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J. Šukys, Ch. Schwab and S. Mishra

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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Abstract We consider stochastic multi-dimensional linear hyperbolic systems of conservation laws. We prove existence and uniqueness of a random weak solution, provide estimates for the regularity of the solution in terms of regularities of input data, and show existence of statistical moments. Bounds for mean square error vs. *expected* work are proved for the Multi-Level Monte Carlo Finite Volume algorithm which is used to approximate the moments of the solution. Using our implementation called ALSVID-UQ, numerical experiments for acoustic wave equation with uncertain uniformly and log-normally distributed coefficients are conducted.

1 Introduction

A number of *linear* phenomenon in physics and engineering are modeled by *linear* hyperbolic systems of conservation laws, i.e. for a given bounded domain $\mathbf{D} \subset \mathbb{R}^d$,

$$\begin{cases} \mathbf{U}_t(\mathbf{x},t) + \sum_{r=1}^{d} \left(\mathbf{A}_r(\mathbf{x})\mathbf{U}(\mathbf{x},t)\right)_{\mathbf{x}_r} = \mathbf{S}(\mathbf{x}), \\ \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x}), \end{cases} \quad \forall (\mathbf{x},t) \in \mathbf{D} \times \mathbb{R}_+. \tag{1}$$

Here, $\mathbf{U} : \mathbf{D} \times \mathbb{R}_+ \to \mathbb{R}^m$ denotes the vector of conserved variables, $\mathbf{A}_r : \mathbb{R}^m \to \mathbb{R}^m$ denote linear maps (linear fluxes), and $\mathbf{S} : \mathbf{D} \to \mathbb{R}^m$ denotes the source term. The

Jonas Šukys
SAM, ETH Zürich, Rämistrasse 101, Switzerland, e-mail: jonas.sukys@sam.math.ethz.ch

Christoph Schwab
SAM, ETH Zürich, Rämistrasse 101, Switzerland, e-mail: christoph.schwab@sam.math.ethz.ch

Siddhartha Mishra
SAM, ETH Zürich, Rämistrasse 101, Switzerland, e-mail: smishra@sam.math.ethz.ch

partial differential equation is augmented with initial data $\mathbf{U}_0 \in \mathbf{D} \to \mathbb{R}^m$. Examples of (1) are acoustic wave equation, equations of linear elasticity and many other.

For the sake of simplicity, only *periodic* Cartesian physical domains $\mathbf{D} = I_1 \times \cdots \times I_d \subset \mathbb{R}^d$ will be considered. However, we would like to note, that most of the results presented can be also extended to the domains with boundaries.

The well-posedness of (1) is analyzed in [8, 9, 10, 13, 23]. However, for non-constant coefficients $\mathbf{A}_r(\mathbf{x})$, the closed form of the analytic solutions is often not available. Efficient numerical schemes for approximating (1) include Finite Volume, Finite Difference, Finite Element and discontinuous Galerkin methods [8, 13, 23].

The *classical* paradigm for designing numerical schemes for approximation of (1) assumes that initial data $\mathbf{U}_0$, source $\mathbf{S}$ and coefficients $\mathbf{A}_r(\mathbf{x})$ are known exactly. However, in most practical situations, measurements (if available at all) of the input data are prone to uncertainty. This uncertainty in input data propagates into the solution of (1). The resulting uncertainties are modeled in a probabilistic manner.

The first aim of this manuscript is to develop an appropriate mathematical framework of random weak solutions for systems of linear conservation laws. We define random weak solutions and provide an existence and uniqueness result, generalizing the classical well-posedness results [8, 9, 10, 13, 23] in the case of uncertain inputs.

The second aim of this manuscript is to present *efficient* numerical methods for approximation of *random* linear hyperbolic systems of conservation laws.

The design of such efficient numerical schemes has seen a lot of activity in recent years, including the stochastic Galerkin based on generalized Polynomial Chaos and stochastic collocation, see references in [17]. Stochastic Galerkin method is highly *intrusive*: existing codes (for deterministic solves) need to be completely reconfigured and are hard to parallelize. Currently these methods are not able to handle even a moderate number of sources of uncertainty (stochastic dimensions).

Another class of methods are the so-called *Monte Carlo* (MC) methods where the underlying deterministic PDE is solved for each statistical *sample* and the samples are combined to ascertain statistical information about the random solution. Although non-intrusive, easy to code and to parallelize, the error convergence rate (with repect to the number of samples M) of $1/2$ requires a large number of "samples" (numerical solves of (1)) in order to ensure low statistical errors.

Such slow convergence has inspired the development of *Multi-Level Monte Carlo* or MLMC methods. They were introduced by S. Heinrich for numerical quadrature [12], developed by M. Giles for Itô SPDE [6], and applied to various SPDEs [3, 5, 20]. In particular, recent papers [15, 17, 18, 19] extended and analyzed the MLMC algorithm for nonlinear conservation laws with random initial data, fluxes and sources. The error analysis of the MLMC method in case of scalar conservation laws [15] showed that statistical moments are approximated with the *same accuracy versus cost ratio as a single deterministic solve on the same mesh*. An optimal static load balancing of [22] enabled us to compute solutions of the multi-dimensional random Euler, magnetohydrodynamics (MHD) and shallow water equations.

We extend the MLMC method to linear hyperbolic systems of balance laws (1) and demonstrate that it constitutes a considerable speed-up over the MC method. In particular, the MLMC-FVM method is shown to converge and our convergence analysis yields an optimal strategy for choosing the number MC samples. With this strategy, under some conditions on the FDM/FVM convergence rate, statistical moments of the random weak solution are approximated with the *same accuracy versus expected computational cost ratio as a single deterministic solve of* (1).

The key differences from the recent papers [15, 17] are the following: 1) we consider *linear systems* of conservation laws, whereas in [15] is restricted to *nonlinear scalar* conservation laws; 2) uncertain coefficients $\mathbf{A}_r(\mathbf{x})$ act as random flux, whereas in [15] only uncertain initial data is considered; 3) since we consider only *linear* systems, unlike in [17], the well-posedness results are available. The efficiency of the algorithm is demonstrated throughout numerical experiments.

The remainder of the manuscript is organized as follows: in Sect. 2, we introduce the concept of random solution and show it to be well-posed. In Sect. 3, numerical schemes are designed and the convergence of the approximation error is investigated. In Sect. 4, we rewrite the acoustic wave equation as a linear hyperbolic system of conservation laws. Numerical experiments are presented in Sect. 5.

2 Linear systems of stochastic hyperbolic conservation laws

Definition 1 (Strong hyperbolicity). In the case ($d = 1$), the linear system of conservation laws (1) is called strongly hyperbolic [10] if $\forall \mathbf{x} \in \mathbf{D}, \exists \mathbf{Q_x} : \mathbb{R}^m \to \mathbb{R}^m$:

$$\sup_{\mathbf{x}\in\mathbf{D}} \|\mathbf{Q}_{\mathbf{x}}^{-1}\| \|\mathbf{Q}_{\mathbf{x}}\| \leq K < \infty, \quad \mathbf{Q}_{\mathbf{x}}^{-1}\mathbf{A}_1(\mathbf{x})\mathbf{Q}_{\mathbf{x}} = \operatorname{diag}(\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^{m\times m}. \tag{2}$$

The extension of the definition of strong hyperbolicity for $d > 1$ is available in [10].

Let V denote an arbitrary Banach space. The following notation will be used:

$$\|\mathbf{U}, \mathbf{S}, t\|_V = \|\mathbf{U}\|_V + t\|\mathbf{S}\|_V, \quad \mathbf{U}, \mathbf{S} \in V,\ t \geq 0. \tag{3}$$

The following result summarizes some of the classical existence and uniqueness results [8, 9, 10, 13, 23] for weak solutions of linear hyperbolic systems (1).

