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SPARSE DETERMINISTIC APPROXIMATION OF BAYESIAN INVERSE PROBLEMS

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Abstract. We present a parametric deterministic formulation of Bayesian inverse problems with input parameter from infinite dimensional, separable Banach spaces. In this formulation, the forward problems are parametric, deterministic elliptic partial differential equations, and the inverse problem is to determine the unknown, parametric deterministic coefficients from noisy observations comprising linear functionals of the solution.

We prove a generalized polynomial chaos representation of the posterior density with respect to the prior measure, given noisy observational data. We analyze the sparsity of the posterior density in terms of the summability of the input data's coefficient sequence. To this end, we estimate the fluctuations in the prior. We exhibit sufficient conditions on the prior model in order for approximations of the posterior density to converge at a given algebraic rate, in terms of the number N of unknowns appearing in the parameteric representation of the prior measure. Similar sparsity and approximation results are also exhibited for the solution and covariance of the elliptic partial differential equation under the posterior. These results then form the basis for efficient uncertainty quantification, in the presence of data with noise.

1. Introduction

Quantification of the uncertainty in predictions made by physical models, resulting from uncertainty in the input parameters to those models, is of increasing importance in many areas of science and engineering. Considerable effort has been devoted to developing numerical methods for this task. The most straightforward approach is sampling uncertain system responses by Monte Carlo simulations. These have the advantage of being conceptually straightforward, but are constrained in terms of efficiency by their $N^{-\frac{1}{2}}$ rate of convergence (N number of samples). In the 1980s the engineering community started to develop new approaches to the problem via parametric representation of the probability space for the input parameters [12, 13] based on the pioneering ideas of Wiener [16]. The use of sparse spectral approximation techniques [15, 7] opens the avenue towards algorithms for computational quantification of uncertainty which beat the asymptotic complexity of Monte Carlo (MC) methods, as measured by computational cost per unit error in predicted uncertainty.

Most of the work in this area has been confined to the use of probability models on the input parameters which are very simple, albeit leading to high dimensional parametric representations. Typically the randomness is described by a (possibly countably infinite) set of independent random variables representing uncertain coefficients in parametric expansions of input data, typically with known closed

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form Lebesgue densities. In many applications, such uncertainty in parameters is compensated for by (possibly noisy) observations, leading to an inverse problem. One approach to such inverse problems is via the techniques of optimal control [2]; however this does not lead naturally to quantification of uncertainty. A Bayesian approach to the inverse problem [10, 14] allows the observations to map a possibly simple prior probability distribution on the input parameters into a posterior distribution. This posterior distribution is typically much more complicated than the prior, involving many correlations and without a useable closed form. The posterior distribution completely quantifies the uncertainty in the system's response, under given prior and structural assumptions on the system and given observational data. It allows, in particular, the Bayesian statistical estimation of unknown system parameters and responses by integration with respect to the posterior measure, which is of interest in many applications.

Monte Carlo Markov chain (MCMC) methods can be used to probe this posterior probability distribution. This allows for computation of estimates of uncertain system responses conditioned on given observation data by means of approximate integration. However, these methods suffer from the same limits on computational complexity as straightforward Monte Carlo methods. It is hence of interest to investigate whether sparse approximation techniques can be used to approximate the posterior density and conditional expectations given the data. Our objective is to study this question in the context of a model elliptic inverse problem. Elliptic problems with random coefficients have provided an important class of model problems for the uncertainty quantification community, see, e.g., [4] and the references therein. In the context of inverse problems and noisy observational data, the corresponding elliptic problem arises naturally in the study of groundwater flow (e.g. [11]) where hydrologists wish to determine the transmissivity (diffusion coefficient) from the head (solution of the elliptic PDE) and hence provides a natural model problem in which to study sparse representations of the posterior distribution.

In Section 2 we recall the Bayesian setting for inverse problems from [14], stating and proving an infinite dimensional Bayes rule adapted to our inverse problem setting in Theorem 2.1. Section 3 formulates the forward and inverse elliptic problem of interest, culminating in an application of Bayes rule in Theorem 3.4. The prior model is built on the work in [3, 5] in which the diffusion coefficient is represented parametrically via an infinite sum of functions, each with an independent uniformly distributed and compactly supported random variable as coefficient. Once we have shown that the posterior measure is well-defined and absolutely continuous with respect to the prior, we proceed to study the analytic dependence of the posterior density in Section 4, culminating in Theorems 4.2 and 4.7. In Section 5 we show how this parametric representation, and analyticity, may be employed to develop sparse polynomial chaos representations of the posterior density, and the key Theorem 5.8 summarizes the achievable rates of convergence. In Section 6 we study a variety of practical issues that arise in attempting to exploit the sparse polynomial representations as realizable algorithms for the evaluation of (posterior) expectations. Throughout we concentrate on the posterior density itself. However we also provide analysis related to the analyticity (and hence sparse polynomial representation) of various functions of the unknown input, in particular the solution to the forward elliptic problem, and tensor products of this function. For the above class of elliptic model problems, we prove that for given data, there exist sparse,

N-term gpc ("generalized polynomial chaos") approximations of this expecation with respect to the posterior (which is written as a density reweighted expectation with respect to the prior) which converge at the same rates afforded by best N-term gpc approximations of the system response to uncertain, parametric inputs. Moreover, our analysis implies that the set Λ_N of the N "active" gpc-coefficients is identical to the set Λ_N of indices of a best N-term approximation of the system's response. It was shown in [5, 6] that these rates are, in turn, completely determined by the the decay rates of the input's fluctuation expansions. We thus show that the machinery developed to describe gpc approximations of uncertain system response may be employed to study the more involved Bayesian inverse problem where the uncertainty is conditioned on observational data. Numerical algorithms which achieve the optimal complexity implied by the sparse approximations, and numerical results demonstrating this will be given in our forthcoming work [1].

2. Bayesian Inverse Problems

Let $G : X \to R$ denote a "forward" map from some separable Banach space X of unknown parameters into another separable Banach space R of responses. We equip X and R with norms $\|\cdot\|_X$ and with $\|\cdot\|_R$, respectively. In addition, we are given $\mathcal{O}(\cdot) : R \to \mathbb{R}^K$ denoting a bounded linear observation operator on the space R of system responses, which belong to the dual space R^* of the space R of system responses. We assume that the data is finite so that $K < \infty$, and equip $\mathbb{R}^K$ with the Euclidean norm, denoted by $|\cdot|$.

We wish to determine the unknown data $u \in X$ from the noisy observations

$$\delta = \mathcal{O}(G(u)) + \eta \tag{2.1}$$

where $\eta \in \mathbb{R}^K$ represents the noise. We assume that realization of the noise process is not known to us, but that it is a draw from the Gaussian measure $\mathcal{N}(0, \Gamma)$, for some positive (known) covariance operator Γ on $\mathbb{R}^K$. If we define $\mathcal{G} : X \to \mathbb{R}^K$ by $\mathcal{G} = \mathcal{O} \circ G$ then we may write the equation for the observations as

$$\delta = \mathcal{G}(u) + \eta. \tag{2.2}$$

We define the least squares functional (also referred to as "potential" in what follows) $\Phi : X \times \mathbb{R}^K \to \mathbb{R}$ by

$$\Phi(u;\delta) = \frac{1}{2}|\delta - \mathcal{G}(u)|_\Gamma^2 \tag{2.3}$$

where $|\cdot|_\Gamma = |\Gamma^{-\frac{1}{2}} \cdot|$ so that

$$\Phi(u;\delta) = \frac{1}{2}\left((\delta - \mathcal{G}(u))^\top \Gamma^{-1}(\delta - \mathcal{G}(u))\right) .$$

In [14] it is shown that, under appropriate conditions on the forward and observation model $\mathcal{G}$ and the prior measure on u, the posterior distribution on u is absolutely contionuous with respect to the prior with Radon-Nikodym derivative given by an infinite dimensional version of Bayes rule. Posterior uncertainty is then determined by integration of suitably chosen functions against this posterior. At the heart of the deterministic approach proposed and analyzed here lies the *reformulation of the forward problem with stochastic input data as an infinite dimensional, parametric deterministic problem*. We are thus interested in expressing the posterior distribution in terms of a parametric representation of the unknown coefficient function u. To this end we assume that, under the prior distribution, this function admits a *parametric representation* of the form

$$u(x) = \bar{a}(x) + \sum_{j\in\mathbb{J}} y_j \psi_j(x) \tag{2.4}$$

where $y = \{y_j\}_{j\in\mathbb{J}}$ is an i.i.d sequence of real-valued random variables $y_j \sim \mathcal{U}(-1,1)$. Here and throughout, $\mathbb{J}$ denotes a finite or countably infinite index set, i.e. either $\mathbb{J} = \{1,2,...,J\}$ or $\mathbb{J} = \mathbb{N}$. All assertions proved in the present paper hold in either case.

To derive the parametric expression of the prior measure μ_0 on y we denote by

$$U = (-1,1)^{\mathbb{J}}$$

the space of all sequences $(y_j)_{j\in\mathbb{J}}$ of real numbers $y_j \in (-1,1)$. Denoting the sub σ-algebra of Borel subsets on $\mathbb{R}$ which are also subsets of $(-1,1)$ by $\mathcal{B}^1(-1,1)$, the pair

$$(U,\mathcal{B}) = \left((-1,1)^{\mathbb{J}}, \bigotimes_{j\in\mathbb{J}} \mathcal{B}^1(-1,1)\right) \tag{2.5}$$

is a measurable space. We equip $(U,\mathcal{B})$ with the uniform probability measure

$$\mu_0(dy) := \bigotimes_{j\in\mathbb{J}} \frac{dy_j}{2} \tag{2.6}$$

which corresponds to bounded intervals for the possibly countably many uncertain parameters. Since the countable product of probability measures is again a probability measure, $(U,\mathcal{B},\mu_0)$ is a probability space. *We assume in what follows that the prior measure on the uncertain input data, parametrized in the form* (2.4)*, is* $\mu_0(dy)$. We add in passing that unbounded parameter ranges as arise, e.g., in lognormal random diffusion coefficients in models for subsurface flow [11], can be treated by the techniques developed here, at the expense of additional technicalities. We refer to [1] for details as well as for numerical experiments.

Define $\Xi : U \to \mathbb{R}^K$ by

$$\Xi(y) = \mathcal{G}(u)\Big|_{u=\bar{a}+\sum_{j\in\mathbb{J}} y_j\psi_j}. \tag{2.7}$$

In the following we view U as a bounded subset in $\ell^\infty(\mathbb{J})$, the Banach space of bounded sequences, and thereby introduce a notion of continuity in U.

Theorem 2.1. *Assume that $\Xi : \bar{U} \to \mathbb{R}^K$ is bounded and continuous. Then $\mu^\delta(dy)$, the distribution of y given δ, is absolutely continuous with respect to $\mu_0(dy)$. Furthermore, if*

$$\Theta(y) = \exp\bigl(-\Phi(u;\delta)\bigr)\Big|_{u=\bar{a}+\sum_{j\in\mathbb{J}} y_j\psi_j}, \tag{2.8}$$

then

$$\frac{d\mu^\delta}{d\mu_0}(y) = \frac{1}{Z}\Theta(y), \quad \textit{where} \quad Z = \int_U \Theta(y)\mu_0(dy). \tag{2.9}$$

Proof. Let ν_0 denote the probability measure on $U \times \mathbb{R}^K$ defined by $\mu_0(dy)\otimes\pi(d\delta)$, where π is the Gaussian measure $\mathcal{N}(0,\Gamma)$. Now define a second probability measure ν on $U \times \mathbb{R}^K$ as follows. First we specify the distribution of δ given y to be

$\mathcal{N}(\Xi(y),\Gamma)$. Since $\Xi(y):\bar{U}\to\mathbb{R}^K$ is continuous and $\mu_0(U)=1$ we deduce that Ξ is μ_0 measurable. Hence we may complete the definition of ν by specifying that y is distributed according to μ_0. By construction, and ignoring the constant of proportionality which depends only on δ, [1]

$$\frac{d\nu}{d\nu_0}(y,\delta)\propto\Theta(y).$$

From the boundedness of Ξ on $\bar{U}$ we deduce that Θ is bounded from below on $\bar{U}$ by $\theta_0>0$ and hence that

$$Z\geq\int_U\theta_0\mu_0(dy)=\theta_0>0$$

since $\mu_0(U)=1$. Noting that, under ν_0, y and δ are independent, Lemma 5.3 in [8] gives the desired result. □

Our aim is to show conditions under which expectations under the posterior measure μ^δ can be approximated within a given error, whilst incurring a cost which grows more slowly than that of Monte Carlo methods. We assume that we wish to compute the expectation of a function $\phi:X\to S$, for some Banach space S. With ϕ, we associate the parametric mapping

$$\Psi(y)=\exp\big(-\Phi(u;\delta)\big)\phi(u)\Big|_{u=\bar{a}+\sum_{j\in\mathbb{J}}y_j\psi_j}:U\to S\,. \tag{2.10}$$

From Ψ we define

$$Z'=\int_U\Psi(y)\mu_0(dy)\in S \tag{2.11}$$

so that the expectation of interest is given by $Z'/Z\in S$. Thus our aim is to approximate Z' and Z more efficiently than can be achieved by Monte Carlo. Typical choices for ϕ in applications might be $\phi(u)=G(u)$, the response of the system, or $\phi(u)=G(u)\otimes G(u)$ which would facilitate computation of the covariance of the response.

In the next sections we will study the elliptic problem and deduce, from known results concerning the parametric forward problem, the joint analyticity of the posterior density $\Theta(y)$, and also $\Psi(y)$, as a function of the parameter vector $y\in U$. From these results, we deduce *sharp estimates on size of domain of analyticity of $\Theta(y)$ (and $\Psi(y)$) as a function of each coordinate y_j, $j\in\mathbb{N}$.*

3. Model Parametric Elliptic Problem

3.1. **Function Spaces.** Our aim is to study the inverse problem of determining the diffusion coefficient u of an elliptic PDE from observation of a finite set of noisy linear functionals of the solution p, given u.

