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by

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Abstract

In the year 1620 the printing office of the University of Prague published a 58-page table containing the values $a_n = (1.0001)^n$ for $0 \leq n \leq 23027$, rounded to 9 decimal digits. This table had been devised and computed about 20 years earlier by the Swiss-born astronomer and watchmaker Jost Bürgi in order to facilitate the multi-digit multiplications and divisions he needed for his astronomical computations. The "Progreß Tabulen", as Bürgi called his tables, are considered to be one of the two independent appearances of the logarithms in the history of mathematics - the other one, due to John Napier (1550-1617), appeared in 1614.

There are only a few copies of the original printing extant: one of them is now in the Astronomisch-Physikalisches Kabinett in Munich. Based on a copy of this original, the terminal digits of all table entries were extracted and compared with the exact values of a_n, a matter of a split second on a modern computer.

In this presentation we give a brief account of the mathematical environment at the end of the 16th century as well as a detailed description of Bürgi's Progreß Tabulen and their application to numerical computations. We will also give a sketch of Bürgi's remarkable life and of his numerous achievements besides the discovery of the logarithms.

Our main purpose, however, is to analyze the numerical errors in Bürgi's table. First of all, there are no systematic errors, e.g. the crux of the table, $1.0001^{23027.0022} = 10$, is correct with all digits given. 91.5% of the table entries are entirely correct, and 7.3% of the values show round-off errors between 0.5 and 1 unit of the least significant digit. The remaining 1.17% table errors are mainly errors of transcription and illegible digits.

Statistics of the round-off errors leads to interesting conclusions concerning Bürgi's algorithms of generating his table and on his handling of the round-off errors, as well as on the computational effort involved.

1 Introduction

The origins of this report date back to 1976, when the existence of an original copy of Jost Bürgi's table of logarithms [2] in the Astronomisch-Physikalisches Kabinett, Munich, was pointed out to the author by the engineer and historian Wolfhard Pohl, Zürich. This original had been found in 1847 by Rudolf Wolf [29], [30] in the Royal Library (Königliche Bibliothek) of Munich [3].

Subsequently, W. Pohl [21] was allowed to copy the entire table, in black and white, such that it became possible to investigate the numerical errors in Bürgi's hand calculations for the first time. This copy is now in possession of the author. In 1994 these results were presented to the mathematical community and commented on the occasion of the International Congress of Mathematicians (Zürich) in the form of a souvenir watch displaying Bürgi's title page as its dial [27], see Fig. 6.

In 1998 the author was invited to present a short history of the discovery of the logarithms [28] at "Slide Rule '98", the Fourth International Meeting of Slide Rule Collectors in Huttwil, Switzerland, organized by H. Joss [10] (October 14 to 16, 1998). Here we present an expanded version of the proceedings article, containing more information on the statistics of Bürgi's round-off errors and on his algorithm for generating the table.

It is amusing to ponder about possible lines of development of 16$^{\text{th}}$ century mathematics if calculators had been available at that time. The construction of tables of logarithms as instruments of numerical calculations would not have been necessary, and the concept of logarithms might have arisen only centuries later, e.g. in connection with the development of calculus by Newton, Leibnitz and Euler. Almost certainly, the slide rule [10], the leading calculating device for three centuries, would have been missed altogether. However, as it happened, in the 16$^{\text{th}}$ century no efficient calculating machines were available, and there was a strong need to develop good algorithms for the more tedious arithmetic operations such as multiplications, divisions, and square roots.

Compared to our time, the scientific environment in the 16$^{\text{th}}$ century was simple. Astronomy was by far the most advanced discipline of science, looking back onto a continuous history of at least 2000 years. With the introduction of the Gregorian Calendar (Pope Gregory XIII) in 1582 the length of the year was defined as 365.97/400 days = 365.2425 days (the modern value of the tropical year is 365.2422 days). Obviously, the astronomers of that time already needed to perform long arithmetic calculations, and the precision of some of their data (as the length of the year) asked for a precision of at least 6 digits.

Nowadays, arithmetic operations with multi-digit numbers are a standard topic in elementary schools all over the world. Paradoxically, due to the ubiquity of calculators, the widespread proficiency in these algorithms may eventually get lost.

2 The Idea of the Logarithms

The historical process of the discovery of the logarithms extended over at least half a century. As we understand it now, the sole purpose of the invention was to speed up multiplications and divisions by means of tables that could be computed once for all (with a huge effort, though). Ironically, this aspect of the logarithms is now all but irrelevant: for modern calculators and computers multiplications pose no bigger problems than additions. The logarithms as a tool for computing (their key role for almost 400 years) have disappeared completely within a decade.

On the other hand, about a century after its original discovery the logarithm function was found to be the indefinite integral of c/x (with an appropriate value of $c > 0$). In this role the logarithm will always keep its importance in all of mathematics.

The idea that leads directly to the mathematical object we now refer to as logarithm can be found already in the works of Archimedes, 287-212 BC, (see, e.g. [26]). However Archimedes missed the final breakthrough, and unfortunately his ideas were only picked up much later. After the French mathematician Nicolas Chuquet (1445-1488) had introduced a good nomenclature for large integers (e.g. 10^6 = million, 10^{12} = byllion), the time was ready for a systematic treatment of large and small (real) numbers. In 1544 Michael Stifel [12] rediscovered what had remained forgotten for 1800 years. He considered what we now call the geometric sequence a_n, with initial element 1 and quotient 2,

$$a_n = 2^n, \quad n = \ldots,\ -2,\ -1,\ 0,\ 1,\ 2, \ldots\ , \tag{1}$$

or written in tabular form as

(2)

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$a_n = 2^n$	...	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	...

In modern language the second row of Stifel's table is a list of the (positive and negative) powers of the **base** $B = 2$.

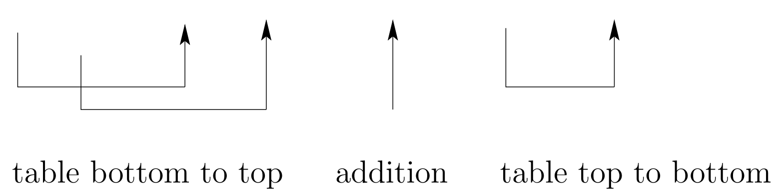
The continuation of the table to the left by fractions was a novel aspect and led directly to the discovery of the general law of multiplying powers of the same base B:

$$B^m \cdot B^n = B^{m+n} \quad \text{or} \quad a_m \cdot a_n = a_{m+n} \quad \text{with} \quad a_n = B^n\,. \tag{3}$$

Stifel became aware of the fact that this law could be exploited for finding the product of the elements a_m, a_n of the table by looking up the element a_{m+n} in the same table. Hence the product of the table elements in positions m, n is the table element in position $m + n$; a multiplication is reduced to 3 table-look-ups and an addition.

Example: $B = 2$

$$8 \cdot 16 = B^3 \cdot B^4 = B^{3+4} = B^7 = 128 \tag{4}$$



The nomenclature still in use today may be understood from M. Stifel's table (2), but was introduced only much later by Napier [19]: the numbers a_n in the bottom row are the principal entries of the table, still referred to as *numeri*; the integers n in the top row are merely used in order to denote the position of a_n within the table: n is called the *logarithm* (greek for the *word*, i.e. the *essence*, of the *number*) of a_n (with respect to the *base* $B = 2$). Due to the "exposed" position of n in the upper row of the table, M. Stifel referred to n as the *exponent*. We still write $a_n = B^n$, with the exponent n in "exposed" position.

3 Jost Bürgi

These few highlights characterize the scientific environment into which Jost Bürgi was born. He was the son of a renowned family in the town of Lichtensteig in the Toggenburg valley (Canton of St. Gallen, Switzerland), born on February 28, 1552. Almost nothing is known about his youth. In his home town Jost Bürgi most likely only received the modest education that was possible in a rural environment. It is conceivable that he entered apprenticeship with his father Lienz Bürgi who was a locksmith. Summaries of the known fragments of Bürgi's early life may be found in [12], [13], [14], [17], [26]. Details on Bürgi's biography were also given by Ph. Schöbi [23]. The most comprehensive study on Bürgi's life and achievements published until now has been made by Fritz Staudacher [24], also showing new aspects of Bürgi's connections and innovations.

Fig. 1. Jost Bürgi at the age of 67. Copper plate print from a drawing by Aegidius Sadeler (1619), taken from Benjamin Bramer: Bericht von Jobsten Bürgi's Geometrischem Triangulations-instrument, in "Apollonius Cattus", 3rd ed., Kassel 1648 ([17], p. 21).

The next known date of Bürgi's life is that in 1579 he was appointed at the court of Duke (Landgraf) Wilhelm IV of Hessen (in the city of Kassel, Germany) as the court watchmaker and "mechanicus". It is not known when Bürgi (Fig. 1) left his home town and how and where he acquired the extraordinary skills that made him eligible for the prestigious appointment in the duke's observatory. Rudolf Wolf [29] speculates that Jost Bürgi might have participated in the construction of the astronomical clock in the dome of Straßburg, carried out 1570-1574 by the clockmakers Isaak and Josua Habrecht from Dießenhofen (Canton of Thurgau, Switzerland). According to another hypothesis, recently brought up by F. Staudacher, the young Bürgi might have acquired his skills in Schaffhausen (Switzerland), where the Habrecht family had built clocks at least until 1572.

From that time on Jost Bürgi's life is relatively well documented, mainly by the numerous precision instruments for geometry and astronomy, and by his astronomical clocks that soon earned world fame. Most famous up to the present day are Bürgi's "Celestial Globes", celestial spheres as they would be seen by an outside observer, with a clockwork inside ([12]). The fame of those masterpieces was so great that the emperor Rudolf II (1552-1612) invited Bürgi to his court in Prague in 1592.

Fig. 2. The little "Himmelsglobus", completed by Bürgi in 1594, to be seen in the Swiss National Museum in Zürich, see [12], [17]. It is an extraordinary piece of early astronomy and precision craftsmanship, 142 mm in diameter and 255 mm of total hight, accurately displaying the motion of the celestial sphere, the sun, and giving the time.

Fig. 3. Bürgi's "Proportional Circkel", a precision compass device invented and built by Jost Bürgi, first mentioned by Levin Hulsius in 1603 (published in 1607). It serves to reduce the scale of an object by a constant factor (from [17], p. 130).

