_5+1)}\right] \tfrac{8\, c^{(3+\max(\gamma_0,\gamma_1)+\gamma_5)} \max(s,1)}{[\min(s,1)]^{-[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5)]}} \\
&\leq \left[2 + 2^{(2p+1)\gamma_7}\right] 8\, c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]} \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+\gamma_7(p\gamma_5+1))]} \\
&\leq 24 \cdot 2^{(2p+1)\gamma_7} c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]} \max(s,1)\, [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+(p\gamma_5+1)\gamma_7)]}.
\end{aligned}
\tag{65}
$$

In the next step we insert (56), (58), (59), (62), (63), (64) and (65) into (47) to obtain for all $t \in (0,h]$, $x \in D_t$ that

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(e^{-\rho t} U(Y_t^x) + \int_0^t e^{-\rho r} \bar{U}(Y_r^x)\, dr\right)\right] - e^{U(x)} \\
&\leq \int_0^t \sqrt{2} \exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\, c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) e^{U(x)} \\
&\quad \cdot \Big[6\rho\, c^{(2p+2+\max(\gamma_0,\gamma_1)+\gamma_5)} [4p]^p \max(s,1) [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad + 20\, [8p]^p c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_0+\gamma_5)} [\max(s,1)]^{(1+\gamma_2)} [\min(s,1)]^{[\min(\gamma_2,1/2)-\alpha(2\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad + \tfrac{47}{2} [8p]^p c^{(2p+4+\max(\gamma_0,\gamma_1)+2\gamma_1+\gamma_5)} [\max(s,1)]^{(2\gamma_3+1)} [\min(s,1)]^{[\min(\gamma_3,1/2)-\alpha(\gamma_0+3\gamma_1+\gamma_5+1)]} \\
&\qquad + \tfrac{1}{2} \left[2^9 p\right]^{2p} c^{2(2p+3+\max(\gamma_0,\gamma_1)+\gamma_1+\gamma_5)} [\max(s,1)]^{(2+2\gamma_3)} [\min(s,1)]^{[\min(\gamma_3,1/2)-2\alpha(\gamma_0+2\gamma_1+\gamma_5+1)]} \\
&\qquad + 4\, [8p]^p c^{(2p+3+\max(\gamma_0,\gamma_1)+\gamma_5)} [\max(s,1)]^{(\gamma_4+1)} [\min(s,1)]^{[\gamma_4-\alpha(\gamma_0+\gamma_1+\gamma_5+1)]} \\
&\qquad + 24 \cdot 2^{(2p+1)\gamma_7} c^{[3+\max(\gamma_0,\gamma_1)+\gamma_5+(p(\gamma_5+2)+1)\gamma_7]} \max(s,1) [\min(s,1)]^{[1/2-\alpha(\gamma_0+\gamma_1+\gamma_5+(p\gamma_5+1)\gamma_7)]}\Big]\, ds \\
&\leq e^{U(x)} \int_0^t \sqrt{2} \exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\, c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \\
&\quad \cdot c^{[6+4p+6\max(\gamma_0,\gamma_1,\gamma_5)+p\gamma_7(\gamma_5+3)]} \left[6\rho\, [4p]^p + 48\, [8p]^p + \tfrac{1}{2}\left[2^9 p\right]^{2p} + 2^{(3p\gamma_7+5)}\right] \\
&\quad \cdot [\max(s,1)]^{\max(1+\gamma_2, 2+2\gamma_3, 1+\gamma_4)} [\min(s,1)]^{[\min(1/2,\gamma_2,\gamma_3,\gamma_4)-\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)]}\, ds.
\end{aligned}
\tag{66}
$$

This implies for all $t \in (0,h]$, $x \in D_t$ that

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\tfrac{U(Y_t^x)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(Y_r^x)}{e^{\rho r}}\, dr\right)\right] \\
&\leq e^{U(x)} + e^{U(x)} \int_0^t \max(\rho,1) \left[2^9 p\right]^{2p} 2^{3p\gamma_7} \exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\, c^{p[\max(\gamma_0,2\gamma_1)+(\gamma_5+4)(\gamma_6+2)]}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \\
&\quad \cdot \frac{c^{[6+4p+6\max(\gamma_0,\gamma_1,\gamma_5)+p\gamma_7(\gamma_5+3)]} [\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\, ds \\
&\leq e^{U(x)} \Bigg[1 + \int_0^t \exp\left(\tfrac{2^{(2p+3)(\gamma_6+2)}\, c^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \\
&\quad \cdot \tfrac{\max(\rho,1) [2pc]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\gamma_5)} [\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\, ds\Bigg] \\
&\leq e^{U(x)} \left[1 + \int_0^t \exp\left(\tfrac{[2c]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}\, s}{[\min(s,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \tfrac{\max(\rho,1)[2pc]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\gamma_5)} [\max(s,1)]^{[\max(\gamma_2,1+2\gamma_3,\gamma_4)+1]}}{[\min(s,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}}\, ds\right].
\end{aligned}
$$

Combining this with (43) proves (42) and thereby finishes the proof of Lemma 2.7. ◻

2.3 Exponential moments for stopped increment-tamed Euler-Maruyama schemes

Using Corollary 2.3 and Lemma 2.7 above, we are now ready to establish exponential moment bounds for a class of stopped increment-tamed Euler-Maruyama schemes in the next theorem.

Theorem 2.8. *Let $T, \gamma, \rho \in [0,\infty)$, $d, m \in \mathbb{N}$, $p, c \in [1,\infty)$, $q \in (1,\infty)$, $\mu \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $U \in C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, $\bar{U} \in C(\mathbb{R}^d,\mathbb{R})$, $\alpha \in \left(0, \frac{1}{2}\min\{\frac{1}{7\gamma+2}, \frac{q-1}{(q+8)\gamma+2}\}\right)$, let $D_h \in \mathcal{B}(\mathbb{R}^d)$, $h \in (0,T]$, be a non-increasing family of sets satisfying $D_h \subseteq \{x \in \mathbb{R}^d : U(x) \leq \frac{c}{h^\alpha}\}$, $\mu|_{D_h} \in C(D_h,\mathbb{R}^d)$, $\sigma|_{D_h} \in C(D_h,\mathbb{R}^{d\times m})$ for* *all $h \in (0,T]$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^h\colon [0,T]\times\Omega\to\mathbb{R}^d$, $h\in(0,T]$, be adapted stochastic processes satisfying*

$$Y_t^h = Y_{nh}^h + \mathbb{1}_{D_h}(Y_{nh}^h)\left[\frac{\mu(Y_{nh}^h)(t-nh)+\sigma(Y_{nh}^h)\,(W_t-W_{nh})}{1+\left\|\mu(Y_{nh}^h)(t-nh)+\sigma(Y_{nh}^h)\,(W_t-W_{nh})\right\|^q}\right] \tag{67}$$

for all $t\in[nh,(n+1)h]\cap[0,T]$, $h\in(0,T]$, $n\in\mathbb{N}_0$ and assume

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + |\bar{U}(x)| \le c\,(1+|U(x)|^\gamma)\,, \quad \tfrac{|\bar{U}(x)-\bar{U}(y)|}{\|x-y\|} \le c\,(1+|U(x)|^\gamma + |U(y)|^\gamma)\,, \tag{68}$$

$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \le \rho\cdot U(x) \tag{69}$$

for all $x,y\in\mathbb{R}^d$ with $x\neq y$. Then it holds for all $t,h\in(0,T]$ that

$$\limsup_{r\searrow 0}\sup_{u\in[0,T]}\mathbb{E}\left[\exp\!\left(\tfrac{U(Y_u^r)}{e^{\rho u}} + \int_0^u \tfrac{\mathbb{1}_{D_r}(Y^r_{\lfloor s\rfloor_r})\,\bar{U}(Y_s^r)}{e^{\rho s}}\,ds\right)\right] \le \limsup_{r\searrow 0}\mathbb{E}\left[e^{U(Y_0^r)}\right] \qquad \textit{and} \tag{70}$$

$$\mathbb{E}\left[\exp\!\left(\tfrac{U(Y_t^h)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(Y^h_{\lfloor s\rfloor_h})\,\bar{U}(Y_s^h)}{e^{\rho s}}\,ds\right)\right] \le \exp\!\left(\tfrac{\max(\rho,1)\,[\min(h,1)]^{[\min(1/2,(q-1)/2-\alpha(q+1)\gamma)-\alpha(7\gamma+2)]}}{\exp\left(-[5cq\max(T,1)]^{9p(q+1)\max(\gamma,1)\max(\gamma,q,2)(\gamma+2)}\right)}\right)\mathbb{E}\left[e^{U(Y_0^h)}\right].$$

Proof of Theorem 2.8. Define $\gamma_0 := \gamma_1 := \gamma_6 := \gamma_7 := \gamma$, $\gamma_2 := \gamma_3 := \frac{q}{2} - \alpha(q+1)\gamma$, $\gamma_4 := \frac{(q-1)}{2} - \alpha(q+1)\gamma$, $\gamma_5 := 0$ and $\hat{c} := \left[16c^{(1+\gamma)}q\max(T,1)\right]^{(q+1)}$. In addition, let $D_s^{(h)} \in \mathbb{R}^d$, $s\in(0,h]$, $h\in(0,T]$, be defined by $D_s^{(h)} := D_h$ for all $s\in(0,h]$, $h\in(0,T]$ and let $\Phi_h \in C^{0,1,2}(\mathbb{R}^d\times[0,h]\times\mathbb{R}^m,\mathbb{R}^d)$, $h\in(0,T]$, be defined by

$$\Phi_h(x,s,y) := x + \mathbb{1}_{D_s^{(h)}}(x)\left[\tfrac{\mu(x)s+\sigma(x)y}{1+\|\mu(x)s+\sigma(x)y\|^q}\right] = x + \mathbb{1}_{D_h}(x)\left[\tfrac{\mu(x)s+\sigma(x)y}{1+\|\mu(x)s+\sigma(x)y\|^q}\right] \tag{71}$$

for all $(x,s,y)\in\mathbb{R}^d\times[0,h]\times\mathbb{R}^m$ and all $h\in(0,T]$. We now verify step by step all assumptions of Lemma 2.7. First of all, we observe that $D_t^{(h)} = D_h \subseteq \{x\in\mathbb{R}^d\colon U(x)\le \frac{c}{h^\alpha}\} \subseteq \{x\in\mathbb{R}^d\colon U(x)\le\frac{c}{t^\alpha}\}$ for all $t\in(0,h]$, $h\in(0,T]$. Next note that $\Phi_h(x,0,0)=0$ for all $x\in\mathbb{R}^d$, $h\in(0,T]$ and that $\Phi_h(x,t,y)=x$ for all $x\in\mathbb{R}^d\backslash D_t^{(h)}$, $t\in(0,h]$, $h\in(0,T]$, $y\in\mathbb{R}^m$. Moreover, observe that (68) ensures that (36) in Lemma 2.7 is fulfilled. Furthermore, note for all $h\in(0,T]$ and all $(x,s,y)\in\mathbb{R}^d\times(0,h]\times\mathbb{R}^m$ that

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial s}\Phi_h\right)(x,s,y)\\
&= \mathbb{1}_{D_h}(x)\left[\frac{\mu(x)\,(1+\|\mu(x)s+\sigma(x)y\|^q) - (\mu(x)s+\sigma(x)y)\,q\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\,\langle\mu(x)s+\sigma(x)y,\mu(x)\rangle}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2}\right]\\
&= \mathbb{1}_{D_h}(x)\left[\mu(x) - \frac{\mu(x)\,\|\mu(x)s+\sigma(x)y\|^q}{1+\|\mu(x)s+\sigma(x)y\|^q} - \frac{q\,(\mu(x)s+\sigma(x)y)\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\,\langle\mu(x)s+\sigma(x)y,\mu(x)\rangle}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2}\right].
\end{aligned}$$

This implies for all $h\in(0,T]$, $s\in(0,h]$, $x\in D_s^{(h)}$, $y\in\mathbb{R}^m$ that

$$\left\|\left(\tfrac{\partial}{\partial s}\Phi_h\right)(x,s,y) - \mu(x)\right\| \le (q+1)\,\|\mu(x)s+\sigma(x)y\|^q\,\|\mu(x)\|\,. \tag{72}$$

Moreover, the inequality $\|\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^m)} \le \sqrt{sr(r-1)/2}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}$ for all $r\in[2,\infty)$, $s\in[0,T]$ shows for all $r\in[2,\infty)$, $h\in(0,T]$, $s\in(0,h]$, $x\in D_s^{(h)}\subseteq D_s$ that

$$\begin{aligned}
\|\mu(x)s+\sigma(x)W_s\|_{L^r(\Omega;\mathbb{R}^d)} &\le cs\,(1+|U(x)|^\gamma) + c\sqrt{sr(r-1)/2}\,(1+|U(x)|^\gamma)\\
&\le c\left(s + c^\gamma s^{(1-\alpha\gamma)}\right) + c\sqrt{r(r-1)/2}\left(s^{1/2} + c^\gamma s^{(1/2-\alpha\gamma)}\right)\\
&\le c^{(1+\gamma)}\max(T,1)\,s^{(1/2-\alpha\gamma)}\left(2+2\sqrt{r(r-1)/2}\right)\\
&\le 2c^{(1+\gamma)}r\max(T,1)\,s^{(1/2-\alpha\gamma)}.
\end{aligned} \tag{73}$$

This together with (72) and the fact $\alpha\gamma<1$ implies for all $h\in(0,T]$, $s\in(0,h]$, $x\in D_s^{(h)}\subseteq D_s$ that

$$\begin{aligned}
\left\|\left(\tfrac{\partial}{\partial s}\Phi_h\right)(x,s,W_s) - \mu(x)\right\|_{L^4(\Omega;\mathbb{R}^d)} &\le (q+1)\,\|\mu(x)s+\sigma(x)W_s\|^q_{L^{4q}(\Omega;\mathbb{R}^d)}\,\|\mu(x)\|\\
&\le (q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{q(1/2-\gamma\alpha)}\,c\,(1+|U(x)|^\gamma)\\
&\le c\,(q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{q(1/2-\gamma\alpha)}\,(1+c^\gamma s^{-\alpha\gamma})\\
&\le c^{(1+\gamma)}\,(q+1)\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{(q/2-\alpha(q+1)\gamma)}\,(s^{\alpha\gamma}+1)\\
&\le 2c^{(1+\gamma)}q\left[8c^{(1+\gamma)}q\max(T,1)\right]^q s^{(q/2-\alpha(q+1)\gamma)}\,2\,[\max(T,1)]\\
&\le \left[8c^{(1+\gamma)}q\max(T,1)\right]^{(q+1)} s^{(q/2-\alpha(q+1)\gamma)} \le \hat{c}s^{\gamma_2}.
\end{aligned} \tag{74}$$

