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Variational forms for the inverses of integral logarithmic operators over an interval

Carlos Jerez-Hanckes[*][†][‡] and Jean-Claude Nédélec[§]

Abstract

We present explicit and exact variational formulations for the weakly singular and hypersingular operators over an open interval as well as for their corresponding inverses. Contrary to the case of a closed curve, these operators no longer map fractional Sobolev spaces in a dual fashion but degenerate into different subspaces depending on their extensibility by zero. We show that a symmetric and antisymmetric decomposition leads to precise coercivity results in fractional Sobolev spaces and characterize the mismatch occurring between associated functional spaces in this limiting case. Moreover, we naturally define Calderón-type identities in each case with potential use as preconditioners.

Key words: Open surface problems; Laplace equation; integral logarithmic equations; boundary integral equations; Calderón projectors.

AMS subject classifications: 45P05, 65N38, 31A10, 46E35

1 Introduction

We analyze the ubiquitous logarithmic singular operators arising when solving via boundary integral equations screen [30, 19, 28], crack or interface problems [4, 24, 25, 31], with piecewise constant coefficients in $\mathbb{R}^2$. In particular, we focus on the associated weakly- and hyper-singular boundary operators as well as on their inverses. In general, solutions over a domain $\mathcal{O} \in \mathbb{R}^2$ with boundary $\partial\mathcal{O}$ can be constructed in terms of boundary data using the single and double layer potentials [29, 17], defined over $\mathbb{R}^2 \setminus \partial\mathcal{O}$ as

$$(\Psi_{\mathrm{SL}}\varphi)(\mathbf{x}) := \int_{\partial\mathcal{O}} \log \frac{1}{\|\mathbf{x}-\mathbf{x}'\|} \varphi(\mathbf{x}')\mathrm{d}\mathbf{x}' , \tag{1}$$

$$(\Psi_{\mathrm{DL}}\alpha)(\mathbf{x}) := \int_{\partial\mathcal{O}} \partial_n \log \frac{1}{\|\mathbf{x}-\mathbf{x}'\|} \alpha(\mathbf{x}')\mathrm{d}\mathbf{x}' , \tag{2}$$

respectively, and where the normal derivative $\partial_n = \mathbf{n} \cdot \nabla_{\mathbf{x}'}$ with $\mathbf{n}$ being the unit normal vector pointing outwards for closed boundaries. After taking Dirichlet and/or Neumann traces of these potentials and imposing boundary conditions, one needs to solve a Fredholm integral equation of either first or second kind. When the boundary is closed, Calderón identities hold even for Lipschitz boundaries with their beneficial properties as preconditioners [9] and Dirichlet and Neumann trace spaces are dual to each other.

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The situation changes drastically when considering open boundaries. Indeed, Calderón identities break down due to the disappearance of the double layer boundary operator (and its adjoint) and the mapping properties of the boundary operators degenerate. Indeed, instead of working on standard Sobolev trace spaces $H^{\pm 1/2}$ (defined below), one must consider the subspaces, commonly denoted as either $\widetilde{H}^{\pm 1/2}$ or $H_{00}^{\pm 1/2}$, obtained for the positive sign by extension by zero (in negative case, by duality with $H^{1/2}$)and which are endowed with a finer topologies [8, 15]. Thus, most existing works tackle the arising integral equations separately.

On one hand, the hyper-singular operator cannot be interpreted as an integral in classical sense and one must regularize or use a variational approach to solve it [7, 3, 27]. Numerically ill-conditioned, the integral equation can be preconditioned using the standard weakly singular operator [18] but the conditioning number still grows logarithmically with mesh size. On the other hand, solutions for the weakly singular operator present strong singularities at the end points of the interval. More precisely, they behave as $1/\sqrt{d}$ where d is the distance to the end points [5, 19, 13]. Consequently, solving this type of first-kind Fredholm equation has received considerable attention in the past as the vast literature proves [1, 14, 6, 26, 10, 12], to name a few. Unfortunately, numerical solution via classic Galerkin schemes with uniform meshes yields ill-conditioned matrices for which preconditioning via standard Calderón projectors perform poorly.