Among the precision instruments we mention the "Proportional Compasses" (Fig. 3), a device invented and built by Bürgi that can be used for proportionally changing the scale of a drawing. Bürgi was not only a skilled watchmaker, astronomer and mathematician, he also knew to organize his private life: when he returned to Kassel in 1593 he had become the owner of the house he had been living in before. It is documented that Bürgi also became a successful real estate agent and banker. One of the few authentic documents in Bürgi's handwriting is a letter (dated Prague, 27.7.1616, signed Jost Bürgi, Uhrmacher) that deals with a loan in the amount of 500 guilders (Gulden) (Fig. 4).

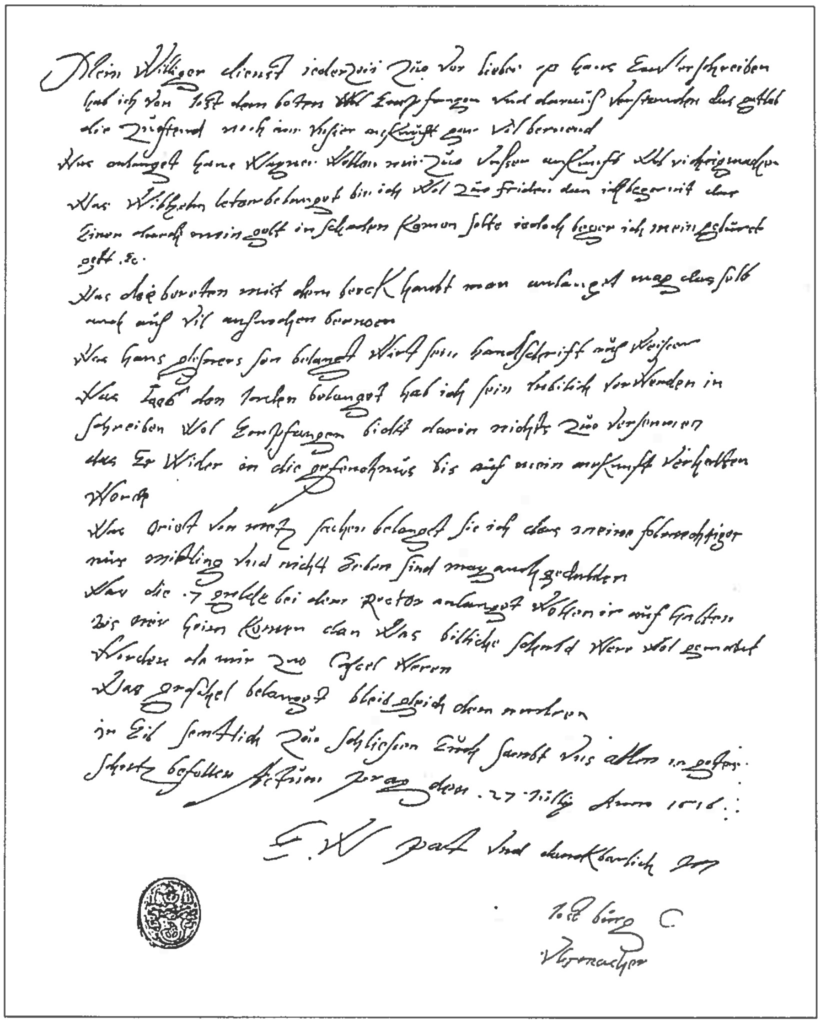


Fig. 4. Handwritten letter by Jost Bürgi, addressed to his attorney "Hans Dickhaut, meinen grossgunstigen freund und paten - Cassel", dated "Actum Prag den 27. jullij Anno 1616" (line 22). Lines 6 and 7 refer to a loan of 500 guilders ("gulde", see the 4th word on line 17): "iedoch beger ich mein gebüret gelt 5c". Original in the Hessisches Staatsarchiv, Marburg an der Lahn, transcript in [5], p. 40.

Important events in Jost Bürgi's life as well as a few other relevant events are summarized by means of the following chronological chart:

		1540		
M. Stifel Arithmetica integra	1544 →			
⋆ John Napier	1550 →			
		1550		
			← 1552	⋆ Jost Bürgi, February 28, in Lichtensteig (St. Gallen, Switzerland)
⋆ Adam Ries	1559 →			
		1560		
† Simon Jacob	1564 →			
† Michael Stifel	1567 →			
		1570		
			← 1570-74	Construction of the astronomical clock in the dome of Straßburg by Isaak and Johann Habrecht, Dießenhofen
⋆ Johannes Kepler	1571 →			
			← 1579	Jost Bürgi Watchmaker of Duke Wilhelm IV of Hessen
		1580		
Introduction of the Gregorian calendar	1582 →			
			← 1585	Bürgi's new metal sextant, worlds first observation watch indicating the second
			← 1588	Raimarus Ursus Dithmarus mentions Bürgi's calculations
		1590		
			← 1592-93	Jost Bürgi's stay in Prague
			← 1594	Bürgi develops and builds small celestial sphere
		1600		
			← 1603	Bürgi's "Proportional compasses" Kepler confidentially edits Bürgi's Arithmetica
			← 1604	Bürgi promoted to Emperor's watchmaker on the Hradschin in Prague Johannes Kepler mentions Jost Bürgi's calculating techniques
Kepler's Astronomia Nova	1609 →			
		1610		
Publication of Napier's tables of natural logarithms	1614 →			
Tables of log sin	1616 →			
† John Napier	1617 →			
Henry Briggs' Chilias Prima	1618 →		← 1618	Window throwing in Prague. Begin of 30-year war
		1620	←	Publication of Bürgi's "Progreß Tabulen" in Prague
Kepler's Chilias Logarithmorum	1629 →			
		1630		
			← 1632	† Jost Bürgi, January 31, in Kassel (age 79 years and 11 months)
	time t		years A.D.	
		1640		

Fig. 5. Chronological chart 1544-1632.

Not only by his scientific achievements, but also according to reports by contemporaries, mainly Kepler [11] and Bramer [1], Bürgi must have been a mathematician of formidable stature. This is even more astounding since Bürgi did not have much of a formal education and must have discovered or rediscovered many relations on his own. Unfortunately, only a small number of Bürgi's theoretical results are documented; most of the evidence is of an indirect nature. As Staudacher [24] discovered in 2012, Bürgi and Kepler were restricted in publishing their knowledge about Bürgi's mathematical innovations by a mutual confidence agreement. A similar agreement existed between Brahe and Kepler.

As an example, Kepler [11] quotes a theorem of Cardano and mentions that Jost Bürgi has announced to be in possession of a proof.

> *"The secant of* 89^0 *plus the tangent of* 89^0 *equals the sum of the sines of all degrees of the semicircle".*

Slightly generalized, and reformulated in modern notation with $N = 180$, $\delta := \pi/N$ this means

$$S(N) := \sum_{k=0}^{N} \sin(k\delta) = \sec\left(\frac{\pi}{2} - \delta\right) + \tan\left(\frac{\pi}{2} - \delta\right).$$

This is by no means an obvious relation, and Kepler's statement sheds some light onto Bürgi's mathematical skills, although, unfortunately, Bürgi's alleged proof has never been found. Using geometric series and the technique of complex numbers, involving $i := \sqrt{-1}$, fully developed only much later by Euler (1707-1783) and others, the proof would look as follows (taking advantage of $\sin(N\delta) = 0$, $e^{iN\delta} = -1$):

$$S(N) = \frac{1}{2i} \sum_{k=0}^{N-1} (e^{ik\delta} - e^{-ik\delta}) = \frac{1}{2i} \left(\frac{2}{1 - e^{i\delta}} - \frac{2}{1 - e^{-i\delta}} \right)$$

$$= \frac{1 + e^{i\delta}}{i(1 - e^{i\delta})} = \frac{e^{\frac{i\delta}{2}} + e^{-\frac{i\delta}{2}}}{\frac{1}{i}(e^{\frac{i\delta}{2}} - e^{-\frac{i\delta}{2}})} = \text{cotan}\,\frac{\delta}{2}\,.$$

Note that Cardano, Kepler, and probably Bürgi missed the simpler form of $S(N) = \tan(89.5^0)$ found above; in fact:

$$\tan\left(\frac{\pi}{2} - \frac{\delta}{2}\right) = \text{cotan}\,\frac{\delta}{2} = \frac{1 + \cos\delta}{\sin\delta} = \sec\left(\frac{\pi}{2} - \delta\right) + \tan\left(\frac{\pi}{2} - \delta\right).$$

Note also that the modern proof heavily uses the exponential function which Bürgi was about to discover (in real numbers only, though).

4 The Progreß Tabulen

There is no question that Bürgi's diversified work in astronomy, triangulation, geometry, clockmaking etc. required extended calculations with multi-digit numbers. It is not known to what extent Bürgi was familiar with M. Stifel's geometric sequence ("progression") $a_n = B^n$ with $B = 2$. However, in his instructions ("Gründlicher Unterricht") [9], p. 27, Bürgi refers to Simon Jacob († 1564), who himself had summarized Michael Stifel's ideas. In any case, Bürgi must have had the clear insight how the power law (3) could be exploited for speeding up multiplications. At the same time he came up with a surprisingly simple and effective means for overcoming the obvious drawback of Stifel's table: the scarcity of the table entries a_n. Bürgi simply chose the base B as an appropriate number close to 1, namely

$$B = 1 + \frac{1}{10000} = 1.0001\,, \tag{5}$$

and tabulated the exponential function with base B,

$$a_n := B^n,\ n = 0, 1, \ldots, 23027, \tag{6}$$

in 9-digit precision [2], [3], [4]. The use of the table for multiplying is exactly the same as in the example of Equ. (4).

In this section we briefly describe the table and comment on the old dispute on the priority for the invention of the logarithms between Jost Bürgi and John Napier. Mathematical considerations on Bürgi's table will be collected in Section 5.