Theorem 1. *Denote* $\mathbf{L}^p(\mathbf{D}) = L^p(\mathbf{D})^m$*,* $\mathbf{W}^{r,\infty}(\mathbf{D}) = W^{r,\infty}(\mathbf{D})^m$*, and assume*

1. *linear system* (1) *is strongly hyperbolic with* $K < \infty$ *in* (2)*,*
2. *there exist* $r_{\mathbf{0}}, r_{\mathbf{S}}, r_{\mathbf{A}} \in \mathbb{N} \cup \{0\}$ *such that:*

$$\mathbf{U}_0 \in \mathbf{W}^{r_{\mathbf{0}},\infty}(\mathbf{D}), \quad \mathbf{S} \in \mathbf{W}^{r_{\mathbf{S}},\infty}(\mathbf{D}), \quad \mathbf{A}_r \in \mathbf{W}^{r_{\mathbf{A}},\infty}(\mathbf{D})^2. \tag{4}$$

Then, for every finite time horizon $T < \infty$*,* (1) *admits a unique weak solution* $\mathbf{U} \in L^\infty(\mathbf{D} \times [0,T])^m$*. Furthermore, for every* $0 \leq t \leq T$*, the following holds:*

$$\|\mathbf{U}(\cdot,t)\|_{\mathbf{L}^2(\mathbf{D})} \leq K\|\mathbf{U}_0, \mathbf{S}, t\|_{\mathbf{L}^2(\mathbf{D})}, \tag{5}$$

$$\|\mathbf{U}(\cdot,t) - \mathbf{V}(\cdot,t)\|_{\mathbf{L}^2(\mathbf{D})} \leq K\|\mathbf{U}_0 - \mathbf{V}_0,\ \mathbf{S}_{\mathbf{U}} - \mathbf{S}_{\mathbf{V}},\ t\|_{\mathbf{L}^2(\mathbf{D})}, \tag{6}$$

$$\mathbf{U} \in C([0,T], \mathbf{W}^{\bar{r},\infty}(\mathbf{D})), \quad \textit{with} \quad \bar{r} = \min\{r_{\mathbf{0}}, r_{\mathbf{S}}, r_{\mathbf{A}}+1\}. \tag{7}$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote a complete probability space and $\mathscr{B}(V)$ a Borel σ-algebra.

Definition 2 (Random field). A V-valued random field is a measurable mapping

$$\mathbf{U} : (\Omega, \mathscr{F}) \to (V,\ \mathscr{B}(V)), \quad \omega \mapsto \mathbf{U}(\mathbf{x}, t, \omega).$$

The stochastic version of the linear system of hyperbolic conservation laws (1) is

$$\begin{cases} \mathbf{U}_t(\mathbf{x},t,\omega) + \displaystyle\sum_{r=1}^{d} \mathbf{A}_r(\mathbf{x},\omega)\mathbf{U}_{\mathbf{x}_r} = \mathbf{S}(\mathbf{x},\omega), \\ \qquad\qquad\qquad \mathbf{U}(\mathbf{x},0,\omega) = \mathbf{U}_0(\mathbf{x},\omega), \end{cases} \quad \forall(\mathbf{x},t) \in \mathbf{D}\times\mathbb{R}_+, \quad \forall\omega\in\Omega. \tag{8}$$

Here, $\mathbf{U}_0$ and $\mathbf{S}$ are $\mathbf{L}^2(\mathbf{D})$-valued random fields $(\Omega,\mathscr{F}) \to (\mathbf{L}^2(\mathbf{D}), \mathscr{B}(\mathbf{L}^2(\mathbf{D})))$. The fluxes $\mathbf{A}_r$ are $\mathbf{L}^\infty(\mathbf{D})^2$-valued random fields $(\Omega,\mathscr{F}) \to (\mathbf{L}^\infty(\mathbf{D})^2, \mathscr{B}(\mathbf{L}^\infty(\mathbf{D})^2))$. We define the following notion of solutions of (8):

Definition 3 (Random weak solution). A $C([0,T],\mathbf{L}^2(\mathbf{D}))$-valued random field $\mathbf{U}$: $\Omega \ni \omega \mapsto \mathbf{U}(\mathbf{x},t,\omega)$ is a random weak solution to the stochastic linear hyperbolic system of conservation laws (8) if it is a weak solution of (1) for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Based on Theorem 1, we obtain the following well-posedness result for (8).

Theorem 2. *In* (8)*, assume that the following holds for some* $k \in \mathbb{N}\cup\{0,\infty\}$*:*

1. (8) *is strongly hyperbolic, with* $\bar{K}_k = \|K(\omega)\|_{L^k(\Omega,\mathbb{R})} < \infty$*,*
2. *there exists non-negative integers* $r_{\mathbf{0}}, r_{\mathbf{S}}, r_{\mathbf{A}} \in \mathbb{N}\cup\{0,\infty\}$ *such that:*

$$\mathbf{U}_0 \in L^k(\Omega, \mathbf{W}^{r_{\mathbf{0}},\infty}(\mathbf{D})), \quad \mathbf{S} \in L^k(\Omega, \mathbf{W}^{r_{\mathbf{S}},\infty}(\mathbf{D})), \quad \mathbf{A}_r \in L^0(\Omega, \mathbf{W}^{r_{\mathbf{A}},\infty}(\mathbf{D})^2), \tag{9}$$

3. *each random field* $\mathbf{A}_r$*,* $r = 1,\dots,d$*, is independent of* $\mathbf{U}_0$ *and* $\mathbf{S}$ *on* $(\Omega,\mathscr{F},\mathbb{P})$.

Then, for $T<\infty$*,* (8) *admits a unique random weak solution*

$$\mathbf{U} : \Omega \to C([0,T],\mathbf{L}^2(\mathbf{D})), \qquad \omega \mapsto \mathbf{U}^\omega(\cdot,\cdot), \quad \forall\omega\in\Omega, \tag{10}$$

where $\mathbf{U}^\omega(\cdot,\cdot)$ *is the solution to the* deterministic *system* (1)*. Moreover,* $\forall t \in [0,T]$*,*

$$\|\mathbf{U}(\cdot,t,\omega)\|_{\mathbf{L}^2(\mathbf{D})} \le K(\omega)\|\mathbf{U}_0(\cdot,\omega),\ \mathbf{S}(\cdot,\omega),\ t\|_{\mathbf{L}^2(\mathbf{D})}, \quad \mathbb{P}\text{-a.s.}, \tag{11}$$

$$\|\mathbf{U}\|_{L^k(\Omega,C([0,T],\mathbf{L}^2(\mathbf{D})))} \le \bar{K}_k\|\mathbf{U}_0,\mathbf{S},t\|_{L^k(\Omega,\mathbf{L}^2(\mathbf{D}))}, \tag{12}$$

with $\|\mathbf{U},\mathbf{S},t\|_{L^k(\Omega,V)} = \|\|\mathbf{U},\mathbf{S},t\|_V\|_{L^k(\Omega,\mathbb{R})}$. We outline the main ideas of the proof.

Proof. We proceed step by step, and using the following lemma:

Lemma 1. *Let* E *be a separable Banach space and* $X : \Omega \to E$ *be an* E*-valued random variable on* $(\Omega,\mathscr{F})$*. Then, mapping* $\Omega \ni \omega \mapsto \|X(\omega)\|_E \in \mathbb{R}$ *is measurable.*