The aforementioned issues can be systematically and holistically addressed when decomposing solutions over the plane into symmetric and antisymmetric parts. By doing so, one obtains: (i) exact characterizations of occurring functional spaces; (ii) mapping properties of the weakly singular and hypersingular operators; and, (iii) explicit and exact variational formulations for the operators as well as for their corresponding inverses are retrieved. Although we carry out the analysis only for the Laplace equation over the unbounded domain surrounding an interval with Dirichlet and Neumann conditions (Section 2) extensions can be immediately be carried out as perturbations. Main results are condensed in Section 3 and proofs are given in Section in 4. These last one are based on the previous observation together with extensions and combinations of many results previously derived in Hölder spaces [22], weighted L^2-spaces [20] and Tchebychev polynomials [16].

2 Preliminaries

2.1 Geometry

Without loss of generality, introduce the canonic splitting of the isotropic $\mathbb{R}^2$ into two half-planes $\pi_\pm := \left\{\mathbf{x} \in \mathbb{R}^2 : x_2 \lesseqgtr 0\right\}$ with interface Γ given by the line $x_2 = 0$. The interface is further divided into the open disjoint segments $\Gamma_c := I \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_c$, where $I := (-1, 1)$. Extension to smooth or Jordan arcs in $\mathbb{R}^2$ can be treated as compact perturbations.

2.2 Notation

Let $\mathcal{O} \subseteq \mathbb{R}^d$, with $d = 1, 2$, be open. We denote by $\mathscr{C}^k(\mathcal{O})$ the space of k-times differentiable continuous functions over $\mathcal{O}$ with $k \in \mathbb{N}_0$. Its subspace of compactly supported functions is $\mathscr{C}_0^k(\mathcal{O})$ and for infinitely differentiable functions we write $\mathscr{D}(\mathcal{O}) \equiv \mathscr{C}_0^\infty(D)$. The space of distributions or linear functionals over $\mathscr{D}(\mathcal{O})$ is $\mathscr{D}'(\mathcal{O})$. Also, let $L^p(\mathcal{O})$ be the standard class of functions with bounded L^p-norm over $\mathcal{O}$. By $\mathbb{S}'(\mathcal{O})$ we denote the Schwartz space of tempered distributions [2, Chapitre 9].

Duality products are denoted by angular brackets, $\langle \cdot, \cdot \rangle$, with subscripts accounting for the duality pairing by stating only the functional space of the second argument. Inner products are denoted by round brackets, $(\cdot, \cdot)$, with integration domains specified by subscripts. Furthermore, operators are denoted in mild calligraphic style and complex conjugates by overline. The adjoint of an operators will be specified by an asterisk.

2.3 Standard Sobolev spaces

For $s \in \mathbb{R}$, $H^s(\mathcal{O})$ denotes standard Sobolev spaces [17, Chapter 3]. Let $s \geq 0$, we say that a distribution belongs to the local Sobolev space $H^s_{\text{loc}}(\mathcal{O})$ if its restriction to every compact set $K \Subset \mathbb{R}^d$ lies in $H^s(K)$. If $s > 0$ and $\mathcal{O}$ Lipschitz, $\widetilde{H}^s(\mathcal{O})$ denotes the space of functions whose extension by zero over a closed domain $\widetilde{O}$ belongs to $H^s(\widetilde{O})$. We make the following identifications:

$$\widetilde{H}^{-1/2}(\mathcal{O}) \;\equiv\; \left(H^{1/2}(\mathcal{O})\right)' \qquad \text{and} \qquad H^{-1/2}(\mathcal{O}) \;\equiv\; \left(\widetilde{H}^{1/2}(\mathcal{O})\right)', \tag{3}$$

and if $\mathcal{O} = \widetilde{O}$, then $\widetilde{H}^{\pm 1/2}(\mathcal{O}) \equiv H^{\pm 1/2}(\mathcal{O})$.

