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Research Report No. 2012-10
April 2012

Seminar für Angewandte Mathematik
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A NOTE ON FRONT TRACKING FOR THE KEYFITZ-KRANZER SYSTEM

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Abstract. A front tracking method is developed for the $n \times n$ symmetric Keyfitz-Kranzer system and convergence of the approximations to the strong generalized entropy solution of the system as defined by Panov [13] is proved. We also present numerical examples and compare the front tracking approximation with the finite difference upwind schemes constructed in [9].

1. Introduction

We consider the Cauchy problem for the $n \times n$ symmetric Keyfitz-Kranzer type system,

$$u_t + (u\phi(|u|))_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1b}$$

where $T > 0$ is given, $u = (u^{(1)}, \dots, u^{(n)}) : \mathbb{R} \times (0,T) \to \mathbb{R}^n$ the unknown and with $|u| := \sqrt{u^{(1)2} + \cdots + u^{(n)2}}$, $u_0 = (u_0^{(1)}, \dots, u_0^{(n)}) \in L^\infty(\mathbb{R}, \mathbb{R}^n)$, the initial data, and $\phi(r) \in C^1(\mathbb{R}_+)$ a scalar function with

$$r\phi(r) \xrightarrow{r \to 0+} 0. \tag{2}$$

System (1) was first considered in [8, 11] as a prototype of a nonstrictly hyperbolic system of conservation laws. In physics, it serves as a model for the elastic string [8], but it also appears in magnetohydrodynamics, where it is for example used to explain certain features of the solar wind [2]. Related systems of equations appear in chromatography [1] or in polymer flooding in porous media [14].

We denote the flux function $F(u) := u\phi(|u|)$. Its Jacobian matrix $A(u) := DF(u)$ is

$$A(u) = \phi(|u|)\,\mathbf{1} + \frac{\phi'(|u|)}{|u|} u \otimes u$$

where $\mathbf{1}$ denotes the $n \times n$ identity matrix. The matrix $A(u)$ is symmetric, therefore its eigenvalues are real and the corresponding collection of eigenvectors is complete, and system (1) is hyperbolic. The eigenvalues of $A(u)$ are $\lambda_1(u) = \phi(|u|) + \phi'(|u|)|u|$ with multiplicity 1 and $\lambda_2(u) = \phi(|u|)$ with multiplicity $n-1$. Due to the presence of eigenvalues with multiplicity > 1, system (1) is not strictly hyperbolic in the sense of Lax [10]. The eigenspaces $E_i(u)$, $i = 1, 2$, corresponding to the eigenvalues $\lambda_i(u)$, are

$$E_1(u) = \operatorname{span}\{u\}, \quad E_2(u) = E_1(u)^\perp,$$

and thus we have for $v_i \in E_i$ with $|v_i| = 1$, denoting $r := u/|u|$,

$$\nabla\lambda_1(u) \cdot v_1 = 2\phi'(r) + \phi''(r)r, \quad \nabla\lambda_2(u) \cdot v_2 = 0. \tag{3}$$

Date: April 12, 2012.

So the first characteristic field is either genuinely nonlinear or linearly degenerate (if $2\phi'(|u|) + \phi''(|u|)|u| = 0$) and the second characteristic field is always linearly degenerate.

Due to the nonlinearity of equation (1a), discontinuities can appear in its solution, no matter how smooth the initial data is. Therefore one seeks a *weak solution* to the equation, that is, one requires the differential equation to be satisfied only in the distributional sense,

$$\int_0^T \int_{\mathbb{R}} u\psi_t + u\phi(|u|)\psi_x \, dx \, dt + \int_{\mathbb{R}} u_0(x)\psi(x,0)\, dx = 0, \quad \forall\, \psi \in C_0^{1,1}(\mathbb{R}\times[0,T)).$$

It is well known that weak solutions are not necessarily unique and therefore additional admissibility criteria have to be imposed to select the relevant solution. In the context of conservation laws, this is usually done by restricting to solutions satisfying in addition an entropy condition, which are therefore called *entropy solutions*. For system (1), the notion of an entropy solution was introduced by Freistühler [3, 4] and by Panov [13]. It is defined as follows:

Definition 1.1. [13] Let $\phi \in C(\mathbb{R}_+)$ satisfy (2). A bounded measurable vector-valued function is called a *strong generalized entropy solution* if the function $r(x,t) = |u(x,t)|$ is the entropy solution of

$$r_t + (\phi(r)r)_x = 0, \quad (x,t) \in \mathbb{R}, \tag{4a}$$

$$r(x,0) = r_0(x) = |u_0(x)|, \quad x \in \mathbb{R}, \tag{4b}$$

that is, (4) is satisfied in the weak sense and in addition it holds for all entropy/entropy flux pairs (p,q), where $p(r)$ is convex and $q(r)$ defined by $q'(r) = (\phi(r)r)'p'(r)$,

$$p(r)_t + q(r)_x \le 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)),$$

and u satisfies

$$\int_0^T \int_{\mathbb{R}} u\psi_t + u\phi(r)\psi_x \, dx \, dt + \int_{\mathbb{R}} u_0(x)\psi(x,0)\, dx = 0, \qquad \forall\, \psi \in C_0^{1,1}(\mathbb{R}\times[0,T)). \tag{5}$$

In [13], Panov proved existence and uniqueness of the entropy solution of (1):

Theorem 1.1. *There exists a unique strong generalized entropy solution* $u \in L^\infty(\mathbb{R}\times(0,T))$ *of* (1) *as in Definition* 1.1. *It can be obtained as the limit of solutions* u^ϵ *in* $L^1_{loc}(\mathbb{R}\times(0,T))$ *of the parabolic equation*

$$u^\epsilon_t + (\phi(|u^\epsilon|)u^\epsilon)_x = \epsilon u^\epsilon_{xx}$$

as $\epsilon \to 0$.

