

Bandlimited shearlet-type frames with nice duals

P. Grohs

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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P. Grohs*

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Abstract

The present paper for the first time constructs a frame/dual frame pair of shearlet type such that both frames possess the distinctive time-frequency localization properties needed in establishing their desirable approximation properties. Our construction is based on a careful pasting together of two bandlimited shearlet Parseval frames associated with two different frequency cones, inspired by domain decomposition methods used primarily for the solution of PDEs.

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1 Introduction

It is by now widely accepted in computational harmonic analysis that the ability to handle anisotropic data in a multiscale fashion requires transforms which possess an anisotropic scaling operation. Several such transformations have enriched the field of applied mathematics in the past few years, including curvelets [4], bandlets [21], shearlets [20] or countourlets [9]. Among these shearlets stand out for several reasons, including their well-adaptedness to digitization, see for example the discussion in [18]. Nevertheless, there exist some problems in constructing nice shearlet frames. In particular so far no construction of a shearlet frame/dual frame pair has been given such that both of these frames possess the time-frequency localization properties required for the transforms to exhibit their desirable properties, such as

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optimal best N -term approximation of cartoon functions (e.g., bivariate piecewise C^2 functions with C^2 singularity curves) [5] It is the purpose of the present paper to close this gap by giving such a construction. In particular we will construct a frame/dual frame pair Σ, Σ' such that both Σ and Σ' are of shearlet type and such that both of these frames constitute a system of parabolic molecules of arbitrary order in the sense of [14]. In particular, this latter property ensures that both of these systems possess the time-frequency localization properties which make shearlets and related transforms so successful, see the results of [14].

First we start by reviewing the constructions of shearlet frames which have been given so far in the literature.

1.1 Previous Work

Let us first briefly describe what we understand as systems of shearlet-type. Classical shearlet constructions are based on the action of anisotropic scaling and shearing to a single bivariate function ψ , e.g.,

$$\sigma_{(i,j,l,k)}(\cdot) = 2^{3j/4} \psi^i (D_{2^j}^i S_{l,j}^i \cdot -k), \quad (1)$$

where $i \in \{1, 2\}$, $j \in \mathbb{N}$, $k \in \mathbb{Z}^2$ and $-2^{j/2} \leq l < 2^{j/2}$, and $D_{2^j}^i, S_{l,j}^i$ are defined as in Section 1.4. Shearlet systems (constructed with appropriate functions ψ^i) share many desirable approximation properties with curvelets [4], which are based on a rotation operation in place of shearing in order to achieve directional selectivity. The distinctive advantage of shearlet constructions over curvelets is the fact that shearing operations can be naturally defined for digital data, while for rotations this is not the case. For us a system of shearlet-type is one, where in (1), the function ψ may also depend on the variables j, l, k to a small extent, see Section 2 for more precise definitions.

Up to now the following different constructions of shearlet-type frames exist.

Bandlimited Parseval frame

Historically, the first construction is given in [15] where bandlimited Parseval frames are described. There, two Parseval frames Σ^1, Σ^2 for the two Hilbert spaces of square integrable functions with frequency support in the horizontal resp. vertical cones with opening angle $\frac{\pi}{2}$ are constructed. In order to aggregate them to form a frame for the full space $L_2(\mathbb{R}^2)$, a simple orthogonal projection onto the corresponding frequency cones is performed on the frame elements. This approach can be problematic since the frame elements which are affected by such a frequency projection completely lose their localization properties in space, compare Figure 1.

Compactly Supported Frames

Recently, compactly supported shearlet frames have been constructed [17, 19, 18]. It can be shown that these constructions provide the desired approximation rates for cartoon images but at present it is not possible to access any properties of the dual frame.

1.2 Motivation

However, for several reasons it can be important to have a construction of a shearlet frame which possesses a dual frame that is also of shearlet type.

Let us give two concrete problems where such a construction is needed. The first one is of a theoretical nature, namely the study of shearlet function spaces. We would like to rigorously establish the fact that the function spaces spanned by a shearlet frame are the same as those spanned by a curvelet frame Γ [6] and similar equivalent constructions as in [2]. The fact that for shearlets we do not in general have information on the dual frame poses a big obstacle in this regard. Assume we are given a shearlet frame Σ which has a dual frame Σ' and that f is a tempered distribution which lies in some curvelet function space, meaning that the sequence $\langle f, \Gamma \rangle$ of curvelet coefficients lies in some discrete sequence space, compare [2]. To show that then also the shearlet coefficient sequence lies in the same sequence space one can write

$$\langle f, \Sigma \rangle = \langle f, \Gamma \rangle^\top \langle \Gamma, \Sigma \rangle$$



Figure 1: Left: indicator function of the unit disc. Right: frequency projection onto the horizontal frequency cone with opening angle $\frac{\pi}{2}$.

and use the fact that the Gramian matrix $\langle \Gamma, \Sigma \rangle$ is almost diagonal. This property only holds if the shearlet frame Σ possesses the same essential time-frequency localization properties as the curvelet frame Γ . To show the converse, e.g., that if the sequence $\langle f, \Sigma \rangle$ lies in a certain sequence space, then also the curvelet coefficient sequence of f lies in the same space, we can proceed similarly, e.g., we can write

$$\langle f, \Gamma \rangle = \langle f, \Sigma \rangle^\top \langle \Sigma', \Gamma \rangle$$

and again conclude that *if the dual frame Σ' possesses the same time-frequency localization properties as Γ* , the Gramian $\langle \Sigma', \Gamma \rangle$ is almost diagonal which shows that $\langle f, \Gamma \rangle$ lies in the same sequence space. This superficial discussion shows that in order to study function spaces associated to shearlets it is imperative to have precise control over the primal, *as well as the dual frame!* Similar considerations related to wavelets and Besov spaces can be found in [11].

