Kolmogorov-Chentsov theorem and differentiability of random fields on manifolds

R. Andreev and A. Lang

KOLMOGOROV–CHENTSOV THEOREM AND DIFFERENTIABILITY OF RANDOM FIELDS ON MANIFOLDS

ROMAN ANDREEV AND ANNIKA LANG

ABSTRACT. A version of the Kolmogorov–Chentsov theorem on sample differentiability and Hölder continuity of random fields on domains of cone type is proved, and the result is generalized to manifolds.

1. INTRODUCTION

Sample regularity of random fields and stochastic processes has been first studied by Kolmogorov in the 1930th, as reported in [22], and extended by Chentsov in [5]. The Kolmogorov–Chentsov theorem states the existence of continuous modifications of stochastic processes and derives bounds on the exponent of sample Hölder continuity. This result has been extended in many ways, for example: for random fields on cubes [2], for random fields on the sphere [12], for random fields on metric spaces [20, 9], and for random fields with values in more general spaces [18, 16]. For a review on the literature and the history of the problem we refer the reader to the introduction of [20].

The first objective of this paper is to reproduce these results for random fields on domains. In fact, we also show sample differentiability under suitable further assumptions on the random fields. The second objective is to extend these results to random fields on manifolds.

In contrast to Hölder continuity of stochastic processes and random fields, sample differentiability has hardly been studied so far, partly because the Brownian motion and the related stochastic processes are almost surely nowhere differentiable. In recent years, however, the question of smoothness of random fields (beyond Hölder continuity) has become more important. In particular, higher spatial smoothness of solutions of stochastic partial differential equations can be exploited to improve the order of convergence of numerical algorithms. Other examples include solutions of random partial differential equations as presented in [4], or lognormal random fields on the sphere in modeling of ice crystals (cf. [17, 12]). Results on existence of sample differentiable modifications for stochastic processes are presented in [6] and [15]. Furthermore, differentiability on $\mathbb{R}^n$ of Gaussian random fields is discussed in [3] and first order sample derivatives are obtained in [21] with a differential and integral calculus in quadratic mean. As a first approach to manifolds, existence of higher order derivatives of isotropic Gaussian random fields on the sphere is shown in relation with the decay of the corresponding angular power spectrum in [12]. To the best of our knowledge, this paper is the first to provide an extension of the Kolmogorov–Chentsov theorem to show sample differentiability of random fields on domains of cone type as well as on manifolds.

The main technical device of our proof is the Sobolev embedding theorem, as was sketched in [7, Proof of Theorem 3.4]. In general, it provides a modification (in space) of the function

in question, and its application to a random field sample-wise may affect the measurability of the random field. We circumvent this complication by showing existence of a sample continuous modification first. Our results are stated in terms of spaces of continuously differentiable functions of fractional order, which characterize the order of differentiability and the "remaining" Hölder continuity of the highest order derivatives.

The paper is organized as follows. In Section 2 we introduce the necessary basics on Sobolev spaces, Sobolev embeddings, manifolds, and random fields, as well as our notation. Section 3 contains our two main results, which state the existence of Hölder continuous and differentiable modifications of random fields on domains of cone type, and on sufficiently smooth manifolds. The proofs can be found in the same section.

2. Preliminaries

In this preparatory section we collect the notions required to obtain our main results in Section 3 in the order that is needed later on. Therefore, we start with the introduction of Sobolev spaces on domains, and recall the appropriate variant of the Sobolev embedding theorem. We move on to define manifolds, and spaces of Hölder continuous and differentiable functions on them. Finally, we introduce random fields on manifolds and associated properties.

We briefly recall the theory of (fractional) Sobolev spaces and spaces of Hölder continuous and differentiable functions on domains. For details, we refer the reader to the standard literature [1, 8, 23].

For any $t > 0$, the integer part $\lfloor t \rfloor \in \mathbb{N}_0$ and the fractional part $\{t\} \in [0, 1)$ of t are uniquely determined by $t = \lfloor t \rfloor + \{t\}$. For a multi-index $\alpha \in \mathbb{N}_0^n$ we set $|\alpha| := \sum_{i=1}^n \alpha_i$.

