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hp-DGFEM for Kolmogorov-Fokker-Planck equations of multivariate Lévy processes*

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hp-DGFEM for Kolmogorov-Fokker-Planck Equations of Multivariate Lévy Processes*

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Abstract We analyze the discretization of non-local degenerate integrodifferential equations arising as so-called forward equations for jump-diffusion processes, in particular in option pricing problems when dealing with Lévy driven stochastic volatility models. Well-posedness of the arising equations is addressed. We develop and analyze stable discretization schemes. The discontinuous Galerkin (DG) Finite Element Method is analyzed. In the DG-FEM, a new regularization of hypersingular integrals in the Dirichlet Form of the pure jump part of infinite variation processes is proposed. Robustness of the stabilized discretization with respect to various degeneracies in the characteristic triple of the stochastic process is proved. We provide in particular an *hp*-error analysis of the DG-FEM and numerical experiments.
Keywords: Discontinuous Galerkin Methods, Feller-Lévy processes, Pure jump processes, Lévy Copulas, Option pricing, Dirichlet Forms, Error analysis

1 Introduction

We consider the discretization of non-local degenerate integrodifferential equations. Such equations arise, for example, in financial modelling with jump processes, cf. [11], when dealing with advanced stochastic volatility models, where the volatility is modeled using a subordinator, cf. [17]. Similar problems arise in the context of pricing derivatives on electricity or other commodities as in this case Ornstein-Uhlenbeck type processes are an appropriate model class and lead to drift dominated equations, cf. [3, 6]. This paper aims at the development and analysis of stable discretization schemes for such equations. We consider the Discontinuous Galerkin method with and without small jump regularization. We derive localization estimates for a large class of processes and obtain sharp estimates for the small jump truncation. Our error analysis is performed in multiple space dimensions. The reason for considering Discontinuous Galerkin discretizations lies in the structure of the equations: Continuous Galerkin Finite Element Methods (CGFEM for short) which are based on continuous, piecewise polynomial functions on simplicial partitions, cf. [27, 32], are not applicable in general as they are well-known to become unstable for operators with dominating drift. The *Discontinuous Galerkin* (DG for short) Finite Element discretizations allow to accurately discretize drift-dominated operators via a judicious choice of the numerical flux to account for dominating drift. However, for discontinuous, piecewise polynomials the Dirichlet Form of the jump part of the process X is, in general, not well-defined, and some form of *jump regularization* is required then. We show that by the so-called *small jump regularization of the stochastic process* X in [10] and the references therein, a Dirichlet form is obtained which remains finite even for Discontinuous Finite Element discretizations, albeit at the expense of introducing an artificial diffusion which depends on the second moments of the jumps of X of size at most ε. The resulting stable DG discretizations of hypersingular integral operators are of independent interest also in other applications.
This paper is organized as follows. We present the necessary preliminaries in Section 2. In Section 3 we discuss the small jump regularization and localization errors. Well-posedness of the arising equations is addressed in

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Section 4. The DG-method is discussed in the subsequent section. We conclude with some remarks on the implementational aspects of the methods and numerical examples in one space dimension.

2 Preliminaries

In this section, the necessary preliminaries are presented. The class of stochastic processes considered in this paper is discussed and the domains of the corresponding generators are defined.

2.1 Lévy processes

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a filtered probability space satisfying the usual assumptions, cf. [24], and $X = (X_t)_{t\geq 0}$ an adapted time-homogeneous Markov process with state space $\mathbb{R}^d$, $d \geq 1$, characterized by the triplet $(\boldsymbol{b}(\boldsymbol{x}), 0, \nu(d\boldsymbol{z}))$ (i.e., a pure jump process):

$$X(t) = X_0 + \int_0^t \boldsymbol{b}(X(s))\, ds + \int_0^t \int_{\mathbb{R}^d} \boldsymbol{z}\widetilde{N}(ds, d\boldsymbol{z}). \tag{2.1}$$

Existence and uniqueness of a solution for the Stochastic Differential Equation (SDE) (2.1) follow from [21, Theorem 9.1] for globally Lipschitz $\boldsymbol{b}(x)$, where $\widetilde{N}(dt, d\boldsymbol{z})$ denotes a compensated Poisson random measure with intensity measure $\nu(d\boldsymbol{z})dt$. The Lévy measure satisfies

$$\int_{\mathbb{R}^d} (|\boldsymbol{z}|^2 \wedge 1)\nu(d\mathbf{z}) < \infty.$$

Remark 2.1. *This setup includes Lévy processes and Lévy driven Ornstein-Uhlenbeck processes. We focus on the harder pure jump case, but the consideration of jump-diffusion processes is also possible in this context.*

Assumption 2.2. *We make the following assumptions on the Lévy measure ν.*

(i) The Lévy measure ν has a density k, i.e., $\nu(d\boldsymbol{z}) = k(\boldsymbol{z})d\boldsymbol{z}$.

(ii) There exist constants $\beta_i^- > 0$, $\beta_i^+ > 1$, $i = 1, \ldots, d$, such that

$$k_i(z) \lesssim \begin{cases} e^{-\beta_i^-|z|}, & \text{if } z < -1 \\ e^{-\beta_i^+|z|}, & \text{if } z > 1 \end{cases},$$

where $k_i(z)$ is the i-th marginal of $k(\boldsymbol{z})$.

(iii) Furthermore we assume that the Lévy density $k(\boldsymbol{z})$ behaves at $\boldsymbol{z} = 0$ similar to an α-stable density $k^0(\boldsymbol{z})$, i.e., there exist constants $C_1, C_2 > 0$ s.t.

$$C_1 k^0(\boldsymbol{z}) \leq k(\boldsymbol{z}) \leq C_2 k^0(\boldsymbol{z}), \quad 0 < |\boldsymbol{z}| < 1.$$

Remark 2.3. *These assumptions can be expressed in terms of marginals of the process and a Lévy copula. We refer to [36, Section 2.3], [32, Section 4.3] and [14, 31] for details and the definition of a Lévy copula. Note that subordinators are excluded by condition (iii).*

We will consider the following boundary value problem: given an appropriate right hand side $f(t, \boldsymbol{x})$, find a suffiently smooth function $u(t, \boldsymbol{x})$ such that

$$\partial_t u(t, \boldsymbol{x}) + \mathcal{L}u(t, \boldsymbol{x}) = f(t, \boldsymbol{x}) \ \text{ in } (0, T) \times \mathbb{R}^d, \quad u(0, \boldsymbol{x}) = P(\boldsymbol{x}), \tag{2.2}$$

where, for sufficiently smooth $u(t, \boldsymbol{x})$,

$$\mathcal{L}u(t, \boldsymbol{x}) := \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(t, \boldsymbol{x}) + c(\boldsymbol{x})u(t, \boldsymbol{x}) - A_J[u](t, \boldsymbol{x}), \tag{2.3}$$

$$A_J[u](t, \boldsymbol{x}) := \int_{\mathbb{R}^d} (u(t, \boldsymbol{x} + \boldsymbol{z}) - u(t, \boldsymbol{x}))\, \nu(d\boldsymbol{z}). \tag{2.4}$$

In the pricing of derivative contracts, $\boldsymbol{x}$ is the vector of log-prices or real-prices and P in (2.2) denotes the payoff function. The operator $\mathcal{L} - cI$ is the infinitesimal generator of the process X. Note that formulation (2.4) is not feasible for general Lévy jump measures in (2.1), but only for finite variation processes. We will approximate the general Kolmogorov equation by this special case via a small jump regularization as described in Section 3. Therefore, we will restrict ourselves to the discretization of Kolmogorov equations corresponding to finite variation processes. Note that the operator A_J in (2.4) is a pseudo differential operator with constant symbol, we refer to the monograph [22] for details on this topic and [25] for analytical properties of the operator A_J.

2.2 Domains of Generators

For the variational formulation, we need to identify the domains of generators and of their Dirichlet forms. As shown in [14, 22, 31], those domains are certain Sobolev spaces in the case of pure jump processes. Therefore we start with the definition of fractional order isotropic spaces. We define for a positive non-integer $\rho \in (0,2)$ and $u \in \mathcal{S}^*(\mathbb{R}^d)$, where $\mathcal{S}^*(\mathbb{R}^d)$ is the space of tempered distributions, the isotropic Sobolev space $H^{\rho/2}(\mathbb{R}^d)$, equipped with the norm $\|\cdot\|_{H^{\rho/2}(\mathbb{R}^d)}$ given by

$$\|u\|^2_{H^{\rho/2}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)^{\rho/2} \left|\hat{u}(\boldsymbol{\xi})\right|^2 \, d\boldsymbol{\xi}, \tag{2.5}$$

denoting by $\hat{u}$ the Fourier transform of u. Similarly for any multi-index $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_d)$, $\rho_i \in (0,2)$, $i = 1, \ldots, d$, anisotropic Sobolev spaces $H^{\boldsymbol{\rho}/2}(\mathbb{R}^d)$ with norm $\|\cdot\|_{H^{\boldsymbol{\rho}/2}(\mathbb{R}^d)}$ given by

$$\|u\|^2_{H^{\boldsymbol{\rho}/\mathbf{2}}(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (1 + \xi_j^2)^{\rho_j/2} \left|\hat{u}(\boldsymbol{\xi})\right|^2 \, d\boldsymbol{\xi}, \tag{2.6}$$

can be defined. We define the following spaces on an open, bounded Lipschitz domain G with boundary Γ

$$\widetilde{H}^{\boldsymbol{\rho}/2}(G) = \left\{ u \in H^{\boldsymbol{\rho}/2}(\mathbb{R}^d),\ u|_{\mathbb{R}^d \setminus \overline{G}} = 0 \right\},$$

where a norm on $\widetilde{H}^{\boldsymbol{\rho}/2}(G)$ is given by $\|u\|_{\widetilde{H}^{\boldsymbol{\rho}/2}(G)} = \|\widetilde{u}\|_{H^{\boldsymbol{\rho}/2}(\mathbb{R}^d)}$ and we denote the zero extension of u outside of G by $\widetilde{u}$. An intrinsic norm on $\widetilde{H}^{\boldsymbol{\rho}/2}(G)$ is given by

$$||u||^2_{\widetilde{H}^{\boldsymbol{\rho}/2}(G)} = ||u||^2_{L^2(G)} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{j=1}^{d} \frac{|\widetilde{u}(\boldsymbol{x}) - \widetilde{u}(\boldsymbol{y})|^2}{|x_j - y_j|^{1+\rho_j}} d\boldsymbol{x} d\boldsymbol{y}. \tag{2.7}$$

Remark 2.4. *Note that one can use the integral over G instead of $\mathbb{R}^d$ in (2.7) for $\rho_j \in (0,2)$, $\rho_j \neq 1$, cf. [27, Section 4.3]. The case $\rho_j = 1$ is different, in fact $\widetilde{H}^{1/2}(G) = H^{\frac{1}{2}}_{00}(G)$ (see [26, Theorem 11.7], [27]). The case $\rho_i = 1$, $i = 1, \ldots, d$, is of special interest in financial modelling, as it arises when using generalized hyperbolic processes, cf. e.g. [13].*

Remark 2.5. *The relation between α in Assumption 2.2 and the Blumenthal-Getoor-index β, cf. [34, Theorem 47.23] or [4], of a Lévy process was studied by [16] for $d = 1$ in (2.2). Korollar I.33 in [16] implies that $\alpha = \beta$ for $0 < \alpha < 2$.*

In the following we briefly outline the standard variational setting for parabolic equations which was applied in, e.g., [31] and [36]. Let $V \subset H$ be two Hilbert spaces with continuous and dense embedding. We identify H with its dual H^* and obtain the Gelfand triplet

$$V \subset H \equiv H^* \subset V^*.$$

The space V is in this setting the domain of a certain bilinear form $\mathcal{A}(\cdot,\cdot)$ associated to an operator A. Let $(A, D(A))$ be a densely defined operator on $H = L^2(\mathbb{R}^d)$ which is negative definite, cf. [22, Definition 4.6.10] and satisfies

$$|(-Au, v)| \leq C \left(-Au, u\right)^{1/2} \left(-Av, v\right)^{1/2},$$

where $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R}^d)$ scalar product and $u,v \in D(A)$. Then we may introduce on $D(A)$ the bilinear form

$$\mathcal{A}(u,v) := (-Au, v).$$

The bilinear form $\widetilde{\mathcal{A}}(\cdot,\cdot)$ given as

$$\widetilde{\mathcal{A}}(u,v) := \mathcal{A}^{\mathrm{sym}}(u,v) + (u,v) = \frac{1}{2}\left(\mathcal{A}(u,v) + \mathcal{A}(v,u)\right) + (u,v)$$

defines a scalar product and we may consider the completion of $D(A)$ with respect to $\widetilde{\mathcal{A}}(\cdot,\cdot)$, which is denoted by $D(\mathcal{A})$. Well-posedness of the following parabolic problem can then be shown: Find $u \in L^2((0,T),V) \cap H^1((0,T),V^*)$ such that

$$\begin{aligned}&(\partial_t u, v)_{V^*,V} + \mathcal{A}(u,v) = (f,v)_{V^*,V}, \forall v \in V, \text{ a.e. in } (0,T),\\ &u(0) = u_0,\end{aligned}$$

with $u_0 \in H$, $f \in L^2((0,T),V^*)$ and $T > 0$. The space $V = D(\mathcal{A})$ is an anisotropic fractional order Sobolev space if the operator A is the infinitesimal generator of a Lévy process, we refer to [31] for further details. For an infinitesimal generator A of a Markov process X the bilinear form $\mathcal{A}(\cdot,\cdot)$ is closely linked to its Dirichlet form, cf. [22, Definition 4.7.21]

Remark 2.6. *Note that pure transport operators do not fit into this framework and have to be analyzed using different techniques.*

Throughout the work we use the generic positive constant C taking different values in different places, it is independent of the mesh width h, the polynomial degree p and the jump truncation threshold ϵ, cf. Section 3. But it may depend on various parameters, such as, s the smoothness of the solution, C_1 and C_2 the shape regularity and quasi-uniformity constants of the triangulation, the drift dominance parameter γ, the penalty parameter α and the dimension of the problem d. Besides, we use the generic constant $C(\epsilon)$ which depends on the same parameters as C and additionally explicitly on ϵ.

3 Small jump regularization and localization

In this section probabilistic results for the small jump regularization and the localization will be presented. These are not based on the parabolic integro-differential equation (PIDE) representation of the option price, but will be useful for the analysis of the PIDE, since the probabilistic estimates can be used to obtain error bounds for the numerical solution of the equation. This will be done at two steps of the discretization. An infinite activity Markov process will be approximated by a finite activity process adding an appropriately scaled diffusion, besides the PIDE formulated on an unbounded domain will be localized to a bounded domain. The rigorous justification of both steps using purely numerical analysis methods without any probabilistic tools is much more tedious and technical.

