Eidgenössische Technische Hochschule Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

n-term approximation rates and Besov regularity for elliptic PDEs on polyhedral domains

M. Hansen

Research Report No. 2012-41
December 2012

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

n-term Approximation Rates and Besov Regularity for Elliptic PDEs on Polyhedral Domains.

Markus Hansen

December 7, 2012

Abstract

We investigate the Besov regularity for solutions of elliptic PDEs. This is based on regularity results in Babuska-Kondratiev spaces. Following the argument of Dahlke and DeVore, we first prove an embedding of these spaces into the scale $B^r_{\tau,\tau}(D)$ of Besov spaces with $\frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}$. This scale is known to be closely related to n-term approximation w.r.to wavelet systems. Ultimately this yields the rate $n^{-r/d}$ for $u \in \mathcal{K}^m_{p,a}(D) \cap H^s_p(D)$ for $r < r^* \leq m$.

In order to improve this rate to $n^{-m/d}$ we leave the scale $B^r_{\tau,\tau}(D)$ and instead consider the spaces $B^m_{\tau,\infty}(D)$. We determine conditions under which the space $\mathcal{K}^m_{p,a}(D) \cap H^s(D)$ is embedded into some space $B^m_{\tau,\infty}(D)$ for some $\frac{m}{d} + \frac{1}{p} > \frac{1}{\tau} \geq \frac{1}{p}$, which in turn indeed yields the desired n-term rate. As an intermediate step we also prove an extension theorem for Kondratiev spaces.

1 Introduction

Ever since the emergence of (adaptive) wavelet algorithms for the numerical computation of solutions to (elliptic) partial differential equations there was also the interest in corresponding rates for n-term approximation rates, since these may be seen as the benchmark rates the optimal algorithm (which at each step would calculate an optimal n-term approximation) would converge with.

Later on, this question was seen to be closely related to the membership in a certain scale of Besov spaces. More precisely a famous result by DeVore, Jawerth and Popov [9] characterizes certain Approximation classes for approximation with respect to $L_p(D)$-norms as Besov spaces $B^r_{\tau,\tau}(D)$ with $\frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}$, where r is the rate of the best n-term approximation.

In another famous article Dahlke and DeVore [2] later used this result to determine n-term approximation rates for the solution of Poisson's equation on general Lipschitz domains. This was done by proving that the solution of $-\Delta u = f$ belongs to Besov spaces $B^r_{\tau,\tau}(D)$ for parameters $r < r^*$, where r^* depends on the Lipschitz-character of the bounded domain $D \subset \mathbb{R}^d$, the dimension d and the regularity of the right-hand side f. In subsequent years this result was extend to more general elliptic operators [3], and for special domains more precise values for r^* were determined [4, 5, 8].

The purpose of this paper now is two-fold. On the one hand, we shall use the ideas of these precursors to re-prove the result for polyhedral domains in two and three space dimensions, this time based on regularity in Babuska-Kondratiev spaces. Here we manage to give a unified treatment to the different cases previously treated separately (polygonal domains in 2D, polyhedral and smooth cones in 3D, edge singularities in 3D). The outcome corresponds to the previous results, which roughly can be summarized as: If the function u admits m weak derivatives with controlled blow-up towards the boundary, then $u \in B^r_{\tau,\tau}(D)$ for every $r < m$. For the n-term approximation this implies that every such function can be approximated at rate $\frac{r}{d} < \frac{m}{d}$.

The second part of the paper then stems from investigating the limiting situation $r = m$. So far, all the previous proofs (and our version as well) fail to cover this case. However, by slightly shifting the point of view, we can close this gap: Instead, inspired by more recent results on n-term approximation for Besov spaces [7, 10], we turn our attention to Besov spaces guaranteeing the rate $n^{-m/d}$. Thus in leaving the "adaptivity scale" $B^r_{\tau,\tau}(D)$, we prove that u belongs to Besov spaces $B^m_{\tau,\infty}(D)$ for certain parameters $0 < \tau < \tau_0$, and we determine a condition under which we have $\frac{1}{\tau_0} < \frac{m}{d} + \frac{1}{p}$ (which in turn implies the mentioned rate $n^{-m/d}$ for approximation in $L_p(D)$).

2 Basic definitions and State of the art

In this section we will fix some notations corresponding to the used wavelet system, recall the definition of Besov and Babuska-Kondratiev spaces, and formulate the regularity and n-term approximation results used later on.

2.1 Wavelets

We are not interested in utmost generality pertaining to the used wavelet system. Instead, for simplicity we will stick to Daubechies' Wavelets, the generalization to compactly supported biorthogonal wavelets constituting Riesz-bases being immediate.
Let ϕ be a univariate scaling function and η the associated wavelet corresponding to Daubechies' construction, where the smoothness of ϕ and η and the number of vanishing moments for η are assumed to be sufficiently large. Let E denote the nontrivial vertices of $[0,1]^d$, and put

$$\psi^e(x_1,\dots,x_d) = \prod_{j=1}^{d} \psi^{e_j}(x_j)\,, \qquad e \in E\,,$$

where $\psi^0 = \phi$ and $\psi^1 = \eta$. Then the set

$$\Psi' = \{\psi^e : e \in E\}$$

generates via shifts and dyadic dilates an orthonormal basis of $L_2(D)$. More precisely, denoting by $\mathcal{D} = \{I \subset \mathbb{R}^d : I = 2^{-j}([0,1]^d + k), j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ the set of all dydadic cubes in $\mathbb{R}^d$, then

$$\left\{\psi_I : I \in \mathcal{D}, \psi \in \Psi'\right\} = \left\{\psi_I = 2^{jd/2}\psi(2^j\cdot - k) : j \in \mathbb{Z}\,, k \in \mathbb{Z}^d\,, \psi \in \Psi'\right\}$$

forms an orthonormal basis in $L_2(\mathbb{R}^d)$. Denote by $Q(I)$ some dyadic cube (of minimal size) such that $\operatorname{supp}\psi_I \subset Q(I)$ for every $\psi \in \Psi'$. Then we clearly have $Q(I) = 2^{-j}k + 2^{-j}Q$ for some dyadic cube Q.
As usual $\mathcal{D}^+$ denotes the dyadic cubes with measure at most 1, and we put $\Lambda' = \mathcal{D}^+ \times \Psi'$. Additionally, we shall need the notation $\mathcal{D}_j = \{I \in \mathcal{D} : |I| = 2^{-jd}\}$. Then we can write every function $f \in L_2(\mathbb{R}^d)$ as

$$f = P_0 f + \sum_{(I,\psi)\in\Lambda'} \langle f, \psi_I\rangle \psi_I\,.$$

Therein $P_0 f$ denotes the orthogonal projector onto the closed subspace S_0, which is the closure in $L_2(\mathbb{R}^d)$ of the span of the function $\Phi(x) = \phi(x_1)\cdots\phi(x_d)$ and its integer shifts $\Phi(\cdot - k)$, $k \in \mathbb{Z}^d$. Later on it will be convenient to include Φ into the set of generators Ψ' together with the notation $\Phi_I := 0$ for $|I| < 1$, and $\Phi_I = \Phi(\cdot - k)$ for $I = k + [0,1]^d$. Then we can simply write

$$f = \sum_{(I,\psi)\in\Lambda} \langle f, \psi_I\rangle \psi_I\,, \qquad \Lambda = \mathcal{D}^+ \times \Psi\,, \quad \Psi = \Psi' \cup \{\Phi\}\,.$$

Remark 1. If not explicitly stated otherwise convergence of wavelet expansions is always understood in $S'(\mathbb{R}^d)$, the space of tempered distributions, or in $L_2(\mathbb{R}^d)$ (since all relevant spaces will be embedded into $L_2(\mathbb{R}^d)$).

2.2 Besov spaces

Besov space can be defined in a number of ways. Here we will need only their characterization in terms of wavelet bases as presented e.g. in [15]. For more detailed information on Besov spaces and related function spaces as well as equivalent definitions we refer to the literature, e.g. [20] and the references given there.

Let $0 < p, q < \infty$ and $r > \max(0, d(\frac{1}{p} - 1))$. Then a function $v \in L_2(\mathbb{R}^d)$ belongs to the Besov space $B^r_{p,q}(\mathbb{R}^d)$ if, and only if

$$\|v|B^r_{p,q}(\mathbb{R}^d)\| = \|P_0 v|L_p(\mathbb{R}^d)\| + \left(\sum_{j=0}^{\infty} 2^{j(r+d(\frac{1}{2}-\frac{1}{p}))q}\left(\sum_{(I,\psi)\in\mathcal{D}_j\times\Psi} |\langle v, \psi_I\rangle|^p\right)^{q/p}\right)^{1/q} < \infty\,.$$

For parameters $q = \infty$ we shall use the usual modification (replacing the outer sum by a supremum), i.e.

$$\|v|B^r_{p,\infty}(\mathbb{R}^d)\| = \|P_0 v|L_p(\mathbb{R}^d)\| + \sup_{j\geq 0} 2^{j(r+d(\frac{1}{2}-\frac{1}{p}))}\left(\sum_{(I,\psi)\in\mathcal{D}_j\times\Psi} |\langle v, \psi_I\rangle|^p\right)^{1/p}.$$

Within the scale $B^\alpha_{\tau,\tau}(\mathbb{R}^d)$ with $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2}$ due to the specific choice of τ this simplifies to

$$\|v|B^\alpha_{\tau,\tau}(\mathbb{R}^d)\| = \|P_0 v|L_\tau(\mathbb{R}^d)\| + \left(\sum_{(I,\psi)\in\Lambda} |\langle v, \psi_I\rangle|^\tau\right)^{1/\tau}.$$

Additionally, we will use spaces $B^{r,\sigma}_{p,q}(\mathbb{R}^d)$, characterized by the quasi-norm

$$\|v|B^{s,\sigma}_{p,q}(\mathbb{R}^d)\| = \|P_0 v|L_p(\mathbb{R}^d)\| + \left(\sum_{j=0}^{\infty} 2^{j(r+\frac{d}{2}-\frac{d}{p})q}(j+1)^{\sigma q}\left(\sum_{(I,\psi)\in\mathcal{D}_j\times\Psi} |\langle v, \psi_I\rangle|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