1. By Theorem 1, the random field in (10) is well defined. Furthermore, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathbf{U}(\cdot,\cdot,\omega)$ is a weak solution of (1).
2. $\forall t \in [0,T]$, $\forall j = 1,\dots,m$, we verify the measurability of the component map $\Omega \ni \omega \mapsto \mathbf{U}_j(\cdot,t,\omega) \in L^2(\mathbf{D})$. Since $L^2(\mathbf{D})$ is a separable Hilbert space, the $\mathscr{B}(L^2(\mathbf{D}))$ is the smallest σ-algebra containing all subsets
$$\{v \in L^2(\mathbf{D}) : \varphi(v) \le \alpha\} \quad : \quad \varphi \in L^2(\mathbf{D}),\ \alpha \in \mathbb{R}.$$
For a fixed $\alpha \in \mathbb{R}$, $\varphi \in L^2(\mathbf{D})$, consider the set $\{\mathbf{U}_j(\cdot,t,\omega) : \varphi(\mathbf{U}_j(\cdot,t,\omega)) \le \alpha\}$. By continuity (5) in $L^2(\mathbf{D})$, since $\mathbf{U}_0, \mathbf{S} \in L^0(\Omega, \mathbf{L}^2(\mathbf{D}))$ and $\mathbf{A}_r \in L^0(\Omega, \mathbf{L}^2(\mathbf{D})^2)$, we obtain $\mathbf{U}_j(\cdot,t,\cdot) \in L^0(\Omega, L^2(\mathbf{D}))$, for every $0 \le t \le T$.
3. (11) follows from (5) and Lemma 1; (12) follows from (11) and assumption 3,
$$\begin{aligned}\|\mathbf{U}\|^k_{L^k(\Omega,C([0,T],\mathbf{L}^2(\mathbf{D})))} &= \mathbb{E}\left[\max_{0\le t\le T} \|\mathbf{U}(\cdot,t,\omega)\|^k_{\mathbf{L}^2(\mathbf{D})}\right] \\ &\le \mathbb{E}\left[K^k(\omega)\|\mathbf{U}_0,\mathbf{S},t\|^k_{\mathbf{L}^2(\mathbf{D})}\right] = \bar{K}^k_k \|\mathbf{U}_0,\mathbf{S},t\|^k_{L^k(\Omega,\mathbf{L}^2(\mathbf{D}))}.\end{aligned}$$

This theorem ensures the existence of the k-th moments $\mathscr{M}^k(\mathbf{U}) \in (\mathbf{L}^2(\mathbf{D}))^k$ [15] of the random weak solution, provided $\mathbf{U}_0, \mathbf{S} \in L^k(\Omega, \mathbf{L}^2(\mathbf{D}))$ and $K \in L^k(\Omega, \mathbb{R})$. □

3 Multi-Level Monte Carlo FVM and FDM methods

3.1 Monte Carlo Method

Assume the hypothesis of Theorem 2 holds for $k \ge 1$, i.e. the unique random weak solution exists and has bounded k-th moments [15]. Fix $M \in \mathbb{N}$ and let $\hat{\mathbf{I}}^i := \{\hat{\mathbf{U}}^i_0, \hat{\mathbf{S}}^i, \hat{\mathbf{A}}^i_1, \dots, \hat{\mathbf{A}}^i_d\}$ be independent, identically distributed (i.i.d.) samples of input data $\hat{\mathbf{I}}(\omega) := \{\hat{\mathbf{U}}_0(\omega), \hat{\mathbf{S}}(\omega), \hat{\mathbf{A}}_1(\omega), \dots, \hat{\mathbf{A}}_d(\omega)\}$. For a fixed time $0 \le t \le T$, the *Monte Carlo (MC)* estimate of the expectation $\mathscr{M}^1(\mathbf{U}) = \mathbb{E}[\mathbf{U}(\cdot,t,\cdot)]$ is given by

$$E_M[\mathbf{U}(\cdot,t,\cdot)] := \frac{1}{M}\sum_{i=1}^{M} \hat{\mathbf{U}}^i(\cdot,t), \tag{13}$$

where $\hat{\mathbf{U}}^i(\cdot,t)$ denotes the M unique random weak solutions of the deterministic linear system of conservation laws (1) with the input data $\hat{\mathbf{I}}^i$. By (11), we have

$$\|E_M[\mathbf{U}]\|_{\mathbf{L}^2(\mathbf{D})} = \frac{1}{M}\left\|\sum_{i=1}^{M}\hat{\mathbf{U}}^i\right\|_{\mathbf{L}^2(\mathbf{D})} \le \frac{1}{M}\sum_{i=1}^{M}\|\hat{\mathbf{U}}^i\|_{\mathbf{L}^2(\mathbf{D})} \overset{(11)}{\le} \frac{1}{M}\sum_{i=1}^{M} K^i \|\hat{\mathbf{U}}^i_0, \hat{\mathbf{S}}^i, t\|_{\mathbf{L}^2(\mathbf{D})}.$$

Using the i.i.d. property of the samples $\{\hat{\mathbf{I}}^i\}_{i=1}^M$ of the random input data $\mathbf{I}(\omega)$, Lemma 1, the linearity of $\mathbb{E}[\cdot]$ and assumption 3 in Theorem 2, we obtain

$$\mathbb{E}\left[\|E_M[\mathbf{U}(\cdot,t,\omega)]\|_{\mathbf{L}^2(\mathbf{D})}\right] = \bar{K}_1\|\hat{\mathbf{U}}_0(\cdot,\omega),\hat{\mathbf{S}}(\cdot,\omega),t\|_{L^k(\Omega,\mathbf{L}^2(\mathbf{D}))} < \infty. \tag{14}$$

The following result states that MC estimates (13) converge as $M \to \infty$.

Theorem 3. *Assume the hypothesis of Theorem 2 is satisfied with $k \geq 2$, i.e. the second moments of the random initial data $\mathbf{U}_0$ source $\mathbf{S}$ and K exist. Then, the MC estimates $E_M[\mathbf{U}(\cdot,t,\omega)]$ in (13) converge to $\mathscr{M}^1(\mathbf{U}) = \mathbb{E}[\mathbf{u}]$ as $M \to \infty$. Furthermore,*

$$\|\mathbb{E}[\mathbf{U}] - E_M[\mathbf{U}](\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \leq M^{\frac{1}{2}}\bar{K}_2\|\mathbf{U}_0,\mathbf{S},t\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))}. \tag{15}$$

Proof. We follow the structure of the analogous proofs in [3]. The M samples $\{\hat{\mathbf{I}}^i\}_{i=1}^M$ are interpreted as realizations of M independent "copies" of $\mathbf{I}(\omega)$ on the probability space $(\Omega,\mathscr{F},\mathbb{P})$, i.e. $\hat{\mathbf{I}}^i = \hat{\mathbf{I}}^i(\omega)$. By $\mathbf{L}^2(\mathbf{D})$ contractivity (6), $\forall 0 \leq t \leq T$, solutions $\hat{\mathbf{U}}(\cdot,t,\omega)$ of any two i.i.d. realizations of $\mathbf{I}(\omega)$ are strongly measurable as $\mathbf{L}^2(\mathbf{D})$-valued functions, hence are independent random fields. By Lemma 1 and by continuity (11), the mapping $\omega \mapsto \|\mathbf{U}(\cdot,t,\omega)\|_{\mathbf{L}^2(\mathbf{D})}$ is measurable. Hence,

$$\begin{aligned}\mathbb{E}\left[\|\mathbb{E}[\mathbf{U}] - E_M[\mathbf{U}](\omega)\|^2_{\mathbf{L}^2(\mathbf{D})}\right] &= \frac{1}{M^2}\mathbb{E}\left[\sum_{i=1}^M \|\mathbb{E}[\mathbf{U}] - \hat{\mathbf{U}}^i(\omega)\|^2_{\mathbf{L}^2(\mathbf{D})}\right] \\ = \frac{1}{M}\mathbb{E}\left[\|\mathbb{E}[\mathbf{U}] - \mathbf{U}\|^2_{\mathbf{L}^2(\mathbf{D})}\right] &= \frac{1}{M}\left(\mathbb{E}\|\mathbf{U}\|^2_{\mathbf{L}^2(\mathbf{D})} - \|\mathbb{E}[\mathbf{U}]\|^2_{\mathbf{L}^2(\mathbf{D})}\right) \leq \frac{1}{M}\mathbb{E}\|\mathbf{U}\|^2_{\mathbf{L}^2(\mathbf{D})}.\end{aligned}$$

Using (11) and assumption 3, we deduce

$$\mathbb{E}\left[\|\mathbb{E}[\mathbf{U}] - E_M[\mathbf{U}](\omega)\|^2_{\mathbf{L}^2(\mathbf{D})}\right] \leq M^{-1}\bar{K}_2^2\mathbb{E}\left[\|\mathbf{U}_0,\mathbf{S},t\|^2_{\mathbf{L}^2(\mathbf{D})}\right],$$

which implies (15) upon taking square roots. □

3.2 Finite Difference and Finite Volume Methods

In considerations of the MC method, we have assumed that the *exact* random weak solutions $\hat{\mathbf{U}}^i(\mathbf{x},t,\omega)$ of (1) are available. In most cases of engineering interest, solutions are approximated by Finite Difference [10] and Finite Volume [13] methods.