3.1 Small jump approximation for Lévy Processes

We consider a Markov process X as defined in (2.1), with a jump measure that satisfies Assumption 2.2. The easiest approach to the approximation of the jump measure consists in a truncation of $\nu(d\boldsymbol{z})$ in a small ball around the origin, i.e., we consider the jump measure $\nu^\epsilon(d\boldsymbol{z}) := \mathbb{1}_{|\boldsymbol{z}|>\epsilon}\nu(d\boldsymbol{z})$, $\nu_\epsilon := \nu - \nu^\epsilon$, with $\epsilon > 0$. We denote the process with characteristic triplet $(\boldsymbol{b}, 0, \nu^\epsilon)$ by Y^ϵ. We can also approximate the small jumps by an appropriately scaled Brownian motion, i.e., we consider the process Z^ϵ with characteristic triplet $(\boldsymbol{b}, Q_\epsilon, \nu^\epsilon)$, where $Q_\epsilon = \int_{\mathbb{R}^d} \boldsymbol{z}\boldsymbol{z}^\top \nu_\epsilon(dz)$. Due to Assumption 2.2.$(iii)$, Q_ϵ is a symmetric positive definite $\mathbb{R}^{d\times d}$-matrix.

The following approximation result for Lévy processes is well known, cf. [10, Theorem 3.1].

Theorem 3.1. *Let X be a Lévy process in $\mathbb{R}^d$ with characteristic triplet $(\boldsymbol{b}, 0, \nu)$ and let the decomposition $\nu = \nu^\epsilon + \nu_\epsilon$ be given. Assume that Q_ϵ is non-singular for every $\epsilon > 0$ and that for every $\delta > 0$ there holds*

$$\int_{(Q_\epsilon^{-1}\boldsymbol{z},\boldsymbol{z})>\delta} (Q_\epsilon^{-1}\boldsymbol{z}, \boldsymbol{z})\nu_\epsilon(d\boldsymbol{z}) \to 0, \quad \textit{as } \epsilon \to 0.$$

Assume further that for some family of non-singular matrices $\{\Sigma_\epsilon\}_{\epsilon\in(0,1]}$ *there holds*

$$\Sigma_\epsilon^{-1} Q_\epsilon \Sigma_\epsilon^{-\top} \to I, \quad \text{as } \epsilon \to 0,$$

where I *denotes the identity matrix in* $\mathbb{R}^d$. *Then for all* $\epsilon \in (0,1]$ *there exists an* $\mathbb{R}^d$*-valued càdlàg process* R^ϵ *and a process* $Z^\epsilon = (Z^{\epsilon,1}, \ldots, Z^{\epsilon,d})$ *with characteristic triplet* $(\boldsymbol{b}, Q_\epsilon, \nu^\epsilon)$ *such that*

$$X_t \overset{(d)}{=} Z_t^\epsilon + R_t^\epsilon,$$

in the sense of equality of finite dimensional distributions.
Furthermore, we have for all $T > 0$, $\sup_{t\in[0,T]} \left|\Sigma_\epsilon^{-1} R_t^\epsilon\right| \overset{(\mathbb{P})}{\to} 0$, *as* $\epsilon \to 0$.

Remark 3.2. *Note that the Assumption on the matrices* Σ_ϵ *can be expressed in terms of the jump measure* ν, *cf. [10, Theorem 2.4].*

Throughout this paper, X in (2.1) will generally not be a Lévy process due to the non-constant drift, therefore a more general result is needed. A weaker convergence result in mean square sense also holds for more general Markov processes, cf. [2, Proposition 3.3].

Theorem 3.3. *Let* X *be an* $\mathbb{R}^d$*-valued Markov process as given in* (2.1)*, then there holds*

$$\mathbb{E}\left[\int_0^T \|X(t) - Z^\epsilon(t)\|^2 \, dt\right] \le C \sum_{i=1}^d \int_{|z_i|<\epsilon} z_i^2 \nu_i(dz_i),$$

for sufficiently small $\epsilon > 0$ *and a constant* C *independent of* ϵ.

For applications to option pricing we are mainly interested in weak convergence estimates.

3.2 Estimates for Lévy processes

Let $X = (X^1, \ldots, X^d)$ be a Lévy process with characteristic triplet $(\boldsymbol{b}, 0, \nu)$, such that ν satisfies Assumption 2.2, where $\boldsymbol{b}$ is chosen such that $e^{X^1}, \ldots, e^{X^d}$ are martingales. Now we consider the process $\widetilde{Z}^\epsilon = (\widetilde{Z}^{\epsilon,1}, \ldots, \widetilde{Z}^{\epsilon,d})$ with characteristic triplet $(\boldsymbol{b}^\epsilon, Q_\epsilon, \nu^\epsilon)$, where $\nu^\epsilon(d\boldsymbol{z})$ and Q_ϵ are chosen as above and $\boldsymbol{b}^\epsilon$ is chosen such that $e^{\widetilde{Z}^{\epsilon,1}}, \ldots, e^{\widetilde{Z}^{\epsilon,d}}$ are martingales. Convergence of X to $\widetilde{Z}^\epsilon$ in an appropriate sense follows from Theorem 3.1.

Lemma 3.4. *Let the payoff function* P *to be globally Lipschitz, then we obtain the following estimate using* $U^\epsilon = X_t + (\boldsymbol{b}^\epsilon - \boldsymbol{b})t$, *where* X *is a Lévy process with characteristic triplet* $(\boldsymbol{b}, 0, \nu)$.

$$|\mathbb{E}[P(\boldsymbol{x} + X_T)] - \mathbb{E}[P(\boldsymbol{x} + U_T^\epsilon)]| \le C \sum_{j=1}^d \int_{-\epsilon}^{\epsilon} |z_j|^3 \nu_j(dz_j), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{3.1}$$

Proof. This estimate can be obtained by Taylor expansion of e^x around 0 and is given in [36, Proposition 8.2.1]. For the one dimensional case we refer to [12, Theorem 5.1]. □

Lemma 3.5. *If* $P \in C^4(\mathbb{R}^d)$ *and* $\overline{\rho} := \max_{i=1,\ldots,d} \rho_i < 1$, *there holds:*

$$\left|\mathbb{E}[P(\boldsymbol{x} + U_T^\epsilon)] - \mathbb{E}[P(\boldsymbol{x} + \widetilde{Z}_T^\epsilon)]\right| \le C \sum_{j=1}^d \int_{-\epsilon}^{\epsilon} |z_j|^3 \nu_j(dz_j), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{3.2}$$

If we only assume $\overline{\rho} < 2$, *then the following estimate holds*

$$\left|\mathbb{E}[P(\boldsymbol{x} + U_T^\epsilon)] - \mathbb{E}[P(\boldsymbol{x} + \widetilde{Z}_T^\epsilon)]\right| \le C \sum_{j=1}^d \int_{-\epsilon}^{\epsilon} |z_j|^2 \nu_j(dz_j), \quad \forall \boldsymbol{x} \in \mathbb{R}^d. \tag{3.3}$$

Proof. This follows using Taylor expansion of P and Jensen's inequality. Note that the existence of first moments of the jump measure (which is a consequence of $\overline{\rho} < 1$) is essentially used in the first part of the proof, cf. [36, Proposition 8.2.3.]. $\square$

Remark 3.6. *Intermediate cases, i.e., $\overline{\rho} < c$ for $c \in (1, 2)$, lead to analogous estimates.*

Finally we obtain the following result from (3.1) - (3.3).

Theorem 3.7. *Let X and $\widetilde{Z}^\epsilon$ be as above and $P \in C^4(\mathbb{R}^d)$, further let $u(t, \boldsymbol{x}) = \mathbb{E}[P(\boldsymbol{x} + X_T)]$ and $u^\epsilon(t, \boldsymbol{x}) = \mathbb{E}[P(\boldsymbol{x} + \widetilde{Z}_T^\epsilon)]$, then the following estimate can be obtained*

$$|u(t, \boldsymbol{x}) - u^\epsilon(t, \boldsymbol{x})| \leq C \begin{cases} \epsilon^{3-\overline{\rho}}, & \forall \boldsymbol{x} \in \mathbb{R}^d, \quad \overline{\rho} \in (0, 1) \\ \epsilon^{2-\overline{\rho}}, & \forall \boldsymbol{x} \in \mathbb{R}^d, \quad \overline{\rho} \in (0, 2) \end{cases}.$$

Remark 3.8. *Note that Theorem 3.7 yields at least quadratic convergence in ϵ for the L^∞-error for finite variation processes and payoffs $P \in C^4(\mathbb{R}^d)$. Using merely a small jump trunctation to approximate the process X without an artificial diffusion would lead to an approximation rate of $\epsilon^{2-\overline{\rho}}$, for all $\overline{\rho} \in (0, 2)$, cf. [36, Corollary 8.2.5].*

Remark 3.9. *The constant C in Theorem 3.7 depends on the tail behavior and the moments of the jump measure as well as the time to maturity T.*

3.3 Estimates for general Markov processes

The described procedure is not directly applicable for more general Markov processes as a solution of the SDE (2.1) is generally not available in closed form. We can use Theorem 3.3 to obtain a weaker error bound.

Lemma 3.10. *Let P be globally Lipschitz and let X and Z^ϵ be as in Theorem 3.3, then the following estimate holds:*

$$|\mathbb{E}[P(X_T + \boldsymbol{x})] - \mathbb{E}[P(Z_T^\epsilon + \boldsymbol{x})]| \leq C \sum_{i=1}^d \int_{|z_i| < \epsilon} z_i^2 \nu_i(d\boldsymbol{z}).$$

Proof. Using the Lipschitz continuity of P and Jensen's inequality, we obtain

$$|\mathbb{E}[P(X_T + \boldsymbol{x})] - \mathbb{E}[P(Z_T^\epsilon + \boldsymbol{x})]| \leq K \sum_{i=1}^d \mathbb{E}\left[\left|Z_T^{\epsilon,i} - X_T^i\right|\right].$$

The result follows from the Cauchy-Schwarz inequality and Theorem 3.3. $\square$

Theorem 3.11. *Let X and Z^ϵ be as above and let P be globally Lipschitz, let further $u(t, \boldsymbol{x}) = \mathbb{E}[P(\boldsymbol{x} + X_T)]$ and $u^\epsilon(t, \boldsymbol{x}) = \mathbb{E}[P(\boldsymbol{x} + Z_T^\epsilon)]$ be as above. Then, as $\epsilon \downarrow 0$ the following estimate can be obtained*

$$|u(t, \boldsymbol{x}) - u^\epsilon(t, \boldsymbol{x})| \leq C\epsilon^{2-\overline{\rho}}, \quad \forall \boldsymbol{x} \in \mathbb{R}^d, \quad \textit{for all } \overline{\rho} \in (0, 2).$$

Proof. This is a direct consequence of Lemma 3.10. $\square$

Remark 3.12. *Note that Theorem 3.11 yields, in contrast to Theorem 3.7, at least linear convergence in ϵ for the L^∞-error for finite variation processes with globally Lipschitz payoffs. An analogous estimate can be obtained if merely a small jump truncation, without regularization, is employed.*

3.4 Localization

In the following we estimate the error due to localization of the Kolmogorov equation. This is necessary as the Galerkin discretization will be performed on the localized problem. It turns out that the localization error decays exponentially with increasing domain under certain assumptions. We assume the payoff P to satisfy the following polynomial growth condition:

$$|P(\boldsymbol{s})| \lesssim \left(\sum_{i=1}^{d} |s_i| + 1\right)^q, \quad \text{for all } \boldsymbol{s} \in \mathbb{R}^d. \tag{3.4}$$

The variable $\boldsymbol{s}$ denotes the state variable in a real price model and the exponential of the state variable in a log-price model. The condition is satisfied for all standard multi-asset options like basket, maximum or best-of options. We consider log-price models with $\log(s_i) = x_i$, $i = 1, \dots, d$, in the following; the estimates for the real price models follow easily.
The unbounded domain $\mathbb{R}^d$ of $\boldsymbol{x}$ will be truncated to a bounded domain $G_R = [-R, R]^d$. In terms of financial modelling, this corresponds to the approximation of an option by the corresponding double barrier option. In the following we will consider two cases. First we will derive a localization error estimate for tempered Lévy market models and then extend this to tempered affine market models.

Theorem 3.13. *Let the payoff function $P : \mathbb{R}^d \to \mathbb{R}$ satisfy* (3.4)*. Further let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν satisfying Assumption 2.2 with $\beta_i^+, \beta_i^- > q$, where $q > 0$ is as in* (3.4)*. Then*

$$|u(t, \boldsymbol{x}) - u_R(t, \boldsymbol{x})| \lesssim e^{-\alpha R + \beta \|\boldsymbol{x}\|_\infty},$$

for $0 < \alpha < \min_i \min(\beta_i^+, \beta_i^-) - q$ and $\beta = \alpha + q$,

$$u_R(t, \boldsymbol{x}) = \mathbb{E}[P(e^{X_T}) \mathbb{1}_{T < \tau_{G_R}} | X_t = \boldsymbol{x}],$$

and $\tau_{G_R} = \inf\{t \geq 0 | X_t \in G_R^c\}$, where G_R^c is the complement set of G_R.

Proof. See [31, Theorem 4.14]. □

There holds a corresponding result for affine models.

Theorem 3.14. *Let X be a Markov process as given in* (2.1) *with a finite variation jump measure, we set $\boldsymbol{b}(\boldsymbol{x}) = (-b_1 x_1, \dots, -b_d x_d)$, for some constants $b_1, \dots, b_d \in \mathbb{R}^+$. Further let ν and P be as in Theorem 3.13, then the following estimate holds:*

$$|u(t, \boldsymbol{x}) - u_R(t, \boldsymbol{x})| \lesssim e^{-\alpha R + \beta \|\boldsymbol{x}\|_\infty}, \tag{3.5}$$

where α, β are given in Theorem 3.13.

