Exponential convergence of the *hp* version of isogeometric analysis in 1D

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Abstract We establish exponential convergence of the hp-version of isogeometric analysis for second order elliptic problems in one spacial dimension. Specifically, we construct, for functions which are piecewise analytic with a finite number of algebraic singularities at a-priori known locations in the closure of the open domain Ω of interest, a sequence $(\Pi_\sigma^\ell)_{\ell\geq 0}$ of interpolation operators which achieve exponential convergence. We focus on localized splines of reduced regularity so that the interpolation operators $(\Pi_\sigma^\ell)_{\ell\geq 0}$ are Hermite type projectors onto spaces of piecewise polynomials of degree $p \sim \ell$ whose differentiability increases linearly with p. As a consequence, the degree of conformity grows with N, so that asymptotically, the interpoland functions belong to $C^k(\Omega)$ for any fixed, finite k. Extensions to two- and to three-dimensional problems by tensorization are possible.

1 Introduction

Isogeometric Analysis (IGA) is an innovative technique for the discretization of partial differential equations which has been proposed by T.J.R. Hughes *et al.* in 2005 in [6]. IGA is gaining a growing interest in different communities: mechanical engineering, numerical analysis and geometric modeling. In its simplest formulation, IGA consists in solving a PDE with a Galerkin technique projecting onto the space of splines. In this paper, we consider IGA in conjunction with the hp- paradigm, i.e.,

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simultaneous h and p refinements. We prove exponential convergence of IGA for a model elliptic PDE with piecewise analytic solutions.

Exponential convergence of piecewise polynomial approximations with a fixed degree of conformity for analytic functions with a point singularity was shown first for free-knot, variable degree spline approximation in [4, 8] and the references there. Inspired by these results, the hp-version of the finite element method (FEM) for the numerical solution of elliptic problems was proposed in the mid 80ies by I. Babuška and B.Q. Guo (see [1] and the references there). Exponential convergence rates $\exp(-b\sqrt{N})$ with respect to the number of degrees of freedom N for the hp version of the standard, C^0-conforming FEM in one dimension were shown by Babuška and Gui in [5] for the model singular solution $u(x) = x^\alpha - x \in H_0^1(\Omega)$ in $\Omega = (0,1)$. This result required σ-geometric meshes with *any subdivision ratio* $\sigma \in (0,1)$ (in particular, for $\sigma = 1/2$ geometric element sequences Ω_i are obtained by successive element bisection towards $x = 0$) while the constant b in the convergence estimate $\exp(-b\sqrt{N})$ depends on the singularity exponent α as well as on σ. In one space dimension, the results were further refined and optimal values of σ as well as estimates on the actual value of b are known. Among all $\sigma \in (0,1)$, the optimal value was shown in [8, 5] to be $\sigma_{\text{opt}} = (\sqrt{2}-1)^2 \approx 0.17$, see, in particular, [5, Theorem 3.2], provided that the geometric mesh refinement is combined with nonuniform polynomial degrees $p_i \geq 1$ in Ω_i which are *s-linear*, i.e., $p_i \sim si$, with the optimal slope s being $s_{\text{opt}} = 2(\alpha - 1/2)$. In this case, the finite element error converges as $\exp(-b\sqrt{N})$ where $b = 1.76\ldots \times \sqrt{(\alpha - 1/2)}$. For the bisected geometric mesh where $\sigma = 1/2$ and for linear polynomial degree distributions with slope $s_{\text{opt}} = 0.39\ldots \times (\alpha - 1/2)$, one has $b = 1.5632\ldots \times \sqrt{(\alpha - 1/2)}$, whereas for $\sigma = 1/2$ and uniform polynomial degree, $b = 1.1054\ldots \times \sqrt{(\alpha - 1/2)}$; see [5, Table 1]. It was left open in [4, 8, 5] if the convergence rate $\exp(-b\sqrt{N})$ is optimal.

In the present paper, we investigate the rate of convergence of the hp version of isogeometric analysis when local splines are used. Indeed, we consider the space of splines, defined on an open knot vector on $[0,1]$, of degree p and of conformity $\lfloor \frac{p-1}{2} \rfloor$ which is proportional to the polynomial degree p.

In this case, the Hermite-type interpolant and the related analytic convergence estimates proposed in [2] are available and are used here to establish exponential convergence of the hp-version of isogeometric FEM for solutions which are piecewise analytic. Indeed, we prove that for piecewise analytic functions in one space dimension. We prove that for all functions in a countably normed space equipped with a family of weighted Sobolev norms which contain in particular the singular functions $u(x) = x^\alpha - x \in H_0^1(\Omega)$, the interpolation error on reduced spline spaces defined on families $\{\mathscr{G}_\sigma^M\}_{M\geq 1}$ of geometric knot meshes is exponentially decreasing at the rate $\exp(-b\sqrt{N})$. We should note that in terms of the number of degrees of freedom, the approximations we consider are linear, i.e. non-adaptive.

