Isotropic Gaussian random fields on the sphere: regularity, fast simulation, and stochastic partial differential equations

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ISOTROPIC GAUSSIAN RANDOM FIELDS ON THE SPHERE: REGULARITY, FAST SIMULATION, AND STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Isotropic Gaussian random fields on the sphere are characterized by Karhunen–Loève expansions with respect to the spherical harmonic functions and the angular power spectrum. The smoothness of the covariance is connected to the decay of the angular power spectrum and the relation to sample Hölder continuity and sample differentiability of the random fields is discussed. Rates of convergence of their finitely truncated Karhunen–Loève expansions in terms of the covariance spectrum are established, and algorithmic aspects of fast sample path generation via fast Fourier transforms on the sphere are indicated. The relevance of the results on sample path regularity for isotropic Gaussian random fields and the corresponding lognormal random fields on the sphere for several models from environmental sciences is indicated. Finally, the stochastic heat equation on the sphere driven by additive, isotropic Wiener noise is considered and strong convergence rates for spectral discretizations based on the spherical harmonic functions are proven.

1. INTRODUCTION

Path regularity of Gaussian random fields (GRFs) on subsets of Euclidean space is well studied, where the spectral theory of these fields is used (see, e.g., [26, 27, 28]). However, the general theory of second order random fields as developed in [26, 27, 28] requires a group structure on the space of realizations. The (practically relevant) case of GRFs indexed by the sphere, which we denote by $\mathbb{S}^2$, (and, more generally, $\mathbb{S}^{2n}$) takes a special role with regard to invariance under (topological) group actions (see, e.g., [19] and the references there for a lucid discussion), so that the general results in [26] do not apply directly. Due to the relevance of GRFs on $\mathbb{S}^2$ in applications, in particular in environmental modeling, it is of some interest to develop a theory of sample path regularity, stochastic partial differential equations, and their numerical analysis. The contribution of some basic results with direct proofs is the purpose of the present paper.

Specifically, we derive the connection between the smoothness of the covariance kernel of an isotropic GRF on $\mathbb{S}^2$ and the decay of its angular power spectrum and characterize its $\mathbb{P}$-a.s. sample path Hölder continuity and sample differentiability. Furthermore, we construct isotropic Q-Wiener processes using isotropic GRFs. We solve the stochastic heat equation

on $\mathbb{S}^2$ driven by isotropic Q-Wiener noise with a series expansion with respect to the spherical harmonic functions. We show that the convergence rate of the fully discrete approximation scheme given by the truncation of the series expansion depends only on the decay of the angular power spectrum and that it is independent of the chosen space and time discretization.

The outline of this paper is as follows: In Section 2 we recapitulate basic definitions of isotropic GRFs on $\mathbb{S}^2$ and of the Karhunen–Loève expansions in spherical harmonic functions of these fields from [10]. A characterization of the the decay of the angular power spectrum of isotropic GRFs in terms of a regularity of the covariance kernel in a scale of weighted Sobolev spaces on $\mathbb{S}^2$ is presented in Section 3. Section 4 contains a version of the Kolmogorov–Chentsov theorem for random fields on $\mathbb{S}^2$ and therefore sample Hölder continuity of random fields is addressed. Sufficient conditions on the angular power spectrum are presented for $\mathbb{P}$-a.s. sample Hölder continuity and differentiability of isotropic GRFs. In Section 5 we approximate isotropic Gaussian random fields by finite truncation of their Karhunen–Loève expansions. We discuss convergence rates of these approximations in p-th moment and in the $\mathbb{P}$-a.s. sense. The topic of Section 6 is the introduction of the practically important case of lognormal random fields. These are crucial in a number of applications, in particular in meteorology and in climate modeling. In this section, we give analogous results to Section 4, i.e., sample path regularity of lognormal random fields in terms of Hölder continuity and differentiability is addressed. Finally, isotropic Q-Wiener processes are introduced in Section 7. We consider the stochastic heat equation on $\mathbb{S}^2$ driven by an isotropic Q-Wiener process and solve the stochastic partial differential equation (SPDE) with spectral methods. We approximate the solution by truncation of the derived spectral representation and show convergence rates in p-th moment as well as $\mathbb{P}$-almost surely. These results are illustrated by numerical examples.

2. Isotropic Gaussian random fields on the sphere

In this section we introduce isotropic Gaussian random fields and their properties. We focus especially on Karhunen–Loève of these random fields. In doing so, we follow closely the introduction of Gaussian random fields in Chapter 5 in [10]. Throughout, we denote by $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and write $\mathbb{S}^2$ for the unit sphere in $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3, \|x\| = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Let $(\mathbb{S}^2, d)$ be the compact metric space with the geodesic metric given by

$$d(x, y) = \arccos\langle x, y\rangle_{\mathbb{R}^3}$$

for all $x, y \in \mathbb{S}^2$. We denote by $\mathcal{B}(\mathbb{S}^2)$ the Borel σ-algebra of $\mathbb{S}^2$.

Definition 2.1. A $\mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable mapping $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called a *real-valued random field* on the unit sphere.

The random field T is called *strongly isotropic* if, for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$, and for $g \in \mathsf{SO}(3)$ the multivariate random variables $(T(x_1), \ldots, T(x_k))$ and $(T(gx_1), \ldots, T(gx_k))$ have the same law, where $\mathsf{SO}(3)$ denotes the group of rotations on $\mathbb{S}^2$.

It is called *n-weakly isotropic* for $n \geq 2$ if $\mathbb{E}(|T(x)|^n) < +\infty$ for all $x \in \mathbb{S}^2$ and if for $1 \leq k \leq n$, $x_1, \ldots, x_k \in \mathbb{S}^2$ and $g \in \mathsf{SO}(3)$,

$$\mathbb{E}(T(x_1)\cdots T(x_k)) = \mathbb{E}(T(gx_1)\cdots T(gx_k)).$$

Furthermore, it is called *Gaussian* if for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$ the multivariate random variable $(T(x_1), \ldots, T(x_k))$ is multivariate Gaussian distributed, i.e., $\sum_{i=1}^{k} a_i T(x_i)$ is a normally distributed random variable for all $a_i \in \mathbb{R}$, $i = 1, \ldots, k$.

In what follows, we focus on real-valued random fields. Similarly to a Gaussian random field (GRF for short) on $\mathbb{R}^d$, $d \in \mathbb{N}$, a GRF on $\mathbb{S}^2$ has the following property proven, e.g., in Proposition 5.10(3) in [10]:

Proposition 2.2. *Let T be a GRF on $\mathbb{S}^2$. Then, T is strongly isotropic if and only if T is 2-weakly isotropic.*

A key role in our analysis and simulation of isotropic GRFs on $\mathbb{S}^2$ is taken by their Karhunen–Loève expansions. To introduce Karhunen–Loève expansions of isotropic GRFs (and the corresponding Q-Wiener processes on $\mathbb{S}^2$ in the formulation of SPDEs on $\mathbb{S}^2$ in Section 7), we first have to define the spherical harmonic functions on $\mathbb{S}^2$ which take a crucial role. We recall that the *Legendre polynomials* $(P_\ell, \ell \in \mathbb{N}_0)$ are for example given by Rodrigues' formula (see, e.g., [21])

$$P_\ell(\mu) := 2^{-\ell} \frac{1}{\ell!} \frac{\partial^\ell}{\partial \mu^\ell} (\mu^2 - 1)^\ell$$

for all $\ell \in \mathbb{N}_0$ and $\mu \in [-1, 1]$. The Legendre polynomials define the *associated Legendre functions* $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ by

$$P_{\ell m}(\mu) := (-1)^m (1 - \mu^2)^{m/2} \frac{\partial^m}{\partial \mu^m} P_\ell(\mu)$$

for $\ell \in \mathbb{N}_0$, $m = 0, \ldots, \ell$, and $\mu \in [-1, 1]$. Here and throughout, we do not separate indices for doubly subscripted functions and coefficients by a comma, with the understanding that the reader will recognize double indices as such. With this in mind, we further introduce the *surface spherical harmonic functions* $\mathcal{Y} := (Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ as mappings $Y_{\ell m} : [0, \pi] \times [0, 2\pi) \to \mathbb{C}$, which are given by

$$Y_{\ell m}(\vartheta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta) e^{im\varphi},$$

for $\ell \in \mathbb{N}_0$, $m = 0, \ldots, \ell$, and $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$, and by

$$Y_{\ell m} := (-1)^m \overline{Y_{\ell - m}},$$

for $\ell \in \mathbb{N}$ and $m = -\ell, \ldots, -1$. By the Peter–Weyl theorem (see, e.g., Proposition 3.29 in [10]), $\mathcal{Y}$ is an orthonormal basis of $L^2(\mathbb{S}^2; \mathbb{C})$ which we abbreviate by $L^2(\mathbb{S}^2)$. Every real-valued function f in $L^2(\mathbb{S}^2)$ admits the spherical harmonics series expansion

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}$$

and the coefficients satisfy (cf., e.g., Remark 3.37 in [10])

$$f_{\ell m} = (-1)^m \overline{f_{\ell - m}},$$

i.e., f can be represented in $L^2(\mathbb{S}^2)$ by the series expansion

$$f = \sum_{\ell=0}^{\infty} \left(f_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} \left(\operatorname{Re} f_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} f_{\ell m} \operatorname{Im} Y_{\ell m} \right) \right).$$

In what follows, we set for $y \in \mathbb{S}^2$

$$Y_{\ell m}(y) := Y_{\ell m}(\vartheta, \varphi),$$

where $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$, i.e., we identify (with a slight abuse of notation) Cartesian and angular coordinates of the point $y \in \mathbb{S}^2$. Furthermore we denote by σ the *Lebesgue measure on the sphere* which admits the representation

$$d\sigma(y) = \sin \vartheta \, d\vartheta \, d\varphi$$

for $y \in \mathbb{S}^2$, $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$.

We define the *spherical Laplacian*, also called *Laplace–Beltrami operator*, in terms of spherical coordinates similarly to Section 3.4.3 in [10] by

$$\Delta_{\mathbb{S}^2} = (\sin \vartheta)^{-1} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \, \frac{\partial}{\partial \vartheta} \right) + (\sin \vartheta)^{-2} \frac{\partial^2}{\partial \varphi^2}.$$

It is well-known (see, e.g., Theorem 2.13 in [13]) that the spherical harmonic functions $\mathcal{Y}$ are the eigenfunctions of $\Delta_{\mathbb{S}^2}$ with eigenvalues $(-\ell(\ell+1), \ell \in \mathbb{N}_0)$, i.e.,

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for all $\ell \in \mathbb{N}_0$, $m = -\ell, \ldots, \ell$. Furthermore it is shown in Theorem 2.42 in [13] that $L^2(\mathbb{S}^2)$ has the direct sum decomposition

$$L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2),$$

where the spaces $(\mathcal{H}_\ell, \ell \in \mathbb{N}_0)$ are spanned by spherical harmonic functions

$$\mathcal{H}_\ell(\mathbb{S}^2) = \mathrm{span}\{Y_{\ell m}, m = -\ell, \ldots, \ell\},$$

i.e., $\mathcal{H}_\ell(\mathbb{S}^2)$ denotes the space of eigenfunctions of $\Delta_{\mathbb{S}^2}$ that correspond to the eigenvalue $-\ell(\ell+1)$ for $\ell \in \mathbb{N}_0$.

The significance of the spherical harmonic functions lies in the fact that every 2-weakly isotropic random field admits a convergent Karhunen–Loève expansion. The following result, which is proven in Theorem 5.13 in [10], makes this precise.

Theorem 2.3. *Let T be a 2-weakly isotropic random field on $\mathbb{S}^2$, then the following statements hold true:*

  (1) T satisfies $\mathbb{P}$-almost surely

$$\int_{\mathbb{S}^2} T(x)^2 \, d\sigma(x) < +\infty.$$

  (2) T admits a Karhunen–Loève expansion

(1) $$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$$

  with

$$a_{\ell m} = \int_{\mathbb{S}^2} T(y) \overline{Y_{\ell m}}(y) \, d\sigma(y),$$

  for $\ell \in \mathbb{N}_0$ and $m \in \{-\ell, \ldots, \ell\}$.