This proves that (38) in Lemma 2.7 is fulfilled. Similarly, it holds for all $h \in (0,T]$, $(x,s,y) \in \mathbb{R}^d \times (0,h] \times \mathbb{R}^m$ that

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial y}\Phi_h\right)(x,s,y) \\
&= \mathbb{1}_{D_h}(x)\left[\frac{\sigma(x)\left(1+\|\mu(x)s+\sigma(x)y\|^q\right)-(\mu(x)s+\sigma(x)y)\,q\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\,(\mu(x)s+\sigma(x)y)^*\,\sigma(x)}{\left(1+\|\mu(x)s+\sigma(x)y\|^q\right)^2}\right] \\
&= \mathbb{1}_{D_h}(x)\left[\sigma(x)-\frac{\sigma(x)\,\|\mu(x)s+\sigma(x)y\|^q}{1+\|\mu(x)s+\sigma(x)y\|^q}-\frac{q\,(\mu(x)s+\sigma(x)y)\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\,(\mu(x)s+\sigma(x)y)^*\,\sigma(x)}{\left(1+\|\mu(x)s+\sigma(x)y\|^q\right)^2}\right].
\end{aligned}$$

This implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_s^{(h)}$, $y \in \mathbb{R}^m$ that

$$\left\|\left(\tfrac{\partial}{\partial y}\Phi_h\right)(x,s,y)-\sigma(x)\right\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq (q+1)\,\|\mu(x)s+\sigma(x)y\|^q\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}. \tag{75}$$

This together with (73) implies for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_s^{(h)} \subseteq D_s$ that

$$\begin{aligned}
\left\|\left(\tfrac{\partial}{\partial y}\Phi_h\right)(x,s,W_s)-\sigma(x)\right\|_{L^8(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))} &\leq (q+1)\,\|\mu(x)s+\sigma(x)W_s\|^q_{L^{8q}(\Omega;\mathbb{R}^d)}\,\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq (q+1)\left[16c^{(1+\gamma)}q\max(T,1)\right]^q s^{q(1/2-\gamma\alpha)}c\left(1+c^\gamma s^{-\alpha\gamma}\right) \\
&\leq c^{(1+\gamma)}\,(q+1)\left[16c^{(1+\gamma)}q\max(T,1)\right]^q s^{[q/2-\alpha(q+1)\gamma]}\,(s^{\alpha\gamma}+1) \\
&\leq 2c^{(1+\gamma)}q\left[16c^{(1+\gamma)}q\max(T,1)\right]^q s^{[q/2-\alpha(q+1)\gamma]}\,2\max(T,1) \\
&\leq \left[16c^{(1+\gamma)}q\max(T,1)\right]^{(q+1)} s^{[q/2-\alpha(q+1)\gamma]} = \hat{c}s^{\gamma_3}.
\end{aligned} \tag{76}$$

This shows that (39) in Lemma 2.7 is fulfilled. In the next step let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_i\colon \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1,\dots,m\}$, be functions satisfying $\psi(z) = \frac{z}{1+\|z\|^q}$ and $\sigma(z) = (\sigma_1(z),\sigma_2(z),\dots,\sigma_m(z))$ for all $z \in \mathbb{R}^d$. Observe that $\psi \in C^2(\mathbb{R}^d,\mathbb{R}^d)$ and that for all $z,u,v \in \mathbb{R}^d$ it holds that

$$\psi'(z)u = \begin{cases} u & : z = 0 \\ \frac{u}{1+\|z\|^q} - \frac{qz\|z\|^{(q-2)}\langle z,u\rangle}{(1+\|z\|^q)^2} & : z \neq 0 \end{cases} \qquad \text{and}$$

$$\begin{aligned}
&\psi''(z)(u,v) \\
&= \begin{cases} 0 & : z = 0 \\ -\frac{q\|z\|^{(q-2)}[u\langle z,v\rangle+v\langle z,u\rangle+z\langle u,v\rangle]}{(1+\|z\|^q)^2} - \frac{q(q-2)\|z\|^{(q-4)}z\langle z,v\rangle\langle z,u\rangle}{(1+\|z\|^q)^2} + \frac{2q^2\|z\|^{2(q-2)}z\langle z,u\rangle\langle z,v\rangle}{(1+\|z\|^q)^3} & : z \neq 0 \end{cases}.
\end{aligned}$$

This implies for all $z,u \in \mathbb{R}^d$ that

$$\psi''(z)(u,u) = \begin{cases} 0 & : z = 0 \\ \frac{2q^2\|z\|^{2(q-2)}z|\langle z,u\rangle|^2}{(1+\|z\|^q)^3} - \frac{q\|z\|^{(q-2)}\left[2u\langle z,u\rangle+z\|u\|^2\right]}{(1+\|z\|^q)^2} - \frac{q(q-2)\|z\|^{(q-4)}z|\langle z,u\rangle|^2}{(1+\|z\|^q)^2} & : z \neq 0 \end{cases}.$$

Hence, we obtain for all $i \in \{1,2,\dots,m\}$, $(x,s,y) \in \mathbb{R}^d \times (0,h] \times \mathbb{R}^m$ that

$$\begin{aligned}
&\frac{\partial^2}{\partial y_i^2}\Big(\psi\big(\mu(x)s+\sigma(x)y\big)\Big) = \frac{\partial}{\partial y_i}\Big(\psi'\big(\mu(x)s+\sigma(x)y\big)\big(\sigma_i(x)\big)\Big) = \psi''\big(\mu(x)s+\sigma(x)y\big)\big(\sigma_i(x),\sigma_i(x)\big) \\
&= \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,2q^2\,\|\mu(x)s+\sigma(x)y\|^{2(q-2)}\,(\mu(x)s+\sigma(x)y)\,|\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle|^2}{\left(1+\|\mu(x)s+\sigma(x)y\|^q\right)^3} \\
&- \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,q\,\|\mu(x)s+\sigma(x)y\|^{(q-2)}\left[2\sigma_i(x)\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle+(\mu(x)s+\sigma(x)y)\|\sigma_i(x)\|^2\right]}{(1+\|\mu(x)s+\sigma(x)y\|^q)^2} \\
&- \frac{\mathbb{1}_{\mathbb{R}^d\setminus\{0\}}(\mu(x)s+\sigma(x)y)\,q\,(q-2)\,\|\mu(x)s+\sigma(x)y\|^{(q-4)}\,(\mu(x)s+\sigma(x)y)\,|\langle\mu(x)s+\sigma(x)y,\sigma_i(x)\rangle|^2}{\left(1+\|\mu(x)s+\sigma(x)\|^q\right)^2}.
\end{aligned} \tag{77}$$

This and the Cauchy-Schwarz inequality show for all $(x,s,y) \in \mathbb{R}^d \times (0,h] \times \mathbb{R}^m$ that

$$
\begin{aligned}
&\sum_{i=1}^{m} \left\| \frac{\partial^2}{\partial y_i^2}\Big(\psi\big(\mu(x)s + \sigma(x)y\big)\Big) \right\| \\
&\leq \frac{2q^2 \left\|\mu(x)s + \sigma(x)y\right\|^{(2q-1)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\left(1 + \left\|\mu(x)s + \sigma(x)y\right\|^q\right)^3} + \frac{3q \left\|\mu(x)s + \sigma(x)y\right\|^{(q-1)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\left(1 + \left\|\mu(x)s + \sigma(x)y\right\|^q\right)^2} \\
&+ \frac{q\,|q-2| \left\|\mu(x)s + \sigma(x)y\right\|^{(q-1)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\left(1 + \left\|\mu(x)s + \sigma(x)\right\|^q\right)^2} \\
&\leq \left[2q^2 + 3q + q\,|q-2|\right] \left\|\mu(x)s + \sigma(x)y\right\|^{(q-1)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}.
\end{aligned} \tag{78}
$$

Consequently, it follows for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_s^{(h)} \subseteq D_s$, $y \in \mathbb{R}^m$ that

$$
\begin{aligned}
\left\|(\Delta_y \Phi_h)(x,s,y)\right\| &\leq \sum_{i=1}^{m} \left\|\left(\tfrac{\partial^2}{\partial y_i^2}\Phi_h\right)(x,s,W_s)\right\| \leq \sum_{i=1}^{m} \left[\left(2q^2 + 3q + q^2\right) \left\|\mu(x)s + \sigma(x)y\right\|^{(q-1)} \left\|\sigma_i(x)\right\|^2\right] \\
&= 3q\,(q+1) \left\|\mu(x)s + \sigma(x)y\right\|^{(q-1)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}.
\end{aligned} \tag{79}
$$

This together with (73) and the fact $2\alpha\gamma < 1$ yields for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_s^{(h)} \subseteq D_s$ that

$$
\begin{aligned}
&\left\|(\Delta_y \Phi_h)(x,s,W_s)\right\|_{L^4(\Omega;\mathbb{R}^d)} \leq 3q\,(q+1) \left\|\mu(x)s + \sigma(x)W_s\right\|^{(q-1)}_{L^{4(q-1)}(\Omega;\mathbb{R}^d)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq 3q\,(q+1) \left\|\mu(x)s + \sigma(x)W_s\right\|^{(q-1)}_{L^{4q}(\Omega;\mathbb{R}^d)} \left\|\sigma(x)\right\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \\
&\leq 3q\,(q+1) \left[8c^{(1+\gamma)} q \max(T,1)\right]^{(q-1)} s^{(q-1)(1/2-\gamma\alpha)}\, c^2 \left(1 + |U(x)|^{\gamma}\right)^2 \\
&\leq 6q\,(q+1)\, c^2 \left[8c^{(1+\gamma)} q \max(T,1)\right]^{(q-1)} s^{(q-1)(1/2-\gamma\alpha)} \left(1 + c^{2\gamma} s^{-2\alpha\gamma}\right) \\
&\leq 12c^{(2+2\gamma)} q^2 \left[8c^{(1+\gamma)} q \max(T,1)\right]^{(q-1)} s^{[(q-1)/2-\alpha(q+1)\gamma]}\, 2\left[\max(T,1)\right] \\
&\leq \left[8c^{(1+\gamma)} q \max(T,1)\right]^{(q+1)} s^{[(q-1)/2-\alpha(q+1)\gamma]} \leq \hat{c} s^{\gamma_4}.
\end{aligned} \tag{80}
$$

This proves that (40) in Lemma 2.7 is fulfilled. Next observe for all $h \in (0,T]$, $s \in (0,h]$, $x \in D_s^{(h)}$, $r \in [1,\infty)$ that

$$
\begin{aligned}
\left\|\Phi_h(x,s,W_s) - x\right\|_{L^r(\Omega;\mathbb{R}^d)} &\leq \min\Big(1, \left\|\mu(x)s + \sigma(x)W_s\right\|_{L^r(\Omega;\mathbb{R}^d)}\Big) \\
&\leq \hat{c} \min\Big(r, 1 + |U(x)|^{\gamma_5}, \left(1 + |U(x)|^{\gamma_5}\right) \left\|\mu(x)s + \sigma(x)W_s\right\|_{L^r(\Omega;\mathbb{R}^d)}\Big).
\end{aligned} \tag{81}
$$

This shows that (41) in Lemma 2.7 is fulfilled. Thus all assumptions of Lemma 2.7 are satisfied. Next define real numbers $\vartheta_h \in (0,\infty)$, $h \in (0,T]$, by

$$
\begin{aligned}
\vartheta_h &:= \exp\!\left(\tfrac{h\,[2\hat{c}]^{4p(\gamma_6+2)\max(\gamma_0,\gamma_1,\gamma_5,2)}}{[\min(h,1)]^{\alpha[(p\gamma_5+1)(\gamma_6+2)+\gamma_0+2\gamma_1]}}\right) \tfrac{\max(\rho,1)\,[2p\hat{c}\max(h,1)]^{6p(\gamma_7+3)\max(1,\gamma_0,\gamma_1,\ldots,\gamma_5)}}{[\min(h,1)]^{[\alpha(2\gamma_0+4\gamma_1+2\gamma_5+(p\gamma_5+1)\gamma_7+2)-\min(1/2,\gamma_2,\gamma_3,\gamma_4)]}} \\
&= \exp\!\left(\tfrac{h\,[2\hat{c}]^{4p(\gamma+2)\max(\gamma,2)}}{[\min(h,1)]^{\alpha[4\gamma+2]}}\right) \tfrac{\max(\rho,1)\,[2p\hat{c}\max(h,1)]^{3p(\gamma+3)\max(2,2\gamma,q)}}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}
\end{aligned} \tag{82}
$$

for all $h \in (0,T]$. Note that the estimates $\alpha\,[4\gamma+2]-1 < 0$ and $\alpha\,[7\gamma+2] - \min(1/2,(q-1)/2-\alpha(q+1)\gamma) < 0$ ensure that the function $(0,T] \ni h \mapsto \vartheta_h \in (0,\infty)$ is non-decreasing and that $\lim_{h\searrow 0}\vartheta_h = 0$. Combining Lemma 2.7 with the fact that $(0,T] \ni h \mapsto \vartheta_h \in (0,\infty)$ is non-decreasing implies for all $h \in (0,T]$, $(t,x) \in (0,h] \times \mathbb{R}^d$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{U(\Phi_h(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(x)\,\bar{U}(\Phi_h(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \leq \left(1 + \int_0^t \vartheta_s\,ds\right) e^{U(x)} \leq (1 + \vartheta_h t)\, e^{U(x)}. \tag{83}
$$

Clearly, this implies for all $h \in (0,T]$, $(t,x) \in (0,h] \times \mathbb{R}^d$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{\mathbb{1}_{D_h}(x)\,U(\Phi_h(x,t,W_t))}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(x)\,\bar{U}(\Phi_h(x,s,W_s))}{e^{\rho s}}\,ds\right)\right] \leq e^{\vartheta_h t + U(x)}. \tag{84}
$$

Corollary 2.3 hence yields for all $h \in (0,T]$, $t \in [0,T]$ that

$$
\mathbb{E}\!\left[\exp\!\left(\tfrac{U(Y_t^h)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(Y^h_{\lfloor r \rfloor_h})\,\bar{U}(Y_r^h)}{e^{\rho r}}\,dr\right)\right] \leq e^{\vartheta_h t}\, \mathbb{E}\!\left[e^{U(Y_0^h)}\right]. \tag{85}
$$

This implies for all $h \in (0,T]$ that

$$\sup_{t\in[0,T]} \mathbb{E}\Bigg[\exp\bigg(\tfrac{U(Y_t^h)}{e^{\rho t}} + \int_0^t \tfrac{\mathbb{1}_{D_h}(Y^h_{\lfloor s \rfloor_h})\,\bar{U}(Y_s^h)}{e^{\rho s}}\,ds\bigg)\Bigg] \leq e^{\vartheta_h T}\,\mathbb{E}\Big[e^{U(Y_0^h)}\Big]. \tag{86}$$

