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Transmission Conditions in Pre-Metric Electrodynamics

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Abstract The goal of this conceptual paper is to establish a comprehensive derivation of the electromagnetic transmission conditions at moving and deforming interfaces, since they are essential for mathematical modelling and numerical simulation. Transmission conditions are part of pre-metric electrodynamics and can therefore be stated on a differentiable manifold without any additional geometric structure. Electromagnetic fields are represented by differential forms or - alternatively - by fields of Poincaré dual multivectors. To this end, the manifold is equipped with a volume form as additional structure.

The paper gives a short introduction about pre-metric electrodynamics, in the four-dimensional relativistic and the 3+1-dimensional setting. Both settings are related diffeomorphically by a so called pre-observer. The transmission conditions are derived from Maxwell's equations in four dimensions and then decomposed, where various representations in terms of traces of differential forms or restrictions of multivector fields emerge. It is shown that motion and deformation is completely captured by a scalar transverse velocity. This enables immediate generalization to tangentially discontinuous velocity fields.

To make the concepts more tangible a simple numerical example is presented. The classical Wilsons experiment in 1913 about the electromagnetic field in a rotating non-conducting cylinder is studied, based on Finite Element analysis. The weak formulation enables assessment of a systematic source of error in the measurement process. This effect can be quantified by numerical experiments.

Keywords Electromagnetic transmission conditions · Moving and deforming interfaces · Pre-metric electrodynamics

Mathematics Subject Classification (2000) 53B50 · 78M10

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1 Introduction

For mathematical modelling and numerical simulation of electromagnetic boundary value and transmission problems, the transmission conditions for electromagnetic fields are of utmost importance. What about moving and deforming bodies?

The standard textbook approach starts from a boundary at rest and employs Lorentz transformations ("frame hopping"), it is valid for uniform translation only [15, p. 130]. A more general analysis either considers Helmholtz' vector flux theorem (see [2] and the references cited therein) or starts from a model in four dimensions, and derives the 3+1-dimensional transmission conditions by a decomposition into "space" and time relative to an observer [3]. Still, usually coordinates are introduced, Ricci calculus is employed, and metric is utilized heavily. On the other hand, the transmission conditions are part of pre-metric electrodynamics, and should therefore be devoid of any metric [6].

In this paper we derive the transmission conditions in a pre-metric setting, based on minimal structures. Space-time is modelled as a bare differentiable manifold without metric or connection, therefore no recourse to the Poincaré group. Electromagnetic fields are modelled primarily as differential forms on the manifold, in some occasions as multivector fields, in any case free of coordinates.

The paper is organized as follows. In Section 2 Maxwell's equations in four dimensions are introduced and the transmission conditions are derived. The transmission conditions will be formulated in terms of pullback of differential forms, or alternatively, in terms of restrictions of multivector fields. The vectorial representation hinges on Poincaré isomorphisms, which require volume form as further structural element. Section 3 introduces a pre-observer which enables decomposition of pre-metric electrodynamics into its familiar 3+1-dimensional form. After briefly discussing Maxwell's equations the decomposition is applied to the transmission conditions. As a new result, it is shown that the transmission conditions do depend on equivalence classes of transversal velocity fields rather than the velocity fields themselves. Again, representations in terms of multivector fields and differential forms are discussed. In Section 4, as an application, the classical Wilsons experiment [7,17] is modelled numerically by finite elements. All proofs are collected in an appendix.

2 Pre-Metric Electrodynamics in Four Dimensions

2.1 Maxwell's Equations in Four Dimensions

In relativistic physics, space-time (M,g) is defined as a time-oriented connected Lorentz manifold [12]. In a pre-metric setting we discard metric g and work with M only, a smooth differentiable manifold, which shall be connected, orientable, and paracompact. Let $n = \dim M = 4$.

We populate the manifold by introducing the space $\mathscr{C}_p(M)$ of p-chains. $\mathscr{Z}_p = \{c \in \mathscr{C}_p : \partial c = 0\}$ is the space of cycles, that are closed chains. We denote by $\mathscr{F}^p(M)$ the space of piecewise smooth differential forms, and by $\chi_p(M)$ the space of smooth multivector fields on M, $\chi_1 = \chi$. Later, the regularity requirements of the differential forms will be relaxed.

Table 1 Electromagnetic field quantities. M denotes space-time, N ”flatland” (Fig. 1), E ”space” (Fig. 2), and S surface (Fig. 3). $\kappa : N \to M$ and $\gamma_t : S \to E$ are immersions, P and P^N are families of decomposition operators that live on M and N, respectively. Each space-time quantity gives rise to a pair of decomposed quantities. The surface charges and currents are represented equivalently by differential forms or multivector fields.

Space-time quantity	Name	Decomposed quantities	Name	Definition
$F \in \mathscr{F}^2(M)$	Electromagnetic field	$B \in \mathscr{F}^2(E,\mathbb{R})$ $E \in \mathscr{F}^1(E,\mathbb{R})$	Magnetic flux density Electric field	$(B,-E) = P_2(F)$
$G \in \mathscr{F}^2(M)$	Electromagnetic excitation	$D \in \mathscr{F}^2(E,\mathbb{R})$ $H \in \mathscr{F}^1(E,\mathbb{R})$	Electric flux density Magnetic field	$(D,H) = P_2(G)$
$\mathscr{J} \in \mathscr{F}^3(M)$	Electric charge current	$\rho \in \mathscr{F}^3(E,\mathbb{R})$ $J \in \mathscr{F}^2(E,\mathbb{R})$	Electric charge density Electric current density	$(\rho,-J) = P_3(\mathscr{J})$
$\mathscr{K} \in \mathscr{F}^2(N)$	Surface charge current	$\Sigma \in \mathscr{F}^2(S,\mathbb{R})$ $K \in \mathscr{F}^1(S,\mathbb{R})$	Surface charge density Surface current density	$(\Sigma,K) = P_2^N(\mathscr{K})$
$L \in \chi_1(\kappa N)$	Surface charge current	$\sigma \in \chi_0(\gamma_t S,\mathbb{R})$ $l \in \chi_1(\gamma_t S,\mathbb{R})$	Surface charge density Surface current density	$(l,\sigma) = P^1(L)$

The relevant electromagnetic field quantities are collected in Table 1. Pre-metric electrodynamics can be formulated in terms of the following fundamental equations

$$\int_z \mathscr{J} = 0 \quad \forall z \in \mathscr{Z}_3(M), \tag{1}$$

$$\int_z F = 0 \quad \forall z \in \mathscr{Z}_2(M). \tag{2}$$

Equation (1) and (2) express the principles of *charge conservation* and *flux conservation*, respectively, (2) is called *Maxwell-Faraday’s law*. From deRham’s theorem we know that (1) implies the existence of $G \in \mathscr{F}^2(M)$ such that

$$\int_{\partial c} G = \int_c \mathscr{J} \quad \forall c \in \mathscr{C}_3(M), \tag{3}$$

this is *Maxwell-Ampère’s* law. Stokes’ theorem can be applied under differentiability assumptions, to deduce from (2) and (3) the local equations

$$\mathrm{d}F = 0, \tag{4}$$

$$\mathrm{d}G = \mathscr{J}. \tag{5}$$

2.2 Transmission Conditions in Four Dimensions

Consider the situation depicted in Fig. 1. Let N be a $(n-1)$-dimensional connected and orientable compact manifold, dubbed ”flatland”. We embed flatland in space-time by a one-parameter family of immersions $\kappa_s : N \to M$, smoothly depending on $s \in \mathbb{R}$, such that $\kappa_s(N)$ are hypersurfaces which fill space-time densely. Let $\kappa = \kappa_0$. Locally, M is divided by the *interface* $\kappa(N)$ into two manifolds with boundary, M^- and M^+, where the direction of increasing parameter s defines a transverse orientation. Since each point in M is contained in

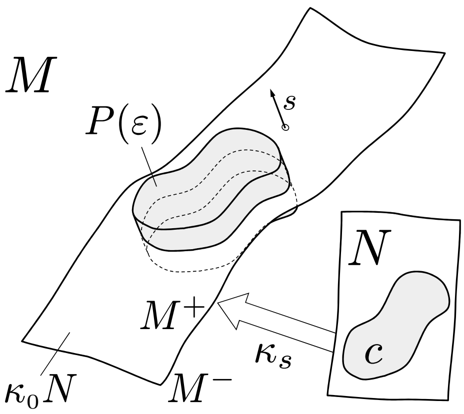

Fig. 1 A three-dimensional manifold N ("flatland") is embedded into space-time M by a one-parameter family of immersions κ_s. The continuity properties of fields at the interface $\kappa_0 N$ can be studied by means of $P(\varepsilon)$, which is defined by extrusion of a 2-chain $c \subset N$ by a small amount $\pm\varepsilon$ ("pillbox approach"). Note that the interface locally divides M into two manifolds with boundary, M^- and M^+, where the direction of increasing parameter s defines a transverse orientation.

exactly one hypersurface κ_s, s can be seen as scalar field on M, $s \in \mathscr{F}^0(M)$, so that $s = \text{const.}$ describes the embedded hypersurfaces $\kappa_s(N)$.