A subset $D \subset \mathbb{R}^n$ is called a *domain* if it is nonempty, open, and connected. If D is a domain, we define for $t \in \mathbb{N}_0$

$$\|f\|_{\bar{C}^t(D)} := \sum_{|\alpha| \leq t} \sup_{x \in D} |\partial^\alpha f(x)|,$$

where $\partial^\alpha := \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$ is the classical partial derivative, and for noninteger $t > 0$

$$\|f\|_{\bar{C}^t(D)} := \|f\|_{\bar{C}^{\lfloor t \rfloor}(D)} + \sum_{|\alpha| \leq \lfloor t \rfloor} \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^{\{t\}}}.$$

For $t > 0$ we define the *Hölder spaces*

$$\bar{C}^t(D) := \left\{ f : D \to \mathbb{R}; f \text{ is } \lfloor t \rfloor \text{ times continuously differentiable and } \|f\|_{\bar{C}^t(D)} < +\infty \right\}.$$

The *Lebesgue space* $L^p(D)$, $p \in [1, \infty)$, comprises all measurable functions $u : D \to \mathbb{R}$ for which $\|u\|_{L^p(D)}^p := \int_D |u(x)|^p \, dx$ is finite. Functions that are equal for almost every $x \in D$ are identified, so that the space $L^p(D)$ endowed with the norm $\|\cdot\|_{L^p(D)}$ becomes a Banach space. For $p \in [1, \infty)$ and $k \in \mathbb{N}$, the *Sobolev space* $W_p^k(D)$ is defined by

$$W_p^k(D) := \{u \in L^p(D) : \partial^\alpha u \in L^p(D) \text{ for all } 0 \leq |\alpha| \leq k\},$$

where ∂^α denotes the distributional partial derivative. Equipped with the norm $\|\cdot\|_{W_p^k(D)}$ given by

$$\|u\|_{W_p^k(D)}^p := \sum_{0 \leq |\alpha| \leq k} \|\partial^\alpha u\|_{L^p(D)}^p,$$

it becomes a Banach space (see e.g. [1, Thm. 3.3]).

Finally, we extend the definition of Sobolev spaces to nonintegers for bounded domains D of cone type following [23, Def. 4.2.3].

Definition 2.1. A bounded domain D is said to be of *cone type* if there exist domains $U_1, \ldots, U_m$, and cones $C_1, \ldots, C_m$, which may be carried over by rotations into the cone of height h

$$K_h := \{x = (x', x_n) \in \mathbb{R}^n : 0 < x_n < h, |x'| < ax_n\}$$

with fixed $a > 0$ such that $\partial D \subset \bigcup_{j=1}^m U_j$ and $(U_j \cap D) + C_j \subset D$ for all $j = 1, \ldots, m$.

An example of a bounded domain of cone type is a bounded *Lipschitz domain*, i.e., the boundary is locally a graph of a Lipschitz continuous function with a Lipschitz continuous inverse.

The Sobolev spaces of fractional smoothness are obtained by setting

$$W_p^s(D) := \{u \in W_p^{\lfloor s \rfloor}(D) : \|u\|_{W_p^s(D)} < \infty\},$$

where

$$\|u\|_{W_p^s(D)}^p := \|u\|_{L^p(D)}^p + \sum_{|\alpha|=\lfloor s \rfloor} \int_{D \times D} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{n + \{s\}p}} \, dx \, dy.$$

The defined norm $\| \cdot \|_{W_p^s(D)}$ is equivalent to the norm induced by the real method of interpolation by Remark 4.4.2/2 in [23]. Together with [23, Theorem 4.6.1(e)], we deduct the following theorem.

Theorem 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded domain of cone type, $1 < p < \infty$, and $t \geq 0$. Then for all $s > t + n/p$ there holds the continuous embedding $W_p^s(D) \hookrightarrow \bar{C}^t(D)$. The embedding is still valid for $s = t + n/p$ if $t \notin \mathbb{N}_0$.*

We generalize the spaces of Hölder continuous and differentiable functions $\bar{C}^t$ to manifolds by imposing these properties on charts. Before doing so, we recapitulate the necessary geometric definitions and properties. For more details on manifolds, we refer the reader to e.g. [11, 13, 14, 24].