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d=1,2$ or 3, with Lipschitz boundary ∂D. Let further $\Big(H,(\cdot,\cdot),\|\cdot\|\Big)$ denote the Hilbert space $L^2(D)$ which we will identify throughout with its dual space, i.e. $H\simeq H^*$.

We define also the space V of variational solutions of the forward problem: specifically, we let $\Big(V,(\nabla\cdot,\nabla\cdot),\|\cdot\|_V\Big)$ denote the Hilbert space $H^1_0(D)$ (everything that follows will hold for rather general, elliptic problems with affine parameter

[1] $\Theta(y)$ is also a function of δ but that we suppress this for economy of notation.

dependence and "energy" space V). The dual space V^* of all continuous, linear functionals on V is isomorphic to the Banach space $H^{-1}(D)$ which we equip with the dual norm to V, denoted $\|\cdot\|_{-1}$. We shall assume for the (deterministic) data $f \in V^*$.

3.2. **Forward Problem.** In the bounded Lipschitz domain D, we consider the following elliptic PDE:

$$-\nabla \cdot \big(u \nabla p\big) = f \quad \text{in} \quad D, \qquad p = 0 \quad \text{in} \quad \partial D. \tag{3.1}$$

Given data $u \in L^\infty(D)$, a weak solution of (3.1) for any $f \in V^*$ is a function $p \in V$ which satisfies

$$\int_D u(x) \nabla p(x) \cdot \nabla q(x) dx =_V \langle v, f \rangle_{V^*} \;\text{ for all }\; q \in V\;. \tag{3.2}$$

For the well-posedness of the forward problem, we shall work under

Assumption 3.1. *There exist constants* $0 < a_{\text{MIN}} \leq a_{\text{MAX}} < \infty$ *so that*

$$0 < a_{\text{MIN}} \leq u(x) \leq a_{\text{MAX}} < \infty, \quad x \in D, \tag{3.3}$$

Under Assumption 3.1, the Lax-Milgram Lemma ensures the existence and uniqueness of the response p of (3.2). Thus, in the notation of the previous section, $R = V$ and $G(u) = p$. Moreover, this variational solution satisfies the a-priori estimate

$$\|G(u)\|_V = \|p\|_V \leq \frac{\|f\|_{V^*}}{a_{\text{MIN}}}\;. \tag{3.4}$$

We assume that the observation function $\mathcal{O} : V \to \mathbb{R}^K$ comprises K linear functionals $o_k \in V^*$, $k = 1, \ldots, K$. In the notation of the previous section, we denote by $X = L^\infty(D)$ the Banach space in which the unknown input parameter u takes values. It follows that

$$|\mathcal{G}(u)| \leq \frac{\|f\|_{V^*}}{a_{\text{MIN}}} \Big(\sum_{k=1}^K \|o_k\|_{V^*}^2 \Big)^{\frac{1}{2}}\;. \tag{3.5}$$

As mentioned in the previous section, we are not only interested in the posterior density Θ itself, but also in certain functionals $\phi(\cdot) : X \mapsto S$ of the system's response $p = G(u)$. For our subsequent error analysis of polynomial chaos approximations of conditional expectations of these functionals, analyticity of the function $\Psi(y)$ defined in (2.10) will be needed. Rather than formulating results for the most general functionals $\phi(\cdot)$ we confine ourselves to p and its m point correlation

$$\phi(u) := (p(u))^{(m)} := \underbrace{p(u) \otimes ... \otimes p(u)}_{m \text{ times}} \in S = V^{(m)} := \underbrace{V \otimes ... \otimes V}_{m \text{ times}}\;. \tag{3.6}$$

Indeed in some cases we consider only $p = 1$, for brevity.

3.3. **Structural Assumptions on Diffusion Coefficient.** As discussed in section 2 we introduce a parametric representation of the random input parameter u via an affine representation with respect to y, which means that the parameters y_j are the coefficients of the function u in the formal series expansion

$$u(x, y) = \bar{a}(x) + \sum_{j \in \mathbb{J}} y_j \psi_j(x), \quad x \in D, \tag{3.7}$$

where $\bar{a} \in L^\infty(D)$ and $\{\psi_j\}_{j\in\mathbb{J}} \subset L^\infty(D)$. We are interested in the effect of approximating the solutions input parameter $u(x,y)$, by truncation of the series expansion (3.7) in the case $\mathbb{J} = \mathbb{N}$, and on the corresponding effect on the forward (resp. observational) map $G(u(\cdot))$ (resp. $\mathcal{G}(u(\cdot))$) to the family of elliptic equations with the above input parameters. In the decomposition (3.7), we have the choice to either normalize the basis (e.g., assume they all have norm one in some space) or to normalize the parameters. It is more convenient for us to do the latter. This leads us to the following assumptions which shall be made throughout:

i) For all $j \in \mathbb{N} : \psi_j \in L^\infty(D)$ and $\psi_j(x)$ is defined for all $x \in D$,

ii)

$$y = (y_1, y_2, \dots) \in U = [-1,1]^{\mathbb{J}}, \tag{3.8}$$

i.e. the parameter vector y in (3.7) belongs to the unit ball of the sequence space $\ell^\infty(\mathbb{J})$,

iii) for each $u(x,y)$ to be considered, (3.7) holds for every $x \in D$ and every $y \in U$.

We will, on occasion, use (3.7) with $\mathbb{J} \subset \mathbb{N}$, as well as with $\mathbb{J} = \mathbb{N}$. We will work throughout under the assumption that the ellipticity condition (3.3) holds uniformly for $y \in U$.

Uniform Ellipticity Assumption: *there exist* $0 < a_{\text{MIN}} \le a_{\text{MAX}} < \infty$ *such that for all* $x \in D$ *and for all* $y \in U$

$$0 < a_{\text{MIN}} \le u(x,y) \le a_{\text{MAX}} < \infty. \tag{3.9}$$

We refer to assumption (3.9) as $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$ in the following. In particular, $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$ implies $a_{\text{MIN}} \le \bar{a}(x) \le a_{\text{MAX}}$ for all $x \in D$, since we can choose $y_j = 0$ for all $j \in \mathbb{N}$. Also observe that the validity of the lower and upper inequality in (3.9) for all $y \in U$ are respectively equivalent to the conditions that

$$\sum_{j\in\mathbb{J}} |\psi_j(x)| \le \bar{a}(x) - a_{\text{MIN}}, \quad x \in D, \tag{3.10}$$

and

$$\sum_{j\in\mathbb{J}} |\psi_j(x)| \le a_{\text{MAX}} - \bar{a}(x), \quad x \in D. \tag{3.11}$$

We shall require in what follows a quantitative control of the relative size of the fluctuations in the representation (3.7). To this end, we shall impose

Assumption 3.2. *The functions* $\bar{a}$ *and* ψ_j *in* (3.7) *satisfy*

$$\sum_{j\in\mathbb{J}} \|\psi_j\|_{L^\infty(D)} \le \frac{\kappa}{1+\kappa}\overline{a}_{\text{MIN}},$$

with $\overline{a}_{\text{MIN}} = \min_{x\in D} \bar{a}(x) > 0$ *and* $\kappa > 0$.

Assumption 3.1 is then satisfied by choosing

$$a_{\text{MIN}} := \overline{a}_{\text{MIN}} - \frac{\kappa}{1+\kappa}\overline{a}_{\text{MIN}} = \frac{1}{1+\kappa}\overline{a}_{\text{MIN}}. \tag{3.12}$$

3.4. **Inverse Problem.** We start by proving that the forward maps $G : X \to V$ and $\mathcal{G} : X \to \mathbb{R}^K$ are Lipschitz.

Lemma 3.3. *If p and $\tilde{p}$ are solutions of* (3.2) *with the same right hand side f and with coefficients u and $\tilde{u}$, respectively, and if these coefficients both satisfy the assumption* (3.3), *then the forward solution map $u \to p = G(u) \in \mathrm{Lip}(X, V)$ and it satisfies*

$$\|p - \tilde{p}\|_V \le \frac{\|f\|_{V^*}}{a_{\text{MIN}}^2} \|u - \tilde{u}\|_{L^\infty(D)}. \tag{3.13}$$

Moreover the forward solution map can be composed with the observation operator to obtain $u \to \mathcal{G}(u) \in \mathrm{Lip}(X, \mathbb{R}^K)$ with Lipschitz dependence on u, i.e.

$$|\mathcal{G}(u) - \mathcal{G}(\tilde{u})| \le \frac{\|f\|_{V^*}}{a_{\text{MIN}}^2} \Big(\sum_{k=1}^K \|o_k\|_{V^*}^2\Big)^{\frac{1}{2}} \|u - \tilde{u}\|_{L^\infty(D)}. \tag{3.14}$$

Proof: Subtracting the variational formulations for p and $\tilde{p}$, we find that for all $q \in V$,

$$0 = \int_D u\nabla p\cdot\nabla q dx - \int_D \tilde{u}\nabla\tilde{p}\cdot\nabla q dx = \int_D u(\nabla p - \nabla\tilde{p})\cdot\nabla q dx + \int_D (u - \tilde{u})\nabla\tilde{p}\cdot\nabla q dx.$$

Therefore $w = p - \tilde{p}$ is the solution of $\int_D u\nabla w \cdot \nabla q = L(q)$ where $L(v) := \int_D (u - \tilde{u})\nabla\tilde{p}\cdot\nabla v$. Hence

$$\|w\|_V \le \frac{\|L\|_{V^*}}{a_{\text{MIN}}},$$

and we obtain (3.13) since it follows from (3.4) that

$$\|L\|_{V^*} = \max_{\|v\|_V = 1} |L(v)| \le \|u - \tilde{u}\|_{L^\infty(D)} \|\tilde{p}\|_V \le \|u - \tilde{u}\|_{L^\infty(D)} \frac{\|f\|_{V^*}}{a_{\text{MIN}}}.$$

Lipschitz continuity of $\mathcal{G} = \mathcal{O} \circ G : X \to \mathbb{R}^K$ is immediate since $\mathcal{O}$ comprises the K linear functionals o_k. Thus (3.13) implies (3.14). □

The next result may be deduced in a straightforward fashion from the preceding analysis:

Theorem 3.4. *Under the* $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$ *and Assumptions 3.2 it follows that the posterior measure $\mu^\delta(dy)$ on y given δ is absolutely continuous with respect to the prior measure $\mu_0(dy)$ with Radon-Nikodym derivative given by* (2.8) *and* (2.9).

Proof. This is a straightforward consequence of Theorem 2.1 provided that we show boundedness and continuity of $\Xi : \bar{U} \to \mathbb{R}^K$ given by (2.7). Boundedness follows from (3.5), together with the boundedness of $\|o_k\|_{V^*}$, under $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$. Let $u, \tilde{u}$ denote two diffusion coefficients generated by two parametric sequences $y, \tilde{y}$ in U. Then, by (3.14) and Assumption 3.2,

$$\begin{aligned} |\Xi(y) - \Xi(\tilde{y})| &\le \frac{\|f\|_{V^*}}{a_{\text{MIN}}^2} \Big(\sum_{k=1}^K \|o_k\|_{V^*}^2\Big)^{\frac{1}{2}} \|u - \tilde{u}\|_{L^\infty(D)} \\ &\le \frac{\|f\|_{V^*}}{a_{\text{MIN}}^2} \Big(\sum_{k=1}^K \|o_k\|_{V^*}^2\Big)^{\frac{1}{2}} \frac{\kappa}{1+\kappa} \overline{a}_{\text{MIN}} \|y - \tilde{y}\|_{\ell^\infty(\mathbb{J})} . \end{aligned}$$

The result follows. □

4. Complex Extension of the Elliptic Problem

As indicated above, one main technical objective will consist in proving analyticity of the posterior density $\Theta(y)$ with respect to the (possibly countably many) parameters $y \in U$ in (3.7) defining the prior, and to obtain bounds on the maximal domains in $\mathbb{C}$ into which $\Theta(y)$ can be continued analytically. Our key ingredients for getting such estimates rely on complex analysis.

It is well-known that the existence theory for the forward problem (3.1) extends to the case where the coefficient function $u(x)$ takes values in $\mathbb{C}$. In this case, the ellipticity assumption (3.3) should be replaced by

$$0 < a_{\text{MIN}} \le \Re(u(x)) \le |\alpha(x)| \le a_{\text{MAX}} < \infty, \quad x \in D. \tag{4.1}$$

and all the above results remain valid with Sobolev spaces understood as spaces of complex valued functions. Throughout what follows, we shall frequently pass to spaces of complex valued functions, without distinguishing these notationally. It will always be clear from the context which coefficient field is implied.

4.1. Notation and Assumptions.

We extend the definition of $u(x, y)$ to $u(x, z)$ for the complex variable $z = (z_j)_{j\ge 1}$ (by using the z_j instead of y_j in the definition of u by (3.7)) where each z_j has modulus less than or equal to 1. Therefore z belongs to the polydisc

$$\mathcal{U} := \bigotimes_{j\in\mathbb{J}} \{z_j \in \mathbb{C} : |z_j| \le 1\} \subset \mathbb{C}^{\mathbb{J}} . \tag{4.2}$$

Note that $\overline{U} \subset \mathcal{U}$. Using (3.10) and (3.11), when the functions $\overline{a}$ and ψ_j are real valued, condition $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$ implies that for all $x \in D$ and $z \in \mathcal{U}$,

$$0 < a_{\text{MIN}} \le \Re(u(x, z)) \le |u(x, z)| \le 2a_{\text{MAX}} , \tag{4.3}$$

and therefore the corresponding solution $p(z)$ is well defined in V for all $z \in \mathcal{U}$ by the Lax-Milgram theorem for sesquilinear forms. More generally, we may consider an expansion of the form,

$$u(x, z) = \overline{a} + \sum_{j\in\mathbb{J}} z_j \psi_j$$

where $\overline{a}$ and ψ_j are complex valued functions and replace $\mathbf{UEA}(a_{\text{MIN}}, a_{\text{MAX}})$ by the following, complex-valued counterpart:

Uniform Ellipticity Assumption in $\mathbb{C}$: *there exist* $0 < a_{\text{MIN}} \le a_{\text{MAX}} < \infty$ *such that for all* $x \in D$ *and all* $z \in \mathcal{U}$

$$0 < a_{\text{MIN}} \le \Re(u(x, z)) \le |u(x, z)| \le a_{\text{MAX}} < \infty. \tag{4.4}$$

We refer to (4.4) as $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$.