The second problem is more of an applied nature. Assume that we would like to compute the evolution of a function (initial condition) f under the action of a wave propagator. The paper [3] shows that by representing the initial condition in a curvelet frame, the associated Gramian matrix of the wave propagator is compressible in the sense of [8]. This implies that if the initial condition to the evolution is sparse in a curvelet frame, then, at least theoretically, it is possible to apply the evolution operator to the curvelet coefficients in linear complexity. Arguably, this is only a theoretical result since it is highly nontrivial how to actually *compute* a compressed version of the evolution operator without deteriorating the complexity, compare [12] for a similar discussion related to wavelet bases and elliptic operator equations. Nevertheless, it is of interest to transfer such a result to the shearlet setting. With a shearlet frame Σ and dual frame Σ' we can write

$$f = \langle f, \Sigma \rangle^\top \Sigma'$$

and let the wave propagator $E(t)$ associated with time t act on the shearlet coefficients of f via

$$\langle E(t)f, \Sigma \rangle = \langle f, \Sigma \rangle^\top \langle E(t)\Sigma', \Sigma \rangle.$$

In order to arrive at results in the spirit of [3] we need to establish that the Gramian $\langle E(t)\Sigma', \Sigma \rangle$ is almost diagonal and that the coefficient sequence $\langle f, \Sigma \rangle$ is sparse. Thus, here again we need to be able to have precise control over the time-frequency localization properties of Σ and its dual Σ' .

1.3 Contributions

The purpose of this paper is to give a construction of a shearlet frame Σ with a shearlet dual Σ' , both having the time-frequency localization properties needed for proving such important results as best N -term approximation of cartoon images [3] or sparsity of Fourier integral operators. In particular we will

show that, given any two Parseval frames of shearlet type for the two frequency cones, one can aggregate them to a frame for $L_2(\mathbb{R}^2)$ such there exists a dual frame which is of the same type. This dual frame will in general *not* be the canonical dual frame which poses no problem. Our construction is completely explicit and is inspired by domain decomposition methods in Finite Element analysis [1, 23].

Outline

The outline is as follows: First in Section 2 we describe mathematically what it means for a system of functions to possess the correct time-frequency localization properties. To this end we make use of the recently introduced concept of *parabolic molecules* [14]. Then in Section 3 we present the main idea of our novel construction. As previously mentioned the inspiration comes from domain decomposition methods for wavelet frames [23] but also from [13], where a similar construction is given for continuous shearlet frames. Essentially our construction is based on patching together shearlet tight frames associated to different frequency cones. The construction of these different tight frames is the subject in Section 4. Section 5 treats the crucial problem of designing suitable weight functions in the frequency domain to patch together the tight frames constructed in Section 4. This is the technically most demanding part of the present paper. In Section 6 we illustrate a few applications of our construction before we close with a discussion of the present work.

1.4 Notation

We shall write $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ the rotation matrix of angle θ and $D_a^1 := \text{diag}(a, \sqrt{a})$, $D_a^2 := \text{diag}(\sqrt{a}, a)$ and $\langle x \rangle := (1 + x^2)^{1/2}$. Furthermore we denote $S_{l,j}^1 = \begin{pmatrix} 1 & 2^{-j/2}l \\ 0 & 1 \end{pmatrix}$ and $S_{l,j}^2 := (S_{l,j}^1)^\top$. We denote \mathcal{S} the space of Schwartz functions on \mathbb{R}^2 and \mathcal{S}' its dual, the space of tempered distributions [22]. For $f \in \mathcal{S}$, the symbol \hat{f} denotes its Fourier transform

$$\hat{f}(\cdot) := \int_{\mathbb{R}^2} f(x) \exp(2\pi i \langle \cdot, x \rangle) dx$$

and extended to $f \in \mathcal{S}'$ by duality. Assume that $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ is a sequence of functions and $\mathbf{c} = (c_\lambda)_{\lambda \in \Lambda}$ a complex valued sequence, where Λ is a discrete index set. Then we denote

$$\mathbf{c}^\top \Psi := \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda.$$

A system Ψ is called a *frame* for a Hilbert space \mathcal{H} if we have the norm equivalence

$$\|f\|_{\mathcal{H}}^2 \sim \inf_{\mathbf{c}^\top \Psi = f} \|\mathbf{c}\|_2^2.$$

If the previous norm equivalence is an equality up to a constant we speak of a *tight frame*, if the constant is 1, we speak of a *Parseval frame*. A second frame Σ' is called *dual frame* of Σ if the identity

$$f = \langle f, \Sigma \rangle^\top \Sigma' \quad \text{for all } f \in \mathcal{H}.$$

See [7] for more information related to frames.

2 Parabolic Molecules

In order to put the various time-frequency localization properties of parabolic transforms like curvelets or shearlets under one roof, the concept of parabolic molecules is introduced in [14]. Roughly speaking a system of parabolic molecules is a frame having equivalent localization properties as e.g. the curvelet frame constructed in [3]. A main result of [14] is that any two such systems are equivalent in terms of their approximation properties. To define parabolic molecules we need to define what we mean by a parameterization.