In Bürgi's table the sequences ("progressions") are arranged in columns with 50 entries per column and 8 columns per page. The index n (i.e. the logarithm) is printed in red (Die Rote Zahl) as $10\,n = 0, 10, 20, \ldots, 500, 510, \ldots$, see Fig. 7. The corresponding table entry $a_n = 1.0001^n$ is printed in black (Die Schwartze Zahl) by omitting the decimal point, i.e. as the integer $10^8 \cdot a_n$. The entire table consisting of 23028 entries thus extends over 58 pages, covering the entire range from $a_n = 1.0000\,0000$ to $a_{23027} = 9.9999\,9779$ (see Fig. 7, Fig. 8). In [15] and [18] the question of the base of the exponential function tabulated by Bürgi is discussed. Bürgi's base B can be identified unambiguously as $B = 1.0001$, the quotient of two consecutive table entries. Therefore it is natural to consider the entry a_n as the nth power of B, which implies $a_0 = 1$. The natural choice for the "red number" corresponding to the table entry $a_n = B^n$ is the exponent n. It can only be speculated why Bürgi used the 10-fold exponents, $10\,n$, instead, e.g. $0, 10, 20, \ldots$ for the beginning of the table (Fig. 7). A possible explanation for Bürgi's increase of the accuracy of n by one decimal digit is that he wanted to suggest (linear) interpolation in the accuracy of one digit (see Section 5). The difference of two consecutive table entries, $a_{n+1} - a_n = a_n/10000$, needed for this operation is readily available. A skilled user like Bürgi can do this division and the subsequent proportionality computation in his head.

In the following, we take the liberty to refer to the table arguments and entries by means of numbers n and a_n lying in the intervals

$$0.0 \le n \le 23027.0 \quad \text{and} \quad 1.00000000 \le a_n < 10.00000000\,,$$

adopting the modern usage of the decimal point.

Bürgi concluded that the table should ideally terminate at the N^{th} entry where

$$a_N = 10.0000\,0000\,.$$

By two refinements of the table on the last page (Fig. 8) he correctly finds the fractional value

$$N = 23027.0022\,,$$

referred to as the "whole Red Number". A more precise value is

$$N := \frac{\log(10)}{\log(1.0001)} = \log_{1.0001}(10) = 23027.00220\,32997\,. \tag{7}$$

The title page (Fig. 6) summarizes the table by listing each 500th entry, as well as the whole Red Number N. In accordance with tabulating the index as $10\,n$, the "whole red number" N is given as $23027\mathring{0}022$ with the superimposed $^\circ$ marking the digit with unit value, which corresponds to $10\,N$ with N from Equ. (7) (see also the final sentence of the legend of Fig. 8).

The arrangement of the entries in a circular dial clearly shows Bürgi's genius since it documents his insight that the next decade, e.g., $[10, 100)$ is a mere repetition in 10-fold size of the current one, e.g., $[1, 10)$. Here Bürgi unknowingly anticipated Eulers's famous relationship $\exp(ix) = \cos x + i \sin x$ between exponential and circular functions.

Example of the use of Bürgi's table (cf. Fig. 6, 8)

In order to illustrate the use of Bürgi's table for multidigit arithmetic we give a (constructed) example merely using the title page (Fig. 6) and later page 1 (reproduced as Fig. 7): **Compute the 5th power of 6.35923131.** The arrows mark the direction from red to black in Bürgi's table.

black	(numerus)		red (logarithm)
$c = 6.35923131$	given data	$\longleftarrow$	$n = 18500.0$
$c^5 = ?$	to be computed	$\longleftarrow$	$5\,n = 92500.0$
10.00000000	the whole red number	$\longleftarrow$	$N = 23027.0022$
10000.00000	10^4	$\longleftarrow$	$4\,N = 92108.0088$
$\frac{c^5}{10^4}$	scaled result	$\longleftarrow$	$5\,n - 4\,N = 391.9912$

The scaling factor 10^4 had to be chosen such that the scaled result is in the range of the table. A crude answer may now be read off page 1 directly: Approximate the logarithm $5\,n - 4\,N$ of the scaled result as $5\,n - 4\,N \approx 392.0 = 350.0 + 42.0$. Then page 1 yields $c^5/10^4 \approx a_{392} = 1.0399?64?$. Due to the illegible digits we have to be satisfied with $c^5 \approx 10399$.

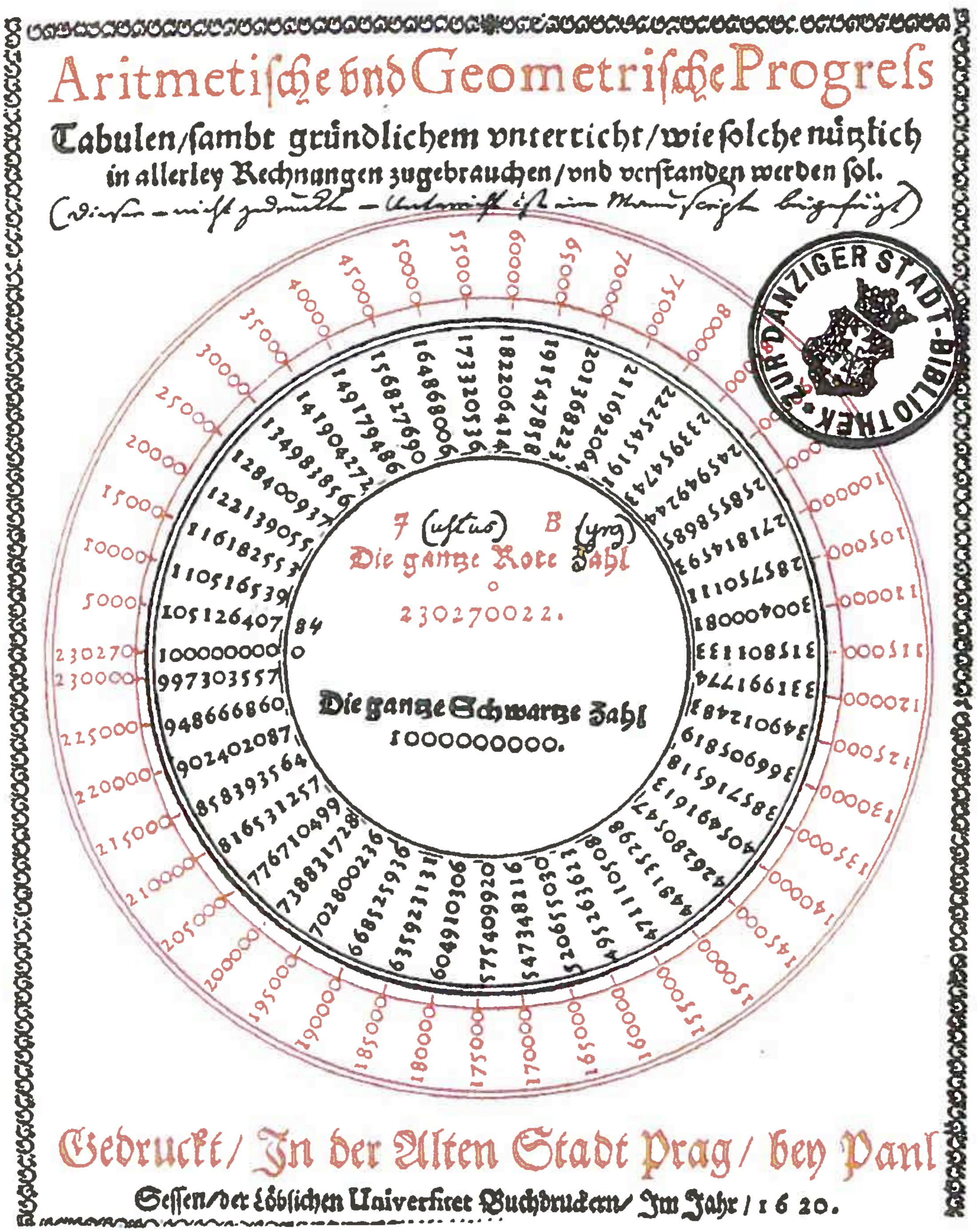


Fig. 6. Title page of Bürgi's Progreß Tabulen, Prague, 1620, as reproduced in [9], [26] from the Danzig (Gdansk, Poland) Copy. It summarizes the table by listing every 500$^{\text{th}}$ table entry and states the logarithm N of 10, the "gantze Rote Zahl", $N := \log 10/\log 1.0001 = 23027.0022$. A typesetting error in the entry for 500.0 (not contained in the table) has been corrected by hand from ...26407 to ...26847.