4.2. Domains of holomorphy.

$\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$ implies that the forward solution map $z \mapsto p(z)$ is strongly holomorphic as a $V-$valued function which is uniformly bounded in certain domains larger than $\mathcal{U}$. For $0 < r \le 2a_{\text{MAX}} < \infty$ we define the open set

$$\mathcal{A}_r = \{z \in \mathbb{C}^{\mathbb{J}} : r < \Re(u(x, z)) \le |u(x, z)| < 2a_{\text{MAX}} \quad \text{for every} \quad x \in D\} \subset \mathbb{C}^{\mathbb{J}} . \tag{4.5}$$

Under $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$, for every $0 < r < a_{\text{MIN}}$ holds $\mathcal{U} \subset \mathcal{A}_r$.

According to the Lax-Milgram theorem, for every $z \in \mathcal{A}_r$ there exists a unique solution $p(z) \in V$ of the variational problem: given $f \in V^*$, for every $z \in \mathcal{A}_r$, find $p \in V$ such that

$$\alpha(z;p,q) = (f,q) \qquad \forall q \in V \ . \tag{4.6}$$

Here the sesquilinear form $\alpha(z;\cdot,\cdot)$ is defined as

$$\alpha\big(z;p,q\big) = \int_D u(x,z)\nabla p \cdot \overline{\nabla q} dx \quad \forall p,q \in V \ . \tag{4.7}$$

The analytic continuation of the solution $p(y)$ to $\mathcal{A}_r$ is the unique solution $p(z)$ of (4.6) which satisfies the a-priori estimate

$$\sup_{z\in\mathcal{A}_r} \|p(z)\|_V \le \frac{\|f\|_{V^*}}{r}. \tag{4.8}$$

The first step of our analysis is to establish strong holomorphy of the forward solution map $z \mapsto p(z)$ in (4.6) with respect to the countably many variables z_j at any point $z \in \mathcal{A}_r$. This follows from the observation that the function $p(z)$ is the solution to the operator equation $A(z)p(z) = f$, where the operator $A(z) \in \mathcal{L}(V,V^*)$ depends in an affine manner on each variable z_j. To prepare the argument for proving holomorphy of the functionals Φ and Θ appearing in (2.8), (2.10) we give a direct proof.

Using Lemma 3.3 we have proved by means a difference quotient argument given in [6], Lemma 4.1 which follows. This lemma, together with Hartogs' Theorem (see, e.g., [9]) and the separability of V, implies strong holomorphy of $p(z)$ as a V-valued function on $\mathcal{A}_r$, stated as Theorem 4.2 below. The proof of this theorem can also be found in [6]; the result will also be obtained as a corollary of the analyticity results for the functionals Ψ, Θ proved below.

Lemma 4.1. *At any $z \in \mathcal{A}_r$, the function $z \mapsto p(z)$ admits a complex derivative $\partial_{z_j} p(z) \in V$ with respect to each variable z_j. This derivative is the weak solution of the problem: given $z \in \mathcal{A}_r$, find $\partial_{z_j} p(z) \in V$ such that*

$$\alpha(z;\partial_{z_j} p(z),q) = L_0(q) := -\int_D \psi_j \nabla p(z) \cdot \overline{\nabla q} dx \ , \qquad \text{for all } q \in V. \tag{4.9}$$

Theorem 4.2. *Under* $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$ *for any $0 < r < a_{\text{MIN}}$ the solution $p(z) = G(u(z))$ of the parametric forward problem is holomorphic as a V-valued function in $\mathcal{A}_r$ and there holds the apriori estimate* (4.8).

We remark that $\mathcal{A}_r$ also contains certain *polydiscs*: for any sequence $\rho := (\rho_j)_{j\ge1}$ of positive radii we define the polydisc

$$\mathcal{U}_\rho = \bigotimes_{j\in\mathbb{J}} \{z_j \in \mathbb{C} : |z_j| \le \rho_j\} = \{z_j \in \mathbb{C} : z = (z_j)_{j\in\mathbb{J}} \ ; \ \ |z_j| \le \rho_j\} \subset \mathbb{C}^{\mathbb{J}} \ . \tag{4.10}$$

We say that *a sequence $\rho = (\rho_j)_{j\ge1}$ of radii is r-admissible* if and only if for every $x \in D$

$$\sum_{j\in\mathbb{J}} \rho_j |\psi_j(x)| \le \Re(\bar{a}(x)) - r. \tag{4.11}$$

If the sequence ρ is r-admissible, then the polydisc $\mathcal{U}_\rho$ is contained in $\mathcal{A}_r$ since on the one hand for all $z \in \mathcal{U}_\rho$ and for almost every $x \in D$

$$\Re(\bar{a}(x,z)) \geq \Re(\bar{a}(x)) - \sum_{j\in\mathbb{J}} |z_j\psi_j(x)| \geq \Re(\bar{a}(x)) - \sum_{j\in\mathbb{J}} \rho_j|\psi_j(x)| \geq r,$$

and on the other hand, for every $x \in D$

$$|a(x,z)| \leq |\bar{a}(x)| + \sum_{j\in\mathbb{J}} |z_j\psi_j(x)| \leq |\bar{a}(x)| + \Re(\bar{a}(x)) - r \leq 2|\bar{a}(x)| \leq 2a_{\text{MAX}}\ .$$

Here we used $|\bar{a}(x)| \leq a_{\text{MAX}}$ which follows from $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$.

Similar to (3.10), the validity of the lower inequality in (4.4) for all $z \in \mathcal{U}$ is equivalent to the condition that

$$\sum_{j\geq 1} |\psi_j(x)| \leq \Re(\bar{a}(x)) - a_{\text{MIN}}, \quad x \in D. \tag{4.12}$$

This shows that the constant sequence $\rho_j = 1$ is r-admissible for all $0 < r \leq a_{\text{MIN}}$.

Remark 4.1. For $0 < r < a_{\text{MIN}}$ there exist r-admissible sequences such that $\rho_j > 1$ for all $j \geq 1$, i.e. such that the polydisc $\mathcal{U}_\rho$ is strictly larger than $\mathcal{U}$ in every variable. This will be exploited systematically below in the derivation of approximation bounds. □

4.3. **Holomorphy of response functionals.** We next show that, for given data δ, the functionals $\mathcal{G}(\cdot)$, $\Phi(u(\cdot);\delta)$ and $\Theta(\cdot)$ depend holomorphically on the parameter vector $z \in \mathbb{C}^{\mathbb{J}}$, on polydiscs $\mathcal{U}_\rho$ as in (4.10) for suitable r-admissible sequences of semiaxes ρ. Our general strategy for proving this will be analogous to the argument for establishing analyticity of the map $z \mapsto G(u(z))$ as a V-valued functions.

We now extend Theorem 4.2 from the solution of the elliptic PDE to the posterior density, and related quantities required to define expectations under the posterior, culminating in Theorem 4.7 and Corollary 4.8. We achieve this through a sequence of lemmas which we now derive.

The following lemma is simply a complexification of (3.5) and (3.14). It implies bounds on $\mathcal{G}$ and its Lipschitz constant in the covariance weighted norm.

Lemma 4.3. *Under* $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$*, for every* $f \in V^* = H^{-1}(D)$ *and for every* $\mathcal{O}(\cdot) \in (V^*)^* \simeq V \to Y = \mathbb{R}^K$ *holds*

$$|\mathcal{G}(u)| \leq \frac{\|f\|_{V^*}}{a_{\text{MIN}}} \Big(\sum_{k=1}^{K} \|o_k\|^2_{V^*}\Big)^{\frac{1}{2}}\ , \tag{4.13}$$

$$|\mathcal{G}(u) - \mathcal{G}(u)| \leq \frac{\|f\|_{V^*}}{a^2_{\text{MIN}}} \|u_1 - u_2\|_{L^\infty(D)} \Big(\sum_{k=1}^{K} \|o_k\|^2_{V^*}\Big)^{\frac{1}{2}}. \tag{4.14}$$

To be concrete we concentrate in the next lemma on computing the expected value of *the pressure* $p = G(u) \in V$ under the posterior measure. To this end we define Ψ with ψ as in (3.6) with $m = 1$. We start by considering the case of a single parameter.

Lemma 4.4. *Let* $\mathbb{J} = \{1\}$ *and take* $\phi = G : U \to V$*. With* $u(x,y)$ *as in* (2.4)*, under* $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$*, the functions* $\Psi : [-1,1] \to V$ *and* $\Theta : [-1,1] \to \mathbb{R}$ *and the potential* $\Phi(u(x,\cdot);\delta)$ *defined by* (2.10)*,* (2.8) *and* (2.3) *respectively, may be extended to functions which are strongly holomorphic on the strip* $\{y + iz : |y| < r/\kappa\}$ *for any* $r \in (\kappa, 1)$.

Proof. We view H, V and $X = L^\infty(D)$ as Banach spaces over $\mathbb{C}$. We extend the equation (3.7) to complex coefficients $u(x,z) = \mathrm{Re}(\overline{a}(x) + z\psi(x)) = \overline{a}(x) + y\psi(x)$ since $z = y + i\zeta$. Note that $\overline{a} + z\psi$ is holomorphic in z since it is linear. Since $\mathrm{Re}(\overline{a} + z\psi) = \overline{a} + y\psi \geq a_{\text{MIN}}$, if follows that, for all $\zeta = \mathrm{Im}(z)$,

$$\mathrm{Re} \int_D u(x)|\nabla p(x) - \nabla \tilde{p}(x)|^2 dx \geq a_{\text{MIN}} \|p - \tilde{p}\|_V^2.$$

We prove that the mapping Ψ and Θ are holomorphic by studying the properties of $G(\overline{a} + z\psi)$ and $\Phi(\overline{a} + z\psi)$ as functions of $z \in \mathbb{C}$. Let $h \in \mathbb{C}$ with $|h| < \epsilon \ll 1$. We show that

$$\lim_{|h| \to 0} h^{-1}\big(p(z+h) - p(z)\big)$$

exists in V (strong holomorphy). Note first that $\partial_z u = \psi$. Now consider p. We have

$$\frac{1}{h}\big(p(z+h) - p(z)\big) = \frac{1}{h}\Big(G\big(\overline{a} + (z+h)\psi\big) - G\big(\overline{a} + z\psi\big)\Big) =: r\ .$$

By Lemma 3.3 we deduce that

$$\|r\|_V \leq \frac{\|f\|_{H^{-1}(D)}}{a_{\text{MIN}}^2} \|\psi\|_{L^\infty(D)}\ .$$

From this it follows that there is a weakly convergent subsequence in V, as $|h| \to 0$. We proceed to deduce existence of a strong limit. To this end, we introduce the sesquilinear form

$$b(p,q) = \int_D u \nabla p \overline{\nabla} q dx\ .$$

Then

$$b\big(G(u), q\big) = (f, q) \quad \forall q \in V\ .$$

For a coefficient function u as in (3.7), the form $b(\cdot,\cdot)$ is equal to the parametric sesquilinear form $\alpha(z; p, q)$ defined in (4.7).

Note that for $z = \bar{a} + y\psi \in \mathbb{R}$ and for real-valued arguments p and q, the parametric sesquilinear form $\alpha(z; p, q)$ coincides with the bilinear form in (3.2). Accordingly, for every $z \in \mathbb{C}^{\mathbb{J}}$ the unique holomorphic extension of the parametric solution $G(u(\bar{a} + y\psi))$ to complex parameters $z = y + i\zeta$ is the unique variational solution of the parametric problem

$$\alpha\big(z; G(\overline{a} + z\psi), q\big) = (f, q), \quad \forall q \in V. \tag{4.15}$$

Assumption $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$ is readily seen to imply

$$\forall p \in V: \quad \mathrm{Re}\big(\alpha(z; p, p)\big) \geq a_{\text{MIN}} \|p\|_V^2\ .$$

If we choose $\delta \in (\kappa, 1)$ and choose $z = y + i\eta$, we obtain, for all ζ and for $|y| \leq \delta/\kappa$

$$\mathrm{Re}\big(\alpha(z; p, p)\big) \geq \overline{a}_{\text{MIN}}(1 - \delta)\|p\|_V^2. \tag{4.16}$$

From (4.15) we see that for such values of $z = y + i\zeta$

$$\begin{aligned}
0 =& \alpha\Big(z; G\big(\overline{a} + z\psi\big), q\Big) - \alpha\Big(z; G\big(\overline{a} + (z+h)\psi\big), q\Big) \\
&+ \alpha\Big(z; G\big(\overline{a} + (z+h)\psi\big), q\Big) - \alpha\Big(z+h; G\big(\overline{a} + (z+h)\psi\big), q\Big) \\
=& \alpha\Big(z; G\big(\overline{a} + z\psi\big) - G\big(\overline{a} + (z+h)\psi\big), q\Big) \\
&- \int_D h\psi \nabla G\big(\overline{a} + (z+h)\psi\big) \overline{\nabla} q dx.
\end{aligned}$$

Dividing by h we obtain that r satisfies, for all $z = y + i\zeta$ with $|y| \le \delta/\kappa$ and every $\zeta \in \mathbb{R}$

$$\forall q \in V: \qquad \alpha\big(z; r, q\big) + \int_D \psi \nabla G\big(\overline{a} + (z+h)\psi\big)\overline{\nabla q} dx = 0\ . \tag{4.17}$$

The second term we denote by $s(h)$ and note that, by Lemma 3.3,

$$|s(h_1) - s(h_2)| \le \frac{1}{a_{\text{MIN}}^2} \|\psi\|_\infty^2 \|f\|_1 \|q\|_V |h_1 - h_2|\ .$$

If we denote the solution r to equation (4.17) by $r_h(\overline{a}; z)$ then we deduce from the Lipschitz continuity of $s(\cdot)$ that $r_h(\overline{a}; z) \to r_0(\overline{a}; z)$ where

$$\alpha(z; r_0, q) = s(0), \quad \forall q \in V.$$

Hence $r_0 = \partial_z G(\overline{a} + z\psi) \in V$ and we deduce that $G : [-1, 1] \to V$ can be extended to a complex-valued function which is strongly holomorphic on the strip $\{y + i\zeta : |y| < \delta/\kappa,\ \zeta \in \mathbb{R}\}$.