Definition 2.1. A parameterization consists of a pair (Λ, Φ_Λ) where Λ is a discrete index set and Φ_Λ is a mapping

$$\Phi_\Lambda : \begin{cases} \Lambda & \rightarrow \mathbb{P}, \\ \lambda \in \Lambda & \mapsto (s_\lambda, \theta_\lambda, x_\lambda). \end{cases}$$

which associate with each $\lambda \in \Lambda$ a scale s_λ , a direction θ_λ and a location $x_\lambda \in \mathbb{R}^2$.

We can now define precisely what we mean by the notion of parabolic molecules.

Definition 2.2. Let Λ be a parameterization. A family $(m_\lambda)_{\lambda \in \Lambda}$ is called a family of parabolic molecules of order (R, M, N_1, N_2) if it can be written as

$$m_\lambda(x) = 2^{3s_\lambda/4} a^{(\lambda)}(D_{2^{s_\lambda}} R_{\theta_\lambda}(x - x_\lambda))$$

such that

$$|\partial^\beta \hat{a}^{(\lambda)}(\xi)| \lesssim \min\left(1, 2^{-s_\lambda} + |\xi_1| + 2^{-s_\lambda/2} |\xi_2|\right)^M \langle \|\xi\| \rangle^{-N_1} \langle \xi_2 \rangle^{-N_2} \quad (2)$$

for all $|\beta| \leq R$. The implicit constants are uniform over $\lambda \in \Lambda$.

The main point of interest for parabolic molecules is that they are all essentially equivalent as far as their approximation properties are concerned, see [14] for more information. In particular in [14] it is shown that for any two systems $(m_\lambda), (m'_{\lambda'})$ of parabolic molecules, their cross Gramian matrix $\langle m_\lambda, m'_{\lambda'} \rangle$ is almost diagonal in a very strong sense.

Importantly, it turns out that shearlet-type systems are instances of parabolic molecules. Consider functions $\varphi, \psi_{j,l,k}^0, \psi_{j,l,k}^1$ satisfy

$$|\partial^\beta \hat{\psi}_{j,l,k}^i(\xi_1, \xi_2)| \lesssim \min(1, |\xi_i|)^M \langle \|\xi\| \rangle^{-N_1} \langle \xi_{3-i} \rangle^{-N_2} \quad (3)$$

for every $\beta \in \mathbb{N}^2$ with $|\beta| \leq R$. Then [14] shows that the shearlet system

$$\Sigma := \{\sigma_\lambda : \lambda \in \Lambda\}, \quad (4)$$

where

$$\Lambda := \left\{ (i, j, l, k) \in \{1, 2\} \times \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}^2 : -2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2} \right\}$$

and

$$\sigma_{(i,j,l,k)}(\cdot) = 2^{3j/4} \psi_{j,l,k}^i(D_{2^j} S_{l,j}^i \cdot -k)$$

constitutes a system of parabolic molecules of order (R, M, N_1, N_2) associated with a nice parameterization.

The main goal of the present paper is to give a (the first) construction of a shearlet frame Σ with a dual frame Σ' , such that both Σ, Σ' are of the form (4) with generators satisfying (3) for arbitrary (R, M, N_1, N_2) .

3 Basic Idea

We define the frequency cones

$$\begin{aligned} \mathcal{C}^1 &:= \left\{ \xi : |\xi_1| \geq \frac{1}{8}, \frac{|\xi_2|}{|\xi_1|} \leq \frac{3}{2} \right\}, & \bar{\mathcal{C}}^1 &:= \left\{ \xi : |\xi_1| \geq \frac{1}{4}, \frac{|\xi_2|}{|\xi_1|} \leq \frac{4}{3} \right\}, \\ \mathcal{C}^2 &:= \left\{ \xi : |\xi_2| \geq \frac{1}{8}, \frac{|\xi_1|}{|\xi_2|} \leq \frac{3}{2} \right\}, & \bar{\mathcal{C}}^2 &:= \left\{ \xi : |\xi_2| \geq \frac{1}{4}, \frac{|\xi_1|}{|\xi_2|} \leq \frac{4}{3} \right\}, \end{aligned}$$

and low-frequency boxes

$$\mathcal{C}^0 := \{\xi : \|\xi\|_\infty \leq 1\}, \quad \bar{\mathcal{C}}^0 := \left\{ \xi : \|\xi\|_\infty \leq \frac{2}{3} \right\}.$$

Let us for now assume that we have Parseval frames Σ^i for the Hilbert spaces $L_2(\mathcal{C}^i)^\vee$ for $i = 0, 1, 2$. In Section 4 we will see how such frames of shearlet type can easily be constructed using standard methods as in [15, 16]. We then find a method to patch together these frames to obtain a shearlet type frame for $L_2(\mathbb{R}^2)$ which also possesses a dual frame of the same type. Our construction is inspired by domain decomposition methods used for the solution of elliptic PDEs, see in particular [23, 1]. In several aspects however our construction crucially differs from known approaches used in domain decomposition methods. For instance, we are dealing with a domain decomposition in the Fourier domain and the decomposition includes infinite subdomains. In the remainder of this section we will lay out the main idea on how three shearlet frames associated to \mathcal{C}^i , $i = 0, 1, 2$ can be patched together using carefully constructed partition-of-unity functions. In later sections we will then elaborate on the details of constructing suitable frames and partition functions which will comprise the main technical part of this paper. First we describe the main idea.