	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	10356179[illegible]
10	10000	11277	15067	21381	30234	41637	55603	....7214[illegible]
20	20001	21328	[illegible]	31574	....4[illegible]437	51991	65904	....8250[illegible]
30	30003	31380	35271	41687	50641	62146	76216	92861
40	40006	41433	45374	51841	60846	72402	86523	103603221
50	50010	51487	[illegible]	[illegible]	71052	....8266[illegible]	96832	13581
60	60015	61543	65584	72158	81259	92918	103107142	23942
70	70021	71599	75691	82309	91467	102603177	17452	34305
80	80028	81656	85799	92468	[illegible]01676	13438	27764	4466
90	90036	91714	95907	101602627	11887	23699	38077	....5503[illegible]
100	100100045	100601773	101106017	12787	22098	33961	48391	....6539[illegible]
110	10055	11834	....1613[illegible]	....[illegible]040	32310	44725	58705	....7576[illegible]
120	20066	21895	26239	33111	42523	54489	69021	80137
130	30078	31957	36351	43274	52738	64755	79338	96501
140	40091	....4[illegible]	46465	....52[illegible]	61953	75021	89656	10370687[illegible]
150	50105	52084	56580	63604	73169	85789	99975	....1724[illegible]
160	60120	62150	66696	73770	83386	95557	103210295	27613
170	70136	72216	76812	83938	....936[illegible]	10270[illegible]827	20616	57986
180	80153	82283	86930	94106	102203[illegible]74	16097	30937	....4836[illegible]
190	90171	92351	....9704[illegible]	101704275	14045	26369	41261	58734
200	100[illegible]0190	[illegible]	[illegible]	14446	24266	....3664[illegible]	51585	....6911[illegible]
210	10210	12491	17289	24617	34488	46915	....61[illegible]10	....[illegible]
220	20231	22562	27411	34790	....4471[illegible]	57100	72237	....8[illegible]855
230	30253	32634	....37[illegible]	44963	54926	67466	82564	10380[illegible]244
240	40276	42707	47657	55135	65162	77742	92897	....[illegible]624
250	50300	52782	57782	65313	75388	88020	103303221	....[illegible]210[illegible]
260	60325	....628[illegible]	....679[illegible]	75490	85616	98709	13557	31357
270	70351	72933	78095	....8566[illegible]	95845	102808579	23583	45770
280	80378	83011	88162	95846	1023062[illegible]4	18860	34210	....5215[illegible]
290	90406	93189	98291	101506[illegible]5	....1630[illegible]	....[illegible]9147	....4154[illegible]	....6[illegible]54
300	100300435	100803168	101308421	16206	26536	39425	54883	....7292[illegible]
310	10465	13248	18552	26387	....3676[illegible]	....4970[illegible]	65219	....8331[illegible]
320	20496	23330	28654	36570	....4700[illegible]	59993	....755[illegible]	[illegible]
330	30528	35412	38817	46754	57237	70279	85293	103904091
340	40562	43496	48950	56939	67473	80566	96232	14481
350	50596	53580	....5[illegible]55	67174	....7771[illegible]	....[illegible]0855	1034[illegible]6571	24873
360	60631	63665	69221	77311	87947	102901144	16917	35265
370	70667	73752	79358	87499	98186	11434	27254	45659
380	....[illegible]704	83839	89496	97687	102408476	21725	....3[illegible]	....5605[illegible]
390	90742	93927	99635	101907[illegible]	18667	32017	....4794[illegible]	06440
400	100407781	100904017	101409775	18061	23905	42310	58285	....768[illegible]6
410	10821	14107	19916	....28[illegible]	39157	....[illegible]604	....6[illegible]63[illegible]	97243
420	....2086[illegible]	24194	....3005[illegible]	38455	49396	62901	....7597[illegible]	....9[illegible]64[illegible]
430	....30[illegible]4	34291	40201	45646	59641	73196	89126	10400804[illegible]
440	40948	44384	50345	58841	69587	83493	99674	....[illegible]44[illegible]
450	50991	54479	60469	69037	80133	....9379[illegible]	103510024	28841
460	61037	64574	70636	79236	90381	103004091	70375	....3924[illegible]
470	....[illegible]	74671	80783	89432	102500630	....14[illegible]	[illegible]	....[illegible]106[illegible]
480	....[illegible]113[illegible]	....8476[illegible]	90931	99631	10880	....2469[illegible]	....4105[illegible]	....6005[illegible]
490	91178	94867	101501080	102009831	....2813[illegible]	....3499[illegible]	51435	....7046[illegible]
500	100501227	101004966	11230	20032	31384	....4[illegible]	61790	80516

Fig. 7. First page of Bürgi's Progreß Tabulen as reproduced in [9], [26] from the Danzig Copy. Most illegible digits are no better in the original. The Munich Copy (see Section 5) is in much better condition. E.g., the illegible entry for 102.0 (3rd entry in the 3rd column) clearly reads as 1.010 25168($= 1.0001^{102}$) in the Munich Copy. The last entry of the page, 1.040 80816, is erroneous by a transmission error. The (supposedly identical) leading entry of the following page, $a_{400} = 1.040\,80869$, is entirely correct.

	228000	228500	229000	229500	230000	230000
0	977556601	982456378	987380714	992329732	[illegible]	997303557
10	···654356	···554623	···475452	···425965	····10000	···403287
20	···752122	···652879	···578200	···528208	····20000	···503027
30	···849897	···751144	···676958	···627461	[illegible]	···602778
40	···947682	···849419	···775726	···726724	····40000	···702538
50	978045477	···547704	···874503	···825996	····50000	···802308
60	···143281	983045999	···973291	···925279	[illegible]	···902088
70	···241096	···144304	985072088	993024572	····70000	998001879
80	···338920	···242618	···170895	···123874	····80000	···101679
90	···436754	···340942	···269712	···223187	[illegible]	···201489
100	···534597	···439276	···368539	···322509	230100000	···301309
110	···632451	···537620	···467376	···421841	····10000	···401139
120	···730314	···635974	···566223	···521183	[illegible]	···500979
130	···828187	···734338	···665080	···620535	····30000	···600829
140	···926070	···832711	···763946	···719898	····40000	···700690
150	979023962	···931094	···862822	···819269	····50000	···800560
160	···121865	984029488	···961709	···918651	····60000	···900440
170	···219777	[illegible]	98[illegible]060605	994018043	····70000	999000330
180	···317699	···226303	···159511	···117445	[illegible]	···100230
190	···415631	···324726	···258427	···216857	····90000	···200140
200	···513572	···423158	···357353	···316278	230200000	···300060
210	···611524	···521601	···456288	···415710	[illegible]	···399990
220	···709485	···620053	···555234	···515152	····20000	···499930
230	···807456	···718515	···654190	···614603	····30000	···599880
240	···905437	···816987	···753155	···714065	····40000	···699840
250	980003427	···915468	···852130	···813536	····50000	···799810
260	···101427	985013960	···951115	···913017	····60000	···899790
270	····99438	···112461	990050111	995012509	[illegible]	999999779
280	···297457	···210973	···149116	···112010	·······10	······879
290	···395487	···309494	···248130	···211521	·······20	······979
300	···493527	···408025	···347155	···311042	········1	·······89
310	···591576	···506565	···446190	···410573	230270028	999999999
320	···689635	···605116	···545235	···510115		
330	···787704	···703677	···644289	···609666		
340	···885783	···802247	···743353	···709227		
350	···983872	···900827	···842428	···808797		
360	981081970	···99417	···941512	···908378		
370	···180078	986099017	991040606	996007969		
380	···278156	···196527	···139711	···107570		
390	···376324	···295247	···238825	···207181		
400	···474462	···393876	···337948	···306801		
410	···572609	···492516	···437082	···406432		
420	···670766	···591165	···536226	···506073		
430	···768934	···689824	···635350	···605723		
440	···867110	···788493	···734543	···705384		
450	···965297	···887172	···833717	···805054		
460	982063494	···985861	···932900	···904735		
470	···161700	987084559	992032093	997004425		
480	···259916	···183268	···131296	···104126		
490	···358142	···281986	···230510	···203836		
500	···456378	···380714	···329732	···303557		

Also enden sich die zwo Summen Zalen in 9. Zyphern/ vnd ist die Rote

230270022 —
230270023 +

Die Schwartze aber ist gantz mit 9. nollen als 1000000000 vnd so dieselben gantzen Zalen/nicht gnug geben mögen / so mag man dieselben 2.3.4.5.6.7.8.9. zusammen addieren.

Fig. 8. Last page (p. 58) of Bürgi's Progreß Tabulen reproduced in black and white from the Munich Copy [3] (Astronomisch-Physikalisches Kabinett). The last column is supplemented by an additional column of "red numbers", indicating a gradual decrease of the table interval in order to interpolate to the final value of 10.0000 0000 (or 9.9999 9999). The logarithm N of 10 is stated in the final comment as lying in the interval $23027.0022 < N < 23027.0023$.

Owing to the transparent structure of Bürgi's table, the illegible digits can easily be reconstructed from $a_{391} = 1.03987243$:

$$a_{392} = a_{391} + \frac{a_{391}}{10000} = 1.03997642\,.$$

For linear interpolation (see also Fig. 15) this value is not even needed:

red	black
391.0	a_{391}
391.9912	$\frac{c^5}{10^4}$
392.0	$a_{391} + \frac{a_{391}}{10000}$

Simple proportionality yields

$$\frac{c^5}{10^4} = a_{391} + 0.9912 \cdot \frac{a_{391}}{10000} = 1.03987243 + 0.00010400 - 0.00000092 = 1.03997551\,.$$

The result $\mathbf{c^5 = 10399.7551}$ is correct with all digits given.

Presently, the existence of two originals of Bürgi's table has been confirmed. Besides the Munich copy [3], which is the basis of this work, an original had been found 1985 in the library of Paul Guldin (1577 - 1643); it is now in the library of the University of Graz, Austria [4] (see Gerlinde Faustmann [7]). Unfortunately, the Danzig original (found 1855 by H. R. Gieswald [9]), which is the source of the well-known reproductions in [13], [26] (Fig. 7, Fig. 8) is lost since Word War II. Rumours that it is in Prague now, and that a fourth original exists in the Vatican have not been confirmed so far. The quality of the printing in all known copies is less than perfect. The abundant illegible digits are not due to imperfections of the facsimile reproductions. The Munich Copy seems to be of better quality than the Danzig copy.

Bürgi was very reluctant in publishing his table. It appeared in print as late as 1620 (Fig. 6), only after Johannes Kepler (1571-1630) had been urging him for a long time to publish it. The reason for the scarcity of the original copies may be that only a few preliminary copies were printed. Wars or financial problems might have interrupted the publishing process.

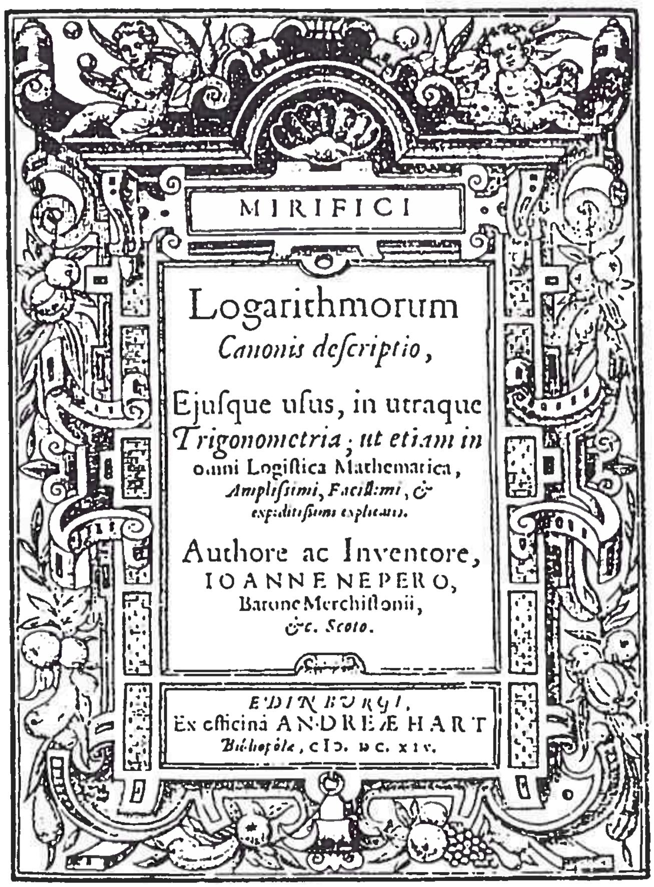
Kepler reported in 1594 that Bürgi was in possession of an efficient method to carry out multiplications and divisions. Even earlier, in 1588, the astronomer Raimarus Ursus Dithmarus (quoted by Rudolf Wolf [30]) reported that Bürgi was using a method to greatly simplify his calculations. It cannot be excluded that these statements mean that Bürgi's tables were operational as early as 1588. It is conceivable, however, that Kepler and Dithmarus refer to the use of trigonometric identities such as

$$\cos(x) \cdot \cos(y) = \frac{1}{2}\,\big(\cos(x+y) + \cos(x-y)\big)$$

for reducing multiplications to additions and table-look-ups. According to many documents, this technique, referred to as *prostapheresis* ("auxiliary separation"), was quite common among human calculators in the 16th century, [9], [14]. In any case, it seems possible for a single human computer to generate the Bürgi table within a few months, as will be explained in Section 5.