We next study the domain of holomorphy of the analytic continuation of the potential $\Phi(\overline{a} + z\psi; d)$ to parameters $z \in \mathbb{C}$. It suffices to consider $K = 1$ noting that then the unique analytic continuation of the potential Φ is given by

$$\Phi(\overline{a} + z\psi; \delta) = \frac{1}{2\gamma^2}\Big(\delta - \mathcal{G}(\overline{a} + z\psi)\Big)^\top \Big(\delta - \mathcal{G}(\overline{a} + z\psi)\Big). \tag{4.18}$$

The function $z \mapsto \mathcal{G}(\overline{a} + z\psi)$ is holomorphic with the same domain of holomorphy as $G(\overline{a} + z\psi)$. Similarly it follows that the function

$$z \mapsto \Big(\delta - \mathcal{G}(\overline{a} + z\psi)\Big)^\top \Big(\delta - \mathcal{G}(\overline{a} + z\psi)\Big)$$

is holomorphic, with the same domain of holomorphy; this shown by composing the relevant power series expansion. From this we deduce that Θ and Ψ are holomorphic, with the same domain of holomorphy. $\square$

So far we have considered the case $\mathbb{J} = \{1\}$. We now generalize. To this end, we pick an arbitrary $m \in \mathbb{J}$ and write $y = (y^\star, y_m)$ and $z = (z^\star, z_m)$.

Assumption 4.5. *There are constants* $0 < \overline{a}_{\text{MIN}} \le \overline{a}_{\text{MAX}} < \infty$ *and* $\kappa \in (0, 1)$ *such that*

$$0 < \overline{a}_{\text{MIN}} \le \overline{a} \le \overline{a}_{\text{MAX}} < \infty, \quad a.e.\ x \in D, \quad \Big\|\|\psi_j\|_{L^\infty(D)}\Big\|_{\ell^1(\mathbb{J})} < \kappa \overline{a}_{\text{MIN}}\ . \tag{4.19}$$

For $m \in \mathbb{J}$, we write (3.7) in the form

$$u(x; y) = \overline{a}(x) + y_m \psi_m(x) + \sum_{j \in \mathbb{J}\setminus\{m\}} y_j \psi_j(x)\ .$$

From Assumption 4.5 we deduce that there are numbers $\kappa_j \le \kappa$ such that

$$\|\psi_j\|_{L^\infty} < \overline{a}_{\text{MIN}} \kappa_j.$$

Hence we obtain, for every $x \in D$ and every $y \in U$ the lower bound

$$\begin{aligned} u(x, y) &\ge \overline{a}_{\text{MIN}}\Big(1 - \big(\kappa - \kappa_m\big) - \kappa_m\Big) \\ &\ge \overline{a}_{\text{MIN}}\Big(1 - \big(\kappa - \kappa_m\big)\Big)\Big(1 - \frac{\kappa_m}{1 - \big(\kappa - \kappa_m\big)}\Big) \\ &\ge a'_{\text{MIN}}(1 - \kappa'_m) \end{aligned}$$

with $a'_{\text{MIN}} = a_{\text{MIN}}(1-\kappa)$ and $\kappa'_m = \kappa_m\Big(1-\big(\kappa-\kappa_m\big)\Big)^{-1} \in (0,1)$. With this observation we obtain

Lemma 4.6. *Let Assumption 4.5 hold and set $U = [-1,1]^{\mathbb{J}}$ and $\phi = G : U \to V$. Then the functions $\Psi : U \to V$ and $\Theta : U \to \mathbb{R}$, as well as the potential $\Phi(u(x,\cdot);\delta) : U \to \mathbb{R}$ admit unique extensions to strongly holomorphic functions on the product of strips given by*

$$\mathcal{S}_\rho := \bigotimes_{j\in\mathbb{J}} \left\{ y_j + iz_j : |y_j| < \delta_j/\kappa'_j, \quad z_j \in \mathbb{R} \right\} \tag{4.20}$$

for any sequence $\rho = (\rho_j)_{j\in\mathbb{J}}$ with $\rho_j \in (\kappa'_j, 1)$.

Proof. Fixing $y^\star$, we view Ψ and Θ as functions of the single parameter y_m. For each fixed $y^\star$, we extend y_m to a complex variable z_m. The estimates preceding the statement of this lemma, together with Lemma 4.4, show that Ψ and Θ are holomorphic in the strip $\{y_m + iz_m : |y_m| < \delta_m/\kappa'_m\}$ for any $\delta_m \in (\kappa'_m, 1)$. Hartogs' theorem [9] and the fact that in separable Banach spaces (such as V) weak holomorphy equals strong holomorphy extends this result onto the product of strips, $\mathcal{S}$. □

We note that the strip $\mathcal{S}_\rho \subset \mathbb{C}^{\mathbb{J}}$ defined in (4.20) contains in particular the polydisc $\mathcal{U}_\rho$ with $(\rho_j)_{\mathrm{J}\in\mathbb{J}}$ where $\rho_j = \delta_j/\kappa'_j$.

4.4. **Holomorphy and bounds on the posterior density.** So far, we have shown that the responses $G(u)$, $\mathcal{G}(u)$ and the potentials $\Phi(u;\delta)$ depend holomorphically on the coordinates $z \in \mathcal{A}_r \subset \mathbb{C}^{\mathbb{J}}$ in the parametric representation $u = \bar{a} + \sum_{j\in\mathbb{J}} z_j\psi_j$. Now we deduce bounds on the analytic continuation of the posterior density $\Theta(z)$ in (2.8) as a function of the parameters z on the domains of holomorphy. We have

Theorem 4.7. *Under* **UEAC**$(a_{\text{MIN}}, a_{\text{MAX}})$ *for the analytic continuation $\Theta(z)$ of the posterior density to the domains $\mathcal{A}_r$ of holomorphy defined in* (4.5)*, i.e. for*

$$\Theta(z) = \exp\Big(-\Phi(u;\delta)|_{u=\bar{a}+\sum_{j\in\mathbb{J}} z_j\psi_j}\Big) \tag{4.21}$$

there holds for every $0 < r < a_{\text{MIN}}$

$$\sup_{z\in\mathcal{A}_r} |\Theta(z)| = \sup_{z\in\mathcal{A}_r} |\exp(-\Phi(u(z);\delta)| \le \exp\left(\frac{\|f\|^2_{V^*}}{r^2}\sum_{k=1}^K \|o_k\|^2_{V^*}\right). \tag{4.22}$$

These analyticity properties, and resulting bounds, can be extended to functions $\phi(\cdot)$ as defined by (3.6), using Lemma 4.6 and Theorem 4.7. This gives the following result.

Corollary 4.8. *Under* **UEAC**$(a_{\text{MIN}}, a_{\text{MAX}})$*, for any $m \in \mathbb{N}$ the functionals $\phi(u) = p^{(m)} \in S = V^{(m)}$ the posterior densities $\Psi(z) = \Theta(z)\phi(u(z))$ defined in* (2.10) *admit analytic continuations as strongly holomorphic, $V^{(m)}$-valued functions with domains $\mathcal{A}_r$ of holomorphy defined in* (4.5)*. Moreoever, for these functionals the analytic continuations of Ψ in* (2.10) *admit the bounds*

$$\sup_{z\in\mathcal{A}_r} \|\Theta(z)(p(z))^{(m)}\|_{V^{(m)}} \le \frac{\|f\|^m_{V^*}}{r^m} \exp\left(\frac{\|f\|^2_{V^*}}{r^2}\sum_{k=1}^K \|o_k\|^2_{V^*}\right). \tag{4.23}$$

5. Polynomial Chaos Approximations of the Posterior

Building on the results of the previous section, we now proceed to approximate $\Theta(z)$, viewed as a holomorphic functional over $z \in \mathbb{C}^{\mathbb{J}}$, by so-called *polynomial chaos representations*. Exactly the same results on analyticity and on N-term approximation of $\Psi(z)$ hold. We omit details for reasons of brevity of exposition and confine ourselves to establishing rates of convergence of N-term truncated representations of the posterior density Θ. The results in the present section are, in one sense, *sparsity results* on the posterior density Θ. On the other hand, such N-term truncated gpc representations of Θ are, as we will show in the next section, computationally accessible once sparse truncated adaptive forward solvers of the parametrized system of interest are available. Such solvers are indeed available (see, e.g., [7, 3] and the references therein), so that the abstract approximation results in the present section have a substantive constructive aspect. Algorithms based on Smolyak-type quadratures in U which are designed based on the present theoretical results will be developed and analyzed in [1]. In this section we analyze the convergence rate of *N-term truncated Legendre gpc-approximations of* Θ and, with the aim of a *constructive N-term approximation* of the posterior $\Theta(y)$ in U in Section 6 ahead, we analyze also N-term truncated *monomial gpc-approximations* of $\Theta(y)$.

5.1. **gpc Representations of Θ.**

With the index set $\mathbb{J}$ from the parametrization (3.7) of the input, we associate the countable index set

$$\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{J}} : |\nu|_1 < \infty\} \tag{5.1}$$

of multiindices where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We remark that sequences $\nu \in \mathcal{F}$ are finitely supported even for $\mathbb{J} = \mathbb{N}$. For $\nu \in \mathcal{F}$, we denote by $\mathbb{I}_\nu = \{j \in \mathbb{N} : \nu_j \neq 0\} \subset \mathbb{N}$ the "support" of $\nu \in \mathcal{F}$, i.e. the finite set of indices of entries of $\nu \in \mathcal{F}$ which are non-zero, and by $\aleph(\nu) := \#\mathbb{I}_\nu < \infty$, $\nu \in \mathcal{F}$ the "support size" of ν, i.e. the cardinality of $\mathbb{I}_\nu$.

For the deterministic approximation of the posterior density $\Theta(y)$ in (2.8) we shall use tensorized polynomial bases similar to what is done in so-called "polynomial chaos" expansions of random fields. We shall consider two particular polynomial bases, Legendre and monomial bases.

5.1.1. *Legendre Expansions of* Θ.

Since we assumed that the prior measure $\mu_0(dy)$ is built by tensorization of the uniform probability measures on $(-1,1)$, build the bases by tensorization as follows: let $L_k(z_j)$ denote the k^{th} Legendre polynomial of the variable $z_i \in \mathbb{C}$, normalized such that

$$\int_{-1}^{1} (L_k(t))^2 \frac{dt}{2} = 1, \quad k = 0, 1, 2, \ldots \tag{5.2}$$

Note that $L_0 \equiv 1$. The Legendre polynomials L_k in (5.2) are extended to tensor-product polynomials on U via

$$L_\nu(z) = \prod_{j \in \mathbb{J}} L_{\nu_j}(z_j), \quad z \in \mathbb{C}^{\mathbb{J}},\ \nu \in \mathcal{F}\ . \tag{5.3}$$

The normalization (5.2) implies that the polynomials $L_\nu(z)$ in (5.3) are well-defined for any $z \in \mathbb{C}^{\mathbb{J}}$ since the finite support of each element of $\nu \in \mathcal{F}$ implies that L_ν

in (5.3) is the product of only finitely many nontrivial polynomials. It moreover implies that the set of tensorized Legendre polynomials

$$\mathbb{P}(U, \mu_0(dy)) := \{L_\nu : \nu \in \mathcal{F}\} \tag{5.4}$$

forms a countable orthornomal basis in $L^2(U, \mu_0(dy))$. This observation suggests, by virtue of Lemma 5.1, the use of mean square convergent gpc-expansions to represent Θ and Ψ. Such expansions can also serve as a basis for sampling of these quantities with draws that are equidistributed with respect to the prior μ_0.

Lemma 5.1. *The density $\Theta : U \to \mathbb{R}$ is square integrable with respect to the prior $\mu_0(dy)$ over U, i.e. $\Theta \in L^2\big(U, \mu_0(dy)\big)$. Moreover, if the functional $\phi(\cdot) : U \to S$ in* (2.10) *is bounded, then*

$$\int_U \|\Psi(y)\|_S^2 \mu_0(dy) < \infty,$$

i.e. $\Psi \in L^2\big(U, \mu_0(dy); S\big)$.