We define the *trace* of a differential form to be its pullback to flatland,

$$\mathbf{t} : \mathscr{F}^p(M) \to \mathscr{F}^p(N) : \omega \mapsto \mathbf{t}\omega = \kappa^*\omega. \tag{6}$$

From the point of view of space-time, the interface might happen to be a surface of discontinuity for the fields. Therefore we define single-sided trace operators as well,

$$\mathbf{t}^\pm : \mathscr{F}^p(M^\pm) \to \mathscr{F}^p(N) : \omega \mapsto \mathbf{t}^\pm\omega = \lim_{s \to \pm 0} \kappa_s^*\omega. \tag{7}$$

Our goal is to study the continuity properties of the quantities F and G. To this end, we apply the classical "pillbox approach". Pick a chain $c \in \mathscr{C}_2(N)$, send it to M and extrude it into both directions by a small parameter $\pm\varepsilon$. This yields the pillbox

$$P(\varepsilon) = \text{extr}(\kappa_s c, -\varepsilon \le s \le \varepsilon). \tag{8}$$

For the boundary of the pillbox,

$$\partial P(\varepsilon) = \kappa_\varepsilon c - \kappa_{-\varepsilon} c - \gamma(\varepsilon),$$

where $\gamma(\varepsilon) = \text{extr}(\kappa_s \partial c, -\varepsilon \le s \le \varepsilon)$. From Maxwell-Faraday's law (2) we find

$$\int_c (\kappa_\varepsilon^* F - \kappa_{-\varepsilon}^* F) - \int_{\gamma(\varepsilon)} F = 0.$$

Taking the limit $\varepsilon \to 0$ yields

$$(\mathbf{t}^+ - \mathbf{t}^-)F = 0, \tag{9}$$

since the integral over $\gamma(\varepsilon)$ vanishes in the limit, and c can be chosen arbitrarily from $\mathscr{C}_2(N)$. The electromagnetic field is said to be continuous across hypersurfaces. Differential forms with this property are called d-*conforming*. In numerical methods, this type of essential transmission condition is either included in the construction of the discrete spaces, or it is enforced by Lagrangian multipliers.

For the electromagnetic excitation, an additional term

$$\int_{P(\varepsilon)} \mathscr{J} \tag{10}$$

occurs, due to the right hand side of Maxwell-Ampère's law (3). Again, this term vanishes for $\varepsilon \to 0$ due to the regularity of $\mathscr{J}$. However, it happens that charges or currents are concentrated in thin surface layers, whose transversal extension can be regarded as zero for macroscopic modelling purposes. To account for such surface charges or currents, we equip flatland with an electric surface charge-current. It is described by a differential form $\mathscr{K} \in \mathscr{F}^2(N)$. We assign to $\mathscr{K}$ a singular electric charge-current $\mathscr{J}_{\kappa N}$ that lives in M by defining its integral over chains $c \in \mathscr{C}^3(M)$ as follows,

$$\int_c \mathscr{J}_{\kappa N} = \begin{cases} \int_{c'} \mathscr{K} & \text{if} \quad \exists c' \in \mathscr{C}_2(N) : \kappa c' = c \cap \kappa N, \\ 0 & \text{else.} \end{cases} \tag{11}$$

The singular differential form defined this way is called a *de Rham current*. It is easy to see that

$$\int_z \mathscr{J}_{\kappa N} = 0 \quad \forall z \in \mathscr{Z}_3(M) \quad \Leftrightarrow \quad \int_{z'} \mathscr{K} = 0 \quad \forall z' \in \mathscr{Z}_2(N).$$

Charge conservation (1) in space-time demands for charge-conservation in flatland,

$$\mathrm{d}^N \mathscr{K} = 0, \tag{12}$$

where d^N denotes the exterior derivative in N.
Returning to the pillbox $P(\varepsilon)$ we find that the additional term (10) evaluates to

$$\int_{P(\varepsilon)} \mathscr{J}_{\kappa N} = \int_c \mathscr{K}. \tag{13}$$

From (13) we conclude that the transmission condition for the electromagnetic excitation reads

$$(\mathbf{t}^+ - \mathbf{t}^-) G = \mathscr{K}. \tag{14}$$

At interfaces which carry an electric surface charge-current, the electromagnetic excitation exhibits a step discontinuity.

2.3 Alternative Representation of the Transmission Conditions

In the sequel, we need the *contraction operator* $\mathbf{i}(\mathbf{u}) : \mathscr{F}^p \to \mathscr{F}^{p-q}, \mathbf{u} \in \chi_q(M), 0 \leq q \leq p$, and the *multiplication operator* $\mathbf{j}(\mu) : \mathscr{F}^p \to \mathscr{F}^{p+q}, \mu \in \mathscr{F}^q(M), 0 \leq q \leq n-p$. The multiplication operator is defined by exterior product, $\mathbf{j}(\mu)\omega = \mu \wedge \omega$. Both operators can also be defined in the dual sense, by $\mathbf{i}(\mu) : \chi_p \to \chi_{p-q}, \mu \in \mathscr{F}^q(M), 0 \leq q \leq p, \mathbf{j}(\mathbf{u}) : \chi_p \to \chi_{p+q}, \mathbf{u} \in \chi_q(M), 0 \leq q \leq n-p$,. For details of the definitions see [5, p. 116]. For 1-forms and vector fields there holds the identity

$$\mathbf{i}(\mathbf{u}) \circ \mathbf{j}(\mu) + \mathbf{j}(\mu) \circ \mathbf{i}(\mathbf{u}) = \big(\mathbf{i}(\mathbf{u})\mu\big) \mathrm{Id}_{\mathscr{F}^p(M)}, \quad \mathbf{u} \in \chi(M), \mu \in \mathscr{F}^1(M). \tag{15}$$

Space-time M is orientable, there exists a non-zero *volume form* $\Omega \in \mathscr{F}^n(M)$. It is not uniquely defined, but can be scaled by a smooth positive function or sign reversed, which corresponds to reversal of orientation. In a metric setting, for given orientation, there exists a unique *metric volume form*.

With the help of the volume form we introduce the *Poincaré isomorphisms* [5, p. 151],

$$D^p : \quad \chi_p(M) \to \mathscr{F}^{n-p}(M) \quad D^p \mathbf{w} = \mathbf{i}(\mathbf{w})\Omega, \tag{16a}$$

$$D_p : \quad \mathscr{F}^p(M) \to \chi_{n-p}(M) \quad D_p \omega = \mathbf{i}(\omega)\mathbf{Z}, \tag{16b}$$

where $\mathbf{Z} \in \chi_n(M)$ is uniquely defined by $\mathbf{i}(\Omega)\mathbf{Z} = 1$. Important properties of the Poincaré isomorphisms are

$$D^{n-p} \circ D_p \quad = (-1)^{p(n-p)} \mathrm{Id}_{\mathscr{F}^p(M)}, \tag{17}$$

$$D_{p+1} \circ \mathbf{j}(\mu) = (-1)^p \quad \mathbf{i}(\mu) \circ D_p, \quad \mu \in \mathscr{F}^1(M), \tag{18}$$

$$D_{p-1} \circ \mathbf{i}(\mathbf{u}) \ = (-1)^{p-1} \mathbf{j}(\mathbf{u}) \circ D_p, \quad \mathbf{u} \in \chi(M). \tag{19}$$

Further relations can be obtained by exchanging $\mathbf{u}$ with μ and all upper and lower indices.

We define the space of *tangential* multivector fields $(\chi_p)_{\|}(M) = \{\mathbf{w} \in \chi_p(M) : \mathbf{i}(\mathrm{d}s)\mathbf{w} = 0\}$.

We will now define an operator which assigns to each p-form a tangential $(n-1-p)$-multivector field as follows

$$\mathrm{prox}_p = D_{p+1} \circ \mathbf{j}(\mathrm{d}s) : \quad \mathscr{F}^p(M) \to (\chi_{n-1-p})_{\|}(M). \tag{20}$$

The tangential property can be easily seen from (18). To better understand the action of prox_p let us for a moment think of an Euclidean manifold E with $n = 3$ and consider the magnetic flux density B and the magnetic field H, compare Table 1. We can define *Euclidean vector proxies* $\overrightarrow{B}$ and $\overrightarrow{H}$ to represent the 2-form B and 1-form H, respectively [8]. If we pick the metric volume element, then we have $\mathrm{prox}_2 B = \overrightarrow{n} \cdot \overrightarrow{B}$ and $\mathrm{prox}_1 H = \overrightarrow{n} \times \overrightarrow{H}$, [16]. The family of prox operators generalizes this notion for arbitrary n and p in a metric-free context.