Definition 3.1. *Pick partition functions $\gamma^i, \chi^i \in C^\infty(\mathbb{R}^2)$, $i = 0, 1, 2$ satisfying the following properties:*

- (i) *supp $\gamma^i \subset \mathcal{C}^i$ and supp $\chi^i \subset \bar{\mathcal{C}}^i$ for $i = 0, 1, 2$,*
- (ii) *$\chi^i \geq 0$ for $i = 0, 1, 2$ and $\sum_{i=0}^2 \chi^i = 1$,*
- (iii) *$\frac{\chi^i}{\gamma^i} \lesssim 1$ for $i = 0, 1, 2$.*

Consider the two systems

$$\Sigma := \bigcup_{i=0}^2 \left(\gamma^i \hat{\Sigma}^i \right)^\vee \quad \text{and} \quad \Sigma' := \bigcup_{i=0}^2 \left(\frac{\chi^i}{\gamma^i} \hat{\Sigma}^i \right)^\vee. \quad (5)$$

The crucial observation is the following result:

Proposition 3.2. *The system Σ , constitutes a frame for $L_2(\mathbb{R}^2)$ with a dual frame given by Σ' .*

Proof. We first show that Σ constitutes a frame which is equivalent to the validity of

$$\|f\|_2^2 \sim \inf_{\mathbf{c}^\top \Sigma = f} \|\mathbf{c}\|_2^2, \quad (6)$$

which, due to the fact that the Σ^i 's constitute frames for $L_2(\mathcal{C}^i)^\vee$, follows from

$$\|f\|_2^2 \sim \inf_{\substack{f = \sum_i f^i \\ \frac{f^i}{\gamma^i} \in L_2(\mathcal{C}^i)}} \sum_{i=0}^2 \left\| \frac{\hat{f}^i}{\gamma^i} \right\|_2^2, \quad (7)$$

which is what we will show. Note that the assumption $\frac{f^i}{\gamma^i} \in L_2(\mathcal{C}^i)$ implies that \hat{f}^i is supported in \mathcal{C}^i . We start by showing the estimate

$$\|f\|_2^2 \lesssim \inf_{\substack{f = \sum_i f^i \\ \frac{f^i}{\gamma^i} \in L_2(\mathcal{C}^i)}} \sum_{i=0}^2 \left\| \frac{\hat{f}^i}{\gamma^i} \right\|_2^2.$$

To this end, assume that $f = \sum_{i=0}^2 f^i$. Then we have

$$\|f\|_2^2 \lesssim \sum_{i=0}^2 \|f^i\|_2^2 = \sum_{i=0}^2 \|\hat{f}^i\|_2^2 \lesssim \sum_{i=0}^2 \left\| \frac{\hat{f}^i}{\gamma^i} \right\|_2^2,$$

the last inequality following from the general fact that

$$\|\gamma^i g\|_2^2 \lesssim \|g\|_2^2, \quad \text{for } g \in L_2(\mathcal{C}^i), \quad (8)$$

which holds since γ^i is bounded. This shows the upper inequality in (7).

We turn to the estimate from below, i.e.

$$\|f\|_2^2 \gtrsim \inf_{\substack{f=\sum_i f^i \\ \frac{\hat{f}^i}{\gamma^i} \in L_2(\mathcal{C}^i)}} \sum_{i=0}^2 \left\| \frac{\hat{f}^i}{\gamma^i} \right\|_2^2. \quad (9)$$

Here, the functions χ^i will come into play. Indeed, due to Property (ii) in Definition 3.1 we can write

$$f = \sum_{i=0}^2 f^i \quad \text{where} \quad f^i := (\chi^i \hat{f})^\vee. \quad (10)$$

Furthermore, due to Property (iii) in Definition 3.1 we have that

$$\frac{\hat{f}^i}{\gamma^i} \in L_2(\mathcal{C}^i),$$

which makes (10) a valid decomposition for (9). To show (9) we write

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \int (\sum_i \chi^i)^2 |\hat{f}|^2 \geq \sum_i \int (\chi^i \hat{f})^2 = \sum_i \|f^i\|_2^2 \gtrsim \sum_i \left\| \frac{\hat{f}^i}{\gamma^i} \right\|_2^2,$$

the first inequality follows from the positivity of the functions χ^i , the second one from (8). This proves (7) and hence that Σ is indeed a frame for $L_2(\mathbb{R}^2)$. A similar argument establishes that Σ' is also a frame for $L_2(\mathbb{R}^2)$.

It remains to show that Σ' is a dual frame of Σ . This turns out to be quite simple: We have

$$\left\langle f, (\gamma^i \hat{\Sigma}^i)^\vee \right\rangle^\top \hat{\Sigma}^i = \gamma^i \hat{f},$$

since Σ^i constitutes a Parseval frame for $L_2(\mathcal{C}^i)^\vee$. Therefore

$$\sum_{i=0}^2 \left\langle f, (\gamma^i \hat{\Sigma}^i)^\vee \right\rangle^\top \frac{\chi^i}{\gamma^i} \hat{\Sigma}^i = \hat{f},$$

which shows the desired statement. \square

The challenge is now to come up with functions γ^i, χ^i such that the resulting frames Σ, Σ' are “nice” in a certain way. More specifically, we would like to wind up with two systems of parabolic molecules which, among other things, would imply that the shearlet coefficients $\langle f, \Sigma \rangle$ characterize exactly the same function spaces as curvelet coefficients.