Therein the additional term $(j+1)^\sigma$ is of logarithmic order, hence the spaces are usually referred to as *Besov spaces of logarithmic smoothness*. In turn, these spaces are special cases of so-called *function spaces of generalized smoothness*; we refer e.g. to [16] or the survey [12].
Apart from these spaces on $\mathbb{R}^d$, for our main interest in boundary value problems for elliptic PDEs we also need to consider function spaces on domains. The easiest way to introduce these is via restriction, i.e.

$$B^s_{p,q}(D) = \left\{f \in \mathcal{D}'(D) : \exists\, g \in B^s_{p,q}(\mathbb{R}^d)\,, g\big|_D = f\right\}, \qquad \|f|B^s_{p,q}(D)\| = \inf_{g|_D = f} \|g|B^s_{p,q}(\mathbb{R}^d)\|\,.$$

Alternative (different or equivalent) versions of this definition can be found, depending on possible additional properties for the distributions g (most often referring to their support). We refer to the monograph [21] for details and references.
The only aspect we need of these spaces is the existence of continuous linear extension operators, i.e. mappings $\mathcal{E} : B^s_{p,q}(D) \longrightarrow B^s_{p,q}(\mathbb{R}^d)$, possibly depending on the parameters s, p, q and, of course, on the domain D. In this respect, Rychkov [18] gave a final answer for Lipschitz domains: There exists a universal extension operator, i.e. an operator $\mathcal{E} : B^s_{p,q}(D) \longrightarrow B^s_{p,q}(\mathbb{R}^d)$ simultaneously for all parameter triples (s, p, q). In particular, due to $B^s_{2,2}(D) = H^s(D)$ this also covers extensions of Sobolev spaces. In the sequel $\mathcal{E}$ will always denote such an extension operator (note that particularly for the H^s-scale there are several more such extension operators, we only mention the one due to Stein [19, Chap. 3]).

2.3 Babuska-Kondratiev spaces

As mentioned in the introduction our interest stems from elliptic boundary value problems such as (2.1) below. It is nowadays classical knowledge that the regularity of the solution depends not only on the one of the cofficient a and right-hand side f, but also on the regularity/roughness of the boundary of the considered domain. While for smooth coefficients A and smooth boundary we have $u \in H^{s+2}(D)$ for $f \in H^s(D)$, it is well-known that this becomes false for more general domains. In particular, if we only assume D to be a Lipschitz domain, then it was shown in [11] that in general we only have $u \in H^s$ for all $s < 3/2$ for the solution of the Poisson equation, even for smooth right-hand side f. This behaviour is caused by singularities near the boundary.
To obtain similar shift theorems as for smooth domains, a possible approach is to adapt the function spaces. To compensate possible singularities one includes appropriate weights. For polyhedral domains,

this idea has lead to the following definition of the Babuska-Kondratiev spaces $\mathcal{K}^m_{p,a}(D)$: If the function u admits m weak derivatives, we consider the norm

$$\|u|\mathcal{K}^m_{p,a}(D)\|^p = \sum_{|\alpha|\le m} \int_D |\rho(x)^{|\alpha|-a}\partial^\alpha u(x)|^p \, dx \,,$$

where $a \in \mathbb{R}$ is an additional parameter, and the weight function $\rho : D \longrightarrow [0,1]$ is the smooth distance to the singular set of D. This means ρ is a smooth function, and in the vicinity of the singular set it is equal to the distance to that set. In 2D this singular set consists exactly of the vertices of the polygon, while in 3D it consists of the vertices and edges of the polyhedra. In case $p = 2$ we simply write $\mathcal{K}^m_a(D)$. Within this scale of function spaces, a regularity result for boundary value problems for elliptic PDEs can be formulated as follows, see [1] and the references given there:

Proposition 1. Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, $d = 2, 3$. Consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,, \tag{2.1}$$

where $A = (a_{i,j})_{i,j=1}^d$ is symmetric and

$$a_{i,j} \in \mathcal{W}^m_\infty = \big\{v : D \longrightarrow \mathbb{C} : \rho^{|\alpha|}\partial^\alpha v \in L_\infty(D)\,, |\alpha| \le m\big\}\,, \qquad 1 \le i,j \le d\,.$$

Let the bilinear form

$$B(v,w) = \int_D \sum_{i,j} a_{i,j}(x)\partial_i v(x)\partial j w(x) dx$$

satisfy

$$|B(v,w)| \le R\|v|H^1(D)\| \cdot \|w|H^1(D)\| \qquad \text{and} \qquad r\|v|H^1(D)\|^2 \le B(v,v)$$

for some constants $0 < r \le R < \infty$. Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$ and any $f \in \mathcal{K}^{m-1}_{a-1}(D)$ the problem (2.1) admits a uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1}(D)$, and it holds

$$\|u|\mathcal{K}^{m+1}_{a+1}(D)\| \le C\,\|f|\mathcal{K}^{m-1}_{a-1}(D)\|$$

for some constant $C > 0$ independent of f.

We restrict ourselves in this presentation to this simplified situation. In the literature there are further results of this type, either treating different boundary conditions, or using slightly different scales of function spaces. We particularly refer to [13, 14], where they showed that under appropriate conditions on A the result in Proposition 1 holds for all a except for countably many values.

Remark 2. We note that in the sequel we will always have the restriction $a \ge 0$. This is a natural one, since for $a < 0$ the space $\mathcal{K}^m_{p,a}(D)$ contains functions not belonging to $L_p(D)$, for example functions which behave towards a vertex singularity like ρ^α for some $-d+a < \alpha < -d$. But this kind of function is no longer locally integrable, and thus cannot be identified with a (tempered) distribution, whereas Besov spaces are defined as spaces of (tempered) distributions.

We finally shall add a comment on the possible domains D: While before and also in the sequel we will only refer to polyhedral domains, the analysis carries over without change to Lipschitz domains with polyhedral structure. Domains with polyhedral structure were seen to be a natural relaxation of polyhedra, for example replacing the flat faces of polyhedra by smooth surfaces. For precise definitions we refer to [6, 17]. As we shall see in the proofs, the only fact needed about the boundary ∂D are certain combinatorial aspects (counting the number of relevant wavelet coefficients), and these remain unchanged so long as the boundary remains Lipschitz; moreover, also Proposition 1 holds for this more general setting.

2.4 n-term approximation

The (error of the) best n-term approximation is defined as

$$\sigma_n\big(u, L_p(D)\big) = \inf_{\Gamma\subset\Lambda:\#\Gamma\le n}\inf_{c_\gamma}\Big\|u - \sum_{\gamma=(I,\psi)\in\Gamma} c_\gamma\psi_I\Big|L_p(D)\Big\|\,,$$

i.e. as the name suggests we consider the best approximation by linear combinations of the basis functions consisting of at most n terms. As shown in [9] the decay of this quantity is closely related to Besov spaces. More specifically, DeVore, Jawerth and Popov proved

$$\sum_{n=0}^{\infty}\Big((n+1)^{\alpha/d}\sigma_n\big(u, L_p(\mathbb{R}^d)\big)\Big)^\tau\frac{1}{n+1} < \infty \iff u\in B^\alpha_{\tau,\tau}(\mathbb{R}^d)\,,\qquad \frac{1}{\tau} = \frac{\alpha}{d}+\frac{1}{p}\,.$$

However, when discussing the optimal convergence rate for adaptive algorithms this result is slightly stronger than required. We are rather interested in conditions on u that simply guarantee a certain decay rate, i.e. we only need to have

$$\sup_{n\ge 0}(n+1)^{\alpha/d}\sigma_n\big(u, L_p(\mathbb{R}^d)\big) < \infty\,.$$

This implies that the "adaptivity scale" $B^\alpha_{\tau,\tau}(\mathbb{R}^d)$ considered so far might not be the optimal choice. Moreover, this result neglects the additional knowledge that the functions of interest belong to function spaces related to the bounded domain D. In [7, 10] the rates for approximation of functions from the full scale of Besov spaces $B^s_{\tau,q}(D)$ and $B^s_{\tau,q}(\mathbb{R}^d)$ were calculated. For our purposes, we only need the following result from [7, Theorem 7]: If $s > d(\frac{1}{\tau}-\frac{1}{p})$ for $0<\tau\le p$, then

$$\sigma_n\big(u, L_p(D)\big) \lesssim n^{-s/d}\|u|B^s_{\tau,q}(D)\|\,,\qquad u\in B^s_{\tau,q}(D)\,, \tag{2.2}$$

independent of the microscopic parameter q. A similar estimate is true for approximation in the energy norm, i.e. in the norm of the space $H^1(D)$, and more generally in the norm of $W^1_p(D)$,

$$\sigma_n\big(u, W^1_p(D)\big) \lesssim n^{-(s-1)/d}\|u|B^s_{\tau,q}(D)\|\,,\qquad u\in B^s_{\tau,q}(D)\,. \tag{2.3}$$

3 Regularity in the adaptivity scale $B^r_{\tau,\tau}(D)$

As previously mentioned, we now adapt the arguments first given in [2]. The result itself is decoupled from the regularity results for elliptic PDEs, thus we formulate it as an embedding theorem.

Theorem 1. Let D be some bounded polyhedral domain in $\mathbb{R}^d$. Then we have a continuous embedding

$$\mathcal{K}^m_{p,a}(D)\cap B^s_{p,p}(D)\hookrightarrow B^r_{\tau,\tau}(D)\,,\qquad \frac{1}{\tau}=\frac{r}{d}+\frac{1}{p}\,,\qquad 1<p<\infty\,,$$

for all $0\le r<\min(m, \frac{sd}{d-1})$ and $a>\frac{\delta}{d}r$. Therein δ denotes the dimension of the singular set (i.e. $\delta=0$ if there are only vertex singularities, $\delta=1$ if there are edge and vertex singularities etc.).

Proof. Clearly, for $r=0$ the result is trivial, thus we always assume $r>0$ and hence $0<\tau<p$.