Proof. Since Φ is positive it follows that $\Theta(y) \in [0,1]$ for all $y \in U$ and the first result follows because μ_0 is a probability measure. Now define $K = \sup_{y\in U} |\phi(y)|$. Then $\sup_{y\in U} \|\Psi(y)\|_S \le K$ and the second result follows similarly, again using that μ_0 is a probability measure. $\square$

Remark 5.1. It is a consequence of (3.4) that in the case where $\phi(u) = G(u) = p \in V$ we have $\|\Psi(y)\|_V \le \|f\|_{V^*}/a_{\text{MIN}}$ for all $y \in U$. Thus the second assertion of Lemma 5.1 holds for calculation of the expectation of the pressure under the posterior distribution on u, the concrete case which we concentrate on here. $\square$

Since $\mathbb{P}(U, \mu_0(dy))$ in (5.4) is a countable orthonormal basis of $L^2(U, \mu_0(dy))$, the density $\Theta(y)$ of the posterior measure given data $\delta \in Y$, and the posterior reweighted pressure $\Psi(y)$ can be represented in $L^2(U, \mu_0(dy))$ by (parametric and deterministic) generalized Legendre polynomial chaos expansions. We start by considering the scalar valued function $\Theta(y)$.

$$\Theta(y) = \sum_{\nu\in\mathcal{F}} \theta_\nu L_\nu(y) \quad \text{in} \quad L^2(U, \rho(dy)) \tag{5.5}$$

where the gpc expansion coefficients θ_ν are defined by

$$\theta_\nu = \int_U \Theta(y) L_\nu(y) \mu_0(dy) \;, \quad \nu \in \mathcal{F} \;. \tag{5.6}$$

By Parseval's equation and the normalization (5.2), it follows immediately from (5.5) and Lemma 5.1 with Parseval's equality that the second moment of the posterior density with respect to the prior

$$\|\Theta\|^2_{L^2(U,\mu_0(dy))} = \sum_{\nu\in\mathcal{F}} |\theta_\nu|^2 \tag{5.7}$$

is finite.

5.1.2. *Monomial Expansions of* Θ. We next consider expansions of the posterior density Θ with respect to monomials

$$y^\nu = \prod_{j\geq 1} y_j^{\nu_j}, \quad y \in U, \quad \nu \in \mathcal{F} .$$

Once more, the infinite product is well-defined since, for every $\nu \in \mathcal{F}$, it contains only $\aleph(\nu)$ many nontrivial factors. By Lemma 4.6 and Theorem 4.7, the posterior density $\Theta(y)$ admits an analytic continuation to the product of strips $\mathcal{S}_\rho$ which contains, in particular, the polydisc $\mathcal{U}_\rho$. In U, $\Theta(y)$ can therefore be represented by a monomial expansion with uniquely determined coefficients $\tau_\nu \in V$ which coincide, by uniqueness of the analytic continuation, with the Taylor coefficients of Θ at $0 \in U$:

$$\forall y \in U: \quad \Theta(y) = \sum_{\nu\in\mathcal{F}} \tau_\nu y^\nu , \quad \tau_\nu := \frac{1}{\nu!}\partial_y^\nu \Theta(y) \mid_{y=0} . \tag{5.8}$$

5.2. **Best N-term Approximations of Θ.** Evaluation of expectations under the posterior requires evaluation of integrals (2.11). Numerical efficiency of this step requires efficient numerical realization of the forward map for given samples of the unknown u and, more importantly, to use as few samples as possible. In the context of MCMC, various strategies of reducing the sample number whilst retaining accuracy in the estimate of (2.11) are available. Here, however, our strategy a different one: we approximate the spectral representation (5.5) by truncating it to a finite number N of significant terms. By (5.7), the coefficient sequence $(\theta_\nu)_{\nu\in\mathcal{F}}$ must necessarily decay. If this decay is sufficiently strong, possibly high convergence rates of approximations of the integral (2.11) result. The following classical result from approximation theory makes these heuristic considerations precise: denote by $(\gamma_n)_{n\in\mathbb{N}}$ a (generally not unique) decreasing rearrangement of the sequence $(|\theta_\nu|)_{\nu\in\mathcal{F}}$. Then, for any summability exponents $0 < \sigma \leq q \leq \infty$ and for any $N \in \mathbb{N}$ holds

$$\left(\sum_{n>N} \gamma_n^q\right)^{\frac{1}{q}} \leq N^{-(\frac{1}{\sigma}-\frac{1}{q})} \left(\sum_{n\geq 1} \gamma_n^\sigma\right)^{\frac{1}{\sigma}} . \tag{5.9}$$

5.2.1. $L^2(U;\mu_0)$ *Approximation.* Denote by $\Lambda_N \subset \mathcal{F}$ a set of indices $\nu \in \mathcal{F}$ corresponding to N largest gpc coefficients $|\theta_\nu|$ in (5.5), and denote by

$$\Theta_{\Lambda_N}(y) := \sum_{\nu\in\Lambda_N} \theta_\nu L_\nu(y) \tag{5.10}$$

the Legendre expansion (5.5) truncated to this set of indices. Using (5.9) with $q = 2$, Paseval's equation (5.7) and $0 < \sigma < 2$ we obtain for all N

$$\|\Theta(z) - \Theta_{\Lambda_N}(z)\|_{L^2(U,\mu_0(dy))} \leq N^{-s}\|(\theta_\nu)\|_{\ell^\sigma(\mathcal{F})}, \quad s := \frac{1}{\sigma} - \frac{1}{2} . \tag{5.11}$$

We infer from (5.11) that *a mean-square convergence rate $s > 1/2$ of the approximate posterior density Θ_{Λ_N} can be achieved if $(\theta_\nu) \in \ell^\sigma(\mathcal{F})$ for some $0 < \sigma < 1$.*

5.2.2. *$L^1(U;\mu_0)$ and pointwise Approximation of* Θ. The analyticity of $\Theta(y)$ in $\mathcal{U}_\rho$ implies that $\Theta(y)$ can be represented by the Taylor exansion (5.8). This expansion is unconditionally summable in U and, for any sequence $\{\Lambda_N\}_{N\in\mathbb{N}} \subset \mathcal{F}$ which exhausts $\mathcal{F}$ [2], the corresponding sequence of N-term truncated partial Taylor sums

$$T_{\Lambda_N}(y) := \sum_{\nu\in\Lambda_N} \tau_\nu y^\nu \tag{5.12}$$

converges pointwise in U to Θ. Since for $y\in U$ and $\nu\in\mathcal{F}$ we have $|y^\nu|\leq 1$, for any $\Lambda_N\subset\mathcal{F}$ of cardinality not exceeding N holds

$$\sup_{y\in U}|\Theta(y)-T_{\Lambda_N}(y)| = \sup_{y\in U}\left|\sum_{\nu\in\mathcal{F}\setminus\Lambda_N}\tau_\nu y^\nu\right| \leq \sum_{\nu\in\mathcal{F}\setminus\Lambda_N}|\tau_\nu|\ . \tag{5.13}$$

Similarly, we have

$$\|\Theta - T_{\Lambda_N}\|_{L^1(U,\mu_0)} = \left\|\sum_{\nu\in\mathcal{F}\setminus\Lambda_N}\tau_\nu y^\nu\right\|_{L^1(U,\mu_0)} \leq \sum_{\nu\in\mathcal{F}\setminus\Lambda_N}|\tau_\nu|\,\|y^\nu\|_{L^1(U,\mu_0)}\ .$$

For $\nu\in\mathcal{F}$, we calculate

$$\|y^\nu\|_{L^1(U,\mu_0)} \quad = \quad \int_{y\in U}|y^\nu|\mu_0(dy) = \frac{1}{(\nu+\mathbf{1})!}$$

so that we find

$$\|\Theta - T_{\Lambda_N}\|_{L^1(U,\mu_0)} \leq \sum_{\nu\in\mathcal{F}\setminus\Lambda_N}\frac{|\tau_\nu|}{(\nu+\mathbf{1})!}\ . \tag{5.14}$$

5.2.3. *Summary.* There are, hence, two main issues to be addressed to employ the approximations i) establishing the summability of the coefficient sequences in the series (5.5), (5.8) and ii) finding algorithms which locate sets $\Lambda_N\subset\mathcal{F}$ of cardinality not exceeding N for which the truncated partial sums preserve the optimal convergence rates and, once these sets are localized, to determine the N "active" coefficients θ_ν or τ_ν, preferably in close to $O(N)$ operations. In the remainder of this section, we address i) and consider ii) in the next section.

5.3. **Sparsity of the posterior density Θ.** The analysis in the previous section shows that the convergence rate of the truncated gpc-type approximations (5.10), (5.12) on the parameterspace U is determined by the σ-summability of the corresponding coefficient sequences $(|\theta_\nu|)_{\nu\in\mathcal{F}}$, $(|\tau_\nu|)_{\nu\in\mathcal{F}}$. We now show that summability of Legendre and Taylor coefficient sequences in the expansions (5.5), (5.8) is determined by that of the sequence $(\|\psi_j\|_{L^\infty(D)})_{j\in\mathbb{N}}$ in the input's fluctuation expansion (3.7). Throughout, Assumptions 3.1 and 3.2 will be required to hold. We formalize the decay of the ψ_j in (2.4) by

Assumption 5.2. *There exists $0<\sigma<1$ such that for the parametric representations* (3.7), (2.4) *it holds that*

$$\sum_{j=1}^\infty \|\psi_j\|^\sigma_{L^\infty(D)} < \infty\ . \tag{5.15}$$

[2] We recall that a sequence $\{\Lambda_N\}_{N\in\mathbb{N}}\subset\mathcal{F}$ of index sets Λ_N whose cardinality does not exceed N exhausts $\mathcal{F}$ if any finite $\Lambda\subset\mathcal{F}$ is contained in all Λ_N for $N\geq N_0$ with N_0 sufficiently large.

5.3.1. *Complex extension of the parametric problem.* To estimate $|\theta_\nu|$ in (5.10), we shall use the holomorphy of solution to the (analytic continuation of the) parametric deterministic problem: let $0 < K < 1$ be a constant such that

$$K \sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min}}{8}. \tag{5.16}$$

Such a constant exists by Assumption 5.15. For K selected in this fashion, we next choose an integer J_0 such that

$$\sum_{j>J_0} \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min} K}{24(1+K)}.$$

Let $E = \{1, 2, \dots, J_0\}$ and $F = \mathbb{N} \setminus E$. We define

$$|\nu_F| = \sum_{j>J_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$ we define a ν-dependent radius vector $\mathbf{r} = (r_m)_{m\in\mathbb{J}}$ with $r_m > 0$ for all $m \in \mathbb{J}$ as follows:

$$r_m = K \text{ when } m \le J_0 \text{ and } r_m = 1 + \frac{a_{\min}\nu_m}{4|\nu_F|\|\psi_m\|_{L^\infty(D)}} \text{ when } m > J_0, \tag{5.17}$$

where we make the convention that $\frac{|\nu_j|}{|\nu_F|} = 0$ if $|\nu_F| = 0$. We consider the open discs $\mathcal{U}_m \subset \mathbb{C}$ defined by

$$[-1,1] \subset \mathcal{U}_m := \{z_m \in \mathbb{C} : |z_m| < 1 + r_m\} \subset \mathbb{C}. \tag{5.18}$$

We will extend the parametric deterministic problem (4.6) to parameter vectors z in the polydiscs

$$\mathcal{U}_{1+\mathbf{r}} := \bigotimes_{m\in\mathbb{J}} \mathcal{U}_m \subset \mathbb{C}^{\mathbb{J}}. \tag{5.19}$$

To do so, we invoke the analytic continuation of the parametric, deterministic coefficient function $a(x,y)$ in (3.7) to $z \in \mathcal{U}$ which is for such z *formally* given by

$$a(x,z) = \bar{a}(x) + \sum_{m\in\mathbb{J}} \psi_m(x) z_m.$$

We verify that this expression is meaningful for $z \in \mathcal{U}_{\mathbf{r}}$: we have, for almost every $x \in D$,

$$\begin{aligned}
|a(x,z)| &\le \bar{a}(x) + \sum_{m\in\mathbb{J}} |\psi_m(x)|(1+r_m) \\
&\le \operatorname*{ess\,sup}_{x\in D} |\bar{a}(x)| + \sum_{m=1}^{J_0} \|\psi_m\|_{L^\infty(D)}(1+K) \\
&\quad + \sum_{m>J_0} \Big(2 + \frac{a_{\min}\nu_m}{4|\nu_F|\|\psi_m\|_{L^\infty(D)}}\Big)\|\psi_m\|_{L^\infty(D)} \\
&\le \|\bar{a}\|_{L^\infty(D)} + 2\sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}\ .
\end{aligned}$$

5.3.2. *Estimates of the θ_ν.*

Proposition 5.3. *There exists a constant $C > 0$ such that, with the constant $K \in (0,1)$ in (5.16), for every $\nu \in \mathcal{F}$ the following estimate holds*

$$|\theta_\nu| \le C\Big(\prod_{m\in\mathbb{I}(\nu)} \frac{2(1+K)}{K}\eta_m^{-\nu_m}\Big), \tag{5.20}$$

where $\eta_m := r_m + \sqrt{1+r_m^2}$ with r_m as in (5.17).