(3) *The series expansion* (1) *converges in* $L^2(\Omega \times \mathbb{S}^2; \mathbb{R})$, *i.e.,*

$$\lim_{L \to \infty} \mathbb{E}\Big(\int_{\mathbb{S}^2} (T(y) - \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(y))^2 \, d\sigma(y)\Big) = 0.$$

(4) *The series expansion* (1) *converges in* $L^2(\Omega; \mathbb{R})$ *for all* $x \in \mathbb{S}^2$, *i.e., for all* $x \in \mathbb{S}^2$

$$\lim_{L \to \infty} \mathbb{E}\big((T(x) - \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x))^2\big) = 0.$$

This result implies that every 2-weakly isotropic random field is an element of $L^2(\Omega; L^2(\mathbb{S}^2))$. For the efficient computational simulation of 2-weakly isotropic Gaussian random fields, which we will call in the following just isotropic Gaussian random fields, we will exploit special properties of the random coefficients $\mathbb{A} = (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$. It turns out that the properties are similar to those of invariant GRFs on the torus with Fourier series expansions (see, e.g., [8]). First of all, we have by Remark 6.4, Proposition 6.6, and Equation (6.6) in [10] the following lemma:

Lemma 2.4. *Let T be a strongly isotropic random field on $\mathbb{S}^2$. The elements of the sequence $\mathbb{A}$ are, except for a_{00}, centered random variables, i.e., $\mathbb{E}(a_{\ell m}) = 0$ for all $\ell \in \mathbb{N}$ and $m = -\ell, \ldots, \ell$. Furthermore they are complex-valued random variables that satisfy*

$$\mathbb{E}(a_{\ell_1 m_1} \overline{a_{\ell_2 m_2}}) = A_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2},$$

for $\ell_1, \ell_2 \in \mathbb{N}$ and $m_i = -\ell_i, \ldots, \ell_i$, $i = 1, 2$, where $\delta_{nm} = 1$ if $n = m$ and zero else. For the first element a_{00}, it holds that

$$\mathbb{E}(a_{00} \overline{a_{\ell m}}) = (A_0 + \mathbb{E}(a_{00})^2)\delta_{0\ell}\delta_{0m}.$$

The sequence of nonnegative real numbers $(A_\ell, \ell \in \mathbb{N}_0)$ is called the angular power spectrum *of T.*

The random variables $a_{\ell m}$ and $a_{\ell-m}$ satisfy for $\ell \in \mathbb{N}$ and $m = 1, \ldots, \ell$ that

$$a_{\ell m} = (-1)^m \overline{a_{\ell-m}}.$$

In the case of interest in this manuscript that T is an isotropic GRF, Theorem 6.12 in [10] implies that $\mathbb{A}_+ = (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, complex-valued, Gaussian random variables. By Proposition 6.8 in [10], the elements of $\mathbb{A}_+$ for $m \neq 0$ satisfy that $\operatorname{Re} a_{\ell m}$ and $\operatorname{Im} a_{\ell m}$ are symmetric random variables that are equal in law, uncorrelated, i.e., $\mathbb{E}(\operatorname{Re} a_{\ell m} \operatorname{Im} a_{\ell m}) = 0$, and that have variance

$$\mathbb{E}((\operatorname{Re} a_{\ell m})^2) = \mathbb{E}((\operatorname{Im} a_{\ell m})^2) = A_\ell/2 \,.$$

By Lemma 2.4 the elements of $\mathbb{A}$ that are not in $\mathbb{A}_+$ can be calculated by the formula

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell-m}, \quad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell-m},$$

for $\ell \in \mathbb{N}$ and $m = -\ell, \ldots, -1$. Furthermore we deduce from Proposition 6.11, Proposition 6.6, and Equation (6.12) in [10] and Lemma 2.4 in this manuscript that $\operatorname{Re} a_{\ell 0}$ is $\mathcal{N}(0, A_\ell)$ distributed, i.e., it is normally distributed with mean zero and variance A_ℓ, and $\operatorname{Im} a_{\ell 0} = 0$ for $\ell \in \mathbb{N}$ and that $\operatorname{Re} a_{00}$ is $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$ distributed while $\operatorname{Im} a_{00} = 0$.

So, in conclusion, we have the following corollary.

Corollary 2.5. *Let T be a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$. Then T admits the Karhunen–Loève expansion*

$$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

where $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ is the sequence of spherical harmonic functions and the sequence $\mathbb{A} = (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ is a sequence of complex-valued, centered, Gaussian random variables with the following properties:

    (1) $\mathbb{A}_+ = (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent, complex-valued Gaussian random variables.

    (2) The elements of $\mathbb{A}_+$ with $m > 0$ satisfy $\operatorname{Re} a_{\ell m}$ and $\operatorname{Im} a_{\ell m}$ are independent and $\mathcal{N}(0, A_\ell/2)$ distributed.

    (3) The elements of $\mathbb{A}_+$ with $m = 0$ are real-valued and the elements $\operatorname{Re} a_{\ell 0}$ are $\mathcal{N}(0, A_\ell)$ distributed for $\ell \in \mathbb{N}$ while $\operatorname{Re} a_{00}$ is $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$ distributed.

    (4) The elements of $\mathbb{A}$ with $m < 0$ are deduced from those of $\mathbb{A}_+$ by the formulae

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell - m}, \quad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell - m}.$$

3. Decay of the angular power spectrum

The error in a κ-term truncation of the Karhunen–Loève expansion of an isotropic GRF T on $\mathbb{S}^2$ is closely related to the decay of the angular power spectrum of T. As we show next, the decay of the angular power spectrum is characterized by the behavior of the covariance kernel function that characterizes the isotropic GRF T. Often the kernel function k_T is prescribed in applications.

To specify this relation, we start with the definition of the kernel k_T of the covariance of an isotropic Gaussian random field with prescribed angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. It is given for $x, y \in \mathbb{S}^2$ by the formula

$$k_T(x, y) := \mathbb{E}(T(x)T(y)) = \sum_{\ell=0}^{\infty} A_\ell \sum_{m=-\ell}^{\ell} Y_{\ell m}(x)\overline{Y_{\ell m}}(y) = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell(\langle x, y \rangle_{\mathbb{R}^3}).$$

We observe that the covariance kernel k_T just depends on the inner product resp. the (spherical) distance. Accordingly, we denote by $k : [0, \pi] \to \mathbb{R}$ the kernel as a function of the distance $r = d(x, y)$, i.e.,

$$k(r) := \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos r),$$

for $r \in [0, \pi]$. A third way to look at the kernel is in terms of the inner product $\langle x, y \rangle_{\mathbb{R}^3}$. Therefore, we define $k_I : [-1, 1] \to \mathbb{R}$ by

$$k_I(\mu) := k(\arccos \mu)$$

for all $\mu \in [-1, 1]$. This implies overall for $x, y \in \mathbb{S}^2$ that

$$k_T(x, y) = k(d(x, y)) = k_I(\langle x, y \rangle_{\mathbb{R}^3}).$$

In the following Proposition, we give a relation between the weak weighted differentiability of the kernel k_I and the decay of the corresponding angular power spectrum in terms of summability.

Proposition 3.1. *For every $n \in \mathbb{N}_0$, it holds that the sequence $(\ell^{n+1/2} A_\ell, \ell \geq n)$ is in $\ell^2(\mathbb{N}_0)$ if and only if $(1 - \mu^2)^{n/2} \frac{\partial^n}{\partial \mu^n} k_I(\mu)$, $\mu \in (-1, 1)$ is in $L^2(-1, 1)$, i.e.,*

$$\frac{1}{(4\pi)^2} \sum_{\ell \geq n} A_\ell^2 \frac{2\ell + 1}{2} \ell^{2n} < +\infty$$

if and only if

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu < +\infty.$$

Proof. Let us observe first that by definition

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu$$

$$= \int_{-1}^{1} \left(\sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} \frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right)^2 (1 - \mu^2)^n \, d\mu$$

$$= \sum_{\ell,\ell'=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} A_{\ell'} \frac{2\ell' + 1}{4\pi} \int_{-1}^{1} \left(\frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right) \left(\frac{\partial^n}{\partial \mu^n} P_{\ell'}(\mu) \right) (1 - \mu^2)^n \, d\mu.$$

By $(P_\ell^{(\alpha,\beta)}, \ell \in \mathbb{N}_0)$ we denote the Jacobi polynomials given, e.g., by Rodrigues' formula

$$P_\ell^{(\alpha,\beta)}(\mu) := \frac{(-1)^\ell}{2^\ell \ell!} (1 - \mu)^{-\alpha} (1 + \mu)^{-\beta} \frac{\partial^\ell}{\partial \mu^\ell} \left((1 - \mu)^\alpha (1 + \mu)^\beta (1 - \mu^2)^\ell \right)$$

for $\ell \in \mathbb{N}_0$, $\alpha, \beta > -1$, and $\mu \in [-1, 1]$. They satisfy that

$$\frac{\partial}{\partial \mu} P_\ell^{(\alpha,\beta)}(\mu) = \frac{1}{2}(\ell + \alpha + \beta + 1) P_{(\ell-1)}^{(\alpha+1,\beta+1)}(\mu).$$

Since Legendre polynomials are particular instances of Jacobi polynomials for $\alpha = \beta = 0$, we conclude by recursion that

$$\frac{\partial^n}{\partial \mu^n} P_\ell(\mu) = \frac{\partial^n}{\partial \mu^n} P_\ell^{(0,0)}(\mu) = \frac{(\ell + n)!}{2^n \, \ell!} P_{(\ell-n)}^{(n,n)}(\mu)$$

for every $n \leq \ell$. This implies that

$$\int_{-1}^{1} \left(\frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right) \left(\frac{\partial^n}{\partial \mu^n} P_{\ell'}(\mu) \right) (1 - \mu^2)^n \, d\mu$$

$$= \int_{-1}^{1} \frac{(\ell + n)!}{2^n \, \ell!} P_{(\ell-n)}^{(n,n)}(\mu) \frac{(\ell' + n)!}{2^n \, \ell'!} P_{(\ell'-n)}^{(n,n)}(\mu)(1 - \mu)^n (1 + \mu)^n \, d\mu$$

$$= \delta_{\ell\ell'} \frac{2}{2\ell + 1} \frac{(\ell + n)!}{(\ell - n)!},$$

where the last equation follows from the orthogonality of the Jacobi polynomials (see, e.g., [21]) and

$$\int_{-1}^{1} \left(P_{(\ell-n)}^{(n,n)}(\mu) \right)^2 (1 - \mu)^n (1 + \mu)^n \, d\mu = \frac{2^{2n+1}}{2\ell + 1} \frac{\ell! \, \ell!}{(\ell - n)! \, (\ell + n)!}.$$

In conclusion we have shown that

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu = \sum_{\ell=n}^{\infty} A_\ell^2 \frac{2\ell + 1}{2(4\pi)^2} \frac{(\ell + n)!}{(\ell - n)!},$$

since for $n > \ell$ the n-th derivative of P_ℓ vanishes. To finish the proof it remains to show that for $n \leq \ell$ there exist constants $c_1(n)$ and $c_2(n)$ such that

$$c_1(n)\ell^{2n} \leq \frac{(\ell + n)!}{(\ell - n)!} \leq c_2(n)\ell^{2n}.$$

This follows from Stirling's inequalities

$$\sqrt{2\pi} \, \ell^{\ell+1/2} e^{-\ell} \leq \ell! \leq e \cdot \ell^{\ell+1/2} e^{-\ell}, \quad \ell \in \mathbb{N}$$

by writing

$$\frac{(\ell + n)^{\ell+n}}{(\ell - n)^{\ell-n}} = \ell^{\ell+n-(\ell-n)} \frac{(1 + n/\ell)^{\ell(1+n/\ell)}}{(1 - n/\ell)^{\ell(1-n/\ell)}}$$

and by using the properties of the exponential function. $\square$

Proposition 3.1 provides a necessary and sufficient criterion for the weighted 2-summability of the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ in terms of weighted square integrability of the n-th weak derivatives of k_I with respect to the weight function $(1 - \mu^2)^n$. It can be formalized in the framework of weighted Sobolev spaces, which we will cover in what follows.