4 Construction of the Frames Σ^i

First we need to spend some time to construct the frames $\Sigma^i, i = 0, 1, 2$. This is more or less standard. Start with a wavelet ψ_1 which satisfies

$$\sum_{j \in \mathbb{Z}} \left| \hat{\psi}_1(2^{-j} \xi) \right|^2 = 1, \quad \text{for } \xi \in \mathbb{R} \quad (11)$$

and

$$\text{supp } \hat{\psi}_1 \subset \left[-\frac{1}{4}, -\frac{1}{32} \right] \cup \left[\frac{1}{32}, \frac{1}{4} \right]. \quad (12)$$

We also need a frequency window function ψ_2 which satisfies

$$\sum_{k=-2}^2 \left| \hat{\psi}_2(\xi - k) \right|^2 = 1 \quad \text{for } \xi \in \left[-\frac{3}{2}, \frac{3}{2} \right], \quad (13)$$

and

$$\text{supp } \hat{\psi}_2 \subset \left[-\frac{3}{2}, \frac{3}{2}\right]. \quad (14)$$

Define

$$\hat{\psi}^1(\xi) := \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right), \quad \hat{\psi}^2(\xi_1, \xi_2) := \hat{\psi}^1(\xi_2, \xi_1).$$

Using these functions we can construct shearlet systems Σ^1, Σ^2 via

$$\Sigma^1 := \left\{2^{2j/4}\hat{\psi}^1(D_{2^j}S_{j,l} \cdot -k) : j \geq 0, -2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}, k \in \mathbb{Z}^2\right\}, \quad (15)$$

and Σ^2 defined similarly.

Lemma 4.1. *The system Σ^i constitutes a Parseval frame for $L_2(\mathcal{C}^i)^\vee$ with $i = 1, 2$.*

Proof. Consider the case $i = 1$ (the case $i = 2$ is the same). We need to show that

$$\sum_{j \geq 0} \sum_{-2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}} \left| \hat{\psi}^1(D_{2^{-j}}S_{j,-l}^\top \xi) \right|^2 = 1 \quad \text{for all } \xi \in \mathcal{C}^1. \quad (16)$$

Then the assertion follows from standard arguments (using that $\text{supp } \hat{\psi}^1 \subset [-\frac{1}{2}, \frac{1}{2}]^2$). To show (16) we write

$$\begin{aligned} \sum_{j \geq 0} \sum_{-2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}} \left| \hat{\psi}^1(D_{2^{-j}}S_{j,-l}^\top \xi) \right|^2 &= \sum_{j \geq 0} \left| \hat{\psi}_1(2^{-j}\xi_1) \right|^2 \sum_{-2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}} \left| \hat{\psi}_2\left(2^{j/2}\frac{\xi_2}{\xi_1} + l\right) \right|^2 \\ &= 1 \quad \text{for } \xi \in \mathcal{C}^1, \end{aligned}$$

because of (11) and (13) and the support properties of $\hat{\psi}_1$ and $\hat{\psi}_2$. \square

The construction of a translation-invariant Parseval frame for $L_2(\mathcal{C}^0)^\vee$ is well-known and will not be detailed here.

5 Construction of the Partition Functions

Here we construct suitable partition functions and prove that the resulting systems Σ, Σ' are parabolic molecules of arbitrary order. The construction turns out to be somewhat delicate and we apologize for the heavy notation.

We start with the function γ^0 which can be quite arbitrary with $\text{supp } \gamma^0 \subset \mathcal{C}^0$. The functions γ^i , $i = 1, 2$ are constructed using two smooth, univariate, real-valued functions g_1, g_2 satisfying

$$\text{supp } g_1 \subset \left[\frac{1}{8}, \infty\right) \quad \text{and} \quad \text{supp } g_2 \subset \left[-\frac{3}{2}, \frac{3}{2}\right]. \quad (17)$$

Now define

$$\gamma^1(\xi) := g_1(\xi_1)g_2\left(\frac{\xi_2}{\xi_1}\right), \quad \gamma^2(\xi_1, \xi_2) := \gamma^1(\xi_2, \xi_1). \quad (18)$$

clearly we have $\text{supp } \gamma^i \subset \mathcal{C}^i$ for all $i = 0, 1, 2$. The construction of the functions χ^i is slightly more complicated. We start again with two nonnegative, nondecreasing functions h_1, h_2 satisfying

$$\text{supp } h_1 \subset \left[\frac{1}{4}, \infty\right), \quad h_1|_{[\frac{1}{2}, \infty)} \equiv 1, \quad \text{supp } h_2 \subset \left[-\frac{4}{3}, \frac{4}{3}\right], \quad h_2|_{[-\frac{3}{4}, \frac{3}{4}]} \equiv 1. \quad (19)$$

Define

$$\chi^1(\xi) := h_1(\xi_1)h_2\left(\frac{\xi_2}{\xi_1}\right), \quad \chi^2(\xi) := h_1(\xi_2)\left(1 - h_2\left(\frac{\xi_2}{\xi_1}\right)\right). \quad (20)$$

Further, we define

$$\chi^0 := 1 - \chi^1 - \chi^2. \quad (21)$$

Lemma 5.1. *We have*

$$\text{supp } \chi^i \subset \bar{C}^i, \quad \text{for } i = 0, 1, 2.$$

Proof. The case $i = 1, 2$ is quite clear, so we only show that $\text{supp } \chi^0 \subset \bar{C}^0$. Assume that $\xi \in \mathbb{R}^2 \setminus \bar{C}^0$. Then we can distinguish three cases as follows:

Case 1:

$$|\xi_1| \geq \frac{2}{3} \text{ and } |\xi_2| \geq \frac{1}{2} \quad \text{or} \quad |\xi_2| \geq \frac{2}{3} \text{ and } |\xi_1| \geq \frac{1}{2}.$$

In this case we have $h_1(\xi_1) = h_1(\xi_2) = 1$ and therefore

$$\chi^1(\xi) + \chi^2(\xi) = h_2\left(\frac{\xi_2}{\xi_1}\right) + 1 - h_2\left(\frac{\xi_2}{\xi_1}\right) = 1,$$

and hence $\chi^0(\xi) = 0$.