Proof. The idea of the proof is to reduce this problem to the setting discussed in Theorem 3.13. We make use of the explicit solvability of the SDE (2.1) in this special case. The solution for this SDE is given as:

$$X_i(t) = X_{i,0} e^{-t b_i} + \int_0^t e^{-(t-u) b_i} dL_i(u), \quad i = 1, \dots, d. \tag{3.6}$$

The process $X_i(t)$ can be estimated pathwise as follows:

$$|X_i(t)| \leq | X_{i,0}| + \max\left(\int_0^t dL_i^+(u), -\int_0^t dL_i^-(u)\right). \tag{3.7}$$

Therefore we obtain the following estimate:

$$|u(t, \boldsymbol{x}) - u_R(t, \boldsymbol{x})| = \mathbb{E}[P(e^{X_T}) \mathbb{1}_{\{T \geq \tau_{G_R}\}} | X_t = \boldsymbol{x}] \leq \mathbb{E}[e^{q M_T} \mathbb{1}_{\{M_T > R\}} | X_t = \boldsymbol{x}],$$

where $M_T = \sup_{s\in[t,T]} \|X_s\|_\infty$. Using (3.7) it follows:

$$\mathbb{E}[e^{qM_T}\mathbb{1}_{\{M_T>R\}}|X_t = \boldsymbol{x}] \leq \mathbb{E}[e^{q\widetilde{M}_T}\mathbb{1}_{\{\widetilde{M}_T>R\}}|X_t = \boldsymbol{x}],$$

for $\widetilde{M}_T = \|X_t\|_\infty + \sup_{s\in[t,T]} \max\{L^+(s), -L^-(s)\}$.

$$\mathbb{E}[e^{q\widetilde{M}_T}\mathbb{1}_{\{\widetilde{M}_T>R\}}|X_t = \boldsymbol{x}] \leq \mathbb{E}[e^{q\widetilde{M}_T^+}\mathbb{1}_{\{\widetilde{M}_T^+>R\}}|X_t = \boldsymbol{x}] + \mathbb{E}[e^{q\widetilde{M}_T^-}\mathbb{1}_{\{\widetilde{M}_T^->R\}}|X_t = \boldsymbol{x}] \tag{3.8}$$

Both terms in (3.8) can be estimated analogously to Theorem 3.13, which yields the claimed result. □

Remark 3.15. *Similar results can also be obtained for more general drift functions. E.g. assuming* $(b_1, \ldots, b_d) \in \mathbb{R}^d$*, leads to an analogous estimate to* (3.5) *under stricter assumptions on* β_i^+*,* β_i^-*,* $i = 1, \ldots, d$*, and different constants* α *and* β*.*

4 Well-posedness of the Kolmogorov equations

The well-posedness of the arising equations is addressed in this section. Abstract existence and uniqueness results are presented. In several particular cases, such as pure diffusion, resp. pure jump, a characterization of the domain of the generator is given.

4.1 Abstract results

In the following we consider the localized problem on a bounded domain G with Lipschitz boundary ∂G. We impose the following conditions on the coefficients.

Assumption 4.1. *Let the coefficients in* (2.2) *satisfy:*

1. $\boldsymbol{b} \in [W^{1,\infty}(G)]^d$.
2. *There exists a positive constant* $c_{\max}$ *such that* $c_{\max} \geq |c(\boldsymbol{x})|\ \ \forall \boldsymbol{x} \in G$.
3. *Let* $a_J^G(\cdot,\cdot)$ *denote the bilinear form of the jump part of the generator of* X*, i.e.* $a_J^G(u, v) := (A_J^G[u], v) := (A_J[\widetilde{u}], \widetilde{v})$*,* $u, v \in D(a_J^G)$*, with a jump measure* $\nu(d\boldsymbol{z})$ *that satisfies Assumption 2.2 with order* $\boldsymbol{\rho}$*,* $\overline{\rho} := \max_{i=1,\ldots,d} \rho_i < 2$ *and* $\underline{\rho} := \min_{i=1,\ldots,d} \rho_i \geq 0$.
4. *There exists a positive constant* γ *s.t.*

$$\forall x \in \overline{G}: \quad c(\boldsymbol{x}) - \frac{1}{2}\operatorname{div}(\boldsymbol{b}(\boldsymbol{x})) > \gamma > 0\,. \tag{4.1}$$

The strong formulation of the localized problem reads:

$$\begin{aligned} &\partial_t u(t, \boldsymbol{x}) + Su(t, \boldsymbol{x}) - A_J^G[u](t, \boldsymbol{x}) = f(t, \boldsymbol{x}) \quad \text{on } (0, T) \times G, \\ &u(0, \boldsymbol{x}) = P(\boldsymbol{x}) \text{ on } G, \qquad u(t, \boldsymbol{x}) = 0 \text{ on } (0, T) \times \Gamma_-, \end{aligned} \tag{4.2}$$

where we set $\Gamma_- := \{\boldsymbol{x} \in \partial G : \boldsymbol{b}(\boldsymbol{x}) \cdot \boldsymbol{n} < 0\}$, $\boldsymbol{n}$ is the exterior unit normal vector to G and we define

$$Su(t, \boldsymbol{x}) := \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(t, \boldsymbol{x}) + c(\boldsymbol{x})u(t, \boldsymbol{x}). \tag{4.3}$$

We obtain the following result. Consider the inner product $(w, v)_H$ for $w, v \in H^1(G)$ given by

$$(w, v)_H := (w, v)_{L^2(G)} + \langle w, v\rangle_{\partial G} + (w, v)_{\widetilde{H}^{\boldsymbol{\rho}/2}(G)},$$

where $\langle w, v\rangle_{\partial G} = \int_{\partial G} |n(s) \cdot b(s)|\, w(s)v(s)\, ds$. We denote by H the closure of $H^1(G)$ in the norm $\|w\|_H := \sqrt{(w, w)_H}$.

Theorem 4.2. *Let* (4.2) *satisfy Assumption 4.1. Then a Gårding inequality and continuity of the bilinear form corresponding to* $\mathcal{L}u$ *holds on a certain Hilbert space* $\widetilde{V} = \widehat{H}(G)$ *which is a subspace of* H*. This implies the well-posedness of problem* (4.2) *on* $L^2((0,T);\widetilde{V}) \cap H^1((0,T);\widetilde{V}^*)$.

Proof. The result follows from [17, Theorem 3.16], [20, Theorem 2] and [28, Theorem 1.4.1]. □

An alternative approach to the proof of well-posedness is the use of semigroup theory. We consider the weak space-time formulation:

$$\int_0^T \int_G \left(\partial_t u(t,\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(t,\boldsymbol{x}) - A_J^G[u](t,\boldsymbol{x}) + c(\boldsymbol{x})u(t,\boldsymbol{x})\right) v(t,\boldsymbol{x}) d\boldsymbol{x}dt = \int_0^T \int_G f(t,\boldsymbol{x})v(t,\boldsymbol{x})d\boldsymbol{x}dt, \quad (4.4)$$

for all $v \in W$, where

$$\begin{aligned} W &:= \left\{ u \in L^2\left((0,T)\times G\right),\ \partial_t u + \boldsymbol{b}\cdot\nabla u + cu \in L^2\left((0,T)\times G\right), u(0) \in L^2(G), u|_{\Gamma_-} \in L^2\left((0,T)\times\Gamma_-\right) \right\} \\ &\cap \quad L^2\left((0,T), \widetilde{H}^{\boldsymbol{\rho/2}}(G)\right), \end{aligned}$$

equipped with the norm $\|\cdot\|_W$ defined by

$$\begin{aligned} ||u||_W^2 &= \int_0^T \int_G u^2(t,\boldsymbol{x})d\boldsymbol{x}dt + \int_0^T \int_G \left(\partial_t u + \boldsymbol{b}\cdot\nabla u + cu\right)^2(t,\boldsymbol{x})d\boldsymbol{x}dt + \int_G u^2(0,\boldsymbol{x})d\boldsymbol{x} \\ &+ \int_0^T \int_{\Gamma_-} u^2(t,s)dsdt + \int_0^T ||u(t,\cdot)||^2_{\widetilde{H}^{\rho/2}(G)}dt. \end{aligned}$$

We shall use the following Theorem from [1], where we denote by $\mathrm{Lip}(G,\mathbb{R})$ the class of real valued functions which are Lipschitz continuous in the domain $G \subseteq \mathbb{R}^d$.

Theorem 4.3. *Let us assume* $F \in Lip(G,\mathbb{R})$ *and* $div(F) \in L^\infty(G)$*; further let the operator* K *be given as*

$$Ku = -\nabla \cdot (F(\boldsymbol{x})u(\boldsymbol{x})),$$

with u *in the domain* $D(K) = \left\{u \in L^2(G) : Ku \in L^2(G), u|_{\Gamma_-} = 0\right\}$*, where* Γ *is assumed to be piecewise* C^1*. Then* $(K, D(K))$ *is the generator of a* C_0 *semigroup of contractions.*

Proof. See [1, Theorem 4.4]. □

Thus the transport as well as the jump operator generate C_0 semigroups, using Assumption 4.1, therefore the mild solution $u \in C([0,T], L^2(G))$ of problem (4.2) is well-defined (cf. [29, Section 4.2, Definition 2.3]), i.e.,

$$u(t,\cdot) = \exp\left(t(S - A_J)\right) P(\cdot) + \int_0^t \exp\left(s(S - A_J)\right) f(t-s,\cdot)\, ds. \quad (4.5)$$

Uniqueness of suffiently smooth solutions follows from linearity of the Partial Integro Differential Equation (PIDE) and the following estimate, which can be obtained choosing $v = u$ and $f = 0$ in (4.4)

$$0 \le \frac{1}{2}\|u(t,\cdot)\|^2_{L^2(G)} + C\|u\|^2_W \le 0.$$

Remark 4.4. *If one assumes* $u(0) \in D(S - A_J)$ *and* $f \in C^1([0,T], L^2(G))$*, then a unique strong solution on* $[0,T)$ *can be obtained due to [29, Section 4.2, Corollary 2.5].*

Remark 4.5. *Using semigroup theory we are not restricted to bounded domains* G*, but we can consider the problem on the whole space* $\mathbb{R}^d$*; in this case the boundary conditions on* Γ_- *are void. Note that the integrability condition* $P \in L^2(\mathbb{R}^d)$ *is strong and not satisfied in many financial applications. Therefore we can either work in weighted Sobolev spaces, cf. [27], an approach we do not pursue in this work, or we can use a localization argument to reduce the problem to a bounded domain and control the truncation error under certain assumptions by a probabilistic argument, cf. Section 3.4.*

Under certain conditions on the Lévy driving process, we can obtain stronger regularity results. We will examine several special cases in the following.

4.2 Infinite variation processes

In the following we assume $\underline{\rho} \geq 1$. Let the bilinear form $\mathcal{A}(\cdot,\cdot)$ be given as

$$\begin{aligned}
\mathcal{A}(u,v) &:= \int_{\mathbb{R}^d} \mathcal{L}u(\boldsymbol{x})\, v(\boldsymbol{x}) d\boldsymbol{x} \qquad (4.6)\\
&= \int_{\mathbb{R}^d} \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x})\, v(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^d} c(\boldsymbol{x}) u(\boldsymbol{x})\, v(\boldsymbol{x}) d\boldsymbol{x} - \int_{\mathbb{R}^d} A'_J[u(\boldsymbol{x})]\, v(\boldsymbol{x}) d\boldsymbol{x},\\
A'_J[u](\boldsymbol{x}) &:= \int_{\mathbb{R}^d} (u(\boldsymbol{x}+\boldsymbol{z}) - u(\boldsymbol{x}) - \boldsymbol{z}\cdot \nabla u(\boldsymbol{x}))\, \nu(d\boldsymbol{z}), \qquad (4.7)
\end{aligned}$$

for $u, v \in D(\mathcal{A})$. We consider the following problem: find $u \in V$ such that $\forall v \in V$, $V = D(\mathcal{A})$ it holds

$$\mathcal{A}(u,v) = \int_G f(\boldsymbol{x})\, v(\boldsymbol{x}) d\boldsymbol{x}.$$

4.2.1 Gårding inequality

Let us denote by $\widehat{v}$ the Fourier transform of v, i.e., $\widehat{v} = \mathcal{F}v$. Using [31, Proposition 3.10], we obtain for $a'_J(\cdot,\cdot)$ with a jump measure ν satisfying Assumption 2.2, for any $u \in D(a'_J)$

$$-a'_J(u,u) = -(A'_J[u],u) \geq C_G^+ ||u||^2_{\boldsymbol{\rho}/2} - C_G^- ||u||_0^2, \qquad (4.8)$$

with $C_G^+ > 0$, $C_G^- \geq 0$, and where we denote by $||\cdot||_{\boldsymbol{\rho}/2}$ and $||\cdot||_0$ the norm in $\widetilde{H}^{\boldsymbol{\rho}/2}(G)$ and $L^2(G)$, respectively. See [31, Section 4] for further details.
Let us now consider the drift and reaction term in (4.6). Due to condition (4.1), it holds for $u \in \{u \in L^2(G) | Su \in L^2(G), u|_{\Gamma_-} = 0\}$, with S as in (4.3)

$$\begin{aligned}
\int_G \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x})\, u(\boldsymbol{x}) d\boldsymbol{x} + \int_G c(\boldsymbol{x}) u^2(\boldsymbol{x})\, d\boldsymbol{x}, &= \int_G c(\boldsymbol{x}) u^2(\boldsymbol{x}) + \frac{1}{2}\int_G \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla u^2(\boldsymbol{x}) d\boldsymbol{x}\\
&= \int_G \left(c(\boldsymbol{x}) - \frac{1}{2}\operatorname{div}(\boldsymbol{b}(\boldsymbol{x}))\right) u^2(\boldsymbol{x}) d\boldsymbol{x} \geq \gamma ||u||_0^2. \qquad (4.9)
\end{aligned}$$

Thus we obtain the following result.

Theorem 4.6. *If Assumption 4.1 holds, the bilinear form $\mathcal{A}(\cdot,\cdot)$ satisfies a Gårding inequality on V.*

Proof. The result follows from (4.8) and (4.9). □

Remark 4.7. *We do not require the assumption $\boldsymbol{\rho} \geq 1$ for Theorem 4.6.*

Remark 4.8. *A similar estimate to (4.8) can be obtained for the localized problem using the Sobolev-Slobodecki norm. Splitting the bilinear form $a_J^G(\cdot,\cdot)$ into its symmetric ($a_J^{G,s}(\cdot,\cdot)$) and antisymmetric part (see [31], Section 4.4), we obtain for $u \in V$*

$$-a_J^G(u,u) = -a_J^{G,s}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\widetilde{u}(\boldsymbol{x}) - \widetilde{u}(\boldsymbol{y}))^2 \left(k(\boldsymbol{x}-\boldsymbol{y}) + k(\boldsymbol{y}-\boldsymbol{x})\right) d\boldsymbol{y} d\boldsymbol{x}. \qquad (4.10)$$

Without loss of generality, we impose $G \subset B(0;\frac{1}{2}) \subset \mathbb{R}^d$. Then we obtain

$$-a_J^G(u,u) \geq C\left(||\widetilde{u}||^2_{\boldsymbol{\rho}/2} - ||\widetilde{u}||_0^2\right).$$

The constant C depends only on C_1 and C_2 in Assumption 2.2.