If $\mathbf{U}_0$ and $\mathbf{S}$ are continuous (then solution $\mathbf{U}$ is also continuous), conventional Finite Difference methods [10, 23] can be used where spatial and temporal derivatives in (1) are approximated by upwinded difference quotients. For discontinuous $\mathbf{U}_0$ and $\mathbf{S}$, (then solution $\mathbf{U}$ is also discontinuous) we present Finite Volume Method.

Let $\mathscr{T} = \mathscr{T}^1 \times \cdots \times \mathscr{T}^d$ denote a uniform axiparallel quadrilateral mesh of the domain $\mathbf{D}$, consisting of identical cells $C_{\mathbf{j}} = C_{\mathbf{j}_1} \times \cdots \times C_{\mathbf{j}_d}$, $\mathbf{j}_r = 1,\ldots,\#\mathscr{T}^r$.

Assume *mesh widths* are equal in each dimension, i.e. $\Delta x := \frac{|I_1|}{\#\mathscr{T}_1} = \cdots = \frac{|I_d|}{\#\mathscr{T}_d}$. Define the approximations to cell averages of the solution $\mathbf{U}$ and source term $\mathbf{S}$ by

$$\mathbf{U}_{\Delta x}(\mathbf{x},t) = \mathbf{U}_{\mathbf{j}}(t) \approx \frac{1}{|C_{\mathbf{j}}|}\int_{C_{\mathbf{j}}} \mathbf{U}(\mathbf{x},t)d\mathbf{x}, \quad \forall \mathbf{x} \in C_{\mathbf{j}}, \quad \mathbf{S}_{\mathbf{j}} \approx \frac{1}{|C_{\mathbf{j}}|}\int_{C_{\mathbf{j}}} \mathbf{S}(\mathbf{x})d\mathbf{x}.$$

Then, a semi-discrete finite volume scheme [13] for approximating (1) is given by

$$\partial_t \mathbf{U}_{\mathbf{j}}(t) = -\sum_{r=1}^{d} \frac{1}{\Delta x}\left(\mathbf{F}^r_{\mathbf{j}+\frac{1}{2}} - \mathbf{F}^r_{\mathbf{j}-\frac{1}{2}}\right) - \mathbf{S}_{\mathbf{j}}, \tag{16}$$

where *numerical fluxes* $\mathbf{F}^r_{\cdot}$ are defined by using (approximate) solutions of local Riemann problems (in direction r) at each cell interface. High order accuracy is achieved by using non-oscillatory TVD, ENO, WENO methods [7, 11]. Strong stability preserving Runge-Kutta methods are used for time integration.

Assumption 1 *We assume that the abstract FDM or FVM scheme* (16) *satisfies*

$$\|\mathbf{U}^n_{\Delta x}\|_{\mathbf{L}^2(\mathbf{D})} \le K\|\mathbf{U}^0_{\Delta x}\|_{\mathbf{L}^2(\mathbf{D})}, \tag{17}$$

and the approximation error convergences (as $\Delta x \to 0$) with rate $s > 0$, i.e.

$$\|\mathbf{U}_0 - \mathbf{U}^0_{\Delta x}\|_{\mathbf{L}^2(\mathbf{D})} \le C\Delta x^s \|\mathbf{U}_0\|_{\mathbf{H}^s(\mathbf{D})}, \quad \|\mathbf{S} - \mathbf{S}_{\Delta x}\|_{\mathbf{L}^2(\mathbf{D})} \le C\Delta x^s \|\mathbf{S}\|_{\mathbf{H}^s(\mathbf{D})}, \tag{18}$$

$$\|\mathbf{U}(t^n) - \mathbf{U}^n_{\Delta x}\|_{\mathbf{L}^2(\mathbf{D})} \le C\Delta x^s \left(\|\mathbf{U}_0,\mathbf{S},t^n\|_{\mathbf{H}^s(\mathbf{D})} + t^n\|\mathbf{U}_0,\mathbf{S},t^n\|_{\mathbf{L}^2(\mathbf{D})}\right), \tag{19}$$

provided $\Delta t = \Delta x/\lambda$. $C,\lambda > 0$ are independent of Δx. $\mathbf{H}^s(\mathbf{D})$ denotes $W^{s,2}(\mathbf{D})^m$.

Assumption 1 is satisfied by many standard FDM and FVM (for small s) schemes, we refer to [8, 9, 10, 13, 23] and the references therein. For q-th order (formally) accurate schemes, $q \in \mathbb{N}$, the convergence estimate (19) holds [10, 13] with

$$s = \min\{q, \bar{r}\} \quad \text{(FDM)}, \qquad s = \min\{q, \max\{\min\{2,q\}/2, \bar{r}\}\} \quad \text{(FVM)}. \tag{20}$$

The computational work of FDM/FVM for a time step and for a complete run is

$$\mathrm{Work}^{\mathrm{step}}_{\Delta x} = B\Delta x^{-d}, \quad \mathrm{Work}_{\Delta x} = \mathrm{Work}^{\mathrm{step}}_{\Delta x}\frac{T}{\Delta t} = \lambda T B \Delta x^{-(d+1)}, \tag{21}$$

where $B > 0$ is independent of Δx and Δt. However, in the *random* case (8), the computational work (21) of FDM/FVM for one complete run depends on the particular realization of the coefficient $c(\cdot,\omega)$: due to the CFL condition ensuring the numerical stability of the explicit time stepping, the number of time steps $N(\Delta x,\omega)$ depends on the speed λ of the fastest moving wave, where $\lambda(\omega) = \sum_{r=1}^{d} \|\sigma^r_{\max}(\cdot,\omega)\|_{L^\infty(\mathbf{D})}$,

$$N(\Delta x,\omega) = \lambda(\omega)T/\Delta x = T/\Delta x \sum_{r=1}^{d} \|\sigma^r_{\max}(\cdot,\omega)\|_{L^\infty(\mathbf{D})}, \quad \forall \omega \in \Omega. \tag{22}$$

Here, $\sigma^r_{\max} = \max\{\sigma^r_1,\dots,\sigma^r_m\}$, where $\sigma^r_1(\cdot,\omega),\dots,\sigma^r_m(\cdot,\omega)$ are the eigenvalues of $\mathbf{A}_r(\cdot,\omega)$ and correspond to the directional speeds of the wave propagation at $\mathbf{x} \in \mathbf{D}$.

3.3 MC-FDM and MC-FVM Schemes

MC-FDM or MC-FVM algorithm consists of the following three steps:

1. **Sample:** We draw M independent identically distributed (i.i.d.) input data and source samples $\mathbf{I}^i$ with $i = 1, 2, \ldots, M$ from the random fields $\mathbf{I}(\cdot, \omega)$ and approximate these by piecewise constant cell averages.
2. **Solve:** For each realization $\mathbf{I}^i$, the underlying balance law (1) is solved numerically by the Finite Volume/Difference Method (16). Denote the solutions by $\mathbf{U}^{i,n}_{\Delta x}$.
3. **Estimate Statistics:** We estimate the expectation of the random solution field with the sample mean (ensemble average) of the approximate solution:

$$E_M[\mathbf{U}^n_{\Delta x}] := \frac{1}{M} \sum_{i=1}^{M} \mathbf{U}^{i,n}_{\Delta x}. \tag{23}$$

   Higher statistical moments can be approximated analogously under suitable statistical regularity of the underlying random entropy solutions [15].