2.4 Traces

Define restrictions over the half-planes:

$$u^\pm \;:=\; u|_{\pi_\pm}.$$

As customary, we introduce the *interior trace operators* $\gamma^\pm : \mathscr{D}(\overline{\pi}_\pm) \to \mathscr{D}(\Gamma)$ as

$$\gamma^\pm u \;:=\; \lim_{\epsilon \to 0^\pm} u(x_1, \epsilon) \;=\; \gamma^\pm u^\pm. \tag{4}$$

If $s > 1/2$, the operators $\gamma^\pm$ have unique extensions to bounded linear operators $H^s_{\text{loc}}(\pi_\pm) \to H^{s-1/2}_{\text{loc}}(\Gamma)$ [17, Chapter 3]. Furthermore, one can define the trace over a bounded subdomain Γ_b in the following way:

Theorem 2.1 (Trace theorem)**.** *Let $\Gamma_b \subset \Gamma$ be bounded. Then, we denote by $\gamma_b^\pm$ the trace operator:*

$$\begin{array}{rcccl} \gamma_b^\pm \;:\; \mathscr{D}(\overline{\pi}_\pm) & \longrightarrow & \mathscr{D}(\Gamma_b) & & \\ u^\pm & \longmapsto & \gamma_b^\pm u^\pm & = & \gamma^\pm u^\pm|_{\Gamma_b} \end{array} \tag{5}$$

If $s > 1/2$, a unique extension to a bounded linear operator $\gamma_b^\pm : H^s_{loc}(\pi_\pm) \to H^{s-1/2}(\Gamma_b)$ can be obtained by density of $\mathscr{D}(\overline{\pi}_\pm)$ in $H^s(\pi_\pm)$.

The symbol $[\gamma] := \gamma^+ - \gamma^-$ represents the jump operator across Γ. In the case of Γ, being a non-orientable manifold of codimension one, we assume $\mathbf{n}$ pointing along the positive x_2-axis, i.e. $\hat{\mathbf{x}}_2$.

2.5 Weighted Sobolev spaces

Since the problem domain is unbounded, one usually works in either local Sobolev spaces or in the following weighted Sobolev space :

$$W^{1,-1}(\Omega) \;=\; \left\{ u \in \mathscr{D}'(\Omega) \;:\; \frac{u}{(1+r^2)^{1/2}\log(2+r^2)} \;\in\; L^2(\Omega),\; \nabla u \in L^2(\Omega) \right\}, \tag{6}$$

which coincides with the standard $H^1_{\text{loc}}(\Omega)$ for a bounded part of Ω and avoid specifying behaviors at infinity [23]. Furthermore, these weighted spaces are Hilbert whereas local Sobolev spaces are only of Fréchet-type. We also define the subspace:

$$W_0^{1,-1}(\Omega) \;=\; \left\{ u \in W^{1,-1}(\Omega) \;:\; \gamma_m^\pm u \;=\; 0 \right\} \;. \tag{7}$$

The following Lemma will be useful:

Lemma 2.2 ([23], Section 2.5.4)**.** *Define the semi-norm:*

$$|u|^2_{1,-1,\Omega} \;:=\; \int_\Omega |\nabla u|^2 \;\mathrm{d}\mathbf{x}\;. \tag{8}$$

Then, there exists $c > 0$ such that

$$\|u\|_{W_0^{1,-1}(\Omega)} \;\le\; c\,|u|_{1,-1,\Omega} \quad \forall\, u \in W_0^{1,-1}(\Omega). \tag{9}$$

Moreover, the semi-norm constitutes a norm on the space $W^{1,-1}(\Omega)/\mathbb{C}$. Specifically, there exists $c > 0$ such that

$$\inf_{p\in\mathbb{C}} \|u \,-\, p\|_{W^{1,-1}(\Omega)} \;\le\; c\,|u|_{1,-1,\Omega} \quad \forall\, u \in W^{1,-1}(\Omega)\;. \tag{10}$$

Now, traces along Γ for elements in $W^{1,-1}(\Omega)$ lie in the usual $H^{1/2}_{\mathrm{loc}}(\Gamma)$, and their restriction to a bounded Γ_c generates the subspace $H^{1/2}(\Gamma_c)$. Lastly, let us introduce the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ as the subspace of $\widetilde{H}^{-1/2}(\Gamma_c)$ distributions with zero mean value, i.e.

$$\widetilde{H}_0^{-1/2}(\Gamma_c) \;=\; \left\{\varphi \in \widetilde{H}^{-1/2}(\Gamma_c) \;:\; \langle \varphi\,,\, 1\rangle_{H^{1/2}(\Gamma_c)} \;=\; 0\right\}\;. \tag{11}$$

which is related to the compatibility condition for Neumann problems.

