High order Galerkin approximations for parametric second order elliptic partial differential equations

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VICTOR NISTOR   AND CHRISTOPH SCHWAB

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ABSTRACT. Let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with piecewise smooth boundary ∂D and let U be an open subset of a Banach space Y. We consider a parametric family P_y of uniformly strongly elliptic, parametric second order partial differential operators P_y on D in divergence form, where the parameter y ranges in the parameter domain U so that, for a given set of data f_y, the solution u and the coefficients of the parametric boundary value problem $P_y u = f_y$ are functions of $(x, y) \in D \times U$. Under suitable regularity assumptions on these coefficients and on the source term f, we establish a regularity result for the solution $u : D \times U \to \mathbb{R}$ of the parametric, elliptic boundary value problem $P_y u(x, y) = f_y(x) = f(x, y)$, $x \in D$, $y \in U$, with mixed Dirichlet-Neumann boundary conditions. Let $\partial D = \partial_d D \cup \partial_n D$ denote decomposition of the boundary into a part on which we assign Dirichlet boundary conditions and the part on which we assign Neumann boundary conditions. We assume that $\partial_d D$ is a finite union of closed polygonal subsets of the boundary such that no adjacent faces have Neumann boundary conditions (*i. e.,* there are no Neumann-Neumann corners or edges). Our regularity and well-posedness results are formulated in a scale of weighted Sobolev spaces $\mathcal{K}_{a+1}^{m+1}(D)$ of Kondrat'ev type in D. We prove that the parametric, elliptic PDEs $(P_y)_{y \in U}$ admit a shift theorem which is uniform in the parameter sequence $y \in U$. Specifically, if coefficients $a_{pq}^{ij}(x, y)$ depend on the parameter sequence $y = (y_k)_{k \geq 1}$ in an affine fashion, ie. $a_{pq}^{ij} = a_{pq0}^{\bar{ij}} + \sum_{k \geq 1} y_k \psi_{pqk}^{ij}$, and if the sequences $\|\psi_{pqk}^{ij}\|_{W^{m,\infty}(D)}$ are p-summable for some $0 < p < 1$, then the parametric solution u_y admit an expansion into tensorized Legendre polynomials $L_\nu(y)$ such that the corresponding coefficient sequence $\mathbf{u} = (u_\nu) \in \ell^p(\mathcal{F}; \mathcal{K}_{a+1}^{m+1}(D))$. Here, we denote by $\mathcal{F} \subset \mathbb{N}_0^{\mathbb{N}}$ the set of sequences $\{k_n\}_{n \in \mathbb{N}}$ with $k_n \in \mathbb{N}_0$ with only finitely many non-zero terms, and by $Y = \ell^\infty(\mathbb{N})$ and $U = B_1(Y)$, the open unit ball of Y. We identify the parametric solution u with its coefficient vector $\mathbf{u} = (u_\nu)_{\nu \in \mathcal{F}}$, $u_\nu \in V$, in the "polynomial chaos" expansion with respect to tensorized Legendre polynomials on U. We also show quasi-optimal algebraic orders of convergence for Finite Element approximations of the parametric solutions $u(y)$ from suitable Finite Element spaces in two and three dimensions.

Let $t = m/d$ and $s = 1/p - 1/2$ for some $p \in (0, 1]$ such that $\mathbf{u} = (u_\nu) \in \ell^p(\mathcal{F}; \mathcal{K}_{a+1}^{m+1}(D))$. We then show that, for each $m \in \mathbb{N}$, exists a sequence $\{S_\ell\}_{\ell \geq 0}$ of nested, finite dimensional spaces $S_\ell \subset L^2(U, \mu; V)$ such that $M_\ell = \dim(S_\ell) \to \infty$ and such that the Galerkin projections $u_\ell \in S_\ell$ of the solution u onto S_ℓ satisfy

$$\|u - u_\ell\|_{L^2(U,\mu;V)} \leq C \dim(S_\ell)^{-\min\{s,t\}} \|f\|_{H^{m-1}(D)} \ .$$

The sequence S_ℓ is constructed using a nested sequence $V_\mu \subset V$ of Finite Element space in D with graded mesh refinements toward the singular boundary points of the domain D as in [7, 9, 27]. Our sequence V_μ is independent of y. Each subspace S_ℓ is then defined by a finite subset $\Lambda_\ell \subset \mathcal{F}$ of "active polynomial chaos" coefficients $u_\nu \in V$, $\nu \in \Lambda_\ell$ in the Legendre chaos expansion of u which, in turn, are approximated by $v_\nu \in V_{\mu(\ell,\nu)}$ for each $\nu \in \Lambda_\ell$, with a suitable choice of $\mu(\ell, \nu)$.

Contents

INTRODUCTION

We study the Finite Element approximations of strongly elliptic, parametric mixed boundary value problems on a domain $D \subset \mathbb{R}^d$, $d = 2, 3$. The domain D will be assumed to be piecewise smooth and bounded. The parameter space U will be an open subset of a Banach space Y. Thus, for each $y \in U$, we are given a second order, uniformly strongly positive, parametric partial differential operator P_y on D. The solution u and the coefficients of the parametric operator $P = (P_y)_{y \in U}$ are functions of $(x, y) \in D \times U$.

Under suitable regularity assumptions on these coefficients and on the source term f, we establish in Section 3 a regularity and well-posedness result for the solution $u : D \times U \to \mathbb{R}$ of the parametric, elliptic boundary value problem $P_y u(x, y) = f(x, y)$, $x \in D$, $y \in U$, with mixed Dirichlet-Neumann boundary conditions. We assume that our boundary conditions are such that no adjacent faces have Neumann boundary conditions (*i. e.*, there are no Neumann-Neumann corners or edges). Our regularity result is formulated in a scale of weighted Sobolev spaces $\mathcal{K}_{a+1}^{m+1}(D)$ of Kondrat'ev type in D, for which we prove that our elliptic PDEs $(P_y)_{y \in U}$ admit a shift theorem. We show how this regularity result leads to algebraic orders of convergence for the Finite Element approximations of u in two dimensions. Our main applications are the parametric diffusion equations or the Lamé-Navier equations of linearized elasticity with inhomogeneous coefficients and with Dirichlet or suitable mixed boundary conditions.

Let μ be a probability measure on U and let $V \subset H^1(D)$ be a closed subspace in which our elliptic PDEs $(P_y)_{y \in U}$ admit a unique solution. We define $\mathcal{F} \subset \mathbb{N}_0^{\mathbb{N}}$ to be the space of sequences that have only finitely many non-zero terms. Let us assume that $Y = \ell^\infty(\mathbb{N})$ and $U = B_1(Y)$, the open unit ball of Y. Then we may identify our parametric solution u with its coefficient vector $(u_\nu)_{\nu \in \mathcal{F}}$, $u_\nu \in V$, in the "polynomial chaos" expansion with respect to tensorized Legendre polynomials on U (indexed by the countable set $\mathcal{F}$). Let $t = m/d$, $d = 2, 3$, $m \geq 1$ fixed, and $s = 1/p - 1/2$ for some $p \in (0, 1]$ such that $u = (u_\nu)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F}; \mathcal{K}_{a+1}^{m+1}(D))$. We then construct a sequence of finite dimensional spaces $S_\ell \subset L^2(U, \mu; V)$, $\ell = 1, 2, ...$, based on polynomials of degree m, such that $\dim(S_\ell) \to \infty$ and prove that the $L^2(U, \mu; V)$ Galerkin projections $u_\ell \in S_\ell$ of the solution u onto S_ℓ satisfy

$$\|u - u_\ell\|_{L^2(U,\mu;V)} \leq C \dim(S_\ell)^{-\min\{s,t\}} \|f\|_{H^{m-1}(D)} .$$

For s large (i.e for $p > 0$ close to 0), we therefore recover the optimal rate of convergence $\dim(S_\ell)^{-t} = \dim(S_\ell)^{-m/d}$, where $d = 2, 3$ is the dimension of our domain D, as in the non-parametric case, thus completely removing the "curse of dimensionality".

The structure of the subspaces $S_\ell \subset L^2(U, \mu; V)$ is as follows: let $V_\mu \subset V$ denote a dense, nested sequence of finite dimensional subspaces in V. In concrete applications, we shall use continuous, piecewise polynomial functions on meshes with suitable mesh refinement towards the singular boundary points of the domain D, as in [5, 7, 9, 27, 32]. Then, each subspace S_ℓ is defined by a certain finite subset $\Lambda_\ell \subset \mathcal{F}$ of ℓ largest (when measured in the norm $\| \circ \|_{\mathcal{K}_{a+1}^{m+1}(D)}$) polynomial coefficients u_ν, $\nu \in \Lambda_\ell$ in the Legendre chaos expansion of u such that for each $v \in S_\ell$ it holds $v_\nu \in V_{\mu(\ell,\nu)}$ for a suitable choice of the sequence $\mu(\ell, \nu)$. Alternatively, there exist *a priori* bounds on the norms $\|u_\nu\|_{\mathcal{K}_{a+1}^{m+1}(D)}$ of the form $C_\nu \|f\|_{\mathcal{K}_{a-1}^{m-1}(D)}$, with

C_ν independent of f (and hence independent of u as well). Then we choose the set $\Lambda_\ell \subset \mathcal{F}$ to consist of the ℓ largest coefficients C_ν. We have

$$(1) \qquad\qquad S_\ell = \bigoplus_{\nu \in \Lambda_\ell} V_{\mu(\ell,\nu)} \otimes \{L_\nu\} \; .$$

The paper is organized as follows. In Section 1 we formulate our parametric partial differential boundary value problem and introduce some of our main assumptions. We also discuss the needed notions of ellipticity and positivity for families of operators and derive some consequences. In Section 2, we recapitulate regularity and well-posedness results for the non-parametric, elliptic problem from [7, 27, 9], the main result being Theorem 2.2. This theorem is then generalized to families in the following section, thus yielding our main regularity and well-posendess result for parametric families of uniformly strongly elliptic partial differential equations, namely Theorem 3.2. As mentioned above, this result is formulated in weighted Sobolev spaces (the so called "Babuška-Kondratiev" spaces). In Section 4, we use the results of Section 3 to obtaine some improved estimates on the parametric derivatives $\partial_y^\nu u_y$ ($y \in U$ is the parameter). Then, in Section 5, we apply Theorem 4.4 to study the so called "best N-term approximation" of the parametric solution u_y. In Section 6, we use spatial discretization to obtain a full discretization of the equation $P_y u_y = f$ in two dimensions (that is D a polygon), thus proving our main theorem in two dimensions, which yields quasi-optimal algebraic rates of convergence for the approximations of u with the Finite Element Method. In the last section, Section 7, we discuss how to extend our results to three dimensions and how to choose the aproximation spaces in a more convenient way.

We shall use the following notation: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathcal{F} \subset \mathbb{N}_0^{\mathbb{N}}$, we shall denote the set of sequences of nonnegative integers with all but finitely many ν_k equal to zero. We let $0! := 1$, as usual. For $\nu \in \mathcal{F}$, we define $|\nu| := \sum_{k=1}^{\infty} \nu_k < \infty$ and $\nu! = \nu_1! \nu_2! \ldots = \prod_k \nu_k! = \prod_{\nu_k > 0} \nu_k! < \infty$, using the convention that $0! = 1$. Throughout, D shall denote a bounded, piecewise smooth Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$ (or a finite union of such domains) and U will denote a "parameter domain" to be specified. By x we denote spatial coordinates of points in D, and by y parameter vectors in U. With the parameter space U, we associate a sigma algebra $\mathcal{A}$. On the measurable space $(U, \mathcal{A})$ we assume given a probability measure μ so that $(U, \mathcal{A}, \mu)$ becomes a probability space, which we assume to be complete, for simplicity. We agree to identify functions that differ only on a set of measure zero. For $1 \le p < \infty$, we denote by $L^p(U, \mu)$ the Banach space of measurable, real-valued functions on U that are p-integrable with respect to the measure μ, endowed with the usual norm $\| \circ \|_{L^p(U,\mu)}$. In the case $p = \infty$, we denote by $L^\infty(U)$ the set of measurable, real-valued functions that are essentially bounded on U, with norm $\|u\|_{L^\infty(U)} = \inf_{\mu(N)=0} \sup_{y \in U \smallsetminus N} |u(y)|$.

1. Ellipticity, positivity, solvability for parametric families

In this section we formulate our parametric partial differential boundary value problem and introduce some of our main assumptions. We also discuss the needed notions of ellipticity and positivity for families of operators and derive some consequences.

1.1. **Notation and assumptions.** By $D \subset \mathbb{R}^d$, $d = 2, 3$, we shall denote a connected, bounded Lipschitz domain. The main examples are a polygonal domain in the plane (when $d = 2$) or a polyhedral domain in space, such as a cube or a prism (when $d = 3$).