Step 1: To start with we note that since a polyhedral domain D in particular is Lipschitz we can extend every $u\in H^s_p(D)$ to some function $\widetilde{u}=\mathcal{E}u\in H^s_p(\mathbb{R}^d)$. Consider first the term $P_0\widetilde{u}$. This can be represented as

$$P_0\widetilde{u} = \sum_{k\in\mathbb{Z}^d}\langle\widetilde{u}, \Phi(\cdot-k)\rangle\Phi(\cdot-k)\,.$$

Since Φ shares the same smoothness and support properties, the coefficients $\langle \widetilde{u}, \Phi(\cdot - k)\rangle$ can be treated exactly like any of the coefficients $\langle \widetilde{u}, \psi_I\rangle$ for $|I| = 1$ in Step 2 (note that below the vanishing moments of ψ^e only become relevant for $|I| < 1$). Thus the claim can be formulated as

$$\Big(\sum_{(I,\psi)\in\Lambda} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \Big)^{1/\tau} \le c \max\big\{ \|u|\mathcal{K}^m_{p,a}(D)\|, \|u|B^s_{p,p}(D)\| \big\} .$$

Step 2: Now put

$$\rho_I = \inf_{x\in Q(I)} \rho(x) ,$$

and consider first the situation $\rho_I > 0$. We recall the following classical approximation result: For every I there exists a polynomial P_I of degree less than m, such that

$$\|\widetilde{u} - P_I|L_p(Q(I))\| \le c_0 |Q(I)|^{m/d} |\widetilde{u}|_{W^m_p(Q(I))} \le c_1 |I|^{m/d} |\widetilde{u}|_{W^m_p(Q(I))}$$

for some constant c_1 independent of I and u. Now recall that ψ_I satisfies moment conditions of order up to m, i.e. it is orthogonal to any polynomial of degree up to $m-1$. Thus we can estimate, using Hölder's inequality,

$$\begin{aligned} |\langle \widetilde{u}, \psi_I\rangle| = |\langle \widetilde{u} - P_I, \psi_I\rangle| &\le \|\widetilde{u} - P_I|L_p(Q(I))\| \cdot \|\psi_I|L_{p'}(Q(I))\| \le c_1 |I|^{m/d} |\widetilde{u}|_{W^m_p(Q(I))} |I|^{\frac{1}{2}-\frac{1}{p}} \\ &\le c_1 |I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}} \rho_I^{-m+a} \Big(\sum_{|\alpha|=m} \int_{Q(I)} |\rho(x)^{m-a} \partial^\alpha \widetilde{u}(x)|^p dx \Big)^{1/p} =: c_1 |I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}} \rho_I^{-m+a} \mu_I . \end{aligned}$$

Now we shall split the index set Λ: For $j \ge 0$ let $\Lambda_j \subset \Lambda$ be the set of all pairs (I,ψ) with $|I| = 2^{-jd}$, and for $k \ge 0$ let $\Lambda_{j,k} \subset \Lambda_j$ contain those (I,ψ) such that

$$k 2^{-j} \le \rho_I < (k+1) 2^{-j} .$$

For $k > 0$ we additionally require $Q(I) \subset D$. Furthermore, we put $\Lambda^0_j = \cup_{k\ge 1} \Lambda_{j,k}$. Then we find using Hölder's inequality

$$\begin{aligned} \sum_{(I,\psi)\in\Lambda^0_j} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau &\le c_1 \sum_{(I,\psi)\in\Lambda^0_j} \big(|I|^{m/d} \rho_I^{-m+a} \mu_I \big)^\tau \\ &\le c_1 \Big(\sum_{(I,\psi)\in\Lambda^0_j} \Big(|I|^{m\frac{\tau}{d}} \rho_I^{(a-m)\tau} \Big)^{\frac{p}{p-\tau}} \Big)^{\frac{p-\tau}{p}} \Big(\sum_{(I,\psi)\in\Lambda^0_j} \mu_I^p \Big)^{\frac{\tau}{p}} . \end{aligned}$$

Since there is a controlled overlap between the cubes $Q(I)$ (i.e. every $x \in \Omega$ is contained in a finite number of cubes $Q(I)$, and this number is bounded by some constant c_2^p independent of x), we can estimate the second factor

$$\begin{aligned} \Big(\sum_{(I,\psi)\in\Lambda^0_j} \mu_I^p \Big)^{\frac{1}{p}} &= \Big(\sum_{(I,\psi)\in\Lambda^0_j} \sum_{|\alpha|=m} \int_{Q(I)} |\rho(x)^{m-a} \partial^\alpha \widetilde{u}(x)|^p dx \Big)^{1/p} \\ &\le c_2 \Big(\sum_{|\alpha|=m} \int_D |\rho(x)^{m-a} \partial^\alpha \widetilde{u}(x)|^p dx \Big)^{1/p} \le c_2 \|u|\mathcal{K}^m_{p,a}(D)\| . \end{aligned}$$

For the first factor we note that by choice of ρ we always have $\rho_I \le 1$, hence the index k is at most 2^j for the sets $\Lambda_{j,k}$ to be non-empty, and the number of elements in each of these sets can be bounded by $c_3 k^{d-1-\delta} 2^{j\delta}$ (where c_3 depends only on D, particularly on the number and precise shape of the singular vertices, edges etc.; this estimate further uses $k \le 2^j$). Then we find

$$\Big(\sum_{(I,\psi)\in\Lambda^0_j} \Big(|I|^{m\frac{\tau}{d}} \rho_I^{(a-m)\tau} \Big)^{\frac{p}{p-\tau}} \Big)^{\frac{p-\tau}{p}} \le \Big(\sum_{(I,\psi)\in\Lambda^0_j} \Big(|I|^{m\frac{\tau}{d}} k^{(a-m)\tau} 2^{-j(a-m)\tau} \Big)^{\frac{p}{p-\tau}} \Big)^{\frac{p-\tau}{p}}$$

$$\leq \Big(\sum_{k=1}^{2^j} \sum_{(I,\psi)\in\Lambda_{j,k}} \Big(2^{-ja\tau} k^{(a-m)\tau}\Big)^{\frac{p}{p-\tau}}\Big)^{\frac{p-\tau}{p}}$$
$$\leq \Big(c_3 2^{-ja\frac{p\tau}{p-\tau}} \sum_{k=1}^{2^j} k^{(a-m)\frac{p\tau}{p-\tau}} k^{d-1-\delta} 2^{j\delta}\Big)^{\frac{p-\tau}{p}} .$$

For this last sum we have to distinguish three cases, according to the value of the exponent (greater, equal or less than -1). We note that

$$(a-m)\frac{p\tau}{p-\tau} + d - 1 - \delta = \frac{p\tau}{p-\tau}\Big(a - m + (d-1-\delta)\Big(\frac{1}{\tau}-\frac{1}{p}\Big)\Big) = \frac{p\tau}{p-\tau}\Big(a - m + r\frac{d-1-\delta}{d}\Big) > -1$$
$$\iff a - m + r\frac{d-1-\delta}{d} > -\frac{1}{\tau} + \frac{1}{p} = -\frac{r}{d} \iff a - m + r\frac{d-\delta}{d} > 0\,,$$

hence we find

$$\Big(\sum_{(I,\psi)\in\Lambda_j^0} \Big(|I|^{m\frac{\tau}{d}} \rho_I^{(a-m)\tau}\Big)^{\frac{p}{p-\tau}}\Big)^{\frac{p-\tau}{p}} \leq c_4 2^{-ja\tau} 2^{j\delta\frac{p-\tau}{p}} \begin{cases} 2^{j(a-m)\tau + j(d-\delta)\frac{p-\tau}{p}}\,, & a - m + r\frac{d-\delta}{d} > 0\,, \\ (j+1)^{\frac{p-\tau}{p}}\,, & a - m + r\frac{d-\delta}{d} = 0\,, \\ 1\,, & a - m + r\frac{d-\delta}{d} < 0\,. \end{cases}$$

Step 3: We now put $\Lambda^0 = \bigcup_{j\geq 0} \Lambda_j^0$. Summing the first line of the last estimate over all j we obtain

$$\sum_{(I,\psi)\in\Lambda^0} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_4 \sum_{j\geq 0} 2^{-jm\tau} 2^{jd\frac{p-\tau}{p}} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau \leq c_5 \|u|\mathcal{K}^m_{p,a}(D)\|^\tau < \infty$$

provided the geometric series converges, thus in the case $a - m + r\frac{d-\delta}{d} > 0$ we find the condition

$$m\tau > d\frac{p-\tau}{p} \iff m > d\frac{p-\tau}{p\tau} = d\Big(\frac{1}{\tau}-\frac{1}{p}\Big) = d\cdot\frac{r}{d} = r\,.$$

Similarly, in case $a - m + r\frac{d-\delta}{d} = 0$ the resulting estimate is

$$\sum_{(I,\psi)\in\Lambda^0} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_4 \sum_{j\geq 0} 2^{-ja\tau} 2^{j\delta\frac{p-\tau}{p}} (j+1)^{\frac{p-\tau}{p}} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau \leq c_5' \|u|\mathcal{K}^m_{p,a}(D)\|^\tau < \infty\,,$$

where the series converges if, and only if

$$a\tau > \delta\frac{p-\tau}{p} \iff a > \delta\Big(\frac{1}{\tau}-\frac{1}{p}\Big) \iff a > \frac{\delta}{d} r \iff m > r\frac{d-\delta}{d} + \frac{\delta}{d} r = r\,,$$

which is exactly the same condition as before. Finally, in case $a - m + r\frac{d-\delta}{d} < 0$ we find

$$\sum_{(I,\psi)\in\Lambda^0} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_4 \sum_{j\geq 0} 2^{-ja\tau} 2^{j\delta\frac{p-\tau}{p}} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau \leq c_5'' \|u|\mathcal{K}^m_{p,a}(D)\|^\tau < \infty\,,$$

where now we obtain the condition

$$a\tau > \delta\frac{p-\tau}{p} \iff a > \delta\frac{p-\tau}{p\tau} = \delta\Big(\frac{1}{\tau}-\frac{1}{p}\Big) = \delta\cdot\frac{r}{d}\,.$$