If $A \subset \mathbb{R}^n$ is any subset, $m \in \mathbb{N}$, and $k \in \mathbb{N} \cup \{0, \infty\}$, then $f : A \to \mathbb{R}^m$ is *k times continuously differentiable* or *of class C^k* if for every $x \in A$ there exists an open $\mathcal{O}_x \subset \mathbb{R}^n$ containing x and $g : \mathcal{O}_x \to \mathbb{R}^m$ of class C^k that coincides with f on A. Such f are collected in $C^k(A; \mathbb{R}^m)$. Looking ahead, in order to avoid technicalities we will only consider manifolds *without manifold boundary*, such as the Euclidean space or a sphere therein. The first step is the definition of an atlas.

Definition 2.3. Let M be a set, $r \in \mathbb{N} \cup \{0, \infty\}$, and $n \in \mathbb{N}$. A *C^r atlas $\mathcal{A}$ on M* is a collection of *charts* (U_i, φ_i), $i \in I$, indexed by an arbitrary set I, satisfying the following:

(1) $U_i \subset M$ and $\bigcup_{i \in I} U_i = M$,
(2) $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ is a bijection and for any $i, j \in I$, $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$,
(3) $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a C^r diffeomorphism for any $i, j \in I$.

Two C^r atlases are called equivalent if their union is again a C^r atlas on M. This indeed defines an equivalence relation on the C^r atlases on M. The union of all atlases in such an equivalence class is again an atlas in the equivalence class, called the *maximal C^r atlas*. The topology induced by any maximal C^r atlas is the empty set together with arbitrary unions of its chart domains.

Definition 2.4. Let $n \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0, \infty\}$. A C^r *n-manifold* M is a set $M \neq \emptyset$ together with a maximal C^r atlas $\mathcal{A}(M)$ such that the induced topology is Hausdorff and paracompact.

Recall that in a Hausdorff topological space distinct points have disjoint open neighborhoods, and a topological space is called paracompact if every open cover admits a locally finite open cover (i.e., for any point, there is an open neighborhood which intersects only finitely many members of the collection) which refines the original cover. Usually, the maximal atlas is not mentioned explicitly and M is understood to be equipped with the induced topology. We will say "a chart on M" to refer to a chart in $\mathcal{A}(M)$. Further, we say that an atlas $\mathcal{A}$ on M is an "atlas for M" if it is compatible with $\mathcal{A}(M)$. Any open subset $U \subset M$ canonically inherits the manifold structure of M.

Definition 2.5. Let M be a C^r *n*-manifold and $k \geq 0$ an integer. A function $f : M \to \mathbb{R}$ is said to be of class C^k, denoted by $f \in C^k(M)$, if $f \circ \varphi^{-1} \in C^k(\varphi(U))$ for every chart (U, φ) on M. The *support* supp f of $f \in C^0(M)$ is the closure of the set $\{x \in M : f(x) \neq 0\}$.

Furthermore, a function $f : M \to \mathbb{R}$ is said to be *continuous* (*locally of class* $\bar{C}^t$) if for any $x \in M$ there exists an open connected subset $V \subset M$, $x \in V$, such that for any chart (U, φ) with $U \subset V$, the composite function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is continuous (of class $\bar{C}^t$).

A useful technical device is the partition of unity defined next.

Definition 2.6. Let M be a C^r *n*-manifold. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of M. A C^r *partition of unity subordinate to* $\mathcal{U}$ is a collection $\{\psi_i\}_{i \in I} \subset C^r(M)$ such that

(1) $0 \leq \psi_i(x) \leq 1$ for all $i \in I$ and $x \in M$,
(2) there exists a locally finite open cover $\{V_i\}_{i \in I}$ of M with supp $\psi_i \subset V_i \cap U_i$,
(3) $\sum_{i \in I} \psi_i(x) = 1$ for all $x \in M$ (where the sum is finite by the previous assertion).

The assumed paracompactness of M implies the existence of such partitions of unity, see [13, Chapter II, Corollary 3.8] or [14, Theorem 1.73], which is stated in the following proposition.