While Bürgi could well have been a daily user of his own tables, similar ideas developed on the other side of the channel and approached their completion. In 1614, John Napier published his own tables of natural logarithms [19] (Fig. 9), after more than 20 years of tedious calculations. In 1616 [20], the same author published an even more advanced table of log-sin values. In his tables Napier managed to grasp many advanced aspects of the natural logarithms. In this respect he was ahead of his contemporaries, but with the laborious computations he paid a high price for it. Neither Napier's nor Bürgi's table was free of errors.

Both authors saw simplification of multiplications and divisions as their main goal. The choice of the basis (Bürgi: 1.0001, Napier: the inverse of the Euler number, $e^{-1} = 0.367879...$) is irrelevant for this application. Since it involves reading the tables in both directions, Napier's approach (equal steps in the numeri, tabulation of a logarithm function) is no better and no worse than Bürgi's (equal steps in the logarithms, tabulation of an exponential function).


MIRIFICI

Logarithmorum
Canonis descriptio,

Ejusque usus, in utraque
Trigonometria; ut etiam in
omni Logistica Mathematica,
Amplissimi, Facillimi, &
expeditissimi explicatio.

Authore ac Inventore,
IOANNE NEPERO,
Barone Merchistonii,
&c. Scoto.

EDINBURGI,
Ex officina ANDREÆ HART
Bibliopolæ, CIϽ. DC. XIV.

Fig. 9. Copper plate on the title page of John Napier's table of logarithms. Edinburgh 1614, from [17], p. 28. This table is a precursor of the log-sin table reprinted in [20].

Bürgi achieved this goal at much lower cost than Napier, actually with the smallest possible effort. Bürgi's table also had the big advantage of simplicity and transparency of the algorithm for generating it. Some authors, [14], [15], [29], [30] suspect that Bürgi's tables might have been operational earlier than Napier's. On the other hand, different opinions (not shared by this author, see Section 5.4) exist, e.g. D. Roegel [22], *Bürgi's "Progress Tabulen" (1620): logarithmic tables without logarithms*. Almost certainly, neither of the human calculators knew about the work of the other. It seems to be fair, therefore, to consider Jost Bürgi and John Napier as the two simultaneous and independent discoverers of the logarithms.

5 Mathematical aspects of Bürgi's Progreß Tabulen

With Bürgi's choice $B = 1.0001$ as the base, the obvious algorithm for generating the table is

$$(8) \qquad a_0 = 1, \quad a_{n+1} = a_n + \frac{a_n}{10000}, \; (n = 0, 1, \ldots, 23027)$$

as follows from (6). The single step is as simple as it could possibly be: in order to calculate the next table entry, augment the current one by its 10000$^{\text{th}}$ part (right-shift by 4 digits). Just do this 23000 times, and you're done.

5.1 Checks

Leading a formidable task like this to a reliable result necessarily requires careful checks for computational errors. The multiplication rule for powers, Equ. (3), provides a simple and effective tool for detecting computational errors.

Bürgi must have taken care of this perfectly; otherwise the "whole red number" N of Equ. (7) could not have been correctly determined to lie in the interval $23027.0022 < N < 23027.0023$ (Fig. 8). Since Bürgi's method of checking is not known, we may ponder about possible checks.

Assume that the final entry of the first page, $a_{400} = 1.04080869$, has been confirmed, e.g. by checking $a_{200} = a_{100}^2$, $a_{400} = a_{200}^2$, etc., or possibly by the binomial formula

$$(1+\varepsilon)^{400} = \sum_{k=0}^{K} \binom{400}{k} \varepsilon^k \; + \; R_k \,, \quad \varepsilon = 10^{-4} \,.$$

With $K = 5$, the result $a_{400} = 1.04080869271$ (all 12 digits correct) is obtained with a negligible computational effort, using paper and pencil only.

Then, every subsequent page can be checked by one long-hand multiplication as follows. Let a_i be the initial entry of any page. Then the final entry, $a_f = a_{i+400}$, must satisfy $a_f = a_i \cdot a_{400}$. If the check fails, one needs to locate the error by testing $a_{i+d} = a_i \cdot a_d$ for some values $d < 400$, e.g. for $d = 200, 100$, etc., and redo (part of) the page.

Another possibility is to take advantage of the entry

$$a_{431} = 1.044040044101 \,,$$

a rather unlikely curiosity. Long-hand multiplications by a_{431} are extremely cheap; they reduce to one quadrupling and a sum of 6 quickly decreasing terms.

5.2 Guard Digits

Besides erroneous calculation, the slow accumulation of round-off errors may be a problem. Here the only remedy is introducing additional digits of precision, called *guard digits*.

Example:

	2 guard digits	**no guard digits**
	$\downarrow\downarrow$	
$a_{12345} =$	3.4364 4765 40	3.4364 4765
	3 4364 48	3 4364
$a_{12346} =$	3.4367 9129 88	3.4367 9129

It is seen that guard digits must be carried along in order to avoid accumulation of round-off errors. Without this precaution (e.g. by strictly rounding the 8$^{\text{th}}$ digits after the decimal point) the erroneously rounded value 3.4367 9129 would have been found for a_{12346} from the correct value for a_{12345}. Fig. 10 shows the accumulated round-off error if the table is computed according to (8) using g guard digits.

We first define our notion of guard digits and describe the modification of the algorithm (8) for simulating rounded calculation with g guard digits. Then, conclusions on the number of guard digits necessary for guaranteeing an accurate table will be drawn.

As before, we denote the exact powers of B by $a_n := B^n$; furthermore, $\widetilde{a}_n$ is the correctly rounded table entry satisfying $|\widetilde{a}_n - a_n| \leq \frac{1}{2}\,\delta$, $\delta = 10^{-8}$. Recall that Bürgi tabulated the integer values $f\,\widetilde{a}_n$ with $f = 1/\delta = 10^8$.

Attaching g guard digits to the table entry $\widetilde{a}_n$ increases its resolution to $1/(f\,G)$ where $G := 10^g$, resulting in the new approximation

$$A_n = \widetilde{a}_n + \frac{1}{f\,G}\,c\,, \quad |c| \leq \frac{G}{2}\,,$$

where c stands for the (positive or negative) integer formed by the guard digits. Now the algorithm (8) becomes

$$f\,G\,A_{n+1} = \text{round}(1.0001\,f\,G\,A_n)\,, \tag{9}$$

where round(x) stands for the integer closest to x. The rounded table values are then obtained as $\widetilde{a}_n = \text{round}(A_n\,f)\,/\,f$.

In the above form the algorithm is not restricted to natural numbers g; any value of $G > 1$ is allowed. In Fig. 10 we have used $G = 1, 3, 10$, 30, 100, 300 (approximately) corresponding to $g = 0$, 0.5, 1, 1.5, 2, 2.5 guard digits. E.g. 1.5 guard digits means that the values of A_n are always rounded a precision of $\frac{1}{30}$, not practical for hand calculations, though.

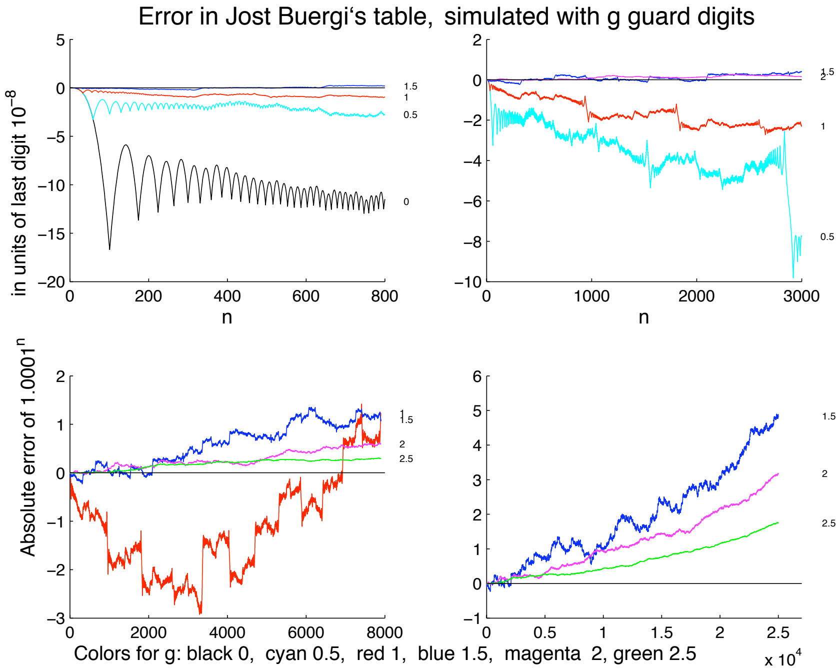


Fig. 10. Plots of $A_n - a_n$ in units of the least significant digit, 10^{-8}, in four intervals of n, as an illustration of the effect of g guard digits. Colours for g: black 0, cyan 0.5, red 1, blue 1.5, magenta 2, green 2.5.