Case 2:

$$|\xi_1| \geq \frac{2}{3} \text{ and } |\xi_2| < \frac{1}{2}.$$

In this case we have $h_1(\xi_1) = 1$ and, since $\frac{|\xi_2|}{|\xi_1|} \leq \frac{3}{4}$, $h_2\left(\frac{\xi_2}{\xi_1}\right) = 1$, which implies that

$$\chi^1(\xi) + \chi^2(\xi) = 1$$

which implies $\chi^0(\xi) = 0$.

Case 3:

$$|\xi_2| \geq \frac{2}{3} \text{ and } |\xi_1| < \frac{1}{2}.$$

In this case we have $h_1(\xi_2) = 1$ and, since $\frac{|\xi_2|}{|\xi_1|} \geq \frac{4}{3}$, $h_2\left(\frac{\xi_2}{\xi_1}\right) = 0$, which implies that

$$\chi^1(\xi) + \chi^2(\xi) = 1$$

which implies $\chi^0(\xi) = 0$. This completes the proof. □

Proposition 5.2. *We have*

$$\chi^i \left((D_{2^j}^i S_{j,l}^i)^\top \xi \right) \in C^\infty \left(\text{supp } \hat{\psi}^i \right), \quad \text{for } i = 1, 2 \quad (22)$$

with norms

$$\left\| \chi^i \left((D_{2^j}^i S_{j,l}^i)^\top \cdot \right) \right\|_{C^N(\text{supp } \hat{\psi}^i)} \leq \gamma_N, \quad N \in \mathbb{N}$$

with constants γ_N independent of j . The same fact is true for γ^i and $\frac{\chi^i}{\gamma^i}$.

Proof. First we define yet another pair of frequency cones, namely

$$\tilde{C}^1 := \left\{ \xi : |\xi_1| \geq \frac{1}{32}, \frac{|\xi_2|}{|\xi_1|} \leq \frac{3}{2} \right\},$$

and

$$\tilde{C}^2 := \left\{ \xi : |\xi_2| \geq \frac{1}{32}, \frac{|\xi_1|}{|\xi_2|} \leq \frac{3}{2} \right\},$$

i = 1: Let us consider the case $i = 1$. In this case we have that

$$\text{supp } \hat{\psi}^1 \subset \tilde{C}^1.$$

We need to study

$$\chi^1 \left(2^j \xi_1, l 2^{j/2} \xi_1 + 2^{j/2} \xi_2 \right) = h_1(2^j \xi_1) h_2 \left(2^{-j/2} \left(l + \frac{\xi_2}{\xi_1} \right) \right).$$

First, note that we can disregard the factor $h_1(2^j \xi_1)$, since whenever $j \geq 2$ and $\xi \in \tilde{\mathcal{C}}^1$, we have $h_1(2^j \xi_1) \equiv 1$. The function $h_2\left(2^{-j/2}\left(l + \frac{\xi_2}{\xi_1}\right)\right)$ on the other hand is clearly in $C^\infty(\tilde{\mathcal{C}}^1)$ which is what we want.

i = 2: We go on and treat the case $i = 2$ which is a bit trickier. The object to study is the function

$$\chi^2\left(l2^{j/2}\xi_2 + 2^{j/2}\xi_1, 2^j\xi_2\right) = h_1(2^j\xi_2) \left(1 - h_2\left(\frac{2^j\xi_2}{l2^{j/2}\xi_2 + 2^{j/2}\xi_1}\right)\right),$$

with

$$\xi \in \text{supp } \hat{\psi}^2 \subset \tilde{\mathcal{C}}^2.$$

Arguing as before we can safely ignore the factor $h_1(2^j\xi_2)$ and we are left to study the factor $h_2\left(\frac{2^j\xi_2}{l2^{j/2}\xi_2 + 2^{j/2}\xi_1}\right)$.

We estimate for $\xi \in \tilde{\mathcal{C}}^2$ (which implies that $|\xi_2| \geq \frac{2}{3}|\xi_1|$)

$$\left|\frac{2^j\xi_2}{l2^{j/2}\xi_2 + 2^{j/2}\xi_1}\right| \geq \frac{2^j|\xi_2|}{l2^{j/2}|\xi_2| + 2^{j/2}|\xi_1|} \gtrsim \frac{2^j|\xi_2|}{l2^{j/2}|\xi_2| + 2^{j/2}|\xi_2|} = \frac{2^j}{(l+1)2^{j/2}} \quad (23)$$

Recall that $\text{supp } h_2 \subset \left[-\frac{4}{3}, \frac{4}{3}\right]$ which by (23) implies that $h_2\left(\frac{2^j\xi_2}{l2^{j/2}\xi_2 + 2^{j/2}\xi_1}\right)$ can only be nonzero on \mathcal{C}^1 if $l \gtrsim 2^{j/2}$. An elementary discussion shows that under these assumptions ($l \gtrsim 2^{j/2}$, $\xi \in \tilde{\mathcal{C}}^2$) the function $\frac{2^j\xi_2}{l2^{j/2}\xi_2 + 2^{j/2}\xi_1}$ possesses uniformly bounded derivatives of arbitrary order which is exactly what we need. The proof for γ^i is the same. \square