Our numerical examples show that exponential convergence rate $\exp(-b\sqrt{N})$, with respect to the number of degrees of freedom N, is attained. We also show that the constant b is considerably larger than in the case of standard hp-finite elements (with constant p), which enforce merely interelement continuity, but not smoothness across interelement boundaries.

2 Model Problem

In the bounded interval $\Omega = (0,1)$, we consider the model Dirichlet problem

$$-(a(x)u')' + c(x)u = f \quad \text{in} \quad \Omega\ , \quad u(0) = 0,\ u(1) = 0\ . \tag{1}$$

We shall consider the Finite Element discretization of (1) based on the (standard) variational formulation. To this end, we introduce the space $V = H^1_0(\Omega)$. Then the variational formulation of (1) reads: find

$$u \in V : \quad a(u,v) = (f,v) \quad \forall v \in V\ , \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is given by

$$a(w,v) = \int_0^1 (a(x)w'v' + c(x)wv)dx$$

and where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ innerproduct. Assuming that $a,c \in \mathsf{L}^\infty(\Omega)$ and positivity of $a(x)$, ie.,

$$\operatorname{ess\,inf}\{a(x) : x \in \Omega\} \geq \underline{a} > 0\ , \qquad \operatorname{ess\,inf}\{c(x) : x \in \Omega\} \geq 0\ , \tag{3}$$

there hold continuity and coercivity, i.e. exists $C(\underline{a}) > 0$ such that, for every $v,w \in V$ holds

$$a(v,v) \geq C(\underline{a})\|v\|^2_{H^1(\Omega)}\ , \quad |a(v,w)| \leq \max\{\|a\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\}\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}\ , \tag{4}$$

and by the Lax-Milgram Lemma, for every $f \in (H^1_0(\Omega))^*$ exists a unique solution of (2).

Let $\{V_\ell\}_{\ell=0}^\infty$ denote a sequence of subspaces $V_\ell \subset V$ of finite dimensions $N_\ell = \dim V_\ell$. Below, we shall be interested in particular in the case when the subspaces are nested, i.e. when

$$V_0 \subset V_1 \subset \ldots \subset V_\ell \subset \ldots \subset V$$

and that V_ℓ are dense, i.e.

$$\forall u \in V : \quad \lim_{\ell\to\infty} \inf_{v_\ell \in V_\ell} \|u - v_\ell\|_{H^1(\Omega)} = 0\ . \tag{5}$$

By (4), for every $\ell \geq 0$, there exists a unique Finite Element solution of the Galerkin approximation of (2) find

$$u_\ell \in V_\ell : \quad a(u_\ell, v) = (f,v) \quad \forall v \in V_\ell \tag{6}$$

which is quasioptimal, i.e there holds

$$\|u - u_\ell\|_{H^1(\Omega)} \leq C \inf_{v_\ell \in V_\ell} \|u - v_\ell\|_{H^1(\Omega)}\ . \tag{7}$$

We next quantify, for particular classes of data f in (1), the regularity of solutions. Subsequently, we shall exhibit choices of FE spaces V_ℓ for which high (exponential) rates of convergence can be achieved.

3 Regularity

We shall be in particular interested in *piecewise analytic* solutions u of (1) which exhibit singularities at $x=0$ (multiple, but finitely many, singularities in $\overline{\Omega}$ could be also considered and everything that follows will apply to this case with straightforward modifications).

To quantify the analytic regularity, for $x \in \Omega = (0,1)$, we consider the weight function $\Phi_\beta(x) = x^\beta$, $\beta \in \mathbb{R}$. For integer $\ell \geq 0$ and for $k = \ell, \ell+1, ...$, we define the weighted seminorms

$$|u|_{H^{k,\ell}_\beta(\Omega)} = \|\Phi_{\beta+k-\ell} D^k u\|_{L^2(\Omega)} \tag{8}$$

and the weighted norms $\|u\|_{H^{k,\ell}_\beta(\Omega)}$ by

$$\|u\|^2_{H^{k,\ell}_\beta(\Omega)} = \begin{cases} \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{k=\ell}^{m} |u|^2_{H^{k,\ell}_\beta(\Omega)} & \text{if} \quad \ell > 0\,, \\ \sum_{k=\ell}^{m} |u|^2_{H^{k,\ell}_\beta(\Omega)} & \text{if} \quad \ell = 0\,. \end{cases} \tag{9}$$

We shall be interested in classes of functions u which are analytic in $(0,1]$ with a point singularity at $x=0$ as follows.