Step 4: Next, we need to consider the sets $\Lambda_{j,0}$. Here the assumption $\widetilde{u} \in B^s_{p,p}(\mathbb{R}^d)$ comes into play once more. We note that $\#\Lambda_{j,0} \leq c_7 2^{j\delta}$, thus we can estimate using Hölder's inequality

$$\sum_{(I,\psi)\in\Lambda_{j,0}} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_7^{\frac{p-\tau}{p}} 2^{j\delta\frac{p-\tau}{p}} 2^{-jd(\frac{1}{p}-\frac{1}{2})\tau} \Big(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^p\Big)^{\frac{\tau}{p}}$$

$$= c_7^{\frac{p-\tau}{p}} 2^{j\delta\frac{p-\tau}{p}} 2^{-js\tau} \Bigg(\sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j(s+\frac{d}{2}-\frac{d}{p})p} |\langle \widetilde{u}, \psi_I\rangle|^p \Bigg)^{\frac{\tau}{p}} .$$

Now summing over j and once more using Hölder's inequality gives

$$\begin{aligned}
\sum_{j\geq 0} \sum_{(I,\psi)\in\Lambda_{j,0}} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau &\leq c_7^{\frac{p-\tau}{p}} \sum_{j\geq 0} 2^{-js\tau} 2^{j\delta\frac{p-\tau}{p}} \Bigg(\sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j(s+\frac{d}{2}-\frac{d}{p})p} |\langle \widetilde{u}, \psi_I\rangle|^p \Bigg)^{\frac{\tau}{p}} \\
&\leq c_7^{\frac{p-\tau}{p}} \Bigg(\sum_{j\geq 0} 2^{-j\frac{sp\tau}{p-\tau}} 2^{j\delta} \Bigg)^{\frac{p-\tau}{p}} \Bigg(\sum_{j\geq 0} \sum_{(I,\psi)\in\Lambda_{j,0}} 2^{j(s+\frac{d}{2}-\frac{d}{p})p} |\langle \widetilde{u}, \psi_I\rangle|^p \Bigg)^{\frac{\tau}{p}} \\
&\leq c_8 \|\widetilde{u}|B^s_{p,p}(\mathbb{R}^d)\|^\tau \lesssim \|u|B^s_{p,p}(D)\|^\tau ,
\end{aligned}$$

under the condition

$$\delta < \frac{sp\tau}{p-\tau} \iff \frac{s}{\delta} > \frac{1}{\tau} - \frac{1}{p} \iff r < \frac{sd}{\delta} .$$

Step 5: Finally, we need to consider those ψ_I whose support intersects ∂D. Then we can estimate similar to Step 4, with δ replaced by $d-1$. This results in the condition

$$\sum_{(I,\psi)\in\Lambda:\operatorname{supp}\psi_I\cap\partial D\neq\emptyset} |I|^{(\frac{1}{p}-\frac{1}{2})\tau} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_9 \|\widetilde{u}|B^s_{p,p}(\mathbb{R}^d)\|^\tau \lesssim \|u|B^s_{p,p}(D)\|^\tau \quad \text{if} \quad r < \frac{sd}{d-1} .$$

Summarily we have proved

$$\|u|B^r_{\tau,\tau}(D)\| \leq \|\widetilde{u}|B^r_{\tau,\tau}(\mathbb{R}^d)\| \lesssim \|u|B^s_{p,p}(D)\| + \|u|\mathcal{K}^m_{p,a}(D)\| ,$$

with constants independent of u. □

Remark 3. The conditions $m > r$ and $a > \frac{\delta}{d}r$ and the cases considered in Steps 2 and 3 (depending on the sign of $a - m + r\frac{d-\delta}{d}$) are not completely independent, though it seems not clear at first glance whether there is potential for improvement.
However, we find that these steps of the argument indeed exclude parameters $r \geq m$ or $a \leq \frac{\delta}{d}r$. In detail: Assuming $r \geq m$ we either have $a - m + r\frac{d-\delta}{d} \geq 0$ which as before leads to the contradicting condition $m > r$, or we have $a - m + r\frac{d-\delta}{d} < 0$, which immediately implies

$$0 > a - m + r\frac{d-\delta}{d} \geq a - r + r\frac{d-\delta}{d} = a - \frac{\delta}{d}r ,$$

thus contradicting the condition $a > \frac{\delta}{d}r$ obtained in Step 3. Thus $r < m$ indeed is necessary for our argument to work.
In a similar way, assuming $a \leq \frac{\delta}{d}r$, again we either have $a - m + r\frac{d-\delta}{d} \geq 0$, which implies

$$\frac{\delta}{d}r \geq a \geq m - r\frac{d-\delta}{d} \Longrightarrow 0 \geq m - r ,$$

and thus contradicts the condition obtained in Step 3, or $a - m + r\frac{d-\delta}{d} < 0$ which obviously once more gives a contradiction to Step 3. Hence we conclude $a > \frac{\delta}{d}r$ to be necessary for our argument.
Whether these restrictions are due to the chosen argument, or the result becomes false otherwise, is not clear.

Remark 4. The result remains true if D is an unbounded domain, but u is a priori known to have compact support. In particular, this refers to D being an infinite cone or some dihedral angle $D = D_\alpha = \{x \in \mathbb{R}^3 : 0 < \vartheta < \alpha\}$, (ρ, ϑ, z) being the cylindrical coordinates in $\mathbb{R}^3$. Since Besov spaces are compatible with localization arguments (i.e. decomposing a domain by using partitions of unity, a function belongs to a Besov space on the original domain if, and only if, every piece belongs to Besov spaces corresponding to the respective subdomains), the respective localization arguments for polyhedral domains carry over to this kind of consideration for Besov regularity.

Remark 5. The above results correspond well to the ones obtained by Dahlke and Dahlke/Sickel: In case $d = 2$ every solution of $-\Delta u = f$ with $f \in H^s(D)$ can be decomposed into a regular part $u_R \in H^{s+2}(D)$ and a singular part u_S with lower Sobolev regularity, but with a special structure (a finite linear combination of special singularity functions which are known explicitly depending on the respective interior angles. In 3D such a decomposition exists only in special cases of the domain D. Dahlke investigated the Besov regularity of the singular part, which in general is significantly higher than its Sobolev regularity, and also much higher than the regularity of the regular part. For the last observation we note that for a bounded Lipschitz domain D we have $H^s(D) = B^s_{2,2}(D) \hookrightarrow B^r_{\tau,\tau}(D)$ for every $r < s$, and the result becomes false for $r = s$ (where as before $\frac{1}{\tau} = \frac{r}{d} + \frac{1}{2}$). Hence, though the singular part might have a higher regularity, the Besov regularity of $u = u_R + u_S$ can in general not exceed $s + 2$. Since $f \in \mathcal{K}^{m-1}_{a-1}$ implies $u \in \mathcal{K}^{m+1}_{a+1}$ and $\mathcal{K}^m_m(D) \hookrightarrow H^m(D)$, choosing $a = m$ (whenever this choice is admissible for the regularity result in Kondratiev spaces) shows that both our and Dahlke's results yield the same Besov regularity for u.
Moreover, the estimates of Steps 3 and 4 essentially reproduce the regularity estimates for the singular functions: Away from the singularities, these functions are smooth (arbitrary high Sobolev regularity), and near the vertex/edge we have a much greater Besov-regularity (Dahlke's 2D-result corresponds to the observation that for only vertex singularities (i.e. $\delta = 0$) in Step 4 we may choose r to be arbitrarily large).
A similar observation is true for the result of Dahlke/Sickel, who investigated polyhedral cones in $d = 3$: Here we obtain exactly the same conditions on the parameters since the spaces $V^l_{p,\beta}$ used therein essentially coincide with the spaces $\mathcal{K}^l_{p,l-\beta}$.

4 Regularity result for spaces $B^m_{\tau,\infty}(D)$

We shall begin this section with reformulations of some estimates in the proof of Theorem 1, as we actually showed a little more than claimed.
Using the notations of that proof, we define an operator P_{int} on $\mathcal{K}^m_{p,a}(D)$ by defining

$$P_{\text{int}} u = \sum_{(I,\psi)\in\Lambda^0} \langle \widetilde{u}, \psi_I\rangle \psi_I \,. \tag{4.1}$$

Then Step 3 can be reformulated as

$$P_{\text{int}} : \mathcal{K}^m_{p,a}(D) \longrightarrow B^r_{\tau,\tau}(D)\,, \qquad \|P_{\text{int}} u|B^r_{\tau,\tau}(D)\| \lesssim \|u|\mathcal{K}^m_{p,a}(D)\|\,. \tag{4.2}$$

With the same arguments, only replacing the summation over j by a supremum, we also find

$$P_{\text{int}} : \mathcal{K}^m_{p,a}(D) \longrightarrow B^m_{\tau,\infty}(D)\,, \qquad \|P_{\text{int}} u|B^m_{\tau,\infty}(D)\| \lesssim \|u|\mathcal{K}^m_{p,a}(D)\|\,. \tag{4.3}$$

(observe that the condition $a - m + r\frac{d-\delta}{d} > 0$ then simply becomes $a - m\frac{\delta}{d} > 0$, which exactly is the assumption on a in case $m = r$). This observation motivated to have a closer look at the case $r = m$ and spaces $B^m_{\tau,\infty}(D)$ (for general τ, i.e. we will not require the relation $\frac{1}{\tau} = \frac{m}{d} + \frac{1}{p}$).
In a similar way, we can reformulate Steps 4 and 5: Defining

$$P_{\text{bd}} u = \sum_{(I,\psi)\in\Lambda\setminus\Lambda^0} \langle \widetilde{u}, \psi_I\rangle \psi_I \,, \tag{4.4}$$

we so far proved

$$P_{\text{bd}} : B^s_{p,p}(D) \longrightarrow B^r_{\tau,\tau}(\mathbb{R}^d)\,, \qquad \|P_{\text{bd}} u|B^r_{\tau,\tau}(\mathbb{R}^d)\| \lesssim \|\widetilde{u}|B^s_{p,p}(\mathbb{R}^d)\| \lesssim \|u|B^s_{p,p}(D)\|\,, \tag{4.5}$$

as long as $r < \frac{d}{d-1}s$. Also here we can be a little more precise: By summing over j without using Hölder's inequality the second time it follows

$$\sum_{j=0}^{\infty}\Bigg(\sum_{(I,\psi)\in\Lambda_{j,0}} |I|^{(\frac{1}{p}-\frac{1}{2})\tau}|\langle\widetilde{u},\psi_I\rangle|^\tau\Bigg)^{\frac{p}{\tau}} \lesssim \|\widetilde{u}|B^{\frac{(d-1)r}{d}}_{p,p}(\mathbb{R}^d)\|^p\,, \quad \text{thus} \quad P_{\text{bd}} : B^{\frac{(d-1)r}{d}}_{p,p}(D) \longrightarrow B^r_{\tau,p}(\mathbb{R}^d)\,,$$

recall $\delta\frac{p-\tau}{p} = \tau\delta(\frac{1}{\tau}-\frac{1}{p}) = \tau\frac{\delta}{d}r$. The second application of Hölder's inequality in Step 4 then simply corresponds to the standard embedding $B^{r+\varepsilon}_{\tau,p}(\mathbb{R}^d) \hookrightarrow B^r_{\tau,\tau}(\mathbb{R}^d)$, $\varepsilon > 0$. In other words: The boundary terms are completely covered by the assumed Besov regularity in $B^s_{p,p}(D)$ respectively Sobolev regularity in $H^s(D)$ (in case $p = 2$).
In the next theorem, we shall have a closer look at the operators $P_{\rm int}$ and $P_{\rm bd}$, the considered function spaces being motivated by the reformulations and observations above.

