ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Finite Elements with mesh refinement for wave equations in polygons

F. Müller and C. Schwab

Research Report No. 2013-11
April 2013

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

- Funding ERC: 247277

Finite Elements with mesh refinement for wave equations in polygons

Fabian Lukas Müller, Christoph Schwab[1]

Seminar for applied Mathematics, Rämistrasse 101, 8092 Zurich, Switzerland

Abstract

Error estimates for the space-semidiscrete approximation of solutions of the wave equation in polygons $G \subset \mathbb{R}^2$ are presented. Based on corner asymptotics of solutions of the wave equation, it is shown that for continuous, simplicial Lagrangian Finite Elements of polynomial degree $p \geq 1$ with either suitably graded mesh refinement or with bisection tree mesh refinement towards the corners of G, the maximal rate of convergence $O(N^{-p/2})$ which is afforded by the Lagrangian Finite Element approximations on quasiuniform meshes for smooth solutions is restored. Dirichlet, Neumann and mixed boundary conditions are considered. Numerical experiments which confirm the theoretical results are presented. Generalizations to nonhomogeneous coefficients and elasticity and electromagnetics are indicated.

Keywords: High order Finite Elements, Wave equation, Regularity, Weighted Sobolev spaces, Method of lines, Local mesh refinement, Graded meshes, Newest vertex bisection

Email addresses: fabian.mueller@sam.math.ethz.ch (Fabian Lukas Müller), christoph.schwab@sam.math.ethz.ch (Christoph Schwab)

[1]Research of Ch. Schwab supported by the European Research Council under Advanced Grant ERC AdG 247277.

1. Introduction

The regularity of elliptic equations in polygonal domains has been studied for several decades, starting with the work by Kondrat'ev [15] and Maz'ya and Plamenevskiĭ [16]. We refer to Maz'ya and Rossmann [17] for a recent account of these results, also in polyhedral domains in $\mathbb{R}^3$ and a comprehensive list of references.

It is well-known that regularity results in scales of Sobolev spaces with weights allow to recover optimal convergence rates for Finite Element Methods (FEM) with local mesh refinement in the vicinity of corners; we refer to Raugel [21] and Babuška et al. [1, 2], and Băcuţă et al. [3] for so-called *graded meshes*, and, more recently, to [7] and the references there for simplicial meshes with bisection tree refinements produced by Adaptive Finite Element Methods (AFEMs).

For evolution problems, in particular for the linear, second order Wave Equation, similar results do not seem to be available. However, corner singularities are known to play a crucial role in the scattering and diffraction of waves. In recent years, results on the regularity of the pure Dirichlet and Neumann problems of solutions of the wave equation in polygonal and in certain polyhedral domains have been proved by Plamenevskiĭ et al. in [20, 10, 11] for the scalar, acoustic Wave Equation, and in [13, 14] for a general class of second order, linear hyperbolic systems. Their results imply that at a fixed time t, $u(\cdot, t)$ belongs to a class of function spaces $H^{2,p+1}_{\delta}$ which appeared already in the study of elliptic equations. Moreover, in these papers explicit formulae for the asymptotics of $u(x, t)$ in the vicinity of corners the polygon G were obtained. Therefore, in principle, approximation results for $H^{2,p+1}_{\delta}$ on several families of locally refined meshes as, for example, in [2], as well as a mesh refinement algorithm presented in [7], may now be applied to the solution of the wave equation. The main result of the present paper is that the space-semidiscrete ("method of lines") type discretization of the wave equation yields optimal convergence rates for solutions with singular asymptotic behaviour in the vicinity of the corners, which are known to typically occur in solutions of the linear, second order wave equation. Although we consider here only the 2nd order, scalar wave equation, we hasten to add that our approximation results are also applicable to singularities which arise in propagation of elastic and electromagnetic waves in polyhedral domains.

The outline of the present paper is as follows. We start with an introduction to the used notations, and the formulation of the scalar wave equation with Dirichlet and Neumann conditions in Section 2. Section 3 contains a review of the regularity theory for the scalar wave equation, starting from the Definitions of weighted Sobolev spaces. In Section 4, we study the FEM-approximation of singular functions, and present two classes of meshes which yield optimal convergence rates in the presence of corner singularities. These results are applied in Section 5 with the decomposition theorem to obtain optimal convergence rates for the space semidiscrete Finite Element approximation of the wave equation in polygonal domains. Finally, in Section 6, we present results of numerical experiments, performed with a very small time-step to approximately "cancel" the influence of the time-stepping error.

2. Problem formulation

On an open, bounded polygonal domain $G \subseteq \mathbb{R}^2$ and for $0 < T_{\max} < \infty$, with boundary $\partial G = \Gamma_D \cup \Gamma_N$ which consists of a finite number of straight segments Γ_i which are partitioned into Dirichlet and Neumann segments, we consider the initial boundary value problem for the scalar wave equation with Dirichlet or Neumann boundary conditions, i.e. we wish to find solutions $u(x, t)$, $(x, t) \in Q :=$

$G \times (0, T_{\max})$ such that

$$\begin{aligned} u_{tt} &= \Delta u + f \quad \text{in} \quad Q\,, \\ u(\cdot, 0) &= u_0 \quad \text{in} \quad G\,, \\ u_t(\cdot, 0) &= v_0 \quad \text{in} \quad G\,, \\ u(\cdot, t) &= 0 \quad \text{on} \quad \Gamma_D \times (0, T_{\max})\,, \\ \tfrac{\partial u}{\partial n} &= g \quad \text{on} \quad \Gamma_N \times (0, T_{\max})\,. \end{aligned} \tag{1}$$

We denote by $H^s(G)$ the usual Sobolev spaces on G, and by $H^1_0(G)$ the subspace of $H^1(G)$ built by functions with vanishing trace. Moreover, given a Hilbert space H, we denote by $H^s(0, T; H)$ the H^s-Bochner space of functions from $[0, T_{\max}]$ to H. We introduce the space V defined as the completion of $\{v \in C^\infty(G) : \ v|_{\Gamma_D} \equiv 0\}$ with respect to the H^1-norm. Evidently,

$$V := \begin{cases} H^1_0(G) & \text{if } \Gamma_N = \emptyset, \\ H^1(G) & \text{if } \Gamma_D = \emptyset. \end{cases}$$

We will also denote by $(\cdot, \cdot)$ the $L^2(G)$ innerproduct, extended to the pair of spaces $V \times V^*$ with duality taken with respect to the "pivot" space $L^2(G)$ by continuity. Applying integration by parts, the mixed initial boundary value problem for the scalar wave equation with homogeneous Dirichlet or Neumann conditions can be written in the following spatial variational form.

$$\begin{aligned} &\text{Find } u \in H^1(0, T_{\max}; V), \text{ such that } \forall t \in [0, T_{\max}] \text{ and } \forall v \in V : \\ &\partial_t^2 (u(\cdot, t), v) + (\nabla u(\cdot, t) \cdot \nabla v) \ = (f(\cdot, t), v)\,, \\ &(u(\cdot, 0), v) = (u_0, v), \\ &\partial_t\,(u(\cdot, 0), v) = (v_0, v), \end{aligned} \tag{2}$$

where $u_0 \in H^1(G)$, $v_0 \in L^2(G)$ and where $f \in L^2(0, T_{\max}; L^2(G))$ are given.

We discretize (2) by the method of lines, using continuous Lagrangian FEM of uniform polynomial degree $p \geq 1$ in the spatial domain G on a family of regular, simplicial triangulations of the domain G, followed by a non-specified discretization method in time. This is well-known to yield optimal convergence rates w.r.t. the mesh size for the semi-discrete formulation, if $u(\cdot, t) \in C^2([0, T_{\max}]; H^{p+1}(G)) \supset H^{2.5}(0, T_{\max}, H^{p+1}(G))$. Necessary conditions for this to be satisfied are $f \in H^3(0, T_{\max}; L^2(G))$, $u_0, v_0 \in V$, and the following compatibility conditions:

$$\frac{\partial^j}{\partial t^j} u(x, 0) \in V, \ j = 0, 1, 2, 3, \text{ and } \frac{\partial^4}{\partial t^4} u(x, 0) \in L^2(G)\,.$$

See, e.g., [26] for a detailed discussion of these compatibility conditions, where also necessary conditions for the regularity $u(\cdot, t) \in H^{p+1}(G)$ for domains G with smooth boundary are derived.

In the case the domain G is a generic bounded polygon in $\mathbb{R}^2$, higher regularity of $u(x, t)$ is only given in suitable scales of weighted Sobolev spaces, ([20, 10, 11]). Therefore, further conditions on the mesh refinement need to be imposed. In Section 4, we will present two types of graded mesh refinements that approximate singular solutions with optimal convergence rates. Our main result will be given in Theorem 5.4 and states that for the space semidiscrete Finite Element approximation of the initial-boundary value problem of the scalar, second order wave equation, the mesh families presented in Section 4 yield optimal convergence rates. Hence, we consider the space-semidiscrete case, and therefore our results are not restricted to specific time-stepping schemes. Numerical experiments which indicate that your theoretical estimates are sharp are presented in the last section. Throughout this paper, we use standard notation: the operators ∇ and Δ will be understood to only operate w.r.t.

the spatial coordinates. By D^k, $k \in \mathbb{N}_0$, we denote the vector of all partial, weak derivatives of order α with $|\alpha| = k$. Hence, given a function $v \in H^s(G)$, we write

$$|\mathsf{D}^k v(x)|^2 := \sum_{|\alpha|=k} \left|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} v(x)\right|^2 ,$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ is a multi-index and $|\alpha| := \alpha_1 + \alpha_2$ and hence

$$\|\mathsf{D}^k v; L^2(G)\|^2 = \int_G |\mathsf{D}^k v(x)|^2 \, dx = |v; H^k(G)|^2 \ .$$

If $\mathcal{T}$ is a regular, simplicial triangulation of G, we denote by $\#\mathcal{T}$ the number of degrees of freedom in $\mathcal{T}$, and by $S^{p,l}(G, \mathcal{T})$ the FE-space of functions $v \in H^l(G)$, such that for all elements $T \in \mathcal{T}$, $v|_{\bar{T}}$ is a polynomial of degree $\leq p$.

3. Regularity

3.1. The geometrical setting

Let $G \subseteq \mathbb{R}^2$ be a bounded polygonal domain with M corners $\mathbf{c}_i$, which we collect in the set

$$C := \{\mathbf{c}_1, \ldots, \mathbf{c}_M\} \ .$$

For each $i = 1, \ldots, M$, denote by $\phi_i \in (0, 2\pi]$ the inner angle at $\mathbf{c}_i$ and

$$R_i := \frac{1}{2} \min_{i' \neq i} |\mathbf{c}_i - \mathbf{c}_{i'}|.$$

Then, we introduce nonintersecting corner neighborhoods $G_i := B_{R_i}(\mathbf{c}_i) \cap G$, where $B_R(\mathbf{c})$ denotes an open ball of radius R with center $\mathbf{c}$. The G_i are disjoint open sectors with vertices $\mathbf{c}_i$. Moreover, we denote for $i = 1, ..., M$ by $\Omega_i := \partial G_i \setminus \partial G$ the respective open circle segments. Inside the segments Ω_i, we introduce an angular coordinate $\vartheta_i \in [0, \phi_i]$ tracing Ω_i.

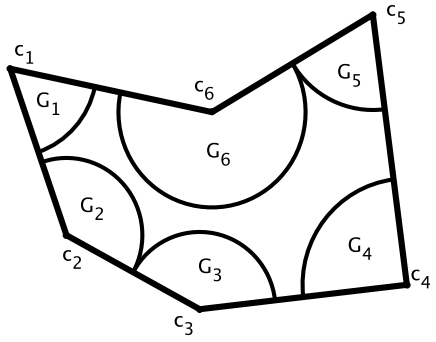

Figure 1: A polygonal domain with one re-entrant corner $\phi_6 > \pi$.