Pick a transverse vector field $\mathbf{n} \in \chi(M)$, normalize it such that $\mathbf{i}(\mathbf{n})\mathrm{d}s = 1$ holds. Then we can define a volume form $\Omega^N \in \mathscr{F}^{n-1}(N)$ in a natural way by

$$\Omega^N = \mathbf{t}\,\mathbf{i}(\mathbf{n})\Omega. \tag{21}$$

It can be shown that Ω^N is independent of the choice of normalized $\mathbf{n}$, see Appendix A.1.

The volume form Ω^N induces Poincaré isomorphisms D_p^N and $(D^p)^N$ on N. The Poincaré isomorphisms exhibit the following pullback property,

$$\mathrm{prox}_p\big|_{s=0} = \kappa_* \circ D_p^N \circ \mathbf{t}. \tag{22}$$

This is a central result, proven in Appendix A.2. It says that we could either take the trace of a form, compute its flatland Poincaré isomorphic multivector field and push it forward into space-time or - equivalently - consider the restriction of the proxy multivector field to embedded flatland. With this result statements about traces in flatland can be equivalently stated in terms of restrictions of proxy multivector fields in space-time.

To work with the vector proxies on surfaces of discontinuity, the following viewpoint is helpful. Assume that $\omega \in \mathscr{F}^p(M)$ is smooth in M, except for the interface κN, where it suffers from a discontinuity. We can restrict ω to the subdomains $M^{\pm}$ and extend the restrictions smoothly beyond κN, yielding $\omega^{\pm}$. This smooth continuation procedure ensures the following property,

$$\mathbf{t}^{\pm}\omega = \mathbf{t}\omega^{\pm}. \tag{23}$$

We define the surface charge-current in its vector field representation by

$$L = \kappa_* D_2^N \mathscr{K} \quad \in \chi(\kappa N), \tag{24}$$

compare Table 1. We would like to emphasize again that the Poincaré isomorphisms depend on the volume form. Therefore, to render L well defined, the volume form Ω needs to be fixed.

With definition (24), we can re-write the transmission conditions (9) and (14), respectively, by using (22) as follows

$$\mathrm{prox}_2(F^+ - F^-)\big|_{s=0} = 0, \tag{25a}$$

$$\mathrm{prox}_2(G^+ - G^-)\big|_{s=0} = L. \tag{25b}$$

It is interesting to write these equations in the language of tensor calculus [9], where the Poincaré isomorphisms are represented by Levi-Civita tensors. From (25) and the definition (20) we obtain

$$\varepsilon^{ijkl}(F^+_{jk} - F^-_{jk})s_{,l} = 0,$$

$$\varepsilon^{ijkl}(G^+_{jk} - G^-_{jk})s_{,l} = L^i.$$

As usual, latin indices run from 0 to 3, and the comma notation stands for partial derivative with respect to the l-th coordinate. Each of the tensorial relations implies four equations, indexed by i.

3 Pre-Metric Electrodynamics in Three plus One Dimensions

3.1 Pre-Observer

The equations of pre-metric electrodynamics have been formulated without recourse to any observer. They constitute an observer independent or *absolute* description of the electromagnetic phenomenon. On the other hand, we must lay down how an observer deduces the *relative* notions from the absolute ones, which means how the observer perceives the properties of the phenomenon through measurements [11, p. 53f]. In relativistic physics, measurements are defined in terms of co-moving locally inertial frames. Without going into details it is clear that the definition of such frames depends on the metric structure, which is not at our disposal here.

In the pre-metric setting we are still able to define mappings which render space-time locally diffeomorphic to a product of a three-dimensional "space" manifold E and the one-dimensional time manifold $\mathbb{R}$. We write "space" when we wish to distinguish the manifold E from some other space. This enables decomposition of the equations of pre-metric electrodynamics into their familiar 3+1-dimensional form. However there is not yet a relation to measurements. In the sequel we call such diffeomorphisms *pre-observers*. The transition to observers involves metric and poses additional restrictions. Since every observer is also a pre-observer, the results are of general validity.

We model "space" as a $(n-1)$-dimensional differentiable manifold E, which shall be connected, orientable and paracompact. The manifold E is embedded in M by a one parameter family of immersions $\Psi_t : E \to M$, smoothly depending on time $t \in \mathbb{R}$, such that $\Psi_t(E)$ are hypersurfaces which fill space-time densely. We receive a foliation of space-time, also called slicing or hypersurface approach [1]. Each point $Q \in E$ is mapped to a smooth curve $\Psi_t(Q)$, which is called the *world line* of Q. Hence a congruence of parameterized curves, densely filling space-time, which define a vector field $\mathbf{u} \in \chi(M)$, the *absolute velocity field* of the pre-observer. For example, in a metric setting one would require $\mathbf{u}$ to be a time-like unit vector field and then call it *four velocity*. Each point in space-time is uniquely contained in one curve of the congruence (its point in "space"), and in one hypersurface of the foliation (its instant in time), see Fig. 2. Therefore t can be seen as scalar *time function* on M, $t \in \mathscr{F}^0(M)$, so that $t = \text{const.}$ describes the embedded hypersurfaces $\Psi_t E$.

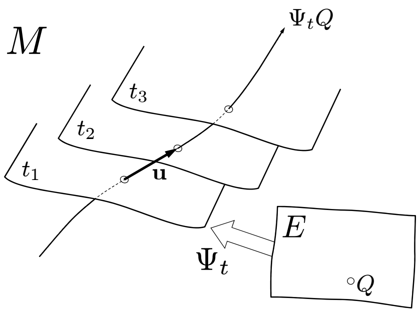

Fig. 2 A pre-observer defines a diffeomorphism between the product manifold $E \times \mathbb{R}$ and space-time M in the following way. A three-dimensional manifold E ("space") is embedded into space-time M by a one-parameter family of Ψ_t. A point Q fixed in E gives rise to a world line $\Psi_t Q$ in M. Hence a congruence of parameterized curves, densely filling space-time, which define a vector field $\mathbf{u}$, the absolute velocity field of the pre-observer.

We denote the space of time-dependent differential forms in E by $\mathscr{F}^p(E,\mathbb{R})$. The pullback $\Psi_t^* : \mathscr{F}^p(M) \to \mathscr{F}^p(E,\mathbb{R})$ maps differential forms from space-time into "space". The pullback of a differential form ω and of its contraction $\mathbf{i}(\mathbf{u})\omega$ with the vector field $\mathbf{u}$, respectively, define the *horizontal* and *transversal* pieces of observation, that are combined into a family of decomposition operators P_p [4],

$$P_p : \mathscr{F}^p(M) \to \mathscr{F}^p \times \mathscr{F}^{p-1}(E,\mathbb{R}) : \omega \mapsto \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} \Psi_t^* \omega \\ \Psi_t^* \mathbf{i}(\mathbf{u})\omega \end{pmatrix}.$$

The usual electromagnetic field quantities in 3+1-dimensions are defined by decomposition, see Table 1.

To include multivector fields we need to extend the definition of the decomposition operators. The identity dual to (15) tells us that each multivector field $\mathbf{w} \in \chi_p(M)$ can be uniquely split according to

$$\mathbf{w} = \mathbf{i}(\mathrm{d}t)\mathbf{j}(\mathbf{u})\mathbf{w} + \mathbf{j}(\mathbf{u})\mathbf{i}(\mathrm{d}t)\mathbf{w},$$

where we made use of $\mathbf{i}(\mathbf{u})\mathrm{d}t = 1$. The multivector fields $\mathbf{i}(\mathrm{d}t)\mathbf{j}(\mathbf{u})\mathbf{w}$ and $\mathbf{i}(\mathrm{d}t)\mathbf{w}$ are tangential to the hypersurfaces $t = \text{const.}$, and can therefore be represented by pushforward of time-dependent multivector fields $\hat{\mathbf{a}} \in \chi_p(E,\mathbb{R})$, and $\hat{\mathbf{b}} \in \chi_{p-1}(E,\mathbb{R})$,

$$\mathbf{w} = (\Psi_t)_* \hat{\mathbf{a}} + \mathbf{j}(\mathbf{u})(\Psi_t)_* \hat{\mathbf{b}}. \tag{26}$$

This defines the decomposition operators P^p for multivector fields,

$$P^p : \chi_p(M) \to \chi_p \times \chi_{p-1}(E,\mathbb{R}) : \mathbf{w} \mapsto \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{pmatrix}.$$

3.2 Maxwell's Equations in 3+1-Dimensions

To derive the usual form of Maxwell's equations in 3+1-dimensions, we need the following result,

$$P_p \circ \mathrm{d} = \begin{pmatrix} \hat{\mathrm{d}} & 0 \\ \frac{\partial}{\partial t} & -\hat{\mathrm{d}} \end{pmatrix} \circ P_p, \tag{27}$$

where $\hat{\mathrm{d}}$ denotes the exterior derivative in E. The first equation is easy to see, since d commutes with pullback, $\Psi_t^* \circ \mathrm{d} = \hat{\mathrm{d}} \circ \Psi_t^*$. For the second equation, we need Cartan's magic formula, $\mathbf{i}(\mathbf{u}) \circ \mathrm{d} = \mathscr{L}_\mathbf{u} - \mathrm{d} \circ \mathbf{i}(\mathbf{u})$, where $\mathscr{L}_\mathbf{u}$ is the Lie derivative. We also note that $\Psi_t^* \circ \mathscr{L}_\mathbf{u} = \partial/\partial t \circ \Psi_t^*$.