4.2.2 Continuity

In order to study the continuity, it is convenient to use the Fourier transform. We consider the term $\int_G Su(\boldsymbol{x})v(\boldsymbol{x})\,d\boldsymbol{x}$. Since $\mathcal{F}\nabla u(\boldsymbol{\xi}) = i\boldsymbol{\xi}\,\widehat{u}(\boldsymbol{\xi})$, it holds

$$\begin{aligned}\left|\int_{\mathbb{R}^d} Su(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x}\right| &\leq ||\boldsymbol{b}||_\infty \left|\int_{\mathbb{R}^d} i\mathbf{1}\cdot\boldsymbol{\xi}\,\widehat{u}(\boldsymbol{\xi})\,\overline{\widehat{v}(\boldsymbol{\xi})}\,d\boldsymbol{\xi}\right| + c_{\max}\|u\|_0\|v\|_0 \\ &\leq C||\boldsymbol{b}||_\infty \left(\int_{\mathbb{R}^d} |\boldsymbol{\xi}|\,|\widehat{u}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right)^{1/2}\left(\int_{\mathbb{R}^d} |\boldsymbol{\xi}|\,|\widehat{v}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\right)^{1/2} + c_{\max}\|u\|_0\|v\|_0\,,\end{aligned}$$

where $\mathbf{1}$ denotes the unit vector in $\mathbb{R}^d$. Since $\rho_j \geq 1,\ j = 1,\cdots,d$, we have

$$\left|\int_G Su(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x}\right| \leq C||u||_{1/2}||v||_{1/2} \leq C||u||_{\boldsymbol{\rho}/2}||v||_{\boldsymbol{\rho}/2}. \tag{4.11}$$

We study the behavior of the jump term. As above it follows from Assumption 2.2 using [31, Proposition 3.5]

$$\left|-\int_G A_J[u](\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x}\right| \leq C||u||_{\boldsymbol{\rho}/2}||v||_{\boldsymbol{\rho}/2}, \tag{4.12}$$

for some positive constant C that depends only on the jump measure.

Theorem 4.9. *The bilinear form* (4.6) *with* $\underline{\rho} \geq 1$, *is continuous on* $V := D(\mathcal{A})$.

Proof. The result follows from (4.11) and (4.12). □

The well-posedness of the parabolic problem: given $f \in L^2((0,T);V^*)$, find $u \in L^2((0,T);V) \cap H^1((0,T);V^*)$, such that

$$(\partial_t u, v) + \mathcal{A}(u,v) = (f,v) \quad \text{a.e. in } (0,T), \qquad u(0,\boldsymbol{x}) = P(\boldsymbol{x}), \tag{4.13}$$

for all $v \in V$, where $\mathcal{A}$ is given as in (4.6), follows from Theorems 4.6 and 4.9.

4.3 Finite variation processes

We now assume that $0 < \underline{\rho} < 1$. Note that in this situation $\overline{\rho} < 1$ ($\overline{\rho} \geq 1$) implies that the corresponding process is of finite (infinite) variation. Therefore (4.11) does not hold. The above steps, leading to Theorem 4.6 and 4.9 still give a Gårding inequality on the space $V = \widetilde{H}^{\boldsymbol{\rho}/2}(G)$ and continuity on the space $\widetilde{H}^{\boldsymbol{\rho}/2}(G) \cap W_{\boldsymbol{b}}$, where $W_{\boldsymbol{b}} = \{w \in L^2(G) \,:\, \boldsymbol{b}\cdot\nabla w \in L^2(G)\}$.

Remark 4.10. *We still rely on the general results presented in Section 4.1 for the well-posedness of the problem.*

Remark 4.11. *If we assume* $\boldsymbol{b} \equiv 0$, *then we obtain the framework discussed in [31]. Therefore well-posedness of the Kolmogorov equation can be proved on the space* $\widetilde{H}^{\boldsymbol{\rho}/2}(G)$.

Remark 4.12. *Note that for Lévy market models, where the drift component is constant, this situation can be achieved using a change of variable (e.g.[27]). The removal of the drift is nontrivial for more general Markov processes.*

4.4 Regularized Kolmogorov equation

We now consider the well-posedness of problem (4.2) after the small jump regularization. We assume the following uniform estimate for the diffusion part of the operator: $Q_\epsilon \geq Q_0$, where Q_0 is a symmetric positive definite matrix. Thus, instead of considering problem (2.2), after localization to Lipschitz domain $G \subset \mathbb{R}^d$, we study the numerical solution of its small jump truncation approximation

$$\begin{aligned} &\partial_t u^\epsilon(t,\boldsymbol{x}) + \mathcal{L}^{G,\epsilon} u^\epsilon(t,\boldsymbol{x}) = f(t,\boldsymbol{x}) \ \text{ in } (0,T)\times G, \\ &u^\epsilon(t,\boldsymbol{x}) = g = 0 \ \text{ in } (0,T)\times\partial G, \quad u^\epsilon(0,\boldsymbol{x}) = P(\boldsymbol{x}) \text{ in } G, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} \mathcal{L}^{G,\epsilon} w(t,\boldsymbol{x}) &= S^\epsilon w(t,\boldsymbol{x}) - A_J^{G,\epsilon}[w](t,\boldsymbol{x}), \\ A_J^{G,\epsilon}[w](t,\boldsymbol{x}) &= \int_{\mathbb{R}^d} \left(\widetilde{w}(t,\boldsymbol{x}+\boldsymbol{z}) - \widetilde{w}(t,\boldsymbol{x})\right)\nu^\epsilon(d\boldsymbol{z}), \\ S^\epsilon w(t,\boldsymbol{x}) &= \boldsymbol{b}^\epsilon(x)\cdot\nabla w(t,\boldsymbol{x}) + c(\boldsymbol{x}) w(t,\boldsymbol{x}) - \frac{1}{2}\mathrm{tr}(Q_\epsilon D^2) w(t,\boldsymbol{x}), \end{aligned}$$

with coefficients that satisfy Assumption 4.1 and ν^ϵ, Q_ϵ and b^ϵ as in Section 3.2. Therefore a weak formulation of (4.14) reads:find $u^\epsilon \in L^2((0,T), H_0^1(G)) \cap H^1((0,T), H^{-1}(G))$ such that

$$(\partial_t u^\epsilon, v) + (S^\epsilon u^\epsilon, v) - (A_J^{G,\epsilon} u^\epsilon, v) = (f,v), \text{ a.e. in } (0,T) \quad u^\epsilon(0) = P(\boldsymbol{x}) \tag{4.15}$$

for all $v \in H_0^1(G)$, where $(\cdot,\cdot)$ denotes the scalar product on $L^2(G)$. We obtain existence of a unique weak solution of the above problem from [31, Theorem 4.8]. If $G = \mathbb{R}^d$, then the domain of the generator is $H^1(\mathbb{R}^d)$.

Remark 4.13. *Sufficiently smooth non-homogeneous Dirichlet boundary conditions can also be considered.*

Remark 4.14. *If the diffusion coefficient Q_ϵ is not positive definite, but only positive semidefinite, we obtain anisotropic Sobolev spaces as the domains of the generators. This corresponds to regularized pure jump processes with a finite activity compound Poisson process in certain directions and infinite activity processes in other directions.*

5 DGFEM for the forward equation

A DG-discretization scheme for the forward equation is described in this section. After introducing the necessary notations, the numerical scheme is presented and analyzed. An error analysis in multiple space dimensions is performed.

5.1 Triangulations

In the following we briefly summarize the requirements that have to be imposed on the triangulation. Let $\mathcal{T}_h$ be a subdivision of G into disjoint open element domains K such that $\overline{G} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$ and each $K \in \mathcal{T}_h$ is an affine image of a fixed master element $\widehat{K}$, i.e. $K = F(\widehat{K})$, where $\widehat{K}$ is the unit simplex. We assume $\mathcal{T}_h$ to be a shape regular, quasi-uniform and simplicial triangulation and

$$\exists\ C_1, C_2 > 0 \,\text{independent of } h \text{ such that } \sup_{K\in\mathcal{T}_h} \frac{h_K}{r_K} < C_1 < +\infty \ \text{ and } \ \sup_{K\in\mathcal{T}_h} h_K < C_2 h_{K'} \ \forall\ K' \in \mathcal{T}_h, \tag{5.1}$$

where h_K and r_K denote the diameter of the element K and the maximum radius of a ball contained in K, $K \in \mathcal{T}_h$, respectively. Moreover, we set $h = \max_{K\in\mathcal{T}_h} h_K$.
We denote by V_h the space of discontinuous piecewise polynomial functions, i.e., $v_h \in V_h$ if and only if $v_h|_K \in \mathbb{P}^p(K)$, $\forall K \in \mathcal{T}_h$, where K is a simplex in $\mathbb{R}^d$ and $\mathbb{P}^p(K)$ is the space of polynomials of total degree p in K. Finally, we assign to the subdivision $\mathcal{T}_h$ the broken Sobolev space of composite order $\boldsymbol{s}$, where $s_K \in \mathbb{N}_0$ are non-negative integers,

$$H^{\boldsymbol{s}}(G,\mathcal{T}_h) = \left\{u \in L^2(G) \ : \ u|_K \in H^{s_K}(K)\ \forall K \in \mathcal{T}_h\right\} \tag{5.2}$$

equipped with the norm

$$\|u\|_{H^s(G,\mathcal{T}_h)} = \left(\sum_{K\in\mathcal{T}_h} \|u\|^2_{H^{s_K}(K)} \right)^{1/2}.$$

Furthermore we use the following notations: $\Gamma_h = \bigcup_{K\in\mathcal{T}_h} \partial K$, $\mathcal{T}_h$ being the considered triangulation and $\Gamma_h^0 = \bigcup_{K\in\mathcal{T}_h} \partial K \cap \partial G$. The average and jump operators are defined as follows: if $e \in \Gamma$ is an edge shared by two elements K_1 and K_2 of $\mathcal{T}_h$ and $\boldsymbol{n}$ is the unit vector normal oriented from K_1 to K_2, then

$$\{v\} = \frac{v|_{K_1} + v|_{K_2}}{2}, \quad [v] = v|_{K_1} - v|_{K_2},$$

otherwise (i.e., $e \in \Gamma \cap \partial G$) we set $\{v\} = [v] = v$.

5.2 DG formulation

The DG semidiscrete formulation of (4.14), with possibly inhomogeneous boundary conditions, reads as follows: for (sufficiently small) jump regularization parameter $\epsilon > 0$, find $u_h^\epsilon(t,\boldsymbol{x}) \in H^1((0,T);V_h)$ such that $\forall\, v_h \in V_h$ it holds

$$\begin{aligned} \int_G \partial_t u_h^\epsilon(t,\boldsymbol{x}) v_h(\boldsymbol{x}) dx + \mathcal{A}^\epsilon_{DG}(u_h^\epsilon(t,\boldsymbol{x}), v_h(\boldsymbol{x})) &= rhs^\epsilon_{DG}(v_h(\boldsymbol{x})), && (5.3)\\ u_h^\epsilon(0,\boldsymbol{x}) &= \Pi_p P(\boldsymbol{x}), && (5.4) \end{aligned}$$

where $\Pi_p P$ is the L^2-projection of the initial condition function P in V_h, and

$$\begin{aligned} \mathcal{A}^\epsilon_{DG}(w,v) &:= d^\epsilon_{DG}(w,v) + t^\epsilon_{DG}(w,v) + r_{DG}(w,v) + j^\epsilon_{DG}(w,v), && (5.5)\\ rhs^\epsilon_{DG}(v) &= \int_G f v d\boldsymbol{x} + bc^\epsilon_{DG}(v). && (5.6) \end{aligned}$$

The bilinear forms $d^\epsilon_{DG}(\cdot,\cdot)$, $t^\epsilon_{DG}(\cdot,\cdot)$, $r_{DG}(\cdot,\cdot)$, $j^\epsilon_{DG}(\cdot,\cdot)$ and the boundary form $bc^\epsilon_{DG}(\cdot)$ are defined as follows for $v, w \in V_h$.

(i) Diffusion term $d^\epsilon_{DG}(\cdot,\cdot)$: for ease of notation we will drop the dependency on time t and space $\boldsymbol{x}$. It holds

$$\begin{aligned} d^\epsilon_{DG}(w,v) &:= \sum_{K\in\mathcal{T}_h} \frac{1}{2}\int_K \nabla w^\top Q_\epsilon \nabla v d\boldsymbol{x} - \sum_{e\in\Gamma_h} \frac{1}{2}\int_e \left\{\nabla w^\top Q_\epsilon \boldsymbol{n}\right\}[v] ds \\ &+ \frac{\beta}{2}\sum_{e\in\Gamma_h}\int_e [w]\left\{\nabla v^\top Q_\epsilon \boldsymbol{n}\right\} ds + \sum_{e\in\Gamma_h}\frac{\alpha}{|e|}\int_e [w][v] ds, \qquad (5.7) \end{aligned}$$

where $\alpha > 0$ is independent of h and ϵ; $\beta = -1$ yields the Symmetric Interior Penalty Galerkin (SIPG) method (which converge only if α is sufficiently large), while $\beta = 1$ gives the Non-Symmetric Interior Penalty Galerkin (NIPG) method. See [33, Chapter 2] for further details. From now on we set $\beta = 1$, i.e., we discretize the diffusion term with the NIPG method.

(ii) Transport term $t^\epsilon_{DG}(\cdot,\cdot)$: following [18], we obtain

$$t^\epsilon_{DG}(w,v) := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{b}^\epsilon \cdot \nabla w\, v d\boldsymbol{x} - \int_{\partial^\epsilon_- K} (\boldsymbol{b}^\epsilon \cdot \boldsymbol{n}_K)[w] v_I ds, \qquad (5.8)$$

where $\boldsymbol{n}_K$ is the normal unit vector exterior to K, v_I (v_O) is the inner (outer) trace of v relative to K and, according to the above definition, $[v] = v_I - v_O$. Moreover we set

$$\partial^\epsilon_- K := \{\boldsymbol{x} \in \partial K \;:\; \boldsymbol{b}^\epsilon \cdot \boldsymbol{n}_K < 0\} \quad \text{and} \quad \partial^\epsilon_+ K := \{\boldsymbol{x} \in \partial K \;:\; \boldsymbol{b}^\epsilon \cdot \boldsymbol{n}_K > 0\}.$$

Notice that in the following we will drop the index ϵ when $\epsilon = 0$, i.e., $\partial_\pm K := \partial^0_\pm K$.