This and the fact $\lim_{h\searrow 0}\vartheta_h = 0$ then show (70). Next observe that the estimate $x \leq \exp\big(x^{1/20}\big)$ for all $x \in [5^{72}, \infty)$ shows for all $h \in (0,T]$ that

$$\begin{aligned}
&\vartheta_h T \\
&= \exp\left(\tfrac{h\,\left[2\left[16c^{(\gamma+1)}q\max(T,1)\right]^{(q+1)}\right]^{4p(\gamma+2)\max(\gamma,2)}}{[\min(h,1)]^{\alpha[4\gamma+2]}}\right) \tfrac{\max(\rho,1)\,T\,\left[2p\max(h,1)\left[16c^{(\gamma+1)}q\max(T,1)\right]^{(q+1)}\right]^{3p(\gamma+3)\max(2,2\gamma,q)}}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}} \\
&\leq \tfrac{\max(\rho,1)\,\exp\left([5cq\max(T,1)]^{8p\max(\gamma,1)(q+1)(\gamma+2)\max(\gamma,2)}\right)[5cpq\max(T,1)]^{6p(q+1)\max(\gamma,1)(\gamma+3)\max(2,2\gamma,q)}}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}} \\
&\leq \tfrac{\max(\rho,1)\,\exp\left([5cq\max(T,1)]^{8p(q+1)\max(\gamma,1)\max(\gamma,2)(\gamma+2)}+[5cpq\max(T,1)]^{3/10p(q+1)\max(\gamma,1)\max(2,2\gamma,q)(\gamma+3)}\right)}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}} \\
&\leq \tfrac{\max(\rho,1)\,\exp\left(2\,[5cq\max(T,1)]^{8p(q+1)\max(\gamma,1)\max(\gamma,q,2)(\gamma+2)}\right)}{[\min(h,1)]^{[\alpha[7\gamma+2]-\min(1/2,(q-1)/2-\alpha(q+1)\gamma)]}}.
\end{aligned} \tag{87}$$

Combining (86) with (87) completes the proof of Theorem 2.8. ☐

The next corollary of Theorem 2.8 considers the case where the sets $D_h \in \mathcal{B}(\mathbb{R}^d)$, $h \in (0,T]$, satisfy $\{x \in D\colon U(x) \leq c\exp\big(|\ln(h)|^{1/2}\big)\}$ for all $h \in (0,T]$ and a Borel measurable set $D \subseteq \mathbb{R}^d$ (see Corollary 2.9 below for details).

Corollary 2.9. *Let $d, m \in \mathbb{N}$, $T, \rho \in [0,\infty)$, $c, q \in (1,\infty)$, $\mu \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $\bar{U} \in C(\mathbb{R}^d,\mathbb{R})$, $D \in \mathcal{B}(\mathbb{R}^d)$, $U \in \cup_{p\in[1,\infty)} C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, let $D_h \in \mathcal{B}(\mathbb{R}^d)$, $h \in (0,T]$, be a non-increasing family of sets satisfying $D_h \subseteq \{x \in D\colon U(x) \leq c\,\exp\big(|\ln(h)|^{1/2}\big)\}$, $\mu|_{D_h} \in C(D_h,\mathbb{R}^d)$, $\sigma|_{D_h} \in C(D_h,\mathbb{R}^{d\times m})$ for all $h \in (0,T]$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^h\colon [0,T]\times\Omega \to \mathbb{R}^d$, $h \in (0,T]$, be adapted stochastic processes satisfying $\sup_{h\in(0,T]}\mathbb{E}\big[e^{U(Y_0^h)}\big] < \infty$ and*

$$Y_t^h = Y_{nh}^h + \mathbb{1}_{D_h}(Y_{nh}^h)\Big[\tfrac{\mu(Y_{nh}^h)(t-nh)+\sigma(Y_{nh}^h)(W_t-W_{nh})}{1+\|\mu(Y_{nh}^h)(t-nh)+\sigma(Y_{nh}^h)(W_t-W_{nh})\|^q}\Big] \tag{88}$$

for all $t \in [nh,(n+1)h]\cap[0,T]$, $h \in (0,T]$, $n \in \mathbb{N}_0$ and assume

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(1+\|x\|^c), \qquad |\bar{U}(x)-\bar{U}(y)| \leq c\,(1+\|x\|^c+\|y\|^c)\,\|x-y\|, \tag{89}$$

$$(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \rho\cdot U(x), \qquad \|x\|^{1/c} \leq c\,(1+U(x)) \tag{90}$$

for all $x, y \in \mathbb{R}^d$. Then $\sup_{h\in(0,T]}\sup_{t\in[0,T]}\mathbb{E}\big[\exp(e^{-\rho t}\,U(Y_t^h) + \int_0^t e^{-\rho s}\,\mathbb{1}_{D_h}(Y^h_{\lfloor s\rfloor_h})\,\bar{U}(Y_s^h)\,ds)\big] < \infty$ and $\limsup_{h\searrow 0}\sup_{t\in[0,T]}\mathbb{E}\big[\exp(e^{-\rho t}\,U(Y_t^h) + \int_0^t e^{-\rho s}\,\mathbb{1}_{D_h}(Y^h_{\lfloor s\rfloor_h})\,\bar{U}(Y_s^h)\,ds)\big] \leq \limsup_{h\searrow 0}\mathbb{E}\big[e^{U(Y_0^h)}\big]$.

Proof of Corollary 2.9. We show Corollary 2.9 through an application of Theorem 2.8. For this define real numbers $\gamma := c\,(c+1)$ and $\alpha := \frac{1}{4}\min\{\frac{1}{7\gamma+2}, \frac{q-1}{(q+8)\gamma+2}\}$ and observe that (89), (90) and the assumption $U \in \cup_{p\in[1,\infty)} C^3_{p,c}(\mathbb{R}^d,[0,\infty))$ ensure that there exist a $p \in [1,\infty)$ and a $\tilde{c} \in [c,\infty)$ such that $U \in C^3_{p,\tilde{c}}(\mathbb{R}^d,[0,\infty))$, such that for all $h \in (0,T]$ it holds that $c\exp\big(|\ln(h)|^{1/2}\big) \leq \tilde{c}\,h^{-\alpha}$ and such that for all $x, y \in \mathbb{R}^d$ with $x \neq y$ it holds that

$$\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + |\bar{U}(x)| \leq \tilde{c}\,(1+|U(x)|^\gamma), \quad |\bar{U}(x)-\bar{U}(y)| \leq \tilde{c}\,(1+|U(x)|^\gamma+|U(y)|^\gamma)\,\|x-y\|. \tag{91}$$

An application of Theorem 2.8 thus completes the proof of Corollary 2.9. ☐

Theorem 2.8 and Corollary 2.9 above establishes exponential integrability properties for a family of stopped increment-tamed Euler-Maruyama approximation schemes. Another interesting class of approximation schemes which might admit exponential integrability properties are certain *rejection- or reflection-type methods*. More formally, let $d, m \in \mathbb{N}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $D_t \in \mathcal{B}(\mathbb{R}^d)$, $t \in (0,T]$, be an appropriate non-increasing family of sets, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion and let $Y^N\colon \{0,1,\ldots,N\}\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be stochastic processes satisfying

$$Y_{n+1}^N = Y_n^N + \mathbb{1}_{\{Y_n^N+\mu(Y_n^N)\frac{T}{N}+\sigma(Y_n^N)(W_{(n+1)T/N}-W_{nT/N})\in D_{T/N}\}}\Big[\mu(Y_n^N)\tfrac{T}{N} + \sigma(Y_n^N)(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}})\Big] \tag{92}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Under suitable additional assumptions we suspect that the stochastic processes Y^N, $N \in \mathbb{N}$, also admit exponential integrability properties. In the setting of the Langevin equation, a similar class of approximation methods has been considered in Bou-Rabee & Hairer [1]. Further related approximation methods have been studied in Milstein & Tretjakov [24]. In [15] (see, e.g., Section 3.6.3 in [15]) several types of appropriately tamed schemes have been investigated. The taming often constitutes by dividing the increment of an Euler-Maruyama step through a possibly large number and thereby decreasing the increment of the tamed scheme (cf., e.g., (3.140), (3.141) and (3.145) in [15]). The larger the number by which we divide the original increment of the Euler-Maruyama step the stronger is the a priori bound that we can expect for the tamed scheme. In particular, if the increment of the Euler-Maruyama step is tamed by an appropriate exponential term, then we might obtain a scheme that admits exponential integrability properties. For instance, consider stochastic processes $Z^N\colon \{0,1,\dots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, satisfying

$$Z_{n+1}^N = Z_n^N + \frac{\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})}{\exp\Big(\big\|\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})\big\|^2\Big)} \tag{93}$$

for all $n\in\{0,1,\dots,N-1\}$ and all $N\in\mathbb{N}$ or, more generally, consider stochastic processes $Z^N\colon\{0,1,\dots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, satisfying

$$Z_{n+1}^N = Z_n^N + \frac{\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})}{\max\Big\{1, \frac{T^\alpha}{N^\alpha}\exp\Big(\frac{T^\beta}{N^\beta}\big\|\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})\big\|^2\Big)\Big\}} \tag{94}$$

for all $n\in\{0,1,\dots,N-1\}$ and all $N\in\mathbb{N}$ and some appropriate $\alpha,\beta\in\mathbb{R}$. Under suitable assumptions it might be the case that schemes of the form (93) and (94) admit exponential integrability properties.

3 Consistency and convergence of a class of stopped and tamed schemes

This subsection analyzes a class of appropriately stopped increment-tamed numerical approximation schemes for SDEs. For this analysis we use the following slightly generalized version of the consistency notion in Definition 3.1 in [15].

Definition 3.1 (Consistency)**.** *Let $T\in(0,\infty)$, $d,m\in\mathbb{N}$, let $D\subseteq\mathbb{R}^d$ be an open set and let $\mu\colon D\to\mathbb{R}^d$ and $\sigma\colon D\to\mathbb{R}^{d\times m}$ be functions. A function $\phi\colon\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ is said to be (μ,σ)-consistent with respect to Brownian motion if for all $t\in(0,T]$ it holds that $\mathbb{R}^d\times\mathbb{R}^m\ni(x,y)\mapsto\phi(x,t,y)\in\mathbb{R}^d$ is Borel measurable and if there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a standard Brownian motion $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ such that for all non-empty compact sets $K\subseteq D$ it holds that*

$$\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\big[\|\sigma(x)W_t - \phi(x,t,W_t)\|\big]\right) = 0 = \limsup_{t\searrow 0}\left(\sup_{x\in K}\big\|\mu(x) - \tfrac{1}{t}\cdot\mathbb{E}[\phi(x,t,W_t)]\big\|\right). \tag{95}$$

In Definition 3.1 in [15], the increment function ϕ is assumed to be Borel measurable while in Definition 3.1 above the increment function ϕ does not need to be Borel measurable in all three arguments $(x,t,y)\in\mathbb{R}^d\times(0,T]\times\mathbb{R}^m$ (see Definition 3.1 for details). In Proposition 3.4 below it is shown under suitable assumptions that if a numerical one-step scheme is consistent in the sense of Definition 3.1, then it converges in probability to the exact solution of the considered SDE (cf. also Corollaries 3.11–3.13 in [15] for strong convergence results based on consistency).

3.1 Consistency of stopped schemes

The next lemma establishes consistency of appropriately stopped numerical approximation schemes.

Lemma 3.2. *Let $T\in(0,\infty)$, $d,m\in\mathbb{N}$, let $D\subseteq\mathbb{R}^d$ be an open set, let $\mu\colon D\to\mathbb{R}^d$ and $\sigma\colon D\to\mathbb{R}^{d\times m}$ be functions, let $\phi\colon\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ be (μ,σ)-consistent with respect to Brownian motion and let $D_t\in\mathcal{B}(\mathbb{R}^d)$, $t\in(0,T]$, be a non-increasing family of sets satisfying $D\subseteq\cup_{t\in(0,T]}\mathring{D}_t$. Then the function $\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\ni(x,t,y)\mapsto\mathbb{1}_{D_t}(x)\cdot\phi(x,t,y)\in\mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion.*

Proof of Lemma 3.2. Throughout this proof let $K\subseteq D$ be an arbitrary non-empty compact subset of D. The fact that K is a compact set together with the assumption that $D\subseteq\cup_{t\in(0,T]}\mathring{D}_t$ ensures that there exist a real number $t_K\in(0,T]$ such that $K\subseteq\mathring{D}_{t_K}$. The fact that the family D_t, $t\in(0,T]$, is non-increasing then shows for all $t\in(0,t_K]$ that

$$\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t - \mathbb{1}_{D_t}(x)\,\phi(x,t,W_t)\big\|\Big] = \tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t - \phi(x,t,W_t)\big\|\Big] \tag{96}$$

and

$$\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\mathbb{1}_{D_t}(x)\,\phi(x,t,W_t)\big]\big\| = \sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\phi(x,t,W_t)\big]\big\|. \tag{97}$$

Combining this with the assumption that ϕ is (μ,σ)-consistent with respect to Brownian motion implies

$$\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t-\mathbb{1}_{D_t}(x)\phi(x,t,W_t)\big\|\Big]\right)=0 \tag{98}$$

and

$$\limsup_{t\searrow 0}\left(\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\mathbb{1}_{D_t}(x)\phi(x,t,W_t)\big]\big\|\right)=0. \tag{99}$$

Combining (98) and (99) with Definition 3.1 completes the proof of Lemma 3.2. □

3.2 Consistency of a class of incremented-tamed Euler-Maruyama schemes

The following lemma proves consistency for a class of increment-tamed Euler-Maruyama approximation schemes for SDEs.