Theorem 2. Let D be some bounded polyhedral domain in $\mathbb{R}^d$ and $0 < \tau < p$, $1 < p < \infty$. Then we have

$$P_{\rm int} : \mathcal{K}^m_{p,a}(D) \longrightarrow \begin{cases} B^m_{\tau,\infty}(D)\,, & m < a + (d-\delta)(\frac{1}{\tau}-\frac{1}{p})\,, \\ B^{m,-\frac{1}{\tau}+\frac{1}{p}}_{\tau,\infty}(D)\,, & m = a + (d-\delta)(\frac{1}{\tau}-\frac{1}{p})\,, \\ B^{a+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})}_{\tau,\infty}(D)\,, & m > a + (d-\delta)(\frac{1}{\tau}-\frac{1}{p})\,, \end{cases}$$

as well as

$$P_{\rm bd} : B^s_{p,q}(D) \longrightarrow B^{s+\frac{1}{\tau}-\frac{1}{p}}_{\tau,q}(\mathbb{R}^d)\,, \qquad 0 < q \le \infty\,,$$

both mappings being bounded linear operators.

Proof. We only show the necessary modifications of the proof of Theorem 1.
Step 2': Using the same notations, we then find using Hölder's inequality

$$\begin{aligned} \sum_{(I,\psi)\in\Lambda^0_j} |\langle \widetilde{u}, \psi_I\rangle|^\tau &\le c_1 \sum_{(I,\psi)\in\Lambda^0_j} \big(|I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}}\rho_I^{-m+a}\mu_I\big)^\tau \\ &\le c_1 \Big(\sum_{(I,\psi)\in\Lambda^0_j} \Big(|I|^{(\frac{m}{d}+\frac{1}{2}-\frac{1}{p})\tau}\rho_I^{(a-m)\tau}\Big)^{\frac{p}{p-\tau}}\Big)^{\frac{p-\tau}{p}} \underbrace{\Big(\sum_{(I,\psi)\in\Lambda^0_j} \mu_I^p\Big)^{\frac{\tau}{p}}}_{\lesssim \|u|\mathcal{K}^m_{p,a}(D)\|^\tau}\,. \end{aligned}$$

For the first factor we proceed as before,

$$\begin{aligned} \Big(\sum_{(I,\psi)\in\Lambda^0_j} \Big(|I|^{\frac{m}{d}+\frac{1}{2}-\frac{1}{p}}\rho_I^{a-m}\Big)^{\frac{p\tau}{p-\tau}}\Big)^{\frac{p-\tau}{p}} &\le \Big(c_3 2^{-j(a+\frac{d}{2}-\frac{d}{p})\frac{p\tau}{p-\tau}} \sum_{k=1}^{2^j} k^{(a-m)\frac{p\tau}{p-\tau}} k^{d-1-\delta} 2^{j\delta}\Big)^{\frac{p-\tau}{p}} \\ &\le c_4 2^{-j(a+\frac{d}{2}-\frac{d}{p})\tau} 2^{j\delta\frac{p-\tau}{p}} \begin{cases} 2^{j(a-m)\tau + j(d-\delta)\frac{p-\tau}{p}}\,, & m-a < (d-\delta)\frac{p-\tau}{p\tau}\,, \\ (j+1)^{\frac{p-\tau}{p}}\,, & m-a = (d-\delta)\frac{p-\tau}{p\tau}\,, \\ 1\,, & m-a > (d-\delta)\frac{p-\tau}{p\tau}\,. \end{cases} \end{aligned}$$

Step 3': Taking the supremum over $j \ge 0$ we obtain from the first line of the last estimate

$$\begin{aligned} \sup_{j\ge0} 2^{j(m+\frac{d}{2}-\frac{d}{\tau})\tau} \sum_{(I,\psi)\in\Lambda^0_j} |\langle \widetilde{u}, \psi_I\rangle|^\tau &\le c_4 \sup_{j\ge0} 2^{j(m+\frac{d}{p}-\frac{d}{\tau})\tau} 2^{-jm\tau} 2^{jd\frac{p-\tau}{p}} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau \\ &= c_4 \sup_{j\ge0} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau = c_4 \|u|\mathcal{K}^m_{p,a}(D)\|^\tau\,, \end{aligned}$$

without any additional condition on τ. Similarly, in case $m - a = (d-\delta)(\frac{1}{\tau}-\frac{1}{p})$ the resulting estimate is

$$\begin{aligned} \sup_{j\ge0} 2^{j(m+\frac{d}{2}-\frac{d}{\tau})\tau} (j+1)^{-\frac{p-\tau}{p\tau}\tau} \sum_{(I,\psi)\in\Lambda^0_j} |\langle \widetilde{u}, \psi_I\rangle|^\tau &\le c_4 \sup_{j\ge0} 2^{j(m-a+(d-\delta)(\frac{1}{p}-\frac{1}{\tau}))\tau} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau \\ &= c_4 \sup_{j\ge0} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau = c_4 \|u|\mathcal{K}^m_{p,a}(D)\|^\tau\,. \end{aligned}$$

Finally, in case $m - a > (d-\delta)(\frac{1}{\tau}-\frac{1}{p})$ we find

$$\sup_{j\ge0} 2^{j(a+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})+\frac{d}{2}-\frac{d}{\tau})\tau} \sum_{(I,\psi)\in\Lambda^0_j} |\langle \widetilde{u}, \psi_I\rangle|^\tau \le c_4 \sup_{j\ge0} 2^{j(a-\delta(\frac{1}{\tau}-\frac{1}{p})\tau} 2^{-ja\tau} 2^{j\delta\frac{p-\tau}{p\tau}\tau} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau$$

$$= c_4 \sup_{j\geq 0} \|u|\mathcal{K}^m_{p,a}(D)\|^\tau = c_4 \|u|\mathcal{K}^m_{p,a}(D)\|^\tau < \infty\,.$$

Step 4': To estimate the boundary part $P_{\rm bd}u$, as before we only need to modify the summation over $j \geq 0$:

$$\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^\tau \leq c_7^{\frac{p-\tau}{p}} 2^{j(d-1)\frac{p-\tau}{p}} \Big(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^p \Big)^{\frac{\tau}{p}},$$

and further for $q < \infty$

$$\begin{aligned}
&\sum_{j\geq 0} 2^{j(s+\frac{1}{\tau}-\frac{1}{p}+d(\frac{1}{2}-\frac{1}{\tau}))q} \Big(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^\tau \Big)^{\frac{q}{\tau}} \\
&\quad \leq c_7^{\frac{p-\tau}{p\tau}q} \sum_{j\geq 0} 2^{j(s+\frac{1}{\tau}-\frac{1}{p}+d(\frac{1}{2}-\frac{1}{\tau}))q} 2^{j(d-1)\frac{p-\tau}{p\tau}q} \Big(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^p \Big)^{\frac{q}{p}} \\
&\quad = c_7^{\frac{p-\tau}{p\tau}q} \sum_{j\geq 0} 2^{j(s+d(\frac{1}{2}-\frac{1}{p}))q} \Big(\sum_{(I,\psi)\in\Lambda_{j,0}} |\langle \widetilde{u}, \psi_I\rangle|^p \Big)^{\frac{q}{p}} \\
&\quad \leq c_7^{\frac{p-\tau}{p\tau}q} \|\widetilde{u}|B^s_{p,q}(\mathbb{R}^d)\|^q \sim \|\widetilde{u}|B^s_{p,q}(\mathbb{R}^d)\|^q \lesssim \|u|B^s_{p,q}(D)\|^q\,,
\end{aligned}$$

which proves $P_{\rm bd}u \in B^{s+\frac{1}{\tau}-\frac{1}{p}}_{\tau,q}(\mathbb{R}^d)$. The result for $q = \infty$ follows by standard modifications. □

Corollary 1. Let $u \in \mathcal{K}^m_{p,a}(D) \cap B^s_{p,\infty}(D)$ for some bounded polyhedral domain $D \subset \mathbb{R}^d$. Suppose $s > \frac{d-1}{d}m$ and $a > \frac{\delta}{d}m$. Then there exists some $0 < \tau_0 \leq p$ such that

$$u \in B^m_{\tau,\infty}(D) \hookrightarrow L_p(D)$$

for all $\tau_* < \tau < \tau_0$, where $\frac{1}{\tau_*} = \frac{m}{d} + \frac{1}{p}$.

Note: While it also holds $u \in B^m_{\tau,\infty}(D)$ for $\tau \leq \tau_*$, these spaces are no longer embedded into $L_p(D)$. From the point of view of n-term approximation, the result then becomes useless (the mere knowledge of this Besov regularity no longer yields any approximation rate). We further note that we always have $H^s_p(D) \hookrightarrow B^s_{p,\infty}(D)$.