(iii) Reaction term $r_{DG}(\cdot,\cdot)$: it holds

$$r_{DG}(w,v) := \int_G c\, w\, v d\boldsymbol{x}. \tag{5.9}$$

(iv) Integrodifferential term $j^{\epsilon}_{DG}(\cdot,\cdot)$: since the jump operator $A_J^{G,\epsilon}$ in (4.14) can be rewritten as

$$A_J^{G,\epsilon}[w](t,\boldsymbol{x}) = \int_{\mathbb{R}^d} (\widetilde{w}(t,\boldsymbol{y}) - \widetilde{w}(t,\boldsymbol{x}))k_\epsilon(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{y}, \tag{5.10}$$

the integrodifferential term is given as

$$j^{\epsilon}_{DG}(w,v) := -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\widetilde{w}(t,\boldsymbol{y}) - \widetilde{w}(t,\boldsymbol{x}))k_\epsilon(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{y}\, \widetilde{v}(\boldsymbol{x})d\boldsymbol{x}. \tag{5.11}$$

(v) The boundary term $bc^{\epsilon}_{DG}(\cdot)$ reads

$$bc^{\epsilon}_{DG}(v) = \sum_{e\in\Gamma_h^0}\int_e \left(\frac{1}{2}\nabla v^\top Q_\epsilon \boldsymbol{n} + \frac{\alpha}{|e|}v\right) g ds - \sum_{K\in\mathcal{T}_h}\int_{\partial^\epsilon_- K\cap\partial G} (\boldsymbol{b}^\epsilon\cdot\boldsymbol{n}_K) g v_I ds.$$

The first term stems from the discretization of the diffusion part, while the second term originates from the transport term. Notice that if $g\equiv 0$ as in (4.14), then $bc^{\epsilon}_{DG}\equiv 0$.

Remark 5.1. *In [19] the authors deal with a DG discretization for the hyperbolic part $\boldsymbol{b}^\epsilon\cdot\nabla u + cu = f$. More precisely, they discretize the bilinear form $t^{\epsilon}_{DG}(\cdot,\cdot) + c_{DG}(\cdot,\cdot)$ as in (5.8) and (5.9) adding the following stabilization term*

$$\delta\sum_{K\in\mathcal{T}_h}\int_K (\boldsymbol{b}^\epsilon\cdot\nabla u + cu)\,(\boldsymbol{b}^\epsilon\cdot\nabla v)d\boldsymbol{x},$$

for $\delta>0$. The consistency of the method is ensured by adding the term $\delta\int_G f(\boldsymbol{b}^\epsilon\cdot\nabla v)d\boldsymbol{x}$ to the right-hand side of the equation. Consistency, stability and an error analysis is provided in [19] with $\delta = Chp^{-1}$, with C independent of h and p. However, numerical results (see [19, Section 5]) show that the scheme without stabilization, i.e., $\delta=0$, is marginally more accurate for $p=1$ and $p=2$. For larger p the stabilized scheme is slightly more accurate.

Remark 5.2. *According to the previous section, $G\subset\mathbb{R}^d$, $d\geq 1$, is a bounded plane faced polyhedral domain with Lipschitz boundary Γ. The above DG formulation is written with integrals over faces of the elements of the mesh, and thus for the case $d>1$. In the one dimensional case, this is to be interpreted as follows: if $K=[a,b]$, then $\partial K = \{a,b\}$ and we set for $v\in\mathbb{P}^p(K)$*

$$\int_a v_I(x)dx = v(a), \qquad \int_b v_I(x)dx = v(b), \qquad \int_a v_I(x)ndx = -v(a), \qquad \int_b v_I(x)ndx = v(b),$$

where $n=1$ (-1) in b (a). Moreover, if h is the length of the interval K, i.e., $h:=b-a$, we replace $|e|$ by h in (5.7).

Remark 5.3. *Let us consider, for simplicity, the one dimensional case, i.e. $d=1$ in (4.14), and the integrodifferential term (5.11): if we denote by $k_\epsilon^{(-1)}$ the antiderivative of k_ϵ, then (5.11) can be rewritten as follows:*

$$\begin{aligned} j^{\epsilon}_{DG}(w,v) &= \int_{\mathbb{R}}\sum_{K\in\mathcal{T}_h}\int_K \widetilde{w}'(y)k_\epsilon^{(-1)}(y-x)dy\,\widetilde{v}(x)dx \\ &- \int_{\mathbb{R}}\sum_{K\in\mathcal{T}_h}\int_{\partial K} (\widetilde{w}_I(s)-\widetilde{w}(x))nk_\epsilon^{(-1)}(s-x)ds\,\widetilde{v}(x)dx, \end{aligned}$$

where the antiderivatives of $k_\epsilon(x)$ are given by

$$k_\epsilon^{(-(i+1))}(x) = \begin{cases} \int_{-\infty}^{x} k_\epsilon^{(-i)}(y)dy & \text{if } x < 0, \\ -\int_{x}^{+\infty} k_\epsilon^{(-i)}(y)dy & \text{if } x > 0, \end{cases} \tag{5.12}$$

and $k_\epsilon^{(0)} = k_\epsilon$. See Section 6 for details.

Remark 5.4. *The discretization of the regularized Lévy density, cf. Section 3.2, can be performed as follows. If we assume a Lévy copula construction for the density $k(z)$ with a 1-homogeneous Lévy copula F, cf. [23], then the jump term $A_J^{G,\epsilon}$ (5.10) can be expressed as follows:*

$$\begin{aligned} A_J^{G,\epsilon} w(\boldsymbol{x}) &= \sum_{i=1}^{d} \int_{\mathbb{R}} \left(\widetilde{w}(t, \boldsymbol{x} + e_i y_i) - \widetilde{w}(t, \boldsymbol{x})\right) k_{\epsilon,i}(y_i)\, dy_i \\ &+ \sum_{j=2}^{d} \sum_{\substack{|I|=j \\ I_1 < \ldots < I_j}} \int_{\mathbb{R}^j} \frac{\partial^j \widetilde{w}}{\partial y^I}(y^I) F^I((U_\epsilon(y_k - x_k))_{k \in I})\, dy^I, \end{aligned} \tag{5.13}$$

where

$$F^I(u^I) = \lim_{a \to \infty} \sum_{(u_j)_{j \in I^c} \in \{-a, \infty\}^{|I^c|}} \left(\prod_{j \in I^c} \mathrm{sign}\, u_j \right) F(u_1, \ldots, u_d),$$

cf. [31, Proposition 4.11]. Therefore Remark 5.3 can be used to discretize the univariate marginal integrals, this corresponds to a Lévy measure with independent marginals. The remaining term in (5.13) *can be discretized analogously.*

5.3 DG stability and error analysis

In the following section we analyze the stability and derive error estimates for the DG semidiscrete formulation (5.3)-(5.4) of (4.14). Let us consider the following notation. We denote by u the smooth solution of problem (4.2), u^ϵ is the smooth solution of problem (4.15) with $\epsilon > 0$ and $u_h^\epsilon \in V_h$ is the solution of problem (5.3)-(5.4) according to the DG discretization.
We prove the consistency of the considered DG scheme in Section 5.3.1, while in Sections 5.3.2 and 5.3.3 we deal with an *a priori* bound and error estimates of the DG solution.

5.3.1 Consistency of the DG scheme

Theorem 5.5. *If u^ϵ is the solution of (4.15), then it satisfies (5.3).*

Proof. Assume for simplicity $g \equiv 0$, i.e., $u^\epsilon|_{\partial G} = 0$. Let $v_h \in V_h$ be a test function. Multiplying (4.14) by v_h and integrating by parts, we obtain

$$(\partial_t u^\epsilon, v_h) + (S^\epsilon u^\epsilon, v_h) - (A_J^{G,\epsilon} u^\epsilon, v_h) = (f, v_h),$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(G)$. Since

$$(A_J^{G,\epsilon} u^\epsilon, v_h) \equiv j_{DG}^\epsilon(u^\epsilon, v_h), \quad (cu^\epsilon, v_h) \equiv r_{DG}(u^\epsilon, v_h) \;\text{ and }\; (f, v_h) \equiv rhs_{DG}(v_h),$$

holds, in order to prove consistency of the method, we have to deal with the diffusion and transport terms. However, the regularity of u^ϵ implies $[u^\epsilon] = 0$ on Γ_h, thus $(\boldsymbol{b}^\epsilon \cdot \nabla u^\epsilon, v_h) = t_{DG}^\epsilon(u^\epsilon, v_h)$. Finally, the consistency of the diffusive part (and thus of the whole formulation) follows from [33, Proposition 2.9]. □

5.3.2 *A priori* bound

Let us assume that $\boldsymbol{b}^\epsilon$ and c satisfy

$$(c_0^\epsilon)^2(\boldsymbol{x}) := c(\boldsymbol{x}) - \frac{1}{2}\mathrm{div}(\boldsymbol{b}^\epsilon(\boldsymbol{x})) > \gamma^\epsilon \geq \widetilde{\gamma} > 0 \tag{5.14}$$

for sufficiently small $\epsilon > 0$ and $\widetilde{\gamma} > 0$. Note that this condition follows from (4.1). Following [18], it holds for all $w \in H^1(G, \mathcal{T}_h)$

$$\begin{aligned} d_{DG}^\epsilon(w,w) + t_{DG}^\epsilon(w,w) + r_{DG}(w,w) \quad &= \quad \sum_{K\in\mathcal{T}_h} \Big(|w|^2_{H^{1,\epsilon}(K)} + ||c_0^\epsilon w||^2_{L^2(K)} + \frac{1}{2}||[w]||^2_{L^2(\partial_-^\epsilon K)} \\ &\quad + \frac{1}{2}||[w]||^2_{L^2(\partial_+^\epsilon K\cap\Gamma_h^0)}\Big) + \sum_{e\in\Gamma_h} \frac{\alpha}{|e|}||[w]||^2_{L^2(e)}, \end{aligned} \tag{5.15}$$

with

$$|w|^2_{H^{1,\epsilon}(K)} := \frac{1}{2}\int_K \nabla w^\top Q_\epsilon \nabla w d\boldsymbol{x}.$$

The above arguments suggest to define for $\epsilon > 0$ sufficiently small and $w \in H^1(G, \mathcal{T}_h)$, the DG norm:

$$\begin{aligned} ||w||^2_{\mathrm{DG}(\epsilon)} \quad &:= \quad \sum_{K\in\mathcal{T}_h} \Big(|w|^2_{H^{1,\epsilon}(K)} + ||c_0^\epsilon w||^2_{L^2(K)} + \frac{1}{2}||[w]||^2_{L^2(\partial_-^\epsilon K)} + \frac{1}{2}||[w]||^2_{L^2(\partial_+^\epsilon K\cap\Gamma_h^0)}\Big) \\ &\quad + \quad \sum_{e\in\Gamma_h} \frac{\alpha}{|e|}||[w]||^2_{L^2(e)}. \end{aligned} \tag{5.16}$$

Let us now consider the term $j_{DG}^\epsilon(\cdot,\cdot)$; for ease of notation, we will omit the dependence on t. Using (4.10), we obtain for $\epsilon > 0$ sufficiently small $j_{DG}^\epsilon(w,w) \geq 0$, and therefore

$$\mathcal{A}_{DG}^\epsilon(w,w) \geq ||w||^2_{\mathrm{DG}(\epsilon)}. \tag{5.17}$$

Considering (5.11), it holds for all $w \in H^1(G, \mathcal{T}_h)$ and all $\epsilon > 0$

$$\begin{aligned} j_{DG}^\epsilon(w,w) \quad &\leq \quad \left|\int_G\int_G (w(\boldsymbol{y}) - w(\boldsymbol{x}))k_\epsilon(\boldsymbol{y}-\boldsymbol{x})d\boldsymbol{y}\, w(\boldsymbol{x})d\boldsymbol{x}\right| \\ &\leq \quad C||k_\epsilon||_{L^\infty(\mathbb{R}^d)}||w||^2_{L^2(G)} \leq C(\epsilon)||w||^2_{\mathrm{DG}(\epsilon)}, \end{aligned} \tag{5.18}$$

i.e.,

$$\mathcal{A}_{DG}^\epsilon(w,w) \leq C(\epsilon)||w||^2_{\mathrm{DG}(\epsilon)}. \tag{5.19}$$

Remark 5.6. *From* (5.18), *it is clear that it is not necessary to add an additional term in the definition of the norm* $||\cdot||_{DG(\epsilon)}$ *to control* $j_{DG}(\cdot,\cdot)$ *once* $\epsilon > 0$ *is fixed. However, if* $\epsilon \to 0$, *then* $C(\epsilon) \to +\infty$ *in (5.19). See Section 5.4 for details on the case* $\epsilon \equiv 0$.

To prove the *a priori* bound, we need the following result.

Lemma 5.7. *Let* $w \in V_h \subset H^2(G, \mathcal{T}_h)$ *and* $K \in \mathcal{T}_h$, *then there exists a constant* $\mathcal{C} > 0$, *independent of* h, p *and dependent on the shape regularity of* $\mathcal{T}_h$, *such that,*

$$\left|\left|\nabla_h w^\top Q_\epsilon \boldsymbol{n}\right|\right|_{L^2(e)} \leq \mathcal{C}ph^{-1/2}\left|\left|\sqrt{\nabla_h w^\top Q_\epsilon \nabla_h w}\right|\right|_{L^2(K)} \quad \forall e \in \partial K.$$

Proof. The result follows from trace inequalities. In fact

$$\left|\left|\nabla_h w^\top Q_\epsilon \boldsymbol{n}\right|\right|^2_{L^2(e)} \leq \mathcal{C}\max\{|Q_\epsilon|\}\int_e \nabla_h w^\top Q_\epsilon \nabla_h w\, d\boldsymbol{x} \leq \frac{\mathcal{C}p^2}{h_K}\max\{|Q_\epsilon|\}\int_K \nabla_h w^\top Q_\epsilon \nabla_h w\, d\boldsymbol{x},$$

where the constant $\mathcal{C}$ is independent of h_K, i.e., the diameter of the element K, the polynomial degree p and $|e|$ (see for example [33, Section 2.1.3] and [18, Section 4.2]). □

Theorem 5.8. *Let u_h^ϵ be the solution of (5.3), then the following* a priori *bound holds:*

$$||u_h^\epsilon(T,\cdot)||^2_{L^2(G)} + \int_0^T ||u_h^\epsilon(t,\cdot)||^2_{DG(\epsilon)}dt \le C\left(||f||^2_{L^2((0,T);\,L^2(G))} + \sum_{e\in\Gamma_h^0}\frac{1}{|e|}||g||^2_{L^2((0,T);\,L^2(e))}\right) + ||\Pi_p P||^2_{L^2(G)}.$$