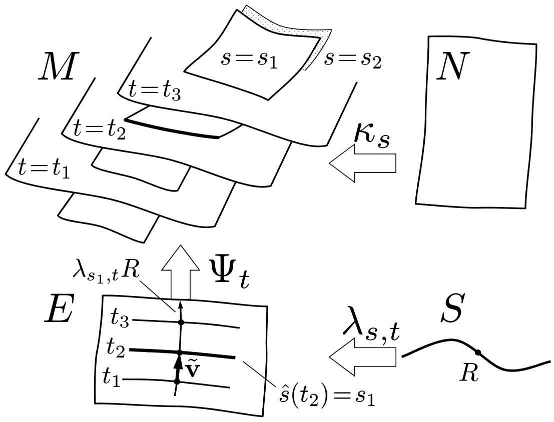

Fig. 3 Relations between space-time M, "flatland" N, "space" E, and surface S.
(i) Consider arbitrary but fixed values for (s,t), say (s_1,t_2). The intersection $\Psi_{t_2}E \cap \kappa_{s_1}N \subset M$ has a pre-image in E which is described implicitly by $\hat{s}(t_2) = s_1$.
(ii) It is assumed that the intersection has the structure of a well-behaved embedded submanifold $\lambda_{s,t} : S \to E$ for all pairs (s,t). For fixed s, say $s = s_1$, and variable t each point $R \in S$ is mapped onto its trajectory $\lambda_{s_1,t}R \subset E$. Therefore the embedding fixes the velocity field $\tilde{\mathbf{v}}$ of the interface.

By applying P_3 to (4) and (5), respectively, taking into account (27) and the definitions from Table (1), we immediately obtain Maxwell's equations in 3+1-dimensions,

$$\begin{aligned} \hat{\mathrm{d}}B &= 0, & \hat{\mathrm{d}}D &= \rho, \\ \hat{\mathrm{d}}E &= -\frac{\partial}{\partial t}B, & \hat{\mathrm{d}}H &= J + \frac{\partial}{\partial t}D. \end{aligned} \tag{28}$$

3.3 Transmission Conditions in Three Plus One Dimensions

To decompose the transmission conditions, we start from the equations (25). This has the advantage that all relevant quantities are defined in space-time M and the decomposition by pre-observer can be applied immediately. Alternatively, one could work with the equations (9) and (14). In this case, the interplay between the pre-observer and flatland had to be studied, which leads to the definition of a second pre-observer for flatland, with related decomposition operator P^N [10].

We define the volume form $\hat{\Omega} \in \mathscr{F}^{n-1}(E,\mathbb{R})$ as the transversal piece of Ω with respect to the pre-observer, up to sign,

$$\hat{\Omega} = (-1)^{n-1}\Psi_t^* \mathbf{i}(\mathbf{u})\Omega. \tag{29}$$

The sign has been chosen in a way that the orientation of M equals the orientation of $E \times \mathbb{R}$. The volume form $\hat{\Omega}$ induces Poincaré isomorphisms $\hat{D}_p$ and $\hat{D}^p$ on E. It is shown in Appendix A.3, eq. (48), that the Poincaré isomorphisms commute with the decomposition operators, up to sign.

Define $\hat{s} = \Psi_t^* s \in \mathscr{F}^0(E,\mathbb{R})$, which characterizes the pre-image of the intersection $\Psi_t E \cap \kappa_s N$ in "space" E, compare Fig. 3 (i). In the sequel, we omit the hat notation if there is no risk of confusing s and $\hat{s}$.

We are now in the position to introduce the operators $\widehat{\mathrm{prox}}$ in E, in complete analogy to (20),

$$\widehat{\mathrm{prox}}_p = \hat{D}_{p+1} \circ \mathbf{j}(\hat{d}s) : \mathscr{F}^p(E,\mathbb{R}) \to \chi_{n-2-p}(E,\mathbb{R}). \tag{30}$$

It is shown in Appendix A.4 that the decomposition of the prox operators reads

$$P^{n-1-p} \circ \mathrm{prox}_p = \begin{pmatrix} (-1)^{n-1-p}\frac{\partial s}{\partial t}\hat{D}_p & (-1)^{n-p}\widehat{\mathrm{prox}}_{p-1} \\ (-1)^{np}\widehat{\mathrm{prox}}_p & 0 \end{pmatrix} \circ P_p. \tag{31}$$

The motion manifests itself through the term $\partial s/\partial t$. For a fixed interface $\partial s/\partial t = 0$, and the prox operators commute with the decomposition operators, up to sign. Actually, the interface to be fixed is sufficient but not necessary for $\partial s/\partial t = 0$. From the pre-observer's point of view each point of the interface follows a certain trajectory in "space", parameterized by time t, hence a velocity vector field at the locus of the interface. Consider its smooth extension $\hat{\mathbf{v}} \in \chi(E,\mathbb{R})$ in a neighbourhood of the interface. In a metric setting with four-velocity $\mathbf{u}$, $\hat{\mathbf{v}}$ would be the ordinary velocity field of the moving interface with respect to the observer.

By definition, the material derivative of s with respect to this velocity field has to vanish

$$\mathscr{L}_{\hat{\mathbf{v}}} s + \frac{\partial s}{\partial t} = \mathbf{i}(\hat{\mathbf{v}})\hat{d}s + \frac{\partial s}{\partial t} = 0.$$

This gives rise to the definition of the scalar transversal velocity

$$v_\perp = \mathbf{i}(\hat{\mathbf{v}})\hat{d}s = -\frac{\partial s}{\partial t}. \tag{32}$$

The motion enters the interface conditions through the transversal velocity only.

Theorem 1 *An interface moving in tangential direction behaves with respect to pre-metric electrodynamics as if it was not moving at all.*

We apply P^1 to (25), taking into account (31) and the definitions in Table (1),

$$\widehat{\mathrm{prox}}_2(B^+ - B^-)\big|_{s=0} = 0, \tag{33a}$$
$$\widehat{\mathrm{prox}}_1(E^+ - E^-)\big|_{s=0} - v_\perp \hat{D}_2(B^+ - B^-)\big|_{s=0} = 0, \tag{33b}$$
$$\widehat{\mathrm{prox}}_2(D^+ - D^-)\big|_{s=0} = \sigma, \tag{33c}$$
$$\widehat{\mathrm{prox}}_1(H^+ - H^-)\big|_{s=0} + v_\perp \hat{D}_2(D^+ - D^-)\big|_{s=0} = l. \tag{33d}$$

This is the general metric independent form of the transmission conditions in 3+1-dimensions, in terms of the proxy multivector fields.
Remarks:

- The equations (33) are valid for arbitrary moving and deforming interfaces, not just for uniform translation. In the absence of metric and connection we could not even tell what is meant by uniform translation.
- The treatment generalizes immediately to *sliding interfaces*, where a discontinuity of the tangential velocity occurs, but the transversal velocity is continuous, by definition.
- The procedure can be easily applied to electromagnetic potentials as well.

We communicate here how the conditions look like if the differential forms are expressed by their Euclidean vector proxies [8], in the setting of special relativity, [15, p. 130],

$$\vec{n}\cdot(\vec{B}^+ - \vec{B}^-)\big|_{s=0} = 0, \tag{34a}$$
$$\vec{n}\times(\vec{E}^+ - \vec{E}^-)\big|_{s=0} - v_\perp(\vec{B}^+ - \vec{B}^-)\Big|_{s=0} = 0, \tag{34b}$$
$$\vec{n}\cdot(\vec{D}^+ - \vec{D}^-)\big|_{s=0} = \sigma, \tag{34c}$$
$$\vec{n}\times(\vec{H}^+ - \vec{H}^-)\big|_{s=0} + v_\perp(\vec{D}^+ - \vec{D}^-)\Big|_{s=0} = l, \tag{34d}$$

where $v_\perp = \overrightarrow{v} \cdot \overrightarrow{n}$. To achieve this form of the equations, the volume form $\hat{\Omega}$ has to be chosen as the Euclidean volume form[1].