We consider the spatial regularity of PDEs whose data depend on a parameter vector y taking value in a "parameter space" U. Following [12], we will assume that U is an open subset of a separable Banach space Y. Beginning with Section 4, we shall assume that $U = B_1(\ell^\infty(\mathbb{N}))$, the unit ball in the space of bounded sequences. This assumption is however not necessary for our parametric regularity results.

By $a_{pq}^{ij} : D \times U \to \mathbb{R}$, $1 \le i, j \le d$, we shall denote bounded, measurable functions satisfying smoothness and other assumptions to be made precise later. Let us denote by $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, d$. We shall then denote by $P = [P_{pq}]$ a $\mu \times \mu$ matrix of parametric differential operators in divergence form

$$(2) \qquad P_{pq}u(x, y) := -\sum_{i,j=1}^{d} \partial_i\big(a_{pq}^{ij}(x, y)\partial_j u(x, y)\big),$$

where $x \in D$ and $y \in U$. We can include also suitable lower order terms, but we chose not to do that, in order not to complicate the notation. The matrix case is needed in order to handle the case of (anisotropic) linear elasticity. The matrix differential operator $P = [P_{pq}]_{p,q=1}^{\mu}$ acts on vector-valued functions $u = (u_q)_{q=1}^{\mu}$ in the usual way

$$(3) \qquad (Pu)_p = \sum_{q=1}^{\mu} P_{pq}u_q, \quad \text{for } u \in \mathcal{C}^\infty(D \times U)^{\mu}.$$

Occasionally, we shall need to specialize P for a particular value of y. We recall that $H^{-1}(D)$ is defined as the dual of $H_0^1(D) := \{u \in H^1(D), u|_{\partial\Omega} = 0\}$ with pivot $L^2(D)$. More precisely, each $f \in L^2(D)$ defines a continuous linear functional on $H_0^1(D)$ by the formula $f(u) = (f, u)$. Then the completion of $L^2(D)$ with respect to the induced dual norm is $H^{-1}(D)$. We shall then write $P_y : \mathcal{C}^\infty(D)^{\mu} \to H^{-1}(D)^{\mu}$ for the induced operator. We emphasize that we allow P to have non-smooth coefficients, so that Pu may not be smooth in general. This explains the choice of domains and ranges in the definition of P_y. Our main examples are parametric, second order operators in divergence form, and the system of linearized, anisotropic elasticity, in which case $a_{pq}^{ij}(x, y) = \mathcal{C}_{ijpq}(x, y) = \mathcal{C}_{pqij}(x, y)$ are the elastic moduli which satisfying the positivity condition (10) ahead.

We use the spaces $H_0^1(D)$ and $H^{-1}(D)$ for vector-valued functions:

$$(4) \qquad V = H_d^1(D) := \{u \in H^1(D)^{\mu} : u = 0 \text{ on } \partial_d D\}$$

and $H_d^{-1}(D)$ to be the ual of $H_d^1(D)$ with pivot space $L^2(D)$ which we identify with its own dual. Note that we assume here, for simplicity of notation, *the same type of boundary conditions for all components u_q of the solution vector u.*

1.2. **Boundary conditions.** We impose suitable mixed boundary conditions that include Dirichlet boundary conditions. More precisely, we shall consider mixed boundary conditions of Dirichlet or Neumann type. To this end, we assume given a closed set $\partial_d D \subset D$, which is a union of polygonal subsets of the boundary and we let $\partial_n D := \partial D \smallsetminus \partial_d D$. In 2D, a vertex that is common to two edges in $\partial_n D$ will be called a *Neumann-Neumann corner* and, in 3D, an edge that is common to two faces in $\partial_n D$ will be called a *Neumann-Neumann* edge. We explicitly allow that $\partial_n D = \emptyset$, but for many of our results, we shall assume that there no adjacent faces with Neumann boundary conditions, that is, we assume that there are no Neumann-Neumann corners or edges, which of course implies that $\mathrm{meas}_{d-1}(\partial_d D) = 0$. The set $\partial_d D$ will be referred to as "Dirichlet boundary" and $\partial_n D$ as "Neumann boundary," according to the type of boundary conditions that we associate to these parts of the boundary. The case of cracks is also allowed, provided that one treats different sides of the crack as *different* parts of the boundary, as in [27], for instance. For each $y \in U$, we then define the *conormal derivatives*

$$(5) \qquad (\nabla_\nu^A u)_p = \sum_{q=1}^\mu \sum_{i,j=1}^d \nu_i a_{pq}^{ij}(x,y)\partial_j u_q(x,y), \quad x \in \partial_n D, \ y \in U$$

where $\nu = (\nu_i)$ is the outward unit normal vector at $x \in \partial_n D$.

1.3. **Ellipticity and positivity for differential operators.** In this subsection we introduce two important properties for our parametric families of differential operators. For $y \in U$, we consider the parametric family of boundary value problems

$$(6) \qquad \begin{cases} Pu(x,y) = f(x,y), & x \in D, \\ u(x,y) = 0, & x \in \partial_d D, \\ \nabla_\nu^A u(x,y) = g(x,y), & x \in \partial_n D \end{cases}$$

where P is as in Equation (3) and ∇_ν^A is as in Equation (5). For most of our results, we shall assume that $g = 0$. The general case, however, is an important intermediate step that will be used in the non-parametric problem in Theorem 2.2.

For any $y \in U$, let us consider the parametric bilinear form $B(y; \cdot, \cdot)$ defined by

$$(7) \quad B(y;v,w) := \int_{x \in D} \sum_{p,q=1}^\mu \Big(\sum_{i,j=1}^d a_{pq}^{ij}(x,y)\partial_i v_p(x,y)\partial_j w_q(x,y) \Big) dx, \quad y \in U \ .$$

Definition 1.1. The family $(P_y)_{y \in U}$ is called *uniformly strictly positive definite* on $V \subset H^1(D)^\mu$ if the coefficients a_{pq}^{ij} are symmetric in i, j and in p, q (that is, $a_{pq}^{ij} = a_{pq}^{ji} = a_{qp}^{ij}$, for all i, j, p, and q), and if there exist $0 < r < R < \infty$ such that for all $y \in U$, and $v, w \in V \subset \mathcal{K}_1^1(D)^\mu$

$$|B(y;v,w)| \le R\|v\|_{H^1(D)}\|w\|_{H^1(D)} \quad \text{and} \quad r\|v\|_{H^1(D)}^2 \le B(y;v,v)$$

Both the definition of the bilinear form B and of the "uniform strictly positive definite" family extend to the case when lower order terms are included in an obvious way. For simplicity of notation, we choose however not to do that.

If U is reduced to a single point, that is, if we deal with the case of a single operator instead of a family, then we say that P is strictly positive definite paper, we shall assume that $(P_y)_{y \in U}$ *is strictly positive definite, (uniformly with respect to the parameter vector y)*. In our applications, the positivity condition will follow

either from the Friedrichs-Poincaré inequality (for scalar operators) or from the Korn inequality (for the elasticity system), under the positivity (10). See Remark 1.3.

In case one is interested only in scalar equations (not in systems), then the assumption that our family P_y is uniformly positive definite can be replaced with the assumption that the family P_y is uniformly strongly elliptic and $c \geq 0$. We now recall the definition of a uniformly strongly elliptic family of differential operators. To this end, let us denote by $|\xi| = \left(\sum_{i=1}^{d} \xi_i^2 \right)^{1/2}$ for any $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

Definition 1.2. The family $(P_y)_{y \in U}$ is called *uniformly strongly elliptic* if the coefficients a_{pq}^{ij} are symmetric in i, j and in p, q and if there exist $0 < r_e < R_e < \infty$ such that for all $x \in D$, $y \in U$, $\xi \in \mathbb{R}^d$, and $\eta \in \mathbb{R}^\mu$

$$(8) \quad r_e |\xi|^2 |\eta|^2 \leq \sum_{p,q=1}^{\mu} \sum_{i,j=1}^{d} a_{pq}^{ij}(x,y) \xi_i \xi_j \eta_p \eta_q$$

$$\leq \Big| \sum_{p,q=1}^{\mu} \sum_{i,j=1}^{d} a_{pq}^{ij}(x,y) \xi_i \xi_j \eta_p \eta_q \Big| \leq R_e |\xi|^2 |\eta|^2 \, .$$

Recall that, if $C = [C_{pq}]$ is a $\mu \times \mu$ matrix, we say that $C \geq 0$ if $C_{pq} = C_{qp}$ and $\sum_{p,q=1}^{\mu} C_{pq} \eta_p \eta_q \geq 0$ for all $\eta = (\eta_p) \in \mathbb{R}^\mu$. Similarly, we write $C \geq D$ if $C - D \geq 0$. Let us denote by $A^{ij} = [a_{pq}^{ij}]$, an $\mu \times \mu$ matrix, and by I_μ the $\mu \times \mu$ the identity matrix. Then Equation (8) can be reformulated as

$$(9) \qquad r_e |\xi|^2 I_\mu \leq \sum_{i,j=1}^{d} A^{ij}(x,y) \xi_i \xi_j \leq \Big| \sum_{i,j=1}^{d} A^{ij}(x,y) \xi_i \xi_j \Big| \leq R_e |\xi|^2 I_\mu.$$

Again, when U is reduced to a single point, then we say that P is *strongly elliptic*.

1.4. **Consequences of positivity.** We now discuss the usual consequences of positivity in our parametric framework.

Remark 1.3. Assume that $\partial_d D$ is not empty (and hence it has positive measure). Examples of operators P that are uniformly strictly positive definite are as follows. Recall that $V = H_d^1(D) := \{u \in H^1(D)^\mu, u = 0 \text{ on } \partial_d D\}$. Assume P is scalar, uniformly strongly elliptic. Then P is uniformly strictly positive. Assume further $\mu = d$, $a_{pq}^{ij} = \mathcal{C}_{ijpq}$ with $\mathcal{C}_{ijpq} = \mathcal{C}_{pqij}$. We assume that P has bounded coefficients and there exists a constant $C > 0$ (independent of y) such that

$$(10) \qquad \sum_{i,j,p,q=1}^{d} \mathcal{C}_{ijpq} E^{ij} E^{pq} \geq C \sum_{i,j=1}^{d} (E^{ij})^2,$$

provided that $E^{ij} = E^{ji}$. Then P is uniformly strictly positive definite. Moreover, in both examples considered, the constants r and F defining the uniform strictly positive property can be determined in terms of the uniform strong ellipticity constants r_e and R_e, the constant in Equation (10) and the constants appearing in the Friedrichs-Poincaré or Korn inequalities. See [28, 27], for instance.

The assumption that $\partial_d D \neq \emptyset$ in the above remark is essential. However, by replacing the operators P in the above remark with $P + \lambda$ for some $\lambda > 0$, the result remains true. The following lemma is standard.