Definition 5.3. *With the appropriate window functions constructed we define the two systems*

$$\Sigma := \left\{ \sigma_{j,l,k}^i : i \in \{1, 2\}, j \in \mathbb{N}, -2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}, k \in \mathbb{Z}^2 \right\}, \quad (24)$$

where

$$\hat{\sigma}_{j,l,k}^i(\cdot) := \gamma^i(\cdot) \hat{\psi}_{j,l,k}^i(\cdot), \quad (25)$$

and

$$\Sigma' := \left\{ \tilde{\sigma}_{j,l,k}^i : i \in \{1, 2\}, j \in \mathbb{N}, -2 \cdot 2^{j/2} \leq l \leq 2 \cdot 2^{j/2}, k \in \mathbb{Z}^2 \right\}, \quad (26)$$

where

$$\hat{\tilde{\sigma}}_{j,l,k}^i(\cdot) := \frac{\chi^i(\cdot)}{\gamma^i(\cdot)} \hat{\psi}_{j,l,k}^i(\cdot). \quad (27)$$

We can now formulate and prove our main result.

Theorem 5.4. *The systems Σ , Σ' constitute two systems of parabolic molecules of arbitrary order. Furthermore, they constitute two mutually dual frames for $L_2(\mathbb{R}^2)$.*

Proof. We need to show (3) for the functions

$$\hat{\sigma}_{j,l,k}^i(\cdot) := \chi^i\left(\left(D_{2^j}^i S_{j,l}^i\right)^\top \cdot\right) \hat{\psi}^i(\cdot)$$

and (R, M, N_1, N_2) arbitrary. But this follows directly from Proposition 5.2 together with the support properties of $\hat{\psi}^i$. The fact that Σ , Σ' constitute mutually dual frames follows from Proposition 3.2 \square

6 Applications

In the present section we would like to briefly elaborate on some implications of our construction, see also [14] for a more detailed discussion of related results. The main importance lies in the fact that with the results of [14], for the first time a clean correspondence principle between curvelet and shearlet results is available. More precisely, due to the fact that the frames constructed in the present paper are parabolic molecules of arbitrary order, by the results in [14], it follows that the Gramian matrices

$$\mathcal{G} := \langle \Sigma, \Gamma \rangle, \quad \mathcal{G}' := \langle \Sigma', \Gamma \rangle,$$

Γ being the curvelet frame from [6], are almost diagonal in a very strong sense. In the following we briefly illustrate to the reader how curvelet properties can be transformed into properties of the frames Σ using the almost-diagonality of the matrices \mathcal{G} , \mathcal{G}' . Since this chapter is only meant to illustrate, we will be a bit sketchy in some places, and leave a more detailed discussion of similar results (in a more general setting) to [14].

6.1 Shearlet Function Spaces

Given our shearlet frames Σ , Σ' it is natural to study the associated approximation spaces

$$S_{p,q}^\alpha := \left\{ f \in \mathcal{S}' : \|f\|_{S_{p,q}^\alpha} < \infty \right\} \quad (28)$$

and

$$\|f\|_{S_{p,q}^\alpha} := \left(\sum_{i,j,l} 2^{\alpha j q} \left(\sum_k |\langle f, \sigma_{i,j,l,k} \rangle|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \quad (29)$$

In particular it is important to know whether these spaces are genuinely new or if they coincide with other, already known spaces. As a matter of fact, it turns out that they coincide with the corresponding curvelet approximation spaces as well as with the spaces $S_{p,q}^\alpha$ as defined in [2]. This follows directly from the almost orthogonality property between any two systems of parabolic molecules as shown in [14]. We do not elaborate on the details here but rather refer to [14] for more information and state the following theorem:

Theorem 6.1. *We have the norm equivalence*

$$\|\cdot\|_{S_{p,q}^\alpha} \sim \|\cdot\|_{R_{p,q}^\alpha}, \quad p, q, \alpha > 0,$$

where $R_{p,q}^\alpha$ denotes the curvelet space introduced in [2, Section 7] with the norm defined as in (29) but the shearlets $\sigma_{i,j,k,l}$ replaced with second generation curvelets as defined in [6].

Proof. Using the fact that Σ , Σ' and the curvelet frame from [6] are all parabolic molecules of arbitrary order, this follows directly from the results in [14]. \square

We hence for the first time have a rigorous statement of the meta theorem frequently heard from experts, stating that curvelet- and shearlet properties are equivalent.

6.2 Cartoon Approximation

A main motivation for the introduction of curvelet or shearlet frame has been the efficient approximation of functions with anisotropic features such as those arising in image data. A popular model for such functions is given by class \mathcal{F} the so-called cartoon images which are (roughly) defined as piecewise C^2 functions with C^2 singularity curves, defined on the unit square in \mathbb{R}^2 [10, 5]. It is well-known that keeping N coefficients in any frame expansion can only give an L_2 -approximation error of at best $\sim N^{-1}$, uniformly for all $f \in \mathcal{F}$ [10]. An important property of curvelets is the fact that they almost attain this optimal rate, namely for any $\varepsilon > 0$ we have

$$\|f - f_N^\Gamma\| \lesssim N^{-1+\varepsilon},$$

where f_N^Γ arises from only keeping the largest N coefficients in the frame expansion $f = \langle f, \Gamma \rangle^\top \Gamma$. Again, the almost diagonality of the Gramian matrix \mathcal{G} immediately implies that an analogous statement holds with Γ replaced by Σ , namely

Theorem 6.2. *For any $\varepsilon > 0$ we have*

$$\|f - f_N^\Sigma\| \lesssim N^{-1+\varepsilon},$$

where f_N^Σ arises from the frame expansion $f = \langle f, \Sigma \rangle^\top \Sigma'$ by keeping only the N largest coefficients in modulus. The implicit constant is uniform in $f \in \mathcal{F}$.