Let us study the case $v_\perp \neq 0$. If we want to employ the transmission conditions for numerical purposes, then the velocity term in (34b, d) is a hindrance, since it cannot be easily recast as the trace of a differential form. It is in general not immediately clear how to discretize terms that lack the interpretation of differential forms.

To remedy this problem we introduce a velocity field $\tilde{\mathbf{v}} \in \chi(E,\mathbb{R})$, subject to the constraint

$$\mathbf{i}(\tilde{\mathbf{v}})\hat{\mathrm{d}}s = v_\perp, \tag{35}$$

not necessarily identical to $\hat{\mathbf{v}}$. Actually, (35) implies the following equivalence relation

$$\tilde{\mathbf{v}} \sim \tilde{\mathbf{v}}' \quad \Leftrightarrow \quad \tilde{\mathbf{v}} - \tilde{\mathbf{v}}' \in \chi_\parallel(E,\mathbb{R}).$$

Two velocity fields are regarded equivalent if they differ at most by a tangential field. Condition (35) ensures that $\tilde{\mathbf{v}} \in [\hat{\mathbf{v}}]$, where $[\hat{\mathbf{v}}]$ denotes the equivalence class of $\hat{\mathbf{v}}$.
Now it holds that

$$v_\perp \hat{D}_p = \widehat{\mathrm{prox}}_{p-1} \circ \mathbf{i}(\tilde{\mathbf{v}}) + (-1)^p \mathbf{j}(\tilde{\mathbf{v}}) \circ \widehat{\mathrm{prox}}_p, \tag{36}$$

see Appendix A.5. Equation (36) allows recasting of (33b, d) into the following form

$$\widehat{\mathrm{prox}}_1 \left(E^+ - E^- \; - \mathbf{i}(\tilde{\mathbf{v}})(B^+ - B^-)\right) \big|_{s=0} = 0, \tag{37a}$$
$$\widehat{\mathrm{prox}}_1 \left(H^+ - H^- + \mathbf{i}(\tilde{\mathbf{v}})(D^+ - D^-)\right) \big|_{s=0} = l - \tilde{\mathbf{v}}\sigma. \tag{37b}$$

Our goal is to return to a differential form representation of the transmission conditions in 3+1-dimensions, by leveraging (22) for $n = 3$ in E. To that end, we assume that the intersection $\Psi_t E \cap \kappa_s N$ is a $(n-2)$-dimensional connected and orientable compact submanifold for all pairs (s,t). We choose a manifold S with these properties to model the surface. Moreover, we choose a two-parameter family of immersions $\lambda_{s,t} : S \to E$, smoothly depending on (s,t), such that $\Psi_t \lambda_{s,t} S = \Psi_t E \cap \kappa_s N$ holds. This also amounts to fixing the velocity field $\tilde{\mathbf{v}}$, compare Fig. 3 (ii).

A time-dependent "spatial" trace operator can be defined by pullback, in analogy to (7),

$$\hat{\mathbf{t}}_t^\pm = \lim_{s \to \pm 0} \lambda_{s,t}^* : \mathscr{F}^p(E^\pm, \mathbb{R}) \to \mathscr{F}^p(S, \mathbb{R}). \tag{38}$$

Define a volume form on S along the same lines as in (21),

$$\Omega^S = \hat{\mathbf{t}}_t \, \mathbf{i}(\hat{\mathbf{n}})\hat{\Omega}, \quad \hat{\mathbf{n}} \in \chi(E,\mathbb{R}), \quad \mathbf{i}(\hat{\mathbf{n}})\hat{\mathrm{d}}s = 1. \tag{39}$$

This provides us with Poincaré isomorphisms D_p^S and $(D^p)^S$ on S, in a natural way, i.e. independent on the specific choice of $\hat{\mathbf{n}}$. Let $\lambda_t = \lambda_{0,t}$ and define $(\Sigma, K) \in \mathscr{F}^2 \times \mathscr{F}^1(S,\mathbb{R})$ by requiring

$$\begin{pmatrix} l - \tilde{\mathbf{v}}\sigma \\ \sigma \end{pmatrix} = (\lambda_t)_* \begin{pmatrix} 0 & D_1^S \\ D_2^S & 0 \end{pmatrix} \begin{pmatrix} \Sigma \\ K \end{pmatrix}. \tag{40}$$

Remarks:

[1] This implies a subtle point, because this means that Ω cannot be chosen at the same time as the Minkowski volume form, but rather differs by the Lorentz factor $\gamma = (1 - \hat{v}^2/c_0^2)^{-1/2}$. If one insists on using the Minkowski volume form in M, to achieve a Lorentz invariant formulation, then the Lorentz factor enters the transmission condition (25b). This has been pointed out in [3, Appendix A]. Such kind of complication is caused by the usage of Euclidean vector proxies, not by the underlying physics.

- Note that $l - \tilde{\mathbf{v}}\sigma$ lies in the range of the $\widehat{\mathrm{prox}}_1$ operator and is therefore tangential, which enables its representation by the surface current density K.
- If non-tangential motion is involved and the interface exhibits a non-zero charge density, l cannot be represented by a differential form that lives in the surface. In this case, l contains a non-tangential convective component, which is eliminated by taking the difference $l - \tilde{\mathbf{v}}\sigma$.
- As for the tangential convective component, there is freedom to balance between l (surface current in a fixed surface) and $\tilde{\mathbf{v}}\sigma$ (surface charges are convected due to the motion of the surface). The split between the two components is fixed by selecting a representative $\tilde{\mathbf{v}}$ for the velocity field.
- This implies that the surface current density form K is in general not well defined with respect to the equivalence class structure, it depends on the choice of the representative $\tilde{\mathbf{v}}$. We write $K(\tilde{\mathbf{v}})$ to emphasize this finding. A canonical choice is of course $\tilde{\mathbf{v}} = \hat{\mathbf{v}}$.
- In the case $v_\perp = 0$ there is no need to introduce $\tilde{\mathbf{v}}$, therefore we have $K = K(0)$, irrespective of possible tangential motion.

From (34a, c), (37) with definition (40) and identity (22) along with (23) we receive a concise differential form representation of the transmission conditions,

$$(\hat{\mathbf{t}}_t^+ - \hat{\mathbf{t}}_t^-)B = 0, \tag{41a}$$

$$(\hat{\mathbf{t}}_t^+ - \hat{\mathbf{t}}_t^-)\big(E - \mathbf{i}(\tilde{\mathbf{v}})B\big) = 0, \tag{41b}$$

$$(\hat{\mathbf{t}}_t^+ - \hat{\mathbf{t}}_t^-)D = \Sigma, \tag{41c}$$

$$(\hat{\mathbf{t}}_t^+ - \hat{\mathbf{t}}_t^-)\big(H + \mathbf{i}(\tilde{\mathbf{v}})D\big) = K(\tilde{\mathbf{v}}). \tag{41d}$$

The same result can be achieved directly from (9) and (14), respectively, without digressing to Poincaré isomorphisms and multivector fields [10], by introducing a pre-observer in N appropriately. However, the equivalence class structure of the velocity fields will then in general not emerge.

4 A Numerical Example

4.1 Setting of the Problem

We pick the classical Wilsons' experiment [7,17] as a simple numerical example, to illustrate the concepts of the previous sections. In the Wilsons' experiment, a hollow rotating cylinder made of a magnetic dielectric is immersed in the homogeneous magnetic field of a solenoid.

To set up a complete electrodynamic model we need to complete pre-metric electrodynamics by the metric elements that we have not discussed so far, since they are not in the scope of this paper. In the setting of special relativity space-time is modelled as Minkowski manifold (M, η), where η is the flat Minkowski metric, with signature $(1, -1, -1, -1)$. We take the Eulerian perspective and describe the experiment from the point of view of a fixed *inertial observer*. In this case, $\eta(\mathbf{u}, \mathbf{u}) = 1$ holds for the absolute velocity throughout, the world lines of the inertial observer are parallel straight lines. Moreover, the hypersurfaces $\Psi_t E$ are parallel hyperplanes perpendicular to the world lines. The axis of revolution shall be at rest with respect to the observer and we pick cylindrical standard coordinates adapted to the situation. The configuration and parameter values can be seen from Fig. 4.