Lemma 1.4. *If the family P_y is uniformly strictly positive definite on a subspace $V \subset H^1(D)^\mu$ that contains $\mathcal{C}_c^\infty(D)^\mu$, then it is uniformly elliptic. More precisely, if $0 < r < R < \infty$ are as in Definition (1.1) of uniform strict positivity, then the uniform ellipticity condition of Equation (8) holds for any $0 < r_e \le r$ and for $R_e \ge R$.*

Proof. Let us assume that P is uniformly strictly positive definite and let $\xi = (\xi_i) \in \mathbb{R}^d$ and $\eta = (\eta_p) \in \mathbb{R}^\mu$ be as in Definition 1.2. Also, choose a smooth function ϕ with compact support in D. We then define the function $\psi \in V = H_d^1(D)$ by the formula

$$\psi(x) = e^{\imath t \xi \cdot x} \phi(x) \eta \in \mathbb{R}^\mu,$$

where $\xi \cdot x = \sum_{k=1}^d \xi_k x_k$. Then

$$(11) \qquad \lim_{t \to \infty} t^{-2} \|\psi\|_{\mathcal{K}_1^1(D)}^2 = \sum_{p=1}^\mu \sum_{j=1}^d \xi_j^2 \eta_p^2 \int_D (\phi(x))^2 dx = |\xi|^2 |\eta|^2 \int_D (\phi(x))^2 dx$$

The coefficients a_{pq}^{ij}, forming the principal symbol of P, are then determined by "oscillatory testing"

$$(12) \qquad \lim_{t \to \infty} t^{-2} (P\psi, \psi) = \int_D \sum_{p,q=1}^\mu \sum_{i,j=1}^d a_{pq}^{ij}(x,y) \xi_i \xi_j \eta_p \eta_q (\phi(x))^2 dx.$$

Substituting Equations (11) and (12) into Equation (8) for $v = w = \psi$, we obtain for all $y \in U$ that

$$r |\xi|^2 |\eta|^2 \int_D (\phi(x))^2 dx \le \int_D \sum_{p,q=1}^\mu \sum_{i,j=1}^d a_{pq}^{ij}(x,y) \xi_i \xi_j \eta_p \eta_q (\phi(x))^2 dx$$

$$\le R |\xi|^2 |\eta|^2 \int_D (\phi(x))^2 dx \ .$$

Since ϕ is an arbitrary compactly supported smooth function on D, it follows that, for all $x \in D$ and $y \in U$,

$$r |\xi|^2 |\eta|^2 \le \sum_{p,q=1}^\mu \sum_{i,j=1}^d a_{pq}^{ij}(x,y) \xi_i \xi_j \eta_p \eta_q \le R |\xi|^2 |\eta|^2 \ .$$

Comparing this last equation to Equation (8), we obtain the desired result. $\qquad \square$

We denote $H_d^1(D) := \{u \in H^1(D),\ u = 0 \text{ on } \partial_d D\}$, as before. Also, by $H_d^{-1}(D)$ we shall denote the dual of $H_d^1(D)$. Of course, if $\partial_d D = \partial D$, then $H_d^1(D) = H_0^1(D)$ and $H_d^{-1}(D) = H^{-1}(D)$. In view of the next proposition, *we shall also assume from now on* that

$$(13) \qquad f_y := f(\,\cdot\,, y) \in H_d^{-1}(D) \quad \text{for all } y \in D \ .$$

Also, for any Banach spaces X_1 and X_2, we shall denote by $\mathcal{L}(X_1, X_2)$ the (Banach) space of continuous, linear maps $T : X_1 \to X_2$ endowed with the operator norm $\|T\| = \sup_{\|x\|_{X_1} = 1} \|Tx\|_{X_2}$. The Lax-Milgram Lemma implies the unique solvability of the parametric problem, for each instance of the parameter vector $y \in U$. To state it, we recall that r denotes the lower bound in the uniform positivity equation (Definition 1.1).

Proposition 1.5. *Assume that, for any fixed $y \in U$, the parametric data $f_y :=$ $f(\cdot, y)$ is in $H_d^{-1}(D)^\mu$. Also, assume that P_y is uniformly striclty positive. Then for every $y \in U$, our family of boundary value problems $P_y u_y = f_y$, $u_y \in H_d^1(D)^\mu$, i.e., Equation* (6) *admits a unique solution $u_y := u(\cdot, y) = P_y^{-1} f_y$. Moreover, there exists a constant $C_D > 0$ such that there holds the apriori estimate*

$$(14) \qquad \|P_y^{-1}\|_{\mathcal{L}(H_d^{-1}; H_d^1)} \leq C_D r^{-1}, \quad 0 < r \leq 1, \ y \in U$$

with C_D depending only on the Poincaré constant of D, and not on r.

The parametric solution $u_y \in H_d^1(D)$ of Proposition 1.5 is then obtained from the usual weak formulation: given $y \in U$, find $u_y \in V := H_d^1(D)$ such that

$$B(y; u_y, w) = (f_y, w), \quad \forall w \in V,$$

where (f_y, w) denotes the $L^2(D)$ inner product.

Nothing is being said in the above proposition about the behaviour of u as a function of the parameter y. It is one of the technical goals of this paper to study this behaviour. To this end, however, we shall need to make more assumptions on f and on the family $(P_y)_{y \in U}$ (or more precisely on the coefficients a_{pq}^{ij} of P).

2. Weighted Sobolev spaces and higher regularity of non-parametric solutions

One of our main goals is to obtain regularity of the solution u both in the space variable x and in the parameter y. It is convenient to split this problem into two parts: regularity in x and regularity in y. We first address regularity in x, which is to a large extent known, but we need a slightly stronger version of the classical results [25, 26, 30]. This leads to Theorem 2.2, which will be then generalized to families in the following section. Let us notice also that the results in this section continue to hold true if one includes lower order terms in the definition of P_y, with obvious changes, as long as the uniform strict positivity condition of Definition (1.1) is extended in an obvious way.

We shall thus assume throughout this subsection we are dealing with a single, non-parametric equation (not with a family), that is, that U is reduced to a single point in this subsection.

To formulate further assumptions on our problem and to state our results, we shall need weighted Sobolev spaces, both of L^2 and of L^∞ type. Let us therefore denote by $\rho : \mathbb{R}^d \to [0, 1]$ a continuous function that is smooth outside the set of singular points of D and is such that $\rho(x)$ is equal to the distance from $x \in \mathbb{R}^d$ to the singular points of the boundary of D when x is close to these points. Thus, ρ is the distance to the vertices of D close to these vertices if $d = 2$ and ρ is the distance to the edges of D close to these edges if $d = 3$. The function ρ will be called the *smoothed distance to the singular points of the boundary.* See, e.g. [34, Chap 6.1] for a construction in general, bounded Lipschitz domains. We can also assume

$$(15) \qquad \|\nabla \rho\| \leq 1,$$

which will be convenient in later estimates, since it will reduce the number of constants (or parameters) in our estimates. We then define the *Babuška-Kondrat'ev spaces*

$$(16) \qquad \mathcal{K}_a^m(D) := \{v : D \to \mathbb{C}, \ \rho^{|\alpha| - a} \partial^\alpha v \in L^2(D), \ \text{ for all } |\alpha| \leq m\}$$

and

(17) $\mathcal{W}^{m,\infty}(D) := \{v : D \to \mathbb{C}, \ \rho^{|\alpha|}\partial^\alpha v \in L^\infty(D), \ \text{for all } |\alpha| \le m\}.$

We shall denote by $\| \cdot \|_{\mathcal{K}_a^m(D)}$ and $\| \cdot \|_{\mathcal{W}^{m,\infty}(D)}$ the resulting natural norms on these spaces. For further reference we note that the definitions of these spaces imply that the multiplication and differentiation maps

(18) $\mathcal{W}^{m,\infty}(D) \times \mathcal{K}_a^m(D) \to \mathcal{K}_a^m(D) \quad \text{and} \quad \partial_i : \mathcal{K}_a^m(D) \to \mathcal{K}_{a-1}^{m-1}(D)$

are continuous.

For any subset $S \subset \partial D$ that is a union of entire edges if $d = 2$ and of entire faces if $d = 3$, we introduce the spaces $\mathcal{K}_{a+1/2}^{m+1/2}(S)$ as the restrictions to S of the functions $u \in \mathcal{K}_{a+1}^{m+1}(D)$. These spaces have intrinsic descriptions [1, 28] similar to the usual Babuška-Kondratiev spaces.

Let $a_{pq}^{ij} \in \mathcal{W}^{m,\infty}(D)$ and $u \in \mathcal{K}_{a+1}^{m+1}(D)^\mu$ (recall U is a point in this subsection). Then

$$f_p := (P_a u)_p := \sum_{q=1}^{\mu} \sum_{i,j=1}^{d} \partial_i(a_{pq}^{ij}\partial_j u_q)$$

is well defined for $m \in \mathbb{N} = \{1, 2, \ldots\}$ and $f \in \mathcal{K}_{a-1}^{m-1}(D)^\mu$. For $m = 0$, we have $f \in \mathcal{K}_{a-1,d}^{-1}(D)$, where $\mathcal{K}_{a-1,d}^{-1}(D)$ is defined as the dual of $\mathcal{K}_{1-a,d}^1(D)$ with pivot $L^2(D)$. See the discussion in the paragraph following Equation (23) below. Let us denote by $\mathcal{L}(\mathcal{K}_{a+1}^{m+1}; \mathcal{K}_{a-1}^{m-1}) = \mathcal{L}(\mathcal{K}_{a+1}^{m+1}(D)^\mu; \mathcal{K}_{a-1}^{m-1}(D)^\mu)$. (We sometimes drop the power μ and the refrence to our domain D when this can cause no confusion.) Then the map

(19) $\mathcal{W}^{m,\infty}(D)^{(d^2+1)\mu^2} \ni a = (a_{pq}^{ij}) \to P_a \in \mathcal{L}(\mathcal{K}_{a+1}^{m+1}; \mathcal{K}_{a-1}^{m-1})$

is continuous for $m > 0$, by Equation (18). The continuity of the map (19) motivates the use of the spaces $\mathcal{W}^{m,\infty}(D)$. We shall denote by

(20) $$\|(a_{pq}^{ij})\|_{\mathcal{W}^{m,\infty}(D)} := \sum_{p,q=1}^{\mu} \sum_{i,j=1}^{d} \|a_{pq}^{ij}\|_{\mathcal{W}^{m,\infty}(D)}$$

the induced norm of the coefficients.

Recall that r is the constant appearing in the definition of uniform positivity (Definition 1.1). We start by introducing some notation. We denote our operator by

(21) $$(Pu)_p := -\sum_{q=1}^{\mu}\left(\sum_{i,j=1}^{d} \partial_i(a_{pq}^{ij}\partial_j u_q) + c_{pq}u_q\right).$$

Define the subspaces with homogenous Dirichlet boundary conditions

(22) $\mathcal{K}_{a+1,d}^{m+1}(D) := \mathcal{K}_{a+1}^{m+1}(D)^\mu \cap \{u = 0 \text{ on } \partial_d D\}$

and consider the family of partial differential operators

(23) $\tilde{P}_{a,m} : \mathcal{K}_{a+1,d}^{m+1}(D) \to \mathcal{K}_{a-1}^{m-1}(D) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D),$

$\tilde{P}_{a,m}u = (Pu, \nabla_\nu^A u|_{\partial_n D}),$

where the conormal derivative $\nabla_\nu^A := \sum_{ij} \nu_i a_{ij}\partial_j$ was introduced in Equation (5) for families. When $m = 0$, we replace the codomain of the operator $\tilde{P}_{a,0}$ with $\mathcal{K}_{a-1,d}^{-1}(D)$, the dual of $\mathcal{K}_{1-a,d}^1(D)$ with pivot $L^2(D)$ and define $P_{a,0}$ in a weak sense

(see the discussion around Equation (2.12) in [27] for more details or the discussion around Equation (20) in [28]).

We can consider positivity in the weighted spaces $\mathcal{K}$ in view of the following well known Lemma, which explains why some of our results require that there be no Neumann-Neumann corners or edges (that is, no two adjacent faces are endowed with Neumann boundary conditions).

Lemma 2.1. *Let $u \in H^1(D)$ satisfy $u = 0$ on $\partial_d D$, the Dirichlet part of the boundary, and assume that there are no Neumann-Neumann corners or edges. Then there exists a constant $C > 0$ that depends only on D and on the choice of boundary conditions such that*

$$\|u\|_{\mathcal{K}_1^1(D)} \leq C\|u\|_{H^1(D)} .$$

This lemma is a simple consequence of the Hardy inequality. See [26, 27, 28, 30] for details.

Recall the constants $0 < r \leq r_e$ in the definition of the uniform strict positivity and uniform strong ellipticity (Definitions 1.1 and 1.2). We now state the main result of this section. The proof of this result requires several intermediate results, and will be completed after we will have proved Lemma 2.5. Recall that D is a piecewise smooth domain in two or three dimensions. For simplicity, we do not treat the case of cracks.

Theorem 2.2. *Assume that U is reduced to a point. Also, assume that P is strictly positive definite on $H_d^1(D)$, that D has no Neumann-Neumann corners or edges, and that $a_{ij} \in \mathcal{W}^{m,\infty}(D)$. Then there exists $0 < \eta$ such that for any $m \in \mathbb{N}_0$ and for any $|a| < \eta$, the map $\tilde{P}_{a,m}$ is boundedly invertible and*

$$(24) \qquad \|\tilde{P}_{a,m}^{-1}\| \leq \tilde{C},$$

where $\tilde{C} = \tilde{C}(D, m, r, a, \|a_{pq}^{ij}\|_{\mathcal{W}^{m,\infty}(D)})$ depends only on the indicated variables.

We now include in the following corollary the typical formulation of the above theorem.

Corollary 2.3. *We use the notation and the assumptions of Theorem 2.2. If $f \in \mathcal{K}_{a-1}^{m-1}(D)^\mu$ and $g \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D)^\mu$, then the problem*

$$Pu = f, \quad \nabla_\nu^A u = g \quad on \quad \partial_n D, \quad u = 0 \quad on \quad \partial_d D,$$

has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(D)^\mu$ such that

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(D)} \leq \tilde{C}\big(\|f\|_{\mathcal{K}_{a-1}^{m-1}(D)} + \|g\|_{\mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D)}\big),$$

with $\tilde{C}$ as in Theorem 2.2. If $a \geq 0$, this solution u is also the unique solution $u \in H_d^1(D)$ provided by Proposition 1.5.