For $n \in \mathbb{N}_0$, let $H^n(-1,1) \subset L^2(-1,1)$ denote the standard Sobolev spaces. We define the function spaces $V^n(-1,1)$ as the closures of $H^n(-1,1)$ with respect to the weighted norms $\| \cdot \|_{V^n(-1,1)}$ given by

$$\|u\|_{V^n(-1,1)}^2 := \sum_{j=0}^{n} |u|_{V^j(-1,1)}^2,$$

where for $j \in \mathbb{N}_0$ the seminorm $| \cdot |_{V^j(-1,1)}$ is defined by

$$|u|_{V^j(-1,1)}^2 := \int_{-1}^{1} \left| \frac{\partial^j}{\partial \mu^j} u(\mu) \right|^2 (1 - \mu^2)^j \, d\mu.$$

With this definition, $(V^n(-1,1), n \in \mathbb{N}_0)$ is a decreasing scale of separable Hilbert spaces, i.e.,

$$L^2(-1,1) = V^0(-1,1) \supset V^1(-1,1) \supset \cdots \supset V^n(-1,1) \supset \cdots$$

By Ehrling's lemma, the norm of $V^n(-1,1)$ is equivalent to the first and the last element of the sum, i.e.,

$$\|u\|_{V^n(-1,1)}^2 \simeq \|u\|_{L^2(-1,1)}^2 + |u|_{V^n(-1,1)}^2$$

for all $u \in V^n(-1,1)$. We will in the sequel not distinguish between these norms by a separate notation.

In what follows we are deriving further equivalent norms of $V^n(-1,1)$ similarly to Proposition 3.1. Therefore let us first observe that any $u \in L^2(-1,1)$ can be expanded in the $L^2(-1,1)$ convergent Fourier–Legendre series

$$u = \sum_{\ell=0}^{\infty} u_\ell \frac{2\ell + 1}{2} P_\ell$$

with

$$u_\ell := \int_{-1}^{1} u(x) P_\ell(x)\, dx$$

for all $\ell \in \mathbb{N}_0$. Setting

$$A_\ell := 2\pi u_\ell,$$

we obtain that

$$u = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} P_\ell,$$

i.e., u is a valid kernel k_I. In conclusion we have proven the following version of Proposition 3.1.

Proposition 3.2. *Let $u \in L^2(-1,1)$ and $n \in \mathbb{N}_0$ be given. Then $u \in V^n(-1,1)$ if and only if*

$$\sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell + 1}{2} (1 + \ell^{2n}) < +\infty,$$

i.e.,

$$\|u\|_{V^n(-1,1)}^2 \simeq \sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell + 1}{2} (1 + \ell^{2n})$$

is an equivalent norm in $V^n(-1,1)$.

In other words we have just shown an isomorphism between the spaces $V^n(-1,1)$ and the weighted sequence spaces $\ell_n := \ell^2((\frac{2\ell+1}{2}(1 + \ell^{2n}), \ell \in \mathbb{N}_0))$, where $(\frac{2\ell+1}{2}(1 + \ell^{2n}), \ell \in \mathbb{N}_0)$ denotes the sequence of weights. Our goal is to extend this isomorphism to spaces $V^\eta(-1,1)$ with $\eta \notin \mathbb{N}_0$. We define for $n < \eta < n + 1$ the interpolation space $V^\eta(-1,1)$ with the real method of interpolation in the sense of [23] by

$$V^\eta(-1,1) = \left(V^n(-1,1), V^{n+1}(-1,1)\right)_{\eta-n,2}$$

equipped with the norms $\| \cdot \|_{V^\eta(-1,1)}$ given by

$$\|u\|_{V^\eta(-1,1)}^2 = \int_0^{\infty} t^{-2(\eta-n)} |K(t,u)|^2 \frac{dt}{t},$$

where the K-functional is defined by

$$K(t,u) = \inf_{u=v+w} \left(\|v\|_{V^k(-1,1)} + t \|w\|_{V^n(-1,1)} \right)$$

for $t > 0$.

The definition of the interpolation spaces ℓ_η for $\eta \notin \mathbb{N}_0$ is done similarly. The *interpolation property* of the spaces (see, e.g., step 4 in proof of Theorem 1.3.3 in [23] or Proposition 2.4.1 in [22]) implies that the spaces $V^\eta(-1,1)$ and ℓ_η are also isomorphic. So the only thing that remains to show to extend Proposition 3.2 to fractional differentiation orders is the equivalence of the norm induced by the real method of interpolation and the sequence of weights with parameter η instead of n. This is done with the interpolation theorem of Stein–Weiss in the following proposition.

Proposition 3.3. *Let $u \in L^2(-1,1)$ and $\eta \in \mathbb{R}_+$ be given. Then $u \in V^\eta(-1,1)$ if and only if*

$$\sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell+1}{2}(1+\ell^{2\eta}) < +\infty,$$

i.e.,

$$\|u\|_{V^\eta(-1,1)}^2 \simeq \sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell+1}{2}(1+\ell^{2\eta})$$

is an equivalent norm in $V^\eta(-1,1)$.

Proof. The proposition is already proven for $\eta \in \mathbb{N}_0$ in Proposition 3.2. So let $n < \eta < n+1$ for some $n \in \mathbb{N}_0$ be given and set $\theta := \eta - n$. Applying the interpolation theorem of Stein–Weiss (see, e.g., Theorem 5.4.1 in [1]), we get that the weights of ℓ_η are given by

$$\left(\frac{2\ell+1}{2}(1+\ell^{2n})\right)^{1-\theta} \left(\frac{2\ell+1}{2}(1+\ell^{2(n+1)})\right)^\theta = \frac{2\ell+1}{2}(1+\ell^{2n})^{1-\theta}(1+\ell^{2(n+1)})^\theta.$$

It remains to show that this is equivalent to $\frac{2\ell+1}{2}(1+\ell^{2\eta})$. But this follows immediately with the observation that the function x^p, $p \in (0,1)$ is concave on $\mathbb{R}_+$ and satisfies $(x+y)^p \geq 2^{p-1}(x^p + y^p)$. $\qquad\square$

For more details on this we refer to Appendix A. In conclusion we have shown in this section that the decay of the angular power spectrum in terms of summability is equivalent to the smoothness of the corresponding kernel k_I in terms of elements of (a scale of) weighted Sobolev spaces.

4. SAMPLE HÖLDER CONTINUITY AND DIFFERENTIABILITY

So far, our analysis of GRFs via the Karhunen–Loève expansion in Section 2 focused on mean square properties. In this section we consider sample properties of isotropic GRFs introduced in Section 2. Specifically, we are interested how the $\mathbb{P}$-almost sure Hölder continuity of isotropic GRFs depends on the decay of the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ which is one possible characterization of isotropic GRFs on $\mathbb{S}^2$ by Theorem 2.3 and Lemma 2.4.

The following lemma relates the decay of the angular power spectrum to the Hölder continuity of the kernel k at zero.

Lemma 4.1. *Let $(A_\ell, \ell \in \mathbb{N}_0)$ be the angular power spectrum of an isotropic GRF on $\mathbb{S}^2$ with*

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$$

for some $\beta \in [0,2]$. Then the corresponding kernel function k satisfies that there exists a constant C_β such that for all $r \in [0,\pi]$

$$|k(0) - k(r)| \leq C_\beta r^\beta,$$

Proof. We observe that $P_\ell(1) = 1$ for all $\ell \in \mathbb{N}_0$ and that the derivative of $P_\ell(x)$ is bounded by $P_\ell'(1)$. Therefore

$$|1 - P_\ell(x)| = \left|\int_x^1 P_\ell'(y)\, dy\right| \leq |1-x|\frac{\ell(\ell+1)}{2}.$$

Furthermore we have that
$$|1 - P_\ell(x)| \leq 2.$$

This implies by interpolation that
$$|1 - P_\ell(x)| \leq \left(|1 - x| \frac{\ell(\ell+1)}{2} \right)^\gamma 2^{1-\gamma} \leq 2|1 - x|^\gamma (\ell(\ell+1))^\gamma$$

for all $\gamma \in [0, 1]$. Using this estimate we obtain that
$$|k(0) - k(r)| \leq \sum_{\ell=0}^\infty A_\ell \frac{2\ell+1}{4\pi} |1 - P_\ell(\cos r)|$$
$$\leq (2\pi)^{-1} |1 - \cos r|^\gamma \sum_{\ell=0}^\infty A_\ell (2\ell+1)(\ell(\ell+1))^\gamma$$

where the series converges if $\sum_{\ell=0}^\infty A_\ell \ell^{2\gamma+1}$, which holds by the made assumptions for all $\gamma \leq \beta/2$. Finally we observe that
$$|1 - \cos r| = \left| \int_0^r \sin x \, dx \right| \leq r \sin r = r \int_0^r \cos x \, dx \leq r^2 \cdot 1,$$

which implies overall with the choice $\beta = 2\gamma$ that
$$|k(0) - k(r)| \leq C_\beta r^\beta,$$

where
$$C_\beta := (2\pi)^{-1} \sum_{\ell=0}^\infty A_\ell (2\ell+1)(\ell(\ell+1))^{\beta/2}.$$

This finishes the proof of the lemma. $\qquad\square$

Lemma 4.1 asserts Hölder continuity of $k(r)$ near $r = 0$ in terms of a ℓ^1 criterion on the angular power spectrum of the isotropic GRF T, while we provided ℓ^2 criteria in Section 3. To relate these criteria we first observe that for $\epsilon > 0$
$$\sum_{\ell=0}^\infty A_\ell \ell^{1+\beta} \leq \zeta(1+\epsilon)^{1/2} \left(\sum_{\ell=0}^\infty A_\ell^2 \ell^{3+2\beta+\epsilon} \right)^{1/2}$$

by the Cauchy–Schwarz inequality, where ζ denotes the Riemann zeta function. This implies with Proposition 3.3 that
$$\sum_{\ell=0}^\infty A_\ell \ell^{1+\beta} < +\infty$$

if the kernel k_I is in $V^\eta(-1, 1)$ for some $\eta > \beta + 1$.

Our next step is to give bounds on moments of $T(x) - T(y)$ for $x, y \in \mathbb{S}^2$ in terms of the geodesic distance $d(x, y)$. We prove the lemma by expressing the moments in terms of the kernel k and by an application of the preceding lemma.

Lemma 4.2. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that*
$$\sum_{\ell=0}^\infty A_\ell \ell^{1+\beta} < +\infty$$

for some $\beta \in [0,2]$, then for all $p \in \mathbb{N}$ there exists a constant $C_{\beta,p}$ such that

$$\mathbb{E}(|T(x) - T(y)|^{2p}) \leq C_{\beta,p}\, d(x,y)^{\beta p}$$

for all $x, y \in \mathbb{S}^2$.

Proof. First note that $T(x) - T(y)$ is a centered Gaussian random variable. Furthermore, if X is a $\mathcal{N}(0, \sigma^2)$-distributed random variable, then

$$\mathbb{E}(|X|^{2p}) = \mathbb{E}(|\sigma Y|^{2p}) = (\sigma^2)^p\, \mathbb{E}(|Y|^{2p}) = \mathbb{E}(X^2)^p c_{2p}$$

for $p \in \mathbb{N}$, where Y is a standard normally distributed random variable and c_{2p} denotes the $2p$-th moment of Y. We also observe that $\mathbb{E}(|T(x) - T(y)|^2)$ can be expressed in terms of k since

$$\begin{aligned}
\mathbb{E}(|T(x) - T(y)|^2) &= \mathbb{E}(T(x)^2) - 2\,\mathbb{E}(T(x)T(y)) + \mathbb{E}(T(y)^2) \\
&= k_T(x,x) - 2k_T(x,y) + k_T(y,y) \\
&= 2\big(k(0) - k(d(x,y))\big).
\end{aligned}$$

Combining the two previous observations, we conclude that

$$\begin{aligned}
\mathbb{E}(|T(x) - T(y)|^{2p}) &= c_{2p}\, \mathbb{E}(|T(x) - T(y)|^2)^p \\
&= 2\, c_{2p}(k(0) - k(d(x,y)))^p \\
&\leq 2\, c_{2p}\, C_\beta^p\, d(x,y)^{\beta p},
\end{aligned}$$

where we applied Lemma 4.1 in the last step. This finishes the proof of the lemma. $\qquad\square$

The following result is a version of the Kolmogorov–Chentsov theorem for random fields with domain $\mathbb{S}^2$. Note that in this result the fields do not have to be Gaussian or isotropic.