2.6 Dirichlet Problems

We consider the Laplace problem with two different Dirichlet conditions $g^\pm$ from above and below on Γ_c. This boundary data lies in the Hilbert space:

$$\mathbb{X} := \left\{\mathbf{g} = (g^+, g^-) \in H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c) : g^+ - g^- \in \widetilde{H}^{1/2}(\Gamma_c)\right\} \tag{12}$$

with norm

$$\|\mathbf{g}\|^2_{\mathbb{X}} := \left\|g^+\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^-\right\|^2_{H^{1/2}(\Gamma_c)} + \left\|g^+ - g^-\right\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}$$

Equivalently, we define the Hilbert space for Neumann data:

$$\mathbb{Y} := \left\{\boldsymbol{\varphi} = (\varphi^+, \varphi^-) \in H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c) : \varphi^+ - \varphi^- \in \widetilde{H}_0^{-1/2}(\Gamma_c)\right\} \tag{13}$$

with similar norm. The Dirichlet problem we consider is:

Problem 2.3. For $\mathbf{g} \in \mathbb{X}$, find $u \in W^{1,-1}(\Omega)$ such that:

$$\begin{cases} -\Delta\, u \;=\; 0 & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} u \;=\; \mathbf{g} & \mathbf{x} \in \Gamma_c. \end{cases} \tag{14}$$

2.6.1 Uniqueness of solutions

Any function u in $W^{1,-1}(\Omega)$ can be split into its restrictions on $\pi_\pm$:

$$u^\pm \;:=\; u|_{\pi_\pm} \;\in\; W^{1,-1}(\pi_\pm) \tag{15}$$

By the trace Theorem 2.1, traces $\gamma^{\pm} u^{\pm} \in H^{1/2}_{\rm loc}(\Gamma)$ are well defined. By definition, if u is solution of Problem 2.3, then $\gamma^{\pm}_m u^{\pm} = g^{\pm}$. Since $\mathbf{g} \in \mathbb{X}$, the jump $[\gamma_c u]$ belongs to $\widetilde{H}^{1/2}(\Gamma_c)$. Furthermore, due to the regularity of the solution in the interior of Ω, it holds

$$\text{null Dirichlet jump:} \qquad [\gamma_f u] \;=\; 0, \tag{16}$$
$$\text{null Neumann jump:} \qquad [\gamma_f \partial_n u] \;=\; 0. \tag{17}$$

By the extension theorem [17], there exists a continuous operator $\mathcal{E}^{+}_{\Gamma} : H^{1/2}(\Gamma_c) \to H^{1/2}(\Gamma)$ extending g^{+} over Γ satisfying

$$\mathcal{E}^{+}_{\Gamma} g^{+} \in H^{1/2}(\Gamma)\;, \qquad \operatorname{supp}\left(\mathcal{E}^{+}_{\Gamma} g^{+}\right) \Subset \Gamma \qquad \text{and} \qquad \left(\mathcal{E}^{+}_{\Gamma} g^{+}\right)\big|_{\Gamma_c} \;=\; g^{+}. \tag{18}$$

On the other hand, $[g] \in \widetilde{H}^{1/2}(\Gamma_c)$ so that its extension by zero $\widetilde{[g]}$, lies in $H^{1/2}(\Gamma)$. Moreover, the extension of g^{+} to Γ is also an extension for g^{-} and we can choose the extension operator of the inferior value as one extension for the trace on Γ of $\mathcal{E}^{+}_{\Gamma} g^{+} \,-\, \widetilde{[g]}$, which we call $\mathcal{E}^{-}_{\Gamma} g^{-}$ which is also continuous.

Now, $\mathcal{E}^{+}_{\Gamma} g^{+}$ and $\mathcal{E}^{-}_{\Gamma} g^{-}$ also admit liftings with compact support in the upper and lower half-planes, respectively, provided by the continuous operators $\mathcal{R}^{\pm} : H^{1/2}(\Gamma) \to W^{1,-1}(\pi_{\pm})$. Define $v^{\pm} \in W^{1,-1}(\pi_{\pm})$ through the operator composition:

$$v^{\pm} \;:=\; \left(\mathcal{R}^{\pm} \circ \mathcal{E}^{\pm}_{\Gamma}\right) g^{\pm} \tag{19}$$

having compact support, i.e., $\operatorname{supp}(v^{\pm}) \Subset \pi_{\pm}$. Now, introduce

$$v \;:=\; v^{\pm} \text{if } \mathbf{x} \in \pi_{\pm}, \tag{20}$$

so that by (16), $v \in W^{1,-1}(\Omega)$. This allows the definition of an operator $\mathcal{A} : \mathbb{X} \to W^{1,-1}(\Omega)$ such that $v := \mathcal{A}\,\mathbf{g}$ for which it holds

$$\|\mathcal{A}\,\mathbf{g}\|_{W^{1,-1}(\Omega)} \;\leq\; C_{\mathcal{A}}\,\|\mathbf{g}\|_{\mathbb{X}} \tag{21}$$

by continuity of all the composing operators. On the other hand, continuity of the trace operators gives the following result