It is seen that guard digits are an absolute necessity: $g = 0$ results in the loss of 1 digit already after 100 steps, just as the theory of random walks predicts. If checks are made after every page, 1 guard digit suffices in the lower part of the table. However, since in the upper part of the table the accuracy requirements are up to 10-fold, two guard digits are needed there in order to guarantee 9-digit accuracy. Two guard digits suffice to keep the accumulated round-off error below 3 units of the last digit. To guarantee an accurate table free of errors occasional checks according to Section 5.1 must be carried out.

5.3 Table errors

In 1976 the Astronomisch-Physikalisches Kabinett in Munich allowed W. Pohl [21] to copy their Progreß Tabula entirely. Based on this excellent material the Bürgi table was analyzed subsequently, see the Internal Note [27].

First, it was established that the table contains no systematic errors by checking the "whole Red Number" N, see Equ. (7). Also, the numbers given on the title page (Fig. 6), were found to be correct in every digit, except for the entries at 12000.0, 16000.0, 19000.0 which have a round-off error of slightly more than a half-unit of the least significant digit.

Assuming that the table contains no large systematic errors the actually given (and possibly erroneous) table values $\widetilde{a}_n$ may be reconstructed from the index n and the terminal digit (TD). In this way it was possible to check the entire table by computer (in a few milliseconds) after having keyed in the 23028 terminal digits. The following counts of table errors $\Delta_n := 10^8(\widetilde{a}_n - B^n)$ were obtained:

Type		counts	%
0	correctly rounded ($\lvert\Delta_n\rvert < 0.5$)	21081	91.54
1	TD rounded to wrong side ($0.5 \le \lvert\Delta_n\rvert < 1$)	1679	7.29
2	TD erroneous ($\lvert\Delta_n\rvert \ge 1$)	87	.38
3	TD illegible	181	.79
		23028	100

The distribution of the 87 large errors (Type 2) is shown in the upper histogram of Fig. 11. Clearly, large errors are more abundant in the upper part of the table, where its relative accuracy is higher (up to 1 more digit of relative accuracy). The distribution of the 181 illegible terminal digits (Type 3) in the lower histogram of Fig. 11 is more or less uniform.

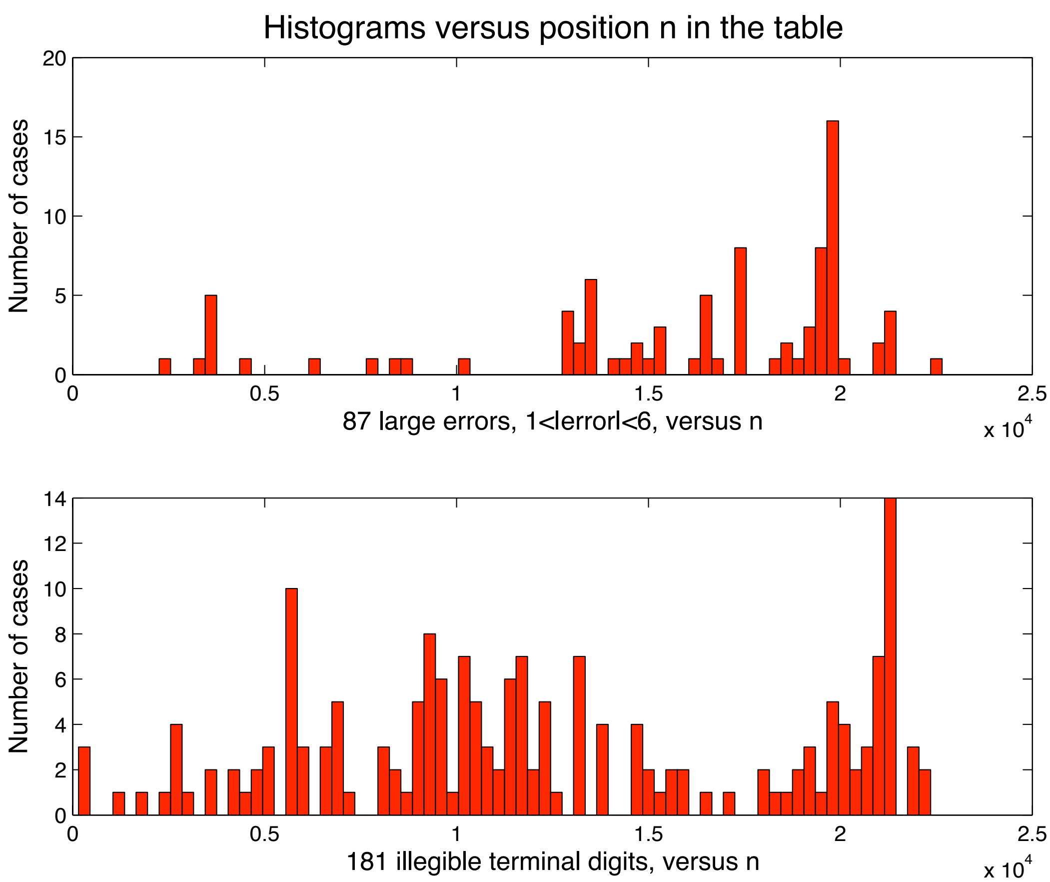


Fig. 11. Histograms of table errors (in units of the least significant digit) versus position n.
Upper plot: 87 large errors, Type 2, 1 to 6 units.
Lower plot: 181 illegible terminal digits, Type 3.

Among the 87 large errors (Type 2) about 25 seem to be errors of transcription (such as $n = 12870$: $\widetilde{a}_n = 3.621671\mathbf{84}$ instead of the correct value $B_n = 3.621671\mathbf{48}$), see Appendix A. For the other 62 cases we have $1 \le |\Delta_n| \le 2.64$. The longest interval of systematically erroneous values is $19890 \le n \le 19904$ with $1.73 \le \Delta_n \le 2.64$ (Appendix B). For the 22759 errors of Types 0 and 1 the histogram given below is also plotted in Fig. 12.

Δ_n	-1		-0.9		-0.8		-0.7		-0.6		-0.5		-0.4		-0.3		-0.2		-0.1		0
cases		10		8		14		45		195		1145		2036		2231		2158		2343	
							272										9913				

Δ_n	0		0.1		0.2		0.3		0.4		0.5		0.6		0.7		0.8		0.9		1
cases		2314		2343		2263		2153		2095		1126		196		48		19		17	
							11168									1406					

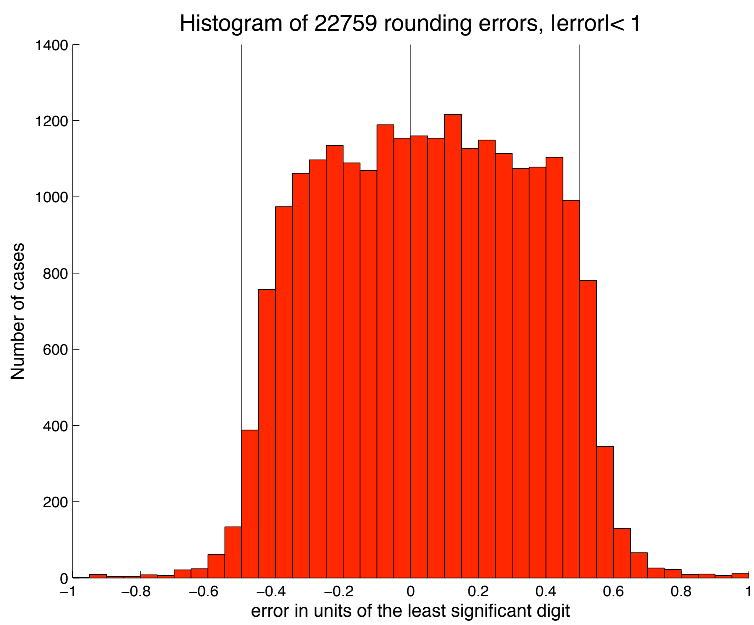


Fig. 12. Histogram of the 22759 rounding errors of Types 0 and 1.

The bias of the small rounding errors towards the positive side seen in Fig. 12 may be a consequence of permanent upwards rounding of the single guard digit 5. This hypothesis may be corroborated by simulations of various rounding algorithms for generating Bürgi's table. The third histogram of Fig. 13, generated by permanent upwards rounding of the single guard digit 5, is an ideal form of the histogram of Bürgi's rounding errors (Fig. 12). In contrast, the symmetric rounding (half-integers to nearest even) produces the symmetric fourth histogram of Fig. 13.

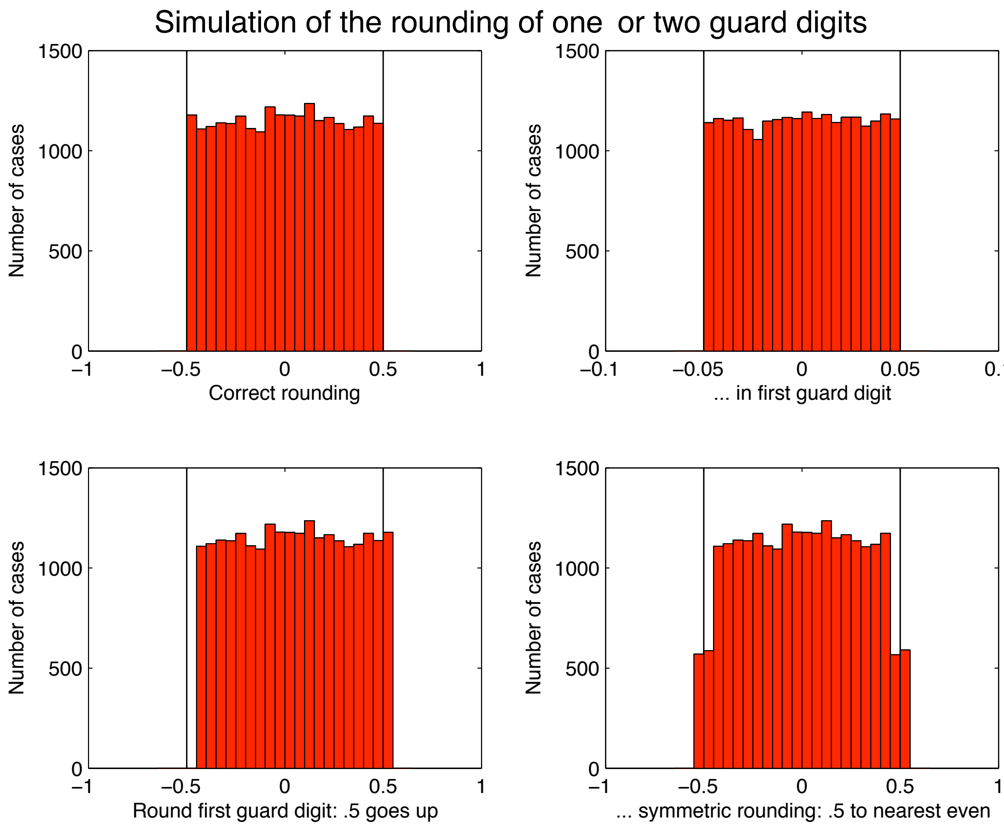


Fig. 13. Histograms of the errors in four simulated rounding models for the entire table (rounding to 9 significant digits).
(1) Direct rounding of the exact values.
(2) Rounding in the first guard digit, absolute errors < 0.05.
(3) Rounding of the first guard digit: 0.5 goes up.
(4) Symmetric rounding of first guard digit: 0.5 to nearest even.