Proof. We can decompose u according to $u = P_{\rm int}u + P_{\rm bd}u|_D$. Then we can consider both terms separately.
First we need to have a closer look at the condition $m - a < (d-\delta)\frac{p-\tau}{p\tau}$:

$$m - a < (d-\delta)\frac{p-\tau}{p\tau} = (d-\delta)\Big(\frac{1}{\tau} - \frac{1}{p}\Big) < (d-\delta)\Big(\frac{1}{\tau_*} - \frac{1}{p}\Big) = (d-\delta)\frac{m}{d}\,.$$

Clearly, the inequality $m - a < (d-\delta)\frac{m}{d}$ is equivalent to the assumption $a > \frac{\delta}{d}m$. Thus for τ sufficiently close to τ_*, the required condition $m - a < (d-\delta)\frac{p-\tau}{p\tau}$ for $P_{\rm int}u \in B^m_{\tau,\infty}(D)$ can be satisfied, and we find $\frac{1}{\tau_0} = \frac{m-a}{d-\delta} + \frac{1}{p}$ (in case $a \geq m$ we just choose $\tau_0 = p$).
Similarly $s > \frac{d-1}{d}m$ implies $s + \frac{1}{\tau_*} - \frac{1}{p} > \frac{d-1}{d}m + \frac{1}{\tau_*} - \frac{1}{p} = m$, thus for τ sufficiently close to τ_* we still have $s + \frac{1}{\tau} - \frac{1}{p} \geq m$. Hence we have $B^{s+\frac{1}{\tau}-\frac{1}{p}}_{\tau,\infty}(\mathbb{R}^d) \hookrightarrow B^m_{\tau,\infty}(\mathbb{R}^d)$, which in turn yields $P_{\rm bd}u|_D \in B^m_{\tau,\infty}(D)$. □

5 An extension argument for Kondratiev spaces

In this section we seek to relax the required Sobolev regularity in Corollary 1. This will be done by modifying the splitting $u = P_{\rm int}u + P_{\rm bd}u$. In what follows we denote by $S \subset \partial D$ the singularity set of

D. Then we recall that the distance function ρ is bounded away from zero on any closed subset of $\overline{D}$ not containing S.
As a first step, instead of P_{bd} we consider the operator $\widetilde{P}_{\mathrm{sing}}$,

$$\widetilde{P}_{\mathrm{sing}} u = \sum_{j\geq 0} \sum_{(I,\psi)\in\Lambda_{j,0}} \langle \widetilde{u}, \psi_I\rangle \psi_I \,,$$

i.e. we take only those terms of wavelets touching the singular set S. Then with the same estimates leading to the properties of P_{bd} in Theorem 2 we obtain

$$\widetilde{P}_{\mathrm{sing}} : B^s_{p,p}(D) \longrightarrow B^{s+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})}_{\tau,p}(D)\,, \qquad 0<\tau<p\,,$$

as a bounded linear operator. Ultimately, the embedding $B^{s+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})}_{\tau,p}(D) \hookrightarrow B^m_{\tau,\infty}(D)$ for some $\tau > \tau_*$ then leads to the condition $s > \frac{\delta}{d} m$.
For the other part $\widetilde{P}_{\mathrm{reg}} u = u - \widetilde{P}_{\mathrm{sing}} u$ we once more want to use the regularity of u in the Kondratiev-scale. However, inspecting the previous proofs yields that this requires modifying the index sets $\Lambda_{j,k}$ to include also those wavelets touching the boundary, and extending the corresponding estimates for wavelet coefficients. This can be done by extending the functions from $\mathcal{K}^m_{p,a}(D)$ to $\mathbb{R}^d$ in a suitable way.
In particular, we first have to define a counterpart of the scale $\mathcal{K}^m_{p,a}(D)$ for functions on $\mathbb{R}^d$. We start with a function η, which is defined on $\mathbb{R}^d$ and smooth on $\mathbb{R}^d \setminus S$, and it is assumed to mimic the distance function ρ, i.e. in a (sufficiently small) neighbourhood of the singularity set S the function η shall be equivalent to the distance to S. Moreover, we suppose that η has values only in the interval $[0,1]$. Then we put

$$\mathcal{K}^m_{p,a}(S) := \big\{ u \text{ measurable} : \eta^{|\alpha|-a}\partial^\alpha u \in L_p(\mathbb{R}^d)\,, |\alpha|\leq m \big\}\,.$$

One possible approach now consists in retracing the steps of Stein's original proof in [19] in order to determine whether his extension operator is also bounded with respect to the $\mathcal{K}^m_a$-norms:

Lemma 5.1. For $a < 3/2$ the extension operator $\mathfrak{E}$ defined in [19, Section 3.2-3.3] is bounded as a mapping $\mathfrak{E} : \mathcal{K}^m_{p,a}(D) \longrightarrow \mathcal{K}^m_{p,a}(S)$.

Proof. For the most part Stein's proof carries over without change, hence we shall mostly be concerned with some necessary modifications.
The first step consists in reducing the problem to smooth functions. It is easily seen that the set $C^\infty(D)$ is dense in $\mathcal{K}^m_a(D)$ for all parameters: Given a function $u \in \mathcal{K}^m_{p,a}(D)$ we can multiply it with a smooth cut-off function, hence we may assume u to have compact support. With standard mollification arguments we see that such a function (and simultaneously its partial derivatives) can be approximated in the L_2-sense by C^∞-functions. Clearly this immediately extends also to approximation w.r.to the $\mathcal{K}^m_a(D)$-norm.
Stein then shows first that the extension operator applied to a smooth function $u \in C^\infty(D)$ yields again a smooth function $\mathfrak{E}u \in C^\infty(\mathbb{R}^d)$. It now remains to check the corresponding norm estimates. Below we shall use Stein's notation.

Step 1: Stein defined the operator $\mathfrak{E}$ on special Lipschitz domains by

$$\mathfrak{E}(x,y) = \int_1^\infty f\big(x, y+\lambda\delta^*(x,y)\big)\psi(\lambda)d\lambda\,, \qquad x\in\mathbb{R}^d, y\in\mathbb{R}\,,$$

where δ^* is a scaled version of the regularized distance to the boundary, and $\psi : [1,\infty) \longrightarrow \mathbb{R}$ is a rapidly decaying continuous function such that $\int_1^\infty \psi(\lambda)d\lambda = 1$ and $\int_1^\infty \lambda^k\psi(\lambda)d\lambda = 0$ for all $k\in\mathbb{N}$. Moreover, $D = \{(x,y)\in\mathbb{R}^{d+1} : y > \varphi(x)\}$ with $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ being Lipschitz continuous.
Now assume $y<0$, and let x^0 be a point with $\varphi(x^0) = 0$. The scaling and further properties of δ^* ensure $2|y| \leq \delta(x^0,y) \leq c\,|y|$, and it follows

$$|\mathfrak{E}f(x^0,y)| \lesssim \int_1^\infty |f(x^0, y+\lambda\delta^*)|\frac{d\lambda}{\lambda^p} = \delta^* \int_{y+\delta^*}^\infty |f(x^0,s)|(s-y)^{-p}\,ds \lesssim |y| \int_{|y|}^\infty |f(x^0,s)|s^{-p}\,ds\,.$$

The essential step now lies in applying Hardy's inequality,

$$\int_0^\infty \left(\int_x^\infty g(y)dy\right)^q x^{r-1}dx \le \left(\frac{q}{r}\right)^q \int_0^\infty \big(yg(y)\big)^q y^{r-1}dy\,,$$

where g is non-negative, $q \ge 1$ and $r > 0$. This shall be applied with $g(s) = f(x^0, s)s^{-2}$; we obtain

$$\begin{aligned}\int_{-\infty}^0 |y|^{-pa}|\mathfrak{E}f(x^0,y)|^p dy &\lesssim \int_{-\infty}^0 |y|^{p-pa}\left(\int_{|y|}^\infty |f(x^0,s)|s^{-2}\,ds\right)^p dy\\ &= \int_0^\infty y^{p-pa}\left(\int_y^\infty |f(x^0,s)|s^{-2}\,ds\right)^p dy\\ &\lesssim \int_0^\infty \big(s^{-1}|f(x^0,s)|\big)^p s^{p-pa}ds = \int_0^\infty |f(x^0,s)|^p s^{-pa}ds\,.\end{aligned}$$

Therein the assumption $r > 0$ for Hardy's inequality corresponds to $a < 3/2$. We observe that in the last integral the term s^{-pa} can again be replaced by $|y|^{-pa}$. Moreover, note that with simple translation arguments, we then obtain for general $x \in \mathbb{R}^d$

$$\int_{-\infty}^{\varphi(x)} |y-\varphi(x)|^{-pa}|\mathfrak{E}f(x,y)|^p dy \lesssim \int_{\varphi(x)}^\infty |f(x,y)|^p |y-\varphi(x)|^{-pa}dy\,.$$

Since $\mathfrak{E}f(x,y) = f(x,y)$ for all $y > \varphi(x)$, we can extend the integration domain on the left-hand side to $\mathbb{R}$. Moreover, if we extend $\delta^*(x,y)$ for $y < \varphi(x)$ as the scaled regularized distance for the special Lipschitz domain $D^c = \mathbb{R}^{d+1} \setminus \overline{D}$, then this choice guarantees $\delta^*(x,y) \sim |y-\varphi(x)|$ for all x and y. Now integration w.r.to $x \in \mathbb{R}^d$ yields the desired estimate

$$\|\delta^{-a}\mathfrak{E}f|L_p(\mathbb{R}^{d+1})\| \lesssim \|\delta^{-a}f|L_p(D)\|$$

for arbitrary special Lipschitz domains $D \subset \mathbb{R}^{d+1}$ with constants only depending on a and D, with δ being the (regularized) distance to ∂D.
For partial derivatives of f similar arguments can be used (as explained in [19]), and no additional restrictions on a occur. Exemplary we show it for some second-order partial derivative, w.l.o.g. $\partial_j^2\mathfrak{E}f$. It holds

$$\partial_j\mathfrak{E}f = \int_1^\infty \partial_j f(\cdots)\psi(\lambda)\,d\lambda + \int_1^\infty \partial_y f(\cdots)\lambda\partial_j\delta^*\psi(\lambda)\,d\lambda$$

and hence

$$\begin{aligned}\partial_j^2\mathfrak{E}f = &\int_1^\infty \partial_j^2 f(\cdots)\psi(\lambda)\,d\lambda + 2\int_1^\infty \partial_j\partial_y f(\cdots)\lambda\partial_j\delta^*\psi(\lambda)\,d\lambda\\ &+ \int_1^\infty \partial_y^2 f(\cdots)(\lambda\partial_j\delta^*)^2\psi(\lambda)\,d\lambda + \int_1^\infty \partial_y^2 f(\cdots)\lambda\partial_j^2\delta^*\psi(\lambda)\,d\lambda\,.\end{aligned} \tag{5.1}$$