Lemma 2.4. *If $u \in W^{1,-1}(\Omega)$ is such that $\gamma^{\pm}_m u = g^{\pm}$ with $(g^{+}, g^{-}) \in \mathbb{X}$, then there exists a real positive constant C_X such that*

$$\|\mathbf{g}\|_{\mathbb{X}} \;\leq\; C_X\,\|u\|_{W^{1,-1}(\Omega)} \tag{22}$$

Proof. Let u be as assumed. By continuity of the trace operators $\gamma^{\pm}_c$, we have

$$\left\|g^{\pm}\right\|_{H^{1/2}(\Gamma_c)} = \left\|\gamma^{\pm}_m u\right\|_{H^{1/2}(\Gamma_c)} \leq C_{\gamma^{\pm}_m}\,\|u\|_{W^{1,-1}(\Omega)}\,.$$

On the other hand, we have $g^{+} - g^{-} \in H^{1/2}_{\rm loc}(\Gamma)$ and $g^{+} - g^{-} \equiv 0$ on Γ_f. Thus $g^{+} - g^{-} \in \widetilde{H}^{1/2}(\Gamma_c)$, and $\mathbf{g} \in \mathbb{X}$, with the above continuity. □

Since by construction $\gamma^{\pm}_c v^{\pm} = g^{\pm}$, it holds $\gamma^{\pm}_m\,(u - v) = 0$ and we can rewrite the above problem with an homogeneous Dirichlet condition:

Problem 2.5. Let v be constructed as just explained. We look for $w = u - v$ in $W^{1,-1}_0(\Omega)$ such that

$$\begin{cases} -\Delta\, w \;=\; f & \mathbf{x} \in \Omega, \\ \gamma^{\pm}_m\, w \;=\; 0 & \mathbf{x} \in \Gamma_c, \end{cases} \tag{23}$$

where $f := \Delta\, v \in \left(W^{1,-1}_0(\Omega)\right)'$.

Proposition 2.1. *Problem 2.5 has a unique solution $w \in W_0^{1,-1}(\Omega)$.*

Proof. From the distributional sense of (23), we first observe

$$-\left\langle \Delta\, w\,,\, w^t \right\rangle_{W_0^{1,-1}(\Omega)} \;=\; \left\langle f\,,\, w^t \right\rangle_{W_0^{1,-1}(\Omega)} \quad \forall\, w^t \in W_0^{1,-1}(\Omega). \tag{24}$$

Now, let B_R be the open ball of radius $R > 0$ centered at zero with boundary ∂B_R and where R is large enough so as to contain the support of f. Let $\Gamma_R := \Gamma \cap B_R$ and $B_R^{\pm} := \pi_{\pm} \cap B_R$ be upper and lower semi-circles with boundaries $\partial B_R^{\pm} = \Gamma_R \cup (\partial B_R \cap \pi_{\pm})$. On the other hand, for every $w^t \in W_0^{1,-1}(\Omega)$ it holds

$$-\left\langle \Delta w\,,\, w^t \right\rangle_{W_0^{1,-1}(B_R^{\pm})} \;=\; \left(\nabla w\,,\, \nabla w^t\right)_{B_R^{\pm}} \;-\; \left\langle \gamma_{\partial B_R^{\pm}} \partial_n w\,,\, \gamma_{\partial B_R^{\pm}} w^t \right\rangle_{H^{1/2}(\partial B_R^{\pm})}, \tag{25}$$