To prove that there is a unique u satisfying (5), in [12], the author defined $v := u/r$. Then v satisfies

$$(Av)_t + (Bv)_t = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \tag{6a}$$

$$v(x,0) = v_0(x) = \frac{u_0(x)}{r_0(x)}, \quad x \in \mathbb{R}, \tag{6b}$$

$$A = A(x,t) = r(x,t), \quad (x,t) \in \mathbb{R}\times(0,T), \tag{6c}$$

$$B = B(x,t) = \phi(r(x,t))r(x,t), \quad (x,t) \in \mathbb{R}\times(0,T). \tag{6d}$$

For this type of equation, we have the following result:

Theorem 1.2. [12] *Let v be a solution of*

$$(Av)_t + (Bv)_t = 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)), \tag{7a}$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{7b}$$

where $A, B \in L^\infty(\mathbb{R} \times (0,T))$ satisfy

$$\operatorname{ess\,lim}_{t\to 0+} A(x,t) = A(x,0) \text{ in } L^1_{loc}(\mathbb{R}), \quad A(x,0) \in L^\infty(\mathbb{R}); \tag{8a}$$

$$|B| \leq N(\epsilon)(A+\epsilon) \text{ a.e. in } \mathbb{R} \times (0,T) \text{ for all } \epsilon > 0, \quad \epsilon N(\epsilon) \xrightarrow{\epsilon\to 0+} 0; \tag{8b}$$

$$A_t + B_x = 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,T)). \tag{8c}$$

Then we have

- *(i) The problem* (7) *has a weak solution $v(x,t)$;*
- *(ii)* $\operatorname{ess\,lim}_{t\to 0+} A(x,t)v(x,t) = A(x,0)v_0(x)$ *in $L^1_{loc}(\mathbb{R})$;*
- *(iii) If $A(x,0)v_0(x) = 0$ a.e. on $\mathbb{R}$, then $A(x,t)v(x,t) = 0$ a.e. on $\mathbb{R} \times (0,T)$ (uniqueness).*

Note that the coefficients A and B in (6) satisfy the conditions (8) in Theorem 1.2 if r is the entropy solution of (4).

Existence and uniqueness of the solution to (1) and in addition L^1_{loc}-continuous dependence of the solution on the initial data u_0 have also been shown by Freistühler [4] using Wagner's transformation theory [15]. Numerical studies of system (1) for the particular case where $\phi(r) = r^2$ have been conducted in [5, 6] and finite difference schemes which can be applied for a large class of functions ϕ have been developed in [9].

1.1. Solution of the Riemann problem.

The Riemann problem for system (1),

$$u_0(x) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0, \end{cases} \tag{9}$$

for $n = 2$ has been solved in [8]. The structure of the solution for a general n is similar, due to the multiplicity of the second eigenvalue. Since the eigenvector corresponding to the first eigenvalue $\lambda_1(u)$ is parallel to the solution u, $v := u/|u|$ does not change along the corresponding composite wave. The second characteristic field is a contact discontinuity, owing to (3), and the elements of the eigenspace corresponding to the second eigenvalue $\lambda_2(u)$ are orthogonal to u. Therefore $r := |u|$ does not change across the contact discontinuity. So the solution of a Riemann problem with left state u_L and right state u_R consists of left, right and middle states separated by shocks, rarefaction waves or contact discontinuities along which only r changes and by contact discontinuities along which only v changes. In particular we have that v does not change across the waves corresponding to $\lambda_1(u)$.

If there is a shock in the first characteristic field, it travels with speed

$$s(u_1, u_2) = \frac{\phi(r_1)r_1 - \phi(r_2)r_2}{r_1 - r_2}, \quad r_1 := |u_1|,\ r_2 := |u_2| \tag{10}$$

for left and right states u_1 and u_2. Again, we see that the shock speed is independent of $v_1 := u_1/r_1$ and $v_2 := u_2/r_2$. On the other hand, the contact discontinuities of the second field travel at speed $s = \phi(r_1) = \phi(r_2)$ (again denoting the left and right states by u_1 and u_2) since the absolute value of u does not change across them.

Example 1.1. The solution of the Riemann problem (1a), (9) with $r_L := |u_L| > r_R := |u_R|$ and $\phi(r) = r^2$ is given by

$$u(x,t) = \begin{cases} u_L, & x < r_L^2 t, \\ \frac{r_L}{r_R} u_R, & r_L^2 t < x < s(u_L,u_R)t, \\ u_R, & x > s(u_L,u_R)t, \end{cases}$$

if $r_L^2 < s(u_L,u_R)$, where $s(u_L,u_R)$ is the shock speed given by (10), and

$$u(x,t) = \begin{cases} u_L, & x < s(u_L,u_R)t, \\ \frac{r_R}{r_L} u_L, & s(u_L,u_R)t < x < r_R^2 t, \\ u_R, & x > r_R^2 t, \end{cases}$$

if $r_L^2 > s(u_L,u_R)$.

We show now that there is at most one contact discontinuity of the second, linearly degenerate wave present in the solution u of the Riemann problem (1a), (9). This means that the function $v := u/|u|$ takes only the two states $v_L = u_L/|u_L|$ and $v_R = u_R/|u_R|$ which are separated by one discontinuity (and it is a constant function if $v_L = v_R$).