We first note $\partial^\alpha\delta^* \le c_\alpha(\delta^*)^{1-|\alpha|}$ for all multiindices α and $|\psi(\lambda)| \le A_k\lambda^{-k}$. For the first term we then find as above for $y < 0$ and $\varphi(x^0) = 0$

$$\left|\int_1^\infty \partial_j^2 f(\cdots)\psi(\lambda)\,d\lambda\right| \le \int_1^\infty \big|\partial_j^2 f(\cdots)\psi(\lambda)\big|\,d\lambda \le A_2\int_1^\infty \big|\partial_j^2 f(\cdots)\big|\,\frac{d\lambda}{\lambda^2} \lesssim |y|\int_{|y|}^\infty |\partial_j^2 f(x^0,s)|s^{-2}\,ds\,, \tag{5.2}$$

and similarly

$$\left|\int_1^\infty \partial_j\partial_y f(\cdots)\lambda\partial_j\delta^*\psi(\lambda)\,d\lambda\right| \le c_j A_3|y|\int_1^\infty \big|\partial_j\partial_y f(\cdots)\big|\,s^{-2}\,ds \tag{5.3}$$

as well as

$$\left|\int_1^\infty \partial_y^2 f(\cdots)(\lambda\partial_j\delta^*)^2\psi(\lambda)\,d\lambda\right| \le c_j^2 A_4|y|\int_1^\infty \big|\partial_y^2 f(\cdots)\big|\,s^{-2}\,ds\,. \tag{5.4}$$

It remains the last integral in (5.1). We re-write $\partial_y f$ as

$$\partial_y f(x^0, y+\lambda\delta^*) = \partial_y f(x^0, y+\delta^*) + \int_{y+\delta^*}^{y+\lambda\delta^*} \partial_y^2 f(x^0,t)\,dt\,.$$

Due to the choice of ψ, i.e. $\int_1^\infty \lambda\psi(\lambda)\,d\lambda = 0$, we then have

$$\int_1^\infty \partial_y^2 f(\cdots)\lambda\partial_j^2\delta^*\psi(\lambda)\,d\lambda = \int_1^\infty \lambda\partial_j^2\delta^*\psi(\lambda)\int_{y+\delta^*}^{y+\lambda\delta^*}\partial_y^2 f(x^0,t)\,dt\,d\lambda\,.$$

This can be estimated by

$$\begin{aligned}\left|\int_1^\infty \partial_y^2 f(\cdots)\lambda\partial_j^2\delta^*\psi(\lambda)\,d\lambda\right| &\lesssim |y|^{-1}A_4\int_1^\infty\Big(\int_{y+\delta^*}^{y+\lambda\delta^*}\big|\partial_y^2 f(x^0,t)\big|\,dt\Big)\lambda^{-3}\,d\lambda\\ &= |y|^{-1}A_4\int_{y+\delta^*}^\infty\Big(\int_{\frac{t-y}{\delta^*}}^\infty\lambda^{-3}\,d\lambda\Big)\big|\partial_y^2 f(x^0,t)\big|\,dt\\ &\sim |y|^{-1}\int_{y+\delta^*}^\infty(\delta^*)^2\big|\partial_y^2 f(x^0,t)\big|\,\frac{dt}{(t-y)^2} \lesssim |y|\int_1^\infty\big|\partial_y^2 f(\cdots)\big|s^{-2}\,ds\,.\end{aligned}$$

For the last integral as well as those in (5.2)–(5.4) we can use analogous arguments as above, we only have to replace $|y|^{-pa}$ by $|y|^{(2-a)p}$. Since the assumption $r>0$ for Hardy's inequality now transfers to $a<7/2$, as announced no additional restrictions on a occur.
Similarly for all other partial derivatives of $\mathfrak{E}f$: After differentiation under the integral every term can be treated separately, and for terms involving lower order derivatives of f we use Taylor expansion and the moment conditions for ψ.

Step 2: The result for special Lipschitz domains in Step 1 now can be used to derive the estimate for general Lipschitz polyhedral domains. For this we only note that the singularity set can be covered by finitely many open sets $U_1,\dots,U_N$ such that on any of these open sets the distance to ∂D is equivalent to the distance to S. This cover of S can be extended with additional finitely many open sets $U_{N+1},\dots,U_M$ to an open cover of $\overline{D}$. On these sets $U_{N+1},\dots,U_M$ the distance function η shall be bounded from below. Finally, we assume that we can associate with every U_i a special Lipschitz domain D_i such that $U_i\cap D = U_i\cap D_i$. With these sets U_i and D_i in hand we are back in the situation of [19, Section 3.3], where it is described how to glue together the extension operators $\mathfrak{E}_i$ (w.r.to the domains D_i) to finally obtain $\mathfrak{E}$ (essentially it is a partition of unity argument for some partition adapted to the domains D_i and the neighbourhoods U_i).
The norm estimates for $\mathfrak{E}$ carry over to our situation without change, we only note that the estimates in Step 1 due to the assumptions on the U_i exactly correspond to estimates for the $\|\cdot|\mathcal{K}^m_a(S)\|$-norm. □

The restriction $a<3/2$ of the above lemma is merely of technical nature. For values $a\geq 3/2$ we can easily define extension operators $\mathfrak{E}_a$ by tracing it back to some fixed valued $a<3/2$.

Lemma 5.2. Let ω be a smooth non-negative function with $\omega(x)=1$ for all $x\in\overline{D}$ and $\operatorname{supp}\omega\subset\widetilde{D}$ for some arbitrary bounded domain $\widetilde{D}$. Then the operator $\mathfrak{E}_a$, defined by

$$\mathfrak{E}_a u = \eta^a\omega\mathfrak{E}\big(\eta^{-a}|_D u\big)\,,$$

is a bounded linear mapping from $\mathcal{K}^m_{p,a}(D)$ into $\mathcal{K}^m_{p,a}(S)$.

Proof. It is easily checked that $\mathfrak{E}_a u$ indeed defines a function on $\mathbb{R}^d$ with $\mathfrak{E}_a u\big|_D = u$. We next observe

$$u\in\mathcal{K}^m_{p,a}(D) \iff \rho^{-a}u\in\mathcal{K}^m_{p,0}(D)\,.$$

This is a consequence of the assumed smoothness of ρ. Note that exactly the same reasoning applies to the spaces $\mathcal{K}^m_{p,a}(S)$. Then the boundedness of $\mathfrak{E}_a$ follows from Lemma 5.1:

$$\big\|\eta^a\omega\mathfrak{E}\big(\eta^{-a}|_D u\big)\big|\mathcal{K}^m_{p,a}(S)\big\| \sim \big\|\omega\mathfrak{E}\big(\eta^{-a}|_D u\big)\big|\mathcal{K}^m_{p,0}(S)\big\| \lesssim \big\|\mathfrak{E}\big(\eta^{-a}|_D u\big)\big|\mathcal{K}^m_{p,0}(S)\big\|$$

$$\lesssim \left\|\eta^{-a}|_D u \middle| \mathcal{K}^m_{p,0}(D)\right\| \lesssim \|u|\mathcal{K}^m_{p,a}(D)\|\,.$$

This proves the claim. □

Below we shall use the notation $\mathcal{E}_a$, where $\mathcal{E}_a = \mathfrak{E}$ for $a < 3/2$ and $\mathcal{E}_a = \mathfrak{E}_a$ otherwise. With this definition at hand, we now put

$$P_{\rm reg}u = \sum_{j\geq 0}\sum_{k>0}\sum_{(I,\psi)\in\Lambda^\eta_{j,k}} \langle \mathcal{E}_a u, \psi_I\rangle \psi_I\,, \qquad \Lambda^\eta_{j,k} = \{(i,\psi)\in\Lambda : 2^{-j}k \leq \eta_I < (k+1)2^{-j}\}\,.$$

We note that we no longer require $Q(I)\subset D$ in the definition of $\Lambda^\eta_{j,k}$. Moreover, we define

$$P_{\rm sing}u = \sum_{j\geq 0}\sum_{(I,\psi)\in\Lambda^\eta_{j,0}} \langle \mathcal{E}_a u, \psi_I\rangle \psi_I\,.$$

This implies

$$u = \mathcal{E}_a u\big|_D = P_{\rm reg}u\big|_D + P_{\rm sing}u\big|_D\,.$$

The wavelet coefficients corresponding to terms with $(I,\psi)\in\Lambda^\eta_{j,k}$ for $k>0$ can now be estimated in exactly the same way as in the proof of Theorems 1 and 2, in particular

$$\sum_{k>0}\sum_{(I,\psi)\in\Lambda^\eta_{j,k}} \mu_I^p \lesssim \sum_{|\alpha|=m}\int_{\mathbb{R}^d} \left|\eta(x)^{m-a}\partial^\alpha(\mathcal{E}_a u)(x)\right|^p dx \leq \|\mathcal{E}_a u|\mathcal{K}^m_{p,a}(S)\|^p \lesssim \|u|\mathcal{K}^m_{p,a}(D)\|^p\,.$$

Corresponding the singular part $P_{\rm sing}u$ we either need to prove boundedness of $\mathcal{E}_a$ as an extension operator on $B^s_{p,\infty}(D)$; or we use the embedding $H^s_p(\mathbb{R}^d)\hookrightarrow B^s_{p,\infty}(\mathbb{R}^d)$ together with the boundedness of $\mathcal{E}_a$ on $H^s_p(D)$ (which in turn is easily seen once more retracing the steps of Stein's proof). Altogether, we have proved

Theorem 3. Let $u\in\mathcal{K}^m_{p,a}(D)\cap H^s_p(D)$ for some bounded polyhedral domain $D\subset\mathbb{R}^d$. Suppose $\min(s,a) > \frac{\delta}{d}m$. Then there exists some $0<\tau_0\leq p$ such that

$$u\in B^m_{\tau,\infty}(D)\hookrightarrow L_p(D)$$

for all $\tau_* < \tau < \tau_0$, where $\frac{1}{\tau_*} = \frac{m}{d} + \frac{1}{p}$.