In G_i, we introduce polar coordinates (r_i, ϑ_i), centered at $\mathbf{c}_i$. To extend this local definition to the entire domain G, we observe that $r_i(x) \in H^k(G_i)$ for all $k \in \mathbb{N}_0$. Hence, for all $k \in \mathbb{N}_0$, it can be extended to a smooth function $\mathfrak{E}_k r_i(x) \in H^k(G)$. Choosing in particular for $\mathfrak{E}_k$ the Stein extension operator (see [25, Chapter 6.2]), the $\mathfrak{E}_k$ do not depend on k, whence we deduce that $r_i(x)$ can be extended to a smooth function $r_i^*(x) \in C^\infty(G)$ on the entire domain G, such that $r_i^*(x) = r_i(x)$ for all $x \in \bar{G}_i$. Moreover, for a given weight exponent $\delta' \in \mathbb{R}$, we define

$$\Psi_{\delta'}(x) := \prod_{i=1}^{M} \left(r_i^*(x)\right)^{\delta'}.$$

Let $\tilde{\chi} : \mathbb{R}_+ \to [0,1]$ be a smooth cut-off function, such that

$$\tilde{\chi}(r) = \begin{cases} 1 & \text{if } r \leq \frac{2}{3} \\ 0 & \text{if } r \geq 1. \end{cases}$$

At each corner $\mathbf{c}_i$, $i = 1, \ldots, M$, we define local cut-offs

$$\chi_i(r) := \tilde{\chi}\left(\frac{r}{R_i}\right),$$

whose supports are fully contained in $(0, R_i)$.

For the time-dependent problem, we introduce the open cylinder $Q := G \times (0, T_{\max})$.

Following [20, 10, 11], we Fourier transform the evolution equation in variational form with respect to the time variable. Let $\gamma > 0$ be fixed, $\sigma \in \mathbb{R}$ be a real parameter, and let $\tau := \sigma - \mathrm{i}\gamma \in \mathbb{C}$. Given a function $w(x,t)$, $x \in \bar{G}, t \in \mathbb{R}$, its Fourier transformation in time onto the complex horizontal line $\mathbb{R} - \mathrm{i}\gamma \ni \tau$ is defined to be

$$\hat{w}(x,\tau) := \mathcal{F}_{t\mapsto\tau}[w(x,t)](x,\tau) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-i\tau t} w(x,t)\, dt.$$

We will consider only $t \in [0, T_{\max}]$, with $T_{\max} < \infty$. In this case, $\hat{w}(x,\tau)$ denotes the Fourier transformation of the Null extension of w outside $[0, T_{\max}]$. If $u(x,t)$ is a solution of (2), then $\hat{u}(x,\tau)$ is a solution of the transformed equation:

$$\begin{gathered} \text{Find } u \in V, \text{ such that } \forall \tau \in \mathbb{R} - i\gamma \text{ and } \forall v \in V : \\ \int_G \left[-\tau^2 \hat{u}(x,\tau)\, \overline{v(x)} + \nabla \hat{u}(x,\tau)\, \overline{\nabla v(x)}\right] dx = \int_G \hat{f}(x,\tau)\, \overline{v(x)}\, dx, \\ \lim_{\sigma\to\infty} |\tau|\, (\hat{u}(\cdot,\tau), v) = (u_0, v) \\ \lim_{\sigma\to\infty} |\tau|^2\, (\hat{u}(\cdot,\tau), v) = (v_0, v)\ . \end{gathered} \tag{3}$$

3.2. Weighted Sobolev spaces

The regularity of u will be described in terms of a scale of weighted Sobolev spaces. We next recall definitions and basic properties of such spaces. Let $\gamma > 0$ and $\omega \in \mathbb{R}$ be given weight parameters, and let $s \in \mathbb{N}_0$ be an integer.

We define the spaces $H^s_\omega(G)$ and $H^s_\omega(G, |\tau|)$ as completions of $C_0^\infty\left(\bar{G} \setminus C\right)$ with respect to the norms $\|v; H^s_\omega(G)\|$ and $\|v; H^s_\omega(G, |\tau|)\|$ which are given, respectively, by

$$\|v; H^s_\omega(G)\|^2 := \sum_{k=0}^{s} \int_G \Psi_{\omega+k-s}(x)^2 \left|\mathsf{D}^k v(x)\right|^2 dx,$$

$$\|v; H^s_\omega(G, |\tau|)\|^2 := \sum_{j=0}^{s} |\tau|^{2j} \|v; H^{s-j}_\omega(G)\|^2\ .$$

In the case $s = 0$, we define $L^2_\omega(G) := H^0_\omega(G)$.

Furthermore, we define the space $V^s_\omega(Q;\gamma)$ as completion of $C_0^\infty\left(\bar{G} \setminus C \times [0, T_{\max}]\right)$ with respect to the following norm:

$$\|w, V^s_\omega(Q;\gamma)\|^2 := \int_{\mathbb{R}} \|\hat{w}(\cdot,\tau); H^s_\omega(G;|\tau|)\|^2 \, d\sigma \, . \tag{4}$$

Remark 3.1. The notation $H^s_\omega(G)$ can give rise to a conflict for $\omega = 0$ and $s = 1$, since $H^1_0(G)$ was defined to be the subspace of $H^1(G)$ of functions with vanishing trace. In the case of $\omega = 0$ and $s = 1$, the weighted Sobolev space $H^s_\omega(G)$ will always be denoted $H^1_{\omega=0}(G)$.

As a first observation, we have the following continuous inclusion:

Lemma 3.2. *Let $\omega \le \omega'$ be real numbers and $G \subseteq \mathbb{R}^2$ be bounded. Then,*

$$L^2_\omega(G) \hookrightarrow L^2_{\omega'}(G) \, .$$

Proof.

$$\begin{aligned}\int_G \Psi_{\omega'}(x)^2 \, |v(x)|^2 \, dx &= \int_G \Psi_{\omega'-\omega}(x)^2 \, \Psi_\omega(x)^2 |v(x)|^2 \, dx \\ &\le \underbrace{\max\left\{\Psi_{\omega'-\omega}(x) : \, x \in \bar{G}\right\}^2}_{=:c} \int_G \Psi_\omega(x)^2 |v(x)|^2 \, dx,\end{aligned}$$

for all $v \in L^2_\omega(G)$, where $c < \infty$, since $\omega' - \omega \ge 0$, and $|G| < \infty$. □

For a better understanding of the space $V^s_\omega(Q;\gamma)$, the following result is useful.

Proposition 3.3. *Let $q, s, s' \in \mathbb{N}_0$ and $G \subseteq \mathbb{R}^2$ be a bounded polygonal domain.*
If, morever, $q + 1 \ge s + s'$, then for all $\gamma > 0$ and $\omega \le -q$ the following inclusion is continuous:

$$V^{q+1}_{\omega+q}(Q;\gamma) \hookrightarrow H^s(0, T_{\max}; H^{s'}(G)). \tag{5}$$

Proof. Let $q \in \mathbb{N}_0$ and $\omega \in \mathbb{R}$ be generic, and let $u \in V^{q+1}_{\omega+q}(Q;\gamma)$. Then,

$$\begin{aligned}\|u; V^{q+1}_{\omega+q}(Q;\gamma)\|^2 &= \int_{\mathbb{R}} \|\hat{u}(\cdot,\tau); H^{q+1}_{\omega+q}(G,|\tau|)\|^2 \, d\sigma \\ &= \sum_{j=0}^{q+1} \int_{\mathbb{R}} |\tau|^{2j} \, \|\hat{u}(\cdot,\tau); H^{q+1-j}_{\omega+q}(G)\|^2 \, d\sigma \\ &= \sum_{j=0}^{q+1} \sum_{k=0}^{q+1-j} \int_{\mathbb{R}} |\tau|^{2j} \, \left\|\Psi_{\omega-1+j+k}(x) \, \mathsf{D}^k \hat{u}(\cdot,\tau); L^2(G)\right\|^2 \, d\sigma \\ &\ge c_{\omega,q,G} \sum_{j=0}^{q+1} \sum_{k=0}^{q+1-j} \int_{\mathbb{R}} |\tau|^{2j} \, \left\|\mathsf{D}^k \hat{u}(\cdot,\tau); L^2(G)\right\|^2 \, d\sigma,\end{aligned}$$

where

$$c_{\omega,q,G} := \min_{0 \le j,k \le q+1} \left(\inf_G \left(\Psi_{\omega-1+j+k}\right)\right)^2 > 0,$$

since $\omega - 1 + j + k \le \omega + q \le 0$, and $|G| < \infty$.

By Plancherel's theorem, for all $0 \le j, k \le q+1$:

$$\int_{\mathbb{R}} |\tau|^{2j} \, \|\mathsf{D}^k \hat{u}(\cdot,\tau); L^2(G)\|^2 \, d\sigma = \int_0^{T_{\max}} e^{-2\gamma t} \left\| \partial_t^j \, \mathsf{D}^k u(\cdot,t); L^2(G) \right\|^2 \, dt$$
$$\ge e^{-2\gamma T_{\max}} \int_0^{T_{\max}} \|\partial_t^j \, \mathsf{D}^k u(\cdot,t); L^2(G)\|^2 \, dt \, .$$

Applying this to our orginal estimate implies

$$\|u; V^{q+1}_{\omega+q}(Q;\gamma)\|^2 \ge e^{-2\gamma T_{\max}} \, c_{\omega,q,G} \sum_{j=0}^{q+1} \sum_{k=0}^{q+1-j} \int_0^{T_{\max}} \left\| \partial_t^j \, \mathsf{D}^k u(\cdot,t); L^2(G) \right\|^2 \, dt \, . \tag{6}$$

Now, we complete the proof if we can find conditions on q and ω such that

$$\|u; V^{q+1}_{\omega+q}(Q;\gamma)\|^2 \ge e^{-2\gamma T_{\max}} \, c_{\omega,q,G} \sum_{j=0}^{s} \sum_{k=0}^{s'} \int_0^{T_{\max}} \left\| \partial_t^j \, \mathsf{D}^k u(\cdot,t); L^2(G) \right\|^2 \, dt \, .$$

Comparing the sums over j, a necessary condition obviously is $q+1 \ge s$. Moreover, for j fixed, in the second sum of (6), the index k runs from 0 to $q+1-j$, whence we deduce the second condition, $q+1-j \ge s'$, $\forall\, j = 0, 1, \ldots, s$. These two conditions together are equivalent to $q+1 \ge s+s'$. □

3.3. Regularity of $u(x,t)$

In this section, we review the the regularity theory and the asymptotic analysis for the wave equation in polygonal domains as developed in [20, 10, 11] and the references there. On each circle segment Ω_i, $1 \le i \le M$, the restriction $\tilde{u}(\vartheta_i, t) := u|_{\Omega_i}(x,t) = u|_{\{r_i = R_i\}}(r_i, \vartheta_i; t)$ either satisfies $\tilde{u}(\vartheta_i, t) = 0$ or $\partial_{\vartheta_i} \tilde{u}(\vartheta_i, t) = 0$ for $\vartheta_i \in \{0, \phi_i\}$. These cases will be considered seperately and lead to the total decomposition in G, obtained by superposition over $i \in \{1, \ldots, M\}$. Such results have been proved in [20], [10], and [11] for pure homogeneous Dirichlet or Neumann conditions. It is expected that these results can be extended to mixed boundary conditions, i.e. where $\Gamma_D \ne \emptyset$ and $\Gamma_N \ne \emptyset$. With an approach which is completely analogous to the techniques in these references, similar results are also expected for the case of mixed boundary conditions; we present the expected results in Subsection 3.6.