Proof. Everything follows from Theorem 2.2, except the very last statement, which follows from $\mathcal{K}_{1+a,d}^1(D) \subset H_d^1(D)$. $\qquad\square$

Although we will not need that, let us notice that the constant η depends only on the values of the coefficients a_{pq}^{ij} at the vertices. In particular, η is independent of m.

It is convenient to organize the proof of Theorem 2.2 as sequence of lemmas. Throughout, we work under the assumptions of Theorem 2.2. In the following lemma we need not assume $d \leq 3$.

Lemma 2.4. *Let us assume that $D \subset \mathbb{R}^d$, $d \geq 1$ arbitrary, is smooth and bounded and that to each component of the boundary it is associated a single type of boundary conditions (either Dirichlet or Neumann). Assume that P is strictly positive definite on $H_d^1(D)$ and let $\|P^{-1}\|$ denote norm of the inverse of the map $P : H_d^1(D) \to H_d^1(D)^* =: H_d^{-1}(D)$. Then there exists a constant $\tilde{C}_1 > 0$ such that*

$$(25) \qquad \|u\|_{H^{m+1}(D)} \leq \tilde{C}_1\big(\|f\|_{H^{m-1}(D)} + \|g\|_{H^{m-1/2}(\partial_n D)}\big),$$

with the constant $\tilde{C}_1 = \tilde{C}_1(D, m, \|P^{-1}\|, \|a_{pq}^{ij}\|_{\mathcal{W}^{m,\infty}(D)})$ depending only on the indicated variables.

Proof. In the case of the pure Dirichlet boundary conditions for an equation and without the explicit bounds in Equation (24), this lemma is a classical result [17], Theorem 5 in Section 6.3 (see also [19, 35]), which is proved using divided differences and the so called "Nirenberg's trick" (see [18, 29]). See also [31] for the role of positivity in deriving higher regularity. Since we are dealing with systems and since we want the more explicit bounds in the above Equation (25), let us now indicate the main steps of the proof. In all the calculations below, all the constants C below will be generic constants that will depend only on the variables on which $\tilde{C}_1$ depends (in particular, the domain D, the order m, the norms $\|P^{-1}\|$ and $\|(a_{pq}^{ij})\|_{\mathcal{W}^{m,\infty}(D)}$).

Step 1. We first use Proposition 1.5 to conclude that $P : H_d^1(D) \to H_d^1(D)^*$ is indeed invertible. This provides the needed estimate for $m = 0$ (which, we recall, has to be interpreted in a weak sense).

Step 2. We also notice that, in view of the invertibility of P for $m = 0$, it suffices to prove

$$(26) \qquad \|u\|_{H^{m+1}(D)} \leq C\big(\|f\|_{H^{m-1}(D)} + \|g\|_{H^{m-1/2}(\partial_n D)} + \|u\|_{H^m(D)}\big) .$$

Indeed, the desired inequality (25) will follow from Equation (26) by induction on m. Since Equation (26) holds for P if, and only if, it holds for $\lambda + P$, in order to prove Equation (26), it is also enough to assume that $\lambda + P$ is strictly positive for some $\lambda \in \mathbb{R}$. In particular, Equation (26) will continue to hold–with possibly different constants–if we add lower order terms to P.

Step 3. We can assume $g = 0$ if $m > 0$ by using the extension theorem for functions in $H^{1/2}(\partial_n D)$ to $H^1(D)$. More precisely, let $E(g) \in H^2(D)$ be a function such that $\nabla_\nu^A E(g) = g$ on $\partial_n D$ and $\|E(g)\|_{H^2(D)} \leq C\|g\|_{H^{1/2}(\partial_n D)}$. Then we apply the estimate (26) to $u - E(g)$, whose conormal derivative vanishes on the Neumann part of the boundary, and then rearrange the terms using the triangle inequality to get the desired relation (26). Thus there will be no need for a further estimate of the normal derivative at the Neumann boundary. Hence we can drop the domain D in the notation for the Sobolev norms (since all the norms will be on D).

Step 4. Let us assume that D is either the half-space $x_d \geq 0$ or the full space $\mathbb{R}^d$. Then we prove Equation (26) for these particular domains and for $g = 0$ by induction on m. As we have noticed, the Equation (26) is true for $m = 0$, since the stronger relation (25) is true in this case. Thus, we shall assume that Equation (26) has been proved for m and for smaller values and we will prove it for $m + 1$. That is, we want to prove

$$(27) \qquad \|u\|_{H^{m+2}(D)} \leq C\big(\|f\|_{H^m(D)} + \|u\|_{H^{m+1}(D)}\big).$$

To this end, let us first write

$$(28) \qquad \|u\|_{H^{m+2}(D)} \leq \sum_{j=1}^{d} \|\partial_j u\|_{H^{m+1}(D)} + \|u\|_{L^2(D)} .$$

We then use our estimate (26) for m and $g = 0$ applied to the function $\partial_j u$. If we are dealing with a half-space, we assume $j < d$. This gives

$$(29) \quad \|\partial_j u\|_{H^{m+1}(D)} \leq \|P\partial_j u\|_{H^{m-1}(D)}$$
$$\leq \|\partial_j f\|_{H^{m-1}(D)} + \|[P,\partial_j]u\|_{H^{m-1}(D)} \leq \|f\|_{H^m(D)} + C\|u\|_{H^{m+1}(D)}$$

since the commutator $[P,\partial_j] = P\partial_j - \partial_j P$ is an operator of order ≤ 2 whose coefficients can be bounded in terms of $\|(a_{pq}^{ij})\|_{\mathcal{W}^{m,\infty}(D)}$. Equation (29), when used in Equation (28), then suffices to conclude our desired estimate (27) in the case that D is the full space.

In the case that D is the half space, we still need to estimate $\|\partial_d u\|_{H^{m+1}}$. To this end, we proceed as in the classical case (with norms in the remainder of this proof understood to be taken over the half-space)

$$(30) \quad \|\partial_d u\|_{H^{m+1}} \leq \sum_{j=1}^{d} \|\partial_j \partial_d u\|_{H^m} + \|\partial_d u\|_{L^2}$$
$$\leq \sum_{j=1}^{d-1} \|\partial_j u\|_{H^{m+1}} + \|\partial_d^2 u\|_{H^m} + \|u\|_{H^1} .$$

The right hand side of the above equation contains only terms that have been estimated in the way we need, except for $\|\partial_d^2 u\|_{H^m}$. Since $m \geq 0$, we can use the relation $Pu = f$ to estimate this term as follows. Let us write $Pu = \sum \partial_i(A^{ij}\partial j u) + cu$, where A^{ij} is the matrix $[a_{pq}^{ij}]$. This gives

$$A^{dd}\partial_d^2 u = f - \sum_{(i,j)\neq(d,d)} A^{ij}\partial_i\partial_j u + Qu,$$

where Q is a first order differential operator. Next we notice that the matrix A^{dd} is invertible by the uniform strong ellipticity condition and moreover $\|(A^{dd})^{-1}\| \leq r_e^{-1}$. Note that by Lemma 1.4, we have $\|P^{-1}\| \leq r_e$, and hence r_e^{-1} is an admissible constant. This gives

$$(31) \qquad \partial_d^2 u = (A^{dd})^{-1}f - \sum_{j=1}^{d-1} B^j\partial_j u + Q_1 u$$

where B^j and Q_1 are first order differential operators with coefficients bounded by admissible constants. Finally, we have

$$(32) \quad \|\partial_d^2 u\|_{H^m} \leq C\big(\|f\|_{H^m} + \sum_{j=1}^{d-1} \|\partial_j u\|_{H^{m+1}} + \|u\|_{H^{m+1}}\big)$$
$$\leq C\big(\|f\|_{H^m} + \|u\|_{H^{m+1}}\big)$$

by Equation (29). Equation (30) and (32) then give

$$(33) \qquad \|\partial_d u\|_{H^{m+1}} \leq C\big(\|f\|_{H^m} + \|u\|_{H^{m+1}}\big) .$$

Combining Equations (33) and (29) with Equation (28) gives then the desired Equation (27) for $g = 0$ and m replaced with $m + 1$.

Step 5. We finally reduce to the case of a half-space or a full space using a partition of unity as in the classical case, as follows. We choose a smooth partition of unity (ϕ_j) on D consisting of functions with small supports. The supports should be small enough so that if the support of ϕ_j intersects the boundary of D, then the boundary can be straightened in a small neighborhood of the support of ϕ_j. Choose also smooth functions ψ_j with small support such that $\psi_j = 1$ in a neighborhood of the support of ϕ_s. Then we choose a change of coordinates that straightens the boundary around the support of ψ_s and we replace our induced operator, still denoted by P, with

$$(34) \qquad L = \psi_j^{1/2} P \psi_j^{1/2} + (1 - \psi_j)^{1/2}(1 - \Delta)(1 - \psi_j)^{1/2}$$

with Δ the (vector) Laplacian. Then $L + I$ is strictly positive definite and hence Step 4 then provides estimates for $\phi_j u$ in terms of $L(\phi_j u) = P(\phi_j u)$. The proof is completed by noticing that the norm $\|u\|_{H^k}$ is equivalent to $\sum_j \|\phi_j u\|_{H^k}$ (see [6] for a proof of this simple fact). $\qquad \square$

We shall need also the following regularity result.

Lemma 2.5. *Let the coefficients a_{ij} be as in Theorem 2.2. Let $a \in \mathbb{R}$ be arbitrary, $m \geq 1$, $f \in \mathcal{K}_{a-1}^{m-1}(D)$, $g \in \mathcal{K}_{a-1/2}^{m-1/2}(D)$, and $u \in \mathcal{K}_{a+1}^1(D)$ satisfy $u = -\sum_{ij} \partial_i(a_{ij}\partial_j u) = f$, $u = 0$ on $\partial_d D$ and $\nabla_\nu^A u = g$. Then $u \in \mathcal{K}_{a+1}^{m+1}(D)$. Moreover*

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(D)} \leq \tilde{C}_3\big(\|f\|_{\mathcal{K}_{a-1}^{m-1}(D)} + \|g\|_{\mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D)} + \|u\|_{\mathcal{K}_{a+1}^m(D)}\big),$$

where $\tilde{C}_3 = \tilde{C}(D, m, r, a, \|a_{pq}^{ij}\|_{\mathcal{W}^{m,\infty}(D)})$ depends only on the indicated variables.

Proof. A suitable partition of unity reduces this result to the case when D is smooth (Lemma 2.4) as in [28], Section 7 (but see also Section 5 of that paper for the definition of the weighted Sobolev function spaces using partitions of unity). $\qquad \square$

See also the explicit partition of unity and proof or regularity in [2]. For polygons, similar proofs using dyadic partitions of unity can be found also in [14, 15, 16, 26], and in other papers. A general partition of unity argument leading to regularity results can be found in [1]. We can now complete the proof of Theorem 2.2.

Proof. (of Theorem 2.2). The result of the theorem (without the explicit bounds) is known in the case when the coefficients a_{ij} are smooth (see [27], for example). We proceed along the lines of the proof of Theorems 3.2 and 3.3 in [27], to which we refer for more details. More precisely, we first establish our result for $m = 0$ and $a = 0$. Then we establish it for $m = 0$ and $|a| < \eta$. Finally, we establish it for all m and $|a| < \eta$.

The assumption that P is positive definite (Definition 1.1) implies

$$r^{-1}(Pv, v) \geq (v, v)_{\mathcal{K}_1^1(D)} := (\nabla v, \nabla v)_{L^2(D)} + (v, v)_{L^2(D)},$$

for all $u \in H_d^1(D)$ and for all $0 < r < r_e$, and hence $H_d^1(D) = \mathcal{K}_{1,d}^1(D)$ [27] (note, however, that the results in [27] apply only when there are no Neumann-Neumann corners or edges). See Lemma 3.5 and the beginning of the proof of Theorems 3.2 and 3.3 in [27] for more details. The Lax-Milgram Lemma then proves our result for $m = 0$ and $a = 0$.

Recall the function ρ used to define the Babuška-Kondratiev spaces, that is, the regularized distance to the singular points of the boundary. Also, recall the domain $\mathcal{K}^{m+1}_{a+1,d}(D)$ of the operator $\tilde{P}_{a,m}$ from Equation (23). Let $m = 0$. The family

$$\rho^a \tilde{P}_{a,0} \rho^{-a} : \mathcal{K}^1_{1,d}(D) \to \mathcal{K}^1_{1,d}(D)^*$$

is defined between the same spaces (unlike the family $P_{a,0}$). It depends continuously on y and in a and is boundedly invertible for $a = 0$. Hence it will be boundedly invertible for $|a| < \eta$, for some $\eta > 0$ and $m = 0$.