In Appendix B the only three sequences of consecutive seriously erroneous entries are listed. The comment "correct transition" expresses the hypothesis that the transition from $\widetilde{a}_{n-1}$ to $\widetilde{a}_n = \text{round}(1.0001 \cdot \widetilde{a}_{n-1})$ was done correctly; presumably, these errors were caused by erroneous transmissions. Appendix C lists a few sequences of consecutive roundings to the wrong side.

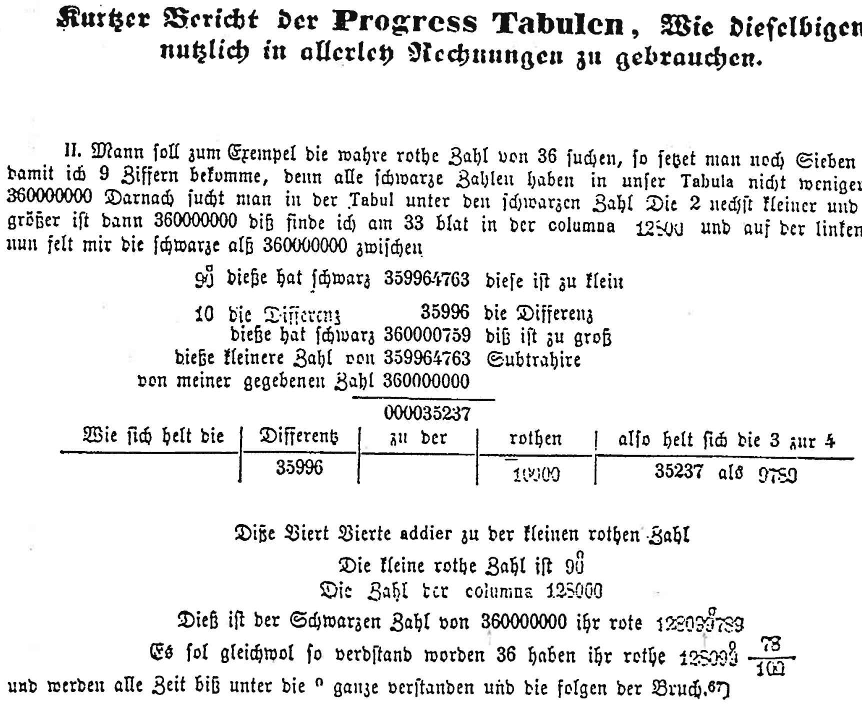
5.4 Linear interpolation

Linear interpolation was suggested by Bürgi himself, and the "user's manual" that was to go with the table does in fact contain instructions for accurate or approximate linear interpolation. Unfortunately, those instructions, announced on the title page (Fig. 6) by the words *sambt gründlichem Unterricht*, "with thorough instruction", have not been published with the table. Only much later, Gieswald rediscovered their manuscript in the archives of Danzig und published it in 1856 [9], [14].

Fig. 14 shows an excerpt from Bürgi's instructions as reproduced in [16]. In the example Bürgi explains how to find the accurate logarithm (to base 1.0001) of a given number a_n,

$$n = \log_{1.0001}(a_n).$$

He therefore shows how to cheaply find logarithms from his table of an exponential function.



Kurtzer Bericht der Progress Tabulen, Wie dieselbigen nutzlich in allerley Rechnungen zu gebrauchen.

II. Mann soll zum Exempel die wahre rothe Zahl von 36 suchen, so setzet man noch Sieben 0 für, damit ich 9 Ziffern bekomme, denn alle schwarze Zahlen haben in unser Tabula nicht weniger alßo 360000000 Darnach sucht man in der Tabul unter den schwarzen Zahl Die 2 nechst kleiner und nechst größer ist dann 360000000 diß finde ich am 33 blat in der columna 12800 und auf der linken seite, nun felt mir die schwarze alß 360000000 zwischen

90 dieße hat schwarz 359964763 diese ist zu klein
10 die Differenz 35996 die Differenz
dieße hat schwarz 360000759 diß ist zu groß
dieße kleinere Zahl von 359964763 Subtrahire
von meiner gegebenen Zahl 360000000
000035237

Wie sich helt die	Differentz	zu der	rothen	also helt sich die 3 zur 4
	35996		10000	35237 als 9789

Diße Viert Vierte addier zu der kleinen rothen Zahl
Die kleine rothe Zahl ist 90
Die Zahl der columna 128000
Dieß ist der Schwarzen Zahl von 360000000 ihr rote 128099789
Es sol gleichwol so verdstand worden 36 haben ihr rothe 128099 $\frac{78}{100}$
und werden alle Zeit biß unter die ⁰ ganze verstanden und die folgen der Bruch.67)

Fig. 14. In the example Bürgi explains backwards reading of the table, i.e. given a black number a_n, find the corresponding red number $n = \log_{1.0001} a_n = \log(a_n)/\log(1.0001)$. In particular $a_n = 3.6$ yields $n = 12809.9789\,1087$. Bürgi gives 9 correct digits (third line from bottom).

For generally discussing linear interpolation in this context, let x be a real variable, and consider the exponential function

$$f(x) := B^x \quad \text{with} \quad B = 1 + \varepsilon, \quad \varepsilon = 0.0001 \tag{10}$$

in the range $0 < x < \log 10/\log(1+\varepsilon)$. Since the table interval is 1, define $n := \text{floor}(x)$, $t := x - n$. Then linear interpolation in the interval $n \le x \le n+1$ yields the approximation

$$f^* = t\,f(n+1) + (1-t)\,f(n) \tag{11}$$

of $f(x)$. Due to the convex curvature of the graph of f the relative interpolation error is nonnegative and roughly bounded by $\varepsilon^2/8$,

$$0 \le (f^* - f(x))/f(x) \le \varepsilon^2/8\,. \tag{12}$$

This implies $0 \le f^* - f(x) \le 1.25 \cdot 10^{-8}$ if $1 \le f(x) \le 10$.

The conclusion is that Bürgi's choice of tabulating the values $f(n)$ with 8 figures after the decimal point is optimal in the sense that the error due to linear interpolation is never more than 1.25 units at the least significant digit. This error bound is reached near the upper end of the table, whereas near the lower end of the table the error is bounded by $0.125 \cdot 10^{-8}$.

The linear interpolation process of the exponential function (10) in the interval $n \leq x \leq n+1$ is sketched in Fig. 15. The tickmarks on the vertical lines indicate the grid of the table entries. $\widetilde{f}$ denotes the exact interpolant of the rounded table entries. In the upper half of the table the positive bias of the interpolation error may almost be compensated by always rounding $\widetilde{f}$ downwards.

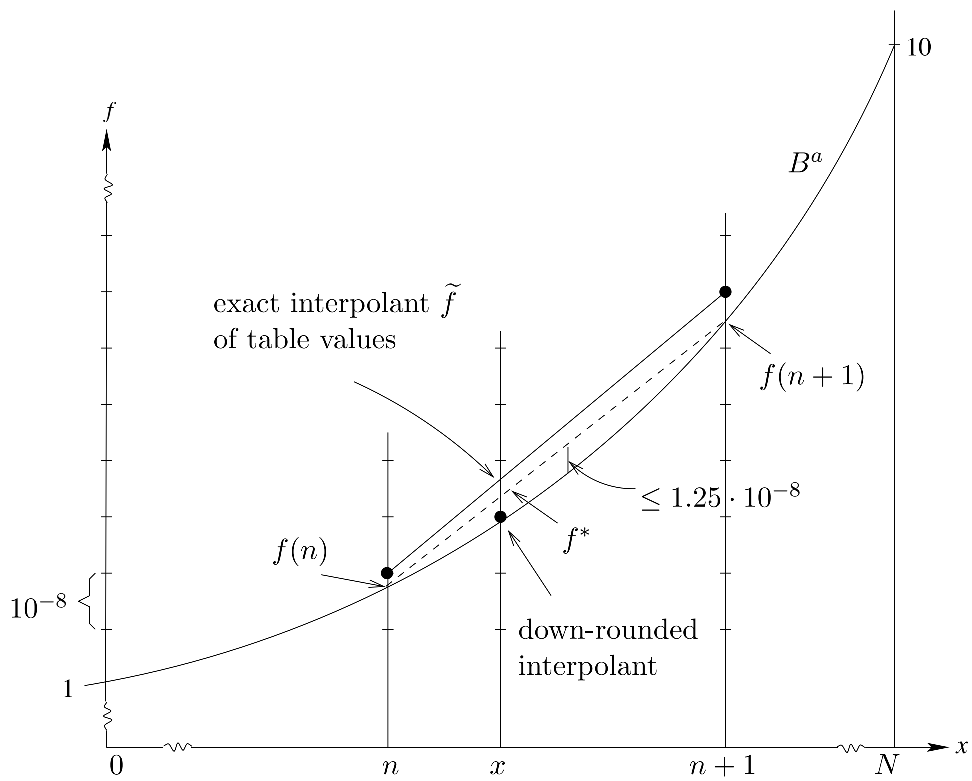


Fig. 15. Linear interpolation.
Solid curve: graph of $f(x) := B^x$, $0 \leq x \leq N$ (Equ. (7)).
Dashed line: linear interpolation polynomial of exact function values for $n \leq x \leq n+1$, with exact interpolant f^*.
Black dots: rounded table entries to be interpolated.
Solid line: linear interpolation polynomial of rounded table entries. The down-rounded approximation of $\widetilde{f}$ is also marked by a black dot.