Lemma 1.3. *Let $u(x,t)$ solve the Riemann problem* (1a), (9)*, and define $v(x,t) := u(x,t)/|u(x,t)|$ and $v_L := u_L/|u_L|$ and $v_R := u_R/|u_R|$. Then v takes the form*

$$v(x,t) = \begin{cases} v_L, & x < \phi(r^*)t, \\ v_R, & x > \phi(r^*)t, \end{cases} \tag{11}$$

for some $r^* \in [\min\{|u_L|,|u_R|\}, \max\{|u_L|,|u_R|\}]$ for all times $t \geq 0$.

Proof. We assume without loss of generality that $v_L \neq v_R$. From Theorem 1.1 we know that there is exactly one u satisfying (5) given that $r := |u|$ is the entropy solution of (4). We can therefore construct the solution of the Riemann problem (1a), (9), by first solving the Riemann problem for r,

$$r_0(x) = \begin{cases} r_L := |u_L|, & x < 0, \\ r_R := |u_R|, & x > 0, \end{cases} \tag{12}$$

with (4) and then use this to find the locations where v changes and obtain $u = r \cdot v$. To solve this Riemann problem, we use the lower (or upper) convex envelope $f_\smile(r)$ ($f_\frown(r)$) of $f(r) := \phi(r)r$ between r_L and r_R if $r_L > r_R$ (or if $r_R > r_L$) (see 2.2 in [7]). We denote $g := f'_\smile$ if $r_L < r_R$ and $g := f'_\frown$ if $r_L > r_R$. Then the solution of the Riemann problem for r is given by

$$r(x,t) = \begin{cases} r_1, & x \leq g(r_1)t, \\ g^{-1}(x/t), & g(r_1)t < x \leq g(r_m)t, \\ r_m, & x > g(r_m)t. \end{cases}$$

If f is not convex between r_L and r_R, its convex envelopes contain linear segments and the function g contains constant segments. So its inverse g^{-1} consists of continuous segments along which r is increasing and which correspond to rarefaction waves, jump discontinuities which correspond to shock waves or contact discontinuities in the r-waves, and segments along which r is constant, these correspond to the values of r between rarefactions and shocks. We denote the locations in which g is not differentiable or discontinuous by r_i, $r_L = r_1 < \cdots < r_i < \cdots < r_m = r_R$ if $r_L < r_R$ and similar $r_R = r_1 < \cdots < r_i < \cdots < r_m = r_L$, if $r_L > r_R$. If g is not continuous at r_i, we define

$$g(r_i^+) = \lim_{r \downarrow r_i} g(r) \quad \text{and} \quad g(r_i^-) = \lim_{r \uparrow r_i} g(r).$$

The states r_i, $i = 1, \dots, m$ in the solution r are separated by shocks, rarefaction waves or contact discontinuities. If $g(r_i^+) = g(r_{i+1}^-)$, r_i and r_{i+1} are separated by a shock wave or a contact discontinuity in the r-wave and $g(r_i^+) = g(r_{i+1}^-) = (f(r_{i+1}) - f(r_i))/(r_{i+1} - r_i)$, otherwise there is a rarefaction wave between them and $g(r) = \phi(r) + \phi'(r)r$ in $r \in [\min\{r_i, r_{i+1}\}, \max\{r_i, r_{i+1}\}]$, so the convex envelope agrees with the function f. Note also that g is nondecreasing along $w(z) := r(x,t)$, $z := x/t \in [g(r_1^+), g(r_m^-)]$ by construction of the convex envelopes.

As observed before, contact discontinuities in the second wave (along which only v changes) travel at speed $\phi(r^*)$, so they are either in places where $g(r_i^-) \le \phi(r^*) \le g(r_i^+)$, $r^* = r_i$ or where $g(r_i^+) \le \phi(r^*) \le g(r_{i+1}^-)$, $r^* \in [\min\{r_i, r_{i+1}\}, \max\{r_i, r_{i+1}\}]$. We must show that there is at most one interval $[\min\{r_i, r_{i+1}\}, \max\{r_i, r_{i+1}\}]$, $i \in \{0, \dots, m\}$ (where we set $r_0 = -\infty$ and $r_{m+1} = \infty$) such that $g(r_i^-) \le \phi(r^*) \le g(r_i^+)$ or $g(r_i^+) \le \phi(r^*) \le g(r_{i+1}^-)$.

We assume by contradiction that there are at least two contact discontinuities σ_1 and σ_2 present in the second wave traveling with speeds $\phi(r^*)$ and $\phi(r^{**})$. Thus there exists r_{i_1} and r_{i_2} such that

$$\phi(r^*) < g(r_{i_1}^-) \le g(r_{i_2}^+) < \phi(r^{**}), \tag{13}$$

so the wave of the second family is slower than the wave of the first family at $x = g((r_{i_1}^-))t$ and faster than the first one at $x = g(r_{i_2}^+)t$. This means that there exists $i_1 \le j < j+1 \le i_2$ such that $\phi(r_j) < g(r_j^-)$ and $\phi(r_{j+1}) > g(r_{j+1}^+)$. If $g(r_j^-) = g(r_{j+1}^+)$, the states r_j and r_{j+1} are separated by a shock wave or a contact discontinuity in the r-wave and $g(r_j^-) = g(r_{j+1}^+) = (f(r_{j+1}) - f(r_j))/(r_{j+1} - r_j)$, so

$$\phi(r_j) < \frac{\phi(r_{j+1})r_{j+1} - \phi(r_j)r_j}{r_{j+1} - r_j} < \phi(r_{j+1}),$$