Finally, let m be arbitrary, $f \in \mathcal{K}^{m-1}_{a-1}(D)$, $g \in \mathcal{K}^{m-1/2}_{a-1/2}(\partial_n D)$, and $u \in \mathcal{K}^1_{a+1}(D)$ satisfy $\tilde{P}_{a,0} u = (f, g)$ (that is, $Pu = f$ and $\nabla^A_\nu u = g$). Then our regularity result, Lemma 2.5 shows that $u \in \mathcal{K}^{m+1}_{a+1}(D)$. This proves that $\tilde{P}_{a,m}$ is surjective, uniformly in $y \in u$. The injectivity of $\tilde{P}_{a,m}$ follows directly from our assumptions since $\tilde{P}_{a,0}$ is injective on $\mathcal{K}^{m+1}_{a+1,d}(D) \subset \mathcal{K}^1_{a+1,d}(D)$ and since $\tilde{P}_{a,m}$ is the restriction of $\tilde{P}_{a,0}$ to $\mathcal{K}^{m+1}_{a+1,d}(D)$. To complete the proof, we just need the explicit estimate of the norm of $\tilde{P}^{-1}_{a,m}$. This follows from Lemma 2.5 and induction on m

$$\begin{aligned}
\|\tilde{P}^{-1}_{a,m}(f,g)\|_{\mathcal{K}^{m+1}_{a+1}(D)} &= \|u\|_{\mathcal{K}^{m+1}_{a+1}(D)} \\
&\leq \tilde{C}_3 \big(\|f\|_{\mathcal{K}^{m-1}_{a-1}(D)} + \|g\|_{\mathcal{K}^{m-1/2}_{a-1/2}(\partial_n D)} + \|u\|_{\mathcal{K}^m_{a+1}(D)} \big) \\
&\leq \tilde{C}^m_3 \big(\|f\|_{\mathcal{K}^{m-1}_{a-1}(D)} + \|g\|_{\mathcal{K}^{m-1/2}_{a-1/2}(\partial_n D)} + \|u\|_{\mathcal{K}^1_{a+1}(D)} \big) \\
&\qquad \leq \tilde{C}^m_3 (1 + \|\tilde{P}^{-1}_{a,0}\|) \big(\|f\|_{\mathcal{K}^{m-1}_{a-1}(D)} + \|g\|_{\mathcal{K}^{m-1/2}_{a-1/2}(\partial_n D)} \big).
\end{aligned}$$

Thus we can take $\tilde{C} = \tilde{C}^m_3 (1 + \|\tilde{P}^{-1}_{a,0}\|)$, which will have the desired dependence on parameters. □

We shall need also the following corollary of the proof of Theorem 2.2.

Corollary 2.6. *We keep the same assumptions and notations as in Theorem 2.2. Let $0 < \eta$ be such that the map $\tilde{P}_{a,0}$ is boundedly invertible for any $|a| < \eta$. Then the map $\tilde{P}_{a,m}$ is boundedly invertible for any $|a| < \eta$ and for any $m \in \mathbb{N}_0$.*

Proof. This result is contained in the proof of Theorem 2.2. □

Here is another useful corollary.

Corollary 2.7. *Let r be the constant appearing in the definition of the uniform positive definite property of the family P_y. Using the same assumptions as in Theorem 2.2, there exists a constant $C_D > 0$ that depends only on D such that $\|\tilde{P}^{-1}_{0,0}\| < C_D r^{-1}$.*

Proof. Again, this is implicit in the application of the Lax-Milgram Lemma in the proof of Theorem 2.2. It can be also obtained from Equation (14) of Proposition 1.5 together with the equivalence of the H^1 and the $\mathcal{K}^1_1$ norms on $\mathcal{K}^1_{1,d}(D)$, Lemma 2.1. □

For a scalar equation on a polygonal domain in two dimensions, the Theorem 2.2 is (essentially) due to Kondratiev [25]. For D a polyhedral domain in $\mathbb{R}^3$, it is (essentially) a result from [8] (in that paper, this result was proved for the Laplace operator). A similar result holds for more general boundary conditions and jump discontinuities in coefficients (transmisssion problems) [27]. Countably sequences of norms were used in [4, 16, 23, 24]. See also [3, 10, 26, 28] for related results.

3. REGULARITY FOR FAMILIES

We will now extend Theorem 2.2 to families of boundary value problems. To this end, we need to introduce smoothness in the variable $y \in U \subset Y$. Our first extension of Theorem 2.2 to families will be to combine that theorem with some functional analysis.

For any Banach space V, we shall denote by $\mathcal{C}_b^k(U; V)$ the space of functions $v : U \to V$ that have k (Gateaux) *continuous, bounded* derivatives in U. We allow also for $k \in \{\infty, \omega\}$. We define $\mathcal{C}_b^\omega(U; V)$ to consist of the *analytic* functions $v \in \mathcal{C}_b^\infty(U; V)$ (we say that v is analytic if every point $y \in Y$ has a non-empty open ball on which the Taylor series of v converges uniformly to v). By dropping the conditions that the function and its first k derivatives are bounded, we obtain the spaces $\mathcal{C}^k(U; V)$.

Let $v \in \mathcal{C}_b^k(U; V)$, $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$ and $i \in \mathbb{N}_0$, $i \leq k$. We shall denote by $D^i v$ the ith (Gateaux) derivative of v. Recall then that $D^i v$ is an element of the Banach space $\mathcal{L}_i(Y; V)$ of continuous, multi-linear functions $L : Y \times Y \times \ldots \times Y \to V$ (i copies of Y). The norm on $\mathcal{L}_i(Y; V)$ is $\|L\|_{\mathcal{L}_i(Y;V)} = \sup_{\|x_j\|=1} \|L(x_1, x_2, \ldots, x_i)\|_V$. Finally, we shall denote for each finite $j \leq k$ by

$$(35) \qquad \|v\|_{\mathcal{C}_b^j(U;V)} := \sup_{y \in U, i \leq j} \|D^i v(y)\|_{\mathcal{L}(Y^i;V)}$$

the natural norm on $\mathcal{C}_b^j(U; V)$, which makes it a Banach space. The topology on $\mathcal{C}_b^k(U; V)$ for $k = \infty$ or $k = \omega$ is then defined by all the norms $\| \cdot \|_{\mathcal{C}_b^j(U;V)}$ with j finite (of course, they are no longer Banach spaces). We would like to point out that in our calculations, we will not use the Gateaux derivatives, but rather their pedestrian form, that is, partial derivatives with respect to the space coordinates. The Gateaux derivatives are however crucially needed in order to define our spaces and in our proofs. We start with the following well known lemma.

Lemma 3.1. *Let Y and V be two Banach spaces.*

   (i) *The map $\mathcal{L}(Y; V) \times Y \ni (T, y) \to Ty \in V$ is analytic.*
   (ii) *Let us denote by $\mathcal{L}(Y; V)_{inv}$ the set of invertible, continuous linear maps $Y \to V$. Then the map*

$$\mathcal{L}(Y, V)_{inv} \ni T \to T^{-1} \in \mathcal{L}(V; Y)$$

   is analytic.

Proof. (i) The composition map is bilinear and continuous and hence analytic. The proof of (ii) is obtained using a Neumann series argument. $\qquad \square$

We are now ready to state our main regularity result for families $(P_y)_{y \in U}$ parametrized by an open subset U of a Banach space. We use the notation introduced in the previous subsection. Recall that we are assuming that the family $(P_y)_{y \in U}$ is uniformly elliptic and that $f_y \in H^{-1}(D)$ for all $y \in U$, so that the solution u is uniquely defined such that $u(\,\cdot\,, y) \in H_0^1(D)$, for all $y \in U$ (Proposition 1.5).

For each $y \in U$, let us denote by $\eta(y) > 0$ the constant associated to P_y by Theorem 2.2. Let us also introduce

$$(36) \qquad \eta := \inf_{y \in U} \eta(y).$$

Recall that r is the constant appearing in the definition of uniform positivity of the family $(P_y)_{y \in U}$.

Theorem 3.2. *Let $m \in \mathbb{N}_0$ and $k_0 \in \mathbb{N}_0 \cup \{\infty, \omega\}$ be fixed. Assume that $a_{ij}, c \in \mathcal{C}_b^{k_0}(U; \mathcal{W}^{m,\infty}(D))$ and that the family P_y is uniformly positive definite. Then $\eta = \inf_{y \in U} \eta(y) > 0$. Let $f \in \mathcal{C}_b^{k_0}(U; \mathcal{K}_{a-1}^{m-1}(D))$, $g \in \mathcal{C}_b^{k_0}(U; \mathcal{K}_{a-1/2}^{m-1/2}(D))$, and $|a| < \eta$. Then the solution u of our family of boundary value problems (6) satisfies $u \in \mathcal{C}_b^{k_0}(U; \mathcal{K}_{a+1}^{m+1}(D))$. Moreover, for each finite $k \le k_0$, there exists a constant $C_{a,m} > 0$ such that*

$$\|u\|_{\mathcal{C}_b^k(U; \mathcal{K}_{a+1}^{m+1})} \le C_{a,m}\big(\|f\|_{\mathcal{C}_b^k(U; \mathcal{K}_{a-1}^{m-1}(D))} + \|g\|_{\mathcal{C}_b^k(U; \mathcal{K}_{a-1/2}^{m-1/2}(D))}\big) .$$

The constant $C_{a,m}$ depends only on r, m, a, k, and the norms of the coefficients a_{ij} in $\mathcal{C}_b^j(U; \mathcal{W}^{m,\infty}(D))$, but not on f or g.

Proof. We start by observing that, for each realization of the coefficient matrix $a_{pq}^{ij}(\cdot, y)$, $y \in U$, and the corresponding parametric, elliptic operator $\tilde{P}_{a,m}$ satisfies the bound (24) uniformly with respect to $y \in U$, by our assumptions on the uniform ellipticity with respect to y. In particular, by Proposition 1.5, the constant η in (36) is positive. For simplicity, we will omit the domain D in notation of the spaces $\mathcal{K}_a^m(D)$ and $\mathcal{K}_{a,d}^m(D) := \mathcal{K}_a^m(D) \cap \{u_{|\partial_d D} = 0\}$ in the rest of this proof. We thus write also $\mathcal{K}_{a-1/2}^{m-1/2} = \mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D)$ with the weight-shift of $-1/2$ indicating that a space of Neumann data on $\partial_n D$ is meant. We consider the operators

$$\tilde{P}_y := (P_y, \nabla_{\nu,y}^A) : \mathcal{K}_{a+1,d}^{m+1} \to \mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}$$

as in the proof of Theorem 2.2, with the conormal derivative ∇_ν^A given by Equation (5) and $\nabla_{\nu,y}^A$ the specialization of this formula at some arbitrary but fixed value of $y \in U$. The idea of the proof is to use first that the function Φ defined by inverting $\tilde{P}_y$:

$$(37) \qquad U \ni y \to \Phi(y) := \tilde{P}_y^{-1} \in \mathcal{L}(\mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}; \mathcal{K}_{a+1}^{m+1})$$

is defined for $a = 0$ and $m = 0$, by Theorem 2.2. We then extend the existence of the map Φ to other values of a and m, as in the proof of Theorem 2.2, using in particular Corollaries 2.3 and 2.6. An application of Lemma 3.1 (i) then will show that $\Phi \in \mathcal{C}_b^k(U; \mathcal{L}(\mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}; \mathcal{K}_{a+1,d}^{m+1}))$ and hence, by (ii) of the same lemma, that $u \in \mathcal{C}_b^k(U; \mathcal{K}_{a+1,d}^{m+1})$.

Theorem 2.2 shows that $\tilde{P}_y : \mathcal{K}_{a+1,d}^{m+1} \to \mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}$ is invertible for $a = 0$, any m, and any y. We shall use this only for $m = 0$. Since $\tilde{P}_y$ depends continuously on y, the family $\tilde{P}_y^{-1}$ also will depend continuously on $y \in U$, by standard properties of bounded operators. Our assumption that the family P_y is uniformly positive definite implies that $\|\tilde{P}_y^{-1}\|$ is bounded on U (see Corollary 2.7). Therefore

$$(38) \qquad \tilde{P}_y^{-1} \in \mathcal{C}_b^0(U; \mathcal{L}(\mathcal{K}_{a+1,d}^{m+1}(D), \mathcal{K}_{a-1}^{m-1}(D) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D)) .$$

Next, the family $\rho^a \tilde{P} \rho^{-a} \in \mathcal{C}_b^0(U; \mathcal{L}(\mathcal{K}_{a+1,d}^{m+1}(D), \mathcal{K}_{a-1}^{m-1}(D) \oplus \mathcal{K}_{a-1/2}^{m-1/2}(\partial_n D))$ is continuous as a function of a. Since for $a = 0$ this family is invertible, to conclude that P_y are all invertible for a in an open interval containing 0 and hence $\eta > 0$ by Corollary 2.6.