This is exemplified in Fig. 16, where we plot the interpolation error in the interval $23000 \leq x \leq 23025$ near the upper end of the table. The correctly rounded table values of $f(x)$ were used as input data. Three rounding strategies were used: (1) $\widetilde{f}$ exact (black),

(2) $\widetilde{f}$ rounded to nearest integer (cyan), (3) $\widetilde{f}$ down-rounded (red). In the upper half of the table Strategy 3 is preferable (still with a slightly positive bias), in the lower half Strategy 2 is better.

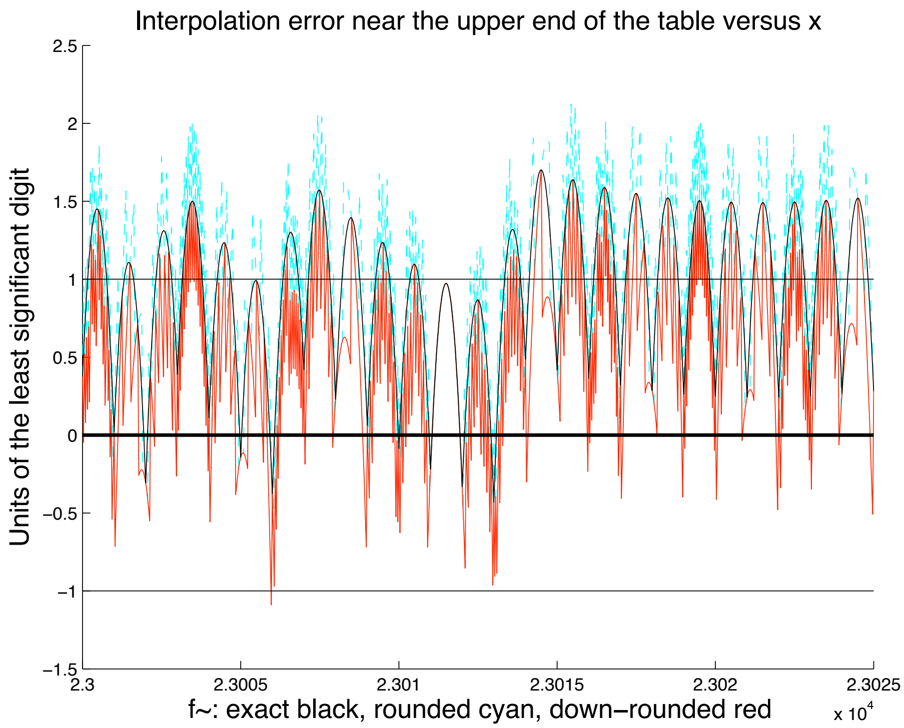


Fig. 16. Interpolation error near the upper end of the table versus x. Black line (Strategy 1): error of the interpolant $\widetilde{f}$ of the values $f(n) = \text{round}(B^n)$. Cyan/Red (Strategies 2/3): errors of correctly rounded/down-rounded interpolants of rounded table entries.

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Appendix A: Summary of isolated large errors

25 computational errors or transcription errors with $|\text{error}| > 10^{-8}$
$a_n = 1.0001^n$, "Bürgi": 4 terminal digits, "Bü-ex": units of last digit

n	$\text{ex} = 10^8 \cdot a_n$	Bürgi	Bü-ex	error	comments
2363	126653925.6	3929	3.38	$6 \to 9$	last digit upside down
4531	157314585.9	4583	-2.92	$6 \to 3$	3 clear
6299	187736370.3	6375	4.70	$0 \to 5$	5 clear but distorted
8795	240958860.1	8867	-6.90		previous entry 240934767
12869	362130935.0	0938	3.02	$5 \to 8$	8 clear
12870	362167148.1	7184		$48 \to 84$	two digits interchanged
12952	365148978.4	8975	-3.40	$8 \to 5$	5 clear
13017	367530057.8	0053	-4.83	$8 \to 3$	3 clear
13335	379404726.6	4724	-2.57	$7 \to 4$	4 clear
13516	386334127.6	4125	-2.57	$8 \to 5$	5 clear
14422	422968657.6	8460	2.40	$658 \to 460$	460 clear
14950	445900325.2	5740			previous entry 445855740
15198	457096350.4	6354	3.60	$50.4 \to 54$	4 clear, guard digit?
16350	512903870.9	3817		$71 \to 17$	two digits interchanged
18504	636177538.0	7535	-2.98	$8 \to 5$	5 clear
18713	649612887.1	2883	-4.14	$8 \to 3$	3 clear
18925	663530995.1	0999	3.85	$5 \to 9$	9 clear
19414	696782369.0	2362	-7.04	$9 \to 2$	2 clear
19464	700274830.1	4839	8.86	$30 \to 39$	39 clear
19574	708019986.3	9988	1.71	$6 \to 8$	8 clear
19822	725797523.4	7537		$23 \to 37$	37 clear
20997	816286346.3	6341	-5.30	$6 \to 1$	1 clear
21255	837619481.6	9480	-1.56	$2 \to 0$	0 clear
21389	848918552.4	8554	1.62	$2 \to 4$	4 clear
22544	952849980.9	9976	-4.87	$81 \to 76$	76 clear

Appendix B: Sequences of consecutive seriously erroneous entries

3 sequences with 29 serious errors, $|\text{error}| > 10^{-8}$
$a_n = 1.0001^n$, "Bürgi": 4 terminal digits, "Bü-ex": units of last digit

n	ex $= 10^8 \cdot a_n$	Bürgi	Bü-ex	error	comments
16387	514805035.1	5034	-1.14		1 unit low
16388	514856515.6	6516	0.35		correctly rounded
16389	514908001.3	7981	-20.30	8001→7981	20 units low
16390	514959492.1	9392	-100.10	9492→9392	100 units low
16391	515010988.0	8888		0888→8888	after transition
16392	515062489.1	0390			correct transition
16393	515113995.4	1897			correct transition
16394	515165506.8	3409			correct transition
16395	515217023.3	7023	-0.35		recovered
16396	515268545.0	8545	-0.05		recovered
16397	515320071.9	0702	0.10	0072→0702	digits 0,7 interchanged
19245	685106271.7	6272	0.30		correctly rounded
19246	685174782.3	4790	7.67	82→90	8 units high
19247	685243299.8	3308	8.19	00→08	correct transition
19248	685311824.1	1832	7.86	24→32	correct transition
19249	685380355.3	0365	9.68	0363→0365	after correct transition
19250	685448893.3	8893	-0.36		recovered, 3 distorted
19251	685517438.2	7438	-0.25		recovered, corr.rounded
19887	730530335.7	0335	-0.68		1 unit low
19888	730603388.7	3388	-0.72		1 unit low
19889	730676449.1	6449	-0.06		correctly rounded
19890	730749516.7	9519	2.30		2 units high
19891	730822591.7	2594	2.35		2 units high
19892	730895673.9	5676	2.09		2 units high
19893	730968763.5	8766	2.52		3 units high
19894	731041860.4	1863	2.64		3 units high
19895	731114964.5	4967	2.46		2 units high
19896	731188076.0	8078	1.96		2 units high
19897	731261194.8	1197	2.15		2 units high
19898	731334321.0	4323	2.03		2 units high
19899	731407454.4	7457	2.60		3 units high
19900	731480595.1	0597	1.86		2 units high
19901	731553743.2	3745	1.80		2 units high
19902	731626898.6	6901	2.42		2 units high
19903	731700061.3	0063	1.73		2 units high
19904	731773231.3	3233	1.73		2 units high
19905	731846408.6	6409	0.40		recovered, corr.rounded

Appendix C: Sequences of consecutive erroneous roundings

5 sequences with 37 harmless errors, $|\text{error}| < 1.6 \cdot 10^{-8}$

$a_n = 1.0001^n$, "Bürgi": 4 terminal digits, "Bü-ex": units of last digit

n	$\text{ex} = 10^8 \cdot a_n$	Bürgi	Bü-ex	error	comments
3501	141918462.1	8463	0.89		1 unit high
3502	141932654.0	2655	1.05		1 unit high
3503	141946847.2	6848	0.78		1 unit high
3504	141961041.9	1043	1.10		1 unit high
3505	141975238.0	5239	0.99		1 unit high
3506	141989435.5	9437	1.47		1 unit high
3507	142003634.5	3635	0.53		1 unit high
3508	142017834.8	7836	1.16		1 unit high
3509	142032036.6	2038	1.38		1 unit high
3510	142046239.8	6240	0.18		recovered
13415	382451984.8	1984	-0.80		1 unit low
13416	382490230.0	0229	-1.00		1 unit low
13417	382528479.0	8478	-1.02		1 unit low
13418	382566731.9	6731	-0.87		1 unit low
13419	382604988.5	4988	-0.55		1 unit low
13420	382643249.0	3248	-1.04		1 unit low
13421	382681513.4	1512	-1.37		1 unit low
13422	382719781.5	9781	-0.52		1 unit low
13423	382758053.5	8053	-0.50		correctly rounded
13424	382796329.3	6328	-1.30		1 unit low
14786	438647557.8	7559	1.25		1 unit high
14787	438691422.5	1423	0.49		correctly rounded
14788	438735291.7	7835293	1.35	87→78	digits 8,7 interchanged
14789	438779165.2	9166	0.82		1 unit high
17389	569058503.4	8505	1.59		1 unit high
17390	569115409.3	5410	0.74		1 unit high
17391	569172320.8	2322	1.20		1 unit high
17392	569229238.0	9239	0.97		1 unit high
17393	569286161.0	6162	1.04		1 unit high
17394	569343089.6	3091	1.43		1 unit high
17395	569400023.9	0025	1.12		1 unit high
17396	569456963.9	6965	1.12		1 unit high
17397	569513909.6	3911	1.42		1 unit high
17398	569570861.0	0862	1.03		1 unit high
17399	569627818.1	7819	0.94		1 unit high
19525	704559346.9	9348	1.13		1 unit high
19526	704629802.8	9804	1.19		1 unit high
19527	704700265.8	0267	1.21		1 unit high
19528	704770735.8	0737	1.19		1 unit high
19529	704841212.9	1214	1.11		1 unit high

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