which implies

$$\phi(r_j) < \phi(r_{j+1}) \quad \text{and} \quad \phi(r_j) > \phi(r_{j+1}).$$

This is a contradiction. If on the other hand $g(r_j^-) < g(r_{j+1}^+)$, the states r_j and r_{j+1} are separated by a rarefaction wave and $g(r) = \phi(r) + \phi'(r)r$ for $r \in [\min\{r_j, r_{j+1}\}, \max\{r_j, r_{j+1}\}]$, so

$$\phi(r_j) < \phi(r_j) + \phi'(r_j)r_j < \phi(r_{j+1}) + \phi'(r_{j+1})r_{j+1} < \phi(r_{j+1}), \tag{14}$$

and $g(r) = \phi(r) + \phi'(r)r$ is strictly increasing in $r \in [r_j, r_{j+1}]$ if $r_j < r_{j+1}$ and striclty decreasing in $[r_{j+1}, r_j]$ if $r_j > r_{j+1}$. Equation (14) implies that $\phi'(r_j) > 0$ and $\phi'(r_{j+1}) < 0$. However, g is strictly increasing in $[r_j, r_{j+1}]$ (if $r_j < r_{j+1}$), this means that

$$g'(r) = 2\phi'(r) + \phi''(r)r > \epsilon, \quad r \in [r_j, r_{j+1}]$$

for some $\epsilon > 0$, and thus

$$rg'(r) = 2r\phi'(r) + \phi''(r)r^2 > \epsilon r, \quad r \in [r_j, r_{j+1}].$$

We notice that $rg'(r) = (r^2\phi'(r))'$, and thus

$$r^2\phi'(r) - r_j^2\phi'(r_j) > \frac{\epsilon}{2}(r^2 - r_j^2), \quad r \in [r_j, r_{j+1}],$$

which implies

$$\phi'(r) > \frac{\frac{\epsilon}{2}(r^2 - r_j^2) + r_j^2\phi'(r_j)}{r^2} > 0, \quad r \in [r_j, r_{j+1}],$$

contradicting (14). In a similar way we find for $r_{j+1} < r_j$,

$$r^2\phi'(r) - r_{j+1}^2\phi'(r_{j+1}) < -\frac{\epsilon}{2}(r^2 - r_{j+1}^2), \quad r \in [r_{j+1}, r_j],$$

for some $\epsilon > 0$, implying $\phi'(r) < 0$ in $r \in [r_{j+1}, r_j]$ and therefore also contradicting (14). Consequently, we have at most one contact discontinuity of the second wave in the solution of the Riemann problem and v has the form (11).

Remark 1.1. One of the strict inequalities in (13) could in fact be an equality. But if this is the case, we could go through the steps above to arrive at the same conclusion.

□

2. Front tracking

The aim is to approximate the strong generalized entropy solution to (1) by a front tracking method for the system

$$r_t + (\phi(r)r)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{15a}$$

$$u_t + (u\phi(r))_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{15b}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{15c}$$

$$r(0,x) = r_0(x) = |u_0(x)|, \quad x \in \mathbb{R}, \tag{15d}$$

with $r_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$. The idea is to first approximate the entropy solution r to the scalar equation (15a) via front tracking and then use this to find an approximation of the solution u to (15b) which satisfies $|u| = r$.

We start by describing the front tracking algorithm for the scalar conservation laws (15a). We denote $M := \|r_0\|_{L^\infty(\mathbb{R})}$ and $\delta > 0$ a small number. We let $r_i = \delta i$, $-M \leq i\delta \leq M$, and discretize the spatial domain by a grid $\{x_j = j\delta, j \in \mathbb{Z}\}$. Then, r_0 is approximated by a piecewise constant function r_0^δ taking in each cell $[j\delta, (j+1)\delta)$ one of the values in $U_\delta := \{r_i \,|\, i \in \mathbb{Z}, r_i \leq M\}$, and the flux function $\phi(r)r := f(r)$ is approximated by a piecewise linear interpolation f^δ,

$$f^\delta(r) = f(r_j) + \frac{f(r_{j+1}) - f(r_j)}{r_{j+1} - r_j}(r - r_j),$$
$$r \in [r_j, r_{j+1}), \quad j \in \mathbb{Z}, |j| \leq M\delta^{-1}. \tag{16}$$

Then we solve the initial value problem

$$r_t + f^\delta(r)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{17a}$$

$$r(x,0) = r_0^\delta(x), \quad x \in \mathbb{R}, \tag{17b}$$

exactly and denote the solution by r^δ. This means that in each step, we solve the Riemann problems between the states of the piecewise constant function, then track the discontinuities, which we will call *r-fronts*, until they interact, solve the emerging Riemann problem and so on. Note that the solution of each Riemann problem is again a piecewise constant function taking values in U_δ because f^δ is piecewise linear with breakpoints $r_i \in U_\delta$. It can be shown that the number of interactions between fronts is finite in $t \in (0,\infty)$, so the process terminates (see e.g. Lemma 2.6 in [7]). Moreover, the solution r^δ of (17) satisfies the Kružkov entropy condition and we have, see [7]

Theorem 2.1. *(i) The solutions r^δ to the differential equation* (17) *are uniformly bounded in δ for all $t \in (0,T)$:*

$$\|r^\delta(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|r_0\|_{L^\infty(\mathbb{R})}, \quad t \in (0,T),$$

(ii) The total variation of r^δ is bounded by the total variation of the initial data for all times $t \in (0,T)$,

$$TV(r^\delta(\cdot,t)) \leq TV(r_0), \quad t \in (0,T),$$

(iii) As the discretization parameter δ goes to zero, the sequence $(r^\delta)_{\delta>0}$ converges in $C((0,T);L^1_{loc}(\mathbb{R}))$ to a function r which is the unique entropy solution of (15a), (15d), *that is*

$$p(r)_t + q(r)_x \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)),$$

for all entropy pairs $(p(r), q(r))$, where p is convex and q defined by $q'(r) = (\phi(r)r)'p'(r)$.