Remark 6. We specialize the above result to the cases $d=2$ and $d=3$ and $p=2$: In case $d=2$ we always have $\delta = 0$, hence there is no restriction on the parameters except for the (almost) trivial ones $s>0$ and $a>0$. In particular, concerning the Sobolev regularity the trivial result $u\in H^1(D)$ whenever $f\in H^{-1}(D)$ is already sufficient.
The case $d=3$ is a little more diverse: Except for special right-hand sides we now have $\delta = 1$, hence there is an upper bound for m, which corresponds to the limited Besov regularity of the corresponding singularity functions proved in [5].
In case of smooth cones or smooth domains except for conical points at the boundary there are analogous regularity results in weighted Sobolev spaces (see [13]), and our argument for the Besov regularity can be transferred (cf. [8]). Then we again have arbitrary Besov regularity for the singular part, and no upper bound for m.

6 Approximation rates for solutions of elliptic boundary value problems

In this final section we shall combine the embedding from Theorem 3 with the n-term approximation rates from (2.2) and (2.3).

Theorem 4. Let D be some bounded polyhedral domain in $\mathbb{R}^d$. Suppose $\min(s,a) > \frac{\delta}{d}m$. Then every function $u \in \mathcal{K}^m_{p,a}(D) \cap H^s_p(D)$ satisfies

$$\sigma_n\big(u; L_p(D)\big) \lesssim n^{-m/d} \|u|B^m_{\tau,\infty}(D)\| \lesssim n^{-m/d} \max\big(\|u|\mathcal{K}^m_{p,a}(D)\|, \|u|H^s_p(D)\|\big)$$

as well as

$$\sigma_n\big(u; W^1_p(D)\big) \lesssim n^{-(m-1)/d} \max\big(\|u|\mathcal{K}^m_{p,a}(D)\|, \|u|H^s_p(D)\|\big)$$

with constants independent of u and n.

As a final step, we now assume u to be the solution of some elliptic boundary value problem.

Theorem 5. Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, and consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,.$$

Under the assumptions of Proposition 1, for a right-hand side $f \in H^{m-1}(D)$ the uniquely determined solution $u \in \mathcal{K}^{m+1}_{a+1}(D)$ can be approximated at the rate

$$\sigma_n\big(u; H^1(D)\big) \lesssim n^{-m/d} \|f|H^{m-1}(D)\|\,,$$

where

$$a \le m < \min\big(\tfrac{d}{\delta}(a+1) - 1\,,\, \tfrac{d}{\delta}s_0 - 1\big)\,.$$

Therein s_0 denotes the Sobolev-regularity of u.

Proof. We just need to check, under which conditions the assumptions of the previous theorem are fulfilled. We have $H^{m-1}(D) \hookrightarrow \mathcal{K}^{m-1}_{a-1}(D)$ whenever $a \le m$. On the other hand, we have the restriction $\frac{\delta}{d}(m+1) < a+1$, which gives the right part of the condition on m. □

Note that often Sobolev regularity statements are of the form: u belongs to Sobolev spaces $H^s(D)$ for all $s < s_0$, and in general $u \notin H^{s_0}(D)$. Then we still have the condition $m + 1 < \frac{d}{\delta}s_0$ in the previous theorem.

Corollary 2. Let $D \subset \mathbb{R}^2$ be a polygon (or more generally a Lipschitz domain with polygonal structure). Let $a_{i,j} \in \mathcal{W}^m_\infty(D)$, $i,j = 1,2$, $A = (a_{i,j})_{i,j=1,2}$, and consider the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \quad \text{in} \quad D\,, \qquad u|_{\partial D} = 0\,, \tag{6.1}$$

for $f \in \mathcal{K}^m_a \cap H^{-1}(D)$. Then it holds

$$\sigma_n\big(u, H^1(D)\big) \lesssim n^{-m/2} \max\big(\{f|H^{-1}(D)\|\,, \|f|\mathcal{K}^{m-1}_{a-1}(D)\|\big)$$

whenever $a > -1$ is a parameter such that (6.1) is uniquely solvable.

Thus in this situation, apart from the basic existence result in $H^1(D)$ we do not need any information about the Sobolev regularity, and similarly, also for the parameter a the only restriction is the availability of an existence result for $f \in \mathcal{K}^{m-1}_{a-1}(D)$.
Similar results hold for other types of boundary conditions, and also for bounded domains $D \subset \mathbb{R}^3$ with smooth boundary except for conical points. More general polyhedral domains in R^3 require additional conditions: On the one hand we need more specific knowledge of the Sobolev-regularity of the solution, and on the other hand we need a to be large enough (which in turn might require more sophisticated existence results than Proposition 1). Nevertheless the resulting conditions improve the ones available so far by replacing the usual factor $\frac{d}{d-1}$ by $\frac{d}{\delta}$.

References

[1] C. Bakuta, A. Mazzucato, V. Nistor, L. Zikatanov (2010): Interface and mixed boundary value problems on n-dimensional polyhedral domains. *Doc. Math.* **15**, 687–745.

[2] S. Dahlke, R. A. DeVore (1997): Besov regularity for elliptic boundary value problems. *Comm. Partial Differential Equations* **22**(1-2), 1–16.

[3] S. Dahlke (1998): Besov regularity for second order elliptic boundary value problems with variable coefficients. *Manuscripta Math.* **95**, 59–77.

[4] S. Dahlke (1999): Besov regularity for elliptic boundary value problems in polygonal domains. *Appl. Math. Lett.* **12**, 31–36.

[5] S. Dahlke (2002): Besov regularity of edge singularities for the Poisson equation in polyhedral domains. *Numer. Linear Algebra Appl.* **9**, 457–466.

[6] M. Dauge (1988): Elliptic boundary value problems in corner domains. Lecture Notes in Mathematics **1341**, Springer, Berlin.

[7] S. Dahlke, E. Novak, W. Sickel (2006): Optimal approximation of elliptic problems by linear and nonlinear mappings II. *J. Complexity* **22**, 549–603.

[8] S. Dahlke, W. Sickel (2009): Besov regularity for the Poisson equation in smooth and polyhedral cones. In: Sobolev spaces in mathematics II. V. Mazýa (Ed.). *Int. Math. Ser. (N.Y.)* **9**, Springer, New York.

[9] R. A. DeVore, B. Jawerth, V. Popov (1992): Compression of wavelet decompositions. *Amer. J. Math.* **114**, 737–785.

[10] M. Hansen, W. Sickel (2011): Best m-term approximation and Lizorkin-Triebel spaces. *JAT* **163**(8), 923–954.

[11] D. Jerison, C. Kenig (1995): The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.* **130**, 161–219.

[12] G. A. Kalyabin, P. I. Lizorkin (1987): Spaces of generalized smoothness. *Math. Nachr.* **133**, 7–32.

[13] V. A. Kozlov, V. G. Maz'ya, J. Rossmann (1997): Elliptic boundary value problems in domains with point singularities. American Mathematical Society, Rhode Island.

[14] V. A. Kozlov, V. G. Maz'ya, J. Rossmann (2000): Spectral problems associated with corner singularities of solutions to elliptic equations. American Mathematical Society, Rhode Island.

[15] Y. Meyer (1992): Wavelets and operators. Cambridge Studies in Advanced Mathmetics, vol. **37**, Cambridge.

[16] S. Moura (2001): Function spaces of generalized smoothness. *Dissertationes Math.* **398**, 1–88.

[17] A. Mazzucato, V. Nistor (2010): Well posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. *Arch. Ration. Mech. Anal.* **195**, 25–73.

[18] V. S. Rychkov (1999): On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains. *J. London Math. Soc.* **60**, 237–257.

[19] E. M. Stein (1970): Singular integrals and differentiability properties of functions. Princeton Mathematical series **30**. Princeton University Press, Princeton, New Jersey.

[20] H. Triebel (2006): Theory of Function spaces III. Birkhäuser, Basel.

[21] H. Triebel (2008): Function spaces on domains. *EMS Tracts on Mathematics* **7**. EMS Publishing House, Zürich.

Research Reports

No.	Authors/Title
12-41	*M. Hansen* n-term approximation rates and Besov regularity for elliptic PDEs on polyhedral domains
12-40	*D. Schötzau, Ch. Schwab, T. Wihler and M. Wirz* Exponential convergence of hp-DGFEM for elliptic problems in polyhedral domains
12-39	*A. Buffa, G. Sangalli and Ch. Schwab* Exponential convergence of the hp version of isogeometric analysis in 1D
12-38	*R. Hiptmair, A. Moiola, I. Perugia and Ch. Schwab* Approximation by harmonic polynomials in star-shaped domains and exponential convergence of Trefftz hp-DGFEM
12-37	*Cl. Schillings and Ch. Schwab* Sparse, adaptive Smolyak algorithms for Bayesian inverse problems
12-36	*R. Hiptmair and L. Kielhorn* BETL A generic boundary element template library
12-35	*S. Mishra, N.H. Risebro, Ch. Schwab and S. Tokareva* Numerical solution of scalar conservation laws with random flux functions
12-34	*R. Hiptmair, Ch. Schwab and C. Jerez-Hanckes* Sparse tensor edge elements
12-33	*R. Hiptmair, C. Jerez-Hanckes and S. Mao* Extension by zero in discrete trace spaces: Inverse estimates
12-32	*A. Lang, S. Larsson and Ch. Schwab* Covariance structure of parabolic stochastic partial differential equations
12-31	*A. Madrane, U.S. Fjordholm, S. Mishra and E. Tadmor* Entropy conservative and entropy stable finite volume schemes for multi-dimensional conservation laws on unstructured meshes
12-30	*G.M. Coclite, L. Di Ruvo, J. Ernest and S. Mishra* Convergence of vanishing capillarity approximations for scalar conservation laws with discontinuous fluxes
12-29	*A. Abdulle, A. Barth and Ch. Schwab* Multilevel Monte Carlo methods for stochastic elliptic multiscale PDEs
12-28	*E. Fonn, Ph. Grohs and R. Hiptmair* Hyperbolic cross approximation for the spatially homogeneous Boltzmann equation