Proposition 2.7. *Let M be a C^r n-manifold. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of M. Then there exists a C^r partition of unity subordinate to $\mathcal{U}$.*

We close the preparatory section by introducing random fields on manifolds. Random fields on domains are defined accordingly. In what follows, let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.8. Let M be a C^r *n*-manifold and $\mathcal{B}(M)$ denote its Borel σ-algebra. A mapping $X : \Omega \times M \to \mathbb{R}$ that is $(\mathcal{F} \otimes \mathcal{B}(M))$-measurable is called a *(real-valued) random field* on the manifold M. A random field Y is a *modification* of a random field X if $P(X(x) = Y(x)) = 1$ for all $x \in M$. Furthermore, a random field X on M is said to be *continuous* (*locally of class* $\bar{C}^t$) if $X(\omega)$ is continuous (locally of class $\bar{C}^t$) for all $\omega \in \Omega$.

We note that if M is endowed with a metric (say, given by a Riemannian metric), and the resulting metric space is separable and locally compact, then measurability of $X(x)$ for all $x \in M$ and continuity in probability of X imply $(\mathcal{F} \otimes \mathcal{B}(M))$-measurability of X (cf. [19]).

3. HÖLDER CONTINUITY AND DIFFERENTIABILITY OF RANDOM FIELDS

This section contains our main results on Hölder continuity and differentiability of random fields. We begin by considering random fields on domains of cone type. As indicated in the introduction, results on sample Hölder continuity on different types of domains are well-known (see, e.g., [2, 7, 20, 10]), but sample differentiability has not been of main interest so

far (see, however, [3, 21] for the available results). We prove sample Hölder continuity and differentiability properties in Theorem 3.1 by revisiting the approach of [7, Proof of Theorem 3.4] via the Sobolev embedding theorem. We then address sample Hölder and differentiability properties of random fields on manifolds in Theorem 3.5.

We now state our version of the Kolmogorov–Chentsov theorem on domains of cone type.

Theorem 3.1. *Let $D \subset \mathbb{R}^n$ be a bounded domain of cone type and $X : \Omega \times D \to \mathbb{R}$ be a random field on D. Assume that there exist $d \in \mathbb{N}_0$, $p > 1$, $\epsilon \in (0, 1]$, and $C > 0$ such that $\partial^\alpha X \in L^p(\Omega \times D)$ and*

$$(1) \qquad \mathbb{E}(|\partial^\alpha X(x) - \partial^\alpha X(y)|^p) \leq C\,|x - y|^{n+\epsilon}$$

for all $x, y \in D$ and any multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq d$. Then X has a modification that is locally of class $\bar{C}^t$ for all $t < d + \min\{\epsilon/p, 1 - n/p\}$.

The proof is given in two steps. In the following lemma we first obtain a continuous modification of X based on [10, Theorem 2.3.1], which we again denote by X. In the second step, we prove Theorem 3.1 by invoking the Sobolev embedding on $X(\omega)$ for all $\omega \in \Omega$. Since $X(\omega)$ is continuous for all $\omega \in \Omega$, this does not modify the random field, and there is no need to prove measurability of a modified field.

Let us start by showing the existence of a continuous modification.

Lemma 3.2. *Under the assumptions of Theorem 3.1, X admits a continuous modification.*

Proof. Observe that the domain D equipped with the usual Euclidean metric $|\cdot - \cdot|$ is a totally bounded pseudometric space in the sense of [10]; indeed, its metric entropy $\mathsf{D}(\epsilon) := \mathsf{D}(\epsilon; D, |\cdot - \cdot|)$ is bounded by $\mathsf{D}(\epsilon) \leq \tilde{C}\epsilon^{-n}$ for all $\epsilon > 0$ and some constant $\tilde{C} > 0$, since the domain D can be embedded into a n-dimensional closed cube of finite diameter. We set $\Psi(r) := Cr^{n+\epsilon}$, where the constant C is provided by (1), and $f(r) := r^{\epsilon/p}$ for $r \geq 0$. The integrals $\int_0^1 r^{-1} f(r)\, dr = p/\epsilon$ as well as $\int_0^1 \mathsf{D}(r)\Psi(2r)f(r)^{-p}\, dr \leq C\tilde{C}2^{n+\epsilon}$ are finite. Therefore, [10, Theorem 2.3.1] shows the existence of a continuous modification of X. $\qquad\square$

Having obtained a continuous modification, we are set to continue with the proof of Theorem 3.1.