Definition 1. We say that $u \in \mathscr{B}^\ell_\beta(\Omega)$ if $u \in \bigcap_{m \geq \ell} H^{m,\ell}_\beta(\Omega)$ and if there exist constants $C_u > 0$, $d_u \geq 1$ such that

$$|u|_{H^{k,\ell}_\beta(\Omega)} \leq C_u d_u^{k-\ell} (k-\ell)! \quad k = \ell, \ell+1, \dots\,. \tag{10}$$

Functions u in the set $\mathscr{B}^\ell_\beta(\Omega)$ are analytic in $(0,1]$ with possibly an algebraic singularity at $x=0$. It follows directly from the definition that for $0 < \beta < 1$ and for $\ell \geq 1$ it holds that $\mathscr{B}^\ell_\beta(\Omega) \subset H^{\ell-1}(\Omega)$.

The spaces $\mathscr{B}^\ell_\beta(\Omega)$ (and closely related spaces $\mathfrak{C}^2_\beta(\Omega)$) were introduced in [1], in plane polygonal domains with curved boundaries.

For the model problem (1), piecewise analyticity of the right hand side f implies corresponding smoothness of the solution u. We have the following precise regularity result.

Theorem 1. *If the coefficient functions a and c in* (1) *are analytic in* $\overline{\Omega}$ *and if* (3) *holds, then for every* $f \in \mathscr{B}^0_\beta(\Omega)$ *for some* $0 < \beta < 1$*, the unique weak solution* $u \in H^1_0(\Omega)$ *of* (1) *belongs to* $\mathscr{B}^2_\beta(\Omega)$.

Proof. The proof of the $\mathscr{B}^2_\beta(\Omega)$ regularity follows from the integral representation of the exact solution u of (1), and from the assumed analyticity of the coefficient functions $a(x)$ and $c(x)$ in $\overline{\Omega}$.

4 *hp*-isogeometric FEM

By the quasioptimality (7), the error in the FE approximations of the solution u is bounded by the best approximation of u in the $H^1(\Omega)$ norm.

We shall be interested in establishing *exponential* convergence rates for approximations of functions $u \in \mathscr{B}^\ell_\beta(\Omega)$ from spaces of piecewise polynomials in Ω, expressed in terms of the number of degrees of freedom, i.e. of their dimension N. As indicated in the introduction, particular attention will be paid to *smoothest hp-approximations*, i.e to Finite Element spaces with substantial extra regularity beyond the (minimal) C^0-interelement regularity which is necessary for conformity $V_\ell \subset V$.

We start by introducing notation for meshes, polynomial degrees and interelement conformity. We denote by $\{\Omega_j : j = 1,...,M\}$ a partition of Ω into open, nonempty intervals such that $\overline{\Omega} = \bigcup_{j=1}^M \overline{\Omega}_j$. We denote $\Omega_j = (x_{j-1}, x_j)$, with the endpoints given by $0 = x_0 < x_1 < x_2 < ... < x_M = 1$. We denote by $h_j = |\Omega_j| = x_j - x_{j-1}$. On Ω_j, we consider spaces of polynomial functions of degree at most $p_j \geq 1$, denoted by $\mathbb{P}_{p_j}(\Omega_j)$. We collect the polynomial degrees p_i in a *degree vector* $\mathbf{p} = \{p_j\}_{j=1}^M$. At the node $x_j = \overline{\Omega_j} \cap \overline{\Omega_{j+1}}$, $j = 1,2,...,M-1$, we enforce *interelement compatibility* of orders $0 \leq k_j \leq p_j \wedge p_{j-1}$ by the condition

$$[\![u^{(m)}]\!](x_j) = 0 \quad m = 0,1,2,...,k_j - 1\ . \tag{11}$$

We combine also the interelement compatibilities into the *conformity vector* $\mathbf{k} = \{k_j\}_{j=1}^{M-1}$ and all elements Ω_j into the mesh $\mathscr{T} = \{\Omega_j\}_{j=1}^M$. Then, we denote

$$S^{\mathbf{p}}_{\mathbf{k}}(\Omega;\mathscr{T}) := \left\{ v \in L^2(\Omega) : v|_{\Omega_j} \in \mathbb{P}_{p_j}(\Omega_j)\ ,\ [\![v^{(m)}]\!](x_j) = 0 \text{ for } j = 1,2,...,M-1,\ m = 0,1,...,k_j - 1 \right\}\ . \tag{12}$$

The number of degrees of freedom in the space $S^{\mathbf{p}}_{\mathbf{k}}(\Omega;\mathscr{T})$ is easily seen to be

$$N = \dim(S^{\mathbf{p}}_{\mathbf{k}}(\Omega;\mathscr{T})) = \sum_{j=1}^M (p_j + 1) - \sum_{j=1}^{M-1} k_j\ .$$