Theorem 4.3 (Kolmogorov–Chentsov theorem)**.** *Let T be a random field on the sphere that satisfies for some $p > 0$ and some $\epsilon \in (0,1]$ that there exists a constant C such that*

$$\mathbb{E}(|T(x) - T(y)|^p) \leq Cd(x,y)^{2+\epsilon p}$$

for all $x, y \in \mathbb{S}^2$. Then there exists a continuous modification of T that is locally Hölder continuous with exponent γ for all $\gamma \in (0, \epsilon)$.

Proof. Let us first construct six charts $(U_i, i = 1, \ldots, 6)$ that cover the sphere by taking the six possible hemispheres given by the coordinate system such that the boundary is a circle of radius r with $r \in (\sqrt{2/3}, 1)$, i.e., we take a bit less than the complete hemispheres but enough to cover the whole sphere. Let the coordinate maps $(\varphi_i, i = 1, \ldots, 6)$ be given by the projection onto the plane that divides the hemispheres, i.e., if U is contained in the northern hemisphere than the corresponding coordinate map φ is given by $\varphi((x_1, x_2, x_3)) := (x_1, x_2)$ for $x = (x_1, x_2, x_3) \in U$ and maps onto the disc $\{x \in \mathbb{R}^2, \|x\|_{\mathbb{R}^2} < r\}$.

For a given chart (U, φ), we have to show that the Euclidean norm in $\mathbb{R}^2$ is equivalent to the metric on $\mathbb{S}^2$, i.e., that there exist constants $C_1, C_2 > 0$ such that for all $x, y \in U$

$$C_1 \|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2} \leq d(x,y) \leq C_2 \|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}$$

or equivalently that

$$C_1 \leq \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}} \leq C_2.$$

We show this estimate for U contained in the northern hemisphere. The calculations for the other five charts are similar and the bounds are the same due to symmetry.

One first calculates that

$$\langle x, y \rangle_{\mathbb{R}^3} = 1 - \frac{1}{2}(\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2 + |x_3 - y_3|^2)$$

and shows that

$$0 \leq |x_3 - y_3|^2 \leq \frac{2r^2}{1 - r^2}\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2.$$

This implies that we can bound the quotient of interest from above and below by

$$\frac{\arccos(1 - \frac{1}{2}\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2)}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}} \leq \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}}$$

$$\leq \frac{\arccos(1 - (\frac{1}{2} + \frac{r^2}{1 - r^2})\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2)}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}},$$

since arccos is a monotonically decreasing function. Let us define $f : [0, 2r) \to \mathbb{R}$ by

$$f(a) := \frac{\arccos(1 - \alpha a^2)}{a}$$

for $a \in (0, 2r)$, where $\alpha = 1/2, 1/2 + r^2/(1 - r^2)$. Then one shows with standard methods from real analysis that f is well-defined on $[0, 2r)$ and monotonically increasing which leads with the observation that $f(0) = \sqrt{2\alpha}$ by l'Hôpital's rule to the conclusion that

$$C_1 := 1 \leq \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}} \leq \frac{\arccos(\frac{2r^4 + 3r^2 - 1}{r^2 - 1})}{2r} =: C_2 < +\infty$$

and finishes the proof of the equivalence of geodetic and Euclidean distances on the sphere and in the charts.

For $a, b \in \varphi(U)$ it holds for the random field on the chart by the made assumptions and the equivalence of the distances that

$$\mathbb{E}(|T(\varphi^{-1}(a)) - T(\varphi^{-1}(b))|^p) \leq Cd(\varphi^{-1}(a), \varphi^{-1}(b))^{2/p+\epsilon} \leq C \cdot C_2^{2/p+\epsilon}\|a - b\|^{2/p+\epsilon}.$$

Since $\varphi(U)$ is a domain in $\mathbb{R}^2$, we obtain by the Kolmogorov–Chentsov theorem for domains (see Theorem 2.1 in [11] or Theorem 4.5 in [16]) that there exists a continuous modification $T_1 \circ \varphi^{-1}$ that is locally Hölder continuous with exponent γ for all $\gamma \in (0, \epsilon)$ and so is T_1 on U due to the smoothness of the coordinate map.

With the same proof we obtain continuous modifications $(T_i, i = 1, \ldots, 6)$ on all charts $(U_i, i = 1, \ldots, 6)$. We glue these together with a smooth partition of unity $(\rho_i, i = 1, \ldots, 6)$ on $\mathbb{S}^2$, which is subordinate to the open covering, (see, e.g., Theorem 1.73 in [9]), by

$$\tilde{T}(x) := \sum_{i=1}^{6} \rho_i(x)T_i(x)$$

for all $x \in \mathbb{S}^2$, where $T_i(x) = 0$ for $x \notin U_i$. Then $\tilde{T}$ is a continuous modification of T that is locally Hölder continuous with the same exponent γ for all $\gamma \in (0, \epsilon)$ due to the smoothness of the partitions of unity. This finishes the proof of the theorem. $\square$

With the made observations, we are now prepared to prove one of the main results of this section which states that if the angular power spectrum of an isotropic Gaussian random field is summable with weights $\ell^{1+\beta}$, than there exists a continuous modification which is Hölder continuous of exponent γ for all $\gamma < \beta/2$.

Theorem 4.4. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that*

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$$

for some $\beta \in [0, 2]$, then there exists a continuous modification of T that is Hölder continuous of exponent γ for all $\gamma < \beta/2$.

Proof. The claim follows by the application of the previous results in the following way: It holds by Lemma 4.2 that for all $p \in \mathbb{N}$ and $x, y \in \mathbb{S}^2$ the random field satisfies

$$\mathbb{E}(|T(x) - T(y)|^{2p}) \leq C_{\beta,p}\, d(x,y)^{\beta p} = C_{\beta,p}\, d(x,y)^{2+(\beta/2-1/p)2p}.$$

Theorem 4.3 implies that there exists a continuous modification that is locally Hölder continuous of exponent γ for all $\gamma < \beta/2 - 1/p$ for any $p \in \mathbb{N}$, i.e., of exponent γ for all $\gamma < \beta/2$. $\quad\square$

Just as an example let us calculate the parameters of $\mathbb{P}$-almost sure Hölder continuity for the two choices of α that we simulate in the following sections. For $\alpha = 3$ we get $\beta < 1$ which implies $\gamma < 1/2$ in Theorem 4.4 and $\alpha = 5$ implies $\beta = 2$ and therefore $\gamma < 1$.

Furthermore as second main result of this section we are interested in the assumptions on the angular power spectrum that imply the existence of differentiable modifications of isotropic GRFs. In particular in the context of approximate, numerical solutions of partial differential equations, regularity properties of samples are essential for the derivation of convergence rates for, e.g., Finite Element or Finite Difference discretizations.

Theorem 4.5. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ and expectation zero. If the angular power spectrum satisfies that*

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$$

for some $\beta > 0$, then there exists a modification of T that is k-times continuously differentiable for all nonnegative integers $k < \beta/2 - 1$.

Proof. Let us first observe that the made assumptions imply that T has a continuous modification by Theorem 4.4. Without loss of generality let T already be the continuous modification, which is an isotropic Gaussian random field with the same parameters and has a Karhunen–Loève expansion with the same parameters by Corollary 2.5.

We observe that the norms in the Sobolev spaces $H^\eta(\mathbb{S}^2) \subset L^2(\mathbb{S}^2)$ for $\eta > 0$ are equivalent to the norm $\|\cdot\|_{\Delta_\eta}$ given by

$$\|u\|_{\Delta_\eta}^2 := \|u\|_{L^2(\mathbb{S}^2)}^2 + \|(-\Delta)^{\eta/2} u\|_{L^2(\mathbb{S}^2)}^2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |u_{\ell m}|^2 \left(1 + (\ell(\ell+1))^\eta\right)$$

for $u \in H^\eta(\mathbb{S}^2)$ with expansion

$$u = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m} Y_{\ell m}.$$

Furthermore it holds that $H^\eta(\mathbb{S}^2)$ is continuously embedded into $C^k(\mathbb{S}^2)$ for all $k < \eta - 1$ by Theorem 6.5 in [25]. Now T satisfies that

$$
\begin{aligned}
\mathbb{E}(\|T\|^2_{\Delta_\eta}) &= \mathbb{E}\left(\sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell |a_{\ell m}|^2 \left(1 + (\ell(\ell+1))^\eta\right)\right) \\
&= \sum_{\ell=0}^\infty \left(A_\ell + 2\sum_{m=1}^\ell A_\ell/2\right)\left(1 + (\ell(\ell+1))^\eta\right) \\
&= \sum_{\ell=0}^\infty A_\ell(\ell+1)\left(1 + (\ell(\ell+1))^\eta\right) \\
&< +\infty
\end{aligned}
$$

for all $\eta \leq \beta/2$ by our assumptions, where we applied Corollary 2.5. Together with a Cauchy sequence argument this implies that the series expansion of T converges in $L^2(\Omega; H^\beta(\mathbb{S}^2))$ and therefore in probability. By the Itô–Nisio theorem [4] the series converges $\mathbb{P}$-almost surely to the same limit. Therefore $T \in H^\beta(\mathbb{S}^2)$ $\mathbb{P}$-almost surely and by the above Sobolev embedding and the continuity of the random field, $T \in C^k(\mathbb{S}^2)$ for all $k < \beta/2 - 1$ $\mathbb{P}$-almost surely. Setting T to zero on the remaining $\mathbb{P}$-null set of Ω, we obtain a modification of T in $C^k(\mathbb{S}^2)$ for all $k < \beta/2 - 1$. This finishes the proof of the theorem. $\qquad\square$

5. Approximation of isotropic Gaussian random fields

Let us approximate and simulate isotropic Gaussian random fields in this section, where we use the properties of the random fields that were introduced in Section 2. In what follows, we consider centered random fields without loss of generality. It is clear by Corollary 2.5 that we can transform the centered, isotropic random field into a field with nonzero expectation by adding the expectation, which is a constant according to Lemma 2.4. To prepare the presentation of the approximation of isotropic GRFs on $\mathbb{S}^2$, we rewrite its series expansions, where we use the properties of the spherical harmonic functions and the structure of real-valued random fields.

Lemma 5.1. *Let T be a strongly isotropic, centered, Gaussian random field. For $\ell \in \mathbb{N}$, $m = 1, \ldots, \ell$, and $\vartheta \in [0, \pi]$, set*

$$
L_{\ell m}(\vartheta) := \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos\vartheta).
$$

Then, for $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$,

$$
T(y) = \sum_{\ell=0}^\infty \sqrt{A_\ell} X^1_{\ell 0} L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^\ell L_{\ell m}(\vartheta)(X^1_{\ell m}\cos(m\varphi) + X^2_{\ell m}\sin(m\varphi))
$$

in law, where $((X^1_{\ell m}, X^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, real-valued, standard normally distributed random variables and $X^2_{\ell 0} = 0$ for $\ell \in \mathbb{N}_0$.