Lemma 3.3. *Let $T\in(0,\infty)$, $q\in[1,\infty)$, $d,m\in\mathbb{N}$, let $D\subseteq\mathbb{R}^d$ be an open set and let $\mu\in\mathcal{L}^0(D;\mathbb{R}^d)$ and $\sigma\in\mathcal{L}^0(D;\mathbb{R}^{d\times m})$ be locally bounded. Then the function $\mathbb{R}^d\times(0,T]\times\mathbb{R}^m\ni(x,t,y)\mapsto\frac{\mu(x)t+\sigma(x)y}{1+\|\mu(x)t+\sigma(x)y\|^q}\in\mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion.*

Proof of Lemma 3.3. It holds for all non-empty compact sets $K\subseteq D$ that

$$\begin{aligned}
&\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\left[\left\|\sigma(x)W_t-\frac{\mu(x)t+\sigma(x)W_t}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right\|\right]\right)\\
&=\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\left[\left\|\frac{\sigma(x)W_t\,\|\mu(x)t+\sigma(x)W_t\|^q-\mu(x)t}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right\|\right]\right)\\
&\leq\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t\,\|\mu(x)t+\sigma(x)W_t\|^q-\mu(x)t\big\|\Big]\right)\\
&\leq\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\big[\,\|\sigma(x)W_t\|\,\|\mu(x)t+\sigma(x)W_t\|^q\,\big]\right)+\limsup_{t\searrow 0}\left(\sqrt{t}\cdot\sup_{x\in K}\|\mu(x)\|\right)\\
&\leq 2^{(q-1)}\cdot\limsup_{t\searrow 0}\left(t^{\left(q-\frac{1}{2}\right)}\left[\sup_{x\in K}\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\,\|\mu(x)\|^q\right]\mathbb{E}\big[\|W_t\|\big]\right)\\
&\quad+2^{(q-1)}\cdot\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\left[\sup_{x\in K}\|\sigma(x)\|^{(1+q)}_{L(\mathbb{R}^m,\mathbb{R}^d)}\right]\mathbb{E}\Big[\|W_t\|^{(1+q)}\Big]\right)=0.
\end{aligned}\tag{100}$$

In addition, note for all non-empty compact sets $K\subseteq D$ that

$$\begin{aligned}
&\limsup_{t\searrow 0}\left(\sup_{x\in K}\left\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\left[\frac{\mu(x)t+\sigma(x)W_t}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right]\right\|\right)\\
&\leq\limsup_{t\searrow 0}\left(\sup_{x\in K}\left\|\mathbb{E}\left[\frac{\mu(x)\,\|\mu(x)t+\sigma(x)W_t\|^q}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right]\right\|\right)+\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\left\|\mathbb{E}\left[\frac{\sigma(x)W_t}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right]\right\|\right)\\
&\leq\limsup_{t\searrow 0}\left(\sup_{x\in K}\mathbb{E}\big[\,\|\mu(x)\|\,\|\mu(x)t+\sigma(x)W_t\|^q\,\big]\right)\\
&\quad+\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\left\|\mathbb{E}\left[\frac{\sigma(x)W_t}{1+\|\mu(x)t+\sigma(x)W_t\|^q}-\sigma(x)W_t\right]\right\|\right)\\
&=\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\left\|\mathbb{E}\left[\frac{\sigma(x)W_t\,\|\mu(x)t+\sigma(x)W_t\|^q}{1+\|\mu(x)t+\sigma(x)W_t\|^q}\right]\right\|\right)\\
&\leq\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\mathbb{E}\big[\,\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\,\|W_t\|\,\|\mu(x)t+\sigma(x)W_t\|^q\,\big]\right)=0.
\end{aligned}\tag{101}$$

Combining (100) and (101) with Definition 3.1 completes the proof of Lemma 3.3. □

3.3 Convergence of stopped increment-tamed Euler-Maruyama schemes

This subsection establishes consistency, convergence in probability, strong convergence and numerically weak convergence of a class of stopped increment-tamed Euler-Maruyama schemes.

3.3.1 Setting

Throughout Subsection 3.3 the following setting is frequently used. Let $T \in (0,\infty)$, $d,m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\mu\colon D \to \mathbb{R}^d$ and $\sigma\colon D \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions and let $X\colon [0,T]\times\Omega \to D$ be an adapted stochastic process with continuous sample paths satisfying

$$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{102}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$.

3.3.2 Convergence in probability of appropriate time-continuous interpolations

The next proposition, Proposition 3.4, is a slight generalization of Theorem 3.3 in [15]. The proof of Proposition 3.4 is entirely analogous to the proof of Theorem 3.3 in [15] and therefore omitted.

Proposition 3.4. *Assume the setting in Subsection 3.3.1, let $\phi\colon \mathbb{R}^d \times (0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be (μ,σ)-consistent and let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings satisfying for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y_0^N = X_0$ and $Y_t^N = Y^N_{\frac{nT}{N}} + \big(\frac{tN}{T} - n\big)\cdot \phi\big(Y^N_{\frac{nT}{N}}, \frac{T}{N}, W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)$. Then it holds for all $\varepsilon \in (0,\infty)$ that $\lim_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0$.*

The next proposition is an extension of Proposition 3.4 and proves convergence in probability of suitable time-continuous interpolations of numerical approximation processes of consistent schemes.

Proposition 3.5. *Assume the setting in Subsection 3.3.1, let $\Psi\colon \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ be a function satisfying that $\mathbb{R}^d \times (0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \Psi(x,t,t,y) \in \mathbb{R}^d$ is (μ,σ)-consistent, let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be stochastic processes with continuous sample paths satisfying for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in (\frac{nT}{N}, \frac{(n+1)T}{N}]$ that $Y_0^N = X_0$ and $Y_t^N = Y^N_{\frac{nT}{N}} + \Psi\big(Y^N_{\frac{nT}{N}}, \frac{T}{N}, t - \frac{nT}{N}, W_t - W_{\frac{nT}{N}}\big)$, assume for all non-empty compact sets $K \subseteq D$ that there exists a $h_K \in (0,T]$ such that for all $h \in (0,h_K]$ it holds that $\sup_{x\in K}\sup_{t\in[0,T]} \|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\|$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable and assume for all non-empty compact sets $K \subseteq D$ that*

$$\limsup_{h\searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \big\|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\big\|\right] = 0. \tag{103}$$

Then it holds for all $\varepsilon \in (0,\infty)$ that $\lim_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0$.

Proof of Proposition 3.5. The triangle inequality implies for all $N \in \mathbb{N}$ that

$$\sup_{t\in[0,T]} \big\|X_t - Y_t^N\big\| \leq \sup_{t\in[0,T]} \Big\|X_t - X_{\lfloor t\rfloor_{T/N}}\Big\| + \sup_{t\in[0,T]} \Big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| + \sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\|. \tag{104}$$

Combining this, Proposition 3.4 and continuity of X_t, $t \in [0,T]$, implies for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \big\|X_t - Y_t^N\big\| \geq \varepsilon\right] \\
&\leq \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \Big\|X_t - X_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\right] + \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \Big\|X_{\lfloor t\rfloor_{T/N}} - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\right] \\
&+ \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\right] \\
&= \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \Big\|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\Big\| \geq \tfrac{\varepsilon}{3}\right].
\end{aligned} \tag{105}$$

It thus remains to prove that $\sup_{t\in[0,T]} \|Y_t^N - Y^N_{\lfloor t\rfloor_{T/N}}\|$ converges to zero in probability as $N \to \infty$. To prove this we define a sequence $D_v \subseteq D$, $v \in \mathbb{N}$, of open sets by $D_v := \{x \in D\colon \|x\| < v \text{ and } \operatorname{dist}(x, D^c) > \frac{1}{v}\}$ for all

$v \in \mathbb{N}$. Then (103) and Markov's inequality show for all $\varepsilon \in (0,\infty)$ and all $N, v \in \mathbb{N}$ that

$$
\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup_{t\in[0,T]} \big\|Y_t^N - Y_{\lfloor t\rfloor_{T/N}}^N\big\| \geq \varepsilon\Big\} \cap \Big\{\forall\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \in D_v\Big\}\Big] \\
&= \limsup_{N\to\infty} \mathbb{P}\Big[\mathbb{1}_{\{\forall\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \in D_v\}}\big(\sup_{t\in[0,T]} \big\|Y_t^N - Y_{\lfloor t\rfloor_{T/N}}^N\big\|\big) \geq \varepsilon\Big] \\
&\leq \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{\{\forall\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \in D_v\}}\big(\sup_{t\in[0,T]} \big\|Y_t^N - Y_{\lfloor t\rfloor_{T/N}}^N\big\|\big)\Big] \\
&= \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{\{\forall\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \in D_v\}}\big(\sup_{t\in[0,T]} \big\|\Psi\big(Y_{\lfloor t\rfloor_{T/N}}^N, \tfrac{T}{N}, t-\lfloor t\rfloor_{T/N}, W_t - W_{\lfloor t\rfloor_{T/N}}\big)\big\|\big)\Big] \\
&\leq \frac{1}{\varepsilon} \limsup_{N\to\infty} \mathbb{E}\Big[\sup_{x\in\overline{D_v}} \sup_{t\in[0,T]} \big\|\Psi\big(x, \tfrac{T}{N}, t-\lfloor t\rfloor_{T/N}, W_t - W_{\lfloor t\rfloor_{T/N}}\big)\big\|\Big] = 0.
\end{aligned}
\tag{106}
$$

In addition, Proposition 3.4 and the continuity of the sample paths of X imply for all $\varepsilon \in (0,\infty)$ that

$$
\begin{aligned}
&\limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup_{t\in[0,T]} \big\|Y_t^N - Y_{\lfloor t\rfloor_{T/N}}^N\big\| \geq \varepsilon\Big\} \cap \Big\{\exists\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \notin D_{2v}\Big\}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\exists\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \notin D_{2v}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y_{\lfloor t\rfloor_{T/N}}^N\big\| < \tfrac{1}{2v}\Big\} \cap \Big\{\exists\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \notin D_{2v}\Big\}\Big] \\
&+ \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\sup_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y_{\lfloor t\rfloor_{T/N}}^N\big\| \geq \tfrac{1}{2v}\Big] \\
&= \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\Big\{\sup_{t\in[0,T]} \big\|X_{\lfloor t\rfloor_{T/N}} - Y_{\lfloor t\rfloor_{T/N}}^N\big\| < \tfrac{1}{2v}\Big\} \cap \Big\{\exists\, t\in[0,T]\colon Y_{\lfloor t\rfloor_{T/N}}^N \notin D_{2v}\Big\}\Big] \\
&\leq \limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\Big[\exists\, t\in[0,T]\colon X_{\lfloor t\rfloor_{T/N}} \notin D_v\Big] \leq \limsup_{v\to\infty} \mathbb{P}\big[\exists\, t\in[0,T]\colon X_t \notin D_v\big] = 0.
\end{aligned}
\tag{107}
$$

Combining (106) and (107) proves that $\sup_{t\in[0,T]} \|Y_t^N - Y_{\lfloor t\rfloor_{T/N}}^N\|$ converges in probability to zero as N tends to infinity. The proof of Proposition 3.5 is thus completed. ∎

3.3.3 Convergence of stopped increment-tamed Euler-Maruyama schemes

Combining Lemma 3.2 and Lemma 3.3 immediately proves the following consistency result.

Corollary 3.6. *Let $T \in (0,\infty)$, $q \in [1,\infty)$, $d, m \in \mathbb{N}$, let $D \subseteq \mathbb{R}^d$ be an open set, let $D_t \in \mathcal{B}(\mathbb{R}^d)$, $t \in (0,T]$, be a non-increasing family of sets satisfying $D \subseteq \cup_{t\in(0,T]} \mathring{D}_t$ and let $\mu \in \mathcal{L}^0(D;\mathbb{R}^d)$ and $\sigma \in \mathcal{L}^0(D;\mathbb{R}^{d\times m})$ be locally bounded. Then the function $\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \frac{\mathbb{1}_{D_t}(x)\,[\mu(x)t+\sigma(x)y]}{1+\|\mu(x)t+\sigma(x)y\|^q} \in \mathbb{R}$ is (μ,σ)-consistent with respect to Brownian motion.*

Combining Corollary 3.6 with Proposition 3.5 shows that the stopped increment-tamed Euler-Maruyama schemes converge in probability. This is the subject of the next result.

Corollary 3.7. *Assume the setting in Subsection 3.3.1, let $q \in [1,\infty)$, let $D_t \in \mathcal{B}(\mathbb{R}^d)$, $t \in (0,T]$, be a non-increasing family of sets satisfying $D \subseteq \cup_{t\in(0,T]} \mathring{D}_t$ and let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be mappings satisfying $Y_0^N = X_0$ and*

$$
Y_t^N = Y_{\frac{nT}{N}}^N + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y_{\frac{nT}{N}}^N\big)\left[\frac{\mu(Y_{\frac{nT}{N}}^N)\big(t-\frac{nT}{N}\big) + \sigma(Y_{\frac{nT}{N}}^N)\big(W_t - W_{\frac{nT}{N}}\big)}{1+\big\|\mu(Y_{\frac{nT}{N}}^N)\big(t-\frac{nT}{N}\big) + \sigma(Y_{\frac{nT}{N}}^N)\big(W_t - W_{\frac{nT}{N}}\big)\big\|^q}\right]
\tag{108}
$$

for all $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Then it holds for all $\varepsilon \in (0,\infty)$ that $\lim_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0$. Moreover, if $p \in (0,\infty)$ satisfies $\limsup_{N\to\infty}\big(\sup_{t\in[0,T]} \mathbb{E}\big[\|Y_t^N\|^p\big]\big) < \infty$, then it holds for all $q \in (0,p)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^q\big]\big) = 0$ and $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^p\big] < \infty$. In addition, if $f\colon C([0,T],\mathbb{R}^d) \to \mathbb{R}$ is a continuous function which satisfies $\limsup_{p\searrow 1} \limsup_{N\to\infty} \mathbb{E}\big[|f(Y^N)|^p\big] < \infty$, then $\lim_{N\to\infty} \mathbb{E}\big[f(Y^N)\big] = \mathbb{E}\big[f(X)\big]$.