Proof For $\nu \in \mathcal{F}$, define θ_ν by (5.6) let $S = \mathbb{I}(\nu)$ and define $\bar{S} = \mathbb{J}\setminus S$. For S denote by $\mathcal{U}_S = \otimes_{m\in S}\mathcal{U}_m$ and $\mathcal{U}_{\bar{S}} = \otimes_{m\in\bar{S}}\mathcal{U}_m$, and by $y_S = \{y_i : i \in S\}$ the extraction from y. Let $\mathcal{E}_m$ be the ellipse in $\mathcal{U}_m$ with foci at ± 1 and semiaxis sum $\eta_m > 1$. Denote also $\mathcal{E}_S = \prod_{m\in\mathbb{I}(\nu)}\mathcal{E}_m$. We can then write (5.6) as

$$\theta_\nu = \frac{1}{(2\pi i)^{|\nu|_0}}\int_U L_\nu(y)\oint_{\mathcal{E}_S}\frac{\Theta(z_S, y_{\bar{S}})}{(z_S - y_S)^{\mathbf{1}}}dz_S d\rho(y).$$

For each $m \in \mathbb{N}$, let Γ_m be a copy of $[-1,1]$ and $y_m \in \Gamma_m$. We denote by $U_S = \prod_{m\in S}\Gamma_m$ and $U_{\bar{S}} = \prod_{m\in\bar{S}}\Gamma_m$. We then have

$$\theta_\nu = \frac{1}{(2\pi i)^{|\nu|_0}}\int_{U_{\bar{S}}}\oint_{\mathcal{E}_S}\Theta(z_S, y_{\bar{S}})\int_{U_S}\frac{L_\nu(y)}{(z_S - y_S)^{\mathbf{1}}}d\rho_S(y_S)dz_S d\rho_{\bar{S}}(y_{\bar{S}}).$$

To proceed further, we recall the definitions of the Legendre functions of the second kind

$$Q_n(z) = \int_{[-1,1]}\frac{L_n(y)}{(z-y)}d\rho(y).$$

Let ν_S be the restriction of ν to S. We define

$$\mathcal{Q}_{\nu_S}(z_S) = \prod_{m\in\mathbb{I}(\nu)} Q_{\nu_m}(z_m).$$

Under the Joukovski transformation $z_m = \frac{1}{2}(w_m + w_m^{-1})$, the Legendre polynomials of the second kind take the form

$$Q_{\nu_m}(\frac{1}{2}(w_m + w_m^{-1})) = \sum_{k=\nu_m+1}^{\infty}\frac{q_{\nu_m k}}{w_m^k}$$

with $|q_{\nu_m k}| \le \pi$. Therefore

$$|\mathcal{Q}_{\nu_S}(z_S)| \le \prod_{m\in S}\sum_{k=\nu_m+1}^{\infty}\frac{\pi}{\eta_m^k} = \prod_{m\in S}\pi\frac{\eta_m^{-\nu_m-1}}{1-\eta_m^{-1}}.$$

We then have

$$
\begin{aligned}
|\theta_\nu| &= \left| \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_{\bar S}} \oint_{\mathcal{E}_S} \Theta(z_S, y_{\bar S}) \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar S}(y_S) \right| \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \int_{U_{\bar S}} \oint_{\mathcal{E}_S} |\Theta(z_S, y_{\bar S})| \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar S}(y_S) \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \|\Theta(z)\|_{L^\infty(\mathcal{E}_S \times U_{\bar S})} \max_{\mathcal{E}_S} |\mathcal{Q}_{\nu_S}| \prod_{m\in S} \mathrm{Len}(\mathcal{E}_m) \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \|\Theta(z)\|_{L^\infty(\mathcal{E}_S \times \mathcal{U}_{\bar S})} \prod_{m\in S} \pi \frac{\eta_m^{-\nu_m - 1}}{1-\eta_m^{-1}} \mathrm{Len}(\mathcal{E}_m) \\
&\le C \prod_{m\in S} \frac{2(1+K)}{K} \eta_m^{-\nu_m},
\end{aligned}
$$

as $\mathrm{Len}(\mathcal{E}_m) \le 4\eta_m$, $\eta_m \ge 1+K$ and as $|\Theta(z)|$ is uniformly bounded on $\mathcal{E}_S \times \mathcal{U}_{\bar S}$ by Theorem 4.7. □

5.3.3. *Summability of the* θ_ν. To show the $l^\sigma(\mathcal{F})$ summability of $|\theta_\nu|$, we use the following result from [5].

Proposition 5.4. *For* $0 < \sigma < 1$ *and for any sequence* $(b_\nu)_{\nu\in\ell^\sigma(\mathcal{F})}$,

$$
\Big(\frac{|\nu|!}{\nu!} b^\nu\Big)_{\nu\in\mathcal{F}} \in l^\sigma(\mathcal{F}) \Longleftrightarrow \sum_{m\ge 1} |b_m| < 1 \quad \textit{and} \quad (b_m)_{m\in\mathbb{N}} \in l^\sigma(\mathbb{N}) \, .
$$

We can now establish the p-summability of the sequence (θ_ν) of Legendre coefficients.

Proposition 5.5. *Under Assumptions 3.1, 3.2, for* $0 < \sigma < 1$ *as in Assumption 5.2,* $\sum_{\nu\in\mathcal{F}} |\theta_\nu|^\sigma$ *is finite.*

Proof We have from Proposition 5.3 that

$$
\begin{aligned}
|\theta_\nu| &\le C \prod_{m\in S} \frac{2(1+K)}{K} (1+r_m)^{-\nu_m} \\
&\le C \Big(\prod_{m\in E, \nu_m\neq 0} \frac{2(1+K)}{K} \eta^{\nu_m} \Big) \Big(\prod_{m\in F, \nu_m\neq 0} \frac{2(1+K)}{K} \Big(\frac{4|\nu_F| \|\psi_m\|_{L^\infty(D)}}{a_{\min}\nu_m} \Big)^{\nu_m} \Big)
\end{aligned}
$$

where $\eta = 1/(1+K) < 1$. Let $\mathcal{F}_E = \{\nu \in \mathcal{F} : \quad \mathbb{I}(\nu) \subset E\}$ and $\mathcal{F}_F = \mathcal{F} \setminus E$. From this, we have

$$
\sum_{\nu\in\mathcal{F}} |\theta_\nu|^\sigma \le C A_E A_F
$$

where

$$
A_E = \sum_{\nu\in\mathcal{F}_E} \prod_{m\in E, \nu_m\neq 0} \Big(\frac{2(1+K)}{K} \Big)^\sigma \eta^{\sigma\nu_m},
$$

and

$$
A_F = \sum_{\nu\in\mathcal{F}_F} \prod_{m\in F, \nu_m\neq 0} \Big(\frac{2(1+K)}{K} \Big)^\sigma \Big(\frac{4|\nu| \|\psi_m\|_{L^\infty(D)}}{a_{\min}\nu_m} \Big)^{\sigma\nu_m}.
$$

We estimate A_E and A_F: for A_E, we have

$$A_E = \left(1 + \left(\frac{2(1+K)}{K}\right)^\sigma \sum_{m\geq 1} \eta^{pm}\right)^{J_0},$$

which is finite due to $\eta < 1$. For A_F, we note that for $\nu_m \neq 0$,

$$\frac{2(1+K)}{K} \leq \left(\frac{2(1+K)}{K}\right)^{\nu_m}.$$

Therefore

$$A_F \leq \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F} \left(\frac{|\nu| d_m}{\nu_m}\right)^{\sigma \nu_m},$$

where

$$d_m = \frac{8(1+K)\|\psi_m\|_{L^\infty(D)}}{K a_{\min}}.$$

With the convention that $0^0 = 1$ we obtain from the Stirling estimate

$$\frac{n! e^n}{e\sqrt{n}} \leq n^n \leq \frac{n! e^n}{\sqrt{2\pi n}}$$

that $|\nu|^{|\nu|} \leq |\nu|! e^{|\nu|}$. Inserting this in the above bound for A_F, we obtain

$$\prod_{m \in F} \nu_m^{\nu_m} \geq \frac{\nu! e^{|\nu|}}{\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\}}.$$

Hence

$$A_F \leq \sum_{\nu \in \mathcal{F}_F} \left(\frac{|\nu|!}{\nu!} d^\nu\right)^\sigma \left(\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\}\right)^\sigma \leq \sum_{\nu \in \mathcal{F}_F} \left(\frac{|\nu|!}{\nu!} \bar{d}^\nu\right)^\sigma,$$

where $\bar{d}_m = e d_m$ and where we used the estimate $e\sqrt{n} \leq e^n$. From this, we have

$$\sum_{m \geq 1} \bar{d}_m \leq \sum_{m \in F} \frac{24(1+K)\|\psi_m\|_{L^\infty(D)}}{K a_{\min}} \leq 1.$$

Since also

$$\|\bar{d}\|_{l^\sigma(\mathbb{N})} < \infty$$

we obtain with Proposition 5.4 the conclusion. $\square$

We now show σ summability of the Taylor coefficients τ_ν in (5.8). To this end, we proceed as in the Legendre case: first we establish sharp bounds on the τ_ν by complex variable methods, and then show σ-summability of $(\tau_\nu)_{\nu \in \mathcal{F}}$ by a sequence factorization argument.

5.3.4. *Bounds on the Taylor coefficients τ_ν.*

Lemma 5.6. *Assume* $\mathbf{UEAC}(a_{\text{MIN}}, a_{\text{MAX}})$ *and that $\rho = (\rho_j)_{j\geq 1}$ is an r-admissible sequence of disc radii for some $0 < r < a_{\text{MIN}}$. Then the Taylor coefficients τ_ν of the parametric posterior density* (5.8) *satisfy*

$$\forall \nu \in \mathcal{F}: \quad |\tau_\nu| \leq \exp\left(\frac{\|f\|_{V^*}^2}{r^2} \sum_{k=1}^K \|o_k\|_{V^*}^2\right) \prod_{j\geq 1} \rho_j^{-\nu_j}. \tag{5.21}$$

Proof For $\nu = (\nu_j)_{j\geq 1} \in \mathcal{F}$ holds $J = \max\{j \in \mathbb{N} : \nu_j \neq 0\} < \infty$. For this J, define $\Theta_{[J]}(z^J) := \Theta(z_1, z_2, ..., z_J, 0, ...)$, i.e. $\Theta_{[J]}(z^J)$ denotes the function of $z^J \in \mathbb{C}^J$ obtained by setting in the posterior density $\Theta(z)$ all coordinates z_j with $j > J$ equal to zero. Then

$$\partial_z^\nu \Theta(z)|_{z=0} = \frac{\partial^{|\nu|}\Theta_{[J]}}{\partial z_1^{\nu_1}...\partial z_J^{\nu_J}}(0, ..., 0) \ .$$

Since the sequence ρ is r-admissible it follows with (4.22) that

$$\sup_{(z_1,\dots,z_J)\in\mathcal{U}_{\rho,J}} |\Theta_{[J]}(z_1, \dots, z_J)| \leq \exp\left(\frac{\|f\|^2_{V^*}}{r^2}\sum_{k=1}^K \|o_k\|^2_{V^*}\right) . \tag{5.22}$$

for all $(z_1, \dots, z_J)$ in the polydisc $\mathcal{U}_{\rho,J} := \otimes_{1\leq j\leq J}\{z_j \in \mathbb{C} : |z_j| \leq \rho_j\} \subset \mathbb{C}^J$. We now prove (5.21) by Cauchy's integral formula. To this end, we define $\tilde\rho$ by

$$\tilde\rho_j := \rho_j + \epsilon \text{ if } j \leq J,\ \ \tilde\rho_j = \rho_j \text{ if } j > J, \quad \epsilon := \frac{r}{2\|\sum_{j\leq J}|\psi_j|\|_{L^\infty(D)}}.$$

Then the sequence $\tilde\rho$ is $r/2$-admissible and therefore $\mathcal{U}_{\tilde\rho} \subset \mathcal{A}_{r/2}$. This implies that for each $z \in \mathcal{U}_{\tilde\rho}$, u is holomorphic in each variable z_j.

It follows that u_J is holomorphic in each variable $z_1, \dots, z_J$ on the polydisc $\otimes_{1\leq j\leq J}\{|z_j| < \tilde\rho_j\}$ which is an open neighbourhood of $\mathcal{U}_{\rho,J}$ in $\mathbb{C}^J$.

We may thus apply the Cauchy formula (e.g. Theorem 2.1.2 of [9]) in each variable z_j:

$$u_J(z_1, \dots, z_J) = (2\pi i)^{-J}\int_{|\tilde z_1|=\tilde\rho_1}\cdots\int_{|\tilde z_J|=\tilde\rho_J}\frac{u_J(\tilde z_1, \dots, \tilde z_J)}{(z_1 - \tilde z_1)\dots(z_J - \tilde z_J)}d\tilde z_1 \dots d\tilde z_J \ .$$

We infer

$$\frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\dots\partial z_J^{\nu_J}}u_J(0, \dots, 0) = \nu!(2\pi i)^{-J}\int_{|\tilde z_1|=\tilde\rho_1}\cdots\int_{|\tilde z_J|=\tilde\rho_J}\frac{u_J(\tilde z_1, \dots, \tilde z_J)}{\tilde z_1^{\nu_1}\dots\tilde z_J^{\nu_J}}d\tilde z_1 \dots d\tilde z_J \ .$$

Bounding the integrand on $\{|\tilde z_1| = \tilde\rho_1\} \times \dots \times \{|\tilde z_J| = \tilde\rho_J\} \subset \mathcal{A}_r$ with (4.22) implies (5.21). □

5.3.5. *σ-summability of the τ_ν.* Proceeding in a similar fashion as in Section 3 of [6], we can prove the σ-summability of the Taylor coefficients τ_ν.

Proposition 5.7. *Under Assumptions 3.1, 3.2 and 5.2, $(\|\tau_\nu\|_V) \in \ell^\sigma(\mathcal{F})$ for $0 < \sigma < 1$ as in Assumption 5.2.*

We remark that under the same assumptions, we also have p-summability of $(\tau_\nu/(\nu + \mathbf{1})!)_{\nu\in\mathcal{F}}$, since

$$\forall \nu \in \mathcal{F} : \quad |\tau_\nu| \leq \frac{|\tau_\nu|}{(\nu + \mathbf{1})!} \ .$$

5.4. **Best N-term convergence rates.** With (5.9), we infer from Proposition 5.5 and from (5.11) convergence rates for "polynomial chaos" type approximations of the posteriori density Θ.