Proof. By Corollary 2.5, T can be represented in the (mean-square convergent) Karhunen–Loève expansion

$$
T = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell a_{\ell m} Y_{\ell m}.
$$

This sum can be rewritten to

$$
\begin{aligned}
T &= \sum_{\ell=0}^{\infty}\Big(a_{\ell 0}Y_{\ell 0} + \sum_{m=1}^{\ell}(a_{\ell m}Y_{\ell m} + a_{\ell -m}Y_{\ell -m})\Big)\\
&= \sum_{\ell=0}^{\infty}\Big(a_{\ell 0}L_{\ell 0}(\vartheta) + \sum_{m=1}^{\ell}(a_{\ell m}Y_{\ell m} + (-1)^{m}\overline{a_{\ell m}}(-1)^{m}\overline{Y_{\ell m}}))\Big)\\
&= \sum_{\ell=0}^{\infty}\Big(a_{\ell 0}L_{\ell 0}(\vartheta) + \sum_{m=1}^{\ell}(a_{\ell m}Y_{\ell m} + \overline{a_{\ell m}Y_{\ell m}}))\Big)\\
&= \sum_{\ell=0}^{\infty}\Big(a_{\ell 0}L_{\ell 0}(\vartheta) + \sum_{m=1}^{\ell}2\,\mathrm{Re}(a_{\ell m}Y_{\ell m})\Big)
\end{aligned}
$$

by Lemma 2.4 and the properties of the spherical harmonic functions. We observe that

$$
Y_{\ell m}(\vartheta,\varphi) = L_{\ell m}(\vartheta)e^{im\varphi} = L_{\ell m}(\vartheta)(\cos(m\varphi) + i\sin(m\varphi))
$$

for $(\vartheta,\varphi) \in [0,\pi] \times [0,2\pi)$ and therefore, by the properties of complex numbers, that

$$
\mathrm{Re}(a_{\ell m}Y_{\ell m}(\vartheta,\varphi)) = L_{\ell m}(\vartheta)\big(\mathrm{Re}\,a_{\ell m}\cos(m\varphi) - \mathrm{Im}\,a_{\ell m}\sin(m\varphi)\big).
$$

Let $((X_{\ell m}^{1}, X_{\ell m}^{2}), \ell \in \mathbb{N}_0, m = 0,\ldots,\ell)$ be a sequence of independent, real-valued, standard normally distributed random variables, then

$$
\mathrm{Re}\,a_{\ell m} = \sqrt{\frac{A_\ell}{2}}X_{\ell m}^{1} \quad\text{and}\quad -\mathrm{Im}\,a_{\ell m} = \mathrm{Im}\,a_{\ell m} = \sqrt{\frac{A_\ell}{2}}X_{\ell m}^{2}
$$

in law for $\ell \in \mathbb{N}$ and $m = 1,\ldots,\ell$ by Corollary 2.5. Furthermore the corollary implies that

$$
a_{\ell 0} = \sqrt{A_\ell}X_{\ell 0}^{1}
$$

for $\ell \in \mathbb{N}_0$. The insertion these observations into the Karhunen–Loève expansion of T completes the proof. $\qquad\square$

For a given sequence $((X_{\ell m}^{1}, X_{\ell m}^{2}), \ell \in \mathbb{N}_0, m = 0,\ldots,\ell)$ as specified in Lemma 5.1, set

$$
T(y) := \sum_{\ell=0}^{\infty}\sqrt{A_\ell}X_{\ell 0}^{1}L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^{\ell}L_{\ell m}(\vartheta)(X_{\ell m}^{1}\cos(m\varphi) + X_{\ell m}^{2}\sin(m\varphi)).
$$

In what follows, we truncate the series expansion in order to implement it and prove its convergence. For $\kappa \in \mathbb{N}$, we set

$$
T^{\kappa}(y) := \sum_{\ell=0}^{\kappa}\sqrt{A_\ell}X_{\ell 0}^{1}L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^{\ell}L_{\ell m}(\vartheta)(X_{\ell m}^{1}\cos(m\varphi) + X_{\ell m}^{2}\sin(m\varphi)),
$$

where $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ and $(\vartheta,\varphi) \in [0,\pi] \times [0,2\pi)$.

Proposition 5.2. *Let the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ decay algebraically with order $\alpha > 2$, i.e., there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the series of approximate random fields $(T^{\kappa}, \kappa \in \mathbb{N})$ converges to the random field T in $L^2(\Omega; L^2(\mathbb{S}^2))$ and the truncation error is bounded by*

$$
\|T - T^{\kappa}\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq \hat{C} \cdot \kappa^{-(\alpha-2)/2}
$$

for $\kappa \geq \ell_0$, where

$$\hat{C}^2 = C \cdot \left(\frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \right).$$

Proof. Since $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, standard normally distributed random variables, the error is equal to

$$\|T - T^\kappa\|_{L^2(\Omega; L^2(\mathbb{S}^2))}$$

$$= \sum_{\ell = \kappa + 1}^\infty \Big(A_\ell \, \mathbb{E}((X_{\ell 0}^1)^2) \|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2$$

$$+ 2A_\ell \sum_{m=1}^\ell \big(\mathbb{E}((X_{\ell m}^1)^2) \| \operatorname{Re} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 + \mathbb{E}((X_{\ell m}^2)^2) \| \operatorname{Im} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 \big) \Big)$$

$$= \sum_{\ell = \kappa + 1}^\infty \Big(A_\ell \|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2 + 2A_\ell \sum_{m=1}^\ell \big(\| \operatorname{Re} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 + \| \operatorname{Im} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 \big) \Big).$$

We observe that $\|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2 = 1$ and $\| \operatorname{Re} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 + \| \operatorname{Im} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 = 1$ for $\ell \in \mathbb{N}_0$ and $m = 1, \ldots, \ell$. Therefore, the sum simplifies to

$$\|T - T^\kappa\|_{L^2(\Omega; L^2(\mathbb{S}^2))} = \sum_{\ell = \kappa + 1}^\infty (2\ell + 1) A_\ell,$$

which is bounded by

$$\sum_{\ell = \kappa + 1}^\infty (2\ell + 1) A_\ell \leq C \sum_{\ell = \kappa + 1}^\infty (2\ell^{-(\alpha - 1)} + \ell^{-\alpha})$$
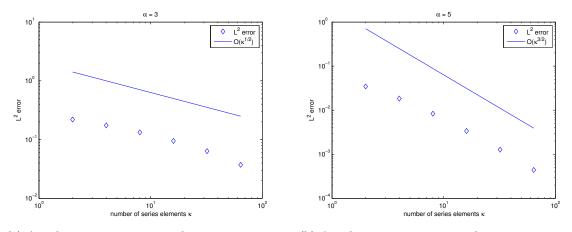
due to the assumed properties of the angular power spectrum. We rewrite the sum and bound it by the corresponding integral which leads to

$$\sum_{\ell = \kappa + 1}^\infty (2\ell^{-(\alpha - 1)} + \ell^{-\alpha}) = \sum_{\ell = 1}^\infty (2(\ell + \kappa)^{-(\alpha - 1)} + (\ell + \kappa)^{-\alpha})$$

$$\leq \int_0^\infty \big(2(x + \kappa)^{-(\alpha - 1)} + (x + \kappa)^{-\alpha} \big) \, dx$$

$$= \Big(\frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \kappa^{-1} \Big) \kappa^{-(\alpha - 2)}.$$

This finishes the proof since κ^{-1} is bounded by 1. $\qquad\square$

In an implementation in MATLAB we verified the theoretical results. We took as "exact" solution the random fields with $\kappa = 2^7$ since for larger κ the elements of the angular power spectrum A_ℓ and therefore the increments were so small that MATLAB failed to calculate the series expansion. Instead of the $L^2(\mathbb{S}^2)$ error in space, we used the maximum over all grid points which is a stronger error. In Figure 1 the results and the theoretical convergence rates are shown for $\alpha = 3, 5$. One observes that the simulation results match the theoretical results in Proposition 5.2.

Since we discussed $\mathbb{P}$-almost sure Hölder continuity in Section 4, we are also interested in $\mathbb{P}$-almost sure convergence rates of the approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$. Therefore,

(a) Angular power spectrum with parameter $\alpha = 3$.   (b) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 1. L^2 error of the approximation of Gaussian random fields with different angular power spectrum and 1000 Monte Carlo samples.

we include the following result on $L^p(\Omega; L^2(\mathbb{S}^2))$ convergence since we need it for optimal pathwise convergence rates of the approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$.

Lemma 5.3. *Let the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ decay algebraically with order $\alpha > 2$, i.e., there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the series of approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$ converges to the random field T in $L^p(\Omega; L^2(\mathbb{S}^2))$ for any finite $p > 0$, and the truncation error is bounded by*

$$\|T - T^\kappa\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-(\alpha-2)/2}$$

for $\kappa \geq \ell_0$, where $\hat{C}_p$ is a constant that depends on p, C, and α.

Proof. For $p \leq 2$ the result follows with Proposition 5.2 and Hölder's inequality. Therefore let us consider $p > 2$ now. We prove the claim for $p = 2m$, $m \in \mathbb{N}$. For all other $p \in \mathbb{R}_+$, the result follows again by Hölder's inequality. So let $m \in \mathbb{N}$, then Corollary 2.17 in [2] states that there exists a constant C_m such that

$$\|T - T^\kappa\|^{2m}_{L^{2m}(\Omega; L^2(\mathbb{S}^2))} \leq C_m \|T - T^\kappa\|^{2m}_{L^2(\Omega; L^2(\mathbb{S}^2))}.$$
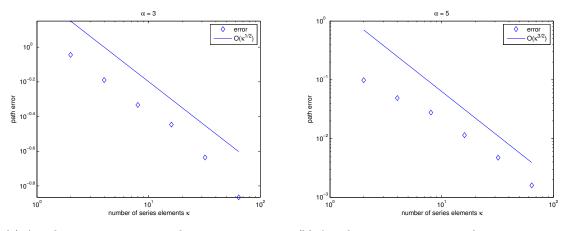
Applying Proposition 5.2 we conclude that

$$\|T - T^\kappa\|_{L^{2m}(\Omega; L^2(\mathbb{S}^2))} \leq (C_m)^{1/(2m)} \hat{C} \cdot \kappa^{-(\alpha-2)/2},$$

where

$$\hat{C}^2 = C \cdot \left(\frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \right),$$

which finishes the proof.   □

We have just shown that the convergence rate does not depend on p. This is necessary to get up to an epsilon the same sample convergence rates as in the p-th moment by the Borel–Cantelli lemma, which we show in what follows.

(a) Angular power spectrum with parameter $\alpha = 3$. (b) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 2. Error of the approximation of a sample of Gaussian random fields with different angular power spectrum.

Corollary 5.4. *Let the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ decay algebraically with order $\alpha > 2$, i.e., there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the series of approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$ converges to the random field T $\mathbb{P}$-almost surely and the truncation error is bounded by*

$$\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}, \quad \mathbb{P}\text{-a.s.}$$

for all $\beta < (\alpha - 2)/2$.

Proof. Let $\beta < (\alpha - 2)/2$, then the Chebyshev inequality and Lemma 5.3 imply that

$$\mathbb{P}(\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \geq \kappa^{-\beta}) \leq \kappa^{\beta p} \, \mathbb{E}(\|T - T^\kappa\|^p_{L^2(\mathbb{S}^2)}) \leq \hat{C}_p^p \kappa^{(\beta - (\alpha-2)/2)p}.$$

For all $p > ((\alpha - 2)/2 - \beta)^{-1}$ the series

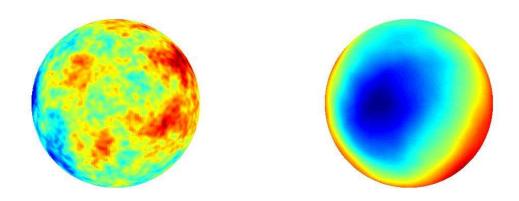$$\sum_{\kappa=1}^{\infty} \kappa^{(\beta - (\alpha-2)/2)p} < +\infty$$

converges and therefore the Borel–Cantelli lemma implies the claim. $\square$

In Figure 2, we show the corresponding error plots to Figure 1 but instead of a Monte Carlo simulation of the approximate $L^2(\Omega; L^2(\mathbb{S}^2))$ error we plotted the error of one sample. The convergence results coincide with the theoretical results in Corollary 5.4.

To give the reader an idea of the structure of the Gaussian random fields in dependence of the decay of the angular power spectrum, we include two samples in Figure 3. Here we chose $A_\ell = (\ell + 1)^{-\alpha}$ for $\ell \in \mathbb{N}_0$ and $\alpha = 3, 5$. Therefore, $A_\ell \leq \ell^{-\alpha}$ for all $\ell \geq 1$, which meets the assumptions of Proposition 5.2. We truncated the series at $\kappa = 100$, since larger κ do not affect the pictures, but the numerical accuracy suffers due to multiplication and addition of very small numbers.

We remark that similarly to fast Fourier transforms, there exist fast transforms for spherical harmonic functions, see, e.g., [12] and the set of C routines *SpharmonicKit* explained in [3].