Remark 6.3. *Actually, the factor $N^{-1+\varepsilon}$ can be slightly improved to $N^{-1} \log(N)^{3/2}$ by repeating the curvelet-arguments in [5] which only use the time-frequency localization properties of the curvelet frame Γ .*

6.3 Wave Propagation

Let $E(t)$ be the propagation operator of a time-dependent wave equation of the form

$$\frac{\partial u}{\partial t}(x, t) + \langle A(x), \nabla u(x, t) \rangle + B(x)u(x, t) = 0,$$

where $u : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $A : \mathbb{R} \rightarrow \mathbb{R}^2$ and $B : \mathbb{R} \rightarrow \mathbb{R}$ with all occurring functions arbitrarily smooth. In [3] it is shown that the Gramian matrix

$$\mathcal{E}^\Gamma := \langle E(t)\Gamma, \Gamma \rangle$$

is sparse in the sense that the discrete operator norms

$$\|\mathcal{E}^\Gamma\|_{l_p \rightarrow l_p}$$

are all bounded for $p > 0$. This opens up a path for studying efficient algorithms for wave propagation based on curvelets. We only scratch the surface of this topic, the interested reader is referred to [5] and the references therein. Due to the fact that the matrices $\mathcal{G}, \mathcal{G}'$ are almost diagonal and therefore also bounded on all l_p spaces, we immediately arrive at the following result.

Theorem 6.4. *Define*

$$\mathcal{E}^\Sigma := \langle E(t)\Sigma, \Sigma' \rangle.$$

then the matrix operator induced by \mathcal{E}^Σ is bounded on l_p for any $p > 0$.

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References

- [1] I. Babuska and J. M. Melenk. The partition of unity method. *International Journal of Numerical Methods in Engineering*, 40:727–758, 1996.
- [2] L. Borup and M. Nielsen. Frame decompositions of decomposition spaces. *Journal of Fourier Analysis and Applications*, 13:39–70, 2007.
- [3] E. Candes and L. Demanet. The curvelet representation of wave propagators is optimally sparse. *Communications in Pure and Applied Mathematics*, 58:1472–1528, 2004.
- [4] E. Candes and D. Donoho. Curvelets - a surprisingly effective nonadaptive representation for objects with edges. In *Curves and Surfaces*. Vanderbilt University Press, 1999.
- [5] E. Candes and D. Donoho. New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities. *Communications on pure and applied mathematics*, 57(2):219–266, 2004.
- [6] E. Candes and D. Donoho. Continuous curvelet transform: II. discretization and frames. *Applied and Computational Harmonic Analysis*, 19(2):198–222, 2005.
- [7] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2003.
- [8] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods for elliptic operator equations – convergence rates. *Mathematics of Computation*, 70:27–75, 1998.
- [9] M. Do and M. Vetterli. The contourlet transform: An efficient directional multiresolution image representation. *IEEE Transactions on Image Processing*, 14:2091–2106, 2005.
- [10] D. Donoho. Sparse components of images and optimal atomic decompositions. *Constructive Approximation*, 17:353–382, 1999.

- [11] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *Journal of Functional Analysis*, 93(1):34–170, 1990.
- [12] T. Gantumur and R. Stevenson. Computation of differential operators in wavelet coordinates. *Mathematics of Computation*, 75:697–709, 2006.
- [13] P. Grohs. Continuous shearlet tight frames. *Journal of Fourier Analysis and Applications*, 17:506–518, 2011.
- [14] P. Grohs and G. Kutyniok. Parabolic molecules. 2011. manuscript.
- [15] K. Guo and D. Labate. Optimally sparse multidimensional representation using shearlets. *SIAM journal on mathematical analysis*, 39(1):298–318, 2008.
- [16] K. Guo, D. Labate, W.-Q. Lim, G. Weiss, and E. Wilson. Wavelets with composite dilations and their MRA properties. *Applied and Computational Harmonic Analysis*, 20(2):202 – 236, 2006.
- [17] P. Kittipoom, G. Kutyniok, and W.-Q. Lim. Construction of compactly supported shearlet frames. *Constructive Approximation*, 2011. to appear.
- [18] G. Kutyniok, J. Lemvig, and W.-Q. Lim. Compactly supported shearlets. In *Approximation Theory XIII (San Antonio, TX, 2010)*. Springer, 2011.
- [19] G. Kutyniok and W.-Q. Lim. Compactly supported shearlets are optimally sparse. *Journal of Approximation Theory*, 2011. to appear.
- [20] D. Labate, W.-Q. Lim, G. Kutyniok, and G. Weiss. Sparse multidimensional representation using shearlets. In *Proceedings of the SPIE*, pages 254–262, 2005.
- [21] E. LePennec and S. Mallat. Sparse geometric image representation with bandelets. *IEEE Transactions on Image Processing*, 14:423–438, 2005.
- [22] W. Rudin. *Functional Analysis*. Mc Graw-Hill, 1991.
- [23] R. Stevenson. Adaptive solution of operator equations using wavelet frames. *SIAM Journal on Numerical Analysis*, 41:1074–1100, 2002.

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