Definition 3.4. On Ω_i, we define an operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}_i(\lambda)$ by

$$\mathfrak{A}_i(\lambda) : H^2(\Omega_i) \to L^2(\Omega_i) \times \mathbb{R}^2 \quad \Phi \mapsto \mathfrak{A}_i(\lambda)\Phi := \left(-\lambda^2 \Phi + \partial^2_{\vartheta_i} \Phi, \; \mathfrak{N}_i \Phi \right), \tag{7}$$

where $\mathfrak{N}_i\Phi$ is the boundary operator on $\partial\Omega_i$, determined by the imposed boundary conditions on the edges that contain $\mathbf{c}_i$. For example, in the case of homogenous Dirichlet boundary conditions on each of the two sides which meet at the vertex $\mathbf{c}_i$, $\mathfrak{N}_i\Phi = (\Phi(0), \, \Phi(\phi_i)) \in \mathbb{R}^2$.

The eigenpairs of $\mathfrak{A}_i$ have decisive influence on the regularity of $u(x,t)$. Concretely, the eigenvalues $\lambda^i_{\pm n}$, such that $\mathfrak{A}_i(\lambda^i_{\pm n})\Phi^i_n = 0$, are given by $\lambda^i_{\pm n} = \mp i\sqrt{\mu^i_n}$, being $0 \le \mu^i_1 \le \mu^i_2 \ldots$ the solutions of the Sturm-Liouville eigenvalue problem

$$-\partial^2_{\vartheta_i} \Phi^i_n = \mu^i_n \Phi^i_n \, , \quad \mathfrak{N}_i \Phi^i_n = (0, \, 0) \, . \tag{8}$$

Definition 3.5. For each $i \in \{1, \dots, M\}$ and all $n \in \mathbb{Z}$, we define the singular functions

$$S_{n,i}(r_i, \vartheta_i) := r_i^{i\lambda_n^i} \Phi_n^i(\vartheta_i) \, .$$

Following [20, 10, 11], we introduce the spaces $DV_{\omega,q}(Q;\gamma)$ and $RV_{\omega,q}(Q;\gamma)$. To this end, we need the notion of a so-called continuation operator $w \mapsto \mathbb{X}w$, that "smoothens" a function w which is nonregular, or not defined in G_i outside $\mathbf{c}_i$. See Section 5 in [14], and Chapter 11.3 in [18].

Definition 3.6. Let $i \in \{1, \dots, M\}$. For $w(x,t) \in L^2(0, T_{\max}; L^2(G))$, we define the operators $\mathbb{X}^i$ and Λ, respectively, by

$$(\mathbb{X}^i w)(x,t) := \mathcal{F}^{-1}_{\tau \mapsto t} \chi_i(|\tau| \, r_i^*(x)) \, \mathcal{F}_{t' \mapsto \tau} w(x,t') \, .$$

and

$$(\Lambda w)(x,t) := \mathcal{F}^{-1}_{\tau \mapsto t} \, |\tau| \, \mathcal{F}_{t' \mapsto \tau} w(x,t') \, .$$

Remark 3.7. Let $w(t) \in H^s(0, T_{\max})$ a real-valued function with $s \geq \frac{1}{2}$. The operator $\mathbb{X}^i$ extends $w(t)$ to a function $\mathbb{X}^i w(x,t)$ which is smooth in $\overline{G_i}$, with value $\mathbb{X}^i w(\mathbf{c}_i, t) = w(t)$ for all t.

Definition 3.8. Let $\gamma > 0$, $\omega \in \mathbb{R}$ and $q \in \mathbb{N}_0$. We define

$$\|w; \mathsf{DV}^N_{\omega,q}(Q;\gamma)\|^2 := \sum_{i=1}^M \left\{ \int_0^{T_{\max}} \partial_t \left\| r^{\omega-1} e^{-\gamma t} \, \mathbb{X}^i u(\cdot,t); L^2(G) \right\|^2 + \right.$$
$$\left. \sum_{k=1}^{q+2} \int_0^{T_{\max}} \left\| r^{\omega-2+k} e^{-\gamma t} \, \mathsf{D}^k \left[\mathbb{X}^i u \right](x,t) \right\|^2 \, dt \right\} + \gamma^2 \|u; V^{q+1}_{\omega+q}(Q;\gamma)\|^2,$$

and the space $DV_{\omega,q}(Q;\gamma)$ as completion of $C_0^\infty(\bar{Q} \setminus C \times [0, T_{\max}])$ w.r. to the norm which is defined by

$$\|w; DV_{\omega,q}(Q;\gamma)\|^2 := \begin{cases} \sum_i \|\mathbb{X}^i w; V^{q+2}_{\omega+q}(Q;\gamma)\|^2 + \|w; V^{q+1}_{\omega+q}(Q;\gamma)\|^2 & \text{if } \Gamma_D \neq \emptyset, \\ \|w; \mathsf{DV}^N_{\omega,q}(Q;\gamma)\|^2 & \text{if } \Gamma = \Gamma_N \, . \end{cases}$$

Furthermore, we define the space $RV_{\omega,q}(Q;\gamma)$, equipped with the norm, defined by

$$\|g; RV_{\omega,q}(Q;\gamma)\|^2 := \sum_{j=0}^q \gamma^{-2j} \|\Lambda^j g; V^{q-j}_{\omega+q-j}(Q;\gamma)\|^2 + \gamma^{-2(1+q)} \|\Lambda^{1-\omega+q} g; V^0_0(Q;\gamma)\|^2 \, .$$

Proceeding as in [20, 10, 11], we obtain the following result:

Theorem 3.9. *Let G be a bounded polygonal domain with M corners and with interior opening angles $\phi_i \in (0, 2\pi]$, $i = 1, \dots, M$. Let $\gamma > 0$, $q \in \mathbb{N}_0$, and $\omega < 1$, such that there is no λ_n^i with $\Im(\lambda_n^i) = \omega - 1$. For each $1 \leq i \leq M$, we define*

$$n^i_{\max,\omega} := \max \left\{ n \in \mathbb{Z} : \; 0 < -\Im(\lambda_n^i) < 1 - \omega \right\}.$$

If the data satisfies $u_0, v_0 \in C_0^\infty(G)$, and if $f \in RV_{\omega,q}(Q;\gamma)$, then the solutions of (1) admit a decomposition of the form

$$u(x,t) = u_r^{\omega,q}(x,t) + \sum_{i=1}^M \chi_i(r_i) u_{s,i}^{\omega,q}(x,t), \tag{9}$$

where the singularities $u^{\omega,q}_{s,i}(x,t)$ are given by

$$u^{\omega,q}_{s,i}(x,t) := \sum_{n=1}^{n^i_{\max,\omega}} d^i_n(x,t) S_{i,n}(r_i,\vartheta_i),$$

where the regular part satisfies $u^{\omega,q}_r \in DV_{\omega,q}(Q;\gamma)$, and

$$d^i_n(x,t) := \left(\sum_{m=0}^{N_j} c_{j,m}(\mathrm{i}r_i\partial_t)^{2m}\right)(\mathbb{X}\check{c}^i_n)(x,t),$$

with a sufficiently large N_j, constants $c_{j,m}$ and, in general,

$$e^{-\gamma t}\check{c}^i_n(t) \in H^{1-\Im(\lambda^i_{-n})-\omega}([0,T_{\max}]) \,. \tag{10}$$

Moreover, if $f \in C^\infty_0(0,T_{\max};C^\infty(\bar{G}))$, then $\check{c}^i_n(t) \in C^\infty([0,T_{\max}])$, and

$$d^i_n(x,t) \in C^\infty([0,T_{\max}],C^\infty(\bar{G})) \,.$$

If f is not smooth a smooth function of time, then (10) is sharp, at least in the scale of spaces $H^s(0,T_{\max})$. See [14], Section 5, and Remark 8.3 in [11].

With Proposition 3.3, we interpret Theorem 3.9 in the following way: choosing suitable parameters ω, q, the solution is represented as the sum of a "sufficiently smooth" function $u^{\omega,q}_r \in H^s(0,T_{\max},H^{s'}(G))$ and, for each order of regularity, a finite number of singular terms. If $s+s'$ is increased, ω and q need to be changed such that $u^{\omega,q}_{s,i}$ contains more summands. To compute $\lambda^i_{\pm n}$ and Φ^i_n for all $i=1,\dots,M$, we consider different boundary conditions. Let $\mathbf{c}_i$ be a vertex of G and let Γ_j and Γ_k be the boundary edges intersecting at $\mathbf{c}_i$.

3.4. Dirichlet boundary conditions

If $\Gamma_j \cup \Gamma_k \subseteq \Gamma_D$, the boundary operator $\mathfrak{N}_i$ on Ω_i is given by $\mathfrak{N}_i\Phi = (\Phi(0), \Phi(\phi_i))$, and the eigenvalue problem (8) admits solutions

$$\mu^i_n = n^2\frac{\pi^2}{\phi_i^2} \,, \text{ and } \Phi^i_n = \sin(n\pi\vartheta_i/\phi_i), \quad \text{for all } n \geq 1,$$

where all eigenvalues are simple. Hence we obtain $\lambda^i_{\pm n} = \mp \mathrm{i}n\frac{\pi}{\phi_i}$, and the singular functions take the explicit form

$$S_{n,i}(r_i,\vartheta_i) = r^{n\pi/\phi_i}\,\sin(n\pi\vartheta_i/\phi_i) \,. \tag{11}$$

3.5. Neumann boundary conditions

If $\Gamma_j \cup \Gamma_k \subseteq \Gamma_D$, the boundary operator $\mathfrak{N}_i$ on Ω_i is given by $\mathfrak{N}_i\Phi = (\partial_{\vartheta_i}\Phi(0), \partial_{\vartheta_i}\Phi(\phi_i))$, and the eigenvalue problem (8) has the solutions

$$\mu^i_n = n^2\frac{\pi^2}{\phi_i^2} \,, \text{ and } \Phi^i_n = \cos(n\pi\vartheta_i/\phi_i), \quad \text{for all } n \geq 0,$$

where all nonzero eigenvalues are simple. Hence we obtain $\lambda^i_{\pm n} = \mp \mathrm{i}n\frac{\pi}{\phi_i}$, and

$$S_{n,i}(r_i,\vartheta_i) = r^{n\pi/\phi_i}\,\sin(n\pi\vartheta_i/\phi_i) \,. \tag{12}$$

3.6. Mixed boundary conditions

If $\Gamma_j \subseteq \Gamma_D$ and $\Gamma_k \in \Gamma_N$ or vice-versa, $\mathfrak{N}_i\Phi = (\Phi(0),\, \partial_{\vartheta_i}\Phi(\phi_i))$ or $\mathfrak{N}_i\Phi = (\partial_{\vartheta_i}\Phi(0),\, \Phi(\phi_i))$. Either way, the eigenvalue problem (8) has the solutions

$$\mu_n^i = (n - 1/2)^2\, \frac{\pi^2}{\phi_i^2}\ , \text{ and } \ \Phi_n^i = \begin{cases} \sin\left((n - 1/2)\, \pi\vartheta_i/\phi_i\right) \\ \cos\left((n - 1/2)\, \pi\vartheta_i/\phi_i\right), \end{cases} \qquad \text{for all } n \geq 1,$$

hence we obtain $\lambda_{\pm n}^i = \mp \mathrm{i}\,(n - 1/2)\, \frac{\pi}{\phi_i}$, and

$$S_{n,i}(r_i, \vartheta_i) = \begin{cases} r^{(n-1/2)\pi\vartheta_i/\phi_i}\ \sin\left((n - 1/2)\, \pi\vartheta_i/\phi_i\right) \\ r^{(n-1/2)\pi\vartheta_i/\phi_i}\ \cos\left((n - 1/2)\, \pi\vartheta_i/\phi_i\right)\ . \end{cases} \tag{13}$$

Remark 3.10.