Theorem 5.8. *If Assumptions 3.1, 3.2 and 5.2 hold then there is a sequence $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathcal{F}$ of index sets with cardinality not exceeding N (depending σ and* *on the data δ) such that the corresponding N-term truncated gpc Legendre expansions Θ_{Λ_N} in* (5.10) *satisfy*

$$\|\Theta - \Theta_{\Lambda_N}\|_{L^2(U,\mu_0(dy))} \leq N^{-(\frac{1}{\sigma}-\frac{1}{2})}\|(\theta_\nu)\|_{\ell^\sigma(\mathcal{F};\mathbb{R})} \,. \tag{5.23}$$

Likewise, for $q = 1, \infty$ and for every $N \in \mathbb{N}$, there exist sequences $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathcal{F}$ of index sets (depending, in general, on σ, q and the data) whose cardinality does not exceed N such that the N-term truncated Taylor sums (5.12) *converge with rate $1/\sigma - 1$, i.e.*

$$\|\Theta - T_{\Lambda_N}\|_{L^q(U,\mu_0(dy))} \leq N^{-(\frac{1}{\sigma}-1)}\|(\tau_\nu)\|_{\ell^\sigma(\mathcal{F};\mathbb{R})} \,. \tag{5.24}$$

Here, for $q = \infty$ the norm $\| \circ \|_{L^\infty(U;\mu_0)}$ is the supremum over all $y \in U$.

6. Approximation of Expectations under the Posterior

Expectations under the posterior given data δ are deterministic, infinite dimensional parametric integrals Z and Z' with respect to the prior measure μ_0, i.e. iterated integrals over the coordinates $y_j \in [-1, 1]$ against a countable product of the uniform probability measures $\frac{1}{2}dy_j$. To render this practically feasible, numerical evaluation of the conditional expectations

$$\bar{u}^\delta = \mathbb{E}_{\mu^\delta}[u] = \int_{y\in U} u(\cdot, y)\Theta(y)\mu_0(dy) \in X \tag{6.1}$$

and

$$\bar{p}^\delta = \mathbb{E}_{\mu^\delta}[p] = \int_{y\in U} p(\cdot, y)\Theta(y)\mu_0(dy) \in V \tag{6.2}$$

is required. More generally, in the evaluation of the conditional expectation of some (multilinear) functional $\phi(u)$ given the data δ, which takes the generic form

$$\overline{\phi(u)}^\delta = \mathbb{E}_{\mu^\delta}[\phi(u)] = \int_{y\in U} \phi(u(\cdot, y))\Theta(y)\mu_0(dy) \in S \tag{6.3}$$

and which takes values in a suitable state space S. We continue to concentrate on the concrete choice $\phi(u) = G(u) = p$ so that $S = V$. With the techniques developed here and with Corollary 4.8, analogous results can also be established for expectations of m point correlations of $G(u)$ as in (3.6).

The computationally most costly step in Bayesian inverse problems is the numerical evaluation of the expectations (6.1) - (6.3). Assuming that the solution of the forward problem (3.1) is only available approximately, these expectations need only be evaluated to an accuracy commensurate with that of the forward solver. One way of doing this is MC sampling of the posteriori density Θ: using N samples of Θ, one approximates the expectations (6.1) - (6.3) by the corresponding sample averages (with each "sample" of Θ requiring the solution of at least one forward problem). Due to Lemma 5.1, the posteriori density has finite second moments with respect to the prior, so that the convergence rate $N^{-1/2}$ of the MC sample averages towards the expectations (6.1) - (6.3) results.

Higher rates of convergence in terms of N may be obtained by exploting the polynomial chaos approximation ideas introduced in the previous section. The first option is to replace MC sampling by a *sparse tensor numerical integration scheme over U* tailored to the regularity afforded by the analytic parameter dependence of the posteriori density on y and of the integrands in (6.1) - (6.3). This approach is not considered here. We refer to [1] for details and numerical experiments.

We show here that the integrals (6.1) - (6.3) allow *semianalytic evaluation* in log-linear complexity with respect to N, the number of terms in a polynomial chaos approximation of the parametric solution of the forward problem (3.1), (2.4).

To this end, we proceed as follows: based on the *assumption* that N-term gpc approximations of the parametric forward solutions $p(x,y)$ of (3.1) is availabe, for example by the algorithms in [3], we show that it is possible to *construct separable N-term approximations* of the integrands in (6.1) - (6.3). The existence of such an approximate posterior density which is "close" to Θ is ensured by Theorem 5.8, provided the (unknown) input data u satisfies certain conditions. *We prove that sets $\Lambda_N \subset \mathcal{F}$ of cardinality at most N which afford the truncation errors* (5.23), (5.24) *can be found in log-linear complexity with respect to* N and, second, *that the integrals* (6.3) *with the corresponding approximate posterior density can be evaluated in such complexity* and, third, we estimate the errors in the resulting conditional expectations.

6.1. **Assumptions and Notation.**

Assumption 6.1. *Given a draw u of the data, an exact forward solve of the governing equation* (3.1) for this draw of data u *is available at unit cost.*

This assumption is made in order to simplify the exposition. All conclusions remain valid if this assumption is relaxed to include an additional Finite Element discretization error; we refer to [1] for details. We shall use the notion of *monotone sets of multiindices*.

Definition 6.2. *A subset $\Lambda_N \subset \mathcal{F}$ of finite cardinality N is called* monotone *if (M1) $\{0\} \subset \Lambda_N$ and if (M2) $\forall 0 \neq \nu \in \Lambda_N$ it holds that $\nu - e_j \in \Lambda_N$ for all $j \in \mathbb{I}_\nu$, where $e_j \in \{0,1\}^{\mathbb{J}}$ denotes the index vector with 1 in position $j \in \mathbb{J}$ and 0 in all other positions $i \in \mathbb{J}\backslash\{j\}$.*

Next, will assume that *a stochastic Galerkin approximation of the entire forward map of the parametric, deterministic solution with certain optimality properties is available.*

Assumption 6.3. *Given a parametric representation* (3.7) *of the unknown data u, a stochastic Galerkin approximation $p_N \in \mathbb{P}_{\Lambda_N}(U,V)$ of the exact forward solution of the governing equation* (3.1) *is available. Here the set $\Lambda_N \subset \mathcal{F}$ is a finite subset of "active" gpc Legendre coefficients whose cardinality does not exceed N. In addition, we assume that the gpc approximation $p_N \in \mathbb{P}_{\Lambda_N}(U,V)$ is quasi optimal in terms of the best N-term approximation, i.e. there exists $C \geq 1$ independent of N such that*

$$\|p - p_N\|_{L^2(U,\mu_0;V)} \leq C N^{-(1/\sigma - 1/2)} \|(\theta_\nu)\|_{\ell^\sigma(\mathcal{F})} \,. \tag{6.4}$$

Here $0 < \sigma \leq 1$ denotes the summability exponent in Assumption 5.2. Note that best N-term approximations satisfy (6.4) *with $C = 1$; we may refer to* (6.4) *as a quasi best N-term approximation property.*

This best N-term convergence rate of sGFEM approximations follows from results in [5, 6], but these results do not indicate as to how sequences of sGFEM approximations which converge with this rate are actually constructed. We refer to [7, 3] and the references there for details on such sGFEM solvers, also including space discretizations. In what follows, we work under Assumptions 6.1, 6.3.

6.2. **Best N-term based approximate conditional expectation.** We first address the rates that can be achieved by the (a-priori not accesssible) best N-term approximations of the posterior density Θ in Theorem 5.8. These rates serve as benchmark rates to be achieved by any constructive procedure.

To derive these rates, we let $\Theta_N = \Theta_{\Lambda_N}$ denote the best N-term Legendre approximations of the posterior density Θ in Theorem 5.8. With (6.4), we estimate

$$\begin{aligned}
\|\bar{p}^\delta - \bar{p}_N^\delta\|_V &= \left\| \int_U (\Theta p - \Theta_N p_N)\, \mu_0(dy) \right\|_V \\
&= \left\| \int_U ((\Theta - \Theta_N) p + \Theta_N (p - p_N))\, \mu_0(dy) \right\|_V \\
&\leq \int_U |\Theta - \Theta_N| \|p\|_V \mu_0(dy) + \|\Theta_N\|_{L^2(U)} \|p - p_N\|_{L^2(U,\mu_0;,V)} \\
&\leq \|\Theta - \Theta_N\|_{L^2(U)} \|p\|_{L^2(U,\mu_0;V)} + \|\Theta_N\|_{L^2(U)} \|p - p_N\|_{L^2(U,\mu_0;V)} \\
&\leq CN^{-(\frac{1}{p}-\frac{1}{2})}\,.
\end{aligned}$$

With $T_N = T_{\Lambda_N}$ denoting a best N-term Taylor approximation of Θ in Theorem 5.8 we obtain in the same fashion the bound

$$\begin{aligned}
\|\bar{p}^\delta - \bar{p}_N^\delta\|_V &= \left\| \int_U (\Theta p - T_N p_N)\, \mu_0(dy) \right\|_V \\
&= \left\| \int_U ((\Theta - T_N) p + T_N (p - p_N))\, \mu_0(dy) \right\|_V \\
&\leq \int_U |\Theta - T_N| \|p\|_V \mu_0(dy) + \|T_N\|_{L^\infty(U)} \|p - p_N\|_{L^1(U,\mu_0;V)} \\
&\leq \|\Theta - T_N\|_{L^1(U,\mu_0)} \|p\|_{L^\infty(U,\mu_0;V)} + \|T_N\|_{L^\infty(U)} \|p - p_N\|_{L^2(U,\mu_0;V)} \\
&\leq CN^{-(\frac{1}{p}-1)}\,.
\end{aligned}$$

We now address question ii) raised at the beginning of Section 5.2, i.e. *the design of practical algorithms for the construction of sequences $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathcal{F}$ such that the best-N term convergence rates asserted in Theorem 5.8 are attained*. We develop the approximation in detail for (6.2), and state the results for (6.1), (6.3) (whose proof is verbatim the same) later.

6.3. **Constructive N-term Approximation of the Potential Φ.** We show that, from the quasi best N-term optimal stochastic Galerkin approximation $u_N \in \mathbb{P}_{\Lambda_N}(U,V)$ and, in particular, from its (monotone) index set Λ_N, a corresponding N-term approximation Φ_N of the potential Φ in (2.3) can be computed. We start by observing that for monotone index sets $\Lambda_N \subset \mathcal{F}$ properties (M1) and (M2) in Definition 6.2 imply

$$\mathbb{P}_{\Lambda_N}(U) = \mathrm{span}\{y^\nu : \nu \in \Lambda_N\} = \mathrm{span}\{L_\nu : \nu \in \Lambda_N\}\,. \tag{6.5}$$

We denote the observation corresponding to the stochastic Galerkin approximation of the system response p_N by $\mathcal{G}_N$, i.e. the mapping

$$U \ni y \mapsto \mathcal{G}_N(u)|_{u=\bar{a}+\sum_{j\in\mathbb{J}} y_j\psi_j} = (\mathcal{O}\circ G_N)(u)|_{u=\bar{a}+\sum_{j\in\mathbb{J}} y_j\psi_j} \tag{6.6}$$

where $G_N(u) = p_N \in \mathbb{P}_{\Lambda_N}(U;V)$. By the linearity and boundedness of the observation functional $\mathcal{O}(\cdot)$ then $\mathcal{G}_N \in \mathbb{P}_{\Lambda_N}(U;\mathbb{R}^K)$; in the following, we assume for simplicity $K=1$ so that $\mathcal{G}_N|_{u=\bar{a}+\sum_{j\in\mathbb{J}} y_j\psi_j} \in \mathbb{P}_{\Lambda_N}(U)$. We then denote by $U \ni u \mapsto \Phi$ the potential in (2.3) and by Φ_N the potential of the stochastic Galerkin approximation $\mathcal{G}_N$ of the forward observation map. For notational convenience, we suppress the explicit dependence on the data δ in the following and assume that the Gaussian covariance Γ of the observational noise η in (2.1) is the identity: $\Gamma = I$. Then, for every $y \in U$, with $u = \bar{a} + \sum_{j\in\mathbb{J}} y_j\psi_j$ the exact potential Φ and the potential Φ_N based on N-term approximation p_N of the forward solution take the form

$$\Phi(y) = \frac{1}{2}(\delta - \mathcal{G}(u))^2, \quad \Phi_N(y) = \frac{1}{2}(\delta - \mathcal{G}_N(u))^2 \; . \tag{6.7}$$

By Lemma 4.6, these potentials admit extensions to holomorphic functions of the variables $z \in \mathcal{S}_\rho$ in the strip $\mathcal{S}_\rho$ defined in (4.20). Since Λ_N is monotone, we may write $p_N \in \mathbb{P}_{\Lambda_N}(U,V)$ and $\mathcal{G}_N \in \mathbb{P}_{\Lambda_N}(U)$ in terms of their (uniquely defined) Taylor expansions about $y = 0$:

$$\mathcal{G}_N(u) = \sum_{\nu\in\Lambda_N} g_\nu y^\nu \; . \tag{6.8}$$

This implies, for every $y \in U$, $\Phi_N(y) = \delta^2 - 2\delta\mathcal{G}_N(y) + (\mathcal{G}_N(y))^2$ where

$$(\mathcal{G}_N(y))^2 = \sum_{\nu,\nu'\in\Lambda_N} g_\nu g_{\nu'} y^{\nu+\nu'} \in \mathbb{P}_{\Lambda_N+\Lambda_N}(U)$$

has a higher polynomial degree and possibly $O(N^2)$ coefficients. Therefore, an exact evaluation of a gpc approximation of the potential Φ_N might incur loss of linear complexity with respect to N. To preserve log-linear in N complexity, we perform a truncation $[\Phi_N]_{\#N}$ of Φ_N, thereby introducing an additional error which, as we show next, is of the same order as the error of gpc approximation of the system's response. The following Lemma is stated in slightly more general form than is presently needed, since it will also be used for the error analysis of the posterior density ahead.