This follows from the convergence results for the front tracking method applied to scalar conservation laws, see [7].

Next, we approximate u_0 by a piecewise constant function u_0^δ such that $|u_0^\delta| = r_0^\delta$ and solve the system

$$u_t + (u\phi(r^\delta))_x = 0, \quad (x,t)\in\mathbb{R}\times(0,T), \tag{18a}$$

$$u(x,0) = u^\delta(x), \quad x\in\mathbb{R}. \tag{18b}$$

Since r^δ is piecewise constant and we assume ϕ to be continuous, the first equation (18a) reduces to the transport equation

$$u_t + \phi(r^\delta)u_x = 0$$

away from the discontinuities of r^δ. Hence in these regions the values of u_0^δ are transported along the lines $x = \phi(r^\delta)t + x_0$ in the (x,t)-plane until they hit a discontinuity in r^δ. Across the r-fronts, $\phi(r)$ changes and therefore also the speed at which the values of u^δ are transported. However, as we have seen in Section 1.1, the quantity $v = u/|u|$ does not change across an r-shock, only the length of the vector u changes to the value of $r = |u|$ on the other side of the shock, so the same should hold for the approximation $v^\delta := u^\delta/|u^\delta|$.

We consider the interaction of a u-front corresponding to a contact discontinuity traveling at speed $\phi(r^\delta)$ with an r-front. We denote the left and right state of the resulting Riemann problem by u_L and u_R respectively and we assume that before the interaction $r_L = |u_L|$ and $r_R = |u_R|$ holds. The r-front has therefore speed $s_r = (\phi(r_L)r_L - \phi(r_R))/(r_L - r_R)$ and the contact discontinuity is propagated with speed $s_{v_R} = \phi(r_R)$ on the right side of the r-front and with speed $s_{v_L} = \phi(r_L)$ on the left side of the r-front. Notice that either

$$s_{v_R} \geq s_r \geq s_{v_L} \quad \text{or} \quad s_{v_L} \geq s_r \geq s_{v_R},$$

which means that contact discontinuities enter the shock only from the one side and leave it on the other side. This implies in particular that the *generalized Lax entropy condition* described in [8] is satisfied at the shock. Furthermore, we never have to resolve the interaction of two contact discontinuities of the second wave, as such interactions cannot occur. Let us assume without loss of generality that $s_{v_R} > s_r$ (the other case is treated similarly). Then the solution of the Riemann problem is, denoting the location of the interaction by (x_0, t_0)

$$u^\delta(x,t) = \begin{cases} u_L, & x - x_0 < s_{v_L}(t-t_0) \\ \frac{r_L}{r_R}u_R, & s_{v_L}(t-t_0) \leq x - x_0 < s_r(t-t_0) \\ u_R, & s_r(t-t_0) < x - x_0. \end{cases} \tag{19}$$

It follows that the Rankine-Hugoniot condtion is satisfied across the shock and that $|u^\delta| = r^\delta$, so (r^δ, u^δ) constructed in this way is a solution of

$$\begin{aligned} r^\delta_t + f^\delta(r^\delta)_x &= 0, \quad (x,t) \in \mathbb{R} \times (0,T),\\ u^\delta_t + (u^\delta \phi(r^\delta))_x &= 0, \quad (x,t) \in \mathbb{R} \times (0,T),\\ u^\delta(x,0) &= u^\delta_0(x), \quad x \in \mathbb{R},\\ r^\delta(x,0) = r^\delta_0(x) &= |u^\delta_0(x)|, \quad x \in \mathbb{R}, \end{aligned}$$

satisfying $|u^\delta(x,t)| = r^\delta(x,t)$ almost everywhere. Moreover, since r^δ satisfies the Kružkov entropy condition, u^δ is the unique strong generalized entropy solution of

$$\begin{aligned} u^\delta_t + (\phi^\delta(|u^\delta|)u^\delta)_x &= 0, \quad (x,t) \in \mathbb{R} \times (0,T),\\ u^\delta(x,0) &= u^\delta_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $\phi^\delta(r) = f^\delta(r)/r$ with f^δ defined by (16), in the sense of Definition 1.1.

2.1. **Convergence.**

Proposition 2.2. *If $u_0 \in L^\infty(\mathbb{R})$ and $r_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, the approximations (r^δ, u^δ) computed by the method described in Section 2 converge in L^∞-weak-* to the unique strong generalized entropy solution $u \in L^\infty(\mathbb{R} \times (0,T))$ of system (1) as defined in Definition 1.1, when the discretization parameter δ goes to zero. If additionally $v_0 := u_0/r_0 \in BV(\mathbb{R})$, the convergence is in $C((0,T); L^1_{loc}(\mathbb{R}))$ and the solution $u(\cdot,t) \in BV(\mathbb{R})$.*

Proof. From Theorem 2.1, we obtain strong convergence of the sequence $(r^\delta)_{\delta>0}$ to the entropy solution r of (15a), (15d). Moreover, since by the second point in Theorem 2.1, $(r^\delta)_{\delta>0}$ is uniformly bounded, the same holds for the sequence $(u^\delta)_{\delta>0}$, since by construction of the method $|u^\delta| = r^\delta$. By Alaoglu's Theorem, we can therefore extract a L^∞-weak-* convergent subsequence, still denoted $(u^\delta)_{\delta>0}$: $u^\delta \overset{*}{\rightharpoonup} u \in L^\infty(\mathbb{R} \times (0,T))$. We show that the limit u is a weak solution of (15b).