If $p_j = p \geq 1$ and if $k_j = k \geq 0$ for all j, we also write $S^p_k(\Omega;\mathscr{T})$ in place of $S^{\mathbf{p}}_{\mathbf{k}}(\Omega;\mathscr{T})$. Then

$$N = \dim(S^p_k(\Omega;\mathscr{T})) = M(p+1) - (M-1)k\ . \tag{13}$$

In the context of approximation of functions $u \in \mathscr{B}^\ell_\beta(\Omega)$, we shall use so-called *geometric meshes* $\mathscr{T}$. We say that a mesh $\mathscr{T}$ is a *geometric mesh on Ω with $M > 1$*

layers if $x_j = \sigma^{M-j}$ for $j = 1,2,...,M$ for a *geometric grading factor* $\sigma \in (0,1)$. We denote such meshes by $\mathscr{T} = \mathscr{G}_\sigma^M$ and note that $h_1 = x_1 = \sigma^{M-1}$ and that for $j \geq 2$ it holds that $h_j = x_j - x_{j-1} = \lambda x_{j-1}$ where $\lambda = (1-\sigma)/\sigma = \sigma^{-1} - 1$ is independent of j. We observe that $h_j = \lambda x_{j-1}$ for $j = 2,3,...,M$.

5 Basic Local Interpolation Operators

We obtain convergence rate estimates by constructing global hp interpolation operators with high conformity $\mathbf{k}$. As usual in FE analysis, these operators will be built from local, i.e. elemental interpolation operators which are constructed and analyzed on the reference element $\Lambda = (-1,1)$, and then transported to the physical elements $\Omega_j \in \mathscr{G}_\sigma^M$ by an affine mapping. We will give two constructions: the first one is based on a spectral-like elemental approximation proposed in [2], whereas the second will be based on a classical, nodal interpolation operator.

For $u \in \mathscr{B}_\beta^\ell(\Omega)$, we construct a family of spectral hp-interpolants based on a construction which was introduced in [2]. These interpolants are based on L^2 projections on spaces of discontinuous polynomials of a certain derivative of the function to be interpolated, and by subsequent enforcement of interelement conformity of order k_i. This interpolant is a generalization of the one proposed in [7, Chapter 3] for the analysis of C^0-conforming hp-FEM. We require the following result which is Corollary 2 in [2]. For simplicity, we will work under the assumption (which could be weakened) that $p_i = p \geq 1$ and $k_i = k > 0$ for all i.

Proposition 1. *Let $\Lambda = (-1,1)$, $p,k,s \geq 0$ integers, with polynomial degree $p \geq \max\{0, 2k-1\}$, and with $\kappa := p-k+1$. Then there exists a quasi-interpolation operator $\hat{\pi}_k^p$ such that, for any function $\hat{u} : \Lambda \mapsto \mathbb{R}$ such that $\hat{u}^{(k)} \in H^s(\Lambda)$, for every $0 \leq s \leq \kappa$ holds the interpolation error bound*

$$\left\| \hat{u}^{(j)} - \left(\hat{\pi}_k^p \hat{u}\right)^{(j)} \right\|_{L^2(\Lambda)}^2 \leq \frac{(\kappa-s)!}{(\kappa+s)!} \frac{(\kappa-(k-j))!}{(\kappa+(k-j))!} \left\| \hat{u}^{(k)} \right\|_{H^s(\Lambda)}^2 , \quad j = 0,1,...,k . \tag{14}$$

Moreover, the interpolating polynomial $\hat{\pi}_k^p \hat{u}$ satisfies

$$\left(\hat{\pi}_k^p \hat{u}\right)^{(j)} (\pm 1) = \hat{u}^{(j)}(\pm 1) , \quad j = 0,1,...,k-1 \tag{15}$$

(with (15) *understood to be void in the case $k = 0$).*

We specialize this general result to maximal smoothness. To avoid fractional bounds for indices, we substitute in (14)

$$p = 2q - 1 , \quad q \geq 1 .$$

Then $1 \leq k$ and $2k - 1 \leq 2q - 1$. Therefore, we may choose in (14) $k = q \geq 1$. This implies that $\kappa = q$ and that $0 \leq s \leq p$. This implies, upon scaling (14) to a generic interval $J = (a,b) \subset \Omega = (0,1)$ of length $h = b - a > 0$ that there holds, for every

$0 \le s \le q$, $0 \le j \le q$, $q \ge 1$, the interpolation error bound

$$\left\| u^{(j)} - \left(\pi_q^{2q-1} u\right)^{(j)} \right\|_{L^2(J)}^2 \le \left(\frac{h}{2}\right)^{2(q+s-j)} \frac{(q-s)!}{(q+s)!} \frac{i!}{(2q-j)!} \left\| u^{(q+s)} \right\|_{L^2(J)}^2 . \quad (16)$$