Lemma 6.4. *Consider two sequences $(g_\nu) \in \ell^\sigma(\mathcal{F})$, $(g'_{\nu'}) \in \ell^\sigma(\mathcal{F}')$, $0 < \sigma \leq 1$. Then*

$$(g_\nu g'_{\nu'})_{(\nu,\nu')\in\mathcal{F}\times\mathcal{F}'} \in \ell^\sigma(\mathcal{F}\times\mathcal{F}')$$

and there holds

$$\|(g_\nu g'_{\nu'})\|^p_{\ell^p(\mathcal{F}\times\mathcal{F}')} \leq \|(g_\nu)\|^\sigma_{\ell^\sigma(\mathcal{F})}\|(g'_{\nu'})\|^\sigma_{\ell^\sigma(\mathcal{F}')} \; . \tag{6.9}$$

Moreover, a best N-term truncation $[\circ]_\#$ of the corresponding polynomials, defined by

$$\left[\left(\sum_{\nu\in\Lambda_N} g_\nu y^\nu\right)\left(\sum_{\nu'\in\Lambda'_N} g'_{\nu'} y^{\nu'}\right)\right]_{\#N} := \sum_{(\nu,\nu')\in\Lambda^1_N} g_\nu g'_{\nu'} y^{\nu+\nu'} \in \mathbb{P}_{\Lambda^1_N}(U) \tag{6.10}$$

where $\Lambda^1_N \subset \mathcal{F}\times\mathcal{F}'$ is a set of index pairs $(\nu,\nu') \in \mathcal{F}\times\mathcal{F}'$ of at most N largest (in absolute value) products $g_\nu g_{\nu'}$, has a pointwise error in U bounded by

$$N^{-(\frac{1}{\sigma}-1)}\|(g_\nu)\|_{\ell^\sigma(\mathcal{F})}\|(g'_{\nu'})\|_{\ell^\sigma(\mathcal{F})} \; . \tag{6.11}$$

Proof. We calculate

$$\begin{aligned}
\|g_\nu g'_{\nu'}\|^\sigma_{\ell^\sigma(\mathcal{F}\times\mathcal{F})} &= \sum_{\nu\in\mathcal{F}}\sum_{\nu'\in\mathcal{F}} |g_\nu g'_{\nu'}|^\sigma = \sum_{\nu\in\mathcal{F}}\left(|g_\nu|^\sigma \sum_{\nu'\in\mathcal{F}} |g'_{\nu'}|^\sigma\right) \\
&= \|(g_\nu)\|^\sigma_{\ell^\sigma(\mathcal{F})}\|(g'_{\nu'})\|^\sigma_{\ell^\sigma(\mathcal{F})} \; .
\end{aligned}$$

Since $(g_\nu g'_{\nu'}) \in \ell^\sigma(\mathcal{F}\times\mathcal{F})$, we may apply (5.9) with (6.9) as follows.

$$\left\| \left[\sum_{\nu\in\Lambda_N} \sum_{\nu'\in\Lambda'_N} g_\nu g'_{\nu'} y^{\nu'+\nu} \right] - \left[\sum_{\nu\in\Lambda_N} \sum_{\nu'\in\Lambda'_N} g_\nu g'_{\nu'} y^{\nu'+\nu} \right]_{\#N} \right\|_{L^\infty(U)}$$

$$\leq \sum_{(\nu,\nu')\in\mathcal{F}\times\mathcal{F}\setminus\Lambda^1_N} |g_\nu g'_{\nu'}| \leq N^{-(\frac{1}{\sigma}-1)} \|(g_\nu)\|_{\ell^\sigma(\mathcal{F})} \|(g'_{\nu'})\|_{\ell^\sigma(\mathcal{F})} \,.$$

□

Applying Lemma 6.4 with $\mathcal{F}' = \mathcal{F}$ and with $(g'_{\nu'})_{\nu'\in\mathcal{F}'} = (g_\nu)_{\nu\in\mathcal{F}}$, we find

$$\begin{aligned} \sup_{y\in U} \left|\Phi_N(y) - [\Phi_N(y)]_{\#N}\right| &= \sup_{y\in U} \left|(\mathcal{G}_N(y))^2 - \left[(\mathcal{G}_N(y))^2\right]_{\#N}\right| \\ &\leq N^{-(\frac{1}{\sigma}-1)} \|(g_\nu)\|^2_{\ell^\sigma(\mathcal{F})} \,. \end{aligned} \tag{6.12}$$

6.4. **Constructive N-term approximation of $\Theta = \exp(-\Phi)$.** With the N-term approximation $[\Phi_N]_{\#N}$, we now define the *constructive approximation* Θ_N *of the posterior density* as follows. We continue to work under Assumption 6.3, i.e. *that N-term truncated gpc-approximations p_N of the forward solution $p(y) = G(u(y))$ of the parametric problem are available which satisfy* (6.4). For an integer $K(N)\in\mathbb{N}$ to be selected below, we define

$$\Theta_N = \sum_{k=0}^{K(N)} \frac{(-1)^k}{k!} \left[([\Phi_N]_{\#N})^k\right]_{\#N} \,. \tag{6.13}$$

We then estimate (all integrals are understood with respect to $\mu_0(dy)$)

$$\begin{aligned} \|\Theta - \Theta_N\|_{L^1(U)} &= \left\| e^{-\Phi} - e^{-[\Phi_N]_{\#N}} + e^{-[\Phi_N]_{\#N}} - \sum_{k=0}^{K(N)} \frac{(-1)^k}{k!} \left[([\Phi_N]_{\#N})^k\right]_{\#N} \right\|_{L^1(U)} \\ &\leq \left\| e^{-\Phi} - e^{-[\Phi_N]_{\#N}} \right\|_{L^1(U)} + \left\| e^{-[\Phi_N]_{\#N}} - \sum_{k=0}^{K(N)} \frac{(-1)^k}{k!} \left[([\Phi_N]_{\#N})^k\right]_{\#N} \right\|_{L^1(U)} \\ &=: I + II \,. \end{aligned}$$

We estimate both terms separately.

For term I, we observe that due to $x = [\Phi_N]_{\#N} - \Phi \geq 0$ for sufficiently large values of N, it holds $0 \leq 1 - e^{-x} \leq x$, so that by the triangle inequality and the bound (6.12)

$$\begin{aligned} I &= \left\| e^{-\Phi}(1 - e^{\Phi - [\Phi_N]_{\#N}}) \right\|_{L^1(U)} \leq \|\Theta\|_{L^\infty(U)} \left\| 1 - e^{-([\Phi_N]_{\#N} - \Phi)} \right\|_{L^1(U)} \\ &\leq \|\Theta\|_{L^\infty(U)} \|\Phi - [\Phi_N]_{\#N}\|_{L^1(U)} \leq C\left(\|\Phi - \Phi_N\|_{L^1(U)} + \|\Phi_N - [\Phi_N]_{\#N}\|_{L^1(U)} \right) \\ &\leq \|p - p_N\|_{L^2(U,V)} + CN^{-(\frac{1}{x}\sigma - 1)} \leq CN^{-(\frac{1}{\sigma}-1)} \end{aligned}$$

where C depends on δ, but is independent of N. In the preceding estimate, we used that $\Phi > 0$ and $0 \leq \Theta = \exp(-\Phi) < 1$ imply

$$\|\Phi - \Phi_N\|_{L^1(U)} \leq \|\mathcal{O}\|_{V^*} \|p - p_N\|_{L^2(U,V)} \left(2|\delta| + \|\mathcal{O}\|_{V^*} \|p + p_N\|_{L^2(U,V)}\right) \,.$$

We turn to term II. Using the (globally convergent) series expansion of the exponential function, we may estimate with the triangle inequality

$$II \le \left\|R_{K(N)}\right\|_{L^1(U)} + \sum_{k=0}^{K(N)} \frac{1}{k!} \left\| ([\Phi_N]_{\#N})^k - \left[([\Phi_N]_{\#N})^k\right]_{\#N} \right\|_{L^1(U)} \tag{6.14}$$

where the remainder $R_{K(N)}$ is

$$R_{K(N)} := e^{-[\Phi_N]_{\#N}} - \sum_{k=0}^{K(N)} \frac{(-1)^k}{k!}([\Phi_N]_{\#N}])^k = \sum_{k=K(N)+1}^{\infty} \frac{(-1)^k}{k!}([\Phi_N]_{\#N}])^k \,. \tag{6.15}$$

To estimate the second term in the bound (6.14) we claim

$$\forall k \in \mathbb{N},\ N \in \mathbb{N}: \quad \left\| ([\Phi_N]_{\#N})^k - \left[([\Phi_N]_{\#N})^k\right]_{\#N} \right\|_{L^\infty(U)} \le N^{-(\frac{1}{\sigma}-1)} \|(g_\nu)\|_{\ell^\sigma(\mathcal{F})}^{2k\sigma} \,. \tag{6.16}$$

We prove (6.16) for arbitrary, fixed $N \in \mathbb{N}$ by induction with respect to k. For $k = 0, 1$, the bound is obvious. Assume now that the bound has been established for all powers up to some $k \ge 2$. Writing $([\Phi_N]_{\#N})^{k+1} = ([\Phi_N]_{\#N})^k [\Phi_N]_{\#N}$ and denoting the sequence of Taylor coefficients of $[\Phi_N]^k$ by $g'_{\nu'}$ with $\nu' \in (\mathcal{F}\times\mathcal{F})^k \simeq \mathcal{F}^{2k}$, we note that by k-fold application of (6.9) it follows $\|(g'_{\nu'})\|_{\ell^\sigma(\mathcal{F}^{2k})}^{\sigma} \le \|(g_\nu)\|_{\ell^\sigma(\mathcal{F})}^{2k\sigma}$. By the definition of $[\Phi_N]_{\#N}$, the same bound also holds for the coefficients of $([\Phi_N]_{\#N})^k$, for every $k \in \mathbb{N}$. We may therefore apply Lemma 6.4 to the product $([\Phi_N]_{\#N})^k [\Phi_N]_{\#N}$ and obtain the estimate (6.16) with $k+1$ in place of k from (6.11). Inserting (6.16) into (6.14), we find

$$\begin{aligned}\sum_{k=0}^{K(N)} \frac{1}{k!} \left\| ([\Phi_N]_{\#N})^k - \left[([\Phi_N]_{\#N})^k\right]_{\#N} \right\|_{L^1(U)} &\le N^{-(\frac{1}{\sigma}-1)} \sum_{k=0}^{K(N)} \frac{1}{k!} \|(g_\nu)\|_{\ell^\sigma(\mathcal{F})}^{2k\sigma} \\ &\le N^{-(\frac{1}{\sigma}-1)} \exp(\|(g_\nu)\|_{\ell^\sigma(\mathcal{F})}^{2\sigma}) \,.\end{aligned} \tag{6.17}$$

In a similar fashion, we estimate the remainder $R_{K(N)}$ in (6.14): as the truncated Taylor expansion $[\Phi_N]_{\#N}$ converges pointwise to Φ_N and to $\Phi > 0$, for sufficiently large N, we have $[\Phi_N]_{\#N} > 0$ for all $y \in U$, so that the series (6.15) is alternating and converges pointwise. Hence its truncation error is bounded by the leading term of the tail sum:

$$\|R_{K(N)}\|_{L^\infty(U)} \le \frac{\|[\Phi_N]_{\#N}\|_{L^\infty(U)}^{K(N)+1}}{(K(N)+1)!} \le \frac{\|(g_\nu)\|_{\ell^1(\mathcal{F})}^{2(K(N)+1)}}{(K(N)+1)!} \tag{6.18}$$

Now, given N sufficiently large, we choose $K(N)$ so that the bound (6.18) is smaller than (6.17), which leads with Stirling's formula in (6.18) to the requirement

$$(K+1)\ln\left(\frac{Ae}{K}\right) \le \ln B - (\frac{1}{\sigma}-1)\ln N \tag{6.19}$$

for some constants $A, B > 0$ independent of K and N (depending on p and on (g_ν)). One verifies that (6.19) is satisfied by selecting $K(N) \simeq \ln N$.

Therefore, under Assumptions 6.1 and 6.3, we have shown how to construct an N-term approximate posterior density Θ_N by summing $K = O(\ln N)$ many terms in (6.13). The approximate posterior density has at most $O(N)$ nontrivial terms, which can be integrated exactly against the separable prior μ_0 over U in complexity that behaves log-linearly with respect to N, under Assumptions 6.1, 6.3: the construction of Θ_N requires K-fold performance of the $[\cdot]_{\#N}$-truncation operation in (6.10) of products of Taylor expansions, with each factor having at most N nontrivial entries, amounting altogether to

$$O(KN\ln N) = O(N(\ln N)^2)$$

operations.

7. Conclusion

So far, our results on sparsity and complexity of the the evaluation of conditional expectations of the unknown response $p(x)$ to the system's input $u(x)$, given the data δ, were based on the assumption that *exact* system responses p, for given u, were available. In computational practice, however, the response p given u is not exactly available, but only approximations of it by means of, for example, a Finite Element discretization. Based on the results in the present work we will, in [1], present a corresponding analysis including the error due to a Finite Element discretization of the forward problem, under slightly stronger hypotheses on the data u and f, however.

Next, we assumed in the present paper that the observation functional $\mathcal{O}(\cdot) \in V^*$ which precludes, in space dimensions 2 and higher, point observations. Once again, results which are completely analogous to those in the present paper hold also for such $\mathcal{O}$, albeit again under stronger hypotheses on u and on f. This will also be elaborated on in [1].

As indicated in [5, 6, 7, 3], the gpc parametrization of the laws of these quantities allow a choice of discretization of each gpc coeffcient of the quantity of interest by sparse tensorization of hierarchic bases in the physical domain D and the gpc basis functions $L_\nu(y)$ resp. y^ν so that the additional discretization error incurred by the discretization in D can be kept of the order of the gpc truncation error with an overall computational complexity which does not exceed that of a single, deterministic solve of the forward problem. These issues will be addressed in [1] as well.

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