To this end, we define for arbitrary test functions $\psi \in C^{1,1}_0(\mathbb{R} \times [0,T))$

$$\begin{aligned} I_\delta(\psi) &= \int_0^T \int_{\mathbb{R}} u^\delta \psi_t + \phi(r^\delta) u^\delta \psi_x \, dx + \int_{\mathbb{R}} u^\delta_0(x)\psi(x,0)\, dx,\\ \widetilde{I}_\delta(\psi) &= \int_0^T \int_{\mathbb{R}} u^\delta \psi_t + \phi(r) u^\delta \psi_x \, dx + \int_{\mathbb{R}} u^\delta_0(x)\psi(x,0)\, dx, \end{aligned}$$

Since u^δ are a weak solutions of (18), $I_\delta(\psi) = 0$ for all $\psi \in C^{1,1}_0(\mathbb{R} \times [0,T))$. Furthermore, because $r^\delta \to r$ strongly in $L^1_{\text{loc}}(\mathbb{R} \times (0,T))$ and $\|u^\delta\|_{L^\infty} \le M$ uniformly in δ, we have by Lebesgue's dominated convergence theorem,

$$\lim_{\delta \to 0} (I_\delta(\psi) - \widetilde{I}_\delta(\psi)) = \lim_{\delta\to 0} \int_0^T \int_{\mathbb{R}} (\phi(r^\delta) - \phi(r)) u^\delta \psi_x \, dx = 0,$$

so $\widetilde{I}_\delta(\psi) \to 0$ as $\delta \to 0$. Using $u^\delta \overset{*}{\rightharpoonup} u$ and the strong convergence $u^\delta_0 \to u_0$ in L^1_{loc}, we deduce,

$$\lim_{\delta \to 0} \widetilde{I}_\delta(\psi) = \int_0^T \int_{\mathbb{R}} u\psi_t + \phi(r) u \psi_x \, dx + \int_{\mathbb{R}} u_0(x)\psi(x,0)\, dx = 0.$$

Thus, (r, u) is a weak solution of (15). The results in [13] imply that the weak solution of (15) for which in addition r is the entropy solution of (15a), is unique and $|u| = r$, so the limit u is the unique entropy solution of (1).

In order to prove the second statement, we assume $\text{TV}(v_0) \le K < \infty$, where $v_0 := u_0/r_0$, if $r_0 \neq 0$, and $v_0 := 0$, if $r_0 = 0$. We set $v^\delta := u^\delta/r^\delta$, if $r_0 \neq 0$, and $v^\delta := 0$, if $r_0 = 0$, so v^δ_0 is a piecewise constant approximation of v_0 and its total variation is bounded by $K + c$ for some $0 < c < \infty$ for δ small enough. We prove that the total variation of $v^\delta(\cdot, t)$ is bounded by the total variation of v_0^δ. By the construction of the method, v^δ is a piecewise constant function and its total variation can only change in places where we solve a Riemann problem. We have solved this Riemann problem in Section 2 and see from (19) that v^δ changes only across one of the discontinuities and that the size of the jump in v is exactly the one present before solving the Riemann problem. So the total variation of v^δ does not change when solving a Riemann problem and therefore stays constant for all t. We observe next that the total variation of u^δ is bounded independently of t, δ. For $t \in (0, T)$ we let t_j, t_{j+1} two consecutive collision times such that $t \in (t_j, t_{j+1}]$ and denote for $t \in (t_j, t_{j+1}]$ by $x_k^j(t)$ the position of the k-th front from the left at time t and by u_k^j the value of u^δ between the fronts located at $x_k^j(t)$ and $x_{k+1}^j(t)$. Then we can write $u^\delta(x, t)$ for $t \in (t_j, t_{j+1}]$ in the form

$$u^\delta(x,t) = \sum_{k=1}^{N_j} (u_k^j - u_{k-1}^j)\chi_{[x_k^j(t),\infty)}(x) + u_1^j$$

denoting by N_j the number of discontinuities in $\mathbb{R} \times (t_j, t_{j+1}]$. So the total variation of u^δ at time t is given as the sum of the jumps across the discontinuities $x_k^j(t)$ (we denote $r_k^j = |u_k^j|$ and $v_k^j = u_k^j/|u_k^j|$, $k = 1, \ldots, N_j$, $(u_k^j)^{(l)}$ the lth component of u_j^k, $l = 1, \ldots, n$, $k = 1, \ldots, N_j$, and the same for v_k^j),

$$\begin{aligned}
\mathrm{TV}(u^\delta(\cdot,t)) &= \sum_{l=1}^{n}\sum_{k} |(u_k^j)^{(l)} - (u_{k-1}^j)^{(l)}| \\
&= \sum_{l=1}^{n}\sum_{k} |r_k^j\,(v_k^j)^{(l)} - r_{k-1}^j\,(v_{k-1}^j)^{(l)}| \\
&\le \sum_{l=1}^{n}\sum_{k} \Big(|(r_k^j - r_{k-1}^j)(v_k^j)^{(l)}| + |r_{k-1}^j||(v_k^j)^{(l)} - (v_{k-1}^j)^{(l)}|\Big) \\
&\le \mathrm{TV}(r^\delta(\cdot,t)) + \|r^\delta(\cdot,t)\|_{L^\infty}\mathrm{TV}(v^\delta(\cdot,t)) \\
&\le \mathrm{TV}(r_0) + \|r_0\|_{L^\infty}(\mathrm{TV}(v_0) + c),
\end{aligned}$$