Since $a_{ij}, c \in \mathcal{C}_b^k(U; \mathcal{W}^{m,\infty}(D))$, Equation (19) gives that

$$U \ni y \to \tilde{P}_y \in \mathcal{C}_b^k(U; \mathcal{L}(\mathcal{K}_{a+1,d}^{m+1}; \mathcal{K}_{a-1}^{m-1}) \oplus \mathcal{K}_{a-1/2}^{m-1/2}).$$

Lemma 3.1 together with Theorem 2.2 then give that the function Φ defined in (37) is as smooth as $\tilde{P}_y$ is, that is,

$$\Phi \in \mathcal{C}^k(U; \mathcal{L}(\mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}; \mathcal{K}_{a+1,d}^{m+1})).$$

(Note that the uniform bound on the inverses in Theorem 2.2 is crucial here.) We need to show that all the derivatives of Φ are bounded. This is seen by induction using the equation

$$(39) \qquad\qquad D^1(\tilde{P}_y^{-1}) = -\tilde{P}_y^{-1} \circ D^1 \tilde{P}_y \circ \tilde{P}_y^{-1}$$

as follows. First of all $\|\tilde{P}_y^{-1}\|_{\mathcal{L}(\mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}; \mathcal{K}_{a+1}^{m+1})}$ is bounded by Theorem 2.2. Let $D^1 P_y$ be the first derivative (or differential) of P with respect to the parameter $y \in U$. By definition, it is a linear map $Y \ni z \to D^1 P(z) \in \mathcal{L}(\mathcal{K}_{a-1}^{m-1} \oplus \mathcal{K}_{a-1/2}^{m-1/2}; \mathcal{K}_{a+1}^{m+1}))$. The first derivative $D^1 P_y^{-1}$ is then the linear map $Y \ni z \to -P_y^{-1} D^1(P_y^{-1} z)$, and hence it is bounded. Further differentiating the formula (39) using the product rule, we obtain the desired boundedness of the derivatives of P_y^{-1} of order $\leq k$.

We have thus proved that $u_y = \tilde{P}_y^{-1} f_y$ is in $\mathcal{C}_b^k(U; \mathcal{K}_{a+1}^{m+1}(D))$. The only thing that is left to prove is that the constant C depends only on m, a, r, k, and the norms of the coefficients a_{ij}. This however also follows by induction from Equation (39) using also Theorem 2.2. $\qquad\square$

For a single equation and in the isotropic case, *i. e.*, when $[a_{ij}]$ is a multiple of the identity matrix and when the domain $D \subset \mathbb{R}^2$ is a polygon with maximum interior opening angle α_{MAX}, one can show (see, e.g. [21, 22]) that the largest η satisfying the above theorem is given by $\eta = \pi/\alpha_{MAX}$.

Remark 3.3. The results of this section continue to hold if we include lower order derivatives in the definition of the operators P_y, with several small, but obvious changes in the proofs.

More precise estimates on the derivatives of u as a function of y will be obtained in the following section.

4. UNIFORM ESTIMATES OF y-DERIVATIVES

We now establish precise estimates on the y-derivatives of the solution u_y. Throughout the remainder of this paper, we will assume that

$$Y = \ell^\infty(\mathbb{N}) \quad \text{and} \quad U := B_1(\ell^\infty(\mathbb{N})),$$

that is, we place ourselves in the setting of [12, 13], whose framework and notation we use throughout the rest of this paper. Note, however, that in [12, 13], only scalar, second order elliptic problems with homogeneous Dirichlet data were considered. We recall that $\mathcal{F} := \mathbb{N}_0^{\mathbb{N}}$, the set of sequences $\nu = (\nu_k)_{k \geq 1}$, $\nu_k \in \mathbb{N}_0$, with all but finitely many of the ν_k equal to zero.

For any function $v : U \to V$, we define then $\partial_y^\nu v$ to be usual partial derivative with respect to the variables that appear in ν, more precisely

$$\partial_y^\nu v(y) = \left(\partial_{y_1}^{\nu_1} \partial_{y_2}^{\nu_2} ... v \right)(y_1, y_2, \ldots), \quad y = (y_1, y_2, \ldots),$$

whenever these derivatives exist.

For ease of notation only, we work in the framework of scalar equations. The general case of systems is completely similar, but the notation is much more complicated. Let us mention, however, that in order to deal with systems, one would have to replace the functions ψ_{ijk} below with matrices ψ_{ijk}^{pq} and use a tensor product matrix norm (the norm of the matrix acting on the indices j and q). Specifically, for anisotropic elasticity, one would need to establish in this setting Lemma 4.3 by controlling the constants in Equation (10) by an estimate similar to Equation (43).

4.1. **Generalized polynomial chaos expansion.** We now explain the framework and one of the main results of [12], which served as one motivation for the present paper. In this subsection, we assume that we are in the case when $c = 0$ and $[a_{ij}] = aI$ (that is, the matrix of coefficients of our operators P_y is a multiple of the identity matrix I_d for all y). We assume that the function $a(x, y)$ is given in terms of some other functions $\bar{a}, \psi_k : D \to \mathbb{R}$ by the formula

$$(40) \qquad a(x, y) = \bar{a}(x) + \sum_{k=1}^{\infty} y_k \psi_k(x),$$

where $\sup_k |y_k| < 1$ (that is (y_k) is in $U = B_1(\ell^{\infty}(\mathbb{N}))$, the open unit ball of $Y := \ell^{\infty}(\mathbb{N})$).

To justify the expansion (40), we assume that $\bar{a}$ and ψ_k have the following three properties:

(1) there exist $0 < \bar{a}_{\min} < \bar{a}_{\max} < \infty$ such that $\bar{a}_{\min} \leq \bar{a}(x) \leq \bar{a}_{\max}$ for all $x \in D$;

(2) we have

$$\sum_{k=1}^{\infty} \|\psi_k\|_{L^{\infty}(D)} < \frac{\kappa}{1 + \kappa} \bar{a}_{\min}$$

for some $\kappa > 0$ fixed.

(3) the sequence $\|\psi_k\|_{L^{\infty}(D)}$ is in $\ell^p(\mathbb{N})$ for some $0 < p \leq 1$;

Recall that $Y = \ell^{\infty}(\mathbb{N})$ and $U := B_1(\ell^{\infty}(\mathbb{N}))$. Then the function $a(x, y)$ is in $\mathcal{C}_b^{\omega}(U; L^{\infty}(D))$, being an affine function of y. Morevover, we have $a(x, y) \geq (1+\kappa)^{-1} \bar{a}_{\min}$, so the parametric family P_y of elliptic operators is uniformly strongly elliptic in U.

In applications, we also make the assumption that the sequence $(\psi_k)_{k \in \mathbb{N}}$, forms a *complete, orthogonal* system in $L^2(D)$. This assumption is needed in order to expand arbitrary coefficient fuctions a in terms of the ψ_k, thus allowing one to treat non-linear examples also (see [12], where the last condition is also justified). This assumption is not needed for the proof of the following theorem, though. See [33] for possible choices of the basis ψ_k and its properties. The following result is Theorem 4.3 in [12].

Theorem 4.1. *For $k \in \mathbb{N}$ and for $\nu \in \mathcal{F}$, let*

$$(41) \qquad b_{k,0} := \|\psi_k\|_{L^{\infty}(D)} / a_{\min}, \quad and \quad b(0)^{\nu} := \prod_{k \geq 1} b_{k,0}^{\nu_k}.$$

For $f_y = f \in H^{-1}(D)$, and for $y \in U$, let u denote the unique parametric solution of $P_y u_y = f$. Then there exists a constant $B > 0$ such that for any $\nu \in \mathcal{F}$ there holds

$$\|\partial_y^{\nu} u\|_{L^{\infty}(U; H^1(D))} \leq B |\nu|! \, b(0)^{\nu} \|f\|_{H^{-1}(D)}.$$

4.2. **Higher regularity.** We extend Theorem 4.1 to more regular f's. Specifically, to $f \in \mathcal{K}^{m-1}_{a-1}(D)$ for $|a|$ small, and to obtain better regularity estimates for u. We still assume f_y to be independent of y, but we consider an anisotropic diffusion. More specifically, the diffusion coefficient $a(x,y)$ is replaced by the matrix $A(x,y) = [a_{ij}(x,y)]$. The functions ψ_k are then replaced with the matrix valued functions $\Psi_k = [\psi_{ijk}]$ and hence

$$(42) \qquad a_{ij}(x,y) = \overline{a}_{ij}(x) + \sum_{k=1}^{\infty} y_k \psi_{ijk}(x), \quad \text{where} \quad y = (y_k)_{k \geq 1} \in U .$$

Recall that we denote by I_d the identity $d \times d$ matrix. Also, we shall denote by $\|T\|_{M_d}$ the usual norm of a $d \times d$-matrix, so that $-\|T\|_{M_d} I_d \leq T \leq \|T\|_{M_d} I_d$. We make the following assumptions:

Assumption 4.2.

(1) there exist $0 < \overline{a}_{\min} < \overline{a}_{\max} < \infty$ such that

$$\forall x \in D : \quad \overline{a}_{\min} I_d \leq A(x) := [\overline{a}_{ij}(x)] \leq \overline{a}_{\max} I_d$$

for all $x \in D$;

(2) the sequence $\Psi_k := [\psi_{ijk}]$ satisfies

$$(43) \qquad \sum_{k=1}^{\infty} \|\Psi_k\|_{L^\infty(D;M_d)} := \sum_{k=1}^{\infty} \sup_{x \in D} \|\Psi_k(x)\|_{M_d} \leq \frac{\kappa}{1+\kappa} \overline{a}_{\min}$$

for some $\kappa > 0$ fixed.

(3) there exists $m \in \mathbb{N}$ such that $\overline{a}_{ij}, \psi_{ijk} \in \mathcal{W}^{m,\infty}(D)$. Morever, for $0 \leq l \leq m$, there exist summability exponents $0 < p_0 \leq p_1 \leq ... \leq p_m < 1$ such that for every $i,j = 1,...,d$ the sequences $(\psi_{ijk})_{k \geq 1}$ in (42) are p_l-summable as sequences in $\mathcal{W}^{l,\infty}(D)$. More precisely, for $l = 0,1,...,m$ and for $1 \leq i,j \leq d$, there holds

$$(44) \qquad \sum_{k \geq 1} \|\psi_{ijk}\|^{p_l}_{\mathcal{W}^{l,\infty}(D)} < \infty .$$

We now record the following simple lemma.

Lemma 4.3. *Under Assumption 4.2, we have $a_{ij} \in \mathcal{C}^\omega_b(U; \mathcal{W}^{m,\infty}(D))$ and that the family $(P_y)_{y \in U}$ is uniformly strongly elliptic with $r_e = (1+\kappa)^{-1}\overline{a}_{\min}$, and hence it is also uniformly strongly positive with $r > Cr_e$, for a constant $C > 0$ independent of $y \in U$. In particular, for each $y \in U$, P_y satisfies the assumptions of Theorem 3.2 with $k_0 = \omega$.*

Proof. By Assumption 4.2 (iii), the sequence of norms $\|\psi_{ijk}\|_{\mathcal{W}^{m,\infty}(D)}$ is p-summable for all i,j and for some $0 < p \leq 1$, it is in particular 1-summable, and hence the series defining $a_{ij}(x,y)$ converges for every $y \in \ell^\infty(\mathbb{N})$. Since a_{ij} depends linearly and continuously on y, it is therefore trivially analytic.