By fixing the space-time metric and constraining the pre-observer we created a specific instance of pre-metric electrodynamics and pre-observer, respectively. Therefore we

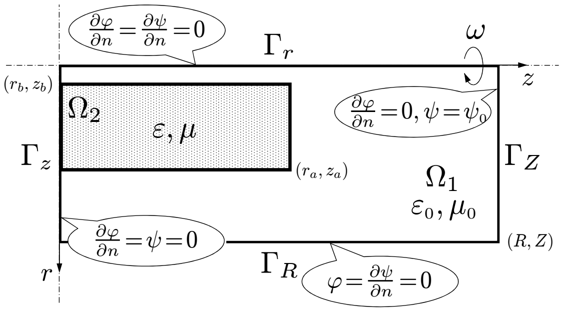

Fig. 4 Wilsons' experiment: A hollow rotating cylinder made of a magnetic dielectric is immersed in the magnetic field of a solenoid. By exploiting the symmetry it is sufficient to consider one quarter Ω_2 of its cross section. A rectangular computational domain $\Omega_1 \cup \Omega_2$ has been defined for Finite Element analysis. The electromagnetic field can be expressed in terms of a electric scalar potential φ and a magnetic scalar potential ψ, with the indicated boundary conditions. The parameters are chosen as follows: magnetic permeability $\mu = 3\,\mu_0$; dielectric constant $\varepsilon = 6\,\varepsilon_0$; angular velocity $\omega = 2\pi \cdot 100\mathrm{s}^{-1}$; cylindrical coordinates (r,ϕ,z); $(R,Z) = (r_a,z_a) + A\,r_b$; $r_b = 10\mathrm{mm}$; $z_b = 0$; $r_a = 18.65\mathrm{mm}$; $z_a = 47.5\mathrm{mm}$; $A \in \{2,10\}$.

know that Maxwell's equations (28) and transmission conditions (41) apply. The solenoid is assumed to be located outside the computational domain and supposed to create a perfect homogeneous field, which shall be taken into account by the boundary conditions. For constant angular velocity the fields are time-independent, so that we have to solve the homogeneous time-independent equations $\hat{\mathrm{d}}B = \hat{\mathrm{d}}E = \hat{\mathrm{d}}D = \hat{\mathrm{d}}H = 0$. Because the computational domain is topologically trivial, the equations for E and H can be solved by introducing the scalar electric and magnetic potentials by $E = -\hat{\mathrm{d}}\varphi$ and $H = -\hat{\mathrm{d}}\psi$, respectively.

The velocity field $\hat{\mathbf{v}}$ is tangential to all boundaries and interfaces and can therefore be removed from transmission and boundary conditions, according to Theorem 1. The motion reveals itself only by the constitutive relations, which can be stated in terms of the metric, the motion and the material properties. A moving dielectric is described by the *Minkowski relations*, see [6, equations (E.4.28), (E.4.29)] for a differential form representation. We are using the constitutive relations in the form $D = D(E,H,\hat{\mathbf{v}})$, $B = B(E,H,\hat{\mathbf{v}})$, in their low-velocity approximation

$$D = \varepsilon\,\hat{*}E + \frac{\lambda}{c^2}\mathbf{j}(\hat{v})H + \mathscr{O}(\frac{\hat{v}^2}{c^2}), \tag{42a}$$

$$B = \mu\,\hat{*}H - \frac{\lambda}{c^2}\mathbf{j}(\hat{v})E + \mathscr{O}(\frac{\hat{v}^2}{c^2}), \tag{42b}$$

where $\hat{*}$ is the Hodge star operator induced by the Euclidean metric $\hat{g}$, $c = (\mu\varepsilon)^{-1/2}$ is the velocity of light in medium, $c_0 = (\mu_0\varepsilon_0)^{-1/2}$ is the velocity of light in empty space,

$$\lambda = 1 - \frac{c^2}{c_0^2} \geq 0, \tag{43}$$

is the *dragging coefficient*, and $\hat{v} \in \mathscr{F}^1(E,\mathbb{R})$ the covariant velocity. It is defined by $\mathbf{i}(\hat{\mathbf{w}})\hat{v} = \hat{g}(\hat{\mathbf{w}},\hat{\mathbf{v}})\ \forall\hat{\mathbf{w}} \in \chi(E,\mathbb{R})$. The velocity provides a coupling between the electric and the magnetic fields. This is due to the fact that a moving electric dipole is also perceived as a magnetic dipole, and vice versa.

From Maxwell's equations, the potential ansatz and the constitutive relations we receive the strong formulation of the problem,

$$\hat{\mathrm{d}}\big(\varepsilon\,\hat{*}\,\hat{\mathrm{d}}\varphi + \frac{\lambda}{c^2}\mathbf{j}(\hat{v})\hat{\mathrm{d}}\psi\big) = 0,$$
$$\hat{\mathrm{d}}\big(\mu\,\hat{*}\,\hat{\mathrm{d}}\psi - \frac{\lambda}{c^2}\mathbf{j}(\hat{v})\hat{\mathrm{d}}\varphi\big) = 0.$$

The boundary conditions are indicated in Fig. 4. The conditions on the boundaries Γ_r and Γ_z are dictated to us by symmetry. The conditions for ψ on the remaining boundaries are chosen in a way that in the absence of the rotating cylinder there is a homogeneous magnetic field in z-direction. The conditions for φ correspond to a grounded shield with radius R that extends to infinity in z-direction.

Since the interface and the boundary are non-smooth, we relax the regularity and seek the solution of the problem in $H^1(\Omega_1 \cup \Omega_2)$, which corresponds to the weak formulation:

Find $(\varphi,\psi) \in H^1_{0,R} \times (H^1_{\psi_0,Z} \cap H^1_{0,z})$ such that

$$\begin{aligned} &\int_0^{2\pi}\int_{\Omega_1\cup\Omega_2} \big(\varepsilon\,\hat{*}\,\hat{\mathrm{d}}\varphi + \frac{\lambda}{c^2}\mathbf{j}(\hat{v})\hat{\mathrm{d}}\psi\big)\wedge\hat{\mathrm{d}}\varphi' = 0,\\ &\int_0^{2\pi}\int_{\Omega_1\cup\Omega_2} \big(\mu\,\hat{*}\,\hat{\mathrm{d}}\psi - \frac{\lambda}{c^2}\mathbf{j}(\hat{v})\hat{\mathrm{d}}\varphi\big)\wedge\hat{\mathrm{d}}\psi' = 0 \end{aligned} \tag{44}$$

holds $\forall\,(\varphi',\psi') \in H^1_{0,R} \times (H^1_{0,Z} \cap H^1_{0,z})$. The notation $H^1_{x,y} = H^1_{x,y}(\Omega_1 \cup \Omega_2)$ means that trace x is prescribed on Γ_y.

In coordinates, the integration in the azimuthal direction can be immediately eliminated.

We have existence and uniqueness by standard arguments: We can show continuity and ellipticity of the bilinear form in the energy norm

$$\|(\varphi,\psi)\|^2 := \frac{1}{2}\int_0^{2\pi}\int_{\Omega_1\cup\Omega_2} (\varepsilon\,\hat{*}\,\hat{\mathrm{d}}\varphi)\wedge\hat{\mathrm{d}}\varphi + (\mu\,\hat{*}\,\hat{\mathrm{d}}\psi)\wedge\hat{\mathrm{d}}\psi.$$

Continuity of the righthand side follows, since the Dirichlet boundary data can be extended continuously. From this we immediately have existence and uniqueness of conforming Galerkin discretization and further the error is bounded by the best approximation error.

4.2 Evaluation of the Dragging Coefficient

From the experimental point of view, the interesting quantity to be observed is the dragging coefficient (43). In pre-relativistic theories and in a more recently raised controversy [13], the coefficient should yield $\lambda' = 1 - \varepsilon_0/\varepsilon < \lambda$. With the parameters given in Fig. 4, $\lambda = 17/18$, $\lambda' = 5/6$.

For an infinite cylinder ($z_a \to \infty$) the problem becomes one-dimensional, and there is an analytical solution available [14][2], from which it can be concluded that in this case

$$\lambda = \frac{2\pi}{\omega}\frac{V(0)}{\Phi(0)}, \quad \Phi(z) = \int_{r_b}^{r_a} B_z 2\pi r\,dr, \quad V(z) = \varphi\big|_{r_a,z} - \varphi\big|_{r_b,z},$$

[2] It is an interesting feature that in this case $D = 0$ throughout. In the constitutive relation (42a), the term containing the electric field is exactly balanced by the velocity term.

Table 2 Numerical estimate for err. The first two digits of the numerical estimate for err become stable for $h \approx 1$mm. The parameter A fixes the size of the computational domain relative to the cylinder, see Fig. 4. The size of the computational domain shows no significant influence on this level.