By (15), the interpolant $\hat{\pi}_q^{2q-1}$ ensures interelement continuity of, roughly speaking, the first $k-1 = q-1 = O(p/2)$ many derivatives of the piecewise polynomial, interpolating function. As increasing conformity of the interpolating function reduces the dimension N of the subspaces $S_{\mathbf{k}}^{\mathbf{p}}(\Omega, \mathscr{T})$, it is readily verified from (13), that conformity of order $O(p/2)$ will imply

$$\dim(S_q^{2q-1}(\Omega, \mathscr{G}_\sigma^M)) = O((M+1)p/2)$$

as $M, p \to \infty$. If, in particular, $M = O(p)$ (as will be the case in hp-FEM), we find $\dim(S_q^{2q-1}(\Omega, \mathscr{G}_\sigma^M)) = O(p^2)$. From (13) it is thus evident that an asymptotic complexity reduction of hp-FEM is only possible for $k \ge p - \overline{k}$, i.e. for *subspaces* $S_k^p(\Omega, \mathscr{G}_\sigma^M)$ *whose conformity orders k equal, up to an absolute gap* $\overline{k}$*, with p*. If this gap is proportional to (any power of) p, $\dim(S_k^p(\Omega, \mathscr{G}_\sigma^M))$ will scale polynomially in p for $M = O(p)$ elements in the mesh $\mathscr{G}_\sigma^M$.

6 Exponential Convergence

We now “assemble” scaled versions of the elementwise quasi-interpolation projectors π_k^p into corresponding global interpolators Π_k^p and prove exponential convergence of these projectors for functions $u \in \mathscr{B}_\beta^2(\Omega)$, from within the hp-FE space $S_q^p(\Omega; \mathscr{G}_\sigma^p)$ where $p = 2q-1 \ge 1$ under the *provision that the geometric grading factor* $0 < \sigma < 1$ *is sufficiently large,*, depending on the constant d_u in (10), following the analysis in [7, Chap. 3]. From these results, exponential convergence of the hp-FEM will follow via (7).

We start by considering a generic element $J = (a,b) \in \mathscr{G}_\sigma^p$ not abutting at the singular support $x = 0$ of u. If $u \in \mathscr{B}_\beta^2(\Omega)$, we have for any such J and any integer $s \ge 0$ that

$$|u|_{H_\beta^{s+1,\ell}(\Omega)}^2 \ge \|u^{(s+1)} \Phi_{\beta+s+1-\ell}\|_{L^2(J)}^2 \ge a^{2(\beta+s+1-\ell)} \|u^{(s+1)}\|_{L^2(J)}^2 .$$

This implies that there holds with $\ell = 2$

$$\|u^{(s+1)}\|_{L^2(J)}^2 \le a^{-2(\beta+s+1-\ell)} |u|_{H_\beta^{s+1,\ell}(\Omega)}^2 . \quad (17)$$

We replace now s with $q+s-1$, $j = 0,1$ and $\ell = 2 > j$ to obtain

$$\|u^{(q+s)}\|_{L^2(J)}^2 \le a^{-2(\beta+q+s-\ell)} |u|_{H_\beta^{q+s,\ell}(\Omega)}^2 . \quad (18)$$

Using this bound for $J = \Omega_i \in \mathscr{G}^p_\sigma$, $2 \le i \le p$, we get from (16) the bound

$$\left\| u^{(j)} - (\pi_q^{2q-1} u)^{(j)} \right\|^2_{L^2(\Omega_i)} \le \left(\frac{h_i}{2}\right)^{2(q+s-j)} \frac{(q-s)!}{(q+s)!(2q-j)!} \|u^{(q+s)}\|^2_{L^2(\Omega_i)}\,, \quad j = 0,1\,. \tag{19}$$

Using here (17), we find with (10) the bound

$$\begin{aligned} \left\| u^{(j)} - (\pi_q^{2q-1} u)^{(j)} \right\|^2_{L^2(\Omega_i)} &\le \left(\frac{h_i}{2}\right)^{2(q+s-j)} \frac{(q-s)!}{(q+s)!(2q-j)!} x_{i-1}^{-2(\beta+q+s-\ell)} |u|^2_{H^{q+s,\ell}_\beta(\Omega)} \\ &\le C \frac{(q-s)!((q+s-\ell)!)^2}{(q+s)!(2q-j)!} x_{i-1}^{-2(\beta+j-\ell)} \left(\frac{\lambda d_u}{2}\right)^{2(q+s-j)}\,. \end{aligned} \tag{20}$$

Since for $i = 2,3,...,M$ it holds $x_i = \sigma^{M-i}$, we can write for $j = 0,1$ and for $\ell = 2$

$$\begin{aligned} &\left\| u^{(j)} - (\pi_q^{2q-1} u)^{(j)} \right\|^2_{L^2(\Omega_i)} \le C \frac{(q-s)!((q+s-\ell)!)^2}{(q+s)!(2q-j)!} \sigma^{2(M-i+1)(\ell-\beta-j)} \left(\frac{\lambda d_u}{2}\right)^{2(q+s-j)} \\ &\le C \underbrace{\frac{(q-s)!((q+s-\ell)!)^2}{(q+s)!(2q-j)!}}_{*} \sigma^{2M(1-\beta)} \left(\frac{\lambda d_u}{2}\right)^{2(q+s-j)} \sigma^{2(1-i)(1-\beta)}\,. \end{aligned}$$

Remark 1. Note that so far, in all error bounds the differentiation order s is an integer. In what follows, we shall use also norms of fractional order s for which the corresponding error bounds can be obtained by classical interpolation arguments.