and addition of both parts yields

$$\begin{aligned} -\left\langle \Delta\, w\,,\, w^t \right\rangle_{W_0^{1,-1}(B_R\cap\Omega)} \;=\; &\left(\nabla w\,,\, \nabla w^t\right)_{B_R\cap\Omega} \;-\; \left\langle \gamma_{\partial B_R} \partial_n w\,,\, \gamma_{\partial B_R} w^t \right\rangle_{H^{1/2}(\partial B_R)} \\ &\sum_{\pm} \mp \left\langle \gamma_R^{\pm} \partial_n w\,,\, \gamma_R^{\pm} w^t \right\rangle_{H^{1/2}(\Gamma_R)}, \end{aligned} \tag{26}$$

where dual space subscripts are dropped for clarity. By definition of $W^{1,-1}(\Omega)$, when R tends to infinity, the term over ∂B_R goes to zero. The remaining boundary term over Γ_R extends now over Γ wherein the splitting into Γ_c and Γ_f holds. By definition $\gamma_m^{\pm} w^t = 0$ and $\gamma_f w^t = \gamma_f^{\pm} w^t$, so that the duality products over Γ_c vanish and the terms over Γ_f yield:

$$-\left\langle \gamma^+ \partial_n w\,,\, \gamma^+ w^t \right\rangle_{\Gamma} + \left\langle \gamma^- \partial_n w\,,\, \gamma^- w^t \right\rangle_{\Gamma} \;=\; -\left\langle [\gamma_f \partial_n w]\,,\, \gamma_f w^t \right\rangle_{\Gamma_f}. \tag{27}$$

By the transmission condition (17), the above contribution disappears to obtain:

$$\Phi_D(w, w^t) \;:=\; \left(\nabla w\,,\, \nabla w^t\right)_{\Omega} \;=\; \left\langle f\,,\, w^t \right\rangle_{\Omega} \qquad \forall\, w^t \;\in\; W_0^{1,-1}(\Omega). \tag{28}$$

The associated bilinear form is continuous and coercive on $W_0^{1,-1}(\Omega)$. Indeed,

$$\Phi_D(w, w) = (\nabla w\,,\, \nabla w)_{\Omega} \;=\; |w|_{1,-1,\Omega}^2 \;\geq\; c^{-2}\, \|w\|_{W_0^{1,-1}(\Omega)}^2 \tag{29}$$

by Lemma 2.2. Thus, by the Lax-Milgram theorem, we have uniqueness of w since f belongs to the dual space of $W_0^{1,-1}(\Omega)$. □

Proposition 2.2. *If $\mathbf{g} \in \mathbb{X}$, then Problem 2.3 has a unique solution in $W^{1,-1}(\Omega)$.*

Proof. Let w^* denote the solution of Problem 2.5. Then, the solution of the original Problem 2.3 is $u^* = w^* + v$ and is independent on the lifting $v \in W^{1,-1}(\Omega)$. Indeed, if we let $u_i^* = w_i^* + v_i$ denote the solution for two different liftings $i = 1, 2$, then it holds

$$\begin{cases} -\Delta\,(u_1^* - u_2^*) \;=\; 0 & \mathbf{x} \in \Omega, \\ \gamma_c^{\pm}\,(u_1^* - u_2^*) \;=\; 0 & \mathbf{x} \in \Gamma_c, \end{cases} \tag{30}$$

which has as unique solution $u_1^* - u_2^* \equiv 0$ by Proposition 2.1. □

2.6.2 Symmetric and Antisymmetric decomposition

Problem 2.3 can be split into two ones, in the following way. To any function u in $W^{1,-1}(\Omega)$, one associates restrictions $u^{\pm}$ on $\pi_{\pm}$ belonging to $W^{1,-1}(\pi_{\pm})$. Denote by $\breve{u}^{\pm} \in W^{1,-1}(\mathbb{R}^d)$ the mirror reflection of $u^{\pm}$ over $\pi_{\mp}$. Then, symmetric and antisymmetric solutions are written as

$$\begin{cases} u_s := \dfrac{\breve{u}^+ + \breve{u}^-}{2}, \\ u_{as} := \dfrac{\breve{u}^+ - \breve{u}^-}{2}, \end{cases} \qquad \text{associated to the data} \qquad \begin{cases} g_s := \dfrac{g^+ + g^-}{2}, \\ g_{as} := \dfrac{g^+ - g^-}{2}. \end{cases} \tag{31}$$