Next, for $y \in U$, we then have

$$rI_d \leq \left(\overline{a}_{\min} - \sum_{k=1}^{\infty} \|\Psi_k\|_{L^\infty(D)}\right) I_d \leq \overline{a}_{\min} I_d - \sum_{k=1}^{\infty} \|\Psi_k\|_{L^\infty(D)} |y_k|$$

$$\leq [\overline{a}_{ij}(x)] + \sum_{k=1}^{\infty} y_k \Psi_k(x) =: A(x,y).$$

The proof that there exists $R > 0$ such that $A(x, y) \leq RI_d$ is analogous. $\qquad \square$

We shall need also the following well known "product rule" formula, which can be proved easily by induction. For any two multi-indices $0 \leq \mu \leq \nu$, $\nu \in \mathcal{F}$, we denote by

$$\binom{\nu}{\mu} := \prod_{\nu_j > 0} \binom{\nu_j}{\mu_j} = \frac{\nu!}{\mu!(\nu - \mu)!},$$

where $\nu! := \prod_{\nu_k > 0} \nu_k!$. Then

$$(45) \qquad \partial_y^\nu (P_y u_y) = \sum_{0 \leq \mu \leq \nu} \binom{\nu}{\mu} (\partial_y^\mu P_y)(\partial_y^{\nu - \mu} u_y).$$

Let $\eta > 0$ be as in Theorem 3.2 (more precisely, as defined in Equation (36)). Also, let the constants $C_{a,m}$ be as in the apriori estimate in Theorem 3.2. Then, for $k, m \in \mathbb{N}$, $\nu \in \mathcal{F}$, and $0 \leq l \leq m$, we define

$$(46) \qquad B_l = C_{a,l} \ , \ b_{k,l} = B_l \sum_{i,j=1}^d \|\psi_{ijk}\|_{\mathcal{W}^{l,\infty}(D)} \ , \ k = 1, 2, \dots ,$$

and, with the convention that $0^0 := 1$, we also introduce the notations

$$b(l) := (b_{k,l})_{k \geq 1} \ \text{ and } \ b(l)^\nu := \prod_{k \geq 1} b_{k,l}^{\nu_k} \ .$$

Theorem 4.4. *Let Assumption 4.2 hold and $|a| < \eta$. Then for any $f \in \mathcal{K}_{a-1}^{m-1}(D)$, and for every $y \in U$, the solution u_y of the parametric, elliptic problem $P_y u_y = f_y = f$ belongs to $\mathcal{C}_b^\omega(U; \mathcal{K}_{a+1}^{m+1}(D))$. Then for any $\nu \in \mathcal{F}$, there holds the following a priori estimate*

$$(47) \qquad \forall \nu \in \mathcal{F}: \quad \|\partial_y^\nu u\|_{L^\infty(U;\mathcal{K}_{a+1}^{l+1}(D))} \leq B_l \, |\nu|! \, b(l)^\nu \, \|f\|_{\mathcal{K}_{a-1}^{l-1}(D)} ,$$

with the constants $B_l > 0$ as defined in Equation (46) (thus B_l depends on a, m, and D, but not on f).

Proof. By Lemma 4.3, our family P satisfies the assumptions of Theorem 3.2, and hence the first part of the statement follows right away from that theorem. In particular, $\eta > 0$. We shall use the following apriori estimate that follows directly from definitions, see Equation (18). For $l = 0, 1, ..., m \in \mathbb{N}$, there exist constants $C_l > 0$ such that

$$(48) \qquad \| \sum_{i,j=1}^d \partial_i (\tilde{a}_{ij} \partial_j v) \|_{\mathcal{K}_{a-1}^{l-1}(D)} \leq C_l \big(\sum_{i,j=1}^d \|\tilde{a}_{ij}\|_{\mathcal{W}^{l,\infty}(D)} \big) \|v\|_{\mathcal{K}_{a+1}^{l+1}(D)} \ .$$

Based on this apriori estimate, we next prove (47) by induction using Equation (45). We show the argument only for the highest order estimate permitted by Assumption 4.2(3), *i. e.*, for $l = m \in \mathbb{N}$, and denote $p_m = p$ in this proof. The cases $l = 0, 1, ..., m - 1$ are shown analogously.

For $|\nu| = 0$, what we need is a bound for $\|u_y\|_{\mathcal{K}_{a+1}^{m+1}}$ that is uniform in $y \in U$, and Theorem 3.2 provides exactly that (note that $b(m)^\nu = 1$ for $|\nu| = 0$). Due to the affine dependence of P_y on the coordinates y_j, $\partial_y^\mu P_y = 0$ if $|\mu| > 1$. Let then

$e_k \in \mathcal{F}$, $|e_k| = 1$, with e_k having 1 on the kth position. If $|\mu| \leq 1$, then either $|\mu| = 0$ or $\mu = e_k$ for some k. If $|\nu| > 0$, Equation (45) then gives

$$0 = \partial_y^\nu (P_y u_y) = \sum_{0 \leq \mu \leq \nu} \binom{\nu}{\mu} (\partial_y^\mu P_y)(\partial_y^{\nu-\mu} u_y)$$

$$= P_y(\partial_y^\nu u_y) + \sum_{\nu_k > 0} \nu_k (\partial_y^{e_k} P_y)(\partial_y^{\nu-e_k} u_y),$$

from which we conclude

$$\partial_y^\nu u_y = -\sum_{\nu_k > 0} \nu_k P_y^{-1}(\partial_y^{e_k} P_y)(\partial_y^{\nu-e_k} u_y).$$

Substituting $\partial_y^{e_k} P_y = \partial_{y_k} P_y = -\sum_{ij} \partial_i(\psi_{ijk} \partial_j)$ and using Equation (48) to estimate the norm of $\partial_y^{e_k} P_y$ and Theorem 3.2 to estimate the norm of P_y^{-1}, we obtain

$$\|\partial_y^\nu u_y\|_{\mathcal{K}_{a+1}^{m+1}(D)} \leq \sum_{\nu_k > 0} \nu_k C_{0,a,m} \Big(\sum_{ij} \|\psi_{ijk}\|_{\mathcal{W}^{m,\infty}(D)}\Big) \|\partial_y^{\nu-e_k} u_y\|_{\mathcal{K}_{a+1}^{m+1}(D)}$$

$$\leq \sum_{\nu_k > 0} \nu_k b_{k,m}(|\nu| - 1)! b(m)^{\nu-e_k} = |\nu|! b(m)^\mu,$$

where the last step is by the induction hypothesis. This completes the proof. $\square$

We will use the estimates in Theorem 4.4 to analyze the approximability of the parametric solution. We proceed in two steps: first, we consider the approximation with respect to $y \in U$ through "the best N-term approximation" and, afterwards, we consider the discretization in the physical domain D. In both cases, we shall use Galerkin projections onto suitable, finite dimensional subspaces of $L^2(U, \mu)$ respectively of $V := H_d^1(D) = \{u \in H^1(D), u = 0 \text{ on } \partial_d D\}$. Here μ is a fixed measure on U. For simplicity, we shall only consider the case when μ is the infinite product of the normalized Lebesgue measures on countably many copies of $[-1, 1]$.

5. Best N-term approximation

We now investigate the consequences of Theorem 4.4 for the so called "best N-term approximation" of the parametric solution u_y. To this end, we follow [12] and define μ to be the infinite product of normalized Legesque measures on $[-1, 1]$. We then consider the expansion of the parametric solution u_y with respect to tensorized polynomial bases of $L^2(U, \mu)$. Let us thus denote by $L_n(t)$ the Legendre polynomial of degree n on $(-1, 1)$, normalized such that

$$\int_{-1}^1 (L_n(t))^2 \frac{dt}{2} = 1, \quad n \in \mathbb{N}_0.$$

Since our normalization gives that $L_0 \equiv 1$, we may define, for $\nu \in \mathcal{F}$ and for $y \in U$, the tensorized Legendre polynomial $L_\nu(y)$ by the infinite product

$$L_\nu(y) = L_{\nu_1}(y_1) L_{\nu_2}(y_2)\ldots = \prod_{j \geq 1} L_{\nu_j}(y_j).$$

Note that due to $L_0 \equiv 1$, for each $\nu \in \mathcal{F}$ the infinite product is well defined and contains only finitely many nontrivial factors. Morever, with this normalization

$$\max_{t \in [-1,1]} |L_n(t)| = \sqrt{2n+1}.$$

Since $\{L_n\}_{n\geq 0}$ is a complete orthonormal set in $L^2([-1,1], dt/2)$, the collection $\mathcal{L} = \{L_\nu(y) : \nu \in \mathcal{F}\}$ is a countable orthonormal basis in $L^2(U, \mu)$. It follows that, for every Hilbert space H, any parametric function $v \in L^2(U, \mu; H)$ admits a Legendre expansion

$$v(y) = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu(y) \in L^2(U, \mu; H),$$

where the Legendre coefficients $v_\nu \in H$ are given by

$$(49) \qquad v_\nu = \int_{y \in U} v(y) L_\nu(y) \mu(dy) \in V, \quad \nu \in \mathcal{F}.$$

Since $\mathcal{L}$ is a countable orthonormal basis of $L^2(U, \mu)$, Parseval's identity gives

$$(50) \qquad \|v\|_{L^2(U,\mu;H)}^2 = \sum_{\nu \in \mathcal{F}} \|v_\nu\|_H^2.$$

Moreover, the Legendre expansion (49) induces an isomorphism between the Bochner space $L^2(U, \mu; H)$ and the space $\ell^2(\mathcal{F}; H)$ of sequences of elements of H whose norms are square summable. By (50), this isomorphism is an isometry. We thus write for $v \in L^2(U, \mu; H)$ as in (49) that $v \in \ell^2(\mathcal{F}; H)$ (with slight abuse of notation).

Theorem 5.1. *Let $f \in \mathcal{K}_{a-1}^{m-1}(D)$, $|a| < \eta$. Under Assumption 4.2, for every $\nu \in \mathcal{F}$ the Legendre coefficients u_ν of the parametric solution belong to $\mathcal{K}_{a+1}^{m+1}(D)$. Moreover, for $l = 0, 1, ..., m$, the sequences $(\|u_\nu\|_{\mathcal{K}_{a+1}^{l+1}(D)})_{\nu \in \mathcal{F}}$ are p_l-summable in the sense that*

$$(51) \qquad (\|u_\nu\|_{\mathcal{K}_{a+1}^{l+1}(D)})_{\nu \in \mathcal{F}} \in \ell^{p_l}(\mathcal{F}), \quad l = 0, 1, ..., m$$

or, equivalently, (cf. (50))

$$(52) \qquad u \in \bigcap_{l=0}^{m} \ell^{p_l}(\mathcal{F}; \mathcal{K}_{a+1}^{l+1}(D)).$$

Proof. Under Assumption 4.2, Lemma 4.3 implies that our parametric family P_y satisfies the assumptions of Theorem 3.2 with $k_0 = \omega$. In particular, $\eta > 0$ and $u \in \mathcal{C}_b^\omega(U; \mathcal{K}_{a+1}^{m+1}(D))$. Hence the Legendre coefficients u_ν of the parametric solution u_y in the expansion

$$(53) \qquad u_y = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu(y)$$

satisfy, for every $\nu \in \mathcal{F}$,

$$(54) \qquad u_\nu = \int_U u_y L_\nu(y) \mu(dy) \in \mathcal{K}_{a+1}^{m+1}(D).$$

Under Assumption 4.2, we have the a-priori estimate (47) of Theorem 4.4. To estimate the size of the Legendre coefficients u_ν, we proceed as in [12]. First, let us recall that the coefficients f_n of the Legendre expansion of a univariate function $f(t)$,

$$f(t) = \sum_{n=0}^{\infty} f_n L_n(t), \quad f_n = \int_{-1}^{1} f(t) L_n(t) \frac{dt}{2}$$

satisfy the bound

$$|f_n| \leq \frac{\beta^n}{n!} \|f^{(n)}\|_{L^\infty(-1,1)}, \quad \beta = 1/\sqrt{3}.$$

In the multivariate case, this implies for $l = 0, 1, ..., m$ the estimates

$$\|u_\nu\|_{\mathcal{K}^{l+1}_{a-1}(D)} \leq \frac{\beta^{|\nu|}}{\nu!} \|\partial_y^\nu u\|_{L^\infty(U, \mathcal{K}^{l+1}_{a-1}(D))} , \quad \nu \in \mathcal{F} .$$

Using the apriori estimate (47), we find the bound

$$\|u_\nu\|_{\mathcal{K}^{l+1}_{a-1}(D)} \leq B_l \frac{|\nu|!}{\nu!} d_l^\nu , \quad \nu \in \mathcal{F} , \quad l = 0, 1, ..., m$$

where the sequences $d_l = (d_{k,l})_{k \geq 1}$ are given by

$$d_{k,l} := \beta b_{k,l} , \quad k = 1, 2, ... , \ l = 0, 1, ..., m .$$

with the sequence $b(m)$ and the constant B_m as in (46) and in the apriori estimate in Theorem 4.4, respectively.