Proof of Theorem 3.1. Assume without loss of generality that the random field X is continuous (otherwise apply Lemma 3.2). Consider arbitrary $0 < \nu < \min\{(n + \epsilon)/p, 1\}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = d$. Since

$$(\omega, x, y) \mapsto \frac{|\partial^\alpha X(\omega, x) - \partial^\alpha X(\omega, y)|^p}{|x - y|^{n+\nu p}}$$

is $(\mathcal{F} \otimes \mathcal{B}(D \times D))$-measurable, we apply Fubini's theorem and hypothesis (1) to obtain

$$\mathbb{E}\left(\int_{D \times D} \frac{|\partial^\alpha X(x) - \partial^\alpha X(y)|^p}{|x - y|^{n+\nu p}}\, dx\, dy\right) = \int_{D \times D} \frac{\mathbb{E}(|\partial^\alpha X(x) - \partial^\alpha X(y)|^p)}{|x - y|^{n+\nu p}}\, dx\, dy$$

$$\leq C \int_{D \times D} |x - y|^{n+\epsilon-(n+\nu p)}\, dx\, dy.$$

The last integral is finite due to $\epsilon - \nu p > -n$. With the $L^p(\Omega \times D)$ integrability assumptions on X and its derivatives of order d, this implies that

$$\mathbb{E}(\|X\|^p_{W_p^{d+\nu}(D)}) = \mathbb{E}(\|X\|^p_{L^p(D)}) + \sum_{|\alpha|=d} \mathbb{E}\left(\int_{D \times D} \frac{|\partial^\alpha X(x) - \partial^\alpha X(y)|^p}{|x - y|^{n+\nu p}}\, dx\, dy\right)$$

is finite, and therefore there exists $\Omega' \in \mathcal{F}$ with $P(\Omega') = 1$ such that $X(\omega) \in W_p^{d+\nu}(D)$ for all $\omega \in \Omega'$. Consider as continuous modification of X the random field $\tilde{X} := \mathbb{1}_{\Omega'} X$, where $\mathbb{1}_{\Omega'}$ is the indicator function of Ω'. By the Sobolev embedding theorem 2.2, we get that $\tilde{X}(\omega) \in \bar{C}^t(D)$ for all $\omega \in \Omega$ and all $t < d + \nu - n/p$. Since $0 < \nu < \min\{(n+\epsilon)/p, 1\}$ was arbitrary, the claim follows. $\qquad\square$

We remark that $1 - n/p$ is positive only for $p > n$. In the case that $n \geq p$, we obtain in Theorem 3.1 only lower sample differentiability order $\lfloor t \rfloor$ than the assumed weak differentiability order d.

Remark 3.3. The assumptions in Theorem 3.1 can be weakened. If $d \neq 0$, it is sufficient that (1) holds for $|\alpha| = d$, and that X has a continuous modification as provided by Theorem 2.3.1 in [10] under the assumption that (1) holds for $\alpha = 0$ and some $\epsilon > 0$.

We apply Theorem 3.1 in the following example to a Brownian motion on the interval, recovering the classical property of Hölder continuity with exponent $\gamma < 1/2$.

Example 3.4. If X is a Brownian motion on the interval $[0, T] \subset \mathbb{R}^1$, $T < +\infty$, then Assumption (1) is satisfied for $\alpha = 0$, any $p \geq 2$, and $\epsilon = p/2 - 1$. Thus X admits a modification that is locally of class $\bar{C}^t$ for any $0 < t < \sup_{p \geq 2}(p/2 - 1)/p = 1/2$, which is the well-known result.

We are now ready to generalize Theorem 3.1 to random fields on manifolds.

Theorem 3.5. *Let M be a C^r n-manifold, $r > 0$, and let $X : \Omega \times M \to \mathbb{R}$ be a random field on M. Assume that there exist $d \in \mathbb{N}_0$, $p > 1$, and $\epsilon \in (0, 1]$ such that for any chart (U, φ) on M with bounded $\varphi(U) \subset \mathbb{R}^n$, there exists $C_\varphi > 0$ such that $X_\varphi := X \circ \varphi^{-1}$ satisfies that $\partial^\alpha X_\varphi \in L^p(\Omega \times \varphi(U))$ and*

$$\mathbb{E}(|\partial^\alpha X_\varphi(x) - \partial^\alpha X_\varphi(y)|^p) \leq C_\varphi |x - y|^{n+\epsilon}$$

for all $x, y \in \varphi(U)$ and any multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq d$. Then X has a modification that is locally of class $\bar{C}^t$ for all $t < d + \min\{\epsilon/p, 1 - n/p\}$ with $t \leq r$.