Similarly, denote by $\gamma_c^{\pm}\partial_n u^{\pm}$ the Neumann boundary value of the restricted solution over each halfplane. Then, normal traces can also be decomposed by parity. Due to the set orientation of the normal $\mathbf{n} \equiv \hat{\mathbf{x}}_2$, they take the form:

$$\begin{cases} (\partial_n)_s u := \frac{1}{2}\hat{\mathbf{x}}_2 \cdot \nabla(\breve{u}^+ - \breve{u}^-), \\ (\partial_n)_{as} u := \frac{1}{2}\hat{\mathbf{x}}_2 \cdot \nabla(\breve{u}^+ + \breve{u}^-), \end{cases} \qquad \text{associated to the values} \begin{cases} u_s = \dfrac{\breve{u}^+ + \breve{u}^-}{2}, \\ u_{as} = \dfrac{\breve{u}^+ - \breve{u}^-}{2}, \end{cases} \tag{32}$$

and we have the associated Green's formula (as $(\nabla u_s \,,\, \nabla v_{as})_\Omega = 0$):

$$(\nabla u \,,\, \nabla v)_\Omega = \langle \gamma_c(\partial_n)_s u \,,\, \gamma_c v_s \rangle_{H^{1/2}(\Gamma_c)} + \langle \gamma_c(\partial_n)_{as} u \,,\, \gamma_c v_{as} \rangle_{\widetilde{H}^{1/2}(\Gamma_c)}\,, \tag{33}$$

for $v \in W^{1,-1}(\mathbb{R}^2)$ split into symmetric and antisymmetric parts. It immediately follows,

Proposition 2.3. *The solution of the Dirichlet isotropic Problem (2.3), is such that its Neumann trace at Γ_c belongs to the space $\mathbb{Y}$. There exists a unique application $\boldsymbol{\mathcal{D}} : \mathbb{X} \to \mathbb{Y}$ associating Dirichlet traces to Neumann traces (Dirichlet-to-Neumann map or DtN). Moreover, the energy inequality holds*

$$\langle \boldsymbol{\mathcal{D}}\,\mathbf{g} \,,\, \mathbf{g} \rangle_{\Gamma_c} \;\geq\; C\,\|\mathbf{g}\|^2_{\mathbb{X}}\,, \tag{34}$$

for $\mathbf{g}$ in $\mathbb{X}$, and where the vector duality product is given by:

$$\langle \boldsymbol{\mathcal{D}}\,\mathbf{g} \,,\, \mathbf{g} \rangle_{\Gamma_c} = \langle \boldsymbol{\mathcal{D}}\,\mathbf{g}_s \,,\, \mathbf{g}_s \rangle_{H^{1/2}(\Gamma_c)} + \langle \boldsymbol{\mathcal{D}}\,\mathbf{g}_{as} \,,\, \mathbf{g}_{as} \rangle_{\widetilde{H}^{1/2}(\Gamma_c)}\,. \tag{35}$$

Proof. By Proposition 2.2, an unique continuous application $\mathcal{T}_D$ exists such that

$$\begin{array}{rccl} \mathcal{T}_D \;: & \mathbb{X} & \longrightarrow & W^{1,-1}(\Omega), \\ & \mathbf{g} & \longmapsto & u = \mathcal{T}_D \mathbf{g}. \end{array} \tag{36}$$

Due to the trace theorem 2.1, one can construct a continuous operator

$$\boldsymbol{\mathcal{D}} \;:=\; \begin{pmatrix} \gamma_c^+ \\ \gamma_c^- \end{pmatrix} \circ \partial_n \circ \mathcal{T}_D \;:\; \mathbb{X} \longrightarrow H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c),$$

belonging to $\mathbb{Y}$ since $\gamma_c^+\partial_n u - \gamma_c^-\partial_n u \in \widetilde{H}_0^{-1/2}(\Gamma_c)$. Parity decomposition follows by taking duality with v split into symmetric and antisymmetric parts using formula (33). $\square$

Corollary 2.6. *If $g^{\pm} =: g \in H^{1/2}(\Gamma_c) \setminus \mathbb{C}$, the corresponding solution of 2.3 in Ω is symmetric with respect to Γ. Moreover, there exists a unique DtN operator $\mathcal{D}_s : H^{1/2}(\Gamma_c) \setminus \mathbb{C} \to \widetilde{H}_0^{-1/2}(\Gamma_c)$. Moreover, the energy inequality holds*

$$\langle \mathcal{D}_s\, g \,,\, g \rangle_{\Gamma_c} \;\geq\; C\,\|g\|^2_{H^{1/2}(\Gamma_c)\setminus\mathbb{C}}\,. \tag{37}$$