1. In the present article, only homogeneous boundary conditions are considered. The generalization of Theorem 3.9 to non-homogeneous Dirichlet boundary conditions is described in [12].
2. We are given $\check{c}_n^i \in H^s(0, T_{\max})$, only with $s > 0$. However, we will need at least $s > 5/2$ for the semidiscrete convergence theorem 5.4. Therefore, the assumptions $u_0, v_0 \in C_0^\infty(G)$, and $f \in C_0^\infty(0, T_{\max}, C^\infty(\bar{G}))$ will be made throughout the following sections. As mentioned above, this guarantees smoothness of $\check{c}_n^i$.

4. Finite Element Approximation of singular functions

In this section, we review several types of mesh refinements on which continuous, piecewise polynomial nodal FEM are known to approximate singular functions $u_{s,i}^{\omega,q}(\cdot, t)$ in the decomposition (9) with optimal convergence rates, for all $t \in [0, T_{\max}]$. Although results of this type are well known (see, e.g., [1, 21, 2] where first order FEM were considered), we present short proofs here, for the readers' convenience and for completeness.

4.1. β-graded meshes

Let $K_0 = \mathrm{conv}\{(1, 0), (0, 0), (0, 1)\}$ be the unit triangle. On K_0, we construct a parametric family of meshes which are graded towards the vertex $(0, 0)$ so as to ensure an optimal rate of convergence of Lagrange interpolating Finite Elements of order $p \geq 1$. The idea of graded node distribution has been presented in one dimension by Rice [23] to improve convergence of splines. Given an integer $m \geq 2$ and a so-called grading parameter $\beta \geq 1$, let

$$z_l := \left(\frac{l}{n}\right)^\beta\ ,\ l = 0, 1, \ldots, m.$$

The nodes of the mesh that lie on the rectangular edges of K_0 are $(z_l, 0)$ and $(0, z_l)$, $l = 0, \ldots, m$. Then, being d_l the diagonal joining $(z_l, 0)$ and $(0, z_l)$, we divide d_l uniformly into $l + 1$ points. This defines all the nodes of a so-called β-graded mesh $\mathcal{T}_{m,\beta}(K_0)$ on K_0.

On the domain G, for each corner $i = 1, \ldots, M$ there exists exactly one triangle $T \subset G_i$ which abuts at corner $\mathbf{c}_i$. The β-graded reference mesh is then mapped via an affine transformation onto each of these triangles, such that the elements become smaller towards $\mathbf{c}_i$, see Figure 2. The mesh family $\{\mathcal{T}_{m,\beta}(K_0), m \in \mathbb{N}\}$ is shape regular with constant κ_β only depending on the grading parameter β.

We begin with some properties of β-graded meshes.

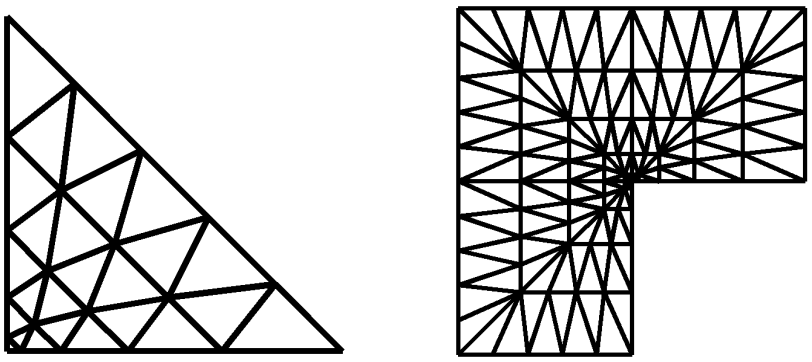

Figure 2: Graded meshes with parameters $n = 5$ and $\beta = 2$. Left: The mesh on the reference patch K_0. Right: A mesh on the L-shaped domain composed by six images of the reference mesh.

Proposition 4.1. *Let $m \geq 2$ be an integer and $\beta \geq 1$. We consider $\mathcal{T}_{m,\beta}(K_0)$.*
Let further be $h(x)$ the piecewise constant function taking the meshwidth h_T as value on each element $T \in \mathcal{T}_{m,\beta}(K_0)$ and denote by $N := \#\mathcal{T}_{m,\beta}(K_0)$.

1. *If $\beta = 1$, then $\mathcal{T}_{m,\beta}(K_0)$ is quasi uniform with meshwidth $h = \frac{1}{m}$.*
2. *There is a constant $C > 0$ which only depends on β, such that $N \leq Cm^2$.*
3. *There holds $h \leq C\beta m^{-1} + O(m^{-2})$, as $m \to \infty$ with some constant $C > 0$ which is independent of m and of β. Hence, for fixed $\beta \geq 1$, $h \to 0$ as $m \to \infty$.*

Definition 4.2. Let $G \subseteq \mathbb{R}^2$ be a bounded polygon with corners $\mathbf{c}_i \in \partial G$.
Let $\delta \in \mathbb{R}$, and $s \geq s_0 \in \mathbb{N}_0$. We define the weighted Sobolev space $H^{s,s_0}_\delta(G)$ as the completion of $C^\infty(\bar{G} \setminus C)$ with respect to the norm

$$\|v; H^{s,s_0}_\delta(G)\|^2 := \|v; H^{s_0-1}(G)\|^2 + \underbrace{\sum_{k=s_0}^{s} \int_G \Psi_{\delta+k-s_0}(x)^2 |\mathsf{D}^k v(x)|^2 \, dx}_{=:|v;H^{s,s_0}_\delta(G)|^2} \tag{14}$$

The case $s_0 = 2$ and $0 \leq \delta < 1$ is especially of our interest. We cite two properties of $H^{s,2}_\delta(G)$, proved in [2] and [24], respectively.

Proposition 4.3. *Let G, r and ϑ be as in Definition 4.2, $\delta \in [0, 1)$, and $s \geq 2$. Then there hold the following assertions.*

1. *The inclusion $H^{s,2}_\delta(G) \hookrightarrow C^0(\bar{G})$ is continuous.*
2. *Let $T_0 \in \mathbb{R}^2$ be a nondegenerate triangle with $(0, 0)$ as a vertex and meshwidth h_{T_0}. Then, there exists a constant $C > 0$ such that $\forall v \in H^{2,2}_\delta(T_0)$:*

$$\|v - I_1 v; H^1(T_0)\| \leq C h^{1-\delta}_{T_0} \|v; H^{2,2}_\delta(T_0)\| \,, \tag{15}$$

where I_1 denotes the linear, nodal interpolant in the three vertices of T_0.

We start with an approximation Theorem for functions in $H^{p+1,2}_\delta(G)$ on certain β-graded meshes in the reference patch.

Proposition 4.4. *Let $\delta \in [0,1)$, $p \in \mathbb{N}$, $v \in H^{p+1,2}_{\delta}(K_0)$, and*

$$\beta > \max\left\{1, \frac{p}{1-\delta}\right\}.$$

We construct a β-graded mesh family on the reference patch K_0 denoted by $\mathcal{T}_{m,\beta}(K_0)$, with total number of vertices $N := \#\mathcal{T}_{m,\beta}(K_0) \overset{m\to\infty}{\longrightarrow} \infty$.

Then, there is a constant $c > 0$, independent of v and N, such that

$$\min_{w \in S^{p,1}(K_0, \mathcal{T}_{m,\beta}(K_0))} \|v - w; H^1(K_0)\| \leq C N^{-p/2} \, |v; H^{p+1,2}_{\delta}(K_0)|, \ N \to \infty \, . \tag{16}$$

Proof. Let $v \in H^{p+1,2}_{\delta}(G)$ and let I_p denote the nodal Lagrangian Finite Element interpolant of degree p on $\mathcal{T}_{m,\beta}(K_0)$.
We denote by T_0 the element containing $(0,0)$ in its closure. Since

$$\|u - I_p u; H^1(K_0)\| = \|u - I_p u; H^1(K_0 \setminus T_0)\| + \|u - I_p u; H^1(T_0)\| \, ,$$

it suffices to prove the claim separately both inside and outside T_0.
We start with estimates in $K_0 \setminus T_0$. In the case $G = K_0$, $\Psi_{\delta'}(x) = r(x)^{\delta'}$ for all $\delta' \in \mathbb{R}$, where r is centered at $(0,0)$. By construction of $\mathcal{T}_{m,\beta}(K_0)$, we have for $T \neq T_0$

$$\Psi_1(x) > c m^{-\beta} \ \forall x \in T \ \text{ and } h_T \leq c m^{-1} \, r^{1-1/\beta}(x).$$

Then, exterior regularity results yield $v|_{K_0 \setminus T_0} \in H^{p+1}(K_0 \setminus T_0)$, and

$$|v - I_p v; H^1(K_0 \setminus T_0)|^2 \leq c \sum_{\substack{T \in \mathcal{T}_{m,\beta}(K_0) \\ T \neq T_0}} \int_T h(x) \, |\mathsf{D}^{p+1} v(x)|^2 \, dx.$$

Therefore,

$$\begin{aligned}
|v - I_p v; H^1(K_0 \setminus T_0)|^2 &\leq c \sum_{\substack{T \in \mathcal{T}_{m,\beta}(K_0) \\ T \neq T_0}} \int_T h(x)^{2p} \, |\mathsf{D}^{p+1} v(x)|^2 \, dx \\
&\leq c \sup_{x \in K_0 \setminus T_0} \left(h(x)^p \, r^{1-p-\delta}(x)\right)^2 \, |u; H^{p+1,2}_{\delta}(K_0 \setminus T_0)|^2 \\
&\leq c' \, m^{-2p} \sup_{x \notin T_0} \left(r^{p - \frac{p}{\beta} + 1 - p - \delta}\right)^2 |u; H^{p+1,2}_{\delta}(K_0 \setminus T_0)|^2 \\
&\leq c'' \, N^{-p} |u; H^{p+1,2}_{\delta}(K_0 \setminus T_0)|^2,
\end{aligned}$$

where the last step is valid if and only if $\beta > \frac{p}{1-\delta}$, and the constants $c, c', c'' > 0$ only depend on β and G.

In elements which are abutting at the corner, $v|_{T_0} \in H^{p+1}(T_0)$ may not be defined. However,

$$\begin{aligned}
\|v - I_p v; H^1(G)\| &\leq \|v - I_1 v; H^1(G)\| + \|I_p v - I_p I_1 v; H^1(G)\| \\
&\leq (1 + \|I_p\|) \|v - I_1 v; H^1(G)\|,
\end{aligned}$$

where, as in [6, §4, Proposition 1], the operator norm $\|I_p\| := \|I_p\|_{H^{1+\varepsilon}(G) \to H^1(G)}$ is finite. Now, (15) yields: there exists $C > 0$ such that for all $v \in H^{p+1,2}_{\delta}(T_0)$ there holds

$$\|v - I_p v; H^1(T_0)\| \leq C h_{T_0}^{1-\delta} \|v; H^{2,2}_{\delta}(T_0)\| \leq C h_{T_0}^{p/\beta} \|v; H^{p+1,2}_{\delta}(T_0)\| \, .$$

By construction of $\mathcal{T}_{m,\beta}$, we have $h_{T_0} \leq \tilde{c}m^{-\beta}$ for some $\tilde{c} > 0$, hence

$$\|v - I_p v; H^1(T_0)\| \leq Cm^{-p}\|v; H^{p+1,2}_\delta(T_0)\| \leq CN^{-p/2}\|v; H^{p+1,2}_\delta(T_0)\| .$$

□

In the considered cases of boundary conditions, it is easily verified that if $\delta \in [0, 1)$ satisfies

$$\delta > 1 - \mathrm{i}\lambda^i_1 \text{ for all } i = 1, \ldots, M,$$

We note in passing that the eigenvalues λ of the operator pencil (7) are purely imaginary, so that multiplication with i will result in $\delta \in \mathbb{R}$. The singular functions $S_{n,i}(r_i, \vartheta_i) := r_i^{i\lambda^i_n}\Phi^i_n(\vartheta_i)$, arising in decomposition (9), then belong to $H^{p+1,2}_\delta(G)$, for all $p \in \mathbb{N}$.
In order to define the notion of a β-graded mesh on polygonal domains, we need to formulate an analogous theorem for domains $G \neq K_0$.