We now analyze the factorial expression $*$.

Lemma 1. *There is a constant $C > 0$ such that for every $q \ge 1$ and for the choice $s = \alpha q$ for some $0 < \alpha < 1$ there holds the estimate*

$$|*| \le C G(\alpha)^q\,, \quad \textit{where} \quad G(\alpha) := \frac{1}{4}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}\,. \tag{21}$$

Here, the factorial expressions in $*$ are continued to noninteger arguments obtained by choosing $s = \alpha q$ using the Gamma function.

Proof. Throughout the proof, $C > 0$ and $\lesssim$ denotes a constant and a bound, respectively, which are independent of s, p and of n. We start by recalling the *Stirling inequalities*

$$\forall n \in \mathbb{N}: \quad \frac{n!e^n}{e\sqrt{n}} \le n^n \le \frac{n!e^n}{\sqrt{2\pi n}}$$

which imply $n! \ge n^n e^{-n}\sqrt{2\pi n} \ge c n^{n+1/2} e^{-n}$, and $n! \le n^n e^{-n} e\sqrt{n} \le c n^{n+1/2} e^{-n}$. We then estimate $(*)$ in the case $j = 1$ as follows:

$$
\begin{aligned}
|(*)| &\leq C2q \frac{(q-s)^{q-s+1/2} e^{-(q-s)} \left[(q+s-2)^{(q+s-2)+1/2} e^{-q-s+2} \right]^2}{(q+s)^{q+s+1/2} e^{-q-s} (2q)^{2q+1/2} e^{-2q}} \\
&\leq Cq \frac{(q-s)^{q-s} (q+s)^{2(q+s)-s}}{(q+s)^{q+s} (2q)^{2q+1/2}} \\
&\leq Cq^{1/2} (q+s)^{-3} \frac{(q-s)^{q-s} (q+s)^{q+s}}{(2q)^{2q}} \ .
\end{aligned}
$$

In this bound, we now *choose* $s = \alpha q$ for some $0 < \alpha < 1$ to be selected.
Then

$$
q^{1/2}(q+s)^{-3} = q^{1/2} q^{-s} (1+\alpha)^{-3} \leq C(\alpha) < \infty \ .
$$

Therefore

$$
\begin{aligned}
|(*)| &\leq C \frac{(q-s)^{q-s}(q+s)^{q+s}}{(2q)^{2q}} = C \frac{(q^2-s^2)^q (q+s)^s (q-s)^{-s}}{(2q)^{2q}} \\
&= C \frac{q^{2q}(1-\alpha^2)^q q^{\alpha q} (1+\alpha)^{\alpha q} q^{-\alpha q} (1-\alpha)^{-\alpha q}}{(2q)^{2q}} \\
&= 2^{-2q} (1-\alpha^2)^q \left(\frac{1+\alpha}{1-\alpha} \right)^{\alpha q} \\
&= \left[\frac{1-\alpha^2}{4} \left(\frac{1+\alpha}{1-\alpha} \right)^{\alpha} \right]^q = \left[\frac{1}{4} (1+\alpha)^{1+\alpha} (1-\alpha)^{1-\alpha} \right]^q = [G(\alpha)]^q \ .
\end{aligned}
$$

We observe that the function $G(\cdot)$ defined in (21) satisfies for $\alpha \in [0,1]$

$$
1/4 \leq G(\alpha) \leq 1 \ , \quad G(0) = \frac{1}{4} \ , \quad \text{and} \quad \lim_{\alpha \to 1^-} G(\alpha) = 1 \ .
$$

Inserting the bound obtained in Lemma 1 into (20), we find

$$
\begin{aligned}
\left\| (u - \Pi_q^{2q-1} u)^{(j)} \right\|^2_{L^2(x_1,1)} &= \sum_{i=2}^{M} \left\| u^{(j)} - (\pi_q^{2q-1} u)^{(j)} \right\|^2_{L^2(\Omega_i)} \\
&\leq C(\alpha, d_u) G(\alpha) q \left(\frac{\lambda d_u}{2} \right)^{2q(1+\alpha)} \left(\frac{\lambda d_u}{2} \right)^{-2j} \sum_{i=2}^{M} \sigma^{2(M+1-i)(1-\beta)} \\
&\leq C(\alpha, \sigma, d_u) \left[G(\alpha) \left(\frac{\lambda d_u}{2} \right)^{2(1+\alpha)} \right]^q \ .
\end{aligned}
$$