Proof of Corollary 3.7. Throughout this proof let $\Psi\colon \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m \to \mathbb{R}^d$ be defined by

$$
\Psi(x,t,s,y) := \mathbb{1}_{D_t}(x)\left[\frac{\mu(x)s + \sigma(x)y}{1+\|\mu(x)s+\sigma(x)y\|^q}\right]
\tag{109}
$$

for all $(x,t,s,y) \in \mathbb{R}^d \times (0,T]^2 \times \mathbb{R}^m$. Then Corollary 3.6 implies that $\mathbb{R}^d \times (0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto \Psi(x,t,t,y) \in \mathbb{R}^d$ is (μ,σ)-consistent. In addition, observe for all non-empty compact sets $K \subseteq D$ that

$$
\begin{aligned}
&\limsup_{h\searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\|\Psi(x,h,t-\lfloor t\rfloor_h, W_t - W_{\lfloor t\rfloor_h})\right\|\right] \\
&= \limsup_{h\searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\| \mathbb{1}_{D_h}(x) \left[\frac{\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})}{1+\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|^q}\right]\right\|\right] \\
&\leq \limsup_{h\searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left(\frac{\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|}{1+\left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|^q}\right)\right] \\
&\leq \limsup_{h\searrow 0} \mathbb{E}\left[\sup_{x\in K}\sup_{t\in[0,T]} \left\|\mu(x)(t-\lfloor t\rfloor_h) + \sigma(x)(W_t - W_{\lfloor t\rfloor_h})\right\|\right] \\
&\leq \left(\limsup_{h\searrow 0} h\right)\left(\sup_{x\in K}\|\mu(x)\|\right) + \left(\limsup_{h\searrow 0}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|W_t - W_{\lfloor t\rfloor_h}\right\|\right]\right)\left(\sup_{x\in K}\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\right) = 0.
\end{aligned}
\tag{110}
$$

Proposition 3.5 hence shows for all $\varepsilon \in (0,\infty)$ that $\lim_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - Y_t^N\| \geq \varepsilon\big] = 0$. The proof of the strong convergence statement in Corollary 3.7 is entirely analogous to the proof of Corollary 3.12 in [15] and thus omitted. It thus remains to prove the weak convergence statement in Corollary 3.7. For this assume that $p \in (1,\infty)$ is a real number, that $N_0 \in \mathbb{N}$ is a natural number and that $f\colon C([0,T],\mathbb{R}^d) \to \mathbb{R}$ is a continuous function with $\sup_{N\in\{N_0,N_0+1,\dots\}} \mathbb{E}\big[|f(Y^N)|^p\big] < \infty$. The fact that $\sup_{t\in[0,T]} \|X_t - Y_t^N\|$ converges in probability to zero as $N \to \infty$ together with, e.g., Lemma 3.10 in [15] proves then that

$$
\mathbb{E}\big[|f(X)|^p\big] < \infty \qquad \text{and} \qquad \forall\, \varepsilon \in (0,\infty)\colon \quad \lim_{N\to\infty} \mathbb{P}\big[|f(X) - f(Y^N)| \geq \varepsilon\big] = 0. \tag{111}
$$

This shows that the family $|f(X) - f(Y^N)|$, $N \in \{N_0, N_0+1, \dots\}$, of random variables is uniformly integrable. Combining this and (111) with, e.g., Corollary 6.21 in Klenke [19] proves that $\lim_{N\to\infty} \mathbb{E}\big[|f(X) - f(Y^N)|\big] = 0$. The proof of Corollary 3.7 is thus completed. ☐

Combining Corollary 3.7 with Corollary 2.9 and Fatou's lemma results in Corollary 3.8. Corollary 3.8 establishes both exponential integrability properties and for any $r \in [0,\infty)$ strong L^r-convergence.

Corollary 3.8. *Let $d,m \in \mathbb{N}$, $T,\rho \in [0,\infty)$, $\kappa \in (0,\infty)$, $c,q \in (1,\infty)$, $\mu \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $\bar{U} \in C(\mathbb{R}^d,[-c,\infty))$, $U \in \cup_{p\in[1,\infty)} C^3_{p,c}(\mathbb{R}^d,[0,\infty))$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $X\colon [0,T]\times\Omega \to D$ be an adapted stochastic process with continuous sample paths satisfying $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$ and $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$ $\mathbb{P}$-a.s. for all $t \in [0,T]$, let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $\tau_N\colon \Omega \to [0,T]$, $N \in \mathbb{N}$, be mappings satisfying $\tau_N = \inf\big(\{n \in \{0, \frac{T}{N}, \frac{2T}{N}, \dots, T\}\colon Y^N_{nT/N} \notin D \text{ or } \|Y^N_{nT/N}\| > \exp(|\ln(T/N)|^{1/2})\} \cup \{T\}\big)$, $Y_0^N = X_0$ and*

$$
Y_t^N = Y^N_{\frac{nT}{N}} + \mathbb{1}_{\left\{Y^N_{nT/N}\in D \text{ and } \|Y^N_{nT/N}\| \leq \exp(|\ln(T/N)|^{1/2})\right\}} \left[\frac{\mu(Y^N_{\frac{nT}{N}})\big(t - \frac{nT}{N}\big) + \sigma(Y^N_{\frac{nT}{N}})\big(W_t - W_{\frac{nT}{N}}\big)}{1 + \left\|\mu(Y^N_{\frac{nT}{N}})\big(t-\frac{nT}{N}\big) + \sigma(Y^N_{\frac{nT}{N}})\big(W_t - W_{\frac{nT}{N}}\big)\right\|^q}\right] \tag{112}
$$

for all $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}$, and assume that $\mu|_D\colon D \to \mathbb{R}^d$ and $\sigma|_D\colon D \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous and that

$$
\|\mu(x)\| + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,(1 + \|x\|^c), \qquad |\bar{U}(x) - \bar{U}(y)| \leq c\,(1 + \|x\|^c + \|y\|^c)\,\|x - y\|, \tag{113}
$$

$$
(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \rho \cdot U(x), \qquad \|x\|^{1/c} \leq c\,(1 + U(x)) \tag{114}
$$

for all $x,y \in \mathbb{R}^d$. Then it holds for all $r \in (0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$, that

$$
\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\,ds\right)\right] \leq \sup_{N\in\mathbb{N}}\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(Y_t^N)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \tfrac{\bar{U}(Y_s^N)}{e^{\rho s}}\,ds\right)\right] < \infty \quad \textit{and that} \tag{115}
$$

$$
\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(X_t)}{e^{\rho t}} + \int_0^t \tfrac{\bar{U}(X_s)}{e^{\rho s}}\,ds\right)\right] \leq \limsup_{N\to\infty}\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\tfrac{U(Y_t^N)}{e^{\rho t}} + \int_0^{t\wedge\tau_N} \tfrac{\bar{U}(Y_s^N)}{e^{\rho s}}\,ds\right)\right] \leq \mathbb{E}\Big[e^{U(X_0)}\Big]. \tag{116}
$$

4 Examples of SDEs with exponential moments

In this section Corollary 3.8 is applied to a number of example SDEs from the literature. To keep this article at a reasonable length, we present the example SDEs here in a very brief way and refer to [15, 3] for references and further details for these example SDEs.

4.1 Setting

Throughout Section 4 the following setting is used. Let $T \in (0,\infty)$, $d,m \in \mathbb{N}$, $\mu \in C(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \in C(\mathbb{R}^d,\mathbb{R}^{d\times m})$, let $D \subseteq \mathbb{R}^d$ be an open set, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, assume that $\mu|_D\colon D\to\mathbb{R}^d$ and $\sigma|_D\colon D\to\mathbb{R}^{d\times m}$ are locally Lipschitz continuous, let $W\colon [0,T]\times\Omega\to\mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $X=(X^1,\ldots,X^d)\colon[0,T]\times\Omega\to D$ be an adapted stochastic process with continuous sample paths satisfying

$$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{117}$$

$\mathbb{P}$-a.s. for all $t\in[0,T]$ and let $Y^N=(Y^{1,N},\ldots,Y^{d,N})\colon[0,T]\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, and $\tau_N\colon\Omega\to[0,T]$, $N\in\mathbb{N}$, be mappings satisfying $Y_0^N=X_0$ and

$$Y_t^N = Y^N_{\frac{nT}{N}} + \mathbb{1}_{\left\{Y^N_{nT/N}\in D\right\}\cap\left\{\|Y^N_{nT/N}\|\le\exp(|\ln(T/N)|^{1/2})\right\}}\left[\frac{\mu(Y^N_{nT/N})\frac{T}{N}+\sigma(Y^N_{nT/N})(W_{(n+1)T/N}-W_{nT/N})}{1+\|\mu(Y^N_{nT/N})\frac{T}{N}+\sigma(Y^N_{nT/N})(W_{(n+1)T/N}-W_{nT/N})\|^2}\right] \tag{118}$$

and $\tau_N=\inf\left\{n\in\{0,1,\ldots,N\}\colon Y^N_{nT/N}\notin D \text{ or } \|Y^N_{nT/N}\|>\exp(|\ln(T/N)|^{1/2})\right\}$ for all $t\in[0,T]$ and all $N\in\mathbb{N}$. Then Corollary 3.7 ensures for all $\varepsilon\in(0,\infty)$ that $\lim_{N\to\infty}\mathbb{P}\big[\sup_{t\in[0,T]}\|X_t-Y_t^N\|\ge\varepsilon\big]=0$.

4.2 Stochastic Ginzburg-Landau equation

In this subsection assume the setting in Subsection 4.1, let $\alpha\in[0,\infty)$, $\beta,\delta\in(0,\infty)$ and assume that $d=m=1$, $D=\mathbb{R}$, $\mu(x)=\alpha x-\delta x^3$ and $\sigma(x)=\beta x$ for all $x\in\mathbb{R}$. If $\varepsilon\in(0,\frac{\delta}{\beta^2}]$, $U,\bar U\in C(\mathbb{R}^d,\mathbb{R})$ fulfill $U(x)=\varepsilon x^2$ and $\bar U(x)=2\varepsilon\,[\delta-\beta^2\varepsilon]\,x^4$ for all $x\in\mathbb{R}$ and if $\mathbb{E}\big[e^{U(X_0)}\big]<\infty$, then it holds for all $x\in\mathbb{R}$ that

$$\begin{aligned}(\mathcal{G}_{\mu,\sigma}U)(x)+\tfrac12\|\sigma(x)^*(\nabla U)(x)\|^2+\bar U(x)&=\varepsilon\left[2x\left[\alpha x-\delta x^3\right]+\beta^2x^2\right]+2\left(\beta\varepsilon\right)^2x^4+\bar U(x)\\&=\varepsilon\left[2\alpha+\beta^2\right]x^2+2\varepsilon\left[\beta^2\varepsilon-\delta\right]x^4+\bar U(x)=\left[2\alpha+\beta^2\right]U(x)\end{aligned} \tag{119}$$

and hence Corollary 3.8 shows for all $r\in(0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t-Y_t^N\|^r\big]\big)=0$ and

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\frac{\varepsilon\,(Y_t^N)^2}{e^{[2\alpha+\beta^2]t}}+\int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\delta-\beta^2\varepsilon]\,(Y_s^N)^4}{e^{[2\alpha+\beta^2]s}}\,ds\right)\right]<\infty, \tag{120}$$

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\frac{\varepsilon\,(Y_t^N)^2}{e^{[2\alpha+\beta^2]t}}+\int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\delta-\beta^2\varepsilon]\,(Y_s^N)^4}{e^{[2\alpha+\beta^2]s}}\,ds\right)\right]\le\mathbb{E}\left[e^{\varepsilon|X_0|^2}\right]<\infty. \tag{121}$$

4.3 Stochastic Lorenz equation with additive noise

In this subsection assume the setting in Subsection 4.1, let $\alpha_1,\alpha_2,\alpha_3,\beta\in[0,\infty)$, assume that $d=m=3$, $D=\mathbb{R}^3$, $\sigma(x)=\sqrt{\beta}I_{\mathbb{R}^3}$ for all $x\in\mathbb{R}^3$ and assume that μ is as in Subsection 4.3 in Cox et al. [3]. If $\varepsilon\in(0,\infty)$, $U,\bar U\in C(\mathbb{R}^3,\mathbb{R})$ fulfill $U(x)=\varepsilon\|x\|^2$ and $\bar U(x)=-3\varepsilon\beta$ for all $x\in\mathbb{R}^3$, if $\vartheta:=\min_{r\in(0,\infty)}\max\{(\alpha_1+\alpha_2)^2/r-2\alpha_1,r-1,0\}\in[0,\infty)$ and if $\mathbb{E}\big[e^{U(X_0)}\big]<\infty$, then it holds for all $x\in\mathbb{R}^3$ that $(\mathcal{G}_{\mu,\sigma}U)(x)+\frac12\|\sigma(x)^*(\nabla U)(x)\|^2+\bar U(x)\le[2\varepsilon\beta+\vartheta]\,U(x)$ (see Subsection 4.3 in Cox et al. [3]) and hence Corollary 3.8 shows for all $r\in(0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t-Y_t^N\|^r\big]\big)=0$, $\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\varepsilon\,\|Y_t^N\|^2\,e^{-[2\varepsilon\beta+\vartheta]t}\big)\big]<\infty$ and

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\frac{\varepsilon\,\|Y_t^N\|^2}{e^{[2\varepsilon\beta+\vartheta]t}}\right)\right]\le\exp\left(\int_0^T\frac{3\varepsilon\beta}{e^{[2\varepsilon\beta+\vartheta]s}}\,ds\right)\mathbb{E}\left[e^{\varepsilon\|X_0\|^2}\right]<\infty. \tag{122}$$

4.4 Stochastic van der Pol oscillator

In this subsection assume the setting in Subsection 4.1, let $\alpha\in(0,\infty)$, $\gamma,\delta,\eta_0,\eta_1\in[0,\infty)$, let $g\colon\mathbb{R}\to\mathbb{R}^{1\times m}$ be a globally Lipschtiz continuous function with $\|g(y)\|^2\le\eta_0+\eta_1y^2$ for all $y\in\mathbb{R}$ and assume that $d=2$, $D=\mathbb{R}^2$, $\mu(x)=\big(x_2,\big(\gamma-\alpha(x_1)^2\big)\,x_2-\delta x_1\big)$ and $\sigma(x)u=(0,g(x_1)u)$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, $u\in\mathbb{R}^m$. If $\varepsilon\in(0,\infty)$, $U,\overline{U}\in C(\mathbb{R}^2,\mathbb{R})$ fulfill $\varepsilon\eta_1\le\alpha$, $U(x)=\varepsilon\|x\|^2$ and $\overline{U}(x)=2\varepsilon\,[\alpha-\varepsilon\eta_1]\,(x_1x_2)^2-\varepsilon\eta_0$ for all $x=(x_1,x_2)\in\mathbb{R}^2$, if $\vartheta:=\min_{r\in(0,\infty)}\max\{|\delta-1|/r+\eta_1,r\,|\delta-1|+2\gamma+4\eta_0\varepsilon\}\in[0,\infty)$ and if $\mathbb{E}\big[e^{U(X_0)}\big]<\infty$, then it holds for all $x\in\mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x)+\frac12\|\sigma(x)^*(\nabla U)(x)\|^2+\overline{U}(x)\le\vartheta\,U(x)$ (see Subsection 4.1 in Cox et al. [3]) and hence Corollary 3.8 shows for all $r\in(0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t-Y_t^N\|^r\big]\big)=0$ and

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\frac{\varepsilon\,\|Y_t^N\|^2}{e^{\vartheta t}}+\int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\alpha-\varepsilon\eta_1]\,|Y_s^{1,N}Y_s^{2,N}|^2}{e^{\vartheta s}}\,ds\right)\right]<\infty, \tag{123}$$

$$\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\left(\frac{\varepsilon\,\|Y_t^N\|^2}{e^{\vartheta t}}+\int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\alpha-\varepsilon\eta_1]\,|Y_s^{1,N}Y_s^{2,N}|^2}{e^{\vartheta s}}\,ds\right)\right]\le\exp\left(\int_0^T\frac{\varepsilon\eta_0}{e^{\vartheta s}}\,ds\right)\mathbb{E}\left[e^{\varepsilon\|X_0\|^2}\right]<\infty. \tag{124}$$