where we have used the second statement in Theorem 2.1 to bound the total variation of r^δ. We can use the bound on the total variation to show that u^δ is Lipschitz continuous in time. To do so, we let $t \in (t_j, t_{j+1}]$ and $r \in (t_i, t_{i+1}]$, with $i \le j$ and $r \le t$. Then

$$\begin{aligned}
\int_{\mathbb{R}} |u^\delta(x,t) - u^\delta(x,t_j)|\,dx &= \int_{\mathbb{R}}\left|\int_{t_j}^{t} \frac{d}{d\tau}u^\delta(x,\tau)\,d\tau\right| dx \\
&\le \int_{\mathbb{R}}\int_{t_j}^{t}\sum_{k=1}^{N_j} |u_k^j - u_{k-1}^j|\big|x_k^{j'}(\tau)\big|\delta_{x=x_k^j(\tau)}\,d\tau\,dx \\
&\le \int_{t_j}^{t}\sum_{k=1}^{N_j} |u_k^j - u_{k-1}^j|\big|x_k^{j'}(\tau)\big|\,d\tau
\end{aligned}$$

The speeds $x_j^{k'}(t)$ are bounded by

$$S := \max_{|z|\le \sup|u^\delta|}\{\lambda_1(z), \lambda_2(z)\} < \infty$$

which is bounded away from infinity because u^δ is bounded uniformly in t, δ and η. Thus,

$$\begin{aligned}\int_{\mathbb{R}} |u^\delta(x,t) - u^\delta(x,t_j)|\,dx &\leq S(t-t_j)\sum_{k=1}^{N_j} |u_k^j - u_{k-1}^j| \\ &\leq S(t-t_j)\,\mathrm{TV}(u^\delta(\cdot,t)) \\ &\leq S(t-t_j)\,\mathrm{TV}(u_0).\end{aligned}$$

Similarly, we find

$$\int_{\mathbb{R}} |u^\delta(x,t_j) - u^\delta(x,t_i)|\,dx \leq S(t_j - t_i)\,\mathrm{TV}(u_0)$$

and

$$\int_{\mathbb{R}} |u^\delta(x,t_i) - u^\delta(x,r)|\,dx \leq S(t_i - r)\,\mathrm{TV}(u_0).$$

Hence

$$\int_{\mathbb{R}} |u^\delta(x,t) - u^\delta(x,r)|\,dx \leq S(t-r)\,\mathrm{TV}(u_0), \tag{20}$$

which is uniform in δ.

Using the uniform L^∞- and total variation bounds for u^δ and the Lipschitz continuity in time, (20), we can apply Kolmogorov's Compactness Theorem to conclude convergence of a subsequence $(u^{\delta_n})_{n\in\mathbb{N}}$ to a limit u with bounded total variation in $C((0,T); L^1_{\mathrm{loc}}(\mathbb{R}))$. As we have shown before, the limit u is the unique strong generalized entropy solution of (1). Since the limit is unique, not only a subsequence, but the whole sequence $(u^\delta)_{\delta>0}$ converges. □

3. Numerical experiments

We test the method described in Section 2 with $\phi(r) = r^2 - 4r + 5.5$. This function has a minimum at $r = 2$, so the ordering of the eigenvalues changes, and in addition we have $(f(r))'' = (\phi(r)r)'' = 0$ at $r = 4/3$, and the flux function is nonconvex. We test the method with the following initial data

$$r_0 = \sin(\pi x) + 1.5, \quad v_0 = \begin{pmatrix}\sin(\pi x)\\ \cos(\pi x)\end{pmatrix}, \quad x \in [-1,1], \tag{21}$$

with periodic boundary conditions. A plot of the wave fronts is shown in Figure 1.

Note that the eigenvalues are nonnegative for all $r \geq 0$. We can therefore use the difference schemes constructed in [9], to compute an approximation to the solution of (1a), (21), and compare the results. We denote for a given $\Delta x, \Delta t > 0$, by $x_j = j\Delta x$, $j \in \mathbb{Z}$, the discretization in space, by $t^n = n\Delta t$, $n = 0, \dots, N$, where $N\Delta t = T$, the discretization in time (Δt small enough such that it satisfies a CFL-condition), and by u_j^n, r_j^n the approximations of the solutions u, $r := |u|$ of (1) in the cell $I_j^n := [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1})$. Then we approximate the solution of (1) by the upwind scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n \phi(|u_j^n|) - u_{j-1}^n \phi(|u_{j-1}^n|)\right), \qquad j \in \mathbb{Z},\ n = 1, \dots, N, \tag{22a}$$

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx, \qquad j \in \mathbb{Z}. \tag{22b}$$