For $0 < \sigma < 1$, we have

$$
\lambda d_u = \left(\frac{1}{\sigma} - 1 \right) d_u < 2
$$

if

$$
1 > \sigma > (1 + 2/d_u)^{-1} > 0 \ . \tag{22}
$$

Choosing $\alpha > 0$ sufficiently small, we find for such σ with (22) that

$$\left[G(\alpha)\left(\frac{\lambda(\sigma)d_u}{2}\right)^{2(1+\alpha)}\right] \le F(\sigma,d_u) < 1\ .$$

We have proved

Theorem 2. *Assume that $u \in \mathscr{B}^2_\beta(\Omega)$ for some $0 < \beta \le 1$, $\Omega = (0,1)$, and that* (10) *holds with some $C_u, d_u > 0$. Then, for any $0 < \sigma < 1$ satisfies* (22)*, there exist $b, C > 0$ such that there holds for all $p \ge 1$*

$$\inf_{v \in S^{\mathbf{p}}_q(\Omega,\mathscr{G}^p_\sigma)} \|u - v\|_{H^1(\Omega)} \le C \exp(-b(\sigma,\beta)p) \tag{23}$$

Here, $\mathbf{p} = (q+1, p, ..., p)$ with $p = 2q-1$ and $q \in \mathbb{N}$. By (13)*, in terms of the number N of degrees of freedom, we have the estimate*

$$\inf_{v \in S^{\mathbf{p}}_q(\Omega,\mathscr{G}^p_\sigma)} \|u - v\|_{H^1(\Omega)} \le C' \exp(-b'(\sigma,\beta)\sqrt{N}) \tag{24}$$

with possibly different constants $b', C' > 0$.

Proof. The assertion follows from the previous bounds on the elementwise interpolation error in $(x_1, 1)$, *for either of the global interpolation operators*, i.e. for $\Pi^{\mathbf{p}}_{\mathbf{k}}$ and a Hardy-Type estimate (e.g. (3.3.68) in [7]) in $\Omega_1 = (0, x_1) = (0, \sigma^p)$.

7 Numerical results

In this section we present numerical results in one space dimension with the aim of demonstrating the validity of the convergence theorems presented in the previous section, and also to add some insight on the numerical behavior of the *hp*-isogeometric method.

Instead of dealing directly with the interpolation operator, we show numerical errors of the Galerkin projection u_h of u onto spline spaces, obtained solving the simple Poisson problem

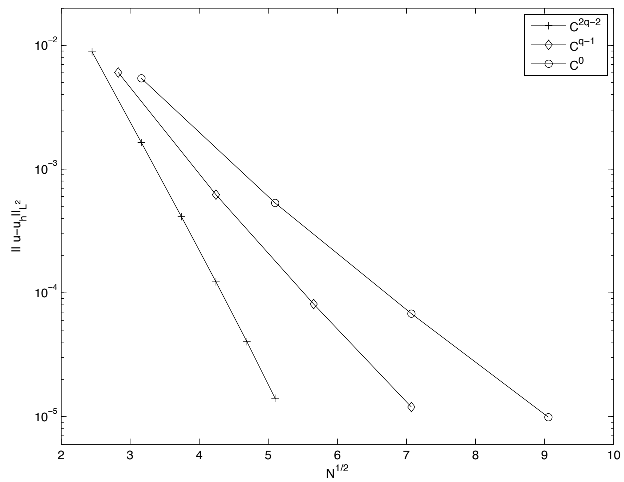
$$-u'' = f \quad \text{in} \quad \Omega = (0,1)\ , \quad u(0) = 0, u(1) = 0, \tag{25}$$

with right hand side f is selected in order to have exact solution $u = x^{0.6} - x$.

We first set $\sigma = 1/2$ and plot in Figure 1 the corresponding L^2-error for the three cases $u_h \in S^{2q-1}_1(\Omega, \mathscr{G}^{2q-1}_\sigma)$, $u_h \in S^{2q-1}_q(\Omega, \mathscr{G}^{2q-1}_\sigma)$, $u_h \in S^{2q-1}_{2q-1}(\Omega, \mathscr{G}^{2q-1}_\sigma)$, that is, for C^0, C^{q-1} (the maximum regularity allowed by our interpolant) and C^{2q-2} continuity respectively. We clearly see exponential convergence $\|u - u_h\|_{L^2} = C\exp(-b\sqrt{N})$ in all cases, with higher b for higher regularity.