Theorem 4. *Assume the hypothesis of Theorem 2 is satisfied with $k \geq 2$, i.e. second moments of the random initial data $\mathbf{U}_0$, source $\mathbf{S}$ and K exist. Under Assumption 1, the MC-FDM/FVM estimate* (23) *satisfies the following error bound,*

$$\begin{aligned} \|\mathbb{E}[\mathbf{U}(t^n)] - E_M[\mathbf{U}^n_{\Delta x}](\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} &\leq M^{\frac{1}{2}} \bar{K}_2 \|\mathbf{U}_0, \mathbf{S}, t\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\ &+ C\Delta x^s \left(\|\mathbf{U}_0, \mathbf{S}, t^n\|_{L^2(\Omega,\mathbf{H}^s(\mathbf{D}))} + t^n \|\mathbf{U}_0, \mathbf{S}, t^n\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \right), \end{aligned} \tag{24}$$

where $C > 0$ is independent of M, K and Δx.

Proof. Firstly, we bound the left hand side of (24) using triangle inequality,

$$\begin{aligned} \|\mathbb{E}[\mathbf{U}(\cdot, t^n)] - E_M[\mathbf{U}^n_{\Delta x}(\cdot, \omega)]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} &\leq \|\mathbb{E}[\mathbf{U}(\cdot, t^n)] - E_M[\mathbf{U}(\cdot, t^n)]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\ &+ \|E_M[\mathbf{U}(\cdot, t^n)] - E_M[\mathbf{U}^n_{\Delta x}(\cdot, \omega)]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} = \mathrm{I} + \mathrm{II}. \end{aligned}$$

Term I is bounded by (15). For term II, by the triangle inequality, by (5) and (19),

$$\begin{aligned} \mathrm{II} &= \|E_M[\mathbf{U}(\cdot, t^n) - \mathbf{U}^n_{\Delta x}(\cdot, \omega)]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\ &\leq \frac{1}{M} \sum_{i=1}^{M} \|\mathbf{U}^i(\cdot, t^n) - \mathbf{U}^i_{\Delta x}(\cdot, t^n, \omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} = \|\mathbf{U}(\cdot, t^n) - \mathbf{U}^n_{\Delta x}(\cdot, \omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\ &\leq C\Delta x^s \left\| \|\mathbf{U}_0, \mathbf{S}, t^n\|_{\mathbf{H}^s(\mathbf{D})} + t^n \|\mathbf{U}_0, \mathbf{S}, t^n\|_{\mathbf{L}^2(\mathbf{D})} \right\|_{L^2(\Omega,\mathbb{R})}. \end{aligned}$$

Finally, (24) is obtained by applying the triangle inequality on the last term. □

To equilibrate statistical and spatio-temporal errors in (24), we need $M = O(\Delta x^{-2s})$. Next, we are interested in the asymptotic behaviour of the error (24) vs. the *expected* computational work. We want to determine the largest convergence rate $\alpha > 0$, s.t.

$$\|\mathbb{E}[\mathbf{U}(t^n)] - E_M[\mathbf{U}^n_{\Delta x}](\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \leq C\left(\mathbb{E}\left[\text{Work}\right]\right)^{-\alpha}.$$

Assuming that the expected fastest wave speed $\bar{\lambda} = \mathbb{E}[\lambda(\omega)]$ in (22) is finite,

$$\mathbb{E}[\text{Work}] = \mathbb{E}\left[M\text{Work}_{\Delta x}(\omega)\right] = \Delta x^{-(d+1+2s)} TB\bar{\lambda} < \infty. \tag{25}$$

Consequently, the asymptotic error bound (29) is satisfied with $\alpha = s/(d+1+2s)$, which is considerably more expensive compared to deterministic rate $\alpha = s/(d+1)$.

3.4 MLMC-FDM and MLMC-FVM Schemes

Given the slow convergence of MC-FDM/FVM, we propose the Multi-Level Monte Carlo methods: MLMC-FDM and MLMC-FVM. The key idea is to simultaneously draw MC samples on a hierarchy of nested grids [15]. There are four steps:

0. **Nested meshes:** Consider *nested* meshes $\{\mathcal{T}_\ell\}_{\ell=0}^{\infty}$ of the domain $\mathbf{D}$ with corresponding mesh widths $\Delta x_\ell = 2^{-\ell}\Delta x_0$, where Δx_0 is the mesh width for the coarsest resolution and corresponds to the lowest level $\ell = 0$.
1. **Sample:** For each level of resolution $\ell \in \mathbb{N}_0$, we draw M_ℓ independent identically distributed (i.i.d) samples $\mathbf{I}^i_\ell$ with $i = 1, 2, \ldots, M_\ell$ from the random input data $\mathbf{I}(\omega)$ and approximate these by cell averages.
2. **Solve:** For each resolution level ℓ and each realization $\mathbf{I}^i_\ell$, the underlying balance law (1) is solved for $\mathbf{U}^{i,n}_{\Delta x_\ell}$ by the FDM/FVM method (16) with mesh width Δx_ℓ.
3. **Estimate solution statistics:** Fix some positive integer $L < \infty$ corresponding to the highest level. Denoting MC estimator (23) with $M = M_\ell$ by E_{M_ℓ}, the expectation of the random solution field $\mathbf{U}$ is estimated by

$$E^L[\mathbf{U}^n_{\Delta x_L}] := \sum_{\ell=0}^{L} E_{M_\ell}[\mathbf{U}^n_{\Delta x_\ell} - \mathbf{U}^n_{\Delta x_{\ell-1}}]. \tag{26}$$

MLMC-FDM/FVM is *non-intrusive* as any standard FDM/FVM codes can be used in step 2. Furthermore, MLMC-FDM/FVM is amenable to *efficient parallelization* [17, 22] as data from different grid resolutions and samples only interacts in step 3.

Theorem 5. *Assume the hypothesis of Theorem 2 is satisfied with $k \geq 2$, i.e. second moments of the random initial data $\mathbf{U}_0$, source $\mathbf{S}$ and K exist. Under Assumption 1, the MLMC-FDM/FVM estimate* (26) *satisfies the following error bound,*

$$\begin{aligned}\left\|\mathbb{E}[\mathbf{U}(t^n)] - E^L[\mathbf{U}^n_{\Delta x_L}](\omega)\right\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} &\leq C_1\left(\Delta x_L^s + \left\{\sum_{\ell=1}^{L} M_\ell^{-\frac{1}{2}}\Delta x_\ell^s\right\}\right) + C_0 M_0^{-\frac{1}{2}}, \\ C_1 &= C\left(\|\mathbf{U}_0, \mathbf{S}, t^n\|_{L^2(\Omega,\mathbf{H}^s(\mathbf{D}))} + t^n\|\mathbf{U}_0, \mathbf{S}, t^n\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))}\right), \\ C_0 &= \bar{K}_2\|\mathbf{U}_0, \mathbf{S}, t^n\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))}.\end{aligned} \tag{27}$$

Proof. Using the triangle inequality, the left hand side of (27) is bounded by

$$\|\mathbb{E}[\mathbf{U}(t^n)] - \mathbb{E}[\mathbf{U}^n_{\Delta x_L}]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} + \|\mathbb{E}[\mathbf{U}^n_{\Delta x_L}] - E^L[\mathbf{U}^n_{\Delta x_L}](\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} = \mathrm{I} + \mathrm{II}.$$