Definition 4.5. Let $G \subseteq \mathbb{R}^2$ be a bounded polygon. Let $m \in \mathbb{N}$ and $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_M)$ with $\beta_i \geq 1$, be given. We construct a $\boldsymbol{\beta}$-graded Mesh $\mathcal{T}_{m,\boldsymbol{\beta}}(G)$ on G.
We assume that there are conforming sets of nondegenerate triangles $\left\{T^0_1, \ldots, T^0_{J_0}\right\}$ in $\bar{G}$, as well as for each $i = 1, \ldots, M$, $\{T^i_1, \ldots, T^i_{J_i}\}$ in $\bar{G}_i$ with common vertex $\mathbf{c}_i$, and satisfying $\mathbf{c}_i \notin \overline{G_i \setminus \bigcup_{j=1}^{J_i} T^i_j}$, such that their union is conforming and covers $\bar{G}$, i.e.

$$\bar{G} = \bigcup_{i=0}^{M}\bigcup_{j=1}^{J_i} \bar{T}^i_j.$$

For all i, j, let $\psi_{i,j} : T^i_j \to K_0$ be the affine map transforming T^i_j to K_0. We construct $\hat{\mathcal{T}}^i_j := \mathcal{T}_{m,\beta_i}(K_0)$ for $i \geq 1$, $\hat{\mathcal{T}}^0_j := \mathcal{T}_{m,1}(K_0)$ on the reference patch, and transport it to $\mathcal{T}^i_j := \psi^{-1}_{i,j}\left(\hat{\mathcal{T}}^i_j\right)$ on T^i_j for all i, j. Merging these meshes together, we finally obtain a (not necessarily conforming) mesh on all of G by

$$\mathcal{T}_{m,\boldsymbol{\beta}}(G) := \bigcup_{i=0}^{M}\bigcup_{j=1}^{J_i} \mathcal{T}^i_j.$$

Remark 4.6. The mesh $\mathcal{T}_{m,\boldsymbol{\beta}}(G)$ is conforming, if and only if all pairs of adjacent triangles T_j and $T_{j'}$, with local numberings of the shared edge e_j and $e_{j'}$, satisfy at least one of the following conditions:

1. $\beta_j = \beta_{j'}$,
2. The common edge, e_j ($e_{j'}$, respectively), is the local image by ψ_j ($\psi_{j'}$), of the hypotenuse of K_0 passing through $(1, 0)$ and $(0, 1)$.

We comment on condition (2). By construction of $\mathcal{T}_{m,\beta_j}(K_0)$, the diagonal d passing through $(1, 0)$ and $(0, 1)$ is the only edge of K_0 which is uniformly subdivided. Therefore, if $\beta_j \neq \beta_{j'}$, we can merge affine images of the patch meshes $\mathcal{T}_{m_j,\beta_j}(K_0)$ and $\mathcal{T}_{m_{j'},\beta_{j'}}(K_0)$ only if the common edge of the respective patches is the image of d, and if $m_j = m_{j'}$.

The transformation rule with bijective, affine maps between nondegenerate triangles implies the main result of this subsection:

Theorem 4.7. *Let $G \subseteq \mathbb{R}^2$ be a bounded polygon and let*

$$\delta_i > 1 - \mathrm{i}\lambda_1^i,\ i = 1, \dots, M,\ \textit{and}\ \ \delta := \max_i \delta_i\ .$$

Given a degree $p \in \mathbb{N}$, grading parameters $\beta_i > \max\left\{1, \frac{p}{1-\delta_i}\right\}$, and $m \in \mathbb{N}$, we construct a $\boldsymbol{\beta}$-graded Mesh $\mathcal{T}_{m,\boldsymbol{\beta}}(G)$ with m layers of shape regular elements as in Definition 4.5.
For each $0 \le \delta < 1$ and $v \in H^{p+1,2}_{\delta}(G)$, there is a constant $C > 0$, independent of v and $N := \#\mathcal{T}_{m,\boldsymbol{\beta}}(G)$, such that

$$\min_{w \in S^{p,1}(G,\mathcal{T}_{m,\boldsymbol{\beta}}(G))} \|v - w; H^1(G)\| \le CN^{-p/2}\|v; H^{p+1,2}_{\delta}(G)\|\ .$$

Note that $N \to \infty$ and $h \to 0$, as $m \to \infty$.

4.2. Meshes with binary tree structure

A disadvantage of β-graded meshes for some applications (e.g. for multilevel iterative solvers) is that the corresponding finite element spaces are not nested, i.e. for $1 \le m < m'$,

$$\dim\left(S^{p,1}(G,\mathcal{T}_{m,\boldsymbol{\beta}})\right) < \dim\left(S^{p,1}(G,\mathcal{T}_{m',\boldsymbol{\beta}})\right) \not\Rightarrow S^{p,1}(G,\mathcal{T}_{m,\boldsymbol{\beta}}) \subset S^{p,1}(G,\mathcal{T}_{m',\boldsymbol{\beta}})\ .$$

In this subsection, we will define a family of nested, locally refined finite element spaces with optimal approximation properties. It is constructed upon a given initial mesh which is then refined. The refinement algorithm has been introduced and investigated in [7].
On triangulations of the domain G, we assume that all the simplices of the triangulations are nondegenerate and oriented the same way.

Let $T := (z_0, z_1, z_2) \subseteq \mathbb{R}^2$ be a 2-simplex. We recall a unique way to split T up into two subsimplices T_1 and T_2, called children of T which were introduced by Bänsch [4]. A new vertex $\bar{z}$ is obtained by bisection of the refinement edge $\overline{(z_0, z_2)}$, i.e.

$$\bar{z} := \frac{1}{2}(z_0 + z_2)\,.$$

This induces two smaller simplices T_1 and T_2, with numbering

$$T_1 := (z_2, \bar{z}, z_1) \text{ and } T_2 := (z_1, \bar{z}, z_0).$$

By this unique numbering, the children of T inherit the orientation of T.

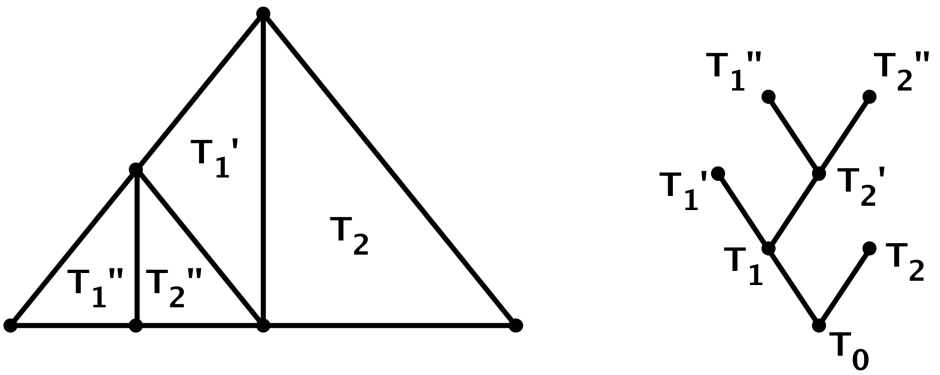

Figure 3: Repeated bisection of a simplex T and corresponding binary tree.

We next turn to the recurrent bisection of a given initial 2-simplex $T_0 := (z_0, z_1, z_2)$. Let $(T_1, T_2) =$ `BISECT`(T) be a function that returns the children of T after performing one bisection. The input T can either be T_0 or an output of any previous application of `BISECT`. We can associate to T_0 an infinite binary tree $\mathbb{F}(T_0)$, induced by recurrent bisection of T_0 and its descendants. The nodes of $\mathbb{F}(T_0)$ correspond uniquely to simplices generated by repeated application of `BISECT` to T_0. Each node in the tree has two successors corresponding to its children.

Definition 4.8. Let T_0 be a 2-simplex and $T \in \mathbb{F}(T_0)$ a node in its associated binary tree. The generation of T is defined to be the number of its ancestors in the tree, i.e. the number of bisections needed to obtain T from T_0. We denote it by $g(T)$.

Now, let $\mathcal{T}_0$ be a conforming triangulation of G. For each $T_0 \in \mathcal{T}_0$, we obtain a infinite binary tree corresponding to all possible bisections of T_0 and its descendants. It makes sense to introduce the set of all such trees, corresponding to all possible bisection refinements of $\mathcal{T}_0$.

Definition 4.9. For a conforming triangulation $\mathcal{T}_0$ of G, we define its master forest $\mathbb{F}(\mathcal{T}_0)$ by

$$\mathbb{F} := \mathbb{F}(\mathcal{T}_0) := \bigcup_{T \in \mathcal{T}_0} \mathbb{F}(T).$$

The generation of a node $T \in \mathbb{F} \cap \mathbb{F}(T_0)$, for some $T_0 \in \mathcal{T}_0$, is defined to be $g(T)$ in $\mathbb{F}(T_0)$.
A subset $\mathcal{F} \subseteq \mathbb{F}$ is called finite forest, if and only if

1. $\mathcal{T}_0 \subseteq \mathcal{F}$,
2. All nodes in $\mathcal{F} \setminus \mathcal{T}_0$ have a predecessor in $\mathbb{F}(\mathcal{T}_0)$,
3. All nodes of $\mathcal{F}$ have either two or no successors,
4. $\sup_{T \in \mathcal{F}} g(T) < \infty$.

Given a finite forest $\mathcal{F}$, we call its nodes without successors leaves. Any finite forest $\mathcal{F}$ induces a triangulation of G

$$\mathcal{T}(\mathcal{F}) := \{T : \ T \text{ is a leaf of } \mathcal{F}\},$$

obtained by bisection of elements in $\mathcal{T}_0$. Conversely, any triangulation $\mathcal{T}$ obtained by bisection of certain elements in $\mathcal{T}_0$ induces a finite forest $\mathcal{F}(\mathcal{T})$.
Given two finite forests $\mathcal{F}$ and $\mathcal{F}'$ with triangulations $\mathcal{T}$ and $\mathcal{T}'$, we call $\mathcal{T}'$ a refinement of $\mathcal{T}$, if $\mathcal{F} \subseteq \mathcal{F}'$.

Since there is a 1-1 correspondence between triangulations of G obtained by refinements of $\mathcal{T}_0$ and finite forests in $\mathbb{F}(\mathcal{T}_0)$, it makes sense to call both finite forests and triangulations obtained from $\mathcal{T}_0$ refinements of each other.

Lemma 4.10. *([19]) Let $\mathcal{T}_0$ be an conforming triangulation of G and let $\mathbb{F}$ be its master forest. Then, there is a constant $0 < c < \infty$ solely depending on $\mathcal{T}_0$, such that*

$$\sup_{T \in \mathbb{F}(T_0)} \frac{\overline{h}_T}{\underline{h}_T} < c,$$

where $\overline{h}_T$ and $\underline{h}_T$ denote the circumradius and the inradius of T.

Generally, a refinement of $\mathcal{T}_0$ is not conforming. In fact, bisection refinement of only one element $T = (z_0, z_1, z_2) \in \mathcal{T}_0$ creates a hanging node on the refinement edge $\overline{z_0 z_2} := S$, if $S \nsubseteq \partial G$. To overcome this fact, the unique element $T' := (z_0', z_1', z_2')$ such that $\bar{T} \cap \bar{T}' = S$ needs to be refined as well. Now,

this only yields a conforming refinement if the local numbering of T' is such, that $S = \overline{z_0' z_2'}$. If this is not the case, the child $T'' := (z_0'', z_1'', z_2'')$ of T', such that $\bar{T} \cap \bar{T}'' = S$ finally carries a vertex labelling such that $S = \overline{z_0'' z_2''}$.