(a) Angular power spectrum with parameter $\alpha = 3$.   (b) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 3. Two samples of isotropic Gaussian random fields with different angular power spectrum and truncation at $\kappa = 100$.

These allow to simulate isotropic Gaussian random fields with the suggested approximations efficiently also for large choices of κ.

6. LOGNORMAL ISOTROPIC GAUSSIAN RANDOM FIELDS

In this section we consider lognormal random fields on $\mathbb{S}^2$, i.e., if T is an isotropic Gaussian random field on $\mathbb{S}^2$ then we are interested in $\exp(T)$ given by $\exp(T(x))$ for all $x \in \mathbb{S}^2$. These random fields are especially of interest when modeling Saharan dust particles (see, e.g., [15]), feldspar particles (cf., [24]), and ice crystals (cf., [14]). We show in the following that the sample regularity of a lognormal random field is the same as that of the underlying Gaussian random field. This is done by first proving regularity in $L^p(\Omega;\mathbb{R})$ and then applying the Kolmogorov–Chentsov theorem similarly to Section 4.

Lemma 6.1. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that $A_\ell \leq C\ell^{-\alpha}$ for all $\ell \in \mathbb{N}$, some $\alpha > 2$, and some constant C, then for all $p \in \mathbb{N}$ and $\beta < \alpha - 2$, $\beta \leq 2$ there exists a constant $C_{\beta,p}$ such that*

$$\| \exp(T(x)) - \exp(T(y)) \|_{L^p(\Omega;\mathbb{R})} \leq 2 \exp(p\,k(0)) C_{\beta,p}\, d(x,y)^{\beta/2}$$

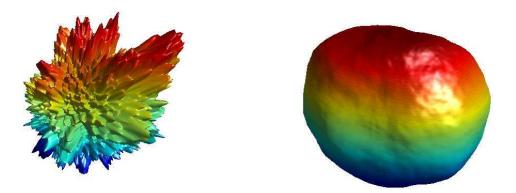*for all $x, y \in \mathbb{S}^2$.*

Proof. Let us first observe that for $a, b \in \mathbb{R}$ it holds that

$$|e^a - e^b| = \left| \int_b^a e^z\, dz \right| \leq |a - b| \max\{e^a, e^b\} \leq |a - b|(e^a + e^b).$$

This implies for $x, y \in \mathbb{S}^2$ that

$$\| \exp(T(x)) - \exp(T(y)) \|_{L^p(\Omega;\mathbb{R})}^p \leq \mathbb{E}\Big(\big(\exp(T(x)) + \exp(T(y)) \big)^p |T(x) - T(y)|^p \Big)$$

$$\leq \mathbb{E}\Big(\big(\exp(T(x)) + \exp(T(y)) \big)^{2p} \Big)^{1/2} \cdot \mathbb{E}(|T(x) - T(y)|^{2p})^{1/2},$$

(a) Angular power spectrum with parameter $\alpha = 3$.   (b) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 4. Corresponding lognormal samples to Figure 3 with $\kappa = 100$.

where we applied Hölder's inequality in the last step. By Lemma 4.2 the second term is bounded by

$$\mathbb{E}(|T(x) - T(y)|^{2p})^{1/2} \leq C_{\beta,p}^p d(x,y)^{p\beta/2}$$

for any $\beta < \alpha - 2$, $\beta \leq 2$. The first term satisfies that

$$\mathbb{E}\Big(\big(\exp(T(x)) + \exp(T(y))\big)^{2p} \Big)^{1/2} \leq 2^{(2p-1)/2} \Big(\mathbb{E}(\exp(2pT(x))) + \mathbb{E}(\exp(2pT(y))) \Big)^{1/2}.$$

Since $T(x)$ and $T(y)$ are real-valued Gaussian random variables with expectation zero and variance $k(0)$, the moment generating function is given by

$$\mathbb{E}(\exp(2pT(x))) = \exp(2p^2 k(0)),$$

which implies that

$$\mathbb{E}\Big(\big(\exp(T(x)) + \exp(T(y))\big)^{2p} \Big)^{1/2} \leq 2^{(2p-1)/2} 2^{1/2} \exp(p^2 k(0)) = 2^p \exp(p^2 k(0)).$$

Therefore we overall conclude that

$$\| \exp(T(x)) - \exp(T(y)) \|_{L^p(\Omega;\mathbb{R})} \leq 2 \exp(p\, k(0)) C_{\beta,p} d(x,y)^{\beta/2},$$

which finishes the proof. $\square$

The lemma enables us to conclude that the lognormal random field of an isotropic Gaussian random field T has the same sample Hölder continuity properties as T.

Corollary 6.2. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell \in \mathbb{N}$, some $\alpha > 2$, and some constant C, then there exists a modification of $\exp(T)$ that is Hölder continuous of exponent s for all $s < (\alpha - 2)/2$, $s \leq 1$.*

Proof. The proof is the same as the one of Theorem 4.4, where we apply Lemma 6.1 instead of Lemma 4.2. $\square$

In Figure 4 we took the Gaussian random field samples that are shown in Figure 3 and plotted the deformed sphere with the corresponding lognormal radius which is done when modeling dust or feldspar particles resp. ice crystals.

In Theorem 4.5 we have shown the existence of k-times continuously differentiable modifications of isotropic GRFs depending on the convergence of the corresponding angular power spectrum. The compactness of the unit sphere, the smoothness of the exponential function, and the chain rule imply as a direct consequence that the same properties hold for the corresponding lognormal random fields.

Corollary 6.3. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ and expectation zero. If the angular power spectrum satisfies that*

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$$

for some $\beta > 0$, then there exists a modification of the corresponding lognormal random field $\exp(T)$ that is k-times continuously differentiable for all nonnegative integers $k < \beta/2 - 1$.

7. Stochastic partial differential equations on the sphere

In this section we consider the heat equation on the sphere with additive Q-Wiener noise as an example of a stochastic partial differential equation (SPDE) on $\mathbb{S}^2$. To discuss stochastic partial differential equations we first introduce Q-Wiener processes on the sphere.

To this end let us consider Q-Wiener processes that take values in $L^2(\mathbb{S}^2)$ and that are isotropic in space. Then, by Lemma 5.1 and by the construction of Q-Wiener processes out of GRFs as was done in an abstract setting, e.g., in [2, 17], a Q-Wiener process taking values in $L^2(\mathbb{S}^2)$ can be characterized by the Karhunen–Loève expansion

$$
\begin{aligned}
W(t,y) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(t) Y_{\ell m}(y) \\
&= \sum_{\ell=0}^{\infty} \sqrt{A_\ell} \beta_{\ell 0}^1(t) Y_{\ell 0}(y) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} (\beta_{\ell m}^1(t) \operatorname{Re} Y_{\ell m}(y) + \beta_{\ell m}^2(t) \operatorname{Im} Y_{\ell m}(y)) \\
&= \sum_{\ell=0}^{\infty} \sqrt{A_\ell} \beta_{\ell 0}^1(t) L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(\beta_{\ell m}^1(t) \cos(m\varphi) + \beta_{\ell m}^2(t) \sin(m\varphi)),
\end{aligned}
$$

where $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ and $((\beta_{\ell m}^1, \beta_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, real-valued Brownian motions and $\beta_{\ell 0}^2 = 0$ for $\ell \in \mathbb{N}_0$ and $t \in \mathbb{R}_+$. The covariance operator Q is characterized similarly to the introduction in [7] by

$$
\begin{aligned}
QY_{LM} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}((W(1), Y_{LM})_H \overline{(W(1), Y_{\ell m})_H}) Y_{\ell m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}(a_{LM}(1) \overline{a_{\ell m}(1)}) Y_{\ell m} \\
&= A_L Y_{LM}
\end{aligned}
$$

for $L \in \mathbb{N}_0$ and $M = -L, \ldots, L$, i.e., the eigenvalues of Q are characterized by the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ and the eigenfunctions are the spherical harmonic functions.

Let us calculate $\|W(t)\|_{L^2(\Omega;L^2(\mathbb{S}^2))}$ for $t \in \mathbb{R}_+$ next. It holds similarly to the proof of Proposition 5.2 that

$$\|W(t)\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2 = \sum_{\ell=0}^{\infty} \Big(A_\ell \, \mathbb{E}((\beta_{\ell 0}^1(t))^2)\|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2$$

$$+ 2A_\ell \sum_{m=1}^{\ell} \big(\mathbb{E}((\beta_{\ell m}^1(t))^2)\|\operatorname{Re} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2 + \mathbb{E}((\beta_{\ell m}^2(t))^2)\|\operatorname{Im} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2 \big) \Big)$$

$$= t \sum_{\ell=0}^{\infty} (2\ell+1)A_\ell = t \operatorname{Tr} Q.$$

This expression is finite for any finite $t \in \mathbb{R}_+$, if

$$\sum_{\ell=0}^{\infty} A_\ell \ell < +\infty.$$

With the made introductions of Q-Wiener processes and the Laplace operator on the sphere in Section 2, we are now able to write down the stochastic heat equation

(2) $$dX(t) = \Delta_{\mathbb{S}^2} X(t)\, dt + dW(t)$$

with initial condition $X(0) = X_0 \in L^2(\Omega; L^2(\mathbb{S}^2))$, where $t \in \mathbb{T} = [0,T]$, $T < +\infty$.

Looking for solutions in $L^2(\mathbb{S}^2)$, we rewrite Equation (2) to

$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + \int_0^t dW(s) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + W(t)$$

and further, since the spherical harmonic functions $\mathcal{Y}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$ and are eigenfunctions of $\Delta_{\mathbb{S}^2}$, we have that

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big((X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} + \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)} \Delta_{\mathbb{S}^2} Y_{\ell m}\, ds + a_{\ell m}(t) Y_{\ell m} \Big)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big((X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\, ds + a_{\ell m}(t) \Big) Y_{\ell m}.$$

This is equivalent to solve for all $\ell \in \mathbb{N}_0$ and $m = -\ell, \ldots, \ell$ the stochastic (ordinary) differential equation

$$(X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} = (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\, ds + a_{\ell m}(t).$$

The variations of constants formula yields

$$(X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} = e^{-\ell(\ell+1)t}(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} + \int_0^t e^{-\ell(\ell+1)(t-s)}\, da_{\ell m}(s).$$

So overall, the solution of the stochastic heat equation (2) reads

$$X(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} + \int_0^t e^{-\ell(\ell+1)(t-s)} \, da_{\ell m}(s) \right) Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \left(\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} + \sqrt{A_\ell} \left(\int_0^t e^{-\ell(\ell+1)(t-s)} \, d\beta_{\ell 0}^1(s) Y_{\ell 0} \right. \right.$$

$$\left. \left. + \sqrt{2} \sum_{m=1}^{\ell} \left(\int_0^t e^{-\ell(\ell+1)(t-s)} \, d\beta_{\ell m}^1(s) \operatorname{Re} Y_{\ell m} + \int_0^t e^{-\ell(\ell+1)(t-s)} \, d\beta_{\ell m}^2(s) \operatorname{Im} Y_{\ell m} \right) \right) \right)$$

$$=: \sum_{\ell=0}^{\infty} X_\ell(t),$$

and we choose the sequence of stochastic processes $(X_\ell, \ell \in \mathbb{N}_0)$ accordingly. Each process in this sequence satisfies the recursion formula

$$X_\ell(t+h) = e^{-\ell(\ell+1)h} X_\ell(t) + \sqrt{A_\ell} \left(\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell 0}^1(s) Y_{\ell 0} \right.$$

$$+ \sqrt{2} \sum_{m=1}^{\ell} \left(\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^1(s) \operatorname{Re} Y_{\ell m} \right.$$

$$\left. \left. + \int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^2(s) \operatorname{Im} Y_{\ell m} \right) \right).$$

Similarly to [5] we observe that by the Itô formula (see, e.g., [6])

$$(3) \qquad \int_0^t e^{-\ell(\ell+1)(t-s)} \, d\beta_{\ell m}^i(s)$$

is normally distributed with mean zero and variance $(2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)t})$ for $\ell \in \mathbb{N}$, $m = 1, \dots, \ell$, and $i = 1, 2$. This implies that

$$\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^i(s) \sim \mathcal{N}(0, \sigma_{\ell h}^2),$$

where

$$\sigma_{\ell h}^2 := (2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)h}).$$

For $\ell = 0$ we have no convolution integral and therefore the distribution of the expression is that of (the increment of) a standard Brownian motion, i.e., $\sigma_{0h}^2 = h$.