4.5 Stochastic Duffing-van der Pol oscillator

In this subsection assume the setting in Subsection 4.1, let $\eta_0, \eta_1, \alpha_1 \in [0,\infty)$, $\alpha_2, \alpha_3 \in (0,\infty)$, let $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ be a globally Lipschitz continuous function with $\|g(y)\|^2 \leq \eta_0 + \eta_1 y^2$ for all $y \in \mathbb{R}$ and assume that $d = 2$, $D = \mathbb{R}^2$, $\mu(x) = \big(x_2, \alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3\big)$ and $\sigma(x)u = (0, g(x_1)u)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$. If $\varepsilon \in (0,\infty)$, $U, \bar{U} \in C(\mathbb{R}^2,\mathbb{R})$ fulfill $\varepsilon\eta_1 \leq \alpha_3$, $U(x_1,x_2) = \varepsilon\big[\frac{(x_1)^4}{2} + \alpha_1 (x_1)^2 + (x_2)^2\big]$ and $\bar{U}(x) = 2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,(x_1x_2)^2 - \varepsilon\eta_0 - \frac{\varepsilon|0\vee(\eta_1 - 2\alpha_1(\varepsilon\eta_0+\alpha_2))|^2}{4(\varepsilon\eta_0+\alpha_2)}$ for all $x = (x_1,x_2) \in \mathbb{R}^2$ and if $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, then it holds for all $x \in \mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq 2\,(\varepsilon\eta_0 + \alpha_2)\,U(x)$ (see Subsection 4.2 in Cox et al. [3]) and hence Corollary 3.8 shows for all $r \in (0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\!\left(\frac{\frac{\varepsilon}{2}\big|Y_t^{1,N}\big|^4 + \varepsilon\alpha_1\big|Y_t^{1,N}\big|^2 + \varepsilon\big|Y_t^{2,N}\big|^2}{e^{2t[\varepsilon\eta_0+\alpha_2]}} + \int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,\big|Y_s^{1,N}Y_s^{2,N}\big|^2}{e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\right)\right] < \infty, \tag{125}$$

$$\begin{aligned}&\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\exp\!\left(\frac{\frac{\varepsilon}{2}\big|Y_t^{1,N}\big|^4 + \varepsilon\alpha_1\big|Y_t^{1,N}\big|^2 + \varepsilon\big|Y_t^{2,N}\big|^2}{e^{2t[\varepsilon\eta_0+\alpha_2]}} + \int_0^{t\wedge\tau_N}\frac{2\varepsilon\,[\alpha_3 - \varepsilon\eta_1]\,\big|Y_s^{1,N}Y_s^{2,N}\big|^2}{e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\right)\right]\\&\leq \exp\!\left(\int_0^T \frac{\varepsilon\eta_0}{e^{2s[\varepsilon\eta_0+\alpha_2]}} + \frac{\varepsilon\,|0\vee(\eta_1 - 2\alpha_1[\varepsilon\eta_0+\alpha_2])|^2}{4\,[\varepsilon\eta_0+\alpha_2]\,e^{2s[\varepsilon\eta_0+\alpha_2]}}\,ds\right)\mathbb{E}\left[e^{\frac{\varepsilon}{2}|X_0^1|^4 + \varepsilon\alpha_1|X_0^1|^2 + \varepsilon|X_0^2|^2}\right] < \infty.\end{aligned} \tag{126}$$

4.6 Experimental psychology model

In this subsection assume the setting in Subsection 4.1, let $\alpha, \delta \in (0,\infty)$, $\beta \in \mathbb{R}$ and assume that $d = 2$, $m = 1$, $D = \mathbb{R}^2$, $\mu(x_1,x_2) = \big((x_2)^2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_1, -x_1x_2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_2\big)$ and $\sigma(x_1,x_2) = (-\beta x_2, \beta x_1)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$. If $\varepsilon \in (0,\infty)$, $q \in [3,\infty)$, $U \in C(\mathbb{R}^2,\mathbb{R})$ fulfill $U(x) = \varepsilon\,\|x\|^q$ for all $x \in \mathbb{R}^2$ and if $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, then it holds for all $x \in \mathbb{R}^2$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*\,(\nabla U)(x)\|^2 = 0$ (see Subsection 4.7 in Cox et al. [3]) and hence Corollary 3.8 shows for all $r \in (0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$, $\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\varepsilon\|Y_t^N\|^q\big)\big] < \infty$ and $\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[\exp\big(\varepsilon\|Y_t^N\|^q\big)\big] \leq \mathbb{E}[\exp(\varepsilon\|X_0\|^q)]$.

4.7 Stochastic SIR model

In this subsection assume the setting in Subsection 4.1, let $\alpha,\beta,\gamma,\delta \in (0,\infty)$ and assume that $d = 3$, $m = 1$, $D = (0,\infty)^3$, $\mu(x_1,x_2,x_3) = (-\alpha x_1x_2 - \delta x_1 + \delta, \alpha x_1x_2 - (\gamma+\delta)x_2, \gamma x_2 - \delta x_3)$, $\sigma(x_1,x_2,x_3) = (-\beta x_1x_2, \beta x_1x_2, 0)$ for all $x = (x_1,x_2,x_3) \in D$ and $\mu(x) = \sigma(x) = 0$ for all $x \in D^c$. In this example $D = (0,\infty)^3 \neq \mathbb{R}^3$ is a strict subset of the $\mathbb{R}^3$. This is why the application of Corollary 3.8 is a bit more subtle in this example; cf. Section 4.6 in [15]. More formally, let $\varepsilon \in (0,\infty)$, $\hat{\varepsilon} \in (0, \frac{4\varepsilon\delta}{\gamma}]$, let $\phi\colon \mathbb{R} \to [0,1]$ and $\psi\colon \mathbb{R}^2 \to [0,1]$ be two infinitely often differentiable functions with $\phi(x) = 0$ for all $x \in (-\infty,0]$ with $\phi(x) = 1$ for all $x \in [1,\infty)$ and with $\psi(x_1,x_2) = \phi(x_1)\cdot\phi(-x_2) + \phi(-x_1)\cdot\phi(x_2)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$ and let $U, \bar{U} \in C(\mathbb{R}^3,\mathbb{R})$ be given by $U(x) = \varepsilon\left[\frac{5}{2} + (x_1+x_2)^2 - 2\cdot x_1\cdot x_2\cdot\psi(x_1,x_2)\right] + \hat{\varepsilon}\,[x_3]^2$ and $\bar{U}(x) = -2\varepsilon\delta$ for all $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. Then note for all $x = (x_1,x_2,x_3) \in D$ that

$$\begin{aligned}&(\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) = (\mathcal{G}_{\mu,\sigma}U)(x) + \tfrac{1}{2}\left|\langle\sigma(x),(\nabla U)(x)\rangle\right|^2 + \bar{U}(x)\\&= (\mathcal{G}_{\mu,\sigma}U)(x) + \bar{U}(x) = 2\varepsilon\,[x_1+x_2]\,[-\delta x_1 + \delta - (\gamma+\delta)x_2] + 2\hat{\varepsilon}x_3\,[\gamma x_2 - \delta x_3] + \bar{U}(x)\\&= -2\varepsilon\delta\,[x_1+x_2]\,[x_1+x_2-1] - 2\varepsilon\gamma\,[x_1+x_2]\,x_2 + 2\hat{\varepsilon}x_3\,[\gamma x_2 - \delta x_3] + \bar{U}(x)\\&= -2\varepsilon\delta\,[x_1+x_2-1]^2 - 2\varepsilon\delta\,[x_1+x_2] - 2\varepsilon\gamma\,[x_1+x_2]\,x_2 - 2\hat{\varepsilon}\delta\,[x_3]^2 + 2\varepsilon\delta + 2\hat{\varepsilon}\gamma x_2x_3 + \bar{U}(0)\\&\leq \bar{U}(0) + 2\varepsilon\delta - 2\varepsilon\gamma\,[x_2]^2 - 2\hat{\varepsilon}\delta\,[x_3]^2 + [2\sqrt{\varepsilon\gamma}x_2]\left[\tfrac{\hat{\varepsilon}\sqrt{\gamma}x_3}{\sqrt{\varepsilon}}\right] \leq \bar{U}(0) + 2\varepsilon\delta + \hat{\varepsilon}\left[\tfrac{\hat{\varepsilon}\gamma}{2\varepsilon} - 2\delta\right][x_3]^2 \leq 0.\end{aligned} \tag{127}$$

Combining Corollary 3.8 with Section 4.6 in [15] hence implies that if $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, then it holds for all $r \in (0,\infty)$ that $\lim_{N\to\infty}\big(\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$, $\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\Big[e^{\frac{\varepsilon}{2}|Y_t^{N,1}|^2 + \frac{\varepsilon}{2}|Y_t^{N,2}|^2 + \hat{\varepsilon}|Y_t^{N,3}|^2}\Big] < \infty$ and $\limsup_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[e^{U(Y_t^N) - 2\varepsilon\delta(t\wedge\tau_N)}\big] \leq \mathbb{E}\big[e^{U(X_0)}\big] < \infty$.

4.8 Langevin dynamics

In this subsection assume the setting in Subsection 4.1, let $\beta,\gamma \in (0,\infty)$, $V \in \cup_{p,c\in[0,\infty)} C^3_{p,c}(\mathbb{R}^m,[0,\infty))$ and assume that $\limsup_{r\searrow 0}\sup_{z\in\mathbb{R}^m}\frac{\|z\|^r}{1+V(z)} < \infty$, $d = 2m$, $D = \mathbb{R}^d$, $\mu(x) = (x_2, -(\nabla V)(x_1) - \gamma x_2)$ and $\sigma(x)u = (0, \sqrt{\beta}u)$ for all $x = (x_1,x_2) \in \mathbb{R}^{2m}$, $u \in \mathbb{R}^m$. If $\varepsilon \in (0, \frac{2\gamma}{\beta}]$, $U, \bar{U} \in C(\mathbb{R}^{2m},\mathbb{R})$ fulfill $U(x) = \varepsilon\,V(x_1) + \frac{\varepsilon}{2}\,\|x_2\|^2$ and $\bar{U}(x) = \varepsilon\big[\gamma - \frac{\varepsilon\beta}{2}\big]\|x_2\|^2 - \frac{\varepsilon\beta m}{2}$ for all $x = (x_1,x_2) \in \mathbb{R}^{2m}$ and if $\mathbb{E}\big[e^{U(X_0)}\big] < \infty$, then it holds for all $x \in \mathbb{R}^{2m}$ that $(\mathcal{G}_{\mu,\sigma}U)(x) + \frac{1}{2}\|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) = 0$ (see Subsection 4.4 in Cox et al. [3]) and hence Corollary 3.8

shows for all $r \in (0,\infty)$ that $\lim_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\varepsilon\, V(Y_t^{1,N}) + \tfrac{\varepsilon}{2}\, \|Y_t^{N,2}\|^2 + \int\limits_0^{t\wedge\tau_N} \varepsilon \left[\gamma - \tfrac{\varepsilon\beta}{2}\right] \|Y_s^{N,2}\|^2 \, ds \bigg)\bigg] < \infty, \tag{128}$$

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\varepsilon\, V(Y_t^{1,N}) + \tfrac{\varepsilon}{2}\, \|Y_t^{N,2}\|^2 + \int\limits_0^{t\wedge\tau_N} \varepsilon \left[\gamma - \tfrac{\varepsilon\beta}{2}\right] \|Y_s^{N,2}\|^2 \, ds \bigg)\bigg] \leq \mathbb{E}\Big[e^{\frac{\varepsilon\beta m t}{2} + \varepsilon V(X_0^1) + \frac{\varepsilon}{2}\|X_0^2\|^2}\Big].$$

4.9 Brownian dynamics (Overdamped Langevin dynamics)

In this subsection assume the setting in Subsection 4.1, let $\beta \in (0,\infty)$, $\eta_0 \in [0,\infty)$, $\eta_1 \in \mathbb{R}$, $\eta_2 \in [0, \frac{2}{\beta})$, $V \in \cup_{p,c\in[0,\infty)} C^3_{p,c}(\mathbb{R}^d, [0,\infty))$ and assume that $\limsup_{r\searrow 0} \sup_{z\in\mathbb{R}^d} \frac{\|z\|^r}{1+V(z)} < \infty$, $d = m$, $D = \mathbb{R}^d$, $\mu(x) = -(\nabla V)(x)$, $\sigma(x) = \sqrt{\beta} I_{\mathbb{R}^d}$ and $(\triangle V)(x) \leq \eta_0 + 2\eta_1 V(x) + \eta_2 \|(\nabla V)(x)\|^2$ for all $x \in \mathbb{R}^d$. If $\varepsilon \in (0, \frac{2}{\beta} - \eta_2]$, $U, \bar{U} \in C(\mathbb{R}^d, \mathbb{R})$ fulfill $U(x) = \varepsilon V(x)$ and $\bar{U}(x) = \varepsilon\,(1 - \frac{\beta}{2}(\eta_2 + \varepsilon))\, \|(\nabla V)(x)\|^2 - \frac{\varepsilon\beta\eta_0}{2}$ for all $x \in \mathbb{R}^d$, then it holds for all $x \in \mathbb{R}^d$ that $(\mathcal{G}_{\mu,\sigma} U)(x) + \frac{1}{2}\, \|\sigma(x)^*(\nabla U)(x)\|^2 + \bar{U}(x) \leq \beta\eta_1 U(x)$ (see Subsection 4.5 in Cox et al. [3]) and hence Corollary 3.8 shows for all $r \in (0,\infty)$ that $\lim_{N\to\infty} \big(\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - Y_t^N\|^r\big]\big) = 0$ and

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{\varepsilon V(Y_t^N)}{e^{\beta\eta_1 t}} + \int\limits_0^{t\wedge\tau_N} \tfrac{\varepsilon\left[1 - \frac{\beta}{2}(\eta_2+\varepsilon)\right]}{e^{\beta\eta_1 s}} \|(\nabla U)(Y_s^N)\|^2 \, ds \bigg)\bigg] < \infty, \tag{129}$$

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\tfrac{\varepsilon V(Y_t^N)}{e^{\beta\eta_1 t}} + \int\limits_0^{t\wedge\tau_N} \tfrac{\varepsilon\left[1 - \frac{\beta}{2}(\eta_2+\varepsilon)\right]}{e^{\beta\eta_1 s}} \|(\nabla U)(Y_s^N)\|^2 - \tfrac{\varepsilon\beta\eta_0}{2e^{\beta\eta_1 s}} \, ds \bigg)\bigg] \leq \mathbb{E}\Big[e^{\varepsilon V(X_0)}\Big] < \infty. \tag{130}$$