We estimate term I and II separately. By linearity of the expectation, term I equals

$$\mathrm{I} = \|\mathbb{E}[\mathbf{U}(\cdot,t^n,\omega) - \mathbf{U}^n_{\Delta x_L}(\cdot,\omega)]\|_{\mathbf{L}^2(\mathbf{D})} = \|\mathbf{U}(\cdot,t^n,\omega) - \mathbf{U}^n_{\Delta x_L}(\cdot,\omega)\|_{L^1(\Omega,\mathbf{L}^2(\mathbf{D}))},$$

which can be bounded by (19). Using MLMC definition (26), linearity of mathematical expectation, and the MC bound (15), term II is bounded by

$$\begin{aligned}
\mathrm{II} &\le \left\| \sum_{\ell=0}^{L} \mathbb{E}[\mathbf{U}^n_{\Delta x_\ell}(\cdot,\omega) - \mathbf{U}^n_{\Delta x_{\ell-1}}(\cdot,\omega)] - E_{M_\ell}[\mathbf{U}^n_{\Delta x_\ell}(\cdot,\omega) - \mathbf{U}^n_{\Delta x_{\ell-1}}(\cdot,\omega)] \right\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\
&\le \sum_{\ell=0}^{L} \|\mathbb{E}[\mathbf{U}^n_{\Delta x_\ell}(\cdot,\omega) - \mathbf{U}^n_{\Delta x_{\ell-1}}(\cdot,\omega)] - E_{M_\ell}[\mathbf{U}^n_{\Delta x_\ell}(\cdot,\omega) - \mathbf{U}^n_{\Delta x_{\ell-1}}(\cdot,\omega)]\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \\
&\le M_0^{-\frac{1}{2}} \|\mathbf{U}^n_{\mathscr{T}_0}(\cdot,\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} + \sum_{\ell=1}^{L} M_\ell^{-\frac{1}{2}} \|\mathbf{U}^n_{\Delta x_\ell}(\cdot,\omega) - \mathbf{U}^n_{\Delta x_{\ell-1}}(\cdot,\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))}.
\end{aligned}$$

The first term is bounded by (17); the detail terms $\mathbf{U}^n_{\Delta x_\ell} - \mathbf{U}^n_{\Delta x_{\ell-1}}$ are bounded by

$$\|\mathbf{U}^n_{\Delta x_\ell} - \mathbf{U}^n_{\Delta x_{\ell-1}}\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \le \|\mathbf{U} - \mathbf{U}^n_{\Delta x_\ell}\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} + \|\mathbf{U} - \mathbf{U}^n_{\Delta x_{\ell-1}}\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))}.$$

Using (19), detail terms can be further bounded by

$$\|\mathbf{U}^n_{\Delta x_\ell} - \mathbf{U}^n_{\Delta x_{\ell-1}}\|_{L^2(\Omega,\mathbf{L}^2(\mathbf{D}))} \le C\Delta x_\ell^s \left\| \|\mathbf{U}_0,\mathbf{S},t^n\|_{\mathbf{H}^s(\mathbf{D})} + t^n \|\mathbf{U}_0,\mathbf{S},t^n\|_{\mathbf{L}^2(\mathbf{D})} \right\|_{L^2(\Omega,\mathbb{R})}.$$

Using triangle inequality and summing over all levels $\ell > 0$, bound (27) follows. □

To equilibrate the statistical and the spatio-temporal errors in (27), we require

$$M_\ell = O(2^{2(L-\ell)s}), \quad 0 \le \ell \le L. \tag{28}$$

Notice that (28) implies that the largest number of MC samples is required on the coarsest mesh level $\ell = 0$, whereas only a few MC samples are needed for $\ell = L$. Next, we are interested in the largest $\alpha > 0$ and smallest $\beta > 0$, such that:

$$\|\mathbb{E}[\mathbf{U}(t^n)] - E^L[\mathbf{U}^n_{\Delta x_L}](\omega)\|_{L^2(\Omega,\mathbf{L}^2(\mathbb{R}))} \le C\,(\mathbb{E}\,[\mathrm{Work}])^{-\alpha} \log(\mathbb{E}[\mathrm{Work}])^{\beta}. \tag{29}$$

Assuming that $\bar{\lambda} = \mathbb{E}[\lambda(\omega)]$ in (22) is finite and using (21) with (28),

$$\begin{aligned}
\mathbb{E}[\mathrm{Work}] &= \mathbb{E}\left[\sum_{\ell=0}^{L} M_\ell \mathrm{Work}_{\Delta x_\ell}(\omega)\right] = \sum_{\ell=0}^{L} M_\ell \mathbb{E}\left[\mathrm{Work}_{\Delta x_\ell}(\omega)\right] \\
&= \sum_{\ell=0}^{L} M_\ell T B \bar{\lambda} \Delta x_\ell^{-(d+1)} = T B \bar{\lambda} \sum_{\ell=0}^{L} M_\ell \Delta x_\ell^{-(d+1)}.
\end{aligned} \tag{30}$$

The last term in (30) was already estimated in [17]. Since the *expectation* of computational work is obtain from the *deterministic* computational work by scaling with a problem dependent constant $\bar{\lambda}$, the *asymptotic* error vs. expected computation work estimate (29) remain analogous to the estimates derived in [17],

$$(\alpha,\beta) = \begin{cases} \left(\min\left\{\frac{1}{2}, \frac{s}{d+1}\right\}, 1\right) & \text{if } s \neq (d+1)/2, \\ \left(\frac{1}{2}, \frac{3}{2}\right) & \text{if } s = (d+1)/2. \end{cases} \tag{31}$$

Finally, we would like to note that bounds (13), (23) and (26) can be easily generalized (all steps in proofs are analogous) for higher moments ($k > 1$).

4 Acoustic isotropic wave equation as linear hyperbolic system

The *stochastic isotropic* linear acoustic wave equation is given by

$$\begin{cases} p_{tt}(\mathbf{x},t,\omega) - \nabla\cdot(c(\mathbf{x},\omega)\nabla p(\mathbf{x},t,\omega)) = f(\mathbf{x},\omega), \\ \qquad p(\mathbf{x},0,\omega) = p_0(\mathbf{x},\omega), \qquad \mathbf{x}\in\mathbf{D},\ t>0,\ \omega\in\Omega, \\ \qquad p_t(\mathbf{x},0,\omega) = p_1(\mathbf{x},\omega), \end{cases} \tag{32}$$

where p is the acoustic pressure. Since in most cases, the initial data p_0, p_1 and the coefficient c are *not* known exactly, they are modeled as random fields, i.e. $c \in L^0(\Omega, W^{r_c,\infty}(\mathbf{D}))$ with $\mathbb{P}[c(\mathbf{x},\omega) > 0, \forall\mathbf{x}\in\mathbf{D}] = 1$, $p_0, p_1 \in L^k(\Omega, \mathbf{W}^{r_0,\infty}(\mathbf{D}))$ and $f \in L^k(\Omega, \mathbf{W}^{r_f,\infty}(\mathbf{D}))$. Wave equation (32) is equivalent to the (one of the many) following *system* of $d+1$ *first order* conservation laws (equations of acoustics)

$$\begin{cases} p_t(\mathbf{x},t,\omega) - \nabla\cdot(c(\mathbf{x},\omega)\mathbf{u}(\mathbf{x},t,\omega)) = tf(\mathbf{x},\omega), \\ \qquad \mathbf{u}_t(\mathbf{x},\omega) - \nabla p(\mathbf{x},\omega) = 0, \\ \qquad p(\mathbf{x},0,\omega) = p_0(\mathbf{x},\omega), \qquad \mathbf{x}\in\mathbf{D},\ t>0,\ \omega\in\Omega, \\ \qquad \mathbf{u}(\mathbf{x},0,\omega) = \mathbf{u}_0(\mathbf{x},\omega), \end{cases} \tag{33}$$

To verify equivalence of (33) and (32), differentiate the first equation of (33) in time:

$$f = p_{tt} - \nabla\cdot(c(\mathbf{x},\omega)\mathbf{u}_t) = p_{tt} - \nabla\cdot(c(\mathbf{x},\omega)\nabla p).$$

For simplicity, only stationary initial data will be considered, i.e. $\mathbf{u}_0 \equiv 0$. Linear system (33) can be written as system of conservation laws (8), with $m = d+1$,

$$\mathbf{U} = \begin{bmatrix} p \\ \mathbf{u} \end{bmatrix}, \quad \mathbf{U}_0 = \begin{bmatrix} p_0 \\ \mathbf{u}_0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} tf \\ 0 \end{bmatrix}, \quad \mathbf{A}_r(\mathbf{x},\omega) \in \mathbb{R}^{(d+1)\times(d+1)}. \tag{34}$$