Proof. Considering (5.17), it holds

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_h^\epsilon||^2_{L^2(G)} + ||u_h^\epsilon||^2_{\mathrm{DG}(\epsilon)} &\le \int_G \partial_t u_h^\epsilon\ u_h^\epsilon dx + d^\epsilon_{DG}(u_h^\epsilon,u_h^\epsilon) + t^\epsilon_{DG}(u_h^\epsilon,u_h^\epsilon) + j^\epsilon_{DG}(u_h^\epsilon,u_h^\epsilon)\\ &= rhs_{DG}(u_h^\epsilon) + bc^\epsilon_{DG}(u_h^\epsilon)\\ &\le ||f||_{L^2(G)}||u_h^\epsilon||_{L^2(G)} + \sum_{e\in\Gamma_h^0}||g||_{L^2(e)}\left(\frac{||\left(\nabla u_h^\epsilon\right)^\top Q_\epsilon \boldsymbol{n}||_{L^2(e)}}{2} + ||\max\{-\boldsymbol{b}^\epsilon\cdot\boldsymbol{n},\,0\}\,u_h^\epsilon||_{L^2(e)} + \frac{\alpha}{|e|}\,||u_h^\epsilon||_{L^2(e)}\right),\end{aligned}$$

where $\boldsymbol{n}$ is the exterior normal unit vector. From Lemma 5.7, we obtain

$$\frac{1}{2}\frac{d}{dt}||u_h^\epsilon||^2_{L^2(G)} + ||u_h^\epsilon||^2_{\mathrm{DG}(\epsilon)} \le \left(\sqrt{\frac{1}{\gamma^\epsilon}}||f||_{L^2(G)} + \sum_{e\in\Gamma_h^0}\widehat{C}_e\ ||g||_{L^2(e)}\right)||u_h^\epsilon||_{\mathrm{DG}(\epsilon)},$$

with $\widehat{C}_e = \max\left(\frac{\mathcal{C}p}{2\sqrt{h}},\ \sqrt{\frac{|e|}{\alpha}}\max\left\{\max_{\boldsymbol{x}\in e}\left(-\boldsymbol{b}^\epsilon(\boldsymbol{x})\cdot\boldsymbol{n}\right),\,0\right\},\ \sqrt{\frac{\alpha}{|e|}}\right)$, where $\mathcal{C}$ is the constant in Lemma 5.7. Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_h^\epsilon||^2_{L^2(G)} + ||u_h^\epsilon||^2_{\mathrm{DG}(\epsilon)} &\le C\left(||f||_{L^2(G)} + \left(\sum_{e\in\Gamma_h^0}\frac{1}{|e|}||g||^2_{L^2(e)}\right)^{1/2}\right)||u_h^\epsilon||_{\mathrm{DG}(\epsilon)}\\ &\le C^2\left(||f||^2_{L^2(G)} + \sum_{e\in\Gamma_h^0}\frac{1}{|e|}||g||^2_{L^2(e)}\right) + \frac{1}{2}||u_h^\epsilon||^2_{\mathrm{DG}(\epsilon)},\end{aligned}$$

where $C = \max\left\{\max_{e\in\Gamma_h^0}\left(\widehat{C}_e\sqrt{|e|}\right),\ \sqrt{\frac{1}{\gamma^\epsilon}}\right\}$. We notice that C is independent of h, for sufficiently small h, due to the mesh regularity properties (5.1). Thus

$$\frac{d}{dt}||u_h^\epsilon||^2_{L^2(G)} + ||u_h^\epsilon||^2_{\mathrm{DG}(\epsilon)} \le 2C^2\left(||f||^2_{L^2(G)} + \sum_{e\in\Gamma_h^0}\frac{1}{|e|}||g||^2_{L^2(e)}\right)$$

and the claimed result is obtained integrating in time and setting $u_h^\epsilon(0,\cdot) = \Pi_p P(\cdot)$. □

5.3.3 *A priori* error estimate

We want to estimate $||u - u_h^\epsilon||$ in a suitable norm. We estimate it as follows:

$$||u - u_h^\epsilon|| \le \underbrace{||u - u^\epsilon||}_{(a)} + \underbrace{||u^\epsilon - u_h^\epsilon||}_{(b)}.$$

The term (a) can be estimated using Theorem 3.11, while the term (b) depends on the DG approximation. In order to prove an *a priori* error estimate for $||u^\epsilon - u_h^\epsilon||$, we need the following Lemma.

Lemma 5.9. *Suppose that $K\in\mathcal{T}_h$ is a shape regular d-simplex or a shape regular d-parallelepiped of diameter h_K. Suppose further that $u|_K\in H^r(K)$, $r\ge 2$. Then there exists a projection $\Pi_{(p,K)}$ on the space of the* *polynomials of degree p in K such that, for $s \geq 1$, $p \geq 1$, s integer*

$$||w - \Pi_{(p,K)} w||_{L^2(K)} \leq C \frac{h_K^{\min(p+1,s)}}{p^s} ||w||_{H^s(K)},$$

$$||\nabla \left(w - \Pi_{(p,K)} w \right)||_{L^2(K)} \leq C \frac{h_K^{\min(p+1,s)-1}}{p^{s-1}} ||w||_{H^s(K)},$$

$$||w - \Pi_{(p,K)} w||_{L^2(\partial K)} \leq C \frac{h_K^{\min(p+1,s)-\frac{1}{2}}}{p^{s-\frac{1}{2}}} ||w||_{H^s(K)},$$

$$||\nabla \left(w - \Pi_{(p,K)} w \right)||_{L^2(\partial K)} \leq C \frac{h_K^{\min(p+1,s)-\frac{3}{2}}}{p^{s-\frac{3}{2}}} ||w||_{H^s(K)}.$$

Moreover, for $0 < l < 1$, it holds

$$||w - \Pi_{(p,K)} w||_{H^l(K)} \leq C \frac{h_K^{\min(p+1,s)-l}}{p^{s-l}} ||w||_{H^s(K)}.$$

Proof. See [8], [18, Lemma 4.4] and [35, Section 4.5]. □

Remark 5.10. *In each estimate, s can be chosen differently, if w is sufficiently smooth.*

Remark 5.11. *Lemma 5.9 can be extended to the case $0 < s < 2$. In this case, the right-hand side of the interpolation estimates contains the term $||w||_{H^s(\delta_K)}$, where δ_K is the union of the element K with its neighbor elements, i.e. $\delta_K = \left\{ K' \in \mathcal{T}_h : \overline{K'} \cap \overline{K} \neq \emptyset \right\}$, e.g., all the elements of the triangulation $\mathcal{T}_h$ that share an edge ($d = 2$) or vertex ($d = 3$) with K.*

For ease of notation, we set $w(t) := w(t, \cdot)$. We define the DG norm as follows:

$$|||w(t)|||^2_{\mathrm{DG}(\epsilon)} = ||w(t)||^2_{L^2(G)} + \int_0^t ||w(s)||^2_{\mathrm{DG}(\epsilon)} ds \quad \forall t \in (0, T).$$

Moreover, we assume that the following condition holds.

Assumption 5.12.

$$\boldsymbol{b}(\boldsymbol{x}) \cdot \nabla_h v_h \in V_h \quad \forall v_h \in V_h. \tag{5.20}$$

This condition will be further discussed in Remark 5.14. We are now able to prove the following result.

Theorem 5.13. *Let u^ϵ and u_h^ϵ be the solutions of (4.15) and (5.3), then $\forall t \in [0, T]$*

$$|||\left(u^\epsilon - u_h^\epsilon\right)(t)|||_{DG(\epsilon)} \leq C(\epsilon) \frac{h^{\min(p+1,s)-1}}{p^{s-\frac{3}{2}}} \left(||u^\epsilon(t)||_{H^s(G,\mathcal{T}_h)} + ||u^\epsilon||_{H^1((0,t);\, H^s(G,\mathcal{T}_h))} \right). \tag{5.21}$$

Proof. Since the scheme is consistent, the DG formulation (5.3) satisfies the orthogonality property

$$\forall t \in (0, T),\ \forall v \in V_h \qquad \int_G \partial_t (u^\epsilon - u_h^\epsilon) v d\boldsymbol{x} + \mathcal{A}^\epsilon_{DG}(u^\epsilon - u_h^\epsilon, v) = 0.$$

Let us consider a suitable projection Π_p onto the space of discontinuous piecewise polynomial functions such that

$$\forall K \in \mathcal{T}_h \quad \left(\Pi_p v\right)|_K = \Pi_{(p,K)}(v|_K).$$

As in [18] we use the L^2 orthogonal projector, i.e., given $w \in L^2(\Omega)$, $(w - \Pi_p w, v_h) = 0 \ \ \forall v_h \in V_h$. We set

$$\eta := u^\epsilon - \Pi_p u^\epsilon \in V \text{ and } \xi := u_h^\epsilon - \Pi_p u^\epsilon \in V_h, \tag{5.22}$$

where obviously $\xi \in V_h$ holds. Using the Galerkin orthogonality and the equality $u^\epsilon - u_h^\epsilon = \eta - \xi$, we obtain

$$\int_G \partial_t \xi \, v d\boldsymbol{x} + \mathcal{A}^\epsilon_{DG}(\xi, v) = \int_G \partial_t \eta v d\boldsymbol{x} + \mathcal{A}^\epsilon_{DG}(\eta, v) \ \ \forall v \in V_h.$$

Thus, setting $v = \xi$ and applying (5.17), we obtain

$$\frac{1}{2}\frac{d}{dt}||\xi||^2_{L^2(G)} + ||\xi||^2_{\mathrm{DG}(\epsilon)} \le \int_G \xi \partial_t \xi d\boldsymbol{x} + \mathcal{A}^\epsilon_{DG}(\xi,\xi) = \int_G \xi \partial_t \eta d\boldsymbol{x} + \mathcal{A}^\epsilon_{DG}(\eta,\xi). \tag{5.23}$$

Let us examine the terms in (5.23) in more detail. For any $\widehat{C_1} > 0$, it holds

$$\begin{aligned}\int_G \partial_t \eta\, \xi d\boldsymbol{x} &\le ||\partial_t\eta||_{L^2(G)}||\xi||_{L^2(G)} \le \frac{\widehat{C_1}}{2}||\xi||^2_{L^2(G)} + \frac{1}{2\widehat{C_1}}||\partial_t\eta||^2_{L^2(G)} \\ &\le \frac{\widehat{C_1}}{2\gamma^\epsilon}||\xi||^2_{\mathrm{DG}(\epsilon)} + \frac{1}{2\widehat{C_1}}||\partial_t\eta||^2_{L^2(G)}.\end{aligned}$$

It follows from [18, Lemma 4.3] and Lemma 5.7 that for any $\widehat{C_2} > 0$

$$d^\epsilon_{DG}(\eta,\xi) \le ||\xi||_{\mathrm{DG}(\epsilon)}\sqrt{\delta_{DG}(\eta)} \le \frac{\widehat{C_2}}{2}||\xi||^2_{\mathrm{DG}(\epsilon)} + \frac{1}{2\widehat{C_2}}\delta_{DG}(\eta) \tag{5.24}$$

with

$$\delta_{DG}(\eta) := ||\sqrt{\alpha_J}[\eta]||^2_{L^2(\Gamma_h)} + \sum_{K\in\mathcal{T}_h} |\eta|^2_{H^{1,\epsilon}(K)} + \mathcal{C}q_\epsilon p^2 h^{-1}\,||[\eta]||^2_{L^2(\partial K)} + q^2_\epsilon \left|\left|\frac{1}{\sqrt{\alpha_J}}\nabla\eta\right|\right|^2_{L^2(\partial K)},$$

where we recall that α_J is the penalization parameter, i.e., $\alpha_J|_e = \frac{\alpha}{|e|}\ \forall e \in \Gamma_h$ and $q_\epsilon = \left|\sqrt{\frac{1}{2}Q_\epsilon}\right|^2_2$ with $|\cdot|_2$ denoting the matrix norm subordinated to the l^2 vector norm on $\mathbb{R}^d$. Notice that $\delta_{DG}(\eta)$ corresponds to the right hand side of [18, Equation (4.8)]. In fact, considering the notation of [18], $d^\epsilon_{DG}(\eta,\xi)$ is equal to $B_a(\eta,\xi) + B_s(\eta,\xi)$ and (5.24) follows along the lines of the proof of [18, Lemma 4.3].
Using [18, Lemma 3.2], we obtain for any $\widehat{C_3} > 0$

$$t^\epsilon_{DG}(\eta,\xi) + c_{DG}(\eta,\xi) \le ||\xi||_{\mathrm{DG}(\epsilon)}\sqrt{\tau_{DG}(\eta)} \le \frac{\widehat{C_3}}{2}||\xi||^2_{\mathrm{DG}(\epsilon)} + \frac{1}{2\widehat{C_3}}\tau_{DG}(\eta), \tag{5.25}$$

where

$$\tau_{DG}(\eta) := \sum_{K\in\mathcal{T}_h} \gamma_1^2||c^\epsilon_0\eta||^2_{L^2(K)} + 2||\eta_I||^2_{L^2(\partial^\epsilon_+ K\cap\Gamma^0_h)} + 2||\eta_O||^2_{L^2(\partial^\epsilon_- K\setminus\Gamma^0_h)},$$

with $\gamma_1 = \sup_{\boldsymbol{x}\in G}\frac{c(\boldsymbol{x}) - \mathrm{div}(b^\epsilon(\boldsymbol{x}))}{(c^\epsilon_0(\boldsymbol{x}))^2}$.
For any $\widehat{C_4} > 0$, reasoning as in (5.18), it holds

$$j^\epsilon_{DG}(\eta,\xi) \le C_5(\epsilon)\left(\frac{\widehat{C_4}}{2}||\xi||^2_{L^2(G)} + \frac{1}{2\widehat{C_4}}||\eta||^2_{L^2(G)}\right) \le C_5(\epsilon)\left(\frac{\widehat{C_4}}{2\gamma^\epsilon}||\xi||^2_{\mathrm{DG}(\epsilon)} + \frac{1}{2\gamma^\epsilon\widehat{C_4}}||\eta||^2_{\mathrm{DG}(\epsilon)}\right).$$

Choosing positive constants $\widehat{C_1}$, $\widehat{C_2}$, $\widehat{C_3}$ and $\widehat{C_4}$ sufficiently small, i.e., such that

$$\frac{\widehat{C_1}}{2\gamma^\epsilon} + \frac{\widehat{C_2}}{2} + \frac{\widehat{C_3}}{2} + \frac{C_5(\epsilon)\widehat{C_4}}{2\gamma^\epsilon} = \frac{1}{2}, \tag{5.26}$$

the following result holds

$$\frac{1}{2}\frac{d}{dt}||\xi||^2_{L^2(G)} + \frac{1}{2}||\xi||^2_{\mathrm{DG}(\epsilon)} \le C\left(||\partial_t\eta||^2_{L^2(G)} + \delta_{DG}(\eta) + \tau_{DG}(\eta) + ||\eta||^2_{\mathrm{DG}(\epsilon)}\right) =: \hat{\xi}[\eta], \tag{5.27}$$

with

$$C(\epsilon) = \max\left(\frac{1}{2\widehat{C_1}}, \frac{1}{2\widehat{C_2}}, \frac{1}{2\widehat{C_3}}, \frac{C_5(\epsilon)}{2\gamma^\epsilon\widehat{C_4}}\right). \tag{5.28}$$