Mesh size h	13.2mm	6.6mm	3.3mm	1.7mm	0.8mm	0.4mm
Large domain, $A = 10$	-3.1%	-2.3%	-1.9%	-1.8%	-1.7%	-1.7%
Small domain, $A = 2$	-3.1%	-2.2%	-1.9%	-1.7%	-1.7%	-1.7%

where ω is the angular velocity, $\Phi(z)$ the magnetic flux through a cross section of the cylinder at z, and $V(z)$ the potential difference between the outer and inner surface of the cylinder at z.

For a cylinder of finite length, the fields become dependent on z. It is suggested in the literature [7], to average over z in this case, i.e.

$$\lambda_{\text{measured}} := \frac{2\pi}{\omega} \frac{\langle V(z)\rangle_0^{z_a}}{\langle \Phi(z)\rangle_0^{z_a}}, \quad \langle f(z)\rangle_a^b = \frac{1}{b-a}\int_a^b f(z)\,dz. \tag{45}$$

It can be shown (see Appendix A.6) that the following relation for the error holds,

$$\text{err} = \frac{\lambda_{\text{measured}}}{\lambda} - 1 = -\left(1 + \frac{z_a \varepsilon \langle V(z)\rangle_0^{z_a}}{(Z - z_a)\varepsilon_0 \langle V(z)\rangle_{z_a}^{Z}}\right)^{-1}. \tag{46}$$

In the actual experiment the inner and outer surfaces of the cylinder are covered by a metallic coating, and the voltage drop V is measured with brushes. The introduction of the metallic coatings yields a modified boundary value problem. Without going into details, it turns out that the error analysis remains valid. Of course, in this situation $V = \langle V(z)\rangle_0^{z_a}$.
Remarks:

- Equation (46) reveals that there is a systematic error, because the true value of λ is always underestimated.
- For $z_a \to Z$ we have err $\to 0$. In this case, the boundary conditions enforce the one-dimensional situation.

4.3 Numerical Experiment

We conducted a Finite Element analysis of (44), with triangular first order standard nodal elements. To model the metallic coatings we enforce additionally the constraint $\partial_z \phi = 0$ on the inner and outer surfaces of the cylinder by the standard Lagrange multiplier technique, e.g. $\int_0^{z_a} \lambda(r_a,z)\partial_z\phi(r_a,z)dz = 0$ for Lagrange multiplier λ. We studied the parameter err for different mesh sizes h and sizes of the computational domain. The results are collected in Table 2 and an exemplary mesh and some equipotentials are displayed in Fig. 5.

To enable comparison we used the same parameters as the Wilsons in their 1913 experiment [17], which exhibited a total error of $+1.6\%$.

Proofs

A.1 Induced Volume Element

First we have

$$\mathbf{t}\mathrm{d}s = \mathrm{d}^N \mathbf{t}s = 0,$$

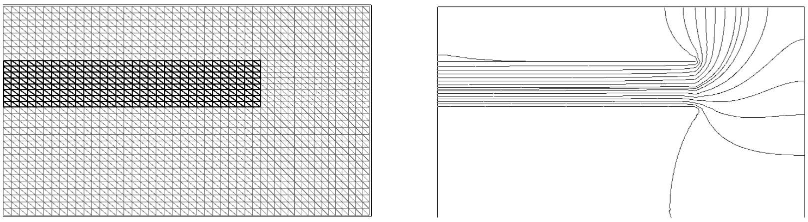

Fig. 5 (i) Left: Mesh for the numerical experiment with mesh size $h = 1.7$mm and small domain, $A = 2$, resulting in 3 234 degrees of freedom. (ii) Right: Equipotentials of the electric scalar potential φ.

because $\mathbf{t} \circ \mathrm{d} = \mathrm{d}^N \circ \mathbf{t}$. Then

$$\mathbf{t} \circ \mathbf{j}(\mathrm{d}s) = 0, \tag{47}$$

since trace distributes over the exterior product.

Let $\tilde{\Omega} = \mathbf{i}(\mathbf{n})\Omega$, $\mathbf{n}$ being a normalized transverse vector field. From (15) we conclude that we can write $\Omega = \mathbf{j}(\mathrm{d}s)\tilde{\Omega}$, independently of $\mathbf{n}$. Moreover, from the definition (21) with (15)

$$\Omega^N = \mathbf{t}\big(\tilde{\Omega} - \mathbf{j}(\mathrm{d}s)\mathbf{i}(\mathbf{n})\tilde{\Omega}\big) = \mathbf{t}(\tilde{\Omega}),$$

again independently of $\mathbf{n}$. □

A.2 Pullback of the Poincaré Isomorphisms

Let $\mathbf{w} = \mathrm{prox}_p \omega$. First consider with normalized transverse vector field $\mathbf{n} \in \chi(M)$ (i.e. $\mathbf{i}(\mathbf{n})\mathrm{d}s = 1$)

$$\begin{aligned} \mathbf{t}\,\mathbf{i}(\mathbf{n})D^{n-1-p}\mathbf{w} &= \mathbf{t}\,\mathbf{i}(\mathbf{n})D^{n-1-p}D_{p+1}\mathbf{j}(\mathrm{d}s)\omega, \\ &= (-1)^{(n-1-p)(p+1)}\mathbf{t}\,\mathbf{i}(\mathbf{n})\mathbf{j}(\mathrm{d}s)\omega \\ &= (-1)^{(n-1-p)(p+1)}\mathbf{t}\omega, \end{aligned}$$

where we used (15), (17), (20), and (47). On the other hand, with (16) and $\mathbf{i}(\mathbf{n}) \circ \mathbf{i}(\mathbf{w}) = \mathbf{i}(\mathbf{w} \wedge \mathbf{n})$

$$\begin{aligned} \mathbf{t}\,\mathbf{i}(\mathbf{n})D^{n-1-p}\mathbf{w} &= \mathbf{t}\,\mathbf{i}(\mathbf{n})\mathbf{i}(\mathbf{w})\Omega \\ &= (-1)^{n-1-p}\mathbf{t}\,\mathbf{i}(\mathbf{w})\mathbf{i}(\mathbf{n})\Omega. \end{aligned}$$

Since $\mathbf{w}$ is tangential, we know that there exists a $\mathbf{w}^N \in \chi_{n-1-p}(N)$ such that

$$\mathbf{w}|_{s=0} = \kappa_* \mathbf{w}^N.$$

With that definition we can continue, along with (16) and (21),

$$\begin{aligned} \mathbf{t}\,\mathbf{i}(\mathbf{n})D^{n-1-p}\mathbf{w} &= (-1)^{n-1-p}\mathbf{i}(\mathbf{w}^N)\mathbf{t}\,\mathbf{i}(\mathbf{n})\Omega \\ &= (-1)^{n-1-p}\mathbf{i}(\mathbf{w}^N)\Omega^N \\ &= (-1)^{n-1-p}(D^{n-1-p})^N\mathbf{w}^N. \end{aligned}$$

By equating the results we receive

$$\begin{aligned} \mathbf{t}\omega &= (-1)^{(n-1-p)(p+1)+n-1-p}(D^{n-1-p})^N\mathbf{w}^N \\ &= (-1)^{(n-1-p)p}(D^{n-1-p})^N\mathbf{w}^N, \end{aligned}$$

and therefore with (17)

$$\begin{aligned} \kappa_* D_p^N \mathbf{t}\omega &= (-1)^{(n-1-p)p}\kappa_* D_p^N (D^{n-1-p})^N\mathbf{w}^N \\ &= \kappa_* \mathbf{w}^N \\ &= \mathbf{w}|_{s=0}, \end{aligned}$$

which completes the proof. □

A.3 Decomposition of the Poincaré Isomorphisms

Let $\mathbf{w} = D_p\omega$, i.e. $\omega = \mathbf{i}(\mathbf{w})\Omega$, $(\hat{\mathbf{a}},\hat{\mathbf{b}}) = P^{n-p}\mathbf{w}$, and $(\hat{\alpha},\hat{\beta}) = P_p\omega$. Take into account (26) and consider

$$\begin{aligned}\hat{\alpha} &= \Psi_t^*\omega \\ &= \Psi_t^*\Big(\mathbf{i}\big((\Psi_t)_*\hat{\mathbf{a}} + \mathbf{j}(\mathbf{u})(\Psi_t)_*\hat{\mathbf{b}}\big)\Omega\Big) \\ &= \Psi_t^*\mathbf{i}\big((\Psi_t)_*\hat{\mathbf{b}}\big)\mathbf{i}(\mathbf{u})\Omega \\ &= (-1)^{n-1}\mathbf{i}\big(\hat{\mathbf{b}}\big)\hat{\Omega} \\ &= (-1)^{n-1}\hat{D}^{n-1-p}\hat{\mathbf{b}},\end{aligned}$$

where $\hat{D}$ denotes Poincaré isomorphism in E, and $\mathbf{i}\big(\mathbf{j}(\mathbf{u})\mathbf{w}\big) = \mathbf{i}(\mathbf{w})\circ\mathbf{i}(\mathbf{u})$, (16), (29) have been used. Along the same lines, with $\mathbf{i}(\mathbf{u})\circ\mathbf{i}(\mathbf{w}) = \mathbf{i}(\mathbf{w}\wedge\mathbf{u})$

$$\begin{aligned}\hat{\beta} &= \Psi_t^*\mathbf{i}(\mathbf{u})\omega \\ &= \Psi_t^*\mathbf{i}(\mathbf{u})\Big(\mathbf{i}\big((\Psi_t)_*\hat{\mathbf{a}} + \mathbf{j}(\mathbf{u})(\Psi_t)_*\hat{\mathbf{b}}\big)\Omega\Big) \\ &= (-1)^{n-p}\Psi_t^*\mathbf{i}\big((\Psi_t)_*\hat{\mathbf{a}}\big)\mathbf{i}(\mathbf{u})\Omega \\ &= (-1)^{p-1}\mathbf{i}\big(\hat{\mathbf{a}}\big)\hat{\Omega} \\ &= (-1)^{p-1}\hat{D}^{n-p}\hat{\mathbf{a}}.\end{aligned}$$