Proof. To obtain the continuous modification we first construct a locally finite atlas with coordinate domains that are bounded and of cone type. On each of these charts, a modification of X is provided by Theorem 3.1. Using a partition of unity we then patch together a modification of X with the desired properties.

For each $x \in M$, let $(\tilde{U}_x, \tilde{\varphi}_x)$ be a chart on M with $x \in \tilde{U}_x$. Let $D_x \subset \tilde{\varphi}_x(\tilde{U}_x)$ be an open ball of positive radius centered at $\tilde{\varphi}_x(x)$. Define $U_x := \tilde{\varphi}_x^{-1}(D_x)$ and $\varphi_x := \tilde{\varphi}_x|_{U_x}$. Let $\mathcal{A} := \{(U_x, \varphi_x) : x \in M\}$ be the resulting atlas for M, which we will index by $\Phi := \{\varphi : (U, \varphi) \in \mathcal{A}\}$.

Now, for each $(U_\varphi, \varphi) \in \mathcal{A}$, the coordinate domain $\varphi(U_\varphi)$ is a bounded domain with smooth boundary, in particular of cone type. With our assumptions on X_φ, Theorem 3.1 provides a modification Y^φ of the random field $X_\varphi : \Omega \times \varphi(U_\varphi) \to \mathbb{R}$ on $\varphi(U_\varphi)$, which is locally of class $\bar{C}^t$ for any fixed $t < d + \min\{\epsilon/p, 1 - n/p\}$, for each $\varphi \in \Phi$.

Let $\{\psi_\varphi\}_{\varphi \in \Phi}$ be a C^r partition of unity subordinate to $\{U_\varphi\}_{\varphi \in \Phi}$, which exists by Proposition 2.7. Define $Y : \Omega \times M \to \mathbb{R}$ by $Y := \sum_{\varphi \in \Phi} \psi_\varphi Y^\varphi \circ \varphi$. Since the covering $\{\operatorname{supp} \psi_\varphi\}_{\varphi \in \Phi}$ is locally finite, the sum is well-defined on a neighborhood of any $x \in M$. Furthermore, Y is a random field on M because all $\varphi \in \Phi$ are C^r diffeomorphisms and therefore at least continuous. Moreover, it is a modification of X by the properties of the partition of unity. Owing to the fact that $r \geq t$, the random field Y is locally of class $\bar{C}^t$. $\qquad\square$

We finish this section with two comments. First, since we only used the assumptions on the random field on the charts to apply Theorem 3.1, it is clear that Remark 3.3 carries over to Theorem 3.5. Second, for an example of random fields on manifolds, we refer the reader to [12]. Therein, isotropic Gaussian random fields on the unit sphere in $\mathbb{R}^3$ are considered and sample regularity is obtained by direct calculations.

Acknowledgment

References

[1] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Robert J. Adler. *The Geometry of Random Fields. Reprint of the 1981 original ed.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010.

[3] Robert J. Adler and Jonathan E. Taylor. *Random Fields and Geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[4] Julia Charrier. Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.*, 50(1):216–246, 2012.

[5] Nikolai N. Chentsov. Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the "heuristic" approach to the Kolmogorov–Smirnov tests. *Theory of Probability & Its Applications*, 1(1):140–144, 1956.

[6] Harald Cramér and M. Ross Leadbetter. *Stationary and Related Stochastic Processes. Sample Function Properties and Their Applications*. John Wiley & Sons, Inc., New York – London – Sydney, 1967.

[7] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*, volume 44 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1992.

[8] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[9] Jørgen Hoffmann-Jørgensen. *Stochastic Processes on Polish Spaces*. Åarhus Universitet, Matematisk Institut, Åarhus, 1991.