Hence, given an initial conforming triangulation $\mathcal{T}_0$ and a subset $\mathcal{M}$ of ("marked") elements, there is a conforming refinement $\mathcal{T}$ of $\mathcal{T}_0$, such that $g(T) > 0$ for all leaves in $\mathcal{F}(\mathcal{T})$ whose roots belong to $\mathcal{M}$, i.e. in which all marked elements are refined. In other words, there is a function $\mathcal{T} = \texttt{REFINE\_MARKED}(\mathcal{T}_0, \mathcal{M})$ that returns said refinement $\mathcal{T}$ in finite time.

Our goal is to find a family of refinements of $\mathcal{T}_0$, such that $u_{s,i}^{\omega,q}(x,t)$ are optimally approximated. We recall an algorithm that refines $\mathcal{T}_0$ by bisection, introduced in [7].

Definition 4.11. Let $G \subseteq \mathbb{R}^2$ be a bounded polygonal domain with a given conforming triangulation $\mathcal{T}_0$. Let $p \in \mathbb{N}$. Let $\lambda := \frac{\pi}{2 \max_i \phi_i}$. Being $\#\mathcal{T}_0$ be the number of degrees of freedom of $\mathcal{T}_0$, we choose parameters $\varepsilon > 0$ and $K \in \mathbb{N}$, such that

$$0 < \varepsilon < (\#\mathcal{T}_0)^{-2} \text{ and } 2^{-\frac{\lambda(K+1)}{p+1}} \leq \varepsilon < 2^{-\frac{\lambda K}{p+1}} .$$

Then, the following algorithm returns a refinement of $\mathcal{T}_0$.

$\mathcal{T}_\varepsilon = \texttt{THRESHOLD}(\mathcal{T}_0, \varepsilon, \lambda, K)$
 $\mathcal{T} := \mathcal{T}_0$
 $\mathcal{M} := \{T \in \mathcal{T};\ h_T > \varepsilon\}$
 % global refinement
 WHILE $\mathcal{M} \neq \emptyset$
  $\mathcal{M} := \{T \in \mathcal{T};\ h_T > \varepsilon\}$
  $\mathcal{T} = \texttt{REFINE\_MARKED}(\mathcal{T}, \mathcal{M})$
 END
 % local graded refinement
 $l := 1$
 WHILE $l < 2K + 1$
  $\mathcal{M} := \left\{ T \in \mathcal{T};\ h_T > \varepsilon 2^{-\frac{l(p+1-\lambda)}{2(p+1)}} \text{ and } \min_i \operatorname{dist}(\mathbf{c}_i, \bar{T}) \leq \sqrt{2}^{-l} \right\}$
  $\mathcal{T} = \texttt{REFINE\_MARKED}(\mathcal{T}, \mathcal{M})$
  $l := l + 1$
 END
RETURN $\mathcal{T}_\varepsilon := \mathcal{T}$

Remark 4.12. THRESHOLD returns a mesh which is graded towards the vertices and exclusively contains elements of size $h \leq \varepsilon$. The grading parameters K and λ are entirely determined by $G, \mathcal{T}_0, p$ and the choice of ε.

As proved in [7], this algorithm returns a mesh family $\{\mathcal{T}_\varepsilon\}_{\varepsilon>0}$ that approximates $u_{s,i}^{\omega,q}(x,t)$ with optimal convergence rates. Concretely, the following Theorem holds:

Theorem 4.13. *([7]) Let $G \subseteq \mathbb{R}^2$ be a bounded, polygonal domain with interior angles $\phi_i \in (0, 2\pi]$, $i = 1, \ldots, M$ at the boundary vertices. Let $p \in \mathbb{N}$, $\lambda := \frac{\pi}{2 \max_i \phi_i}$, and let $v(x)$ be a function such that there is a constant $C > 0$ such that the following conditions are satisfied:*

$$|\mathsf{D}^k v(x)| \leq C \Psi_{\lambda - k}(x) \quad \forall k = \{0, 1, p+1\} . \tag{17}$$

Then, there exists a constant $C > 0$ depending only on the conforming initial triangulation $\mathcal{T}_0$ and on the domain G, such that for any tolerance $\varepsilon > 0$ as in Definition 4.11 and for $\mathcal{T}_\varepsilon := \texttt{THRESHOLD}(\mathcal{T}_0, \varepsilon, \lambda, K)$,

$$\|v - I_p v; H^1(G)\| \leq C(\#\mathcal{T}_\varepsilon - \mathcal{T}_0)^{-p/2} .$$

Remark 4.14.

- Assuming (17), we make use of the explicit form of the singular functions. However, Theorem 4.13 has recently been generalized to functions which belong to certain Besov spaces, in the sense that $u(\cdot, t) \in B^{\alpha+s}_{\tau,\tau}(G)$ in [8]. To see that the functions considered here actually satisfy this condition, we need an inclusion $H^{p+1,2}_\delta(G) \hookrightarrow B^{\alpha+s}_{\tau,\tau}(G)$. This could be done via an intermediate inclusion into Babuška-Kondrat'ev spaces $\mathcal{K}^s_a(G)$, for which an inclusion into $B^{\alpha+s}_{\tau,\tau}(G)$ is known to hold, see [9].

- THRESHOLD, as defined here, uses the same grading intensity (K, λ) towards all the vertices of G. The efficiency of both algorithms (but not the asymptotic convergence rates, only the constants in the work versus accuracy bounds) can be improved by choosing different gradings towards each corner $\mathbf{c}_i$. For the sake of simplicity, these rather obvious generalizations are not treated here.

- The convergence results of Section 4, namely Theorems 4.7 and 4.13, are valid also for more general, so-called "power-logarithmic" singular functions $v(x)$ of the form

$$v(x) = S^i_{n;k,j}(r_i, \vartheta_i) := r_i^{\mathrm{i}\lambda^i_n} \sum_{m=0}^{k} \frac{\log(r_i)^m}{m!} \Phi^i_{n;k,j}(\vartheta_i),$$

  where $m \in \mathbb{N}_0$, and again $\Phi^i_{n;k,j} \in C^\infty([0, \phi_i])$. This class of functions describes the singular functions which appear in the corner asymptotics of general linear hyperbolic systems, see for example [13, 14].
  The modified singular functions $S^i_{n;k,j}(r_i, \vartheta_i)$ satisfy the conditions (17), and moreover, they are contained in $H^{2,p+1}_\delta(G)$ for $[0, 1) \ni \delta > 1 - \mathrm{i}\lambda^i_1$ for $p \in \mathbb{N}$. Therefore, Theorems 4.7 and 4.13 remain valid for this larger class of singular functions. With this remark, in particular all results in the present paper can be adapted to problems with nonhomogeneous coefficients which are sufficiently smooth in $\overline{G}$, or to elliptic systems such as problems of elastodynamics, or to problems of electromagnetic wave propagation in polygonal domains G. Details on this will be given in a forthcoming paper.

5. The FEM semi-discretization

Let $\{\mathcal{T}_h : h > 0\}$ denote a regular family of simplicial meshes with meshwidth h, obtained as described in Section 4, i.e. one of the following assertions is true:

1. There is a $\boldsymbol{\beta}$, $\beta_i \geq \max\{1, p/1-\delta_i\}$ as in Theorem 4.7, such that for each $\mathcal{T}_h$, there exits $m \in \mathbb{N}$ with $\mathcal{T}_h = \mathcal{T}_{m,\boldsymbol{\beta}}$.
2. There is a conforming triangulation $\mathcal{T}_0$ of G such that for each $\mathcal{T}_h$, there exists $\varepsilon > 0$ with

$$\mathcal{T}_h = \texttt{THRESHOLD}(\mathcal{T}_0, \varepsilon, \lambda, K).$$

We further denote the number of vertices in $\mathcal{T}_h$ by $N_h := \#\mathcal{T}_h$.

Given a polynomial degree $p \in \mathbb{N}$, we obtain a sequence of conforming finite element spaces, denoted by

$$V_h := V \cap S^{p,1}(G; \mathcal{T}_h) \, .$$

The dimension $\dim(V_h)$ is finite and tends to ∞, as $h \to 0$.

Definition 5.1. For $l, m \in \mathbb{N}_0$, we say that a tuple $(\omega, q) \in \mathbb{R} \times \mathbb{N}_0$ satisfies the (s, s')-regularity condition on G, if

$$\omega \notin \bigcup_{i=1}^{M} \mathbb{Z} \frac{\pi}{\phi_i}, \quad \omega \leq -q, \quad \text{and} \quad q \geq s + s' - 1. \tag{18}$$

We know from the Proposition 3.3 and Theorem 3.9, that if (ω, q) satisfies the (s, s')-regularity condition, then $u_r^{\omega,q} \in H^s(0, T_{\max}; H^{s'}(G))$.

Theorem 5.2. *Let G be a polygonal domain with interior opening angles $\phi_i \in (0, 2\pi]$, let $T_{\max} > 0$, and let*

$$f \in C_0^\infty(0, T_{\max}; C^\infty(\bar{G})), \text{ and } u_0, v_0 \in C_0^\infty(G). \tag{19}$$

Assume further that $p \in \mathbb{N}$, $\gamma > 0$ and $(\omega, q) \in \mathbb{R} \times \mathbb{N}$ are given such that the $(s, p+1)$-regularity condition on G holds with $s > \frac{3}{2}$.
Then, for all $t \in (0, T_{\max})$, there is a constant $C > 0$ which is independent of N_h, but dependent on $u(\cdot, t)$, such that

$$\min_{v \in V_h} \|\partial_t^j u(\cdot, t) - v; H^1(G)\| \leq C N_h^{-p/2} \text{ for } j = 0, 1 \, , \tag{20}$$

as $N \to \infty$.

Proof. By Theorem 3.9, by the assumptions (19), by the Sobolev embedding, and since (ω, q) satisfy the $(s, p+1)$-regularity condition on G with $s > \frac{3}{2}$, we have the decomposition

$$u(x, t) = u_r^{\omega,q}(x, t) + \sum_{i=1}^{M} u_{s,i}^{\omega,q}(x, t) \, ,$$

where $u_r^{\omega,q} \in C^1([0, T_{\max}]; H^{p+1}(G))$, and where $u_{s,i}^{\omega,q} \in C^\infty([0, T_{\max}]; H^1(G))$. Now, for all $t \in (0, T_{\max})$,

$$\min_{v \in V_h} \|u(\cdot, t) - v; H^1(G)\| \leq \sum_{i=1}^{M} \min_{v \in V_h} \|u_{s,i}^{\omega,q}(\cdot, t) - v; H^1(G)\| + \min_{v \in V_h} \|u_r^{\omega,q}(\cdot, t) - v; H^1(G)\| \, .$$

Since $u_r^{\omega,q}(\cdot, t) \in H^{p+1}(G)$, the second term of the sum is approximated with optimal convergence rates, i.e. (20) holds for $u = u_r^{\omega,q}$. Using Theorems 4.7 and 4.13, we conclude that all $u_{s,i}^{\omega,q}(x, t)$ are approximated with optimal convergence rates for our choice of $\mathcal{T}$, whence the claim follows for $j = 0$. The claim for $j = 1$ follows analogously, since $\partial_t u_{s,i}^{\omega,q}$ also satisfies the assumptions of Theorems 4.7 and 4.13. However, this is only true, because $u_{s,i}^{\omega,q} \in C^\infty([0, T_{\max}]; V)$. □

Let $u(x, t)$ be a solution of (2). We consider solutions $u_h(x, t)$ of the following space semidiscrete initial boundary value problem of the linear, second order wave equation, which is given by

$$\begin{aligned} &\text{Find } u_h \in C^0([0, T_{\max}]; V_h) \text{ such that } \forall v \in V_h \text{ and } t \in [0, T_{\max}] : \\ &\partial_t^2 (u_h(\cdot, t), v) + (\nabla u_h(\cdot, t), \nabla v) = (f(\cdot, t), v) \, , \\ &(u_h, v) = (u_0, v) \, , \\ &\partial_t (u_h(\cdot, 0), v) = (v_0, v) \, . \end{aligned} \tag{21}$$

In fact, choosing the Lagrangian nodal basis $\mathfrak{B}$ of V_h, (21) can be written in matrix form as

$$M\partial_t^2 \mathbf{u}_h(t) + A\mathbf{u}_h(t) = \mathbf{F}(t),\ \mathbf{u}_h(0) := \mathbf{u}_0,\ \mathbf{v}_h(0) := \mathbf{v}_0, \tag{22}$$

where $\mathbf{u}_h(t)$, $\mathbf{F}(t)$, $\mathbf{u}_0$, and $\mathbf{v}_0$ are vectors containing the basis coefficients of $u_h(\cdot,t)$, $f(\cdot,t)$ u_0 and v_0 with respect to $\mathfrak{B}$, which can be identified with values of the respective functions at the grid points.