All elements of $\mathbf{A}_r$ are zero, except $(\mathbf{A}_r(\mathbf{x},\omega))_{1,r+1} = -c(\mathbf{x},\omega)$ and $(\mathbf{A}_r)_{r+1,1} = -1$. Note, that $\mathbf{A}_r$ defines a strongly hyperbolic linear system of conservation laws. This is easily verifiable for $d=1$; there exists an invertible $\mathbf{Q}_\mathbf{x}(\omega)$ diagonalizing $\mathbf{A}$:

$$\mathbf{Q_x}(\omega) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{c}} & -\frac{1}{\sqrt{c}} \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{Q_x}(\omega)\mathbf{A}_1(\mathbf{x},\omega)\mathbf{Q_x}(\omega)^{-1} = \begin{bmatrix} -\sqrt{c} & 0 \\ 0 & \sqrt{c} \end{bmatrix}.$$

Since $\|\mathbf{Q_x}(\omega)\|\|\mathbf{Q_x}^{-1}(\omega)\| = \max\{c^{\frac{1}{2}}, c^{-\frac{1}{2}}\} \leq c^{\frac{1}{2}} + c^{-\frac{1}{2}}$, the uniform boundedness $c, c^{-1} \in L^\infty(\Omega, L^\infty(\mathbf{D}))$ ensures $\bar{K}_\infty < \infty$. For $k < \infty$: $c, c^{-1} \in L^{k/2}(\Omega, L^\infty(\mathbf{D}))$ implies

$$\bar{K}_k^k = \mathbb{E}[K^k(\omega)] \leq \|c(\cdot,\omega)\|^{\frac{k}{2}}_{L^{k/2}(\Omega,L^\infty(\mathbf{D}))} + \|c^{-1}(\cdot,\omega)\|^{\frac{k}{2}}_{L^{k/2}(\Omega,L^\infty(\mathbf{D}))} < \infty. \quad (35)$$

Furthermore, assumption 2 of the Theorem 1 holds with $r_\mathbf{0} = r_0$, $r_\mathbf{S} = r_f$, $r_\mathbf{A} = r_c$. Since the eigenvalues $\{\sigma_1^r, \ldots, \sigma_m^r\}$ of the matrices $\mathbf{A}_r$ are zero except two which are $\pm\sqrt{c(\mathbf{x},\omega)}$, the expected maximum wave speed $\bar{\lambda}$ required in (25) and (30) is

$$\bar{\lambda} = \|c\|_{L^{1/2}(\Omega,L^\infty(\mathbf{D}))} < \infty, \quad \text{if} \quad c \in L^k(\Omega, L^\infty(\mathbf{D})) \quad \text{with} \quad k \geq 1/2. \quad (36)$$

5 Numerical experiments for acoustic isotropic wave equation

Let material coefficient c be given by the Karhunen-Loève expansion

$$c(\mathbf{x},\omega) = \mathbb{E}[c(\mathbf{x},\omega)] \exp\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m}\Psi_m(\mathbf{x})Y_m(\omega)\right), \quad (37)$$

where $\Psi_m(\mathbf{x}) \in L^\infty(\mathbf{D})$ are eigenfunctions satisfying $\|\Psi_m\|_{L^\infty(\mathbf{D})} \leq 1$, $Y_m(\omega)$ are *independent* random variables with zero mean, and $\{\lambda_m\}_{m=1}^\infty \in \ell^{\frac{1}{2}}(\mathbb{N})$ are eigenvalues.

All simulations reported below were performed on Cray XE6 in CSCS [24] with the recently developed massively parallel code ALSVID-UQ [2]. Refer to [22, 17] for the technical description of the implementation and for the linear scaling tests.

5.1 *Propagation of smooth wave with uniform material coefficient*

For physical domain $\mathbf{D} = [0,2]$, consider *deterministic*, smooth ($r_0 = \infty$) initial data

$$p_0(x,\omega) := \sin(\pi x), \quad p_1(x,\omega) \equiv 0, \quad (38)$$

and *stochastic* coefficient $c(\mathbf{x},\omega)$ that is given by KL expansion (37) with identical, *uniformly* distributed $Y_m \sim \mathscr{U}[-1,1]$. We choose eigenvalues $\lambda_m = m^{-2.5}$, eigenfunctions $\Psi_m(\mathbf{x}) = \sin(\pi x)$ and the mean field $\mathbb{E}[c(\mathbf{x},\omega)] \equiv 0.1$. Then both c and c^{-1} are uniformly bounded in Ω: $c(\mathbf{x},\omega), c^{-1}(\mathbf{x},\omega) \in L^\infty(\Omega, L^\infty(\mathbf{D}))$. Hence (35) and (36) holds with any $k \in \mathbb{N}_0 \cup \{\infty\}$. For simulations, KL expansion is truncated up to first 10 terms: $\lambda_m = 0$, $\forall m > 10$. Since $r_0 = \infty$, $r_c \geq 0$, by Theorem 1 the solution $\mathbb{P}$-a.s. has bounded weak derivatives of first order, i.e. $\mathbf{U}(\cdot,\cdot,\omega) \in \mathbf{W}^{\bar{r},\infty}(\mathbf{D})$

with $\bar{r} = 1$. First order accurate FVM scheme ($q = 1$, HLL Rusanov flux [13], FE time stepping) will be used, hence, in (20), $s = \min\{1, \max\{1/2, 1\}\} = 1$. Higher order schemes ($s > 1$) for $d = 1$ case are not useful since $s/(d+1) > 1/2$ in (31).

Results of the MLMC-FVM simulation at $t = 2.0$ are presented in Fig. 1.

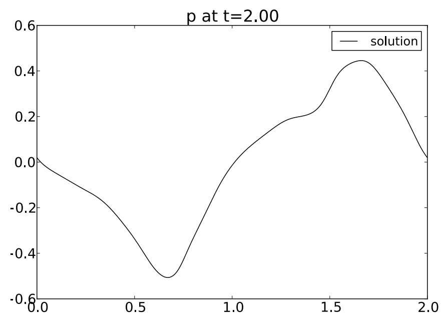

Fig. 1 One sample (left) and mean and variance (right) acoustic pressure $p(x,\omega)$ as in (33).

Using MLMC-FVM approximation from Fig. 1 (computed on 12 levels of resolution with the finest resolution having 16384 cells) as a reference solution $\mathbf{U}_{\text{ref}}$, we run MC-FVM and MLMC-FVM (with $\Delta x_0 = 1/4$) on a series of mesh resolutions from 32 cells up to 1024 cells and monitor the convergence behavior. For $L^2(\Omega,\cdot)$ norms in (24) and (27), the $L^2(\Omega; L^2(\mathbf{D}))$-based relative error estimator from [15] was used. $K = 5$ delivered sufficiently small relative standard deviation σ_K.

In Figure 2, we compare the MC-FVM scheme with $M = O(\Delta x^{-2s})$ and the MLMC-FVM scheme with $M_\ell = M_L 2^{2s(L-\ell)}$, where $M_L = 16$ is chosen as suggested in [15]. Dashed lines indicate *expected* convergence rate slopes proved in Theorems 4 and 5. Theoretical and empirical convergence rates coincide, confirming the robustness of our implementation. MLMC method is observed to be three orders of magnitude faster than MC method. This numerical experiment clearly illustrates the superiority of the MLMC algorithm over the MC algorithm (for $q = 1, s = 1$).

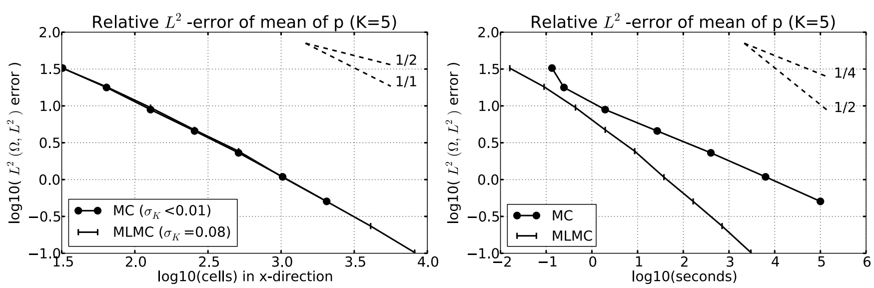

Fig. 2 Convergence of estimated mean for (38). Both MLMC and MC give similar errors for the same spatial resolution. However, MLMC method is 3 orders of magnitude faster than MC.