5 Counterexamples to exponential integrability properties

5.1 An example SDE with finite exponential moments

Let $T \in (0,\infty)$, let $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths and let $X\colon [0,T]\times\Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths satisfying

$$X_t = X_0 - \int_0^t (X_s)^3 \, ds + \int_0^t dW_s \tag{131}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. Moreover, let $\mu, \sigma\colon \mathbb{R} \to \mathbb{R}$ be functions given by $\mu(x) = -x^3$ and $\sigma(x) = 1$ for all $x \in \mathbb{R}$, let $\varepsilon \in (0, \frac{1}{2}]$ satisfy $\mathbb{E}\big[\exp(\varepsilon|X_0|^4)\big] < \infty$ and let $U_\delta\colon \mathbb{R} \to [0,\infty)$, $\delta \in [0,\infty)$, be functions given by $U_\delta(x) = \delta x^4$ for all $x \in \mathbb{R}$ and all $\delta \in [0,\infty)$. Then observe for all $\delta \in [0,\infty)$ and all $x \in \mathbb{R}$ that

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma} U_\delta)(x) + \tfrac{1}{2}\, |\sigma^*(x)(\nabla U_\delta)(x)|^2 \\
&= \langle (\nabla U_\delta)(x), \mu(x)\rangle + \tfrac{1}{2}\operatorname{trace}\big(\sigma(x)\sigma^*(x)(\operatorname{Hess} U_\delta)(x)\big) + \tfrac{1}{2}\, |\sigma^*(x)(\nabla U_\delta)(x)|^2 \\
&= \delta 4x^3 \cdot (-x^3) + \tfrac{1}{2}\delta 12 x^2 + \tfrac{1}{2}\, \big|\delta 4x^3\big|^2 = 6\delta x^2 - (4\delta - 8\delta^2)x^6 = 6\delta x^2 - 4\delta\,(1-2\delta)\, x^6.
\end{aligned} \tag{132}$$

Then Corollary 2.4 in Cox et al. [3] (with $\bar{U}(x) = 4\delta\,(1-2\delta)\, x^6 - 6\delta x^2$ for all $x \in \mathbb{R}$; see also Corollary 3.8 above) implies for all $\delta \in [0,\varepsilon]$ that

$$\sup_{t\in[0,T]} \mathbb{E}\bigg[\exp\bigg(\delta\, |X_t|^4 + \int_0^t 4\delta\,(1-2\delta)\, |X_s|^6 - 6\delta\, |X_s|^2 \, ds \bigg)\bigg] \leq \mathbb{E}\Big[e^{\delta|X_0|^4}\Big] < \infty. \tag{133}$$

In particular, it holds for all $\delta \in [0,\varepsilon] \cap [0, \frac{1}{2})$ that $\sup_{t\in[0,T]} \mathbb{E}\big[\exp\big(\delta |X_t|^4\big)\big] < \infty$.

5.2 Infinite exponential moments for (stopped) Euler schemes

The Euler scheme stopped after leaving certain sets is not suitable for approximating the exponential moments on the left-hand side of (133) as there is at least one Euler step and this results in tails of a normal distribution. Note that in the special case $D_t = \mathbb{R}$, $t \in (0,T]$, the numerical scheme (134) is the Euler scheme for the SDE (131). We also note that Liu and Mao [21] consider a stopped Euler scheme with $D_t = [0,\infty)$, $t \in [0,T]$, for SDEs on the domain $[0,\infty)$.

Lemma 5.1. *Assume the setting in Subsection 5.1, let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a non-increasing family of sets satisfying $\lambda_{\mathbb{R}}(D_T) \cdot \mathbb{P}\big[X_0 \in D_T\big] > 0$ and $\cup_{t\in(0,T]} \mathring{D}_t = \mathbb{R}$ and let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be mappings satisfying $Y_0^N = X_0$ and*

$$Y_t^N = Y_{\frac{nT}{N}}^N + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y_{\frac{nT}{N}}^N\big) \Big(W_t - W_{\frac{nT}{N}} - \big(Y_{\frac{nT}{N}}^N\big)^3 \big(t - \tfrac{nT}{N}\big) \Big) \tag{134}$$

for all $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0, 1, \ldots, N-1\}$ *and all* $N \in \mathbb{N}$. *Then* $\lim_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} \big|X_s - Y_s^M\big| > p\big] = 0$ *and* $\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] = \infty$ *for all* $t \in (0, T]$, $N \in \mathbb{N}$, $p \in (0, \infty)$ *and all* $q \in (2, \infty)$.

Proof of Lemma 5.1. Throughout this proof fix $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $p \in (0, \infty)$ and $q \in (2, \infty)$. Lemma 3.2 implies that the function $\mathbb{R} \times (0, T] \times \mathbb{R} \ni (x, h, y) \mapsto \mathbb{1}_{D_h}(x)\,\big(y - x^3 h\big) \in \mathbb{R}$ is (μ, σ)-consistent with respect to Brownian motion. Then Proposition 3.5 applied to the function $\mathbb{R} \times (0, T]^2 \times \mathbb{R} \ni (x, h, s, y) \mapsto \mathbb{1}_{D_h}(x)\big(y - x^3 s\big) \in \mathbb{R}$ shows that $\lim_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} \big|X_s - Y_s^M\big| > p\big] = 0$. Moreover, since $\mathbb{P}\big[Y_0^N \in D_{\frac{T}{N}}\big] > 0$ and since $D_{\frac{T}{N}}$ has positive Lebesgue measure, we infer that $\mathbb{P}\big[Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}}\big] > 0$. This together with independence of $W_t - W_{\frac{nT}{N}}$ from $Y^N_{\frac{nT}{N}}$ and together with the inequality $|y + x|^q \geq \frac{|y|^q}{2^q} - |x|^q$ for all $x, y \in \mathbb{R}$ yields

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] \geq \mathbb{E}\Big[\mathbb{1}_{\{Y^N_{nT/N}\in D_{T/N}\}} \exp\big(p\,|Y_t^N|^q\big)\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\big(p\,|Y_t^N|^q\big) \,\big|\, Y^N_{\frac{nT}{N}} = x\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\
&= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\Big(p\,\big|W_t - W_{\frac{nT}{N}} + x - x^3\,(t - \tfrac{nT}{N})\big|^q\Big)\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\
&\geq \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_t - W_{\frac{nT}{N}}\big|^q - p\,\big|x - x^3(t - \tfrac{nT}{N})\big|^q\Big)\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \qquad (135) \\
&= \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_{t-\frac{nT}{N}}\big|^q\Big)\Big] \left[\int_{D_{\frac{T}{N}}} \exp\Big(-p\,\big|x - x^3(t - \tfrac{nT}{N})\big|^q\Big)\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big]\right] \\
&= \mathbb{E}\Big[\exp\Big(\tfrac{p}{2^q}\,\big|W_{t-\frac{nT}{N}}\big|^q\Big)\Big]\, \mathbb{E}\Big[\mathbb{1}_{\{Y^N_{nT/N}\in D_{T/N}\}} \exp\Big(-p\,\big|Y^N_{nT/N} - \big[Y^N_{nT/N}\big]^3[t - \tfrac{nT}{N}]\big|^q\Big)\Big] \\
&= \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\Big(\big(\tfrac{Nt - nT}{4N}\big)^{\frac{q}{2}} p\,|y|^q - \tfrac{y^2}{2}\Big)\, dy\; \mathbb{E}\Big[\mathbb{1}_{\{Y^N_{nT/N}\in D_{T/N}\}} \exp\Big(-p\,\big|Y^N_{nT/N} - \big[Y^N_{nT/N}\big]^3[t - \tfrac{nT}{N}]\big|^q\Big)\Big] = \infty
\end{aligned}
$$

where we used in the last step that $\lim_{y\to\infty} |y|^q/y^2 = \infty$. This finishes the proof of Lemma 5.1. ☐

5.3 Infinite exponential moments for a (stopped) linear-implicit Euler scheme

The following lemma shows that the stopped linear-implicit Euler scheme (136) is not suitable for approximating the exponential moments on the left-hand side of (133). Display (136) shows that the linear-implicit Euler scheme (136) with $D_t = \mathbb{R}$ for all $t \in (0, T]$ belongs to the class of balanced implicit methods (choose $c_0(x) = x^2$ and $c_1(x) = 0$ for all $x \in \mathbb{R}$) introduced in Milstein, Platen & Schurz [23].

Lemma 5.2. *Assume the setting in Subsection 5.1, let* $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0, T]$, *be a non-increasing family of sets satisfying* $\lambda_{\mathbb{R}}(D_T) \cdot \mathbb{P}[X_0 \in D_T] > 0$ *and* $\cup_{t\in(0,T]} \mathring{D}_t = \mathbb{R}$ *and let* $Y^N \colon [0, T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, *be mappings satisfying* $Y_0^N = X_0$ *and*

$$
\begin{aligned}
Y_t^N &= Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y^N_{\frac{nT}{N}}\big)\,\Big(W_t - W_{\frac{nT}{N}} - Y_t^N\big(Y^N_{\frac{nT}{N}}\big)^2\,\big(t - \tfrac{nT}{N}\big)\Big) \qquad (136)\\
&= Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y^N_{\frac{nT}{N}}\big)\,\Big(W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3\,\big(t - \tfrac{nT}{N}\big)\Big) + \mathbb{1}_{D_{\frac{T}{N}}}\big(Y^N_{\frac{nT}{N}}\big)\,\big(Y^N_{\frac{nT}{N}}\big)^2\big(Y^N_{\frac{nT}{N}} - Y_t^N\big)\,\big(t - \tfrac{nT}{N}\big)
\end{aligned}
$$

for all $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0, 1, \ldots, N-1\}$ *and all* $N \in \mathbb{N}$. *Then* $\lim_{N\to\infty} \mathbb{E}\big[\sup_{n\in\{0,1,\ldots,N\}} |X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}|^p\big] = 0$ *and* $\mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] = \infty$ *for all* $t \in (0, T]$, $N \in \mathbb{N}$, $p \in (0, \infty)$ *and all* $q \in (2, \infty)$.

Proof of Lemma 5.2. Fix $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $N \in \mathbb{N}$ and $q \in (2, \infty)$ for the rest of this proof. Lemma 3.30 in [15] and Lemma 3.2 imply that the function $\mathbb{R} \times (0, T] \times \mathbb{R} \ni (x, h, y) \mapsto \mathbb{1}_{D_h}(x)\big(\frac{x+y}{1+x^2 h} - x\big) \in \mathbb{R}$ is (μ, σ)-consistent with respect to Brownian motion. Then Proposition 3.5 applied to the function $\mathbb{R} \times (0, T]^2 \times \mathbb{R} \ni (x, h, s, y) \mapsto \mathbb{1}_{D_h}(x)\big(\frac{x+y}{1+x^2 s} - x\big) \in \mathbb{R}$ shows for all $p \in (0, \infty)$ that $\lim_{M\to\infty} \mathbb{P}\big[\sup_{s\in[0,T]} \big|X_s - Y_s^M\big| > p\big] = 0$. In addition, Lemma 2.28 in [15] yields for all $p \in (0, \infty)$ that $\sup_{M\in\mathbb{N}} \big\| \sup_{n\in\{0,1,\ldots,N\}} |Y^M_{\frac{nT}{M}}| \big\|_{L^p(\Omega;\mathbb{R})} < \infty$ and this shows for all $p \in (0, \infty)$ that the family of random variables $\sup_{n\in\{0,1,\ldots,N\}} |X_{\frac{nT}{M}} - Y^M_{\frac{nT}{M}}|^p$, $M \in \mathbb{N}$, is uniformly integrable. Combining this with convergence in probability and, e.g., Corollary 6.21 in Klenke [19] proves for all $p \in (0, \infty)$ that $\lim_{M\to\infty} \mathbb{E}\big[\sup_{n\in\{0,1,\ldots,N\}} |X_{\frac{nT}{M}} - Y^M_{\frac{nT}{M}}|^p\big] = 0$. Moreover, since $\mathbb{P}\big[Y_0^N \in D_{\frac{T}{N}}\big] > 0$ and $D_{\frac{T}{N}}$

has positive Lebesgue measure, we infer that $\mathbb{P}\big[Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}}\big] > 0$. This implies that

$$(137)\quad \begin{aligned} \mathbb{E}\big[\exp(p\,|Y^N_t|^q)\big] &\geq \mathbb{E}\Big[\mathbb{1}_{\{Y^N_{nT/N}\in D_{T/N}\}} \exp\big(p\,|Y^N_t|^q\big)\Big] \\ &= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\big(p\,|Y^N_t|^q\big) \,\big|\, Y^N_{\frac{nT}{N}} = x\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\ &= \int_{D_{\frac{T}{N}}} \mathbb{E}\Big[\exp\Big(p\,\Big|\tfrac{x+W_t-W_{\frac{nT}{N}}}{1+x^2(t-\frac{nT}{N})}\Big|^q\Big)\Big]\, \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] \\ &= \int_{D_{\frac{T}{N}}} \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}} \exp\Big(p\,\Big|\tfrac{x+\sqrt{t-\frac{nT}{N}}\,y}{1+x^2(t-\frac{nT}{N})}\Big|^q - \tfrac{y^2}{2}\Big)\, dy\; \mathbb{P}\Big[Y^N_{\frac{nT}{N}} \in dx\Big] = \infty \end{aligned}$$

where we used in the last step that $\lim_{y\to\infty} |y|^q/y^2 = \infty$. This finishes the proof of Lemma 5.2. □

5.4 Unbounded exponential moments for a (stopped) increment-tamed Euler scheme

Lemma 5.3. *Let $T, q, \delta, \alpha, \beta \in (0,\infty)$ satisfy $q\beta > 2\alpha + 1$, let $f\colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable and locally bounded function, let $D_{h,t} \in \mathcal{B}(\mathbb{R})$, $h, t \in (0,T]$, be sets satisfying $\cup_{r\in(0,T]} \cap_{h\in(0,r]} \cap_{t\in(0,h]} D_{h,t} = \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion with continuous sample paths, let $Y^N\colon [0,T]\times\Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be a family of adapted stochastic processes satisfying*

$$|Y^N_t| \geq \Big[\tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big]\, \mathbb{1}_{\big\{1\leq(t-\frac{nT}{N})^\alpha(W_t-W_{\frac{nT}{N}})\leq 2\big\}\cap\big\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\big\}} \tag{138}$$

$\mathbb{P}$-a.s. for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$, $N \in \mathbb{N}$ and let $X\colon [0,T]\times\Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths and with $\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{P}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| > 1\big] = 0$. Then $\lim_{N\to\infty} \inf_{t\in(0,T]} \mathbb{E}\big[\exp(\delta\,|Y^N_t|^q)\big] = \infty$.