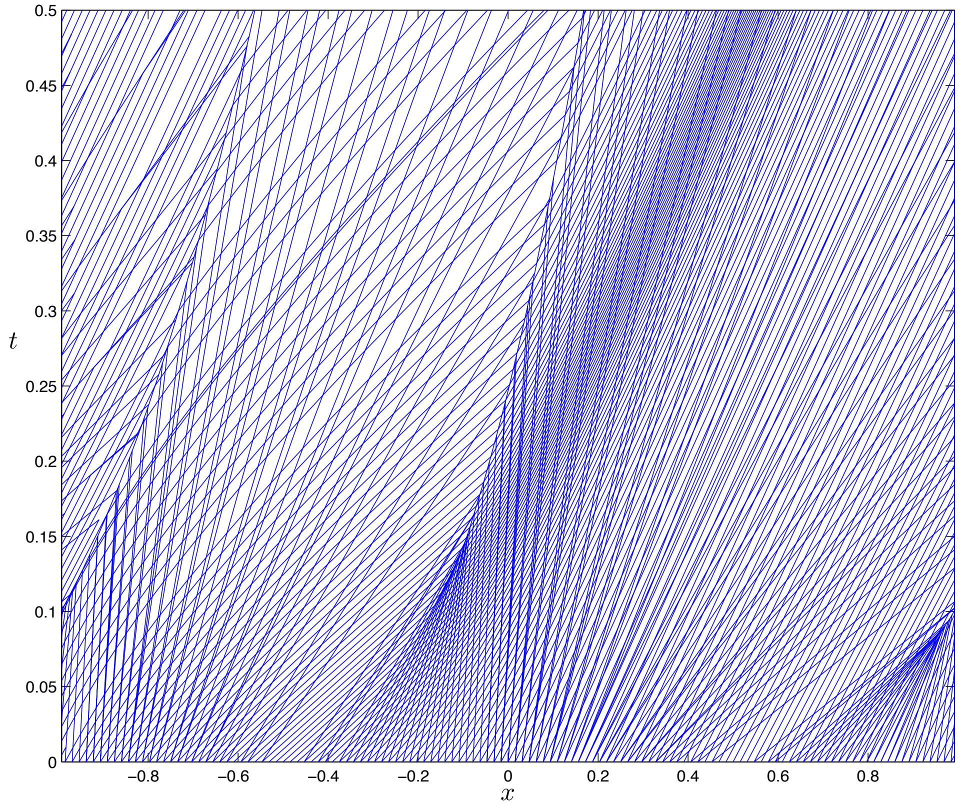

FIGURE 1. The wave fronts in the (x,t)-plane for initial value problem (21) and $\phi(r) = r^2 - 4r + 5.5$, $\delta = 0.02$.

We also considered a scheme based on decoupling of the variables u and r;

(23a) $$r_j^{n+1} = r_j^n - \frac{\Delta t}{\Delta x}(r_j^n \phi(r_j^n) - r_{j-1}^n \phi(r_{j-1}^n)), \qquad j \in \mathbb{Z},\ n = 1, \dots, N,$$

(23b) $$\widetilde{u}_j^{n+1} = \widetilde{u}_j^n - \frac{\Delta t}{\Delta x}(\widetilde{u}_j^n \phi(r_j^n) - \widetilde{u}_{j-1}^n \phi(r_{j-1}^n)), \qquad j \in \mathbb{Z},\ n = 1, \dots, N,$$

(23c) $$\widetilde{u}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx, \qquad j \in \mathbb{Z},$$

(23d) $$r_j^0 = |\widetilde{u}_j^0|, \qquad j \in \mathbb{Z}.$$

In Figure 2, we observe a good agreement of the approximations. Note that the front tracking method has the advantage that it can also be used if the flux function is such that its Jacobian has negative eigenvalues.

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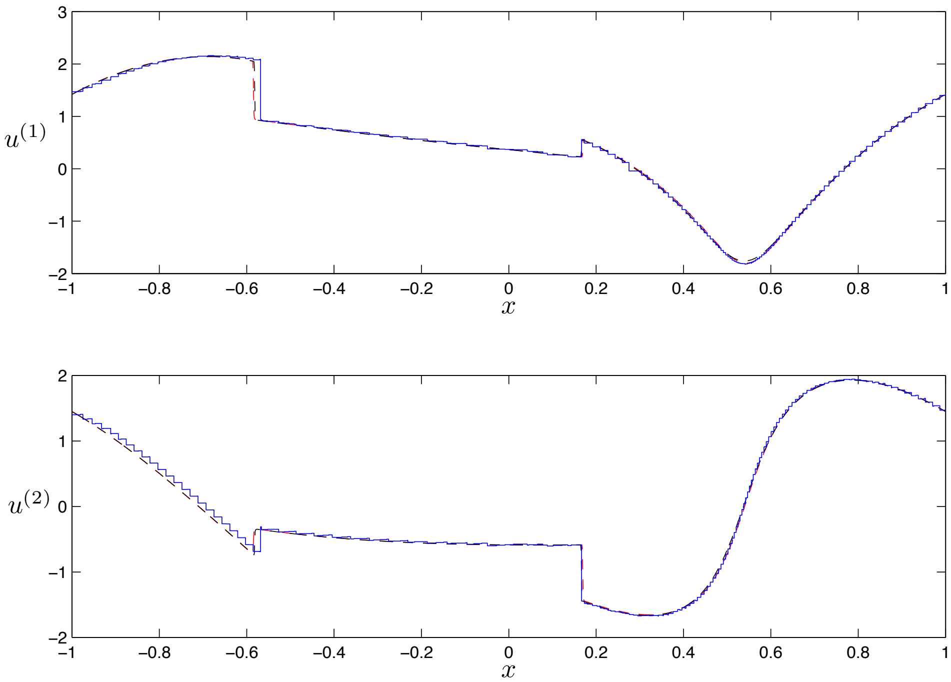

FIGURE 2. The approximations of the solution to (1a), (21) for $\phi(r) = r^2 - 4r + 5.5$ computed by the different methods at time $t = 0.5$. The blue solid line denotes the front tracking approximation ($\delta = 0.01$), the red dashed line the approximation by scheme (22) and the black dashed line the approximation computed with (23). The finite difference approximations have been computed on a mesh with 2000 gridpoints.

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