5.2 Propagation of shock wave with normal material coefficient in

For domain $\mathbf{D} = [0,2]$, consider *deterministic*, discontinuous ($r_0 = 0$) initial data

$$p_0(x,\omega) := 2\chi_{(0.5,1.5)}(x) - 1.0, \quad p_1(x,\omega) \equiv 0. \tag{39}$$

and *stochastic* coefficient $c(\mathbf{x},\omega)$ that is given by KL expansion (37) with identical, *normally* distributed $Y_m \sim \mathcal{N}[0,1]$. We choose eigenvalues $\lambda_m = m^{-2.5}$, eigenfunctions $\Psi_m(\mathbf{x}) = \sin(\pi x)$ and the mean field $\mathbb{E}[c(\mathbf{x},\omega)] \equiv 0.1$. Then, unlike in the *uniform* case before, $c, c^{-1} \notin L^\infty(\Omega, L^\infty(\mathbf{D}))$. However, (35) and (36) holds by

Proposition 1. *Assume* $\{\lambda_m\} \in \ell^{\frac{1}{2}}(\mathbb{N})$. *Then* $c, c^{-1} \in L^k(\Omega, L^\infty(\mathbf{D}))$, $\forall k \in \mathbb{N} \cup \{0\}$.

Proof. Using triangle inequality and $\|\Psi_m\|_{L^\infty(\mathbf{D})} = 1$, we obtain

$$\frac{\|c(\cdot,\omega)\|_{L^\infty(\mathbf{D})}}{\|\mathbb{E}[c(\mathbf{x},\omega)]\|_{L^\infty(\mathbf{D})}} \leq \exp\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m}|Y_m(\omega)|\right) =: \bar{c}(\omega).$$

Since Y_m, $m = 1,\ldots,\infty$ are independent and normally distributed,

$$\mathbb{E}[\bar{c}^k(\omega)] = \prod_{m=1}^{\infty} \mathbb{E}\left[\exp\left(k\sqrt{\lambda_m}|Y_m(\omega)|\right)\right] = \prod_{m=1}^{\infty} \exp\left(\frac{k^2\lambda_m}{2}\right)\left(1 + \operatorname{erf}\left(\frac{k\sqrt{\lambda_m}}{\sqrt{2}}\right)\right),$$

where, using inequalities $\operatorname{erf}(a) \leq \frac{2}{\sqrt{\pi}}a$ and $1 + a \leq \exp(a)$, $\forall a \geq 0$,

$$\prod_{m=1}^{\infty}\left(1 + \operatorname{erf}\left(\frac{k\sqrt{\lambda_m}}{\sqrt{2}}\right)\right) \leq \prod_{m=1}^{\infty}\left(1 + \frac{2}{\sqrt{\pi}}\frac{k\sqrt{\lambda_m}}{\sqrt{2}}\right) \leq \exp\left(\sum_{m=1}^{\infty}\frac{2}{\sqrt{\pi}}\frac{k\sqrt{\lambda_m}}{\sqrt{2}}\right).$$

Finally, $\|c\|_{L^k(\Omega,L^\infty(\mathbf{D}))} = \|\mathbb{E}[c(\mathbf{x},\omega)]\|_{L^\infty(\mathbf{D})}\mathbb{E}[\bar{c}^k(\omega)]^{\frac{1}{k}}$ is bounded by

$$\|\mathbb{E}[c(\mathbf{x},\omega)]\|_{L^\infty(\mathbf{D})} \exp\left(\frac{k}{2}\|\{\lambda_m\}\|_{\ell^1(\mathbb{N})} + \frac{\sqrt{2}}{\sqrt{\pi}}\|\{\sqrt{\lambda_m}\}\|_{\ell^1(\mathbb{N})}\right) < \infty.$$

Proof of $c^{-1} \in L^k(\Omega, L^\infty(\mathbf{D}))$ is analogous. □

Since $r_0 = 0$, by Theorem 1, solution $\mathbf{U}(\omega) \in \mathbf{W}^{\bar{r},\infty}(\mathbf{D})$ is $\mathbb{P}$-a.s. discontinuous ($\bar{r} = 0$). First order accurate ($q_1 = 1$, HLL Rusanov flux [13], FE time stepping) and second order accurate ($q_2 = 2$, HLL Rusanov flux, WENO reconstruction, SSP-RK2 time stepping [13]) FVM schemes will be used; hence, in (20), $s_1 = 1/2$ and $s_2 = 1$. For simulations, KL expansion is truncated up to first 10 terms: $\lambda_m = 0$, $\forall m > 10$. Results of the MLMC-FVM simulation at $t = 2.0$ are presented in Fig. 3.

MLMC-FVM approximation from Fig. 3 (computed on 12 levels of resolution with the finest resolution being on a mesh of 16384 cells) is used as a reference solution $\mathbf{U}_{\text{ref}}$. Additionally to MC, MLMC schemes with $s = s_1$, we consider MC2,

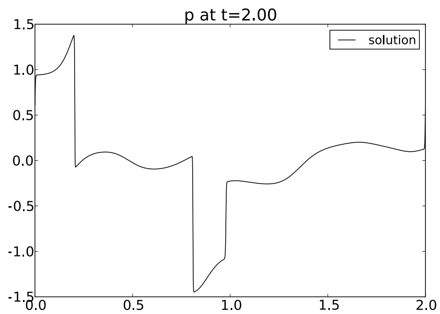

Fig. 3 One sample (left) and mean and variance (right) acoustic pressure $p(x,\omega)$ as in (33).

MLMC2 schemes with $s = s_2$. In Figure 4, we show convergence plots for *variance*; MLMC methods appear to be two orders of magnitude faster than MC methods.

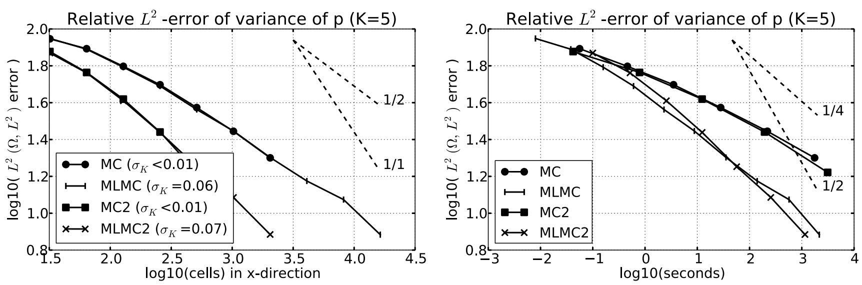

Fig. 4 Convergence of estimated variance for (39). Both MLMC(2) and MC(2) give similar errors for the same spatial resolution. However, MLMC methods are 2 orders of magnitude faster.

6 Conclusion

We consider linear hyperbolic systems of conservation laws in several space dimensions with uncertain input data. The proper notion of random weak solution is formulated and the resulting problem is shown to be well-posed.

We propose Monte Carlo Finite Difference and Finite Volume methods (MC-FDM/FVM). The MC-FDM/FVM are shown to converge to random weak solution but the derived accuracy vs. work estimates prove them to be computationally slow.

Hence, we propose Multi-Level Monte Carlo (MLMC) methods and prove their convergence. MLMC-FDM/FVM are much faster than MC-FDM/FVM and have the same accuracy vs. *expected* work ratio as deterministic FDM/FVM; they are also non-intrusive (existing FDM/FVM solvers can be used) and parallelizable [22].

We present several numerical experiments in one space dimension that reinforce the theory. In particular, the MLMC-FVM method yields about three orders of magnitude speedup versus the MC-FVM method in computing the mean. Furthermore, the speedup is more than two orders of magnitude for computing the variance.

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