[10] Davar Khoshnevisan. *Multiparameter Processes. An Introduction to Random Fields*. Springer Monographs in Mathematics. Springer, New York, 2002.

[11] Wilhelm P. A. Klingenberg. *Riemannian Geometry*, volume 1 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 1995.

[12] Annika Lang and Christoph Schwab. Isotropic Gaussian random fields on the sphere: regularity, fast simulation, and stochastic partial differential equations. arXiv:1305.1170 [math.PR], SAM Report 2013-15, 2012.

[13] Serge Lang. *Fundamentals of Differential Geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer, New York, 1999.

[14] Jeffrey M. Lee. *Manifolds and Differential Geometry*, volume 107 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[15] Michel Loève. *Probability theory. II*, volume 46 of *Graduate Texts in Mathematics*. Springer, New York, fourth edition, 1978.

[16] Katrin Mittmann and Ingo Steinwart. On the existence of continuous modifications of vector-valued random fields. *Georgian Math. J.*, 10(2):311–317, 2003.

[17] Timo Nousiainen and Greg M. McFarquhar. Light scattering by quasi-spherical ice crystals. *Journal of the Atmospheric Sciences*, 61(18):2229–2248, 2004.

[18] Szymon Peszat and Jerzy Zabczyk. *Stochastic Partial Differential Equations with Lévy Noise. An Evolution Equation Approach*, volume 113 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 2007.

[19] Jürgen Potthoff. Sample properties of random fields I: Separability and measurability. *Comm. Stoch. Anal.*, 3(3):143–153, 2009.

[20] Jürgen Potthoff. Sample properties of random fields II: Continuity. *Comm. Stoch. Anal.*, 3(1):331–348, 2009.

[21] Jürgen Potthoff. Sample properties of random fields III: Differentiability. *Comm. Stoch. Anal.*, 4(3):335–353, 2010.

[22] Eugen Slutsky. Qualche proposizione relativa alla teoria delle funzioni aleatorie. *Giorn. Ist. Ital. Attuari*, 8:183–199, 1937.

[23] Hans Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.

[24] Joseph Wloka. *Partial Differential Equations*. Cambridge University Press, Cambridge, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.

(Roman Andreev)
SEMINAR FÜR ANGEWANDTE MATHEMATIK
ETH ZÜRICH
RÄMISTRASSE 101, CH–8092 ZÜRICH, SWITZERLAND.
  E-mail address: `andreevr@math.ethz.ch`

(Annika Lang)
SEMINAR FÜR ANGEWANDTE MATHEMATIK
ETH ZÜRICH
RÄMISTRASSE 101, CH–8092 ZÜRICH, SWITZERLAND.
  E-mail address: `annika.lang@math.ethz.ch`

Recent Research Reports

Nr.	Authors/Title

2013-12   R. Kornhuber and Ch. Schwab and M. Wolf
Multi-Level Monte-Carlo Finite Element Methods for stochastic elliptic variational inequalities

2013-13   X. Claeys and R. Hiptmair and E. Spindler
A Second-Kind Galerkin Boundary Element Method for Scattering at Composite Objects

2013-14   I.G. Graham and F.Y. Kuo and J.A. Nichols and R. Scheichl and Ch. Schwab and I.H. Sloan
Quasi-Monte Carlo finite element methods for elliptic PDEs with log-normal random coefficient

2013-15   A. Lang and Ch. Schwab
Isotropic Gaussian random fields on the sphere: regularity, fast simulation, and stochastic partial differential equations

2013-16   P. Grohs and H. Hardering and O. Sander
Optimal A Priori Discretization Error Bounds for Geodesic Finite Elements

2013-17   Cl. Schillings and Ch. Schwab
Sparsity in Bayesian Inversion of Parametric Operator Equations

2013-18   V. Kazeev and Ch. Schwab
Tensor approximation of stationary distributions of chemical reaction networks

2013-19   K. Schmidt and R. Hiptmair
Asymptotic Boundary Element Methods for Thin Conducting Sheets

2013-20   R. Kruse
Consistency and Stability of a Milstein-Galerkin Finite Element Scheme for Semilinear SPDE

2013-21   J. Sukys
Adaptive load balancing for massively parallel multi-level Monte Carlo solvers