Thus, integrating (5.27), since all the above constants are time-independent, we obtain

$$|||u^\epsilon - u_h^\epsilon|||^2_{\mathrm{DG}(\epsilon)} \quad \leq \quad |||\xi|||^2_{\mathrm{DG}(\epsilon)} + |||\eta|||^2_{\mathrm{DG}(\epsilon)} \leq \int_0^t \widehat{\xi}[\eta](s)ds + |||\eta|||^2_{\mathrm{DG}(\epsilon)}.$$

Therefore the interpolation error estimates in Lemma 5.9 give the claimed result, since $\forall t \in (0,T)$ it holds

$$\begin{aligned}
\int_0^t \widehat{\xi}[\eta](s)ds + |||\eta|||^2_{\mathrm{DG}(\epsilon)} \quad &\leq \quad C(\epsilon)\int_0^t \Big(\sum_{K\in\mathcal{T}_h} \left(1+\frac{p^2}{h}+\frac{1}{h}\right)\int_{\partial K} \eta^2(\tau,s)ds + \int_K \eta^2(\tau,\boldsymbol{x})d\boldsymbol{x} \\
&+ \quad \int_K |\nabla\eta(\tau,\boldsymbol{x})|^2 d\boldsymbol{x} + h\int_{\partial K} |\nabla\eta(\tau,s)|^2 ds + \int_K |\partial_t\eta(\tau,\boldsymbol{x})|^2 d\boldsymbol{x}\Big) d\tau + ||\eta(t)||^2_{L^2(G)} \\
&\leq \quad C(\epsilon)\frac{h^{2\min(p+1,s)-2}}{p^{2s-3}}\left(\int_0^t ||u^\epsilon(\tau)||^2_{H^s(G,\mathcal{T}_h)} + ||\partial_t u^\epsilon(\tau)||^2_{H^s(G,\mathcal{T}_h)} d\tau + ||\eta(t)||^2_{L^2(G)}\right).
\end{aligned}$$

□

Remark 5.14. *Inequality (5.25) depends on Condition (5.20). In [18, Remark 3.13], the authors comment on this condition: if it is violated, then the presented analysis yields an error bound that is still optimal with respect to h but is p-suboptimal. A possible remedy is to supplement the definition of the scheme with a streamline-diffusion term : this restores the hp optimality. However, our numerical results indicate that the DG scheme is hp-optimal even if Assumption 5.12 is violated and no streamline-diffusion stabilization term is added. Assumption 5.12 has been removed in [15, Remark 5.9], replacing* $\boldsymbol{b}$ *by a suitable projection on the space of discontinuous piecewise polynomial functions.*

Remark 5.15. *We choose the stabilization parameter independent of* p*, i.e.,* $\alpha_J|_e = \alpha|e|^{-1}$ *for any* $e \in \Gamma_h$*, with* α *independent of* h *and* p*. From the above proof it is clear that setting* $\alpha_J|_e = \alpha p|e|^{-1}$ *does not affect the hp-convergence order of the error estimate.*

Remark 5.16. *Since* $||k_\epsilon||_\infty \lesssim \epsilon^{-(\overline{\rho}+d)}$ *holds for sufficiently small* ϵ*, and thus* $C_5(\epsilon) \lesssim |G|\epsilon^{-\overline{\rho}-d}$*, we obtain from condition (5.26)* $\widehat{C}_4 \gtrsim \epsilon^{\overline{\rho}+d}$*. Therefore constant* $C(\epsilon)$ *in (5.28) satisfies* $C(\epsilon) \lesssim \left(\epsilon^{\overline{\rho}+d}\right)^{-2}$*, and thus the constant* $C(\epsilon)$ *in (5.21), i.e., in the* a priori *error estimate, satisfies* $C(\epsilon) \lesssim \epsilon^{-(\overline{\rho}+d)}$ *as* $\epsilon \downarrow 0$*.*

Remark 5.17. *The norm* $||\cdot||_{DG(\epsilon)}$*, and thus norm* $|||\cdot|||_{DG(\epsilon)}$*, depend explicitly on* ϵ*. In fact, if* $\epsilon \to 0$ *(and thus* $Q_\epsilon \to 0$*), the* H^1*-part of the norm* $||\cdot||_{DG(\epsilon)}$ *tends to zero. Thus the considered norm becomes weaker. Moreover, as stated in Remark 5.6, when* $\epsilon \to 0$ *we lose control of the jump term.*
If we consider error estimates in the following norms:

$$||u||^2_{DG} := \sum_{K\in\mathcal{T}_h}\left(|u|^2_{H^1(K)} + ||c_0^\epsilon u||^2_{L^2(K)} + \frac{1}{2}||[u]||^2_{L^2(\partial_-^\epsilon K)} + \frac{1}{2}||[u]||^2_{L^2(\partial_+^\epsilon K\cap\Gamma_h^0)}\right) + \sum_{e\in\Gamma_h}\frac{\alpha}{|e|}||[u]||^2_{L^2(e)}, \quad (5.29)$$

and

$$|||u(t)|||^2_{DG} := ||u(t)||^2_{L^2(G)} + \int_0^t |||u(s)|||^2 ds \quad \forall t \in [0,T],$$

and we assume that $\sigma^\epsilon = \max_{1\leq i,j\leq d}\left|(\sqrt{Q_\epsilon})_{ij}\right| < 1$*, then Theorem 5.13 implies*

$$\begin{aligned}
|||\left(u^\epsilon - u_h^\epsilon\right)(t)|||_{DG} \quad &\leq \quad \frac{1}{\sigma^\epsilon}|||\left(u^\epsilon - u_h^\epsilon\right)(t)|||_{DG(\epsilon)} \\
&\leq \quad \frac{C}{\sigma^\epsilon}\frac{h^{\min(p+1,s)-1}}{p^{s-\frac{3}{2}}}\left(||u^\epsilon(t)||_{H^s(G,\mathcal{T}_h)} + ||u^\epsilon||_{H^1((0,t);\,H^s(G,\mathcal{T}_h))}\right).
\end{aligned}$$

Remark 5.18. *The diffusion term in (5.3) has been discretized according to the so-called NIPG-DG method (see [33] and Section 5.2 for this terminology). The results stated in this section hold also for the SIPG method, i.e., setting* $\beta = -1$ *in the DG formulation of the diffusion term. In this case, (5.15) does not hold, since we have to deal with the additional term* $-\int_{\Gamma_h}\{\nabla w^\top Q_\epsilon \boldsymbol{n}\}[w]ds$*. However, using the Cauchy-Schwarz inequality (see [33, Section 2.7.1]), we obtain a lower bound for this additional term.*

5.4 Finite variation processes

To approximate problem (2.2) when $\overline{\rho} \geq 1$, we have to consider that the integral operator A'_J in (4.7) is only well-defined for Lipschitz u, because of the singularity of $k(x)$ in $x = 0$. Thus a Discontinuous Galerkin (DG) discretization is not directly applicable. However, due to the small-jumps regularization, we can consider a DG discretization of the regularized problem (4.14).
Note that for processes with finite variation, i.e., $0 < \overline{\rho} < 1$, the small jump regularization is not necessary to obtain a feasible formulation for the application of a DG discretization. In fact, the jump term $\int_{\mathbb{R}^d} (\phi(x+y) - \phi(x))\, k(y)\, dy$ is not pointwise well-defined for a discontinuous basis function ϕ. This is not necessary for the present algorithm, as a Galerkin formulation with the Dirichlet form of the process is applied and therefore existence of the integral in a weaker sense is sufficient.

Theorem 5.19. *Let A_J be as in (2.4) be an operator of order $\boldsymbol{\rho}$ and let $\overline{\rho} < 1$ hold. Then the following estimate can be proved for $\phi, \psi \in V_h$:*

$$a_J(\phi, \psi) = (A_J[\phi], \psi) < C \left\|\widetilde{\phi}\right\|_{H^{\boldsymbol{\rho}/2}(\mathbb{R}^d)} \left\|\widetilde{\psi}\right\|_{H^{\boldsymbol{\rho}/2}(\mathbb{R}^d)} < \infty.$$

Proof. This follows directly from the continuity of the bilinear form and the embedding $V_h \subset \widetilde{H}^{\boldsymbol{\rho}/2}(G)$. □

Remark 5.20. *Note that an analogous estimate can be obtained for the bilinear form $a^G_J(\cdot,\cdot)$ in the $\widetilde{H}^{\boldsymbol{\rho}/2}(G)$.*

The small jump regularization is not necessary when finite variation Lévy processes are considered. Note that this argumentation does not hold if a finite difference discretization is applied. In this case a pointwise definition of the jump term is necessary and a regularization has to be performed in any case, cf. [12].

5.4.1 DG Formulation

The variational form (5.3) when $\epsilon = 0$, i.e., when no small jump approximation is considered, reads as follows. Find $u_h(t, \boldsymbol{x}) \in H^1((0,T); V_h)$ such that $\forall\, v_h \in V_h$ it holds

$$\int_G \partial_t u_h(t,\boldsymbol{x}) v_h(\boldsymbol{x}) dx + \mathcal{A}_{DG}(u_h(t,\boldsymbol{x}), v_h(\boldsymbol{x})) \quad = \quad rhs_{DG}(v_h(\boldsymbol{x})), \tag{5.30}$$

$$u_h(0,\boldsymbol{x}) \quad = \quad \Pi_p P(\boldsymbol{x}), \tag{5.31}$$

where for $v, w \in V_h$

$$\mathcal{A}_{DG}(w,v) := t_{DG}(w,v) + r_{DG}(w,v) + j_{DG}(w,v), \tag{5.32}$$

$$t_{DG}(w,v) := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{b}\cdot\nabla w\, v d\boldsymbol{x} - \int_{\partial_- K} (\boldsymbol{b}\cdot\boldsymbol{n}_K)[w] v_I ds, \tag{5.33}$$

$$r_{DG}(w,v) := \int_G c\, w\, v d\boldsymbol{x},$$

$$j_{DG}(w,v) := -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\widetilde{w}(t,\boldsymbol{y}) - \widetilde{w}(t,\boldsymbol{x})) k(\boldsymbol{y}-\boldsymbol{x}) d\boldsymbol{y}\, \widetilde{v}(\boldsymbol{x}) d\boldsymbol{x}, \tag{5.34}$$

$$rhs_{DG}(v) = \int_G f v_h d\boldsymbol{x} - \sum_{K\in\mathcal{T}_h} \int_{\partial_- K\cap\partial G} (\boldsymbol{b}\cdot\boldsymbol{n}_K) g v_I ds. \tag{5.35}$$

Notice that (5.32), (5.33) and (5.34) corresponds to (5.5), (5.8) and (5.11), respectively, when $\epsilon = 0$.

5.4.2 *A priori* bound and error estimate

The above formulation is consistent, i.e., the following result holds.

Theorem 5.21. *If u is the solution of (4.2), then it satisfies (5.30).*

Proof. The proof follows along the lines of the proof of Theorem 5.5. □

Let us now assume that $\boldsymbol{b}$ and c satisfy

$$(c_0)^2(\boldsymbol{x}) := c(\boldsymbol{x}) - \frac{1}{2}\mathrm{div}(\boldsymbol{b}(\boldsymbol{x})) > \gamma > 0 \tag{5.36}$$

(see Condition (4.1)). Reasoning as in Section 5.3.2, we define the norm $||\cdot||_{\mathrm{DGFV}}$, for suffiently smooth w,

$$||w||^2_{\mathrm{DGFV}} := ||c_0 w||^2_{L^2(G)} + ||w||^2_{\widetilde{H}^{\rho/2}(G)} + \frac{1}{2}\sum_{K\in\mathcal{T}_h} ||[w]||^2_{L^2(\partial_- K)} + ||[w]||^2_{L^2(\partial_+ K\cap\Gamma_h^0)}. \tag{5.37}$$

Since we have $t_{DG}(w,w) + r_{DG}(w,w) = ||w||^2_{\mathrm{DGFV}} - ||v||^2_{\widetilde{H}^{\rho/2}(G)}$ and we assume $j_{DG}(w,w) \geq C_G^+||u||^2_{\widetilde{H}^{\rho/2}(G)}$, i.e., $C_G^- = 0$ in (4.8), $A_{DG}(w,w) \geq C||w||_{\mathrm{DGFV}}$ holds, for some $C > 0$ and all w, i.e., the bilinear $A_{DG}(\cdot,\cdot)$ form is coercive.

Remark 5.22. *Note that the coercivity assumption on $j_{DG}(\cdot,\cdot)$, i.e., $C_G^- = 0$ in (4.8), is not restrictive as we may use the following transformation $v(t,\boldsymbol{x}) = e^{-tC_G^-}u(t,\boldsymbol{x})$ to obtain a coercive bilinear form in the equation satisfied by v. See [30, Section 11.1.1] for details.*

The following *a priori* bound and error estimate hold.

Theorem 5.23. *Let u_h be the solution of (5.30), then*

$$||u_h(T,\cdot)||^2_{L^2(G)} + \int_0^T ||u_h(t,\cdot)||^2_{DGFV}dt \leq C\left(||f||^2_{L^2((0,T);\,L^2(G))} + \sum_{e\in\Gamma_h^0}\frac{1}{|e|}||g||^2_{L^2((0,T);\,L^2(e))}\right) + ||\Pi_p P||^2_{L^2(G)}.$$

Proof. The proof follows along the lines of the proof of Theorem 5.8. □

Theorem 5.24. *Let us define the DG norm*

$$|||w(t)|||^2_{DGFV} = ||w(t)||^2_{L^2(G)} + \int_0^t ||w(s)||^2_{DGFV}ds \quad \forall t\in(0,T),$$

where, for ease of notation, we set $w(t) := w(t,\cdot)$, and u and u_h be the solution of (4.2) and (5.30), respectively, then

$$|||\,(u-u_h)\,(t)|||_{DGFV} \leq C\frac{h^{\min(p+1,s)-1/2}}{p^{s-1/2}}\left(||u(t)||_{H^s(G,\mathcal{T}_h)} + ||u||_{H^1((0,t);\,H^s(G,\mathcal{T}_h))}\right) \tag{5.38}$$

$\forall t\in[0,T]$.