Proof of Lemma 5.3. Assumption (138) implies that

$$(139)\quad \begin{aligned} &\exp\Big(\delta\,\big|Y^N_t\big|^q\Big) \\ &\geq \exp\Big(\delta\,\Big|\max\Big\{0, \tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big\}\Big|^q\Big)\, \mathbb{1}_{\big\{1\leq(t-\frac{nT}{N})^\alpha(W_t-W_{\frac{nT}{N}})\leq 2\big\}\cap\big\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\big\}} \\ &\geq \exp\Big(\delta\,\Big|\max\Big\{0, \tfrac{2\delta}{(t-\frac{nT}{N})^\beta} - f(Y^N_{\frac{nT}{N}})\Big\}\Big|^q\Big)\, \mathbb{1}_{\big\{1\leq(t-\frac{nT}{N})^\alpha(W_t-W_{\frac{nT}{N}})\leq 2\big\}\cap\big\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\big\}\cap\big\{f(Y^N_{\frac{nT}{N}})\leq\frac{\delta}{(t-\frac{nT}{N})^\beta}\big\}} \\ &\geq \exp\Big(\delta\,\Big|\tfrac{\delta}{(t-\frac{nT}{N})^\beta}\Big|^q\Big)\, \mathbb{1}_{\big\{1\leq(t-\frac{nT}{N})^\alpha(W_t-W_{\frac{nT}{N}})\leq 2\big\}\cap\big\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\big\}\cap\big\{f(Y^N_{\frac{nT}{N}})\leq\frac{\delta}{(t-\frac{nT}{N})^\beta}\big\}} \\ &= \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big)\cdot \mathbb{1}_{\big\{1\leq(t-\frac{nT}{N})^\alpha(W_t-W_{\frac{nT}{N}})\leq 2\big\}}\cdot \mathbb{1}_{\big\{Y^N_{\frac{nT}{N}}\in D_{\frac{T}{N},t-\frac{nT}{N}}\big\}}\cdot \mathbb{1}_{\big\{f(Y^N_{\frac{nT}{N}})\leq\frac{\delta}{(t-\frac{nT}{N})^\beta}\big\}} \end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Taking expectation and exploiting independence of $W_t - W_{\frac{nT}{N}}$ from $Y^N_{\frac{nT}{N}}$ for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$ yields for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$(140)\quad \begin{aligned} &\mathbb{E}\Big[\exp\Big(\delta\,\big|Y^N_t\big|^q\Big)\Big] \\ &\geq \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big)\cdot \mathbb{P}\Big[1 \leq \big(t-\tfrac{nT}{N}\big)^\alpha\big(W_t - W_{\frac{nT}{N}}\big) \leq 2\Big]\cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^\beta \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\ &= \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}}\Big)\cdot \int_1^2 \tfrac{1}{\sqrt{2\pi(t-\frac{nT}{N})^{(2\alpha+1)}}} \exp\Big(-\tfrac{y^2}{2\,(t-\frac{nT}{N})^{(2\alpha+1)}}\Big)\,dy\cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^\beta \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\ &\geq \tfrac{1}{\sqrt{2\pi T^{(2\alpha+1)}}}\cdot \exp\Big(\tfrac{\delta^{(q+1)}}{(t-\frac{nT}{N})^{q\beta}} - \tfrac{2}{(t-\frac{nT}{N})^{(2\alpha+1)}}\Big)\cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^\beta \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \\ &\geq \tfrac{1}{\sqrt{2\pi T^{(2\alpha+1)}}}\cdot \exp\Big(\tfrac{\delta^{(q+1)}N^{qb}}{T^{qb}} - \tfrac{2N^{(2\alpha+1)}}{T^{(2\alpha+1)}}\Big)\cdot \mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^\beta \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big]. \end{aligned}$$

Moreover, path continuity of X implies that there exists a natural number $k \in \mathbb{N}$ such that $\mathbb{P}\big[\sup_{s\in[0,t]} |X_s| \leq k-1\big] \geq \frac{1}{2}$. The assumptions $\cup_{r\in(0,T]} \cap_{h\in(0,r]} \cap_{t\in(0,h]} D_{h,t} = \mathbb{R}$ and $\sup_{x\in[-k,k]} |f(x)| < \infty$ yield that there exists a natural number $N_0 \in \mathbb{N}$ such that $N_0 \geq T\,\big[\frac{1}{\delta}\cdot\sup_{x\in[-k,k]}|f(x)|\big]^{1/\beta}$ and such that $[-k,k] \subseteq \big(\cap_{h\in(0,\frac{T}{N_0}]}$

$\cap_{t\in(0,h]} D_{h,t}) = \big(\cap_{N=N_0}^{\infty} \cap_{h\in(0,\frac{T}{N}]} \cap_{t\in(0,h]} D_{h,t}\big)$. This shows for all $N \in \mathbb{N} \cap [N_0, \infty)$ that

$$\begin{aligned}
[-k,k] &\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t} \cap [-k,k] \colon N \geq T\big[\tfrac{1}{\delta}\cdot \sup\nolimits_{y\in[-k,k]} |f(y)|\big]^{1/\beta}\Big\} \\
&\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t} \cap [-k,k] \colon \big[\tfrac{N}{T}\big]^{\beta} \geq \tfrac{1}{\delta}\cdot |f(x)|\Big\} \\
&\subseteq \bigcap_{h\in(0,\frac{T}{N}]} \bigcap_{t\in(0,h]} \Big\{x \in D_{h,t} \colon |f(x)|\, t^{\beta} \leq \delta\Big\} \subseteq \bigcap_{t\in(0,\frac{T}{N}]} \Big\{x \in D_{\frac{T}{N},t} \colon |f(x)|\, t^{\beta} \leq \delta\Big\} \\
&= \bigcap_{n=0}^{N-1} \bigcap_{t\in(\frac{nT}{N},\frac{(n+1)T}{N}]} \Big\{x \in D_{\frac{T}{N},t-\frac{nT}{N}} \colon |f(x)|\,\big[t - \tfrac{nT}{N}\big]^{\beta} \leq \delta\Big\}.
\end{aligned} \tag{141}$$

This together with monotonicity of $\mathbb{P}$ yields for all $N \in \mathbb{N} \cap [N_0, \infty)$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$\begin{aligned}
&\mathbb{P}\Big[|f(Y^N_{\frac{nT}{N}})|\,\big(t-\tfrac{nT}{N}\big)^{\beta} \leq \delta,\ Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N},t-\frac{nT}{N}}\Big] \geq \mathbb{P}\Big[|Y^N_{\frac{nT}{N}}| \leq k\Big] \\
&\geq \mathbb{P}\Big[|X_{\frac{nT}{N}}| \leq k-1,\ |X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| \leq 1\Big] \geq \frac{1}{2} - \sup_{m\in\{0,1,\ldots,N\}} \mathbb{P}\Big[|X_{\frac{mT}{N}} - Y^N_{\frac{mT}{N}}| > 1\Big].
\end{aligned} \tag{142}$$

Consequently, combining inequalities (140) and (142) together with the assumption $q\beta > 2\alpha + 1$ and together with the assumption $\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{P}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| > 1\big] = 0$ results in

$$\begin{aligned}
&\lim_{N\to\infty} \inf_{t\in(0,T]} \mathbb{E}\Big[\exp\Big(\delta\,\big|Y^N_t\big|^q\Big)\Big] \\
&\geq \lim_{N\to\infty} \left[\tfrac{1}{\sqrt{2\pi T^{2\alpha+1}}} \cdot \exp\Big(\tfrac{\delta^{(q+1)} N^{qb}}{T^{qb}} - \tfrac{2N^{(2\alpha+1)}}{T^{(2\alpha+1)}}\Big) \left(\frac{1}{2} - \sup_{m\in\{0,1,\ldots,N\}} \mathbb{P}\Big[|X_{\frac{mT}{N}} - Y^N_{\frac{mT}{N}}| > 1\Big]\right)\right] = \infty.
\end{aligned} \tag{143}$$

This finishes the proof of Lemma 5.3. ☐

Corollary 5.4. *Assume the setting in Subsection 5.1, let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a non-increasing family of sets satisfying $\cup_{t\in(0,T]} D_t = \mathbb{R}$ and let $Y^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be mappings satisfying $Y^N_0 = X_0$ and*

$$Y^N_t = Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \left[\frac{W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 (t - \frac{nT}{N})}{\max\Big(1, \big(t-\frac{nT}{N}\big)\,\big|W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 (t-\frac{nT}{N})\big|\Big)}\right] \tag{144}$$

for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Then $\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}|^p\big] = 0$ and $\lim_{N\to\infty} \inf_{t\in(0,T]} \mathbb{E}\big[\exp(p\,|Y^N_t|^q)\big] = \infty$ for all $p \in (0,\infty)$ and all $q \in (3,\infty)$.

Proof of Corollary 5.4. Fix $p \in (0,\infty)$ and $q \in (3,\infty)$ for the rest of this proof. Let $\psi \colon \mathbb{R} \times (0,T]^2 \times \mathbb{R} \to \mathbb{R}$ be a mapping given by $\psi(x,h,t,y) = \mathbb{1}_{D_h}(x) \frac{y - x^3 t}{\max(1, t|y - x^3 t|)}$ for all $(x,h,t,y) \in \mathbb{R} \times (0,T]^2 \times \mathbb{R}$. Lemma 3.28 in [15] and Lemma 3.2 yield that the function $\mathbb{R} \times (0,T] \times \mathbb{R} \ni (x,t,y) \mapsto \psi(x,t,t,y) \in \mathbb{R}$ is (μ,σ)-consistent with respect to Brownian motion. Proposition 3.4 hence implies for all $r \in (0,\infty)$ that $\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{P}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}| > r\big] = 0$. Next we combine Theorem 2.13 in [15], Lemma 2.18 in [15], (2.113) in [15] and Corollary 2.9 in [15] to obtain for all $r \in (0,\infty)$ that $\sup_{N\in\mathbb{N}} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\big[|Y^N_{nT/N}|^r\big] < \infty$. As in the proof of Corollary 3.12 in [15] we therefore conclude that

$$\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\Big[\big|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}\big|^p\Big] = 0. \tag{145}$$

For proving the divergence statement in Corollary 5.4, we will apply Lemma 5.3 and first prove inequality (138). For this let $D_{h,t} \in \mathcal{B}(\mathbb{R})$, $h,t \in (0,T]$, be sets given by $D_{h,t} = D_h \cap [-t^{-2/3}, t^{-2/3}]$ for all $h,t \in (0,T]$. Then note that

$$\begin{aligned}
\cup_{r\in(0,T]} \cap_{h\in(0,r]} \cap_{t\in(0,h]} \big(D_h \cap \big[-t^{-2/3}, t^{-2/3}\big]\big) &= \cup_{r\in(0,T]} \cap_{h\in(0,r]} \big(D_h \cap \big[-h^{-2/3}, h^{-2/3}\big]\big) \\
&= \cup_{r\in(0,T]} \big(D_r \cap \big[-r^{-2/3}, r^{-2/3}\big]\big) = \mathbb{R}.
\end{aligned} \tag{146}$$

Next observe for all $(x,h,t,y) \in \mathbb{R} \times (0,T]^2 \times \mathbb{R}$ that

$$\begin{aligned}
|x+\psi(x,h,t,y)| &\geq \left(\tfrac{|y-x^3t|}{\max(1,t|y-x^3t|)} - |x|\right) \cdot \mathbb{1}_{D_{h,t}}(x) \cdot \mathbb{1}_{[1,2]}\big(t|y|\big) \\
&\geq \left(\tfrac{|y|}{\max(1,t|y-x^3t|)} - |x| - |x|^3 t\right) \cdot \mathbb{1}_{D_{h,t}}(x) \cdot \mathbb{1}_{[1,2]}\big(t|y|\big) \\
&\geq \left(\tfrac{|y|}{1+t^2|x|^3+t|y|} - |x| - |x|^3 t\right) \cdot \mathbb{1}_{D_{h,t}}(x) \cdot \mathbb{1}_{[1,2]}\big(t|y|\big) \\
&\geq \left(\tfrac{|y|}{2+t|y|} - |x| - |x|^3 T\right) \cdot \mathbb{1}_{D_{h,t}}(x) \cdot \mathbb{1}_{[1,2]}\big(t|y|\big) \\
&\geq \left(\tfrac{1}{3t} - |x| - |x|^3 T\right) \cdot \mathbb{1}_{D_{h,t}}(x) \cdot \mathbb{1}_{[1,2]}\big(t|y|\big).
\end{aligned} \tag{147}$$

This together with the fact that for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ it holds that $Y_t^N = Y^N_{\frac{nT}{N}} + \psi\big(Y^N_{\frac{nT}{N}}, \frac{T}{N}, t, W_t - W_{\frac{nT}{N}}\big)$ shows for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$ that

$$|Y_t^N| \geq \left[\tfrac{2\delta}{(t-\frac{nT}{N})} - \left[\big|Y^N_{\frac{nT}{N}}\big| + \big|Y^N_{\frac{nT}{N}}\big|^3 T\right]\right] \mathbb{1}_{\left\{1 \leq (t-\frac{nT}{N})(W_t - W_{\frac{nT}{N}}) \leq 2\right\} \cap \left\{Y^N_{\frac{nT}{N}} \in D_{\frac{T}{N}, t-\frac{nT}{N}}\right\}}. \tag{148}$$

Lemma 5.3 can thus be applied with $\alpha = \beta = 1$ and this proves the divergence statement in Corollary 5.4. The proof of Corollary 5.4 is thus completed. ☐

The proof of the following corollary, Corollary 5.5, is analogous to the proof of Corollary 5.4 and therefore omitted.

Corollary 5.5. *Assume the setting in Subsection 5.1, let $D_t \in \mathcal{B}(\mathbb{R})$, $t \in (0,T]$, be a non-increasing family of sets satisfying $\cup_{t\in(0,T]} D_t = \mathbb{R}$ and let $Y^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be mappings satisfying $Y_0^N = X_0$ and*

$$Y_t^N = Y^N_{\frac{nT}{N}} + \mathbb{1}_{D_{\frac{T}{N}}}(Y^N_{\frac{nT}{N}}) \left[\frac{W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 \big(t - \frac{nT}{N}\big)}{1 + \big(t - \frac{nT}{N}\big) \left|W_t - W_{\frac{nT}{N}} - \big(Y^N_{\frac{nT}{N}}\big)^3 \big(t - \frac{nT}{N}\big)\right|}\right] \tag{149}$$

for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Then $\lim_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\big[|X_{\frac{nT}{N}} - Y^N_{\frac{nT}{N}}|^p\big] = 0$ and $\lim_{N\to\infty} \inf_{t\in(0,T]} \mathbb{E}\big[\exp\big(p\,|Y_t^N|^q\big)\big] = \infty$ for all $p \in (0,\infty)$ and all $q \in (3,\infty)$.

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