For the simulation of paths of the solution, we have to compute the solution on a discrete time grid $0 = t_0 < t_1 < \cdots < t_n = T$, $n \in \mathbb{N}$, on which the path of the Brownian motion resp. the stochastic integral (3) is known. The stochastic integral (3) at time t_k, $k = 0, \dots, n$, is equal in law to a sum of weighted, standard normally distributed random variables

$$\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)} \, d\beta_{\ell m}^i(s) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} e^{-\ell(\ell+1)(t_k-s)} \, d\beta_{\ell m}^i(s)$$

$$= \sum_{j=0}^{k-1} e^{-\ell(\ell+1)(t_k-t_{j+1})} \int_{t_j}^{t_{j+1}} e^{-\ell(\ell+1)(t_{j+1}-s)} \, d\beta_{\ell m}^i(s)$$

$$= \sum_{j=0}^{k-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^i(j),$$

where $h_j = t_{j+1} - t_j$, $j = 0, \ldots, n-1$ and $(X_{\ell m}^i(j), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell, i = 1, 2, j = 0, \ldots, n-1)$ is a sequence of independent, standard normally distributed random variables. This enables us to write down the solution of Equation (2) recursively

$$X_\ell(t_{k+1})$$
$$= e^{-\ell(\ell+1)h_k} X_\ell(t_k) + \sqrt{A_\ell} \sigma_{\ell h_k} \left(X_{\ell 0}^1(k) Y_{\ell 0} + \sqrt{2} \sum_{m=1}^{\ell} \left(X_{\ell m}^1(k) \operatorname{Re} Y_{\ell m} + X_{\ell m}^2(k) \operatorname{Im} Y_{\ell m} \right) \right)$$

for all $\ell \in \mathbb{N}_0$ and $k = 0, \ldots, n-1$. Using the notation of Lemma 5.1, we rewrite the recursion to

$$X_\ell(t_{k+1}) = e^{-\ell(\ell+1)h_k} X_\ell(t_k) + \psi_\ell(k)$$
$$= e^{-\ell(\ell+1)t_{k+1}} \sum_{m=-\ell}^{\ell} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} + \sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \psi_\ell(j),$$

where the increments are given by

$$\psi_\ell(j, y) = \sqrt{A_\ell} \sigma_{\ell h_j} \left(X_{\ell 0}^1(j) L_{\ell 0}(\vartheta) + \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \left(X_{\ell m}^1(j) \cos(m\varphi) + X_{\ell m}^2(j) \sin(m\varphi) \right) \right)$$

for $y = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \in \mathbb{S}^2$ and $j = 0, \ldots, n-1$. We observe for later use that

$$\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k-1} h_i} \psi_\ell(j)$$
$$= \sqrt{A_\ell} \left(\left(\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell 0}^1(j) \right) L_{\ell 0}(\vartheta) \right.$$
$$+ \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \left(\left(\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^1(j) \right) \cos(m\varphi) \right.$$
$$\left. \left. + \left(\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^2(j) \right) \sin(m\varphi) \right) \right)$$

and that

$$\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^i(j)$$

is a normally distributed random variable with mean zero and variance

$$(e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j})^2 = \frac{1}{2\ell(\ell+1)} (1 - e^{-2\ell(\ell+1)t_{k+1}}) = \sigma_{\ell t_{k+1}}^2.$$

This implies that we have equality in law of

$$\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \psi_\ell(j, y)$$

$$= \sqrt{A_\ell}\sigma_{\ell t_{k+1}}\Big(X_{\ell 0}^1 L_{\ell 0}(\vartheta) + \sqrt{2}\sum_{m=1}^{\ell} L_{\ell m}(\vartheta)\big(X_{\ell m}^1\cos(m\varphi) + X_{\ell m}^2\sin(m\varphi)\big)\Big),$$

where $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent, standard normally distributed random variables.

To implement the solution, we calculate X_ℓ exactly for finitely many $\ell \in \mathbb{N}_0$ on a finite time and space grid. One way to discretize the sphere is to take an equidistant grid in $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. Then we add the calculated X_ℓ and get an approximate solution, i.e., we simulate the approximate solution X^κ, $\kappa \in \mathbb{N}_0$ by

$$X^\kappa = \sum_{\ell=0}^{\kappa} X_\ell$$

on finitely many time and space points. In what follows let us estimate the mean square error when truncation of the series expansion at $\kappa \in \mathbb{N}$ is done.

Lemma 7.1. *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution X of Equation (2). Furthermore, assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$, and a constant $C > 0$ such that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ satisfies $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the error of the approximate solution X^κ is bounded uniformly in time and independently of the time discretization by*

$$\|X(t) - X^\kappa(t)\|_{L^2(\Omega;L^2(\mathbb{S}^2))} \leq \hat{C} \cdot \kappa^{-\alpha/2}$$

for all $\kappa \geq \ell_0$, where

$$\hat{C}^2 = \|X_0\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2 + C \cdot \Big(\frac{2}{\alpha} + \frac{1}{\alpha+1}\Big).$$

Proof. Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a partition of $[0, t]$ for some $n \in \mathbb{N}$. Since $\mathbb{E}(\psi_\ell(j)) = 0$ for all $\ell \in \mathbb{N}_0$ and $j = 0, \dots, n-1$, we first observe that

(4)
$$\|X(t_n) - X^\kappa(t_n)\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2$$

$$= \Big\|\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n}(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)}Y_{\ell m} + \sum_{\ell=\kappa+1}^{\infty}\sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i}\psi_\ell(j)\Big\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2$$

$$= \Big\|\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n}(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)}Y_{\ell m}\Big\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2$$

$$+ \Big\|\sum_{\ell=\kappa+1}^{\infty}\sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i}\psi_\ell(j)\Big\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2.$$

We define an isotropic Gaussian random field as in Lemma 5.1 by

$$T := \sum_{\ell=0}^{\infty} \sqrt{A_\ell}\sigma_{\ell t_n}\Big(X_{\ell 0}^1 L_{\ell 0}(\vartheta) + \sqrt{2}\sum_{m=1}^{\ell} L_{\ell m}(\vartheta)\big(X_{\ell m}^1\cos(m\varphi) + X_{\ell m}^2\sin(m\varphi)\big)\Big)$$

with angular power spectrum $(A_\ell \sigma^2_{\ell t_n}, \ell \in \mathbb{N}_0)$ and denote similarly to Section 5 by T^κ the truncated series expansion. Then

$$\Big\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \Big\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} = \|T - T^\kappa\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}.$$

The angular power spectrum satisfies with the made assumptions that

$$A_\ell \sigma^2_{\ell t_n} = A_\ell \frac{1}{2\ell(\ell+1)}(1 - e^{-2\ell(\ell+1)t_n}) \le C\ell^{-\alpha} \ell^{-2} \cdot 1 = C\ell^{-(\alpha+2)} \ .$$

With these parameters we apply Proposition 5.2 to the difference of T and T^κ which yields

$$\|T - T^\kappa\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} \le \hat{C}^2 \kappa^{-\alpha} = C \cdot \Big(\frac{2}{\alpha} + \frac{1}{\alpha+1}\Big)\kappa^{-\alpha}.$$

The first term in the last line of (4) is bounded by

$$\Big\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \Big\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}$$

$$= \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-2\ell(\ell+1)t_n} \|(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m}\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} \le e^{-2(\kappa+1)(\kappa+2)t_n} \|X_0\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}.$$

Therefore, it converges faster than any polynomial, especially it can be bounded by $\kappa^{-\alpha}$. So overall we bound

$$\|X(t_n) - X^\kappa(t_n)\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))} \le \Big(C \cdot \Big(\frac{2}{\alpha} + \frac{1}{\alpha+1}\Big) + \|X_0\|^2_{L^2(\Omega;L^2(\mathbb{S}^2))}\Big)\kappa^{-\alpha},$$
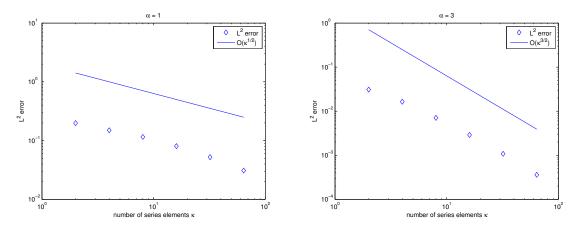
which finishes the proof of the lemma. $\square$

We remark that it is not necessary that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ of the Q-Wiener process decays with $\ell^{-\alpha}$ for $\alpha > 2$ but that it is sufficient to assume that $\alpha > 0$.
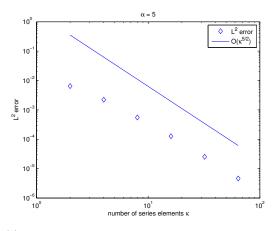
In an implementation in MATLAB we verified the theoretical results of Lemma 7.1. We took as "exact" solution the approximate solution at time $T = 1$ with $\kappa = 2^7$ since for larger κ the elements of the angular power spectrum A_ℓ and therefore the increments were so small that MATLAB failed to calculate the series expansion. We calculated the solution in one time step since we have shown in Lemma 7.1 that the convergence rate is independent of the number of calculated time steps. Instead of the $L^2(\mathbb{S}^2)$ error in space, we used the maximum over all grid points which is a stronger error. In Figure 5 the results and the theoretical convergence rates are shown for $\alpha = 1, 3, 5$. One observes that the simulation results match the theoretical results from Lemma 7.1.

Similarly to the proof of almost sure convergence of approximations of isotropic Gaussian random fields in Section 5, we need a L^p convergence result for the approximation of the solution of the stochastic heat equation to show pathwise convergence. This is proven in the following by a combination of Lemma 5.3 and Lemma 7.1.

Lemma 7.2. *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution X of Equation (2). Furthermore assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$, and a constant $C > 0$ such that the angular power spectrum*

(a) Angular power spectrum with parameter $\alpha = 1$.   (b) Angular power spectrum with parameter $\alpha = 3$.

(c) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 5. L^2 error of approximation of the stochastic heat equation different angular power spectrum of the Q-Wiener process and 100 Monte Carlo samples.

$(A_\ell, \ell \in \mathbb{N}_0)$ *satisfies* $A_\ell \leq C \cdot \ell^{-\alpha}$ *for all* $\ell > \ell_0$. *Then the error of the approximate solution* X^κ *is bounded uniformly in time and independently of the time discretization by*

$$\|X(t) - X^\kappa(t)\|_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-\alpha/2}$$

for all $p > 0$ *and* $\kappa \geq \ell_0$, *where* $\hat{C}_p$ *is a constant that depends on* $\|X_0\|_{L^{\max(p,2)}(\Omega;L^2(\mathbb{S}^2))}$, p, C, *and* α.

Proof. The result follows for $p \leq 2$ with Lemma 7.1 and with Hölder's inequality. So we assume that $p > 2$ from here on. Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a partition of $[0, t]$ for some $n \in \mathbb{N}$. We first observe that

$$\|X(t_n) - X^\kappa(t_n)\|_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n}(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|_{L^p(\Omega;L^2(\mathbb{S}^2))}$$

$$+ \Big\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \Big\|_{L^p(\Omega;L^2(\mathbb{S}^2))}.$$

Similarly to the proof of Lemma 7.1, the second term is equal to the L^p norm of the approximation error of an isotropic Gaussian random field with angular power spectrum $(A_\ell \sigma_{\ell t_n}^2, \ell \in \mathbb{N}_0)$, which satisfies by Lemma 5.3 that

$$\Big\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \Big\|_{L^p(\Omega;L^2(\mathbb{S}^2))} = \|T - T^\kappa\|_{L^p(\Omega;L^2(\mathbb{S}^2))}$$

$$\leq (C_p)^{1/p} C^{1/2} \cdot \Big(\frac{2}{\alpha-2} + \frac{1}{\alpha-1}\Big)^{1/2} \kappa^{-\alpha/2}.$$

Furthermore the first term satisfies similarly to the proof of Lemma 7.1 that

$$\Big\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \Big\|_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq e^{-(\kappa+1)(\kappa+2)t_n} \|X_0\|_{L^p(\Omega;L^2(\mathbb{S}^2))},$$

which converges faster than any polynomial and therefore can be bounded by $\kappa^{-\alpha/2}$. So combining these two estimates, we finish the proof. $\square$

Corollary 7.3. *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution X of Equation (2). Furthermore assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$, and a constant C such that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ satisfies $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the error of the approximate solution X^κ is bounded uniformly in time, independently of the time discretization, and asymptotically in κ by*

$$\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}$$

for all $\beta < \alpha/2$.