Proof. Reasoning as in the proof of Theorem 5.13, we obtain

$$|||u-u_h|||^2_{\mathrm{DGFV}} \quad\leq\quad \int_0^t |\partial_t\eta|^2 + \tau_{DG}(\eta) + ||\eta||^2_{\mathrm{DGFV}}ds$$

and the result follows from interpolation estimates. □

Remark 5.25. *In the case of vanishing reaction and transport terms, we obtain an analogous result to Theorem 5.24, considering the norm $||\cdot||_{DGFV'} := ||\cdot||_{\widetilde{H}^{\rho/2}(G)}$. The following estimate holds in this case*

$$||u-u_h(t)||^2_{L^2(G)} + \int_0^t ||u-u_h(s)||^2_{DGFV'}ds \leq C\frac{h^{\min(p+1,s)-\overline{\rho}/2}}{p^{s-\overline{\rho}/2}}\left(||u(t)||_{H^s(G,\mathcal{T}_h)} + ||u||_{H^1((0,t);\,H^s(G,\mathcal{T}_h))}\right).$$

6 Implementational Aspects

In the following we discuss in more detail some implementational issues for the DG finite element methods for the Lévy generator (2.4). We consider the one dimensional bilinear form:

$$
\begin{aligned}
a_J(u,v) &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\widetilde{u}(x+y)-\widetilde{u}(x))k(y)\widetilde{v}(x)\,dx\,dy \\
&= \int_{\mathbb{R}}\sum_{K\in\mathcal{T}_h}\int_K \widetilde{u}'(y)k^{(-1)}(y-x)dy\,\widetilde{v}(x)dx - \int_{\mathbb{R}}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\widetilde{u}_I(s)-\widetilde{u}(x))nk^{(-1)}(s-x)ds\,\widetilde{v}(x)dx \\
&= \sum_{K'\in\mathcal{T}_h}\int_{\partial K'}\sum_{K\in\mathcal{T}_h}\int_K \widetilde{u}'(y)k^{(-2)}(y-s)dy\,\widetilde{v}(s)nds - \sum_{K'\in\mathcal{T}_h}\int_{K'}\sum_{K\in\mathcal{T}_h}\int_K \widetilde{u}'(y)k^{(-2)}(y-x)dy\,\widetilde{v}'(x)dx \\
&\quad - \int_{\mathbb{R}}\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\widetilde{u}_I(s)-\widetilde{u}(x))nk^{(-1)}(s-x)ds\,\widetilde{v}(x)dx,
\end{aligned}
$$

where $k^{(-i)}(x)$ is given in (5.12). Note that the limits $\lim_{\epsilon\uparrow 0} k^{(-2)}(\epsilon)$ and $\lim_{\epsilon\downarrow 0} k^{(-2)}(\epsilon)$ are finite, but might not coincide, if the process exhibits asymmetric tail behavior. On the other hand $\lim_{\epsilon\to 0} k^{(-1)}(\epsilon)$ does not necessarily exist. Hence, the application of a standard quadrature rule, e.g., Gauss quadrature, is not feasible. We therefore employ a composite Gauss quadrature rule to compute the arising integrals, c.f. [9] and [36] for more details.
We remark that most of the computation time is used for the assembly of the system matrices, which can be trivially parallelized, as the matrix entries are independent of each other.
For the error analysis we have to evaluate fractional Sobolev norms, this was done by a representation of the corresponding function in a Riesz basis of the corresponding Sobolev space.

7 Numerical Examples

In the following we present numerical examples in one space dimension confirming the analytical results obtained in the previous sections. The Lévy measure we use in the following is a CGMY jump measure.

Example 7.1. *We consider the* tempered stable process *(for $c = c_+ = c_-$ also called* CGMY *process in [7] or* KoBoL *in [5]) which has a Lévy density of the form*

$$\nu(dz) = \left(c_+\frac{e^{-\beta_+|z|}}{|z|^{1+\alpha}}1_{\{z>0\}} + c_-\frac{e^{-\beta_-|z|}}{|z|^{1+\alpha}}1_{\{z<0\}}\right)dz, \tag{7.1}$$

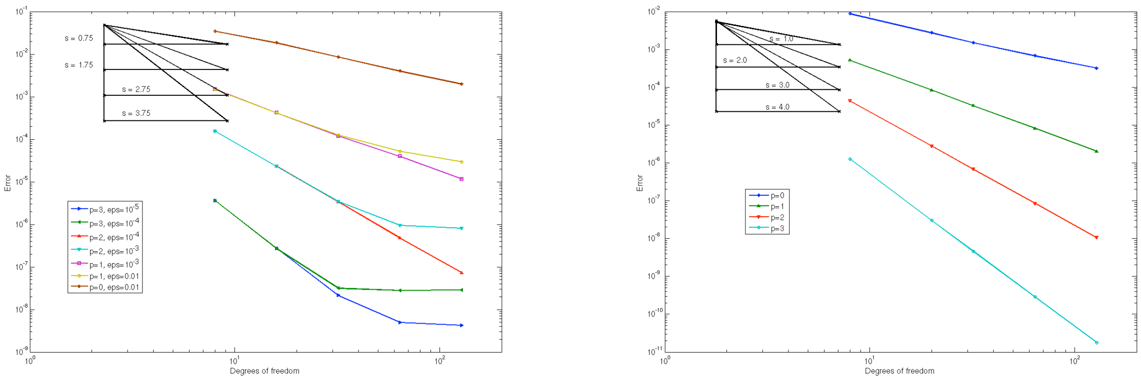
for $c_+, c_-, \beta_+, \beta_- > 0$ and $0 \le \alpha < 2$.

Remark 7.2. *Note that we obtain a finite variation process for $\rho = \alpha < 1$ and an infinite variation process for $\rho = \alpha \ge 1$, with ρ as in Assumption 4.1.*

Different choices for the drift $b(x)$ will be considered in the following test problem:

$$\mathcal{A}u = f \quad \text{for } x \in G = (0,1), \qquad u = 0 \quad \text{for } x \in \partial G = \{0,1\},$$

where $\mathcal{A}$ will be the corresponding Lévy operator and we choose f such that $u = x^2(x-1)^2$.
Let $\mathcal{A}$ be the pure jump operator with a CGMY jump density and parameters given as $\alpha \in \{0, 0.5\}$, $\beta_- = \beta_+ = 5$, $C = 1$. The convergence rates can be observed in Figure 1, where the error has been measured in the $\widetilde{H}^{\alpha/2}((0,1))$-norm. Note that in Figure 1.(a) we additionally employ the small jump approximation as described in Section 3 and approximate the Lévy process Y by Y^ϵ. The figure supports the theoretical results (see Remark 5.25) and it can be observed that the truncation error dominates the discretization error for fine discretization levels and large truncation parameters. In Figure 2 we consider the same equation with a drift term and observe the convergence rates in the DG-norm $||\cdot||_{\mathrm{DGFV}}$, defined in (5.37), when artificial diffusion is not considered. We choose $b(x) = 20 - 20x$. The results are analogous to the driftless case and confirm



(a) CGMY model with $\alpha = 0.5$ and small jump truncation. $\widetilde{H}^{0.25}$-error.

(b) CGMY model with $\alpha = 0$ (Variance Gamma). L^2-error

Figure 1: Convergence rates for different Lévy measures

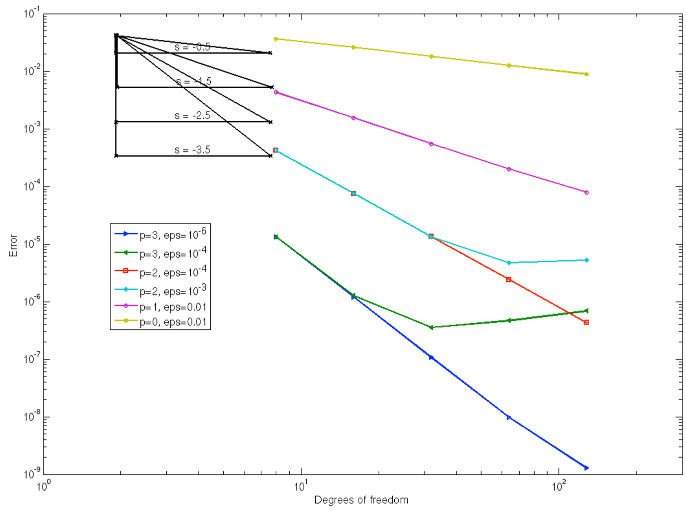


Figure 2: Convergence rates for CGMY jump measure with $\alpha = 0.5$ (drift dominance). DG-error.

the error estimates of Theorem 5.24 when $\epsilon \equiv 0$, i.e., the truncation of the small jumps does not affect the solution. In Figure 3 we consider the same problem adding artificial diffusion. The convergence rates in the DG-norm $||\cdot||_{\mathrm{DG}(\epsilon)}$, defined in (5.16), obtained numerically agree with the theoretical results of Theorem 5.13.

Remark 7.3. *We show the order of convergence of the time-independent problem in Figures 1-3, since for this case an exact solution is known. Theorems 5.13 and 5.24 present* a priori *error estimates for the time-dependent case in the $|||\cdot|||_{DG(\epsilon)}$ and $|||\cdot|||_{DGFV}$ norm, respectively. However, the $||\cdot||_{DG(\epsilon)}$ ($||\cdot||_{DGFV}$) error for the time-independent problem has the same order of convergence of the $|||\cdot|||_{DG(\epsilon)}$ ($|||\cdot|||_{DGFV}$) error of the corresponding time-dependent problem. This can be easily shown along the lines of the proofs of Theorems 5.13 and 5.24.*

Now we consider the dependence of the solution on the regularization parameter ϵ. In the driftless case we observe the behavior presented in Figure 4, which confirms the results of Theorem 3.7 and Remark 3.8. We either only truncate the jump measure on the interval $(-\epsilon, \epsilon)$ or add an appropriately scaled diffusion as

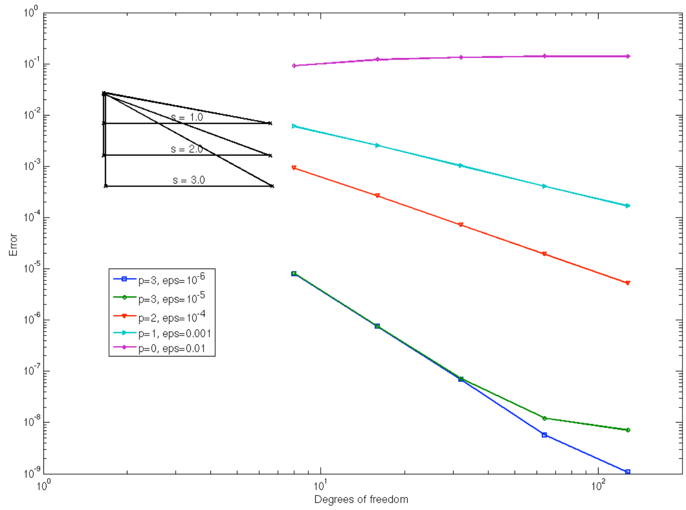
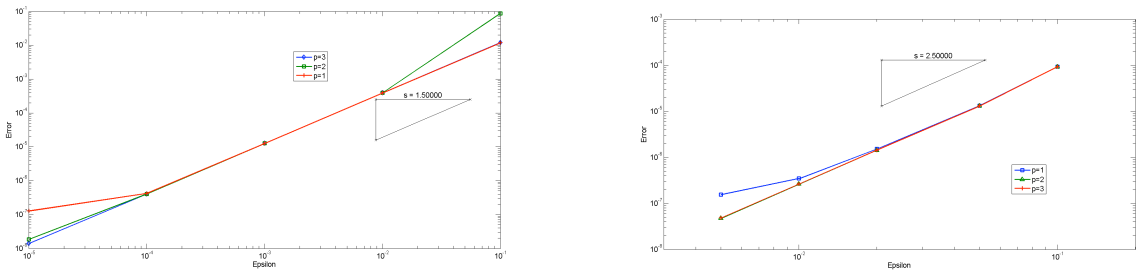


Figure 3: Convergence rates for CGMY jump measure with $\alpha = 0.5$ and artificial diffusion (drift dominance). DG-error.

described in Theorem 3.1. Note that in order to observe a convergence behavior in ϵ, we have to choose a sufficiently fine discretization, such that the discretization error is negligible in comparison with the truncation error. For the more general case we have to refer to the result in Theorem 3.11. We consider the same Lévy



(a) CGMY model with small jump truncation without artificial diffusion.

(b) CGMY model with small jump truncation with artificial diffusion.

Figure 4: Convergence rates in ϵ for CGMY jump measure with $\alpha = 0.5$. L^2-error.

kernel as above with the drift component $b(x) = 20 - 20x$. The numerical results are depicted in Figure 5 and confirm the estimate in Theorem 3.11. The results suggest that the estimates are optimal.

Finally we present a parabolic test case. We consider a pure transport operator with drift $b(x) = 10 - 10x$ and a Lévy operator with the same drift and the Lévy kernel chosen as above (Figure 6).

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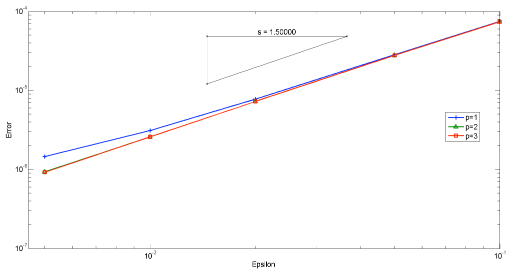
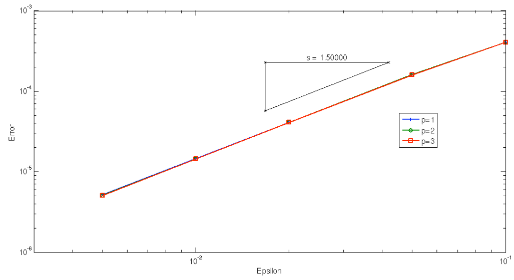


(a) CGMY model with small jump truncation without artificial diffusion.



(b) CGMY model with small jump truncation with artificial diffusion.

Figure 5: Convergence rates in ϵ for CGMY jump measure with $\alpha = 0.5$ (drift dominance). L^2-error.

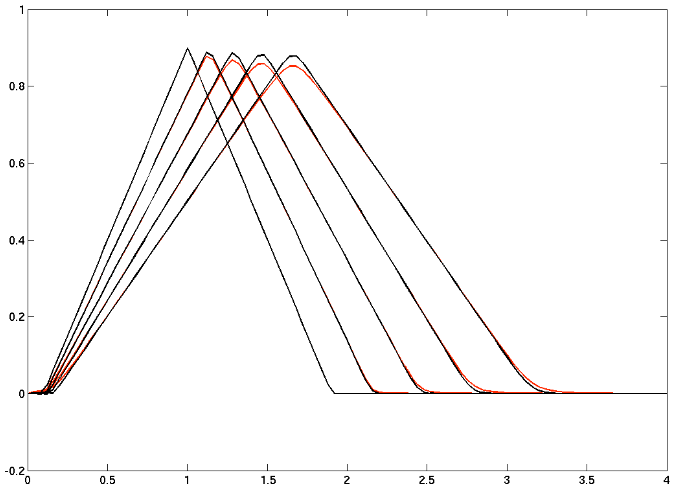


Figure 6: Parabolic test problem. Transport operator (black line). Transport dominated Lévy operator (red line).

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