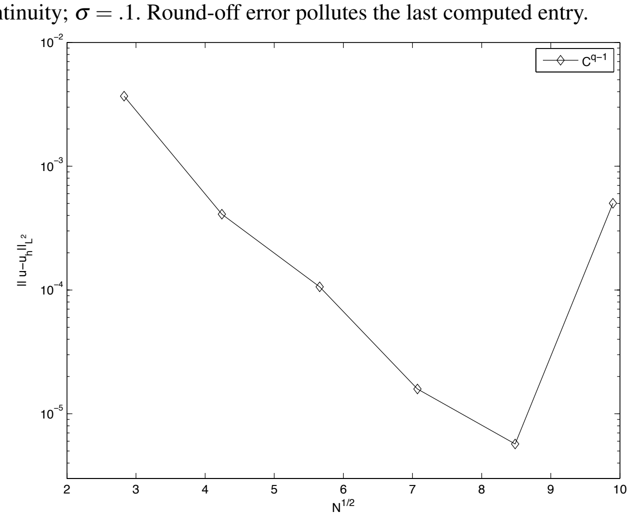
Furthermore, we want to investigate numerically the sharpness of condition (22). It is very easy to see that for our choice $u(x) = x^{-0.6} - x$, the corresponding d_u is equal to 1 and thus, (22) prescribes $\sigma > 1/3$. We present in Figure 2 the error plot

Fig. 1 L^2-error of the Galerkin projection versus degrees-of-freedom number N: comparison between splines and standard finite element approximation.



for $u_h \in S_q^{2q-1}(\Omega, \mathscr{G}_\sigma^{2q-1})$ and $\sigma = .1$. The plot shows exponential convergence, if we exclude the last computed value, where u_h is strongly affected by round-off error. Indeed, for this last plot entry we have $q = 6$, that is degree 13, minimum mesh-size $.1^{12} = 10^{-12}$, and the condition number of the linear system gets $> 10^{17}$.

Fig. 2 L^2-error of the Galerkin projection versus degrees-of-freedom number N for degree $2q-1$ and C^{q-1} continuity; $\sigma = .1$. Round-off error pollutes the last computed entry.



In conclusion, there is numerical evidence that the mesh condition (22) is not necessary to approximate a singular solution in the space $S^{2q-1}_{2q-1}(\Omega, \mathscr{G}^{2q-1}_{\sigma})$. However, we also stress that the B-spline basis is not suitable for a strong geometric grading and high degree, since it induces a severe ill-conditioning of the linear system of equations.

We observe that each of the proposed hp-FEM discretizations converge with exponential rate $\exp(-b\sqrt{N})$. The constant $b > 0$ in the exponential rate depends on σ and on the conformity in the hp-space. In particular, the hp-IGA will afford substantially larger values of b in the exponential convergence bound, leading to several orders of magnitude error reduction at a given budget of N degrees of freedom over standard, C^0-conforming hp-FEM. The increased efficiency is at the expense of a rather large condition number of the (small) linear systems of equations which, presumably, is due to the use of spline-bases in the current implementation of hp-IGA. These work savings by enhanced conformity afforded by the hp-version of isogeometric analysis are expected to be even more pronounced in two and three spatial dimensions, and the unfavourable conditioning of the stiffness matrices is expected to be alleviated by the use of an orthonormal basis. We remark in closing that the present analysis was detailed only for analytic solutions $u(x)$ with a point singularity at $x = 0$. Exactly the same results hold for solutions with a finite number of singularities in $\overline{\Omega}$, provided that the geometric meshes are refined to singular point of $u(x)$; we do not present a detailed statement here.

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References

1. I. Babuška and B.Q. Guo, The *h-p* version of the finite element method for domains with curved boundaries, SIAM J. Numer. Anal., **25**(1988) 837–861.
2. L. Beirao da Veiga, A. Buffa, J. Rivas, G. Sangalli, Some estimates for h-p-k-refinement in Isogeometric Analysis. Numer. Mathem. **118** (2011), no. 2, 271–305.
3. L. Beirao da Veiga, D. Cho, and G. Sangalli. Anisotropic NURBS approximation in isogeometric analysis. Comput. Meth. Appl. Mech. Eng., **209** (2012), 1–11.
4. W. Dahmen and K. Scherer, Best approximation by piecewise polynomials with variable knots and degrees, Journ. Approx. Theory **26** (1979) pp. 1–13.
5. W. Gui and I. Babuška, The *h*, *p* and *h-p* versions of the finite element method in 1 dimension. II. The error analysis of the *h*- and *h-p* versions, Numerische Mathematik **49**(1986) 613–657.
6. T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 4135–4195.
7. C. Schwab, *p*- and *hp*-FEM – Theory and Application to Solid and Fluid Mechanics", Oxford University Press, Oxford, 1998.
8. K. Scherer, On optimal global error bounds obtained by scaled local error estimates, Numerische Mathematik, **36** (1981) pp. 257–277.

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