Theorem 5.2 leads now to the main result. In its proof, we require the elliptic projection $\Pi_h : V \mapsto V_h$.

Definition 5.3. For a closed subspace $V_h \subset V$, the elliptic projection $\Pi_h : V \to V_h$ is defined for any $v \in V$ by

$$\left(\nabla\left(\Pi_h v\right), \nabla w_h(x)\right) = \left(\nabla v, \nabla w_h\right) \quad \forall w_h \in V_h\ .$$

Theorem 5.4. *Let $p \in \mathbb{N}, \gamma > 0$, and assume that f, u_0, v_0 satisfy the assumptions of Theorem 5.2. Moreover, let $u(x,t)$ be the solution of (2), and $u_h(x,t)$ the solution of (21) and (ω, q) satisfy the $(s, p+1)$-regularity condition on G, with $s > \frac{5}{2}$.*
Then, there exists a constant $C > 0$, such that for every $0 \le t \le T_{max}$ holds

$$\begin{aligned}
&\|u(\cdot,t) - u_h(\cdot,t); H^1(G)\| + \|\partial_t u(\cdot,t) - \partial_t u_h(\cdot,t); L^2(G)\| \\
&\qquad \le C \left\{ \|u_0 - u_{0,h}; H^1(G)\| + \|v_0 - v_{0,h}; L^2(G)\| \right. \\
&\qquad + N^{-p/2} \left[\|u(\cdot,t); H^{p+1}(G)\| + \|\partial_t u(\cdot,t); H^{p+1}(G)\| \right. \\
&\qquad \left. \left. + \int_0^t \|\partial_t^2 u(\cdot,s); H^{p+1}(G)\|\, ds \right] \right\}.
\end{aligned} \tag{23}$$

Proof. By the Sobolev embedding, $u \in C^2([0, T_{\max}]; V)$. In that case,

$$\begin{aligned}
&\|u(\cdot,t) - u_h(\cdot,t); H^1(G)\| + \|\partial_t u(\cdot,t) - \partial_t u_h(\cdot,t); L^2(G)\| \\
&\qquad \le C \left\{ \|u_0 - u_{0,h}; H^1(G)\| + \|v_0 - v_{0,h}; L^2(G)\| \right. \\
&\qquad + \|(I - \Pi_h) u(\cdot,t); H^1(G)\| + \|(I - \Pi_h)\partial_t u(\cdot,t); L^2(G)\| \\
&\qquad \left. + \int_0^t \|(I - \Pi_h)\partial_t^2 u(\cdot,s); L^2(G)\|\, ds \right\},
\end{aligned}$$

see e.g. [22, Theorem 8.7-1]. Since $u_{s,i}^{\omega,q}(\cdot,t) \in H_\delta^{p+1,2}(G)$ for $\delta > 1 - \frac{\pi}{\phi_i}$, and $u_{s,i}^{\omega,q}$ satisfies the conditions (17) for all $t \in (0, T_{\max})$, the claim follows from Theorem 5.2, since there exists a constant $C > 0$ which is independent of h such that for all $v \in V_h$,

$$\|v - \Pi_h v; H^1(G)\| \le C \inf_{w \in V_h} \|v - w; H^1(G)\|\ .$$

The same holds for $\partial_t u_{s,i}^{\omega,q}(\cdot,t)$, whence the claim of the theorem follows. □

6. Numerical experiments

In the previous sections, optimal convergence rates were proved for two kinds of meshes, when discretizing (2) with FEM of polynomial order $p \in \mathbb{N}$. We present numerical experiments on an L-shaped domain and on the extreme case of a cracked domain, to illustrate the theoretical results.

6.1. Test 1: Linear FEM on the L-shaped domain

Let G be the L-shaped domain as in Figure 2, with one reentrant corner located at $(0,0)$. The Dirichlet problem for the wave equation on G is numerically solved using a method of lines approach, with space semidiscretization by conforming, Lagrangean Finite Elements with nodal basis functions. For the time discretization, a uniformly stable Crank-Nicolson scheme with uniform timestep $\Delta t > 0$ is used. The time step is chosen so small as to render the time discretization errors negligible in this and all ensuing numerical experiments. We reformulate (22) as a first-order system of ODEs (with slight abuse of notation denoting by $\mathbf{u}_h(t)$ the vector of nodal unkowns as well as their time derivatives and with the obvious meaning for $\hat{A}$ and $\hat{M}$)

$$\partial_t \hat{M}\mathbf{u}_h(t) + \hat{A}\mathbf{u}_h(t) = \hat{\mathbf{F}}(t), \quad \mathbf{u}_h(0) = \hat{\mathbf{u}}_0,$$

and the Crank-Nicolson scheme returns vectors $\hat{\mathbf{u}}^m \simeq \hat{\mathbf{u}}(t_m)$, with $t_m := m\Delta t$, $m = 0, \ldots, {}^{T_{\max}}/_{\Delta t} \in \mathbb{N}$, defined by the iteration

$$\begin{aligned} \hat{\mathbf{u}}^{m=0} &:= \hat{\mathbf{u}}_0 \ , \\ \left(2\hat{M} + \Delta t\hat{A}\right)\hat{\mathbf{u}}^{m+1} &:= \left(2\hat{M} - \Delta t\hat{A}\right)\hat{\mathbf{u}}^m + \Delta t\left(\hat{\mathbf{F}}_{m+1} + \hat{\mathbf{F}}_m\right) \ . \end{aligned}$$

To exhibit the effect of mesh grading near corners, we compute the $L^2(0, T_{\max}; H^1(G))$ norm of the error $u(x,t) - u_h(x,t)$ on uniformly refined meshes, β-graded meshes and on meshes obtained by the procedure `THRESHOLD`.

Let us compute the grading parameter for the β-graded meshes. There is one nonconvex angle $\phi_1 = \frac{3}{2}\pi$ and five convex angles $\phi_i = \frac{\pi}{2}$, $i = 2, \ldots, 6$. Since $p = 1$, $q = 4$, $\omega = -4$ satisfy the required $({}^5/_2, 2)$ regularity condition we have

$$\begin{aligned} n^1_{\max,-4} &= \max\{n \in \mathbb{N} \ : \ {}^{2n}/_3 \leq 5\} = 7 \ , \\ n^i_{\max,-4} &= \max\{n \in \mathbb{N} \ : \ 2n \leq 5\} = 2, \ i = 2, \ldots, 6 \ . \end{aligned}$$

For $i = 1$, we have $\delta_1 > 1 - \frac{2}{3}$. According to Proposition 4.4, a sufficient condition to recover the optimal convergence rate is $\beta_1 > \frac{3}{2}$. At convex corners, we have $\delta_i > -1$. Similarly to Proposition 3.3, it can be proved that for $\delta < 0$, $H^{2,2}_\delta(G) \hookrightarrow H^2(G)$, hence no grading is needed for $p = 1$.

Remark 6.1. This example shows that if p is increased, using a graded mesh with grading factor $\beta > 1$ may become necessary even in the vicinity of convex corners since, in this case, Proposition 4.4 predicts $\beta > \max\{1, {}^p/_2\} > 1$ for $p \geq 2$.

This yields a sequence of discrete times $t_j := j\Delta t$ and discrete solutions $\mathbf{u}^{(j)} \in \mathbb{R}^N$, $j \geq 1$. In order to exhibit the theoretical convergence rates which were proved in the semidiscrete setting, in the present numerical experiments the timestep Δt is chosen so small that the time-discretization error becomes negligibly small compared to the space discretization error. Hence, we expect that, as $N \to \infty$,

$$\|u - u_h; L^2(0, T_{\max}; H^1(G))\|^2 = \int_0^{T_{\max}} \|u(\cdot,t) - u_h(\cdot,t); H^1(G)\|^2 \, dt \lesssim N^{-\rho},$$

where the convergence rate is $\rho > 0$. As an exact solution, we choose (note that ${}^\pi/_\phi = {}^2/_3$),

$$u(x,t) = \sin(\pi t) \sum_{n=1}^{7} r^{2n/3} \sin(2n\vartheta/3) \in C^\infty([0, T_{\max}]; H^{4/3-\delta}(G))$$

for any $\delta > 0$, where (r, ϑ) are polar coordinates centered at the reentrant corner $(0, 0)$. Therefore, standard approximation results for Lagrangian Finite Elements on quasiuniform meshes (see, e.g., [5]) and an interpolation argument predict a convergence rate of $\rho < \frac{1}{3}$ in the case of uniform mesh refinement, while our results predict that the optimal convergence rate $\rho = \frac{1}{2}$ is essentially recovered on the refined meshes, with both bisection as well as graded mesh refinement. In the error computation, the integral in the t variable is computed by the second order trapezoidal rule, and the H^1-errors are computed using a seven node Gauss-type quadrature rule in a triangle (due to J. Radon) which integrates polynomials $p(x_1, x_2)$ of total degree 5 exactly. Figure 4 shows a log-log plot of the discretization errors (with comparison lines to indicate the convergence order).

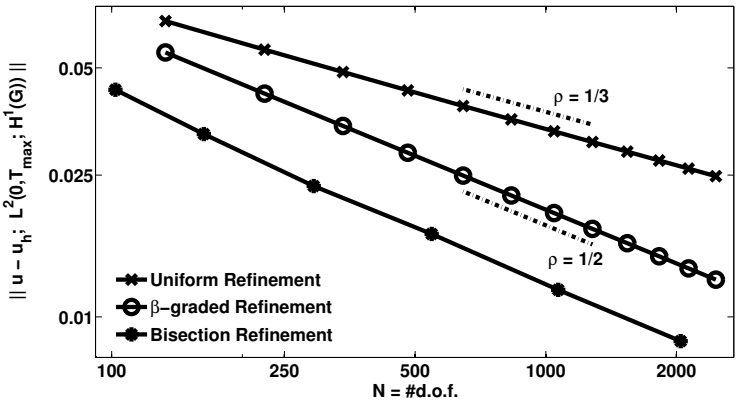

Figure 4: The error $\|u - u_h; L^2(0, T_{\max}; H^1(G))\|$ in Test 1, on a log-log scale. The β-graded mesh is constructed with a partition of G into 24 triangles. The parameters are $\beta_1 = 2, \beta_i = 1, i > 1$, and $m = 3, 4, \ldots, 14$. For the bisection refinement, we choose $\#\mathcal{T}_0 = 11$, and run THRESHOLD with $\varepsilon = 0.01 \cdot 2^k$, $k = 0, -1, \ldots, -5$. The timestepping is done with $\Delta t = 10^{-4}$, and $T_{\max} = 0.25$.

6.2. Test 2: Quadratic FEM on the L-shaped domain

In this example, we consider again the Dirichlet problem on the L-shaped domain, but we discretize now in space with continuous, piecewise quadratic FEM. The mesh grading must be done, according to Proposition 4.4, with $\beta_1 > 3$ to restore optimal convergence rate.