Proof. The proof is similar to the one for isotropic Gaussian random fields in Corollary 5.4 but for completeness we include it here. Let $\beta < \alpha/2$, then the Chebyshev inequality and Lemma 7.2 imply that
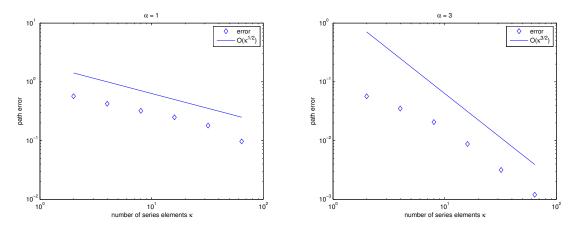
$$\mathbb{P}(\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \geq \kappa^{-\beta}) \leq \kappa^{\beta p} \mathbb{E}(\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)}^p) \leq \hat{C}_p^p \kappa^{(\beta-\alpha/2)p}.$$
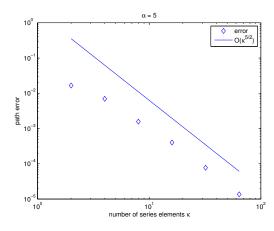
For all $p > (\alpha/2 - \beta)^{-1}$, the series

$$\sum_{\kappa=1}^{\infty} \kappa^{(\beta-\alpha/2)p} < +\infty$$

converges and therefore the Borel–Cantelli lemma implies the claim. $\square$

In Figure 6, we show the corresponding error plots to Figure 5 but instead of a Monte Carlo simulation of the approximate $L^2(\Omega; L^2(\mathbb{S}^2))$ error we plotted the error of one path of the stochastic heat equation. The convergence results coincide with the theoretical results in Corollary 7.3.

(a) Angular power spectrum with parameter $\alpha = 1$.   (b) Angular power spectrum with parameter $\alpha = 3$.

(c) Angular power spectrum with parameter $\alpha = 5$.

FIGURE 6. Error of approximation of a path of the stochastic heat equation different angular power spectrum of the Q-Wiener process.

APPENDIX A. INTERPOLATION SPACES

In this appendix we give a more detailed introduction to interpolation spaces than in Section 3 and show uniqueness of the spaces, i.e., that they are independent of the chosen interpolation couple.

We consider the sequence of spaces $(V^n(-1,1), n \in \mathbb{N}_0)$ that was introduced in Section 3 and start now with the definition of fractional order spaces by the real method of interpolation (see, e.g., [23, Chap. 1]). We observe that for any two integers $k, n \in \mathbb{N}_0$ with $0 \leq k < n$ the pair $(V^k(-1,1), V^n(-1,1))$ is an interpolation couple with $V^n(-1,1) \subset V^k(-1,1) \subset L^2(-1,1)$. For integers $k, m, n \in \mathbb{N}_0$ with $0 \leq k < m < n$ so that $0 < \theta := (m-k)/(n-k) < 1$, we may therefore define the intermediate space at "fine-index" $q \in [1, +\infty]$

$$B_{2,q}^{m,(k,n)}(-1,1) = \left(V^k(-1,1), V^n(-1,1) \right)_{\theta,q}$$

by the real method of interpolation as is introduced in [23, Chap. 1]. Then, these spaces are equipped with the usual norms $\|\cdot\|_{B_{2,q}^{m,(k,n)}(-1,1)}$ given by

$$
\|u\|_{B_{2,q}^m(-1,1)} = \begin{cases} \left(\displaystyle\int_0^\infty t^{-\theta q}|K(t,u)|^q \frac{dt}{t}\right)^{1/q}, & \text{for} \quad 1 \le q < +\infty, \\ \sup_{t>0} t^{-\theta} K(t,u), & \text{for} \quad q = +\infty, \end{cases}
$$

where the K-functional is defined by

$$
K(t,u) = \inf_{u=v+w} \left(\|v\|_{V^k(-1,1)} + t\|w\|_{V^n(-1,1)} \right)
$$

for $t > 0$. We observe that in particular the pair of spaces $(V^n(-1,1), V^{n+1}(-1,1))$ is an interpolation couple for every $n \in \mathbb{N}_0$. Therefore, with $n \in \mathbb{N}_0$ and for $1 \le q \le \infty$, we may extend the family $(B_{2,q}^{m,(k,n)}(-1,1))_{0 \le k < m < n, q \in [1,+\infty]}$ of exact interpolation spaces also to noninteger numbers $s = n + \theta$, $\theta \in (0,1)$, via

$$
B_{2,q}^{n+\theta}(-1,1) := (V^n(-1,1), V^{n+1}(-1,1))_{\theta,q}.
$$

Let us from here on simplify the notation and denote $V^n(-1,1)$ by V^n and $B_{2,q}^{m,(k,n)}(-1,1)$ by $B_{2,q}^{m,(k,n)}$. Our next proposition states that for $q = 2$ and $m \in \mathbb{N}$, the Besov spaces $B_{2,2}^{m,(k,n)}$ are equal to V^m for any choice $k < m < n$.

Proposition A.1. *Let $m \in \mathbb{N}$ be given. For any $k, n \in \mathbb{N}_0$ with $0 \le k < m < n$, it holds that $B_{2,2}^{m,(k,n)} = V^m$.*

Proof. This result is classical (see, e.g., [23], [18] or [20, Chap. 6.5] and the references there). We present the detailed argument here for completeness.

By Proposition 3.2 we already know that the norm in V^m is equivalent to weighted square summability of the coefficients of the Fourier–Legendre expansion. So it is sufficient to show the equivalence of the $B_{2,2}^{m,(k,n)}$-norm and the convergence of the sum for all $0 \le k < m < n$.

Therefore we choose any $k, n \in \mathbb{N}_0$ with $0 \le k < m < n$ and $u \in L^2(-1,1)$. Then u admits the Fourier–Legendre expansion

$$
u = \sum_{\ell=0}^\infty u_\ell \frac{2\ell+1}{2} P_\ell
$$

as has been seen above. Consider now $u \in V^m \cup B_{2,2}^{m,(k,n)} \subset V^k$. We split u into the sum $v + w$ with $v \in V^k$ and $w \in V^n$ and the series expansions

$$
v = \sum_{\ell=0}^\infty (u_\ell - w_\ell)\frac{2\ell+1}{2} P_\ell \quad \text{and} \quad w = \sum_{\ell=0}^\infty w_\ell \frac{2\ell+1}{2} P_\ell.
$$

Then Proposition 3.2 implies that

$$
K(t,u)^2 \simeq \inf_{u=v+w} (\|v\|_{V^k}^2 + t^2\|w\|_{V^n}^2)
$$

$$
\simeq \inf_{u=v+w} \sum_{\ell=0}^\infty \frac{2\ell+1}{2} \left((u_\ell - w_\ell)^2(1 + \ell^{2k}) + w_\ell^2(1 + \ell^{2n}) \right).
$$

We observe further that the infimum over all $u = v + w$ is equal to the infimum over all square summable sequences $(w_\ell)_{\ell \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$, i.e.,

$$K(t,u)^2 \simeq \inf_{(w_\ell)_{\ell \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} G_\ell(u_\ell, w_\ell; t, k, n),$$

where

$$G_\ell(a, d; t, k, n) := (a - d)^2 (1 + \ell^{2k}) + t^2 d^2 (1 + \ell^{2n})$$

is with respect to $d \in \mathbb{R}$ a quadratic polynomial with positive leading coefficient for all $\ell \in \mathbb{N}_0$. For $\ell \in \mathbb{N}_0$, its minimum is attained at

$$d_\ell := \frac{a}{1 + t^2 g_{kn}(\ell)},$$

where

$$g_{kn}(\ell) := \frac{1 + \ell^{2n}}{1 + \ell^{2k}} \geq 1.$$

This implies that

$$K(t,u)^2 \simeq \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \left((u_\ell - d_\ell)^2 (1 + \ell^{2k}) + d_\ell^2 (1 + \ell^{2n}) \right)$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) t^2 \frac{g_{kn}(\ell)}{1 + t^2 g_{kn}(\ell)}$$

and leads with the definition of the norm and the theorem of Fubini–Tonelli to

$$\|u\|_{B_{2,2}^{m,(k,n)}}^2 = \int_0^\infty t^{-2\theta} K(t,u)^2 \frac{dt}{t}$$

$$\simeq \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) \int_0^\infty t^{-(2\theta+1)} \frac{t^2 g_{kn}(\ell)}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt,$$

where $\theta := (m - k)/(n - k) \in (0, 1)$. To finish the proof it remains to show that

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt \simeq \sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell+1}{2} (1 + \ell^{2m})$$

by Proposition 3.2, i.e., we have to prove the equivalence

$$(1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt \simeq 1 + \ell^{2m} = 1 + \ell^{2((1-\theta)k + \theta n)}.$$

Therefore let us split the integral first into

$$\int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt = \int_0^{g_{kn}(\ell)^{-1/2}} \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt + \int_{g_{kn}(\ell)^{-1/2}}^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

and bound the two terms on the right hand side from below and from above by

$$\frac{1}{2} \int_0^{g_{kn}(\ell)^{-1/2}} t^{1-2\theta} \, dt \leq \int_0^{g_{kn}(\ell)^{-1/2}} \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt \leq \int_0^{g_{kn}(\ell)^{-1/2}} t^{1-2\theta} \, dt = \frac{1}{2 - 2\theta} g_{kn}(\ell)^{\theta-1}$$

and

$$\frac{1}{2g_{kn}(\ell)} \int_{g_{kn}(\ell)^{-1/2}}^{\infty} t^{-1-2\theta}\, dt \leq \int_{g_{kn}(\ell)^{-1/2}}^{\infty} \frac{t^{1-2\theta}}{1+t^2 g_{kn}(\ell)}\, dt$$

$$\leq \frac{1}{g_{kn}(\ell)} \int_{g_{kn}(\ell)^{-1/2}}^{\infty} t^{-1-2\theta}\, dt = \frac{1}{2\theta} g_{kn}(\ell)^{\theta-1}.$$

This implies overall that

$$\frac{1}{4(1-\theta)\theta} g_{kn}(\ell)^{\theta-1} \leq \int_0^{\infty} \frac{t^{1-2\theta}}{1+t^2 g_{kn}(\ell)}\, dt \leq \frac{1}{2(1-\theta)\theta} g_{kn}(\ell)^{\theta-1}$$

and moreover that

$$(1+\ell^{2k}) g_{kn}(\ell) \int_0^{\infty} \frac{t^{1-2\theta}}{1+t^2 g_{kn}(\ell)}\, dt \simeq (1+\ell^{2k}) g_{kn}(\ell)^{\theta} = (1+\ell^{2k})^{1-\theta}(1+\ell^{2n})^{\theta}.$$

We observe that the function x^p, $p \in (0,1)$ is concave on $\mathbb{R}_+$ and satisfies $(x+y)^p \geq 2^{p-1}(x^p + y^p)$, which implies finally that

$$(1+\ell^{2k})^{1-\theta}(1+\ell^{2n})^{\theta} \simeq (1+\ell^{2(1-\theta)k})(1+\ell^{2\theta n}) \simeq 1 + \ell^{2((1-\theta)k+\theta n)} = 1 + \ell^{2m}.$$

This concludes the proof. $\qquad\square$

Based on Proposition A.1, it is clear that one can use for every $m \in \mathbb{N}$ in place of $B_{2,2}^{m,(k,n)}$ simply V^m. Moreover, for fractional $\eta = n + \theta$ with $n \in \mathbb{N}_0$ and some $0 < \theta < 1$, we write also V^η in place of $B_{2,2}^\eta$.

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