Based on the estimate (5), it remains to prove the $\ell^{p_l}(\mathcal{F})$-summability of the sequences $(\|u_\nu\|_{\mathcal{K}^{l+1}_{a-1}(D)})_{\nu \in \mathcal{F}}$ for $l = 0, 1, ..., m$. Due to Assumption 4.2,(3) and by the definition of the sequences d_l for $l = 0, 1, ..., m$, we find that $d_l \in \ell^{p_l}(\mathbb{N})$ for the same exponents $0 < p_l < 1$ as in Assumption 4.2 (3). By Assumption 4.2, the sequences d_l also belong to $\ell^1(\mathbb{N})$. Now [12, Theorem 7.2] (with p_l in place of $0 < p < 1$) implies the assertion. $\square$

From the p summability (51) we immediately obtain the following result on convergence rates of N-term approximations of the Legendre series (53). The following theorem is the analogue of Corollary 7.4(i) in [12].

Theorem 5.2. *Under Assumption 4.2, for any $N \in \mathbb{N}$ there exists a set $\Lambda_N \subset \mathcal{F}$ of cardinality not exceeding N such that*

$$(55) \qquad \left\| u - \sum_{\nu \in \Lambda_N} u_\nu L_\nu \right\|_{L^2(U, \mu; H^1_d(D))} \leq C N^{-s_0} , \quad s_l = \frac{1}{p_0} - \frac{1}{2} .$$

Here, the space $H^1_d(D)$ is as in (4) and the constant C is proportional to

$$\left\| \left(\|u_\nu\|_{H^1_d(D)} \right) \right\|_{\ell^{p_l}(\mathcal{F})} .$$

6. Spatial discretization in 2D

So far, we have analyzed then N-term truncation of the Legendre expansion (53) under the *assumption that the coefficient functions $u_\nu \in V$ can be known exactly.* In practice, these coefficients are to be approximated by Finite Element spaces in the domain D. In this section, we analyze the corresponding discretization error. Throughout, we will work under Assumption 4.2. discretization, we consider Finite Element spaces that provide optimal rates of convergence for the non-parametric equation.

We assume in this section that $D \subset \mathbb{R}^2$ is a bounded Lipschitz polygon with straight sides (so $d = 2$). In D, we consider a nested sequence $\{\mathcal{T}_\mu\}_{\mu \geq 0}$ regular, simplicial triangulations in the sense of Ciarlet [11]. We let $V_\mu \subset V$ denote the associated Finite Element space of continuous, piecewise polynomials of degree $m \geq 1$ on the mesh $\mathcal{T}_\mu$. We assume that, for all μ, it holds $V_\mu \subset H^1_d(D)$, *i. e.*, the functions in V_μ vanish on $\partial_d D$.

Let η be as in Theorem 3.2, whose assumptions are satisfied in view of Lemma 4.3 (we are using Assumption 4.2 here). In particular $\eta > 0$. Let us fix $0 < a < \eta$

and let $f \in \mathcal{K}_{a-1}^{m-1}(D)$. As before, we denote by $u \in H_d^1(D)$ the solution of the uniformly strongly elliptic, parametric equation

$$(56) \qquad P_y u_y = - \sum_{ij} \partial_i (\tilde{a}_{ij} \partial_j v) = f \in \mathcal{K}_{a-1}^{m-1}(D),$$

with Dirichlet boundary conditions on $\partial_d D$ and Neumann boundary conditions on $\partial_n D$, as before. We continue to exclude the case of Neumann-Neumann corners and edges. Then $u \in \mathcal{C}_b^\omega(U; \mathcal{K}_{a+1}^{m+1}(D))$, that is, $u_y \in \mathcal{K}_{a+1}^{m+1}(D)$ for each y and the dependence on y is analytic, with bounded derivatives (as a $\mathcal{K}_{a+1}^{m+1}(D)$-valued function) on U, by Theorem 3.2.

It is known from [5, 7, 27, 32] that one can construct nested sequences $\{\mathcal{T}_\mu\}_{\mu \geq 0}$ of regular, simplicial triangulations that depend only on $0 < a < \eta$ with the following properties. For any given $y \in U$, let $u_y^\mu \in V_\mu$ denote the Galerkin approximation of u_y, that is, the Galerkin projection u_y^μ of the parametric solution u_y onto V_μ, which is defined for every $y \in U$ by

$$B(y; u_y^\mu, w) = (f, w) \quad \forall w \in V_\mu \,,$$

where the bilinear form B was defined in Equation (7). Then, for any $m \in \mathbb{N}_0$ there exists a constant $C_m > 0$ such that for every $y \in U$ and for every $\mu \in \mathbb{N}$, there holds (see, e.g. [7, 27])

$$(57) \qquad \begin{cases} \|u_y - u_y^\mu\|_{H^1(D)} \leq C_m \dim(V_\mu)^{-m/d} \|f\|_{\mathcal{K}_{a-1}^{m-1}(D)} \\ \dim(V_\mu) \sim 2^{d\mu}, \quad d = 2, \end{cases}$$

The rate of convergence is then $2^{-m\mu}$.

We next assume that *for given $m \in \mathbb{N}_0$ as in Assumption 4.2, a nested sequence $\{\Lambda_N(m)\}_{N \in \mathbb{N}}$ of finite index sets contained in $\mathcal{F}$ of cardinality not exceeding N has been determined such that $\sum_{\nu \in \Lambda_N(m)} u_\nu L_\nu$ is a best N-term approximation to $u(y)$ in $L^2(U, \mu; \mathcal{K}_{a+1}^{m+1}(D))$.* Note that the sequence $\{\Lambda_N(m)\}_{N \geq 1}$ depends on $m \geq 1$, and possibly differs from the sequence $\{\Lambda_N(0)\}_{N \geq 1}$ obtained from best ℓ-term approximation of $u(y)$ in the $L^2(U, \mu; V)$-norm considered in [12, 13]. We define

$$(58) \qquad \Lambda_N = \Lambda_N(0) \cup \Lambda_N(1) \cup ... \cup \Lambda_N(m) \subset \mathcal{F}, \quad N = 1, 2, ...$$

and observe that $\#\Lambda_N \leq (m+1)N$. Also, the sequence $\{\Lambda_N\}_{N \geq 1}$ defined in (58) implicitly depends on the regularity parameter $m \in \mathbb{N}$ in Assumption 4.2,(3); *throughout what follows, we assume that $m \in \mathbb{N}$ in (58) equal m in Assumption 4.2,(3) and also equals the degree of the Finite Element spaces.*

Recall that, for any $\nu \in \Lambda_N$, Theorem 5.1 we have $u_\nu \in \mathcal{K}_{a+1}^{m+1}(D)$. Hence, there exists *one* sequence $\{V_\mu\}_{\mu \geq 0} \subset V$ of (Finite Element) subspaces of finite dimensions $N_\mu = \dim V_\mu$ such that for *all* $\nu \in \Lambda$, the Finite Element approximations $u_\nu^\mu \in V_\mu$ satisfy

$$(59) \qquad \|u_\nu - u_\nu^\mu\|_{H_d^1(D)} \leq C N_\mu^{-t} \|u_\nu\|_{\mathcal{K}_{a+1}^{m+1}(D)} \,,$$

where $t = m/d$, $d = 2$, and where $N_\mu = \dim V_\mu \to \infty$ as $\mu \to \infty$ with the constant $C > 0$ being independent of μ and of ν. With this observation, using Theorem 5.2, we obtain similarly as in [12, Section 8], case 1, [20]

Theorem 6.1. *Let $D \subset \mathbb{R}^d$, $d = 2$ be a bounded Lipschitz polygon and let $u_y \in \mathcal{K}_{a+1}^{m+1}(D)$, $f \in \mathcal{K}_{a-1}^{m-1}(D)$ be as in Equation (56). Suppose that Assumption 4.2 holds with some $m \in \mathbb{N}$.*

If, in Assumption 4.2, $p_0 = p_m = p \in (0, 1)$, then there exists a nested sequence $\{\Lambda_\ell\}_{\ell \geq 1}$ of sets $\Lambda_\ell \subset \mathcal{F}$ of cardinality $\#\Lambda_\ell < \infty$ and, for each $\ell \in \mathbb{N}$ and each $\nu \in \Lambda_\ell$, there exists a selection of discretization levels $\mu(\ell, \nu) \in \mathbb{N}$ in the hierarchy $\{V_\mu\}_{\mu \geq 0}$ of FE spaces such that

$$(60) \qquad \left\| u - \sum_{\nu \in \Lambda_\ell} u_\nu^{\mu(\ell,\nu)} L_\nu \right\|_{L^2(U,\mu;V)} \leq C M_\ell^{-r} \|f\|_{\mathcal{K}_{a-1}^{m-1}(D)}$$

where $r = \min\{1/p - 1/2, m/d\}$, and where u_ν^μ are as in Equation (59).

If, in Assumption 4.2, $p_0 < p_m < 1$, then the rate r in convergence estimate (60) becomes

$$(61) \qquad r = \min\left(\frac{t(1/p_0 - 1/2)}{t + 1/p_0 - 1/p_m}, t \right) .$$

In (60), M_ℓ denotes the total number of degrees of freedom, defined by

$$M_\ell := \sum_{\nu \in \Lambda_\ell} N_{\mu(\ell,\nu)}$$

denotes the total number of degrees of freedom in the approximation.

7. SPATIAL DISCRETIZATION IN 3D AND OTHER EXTENSIONS

In this section, we assume that D is a polyhedral domain in three dimensions, so $d = 3$. However, we formulate our results so that they remain true in the case $d = 2$ of a polygon. The notation that is not explained is the same as in the previous section.

First of all, we have the following extension of the approximation result of Equation (57). More to the point, we can still construct a sequence $V_\mu \subset V$ such that the Galerkin projections $u_y^\mu \in V_\mu$ satisfy [9]

$$(62) \qquad \begin{cases} \|u_y - u_y^\mu\|_{H^1(D)} \leq C \dim(V_\mu)^{-m/d} \|f\|_{H^{m-1}(D)} \\ \dim(V_\mu) \sim 2^{d\mu}, \quad d = 2, 3, \end{cases}$$

for all y, where the constant $C > 0$ is independent of $y \in U$, and of μ (but depends on m). The rate of convergence is then $2^{-m\mu}$. Note that this time we must assume $f \in H^{m-1}(D)$ and use anisotropic regularity, in addition to the isotropic regularity result $u_y \in \mathcal{K}_{a-1}^{m-1}(D)$. Also, one has to choose the spaces V_μ to consist of functions that vanish near the edges, in order to obtain a nested sequence of spaces.

Hence, there exists again *one* sequence of FE spaces $\{V_\mu\}_{\mu \geq 0} \subset V$ such that *each* $\nu \in \Lambda$, the Finite Element approximations $u_\nu^\mu \in V_\mu$ satisfy

$$(63) \qquad \|u_\nu - u_\nu^\mu\|_{H_d^1(D)} \leq C N_\mu^{-t} \|f\|_{H_{a-1}^{m-1}(D)} ,$$

where $t = m/d$ and where $N_\mu = \dim V_\mu \to \infty$ as $\mu \to \infty$ with the constant $C > 0$ being independent of μ and of ν. Also, let us observe that the error bounds in Theorem 5.2 can be chosen to depend only on $\|f\|_{\mathcal{K}_{a-1}^{m-1}(D)}$ and hence, by using *a priori* bounds for the coefficients u_ν and choosing those coefficients with the first ℓ highest error bounds, we can arrange that the set Λ does not depend on f (and hence it will not depend on u either).

With these observations, we have the following result, that covers both two and three dimensions. Note however that our result in two dimensions in the previous section is more general.

Theorem 7.1. *Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a Lipschitz polyhedron ($d = 3$) or a Lipschitz polygon ($d = 2$). Suppose that Assumption 4.2 holds with some $m \in \mathbb{N}$ and with $0 < p_0 \leq p_1 \leq \ldots \leq p_m < 1$.*

Then there exists a nested sequence $\{\Lambda_\ell\}_{\ell \geq 1}$ of sets $\Lambda_\ell \subset \mathcal{F}$ of cardinality $\#\Lambda_\ell < \infty$ and, for each $\ell \in \mathbb{N}$ and each $\nu \in \Lambda_\ell$, there exists a selection of discretization levels $\mu(\ell, \nu) \in \mathbb{N}$ in the hierarchy $\{V_\mu\}_{\mu \geq 0}$ of FE spaces such that (60) holds, with $\|f\|_{H^{m-1}(D)}$ in place of $\|f\|_{\mathcal{K}_{a-1}^{m-1}(D)}$, and with the rate r given by (61) and with the number M_N of degrees of freedom being as in (60).

We close by remarking that the approximation spaces in Theorems 6.1, 7.1 are

$$(64) \qquad S_\ell = \bigoplus_{\nu \in \Lambda_\ell} V_{\mu(\ell, \nu)} \otimes \{L_\nu\} \,,$$

and $u_\ell := \sum_{\nu \in \Lambda_\ell} u_\nu^{\mu(\nu)} L_\nu \in S_\ell$ is the Galerkin approximation of u.

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