For bisection refinement, we choose ε the same way as in Test 1 (see caption of Figure 4), but have to set $p = 2$, which changes also the choice of the parameter K. As an exact solution, we choose a superposition of singular functions

$$u(x,t) = \sin(\pi t)\left(\operatorname{dist}(x, \mathbf{c}_1)^{2/3} \sin(2\vartheta_1/3) + \sum_{i=1}^{5} \operatorname{dist}(x, \mathbf{c}_i)^2 \sin(2\vartheta_i)\right),$$

which lies in $C^\infty([0, T_{\max}]; H^{4/3-\delta}(G))$ for arbitrary small $\delta > 0$. At each corner $\mathbf{c}_i$, only the terms of least regularity in $u_{s,i}^{\omega,q}(x,t)$ have been considered. Moreover, we did not multiply by a cut-off, since $u_{s,i}^{\omega,q}(\cdot, t) \in C^\infty(\bar{G} \setminus \bar{G}_i)$.

In terms of the number N of interpolation nodes (not vertices) in the triangulation, we expect the convergence rate $\rho < \frac{2}{3}$ on uniformly refined meshes, and the optimal rate $\rho = 1$ on the locally refined meshes. The results are shown in Figure 5.

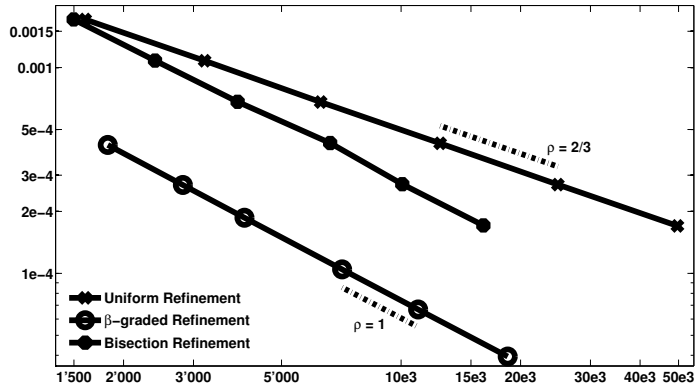

Figure 5: The error $\|u - u_h; L^2(0, T_{\max}; H^1(G))\|$ in Test 2, on a log-log scale. The local mesh refinements are performed with the parameters indicated in the text. The timestep is chosen to be $\Delta t = 10^{-4}$, and $T_{\max} = 0.25$.

6.3. Test 3: Polygonal domain with crack

Polygons with a slit which arise in fracture mechanics as models of a structure with a crack are not Lipschitz domains anymore, but can be represented as finite union of Lipschitz domains. Now, consider the domain $G := (-1, 1)^2 \setminus ((0, 1) \times \{0\})$, with reentrant "corner" $\mathbf{c}_1 = (0, 0)$ and with interior opening angle $\phi_1 = 2\pi$. We discretize again the Dirichlet problem with continuous, piecewise linear FEM. The mesh grading at the convex corners is done as in the previous example. According to Proposition 4.4, $\beta_1 > 2$ must be chosen in order to restore optimal convergence rate. For bisection

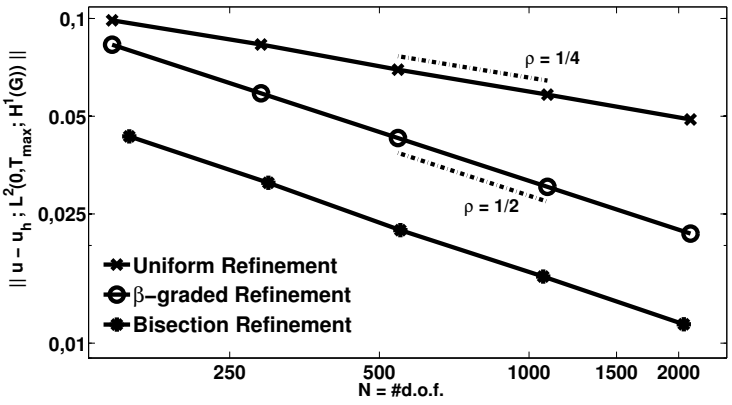

Figure 6: The error $\|u - u_h; L^2(0, T_{\max}; H^1(G))\|$ in Test 3, on a log-log scale. The local mesh refinements are performed with the parameters indicated in the text. The timestep is chosen to be $\Delta t = 10^{-4}$, and $T_{\max} = 0.25$.

refinement, we choose ε the same way as in Test 1 (see caption of Figure 4), but have to adapt $\lambda = \frac{1}{2}$, and therefore also K, as in Definition 4.11. We take the same ε as above. As an exact solution, we choose again the singular function

$$u(x, t) = \sin(\pi t) \sum_{n=1}^{7} r^{n/2} \sin(n\vartheta/2) \in C^\infty([0, T_{\max}]; H^{3/2-\delta}(G))$$

for arbitrary small $\delta > 0$. We expect convergence rate $\rho = \frac{1}{4}$ on uniform meshes, and $\rho = 1/2$ on the locally refined meshes. The results are shown in Figure 6.

References

[1] Babuška, I., 1970. Finite element method for domains with corners. Computing (Arch. Elektron. Rechnen) 6, 264–273.

[2] Babuška, I., Kellogg, R. B., Pitkäranta, J., 1979. Direct and inverse error estimates for finite elements with mesh refinements. Numer. Math. 33, 447–471.

[3] Băcuţă, C., Nistor, V., Zikatanov, L. T., 2005. Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps. Numer. Math. 100 (2), 165–184.
URL http://dx.doi.org/10.1007/s00211-005-0588-3

[4] Bänsch, E., 1991. Local mesh refinement in 2 and 3 dimensions. Impact Comput. Sci. Engrg. 3 (3), 181–191.
URL http://dx.doi.org/10.1016/0899-8248(91)90006-G

[5] Brenner, S. C., Scott, L. R., 2008. The mathematical theory of finite element methods, 3rd Edition. Vol. 15 of Texts in Applied Mathematics. Springer, New York.
URL http://dx.doi.org/10.1007/978-0-387-75934-0

[6] Demkowicz, L., Babuška, I., 2003. p interpolation error estimates for edge finite elements of variable order in two dimensions. SIAM J. Numer. Anal. 41 (4), 1195–1208.
URL http://dx.doi.org/10.1137/S0036142901387932

[7] Gaspoz, F. D., Morin, P., 2009. Convergence rates for adaptive finite elements. IMA J. Numer. Anal. 29 (4), 917–936.
URL http://dx.doi.org/10.1093/imanum/drn039

[8] Gaspoz, F. D., Morin, P., December 2012. Approximation classes for adaptive higher order finite element approximation, accepted at Mathematics of Computation.

[9] Hansen, M., 2012. n-term approximation rates and besov regularity for elliptic pdes on polyhedral domains. Tech. Rep. 2012-41, Seminar for Applied Mathematics, ETH Zürich, Switzerland.
URL http://www.sam.math.ethz.ch/sam_reports/reports_final/reports2012/2012-41.pdf

[10] Kokotov, A. Y., Neittaanmäki, P., Plamenevskiĭ, B. A., 1999. Problems of diffraction by a cone: asymptotic behavior of the solutions near the vertex. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 259 (Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 30), 122–144, 297–298.
URL http://dx.doi.org/10.1023/A:1014492224655

[11] Kokotov, A. Y., Neittaanmäki, P., Plamenevskiĭ, B. A., 2000. The Neumann problem for the wave equation in a cone. J. Math. Sci. (New York) 102 (5), 4400–4428, function theory and applications.
URL http://dx.doi.org/10.1007/BF02672898

[12] Kokotov, A. Y., Plamenevskiĭ, B. A., 1999. A mixed problem for second-order hyperbolic systems in a wedge. In: Nonlinear analysis and related problems (Russian). Vol. 2 of Tr. Inst. Mat. (Minsk). Natl. Akad. Nauk Belarusi, Inst. Mat., Minsk, pp. 98–109.

[13] Kokotov, A. Y., Plamenevskiĭ, B. A., 1999. On the Cauchy-Dirichlet problem for a hyperbolic system in a wedge. Algebra i Analiz 11 (3), 140–195.

[14] Kokotov, A. Y., Plamenevskiĭ, B. A., 2004. On the asymptotic behavior of solutions of the Neumann problem for hyperbolic systems in domains with conical points. Algebra i Analiz 16 (3), 56–98.
URL http://dx.doi.org/10.1090/S1061-0022-05-00862-9

[15] Kondrat'ev, V. A., 1967. Boundary value problems for elliptic equations in domains with conical or angular points. Trudy Moskov. Mat. Obšč. 16, 209–292.

[16] Maz'ya, V. G., Plamenevskiĭ, B. A., 1983. The first boundary value problem for classical equations of mathematical physics in domains with piecewise-smooth boundaries. I. Z. Anal. Anwendungen 2 (4), 335–359.

[17] Maz'ya, V. G., Rossmann, J., 2010. Elliptic equations in polyhedral domains. Vol. 162 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI.

[18] Nazarov, S. A., Plamenevskiĭ, B. A., 1994. Elliptic problems in domains with piecewise smooth boundaries. Vol. 13 of de Gruyter Expositions in Mathematics. Walter de Gruyter & Co., Berlin.
URL http://dx.doi.org/10.1515/9783110848915.525

[19] Nochetto, R. H., Siebert, K. G., Veeser, A., 2009. Theory of adaptive finite element methods: an introduction. In: Multiscale, nonlinear and adaptive approximation. Springer, Berlin, pp. 409–542.
URL http://dx.doi.org/10.1007/978-3-642-03413-8_12

[20] Plamenevskiĭ, B. A., 1998. On the Dirichlet problem for the wave equation in a cylinder with edges. Algebra i Analiz 10 (2), 197–228.

[21] Raugel, G., 1978. Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone. C. R. Acad. Sci. Paris Sér. A-B 286 (18), A791–A794.

[22] Raviart, P.-A., Thomas, J.-M., 1983. Introduction à l'analyse numérique des équations aux dérivées partielles. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris.

[23] Rice, J. R., 1969. On the degree of convergence of nonlinear spline approximation. In: Approximations with Special Emphasis on Spline Functions (Proc. Sympos. Univ. of Wisconsin, Madison, Wis., 1969). Academic Press, New York, pp. 349–365.

[24] Schwab, C., 1998. *p*- and *hp*-finite element methods. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, theory and applications in solid and fluid mechanics.

[25] Stein, E. M., 1970. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J.

[26] Wloka, J., 1987. Partial differential equations. Cambridge University Press, Cambridge, translated from the German by C. B. Thomas and M. J. Thomas.

Recent Research Reports

Nr.	Authors/Title

2013-01 M. Eigel and C. Gittelson and C. Schwab and E. Zander
Adaptive stochastic Galerkin FEM

2013-02 R. Hiptmair and M. Lopez-Fernandez and A. Paganini
Fast Convolution Quadrature Based Impedance Boundary Conditions

2013-03 X. Claeys and R. Hiptmair
Integral Equations on Multi-Screens

2013-04 V. Kazeev and M. Khammash and M. Nip and C. Schwab
Direct Solution of the Chemical Master Equation using Quantized Tensor Trains

2013-05 R. Kaeppeli and S. Mishra
Well-balanced schemes for the Euler equations with gravitation

2013-06 C. Schillings
A Note on Sparse, Adaptive Smolyak Quadratures for Bayesian Inverse Problems

2013-07 A. Paganini and M. López-Fernández
Efficient convolution based impedance boundary condition

2013-08 R. Hiptmair and C. Jerez-Hanckes and J. Lee and Z. Peng
Domain Decomposition for Boundary Integral Equations via Local Multi-Trace Formulations

2013-09 C. Gittelson and R. Andreev and C. Schwab
Optimality of Adaptive Galerkin methods for random parabolic partial differential equations

2013-10 M. Hansen and C. Schillings and C. Schwab
Sparse Approximation Algorithms for High Dimensional Parametric Initial Value Problems