Let us solve for $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ by (17),

$$\begin{aligned}\hat{\mathbf{b}} &= (-1)^{n(p+1)-1}\hat{D}_p\hat{\alpha}, \\ \hat{\mathbf{a}} &= (-1)^{n-p}\hat{D}_{p-1}\hat{\beta},\end{aligned}$$

or in operator notation

$$P^{n-p}\circ D_p = \begin{pmatrix} 0 & (-1)^{(n-p)}\hat{D}_{p-1} \\ (-1)^{n(p+1)-1}\hat{D}_p & 0\end{pmatrix}\circ P_p. \tag{48}$$

Up to sign, the Poincaré isomorphisms commute with the decomposition operators. □

A.4 Decomposition of the prox Operators

As a corollary we need the relation

$$P_{p+1}\circ\mathbf{j}(\mathrm{d}s) = \begin{pmatrix} \mathbf{j}(\hat{\mathrm{d}}s) & 0 \\ \frac{\partial s}{\partial t}\mathrm{Id}_{\mathscr{F}_p(E,\mathbb{R})} & -\mathbf{j}(\hat{\mathrm{d}}s)\end{pmatrix}\circ P_p. \tag{49}$$

To proof this we first note that $\Psi_t^*\circ\mathbf{j}(\mathrm{d}s) = \mathbf{j}(\hat{\mathrm{d}}s)\circ\Psi_t^*$, which yields the first equation, and then

$$\begin{aligned}\Psi_t^*\circ\mathbf{i}(\mathbf{u})\circ\mathbf{j}(\mathrm{d}s) &= \Psi_t^*\big(\mathbf{i}(\mathrm{d}s)\mathbf{u}\big) - \Psi_t^*\circ\mathbf{j}(\mathrm{d}s)\circ\mathbf{i}(\mathbf{u}) \\ &= \Psi_t^*\mathscr{L}_{\mathbf{u}}s - \mathbf{j}(\hat{\mathrm{d}}s)\circ\Psi_t^*\circ\mathbf{i}(\mathbf{u}) \\ &= \frac{\partial s}{\partial t}\Psi_t^* - \mathbf{j}(\hat{\mathrm{d}}s)\circ\Psi_t^*\circ\mathbf{i}(\mathbf{u}),\end{aligned}$$

where we used (15) and $\mathbf{i}(\mathrm{d}s)\mathbf{u} = \mathbf{i}(\mathbf{u})\mathrm{d}s = \mathscr{L}_{\mathbf{u}}s$. From (48) and (49) we obtain the decomposition (31) of the operator prox by matrix multiplication. □

A.5 Equivalence Classes of Velocity Fields

From definition (32) with the identity dual to (15)

$$\begin{aligned} v_\perp \mathrm{Id}_{\chi_{n-1-p}(E,\mathbb{R})} &= \big(\mathbf{i}(\hat{\mathrm{d}}s)\tilde{\mathbf{v}}\big)\mathrm{Id}_{\chi_{n-1-p}(E,\mathbb{R})} \\ &= \mathbf{i}(\hat{\mathrm{d}}s)\circ\mathbf{j}(\tilde{\mathbf{v}})+\mathbf{j}(\tilde{\mathbf{v}})\circ\mathbf{i}(\hat{\mathrm{d}}s). \end{aligned}$$

Therefore with (18) and (19)

$$\begin{aligned} v_\perp \hat{D}_p &= \mathbf{i}(\hat{\mathrm{d}}s)\circ\mathbf{j}(\tilde{\mathbf{v}})\circ\hat{D}_p+\mathbf{j}(\tilde{\mathbf{v}})\circ\mathbf{i}(\hat{\mathrm{d}}s)\circ\hat{D}_p \\ &= \hat{D}_p\circ\mathbf{j}(\hat{\mathrm{d}}s)\circ\mathbf{i}(\tilde{\mathbf{v}})+(-1)^p\mathbf{j}(\tilde{\mathbf{v}})\circ\hat{D}_{p+1}\circ\mathbf{j}(\hat{\mathrm{d}}s), \end{aligned}$$

which proves (36), in connection with definition (30). □

A.6 Evaluation of the Dragging Coefficient

Consider the following manufactured test function

$$\varphi' = \begin{cases} \ln\frac{r_a}{r_b} & 0\le r\le r_b, \\ \ln\frac{r_a}{r} & r_b\le r\le r_a, \\ 0 & r_a\le r\le R. \end{cases}$$

By construction, $\varphi'\in H^1_{0,R}$ is an admissible test function for the weak formulation (44). We have

$$\hat{*}\hat{\mathrm{d}}\varphi' = \begin{cases} d\phi\wedge dz & r_b\le r\le r_a, \\ 0 & \text{otherwise}, \end{cases}$$

and therefore

$$\begin{aligned} \int_0^{2\pi}\int_{\Omega_1\cup\Omega_2}(\varepsilon\,\hat{*}\hat{\mathrm{d}}\varphi)\wedge\hat{\mathrm{d}}\varphi' &= \int_0^{2\pi}\int_{\Omega_1\cup\Omega_2}(\varepsilon\,\hat{*}\hat{\mathrm{d}}\varphi')\wedge\hat{\mathrm{d}}\varphi \\ &= 2\pi\int_0^Z\int_{r_b}^{r_a}\varepsilon\frac{\partial\varphi}{\partial r}\,drdz = 2\pi\int_0^Z\varepsilon V(z)\,dz \\ &= 2\pi z_a\varepsilon\big\langle V(z)\big\rangle_0^{z_a}+2\pi(Z-z_a)\varepsilon_0\big\langle V(z)\big\rangle_{z_a}^Z. \end{aligned}$$

Moreover, we have

$$\mathbf{j}_{\hat{v}}\hat{\mathrm{d}}\psi = -\mathbf{j}_{\hat{v}}H = -\frac{1}{\mu}\mathbf{j}_{\hat{v}}\hat{*}B+\mathscr{O}\left(\frac{\hat{v}^2}{c^2}\right) = \frac{1}{\mu}\hat{*}\mathbf{i}_{\hat{\mathbf{v}}}B+\mathscr{O}\left(\frac{\hat{v}^2}{c^2}\right).$$

We evaluate the second relevant integral, while omitting the higher order velocity terms,

$$\begin{aligned} \int_0^{2\pi}\int_{\Omega_1\cup\Omega_2}\frac{\lambda}{c^2}(\mathbf{j}_{\hat{v}}\hat{\mathrm{d}}\psi)\wedge\hat{\mathrm{d}}\varphi' &= \frac{\lambda}{c^2}\int_0^{2\pi}\int_{\Omega_2}(\mathbf{j}_{\hat{v}}\hat{\mathrm{d}}\psi)\wedge\hat{\mathrm{d}}\varphi' \\ &= \frac{\lambda}{\mu c^2}\int_0^{2\pi}\int_{\Omega_2}(\hat{*}\mathbf{i}_{\hat{\mathbf{v}}}B)\wedge\hat{\mathrm{d}}\varphi' = \varepsilon\lambda\int_0^{2\pi}\int_{\Omega_2}(\mathbf{i}_{\hat{\mathbf{v}}}B)\wedge\hat{*}\hat{\mathrm{d}}\varphi' \\ &= -\varepsilon\lambda\omega\int_0^{z_a}\int_{r_b}^{r_a}B_z\,2\pi r\,drdz = -\varepsilon\lambda\omega\int_0^{z_a}\Phi(z)\,dz \\ &= -z_a\varepsilon\lambda\omega\big\langle\Phi(z)\big\rangle_0^{z_a}. \end{aligned}$$

With (42a) and (45) evaluations of the integrals yield (46). □

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