Proof. Let $\mathbf{g} = (g, g)$ then the difference $g^+ - g^- \equiv 0$ lies trivially in $\widetilde{H}^{1/2}(\Gamma_c)$ and $\mathbf{g} \in \mathbb{X}$. Thus, Proposition 2.3 holds but now the norm is

$$\|\mathbf{g}\|_{\mathbb{X}} = 2\,\|g\|_{H^{1/2}(\Gamma_c)}\,,$$

and the duality product is

$$\sum_{\pm} \left\langle \gamma_m^{\pm}\partial_n \mathcal{T}_D \mathbf{g}\,,\, g^{\pm} \right\rangle_{\Gamma_c} \;=\; 2\,\langle [\gamma_c \partial_n \mathcal{T}_D \mathbf{g}]\;,\, g\rangle_{\Gamma_c}\,, \tag{38}$$

where $\mathcal{T}_D$ is given in (36) and factors two cancel out. We obtain the desired inequality by defining $\mathcal{D}_s := [\gamma_c \partial_n \mathcal{T}_D \mathcal{I}_{2\times 2}]$ where $\mathcal{I}_{n\times n}$ is the identity matrix of dimension n. □

Corollary 2.7. *If $g^{\pm} = \pm g \in \widetilde{H}^{1/2}(\Gamma_c)$, the associated solution of 2.3 is antisymmetric with respect to Γ. Furthermore, there exists a unique DtN operator $\mathcal{D}_{as} : \widetilde{H}^{1/2}(\Gamma_c) \to H^{-1/2}(\Gamma_c)$ Moreover, the energy inequality holds*

$$\langle \mathcal{D}_{as}\, g\,,\, g\rangle_{\Gamma_c} \;\geq\; C\,\|g\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}\,. \tag{39}$$

Proof. Define $\mathbf{g} := (g, -g)$. Then the difference $g^+ - g^-$ lies trivially in $\widetilde{H}^{1/2}(\Gamma_c)$ and $\mathbf{g} \in \mathbb{X}$. Thus, Proposition 2.3 holds but now the norm is

$$\|\mathbf{g}\|_{\mathbb{X}} = 2\,\|g\|_{\widetilde{H}^{1/2}(\Gamma_c)}\,,$$

with duality product

$$\sum_{\pm} \left\langle \gamma_m^{\pm}\partial_n \mathcal{T}_D \mathbf{g}\,,\, g^{\pm} \right\rangle_{\Gamma_c} \;=\; 2\,\langle \gamma_c \partial_n \mathcal{T}_D \mathbf{g}\,,\, g\rangle_{\Gamma_c}\,, \tag{40}$$

so that factors cancel and we obtain the desired inequality. □

2.7 Neumann problems

Problem 2.8. Find $u \in W^{1,-1}(\mathbb{R}^2)$ such that

$$\begin{cases} -\Delta\, u \;=\; 0 & \mathbf{x} \in \Omega, \\ \begin{pmatrix} \gamma_c^+ \partial_n u \\ \gamma_c^- \partial_n u \end{pmatrix} \;=\; \boldsymbol{\varphi} & \mathbf{x} \in \Gamma_c, \end{cases} \tag{41}$$

where $\boldsymbol{\varphi}$ belongs to the space $\mathbb{Y}$ and is defined in distributional sense.

Define $\mathbb{Y}_0$ as the subspace of $\mathbb{Y}$ of functions satisfying

$$\langle [\boldsymbol{\varphi}]\;,\,1\rangle_{\Gamma_c} \;=\; 0\,. \tag{42}$$

Proposition 2.4. *The Neumann Problem 2.8 has a unique solution in the space $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ if and only if $\boldsymbol{\varphi} \in \mathbb{Y}_0$.*

Proof. We have the following variational formulation:

$$\Phi_N(u, v) \;=\; (\nabla u\,,\,\nabla v)_{\mathbb{R}^2} \;=\! \sum_{\pm} \pm \left\langle \varphi^{\pm}\,,\,\gamma^{\pm} v \right\rangle_{\Gamma_c}, \qquad \forall\, v \in W^{1,-1}(\mathbb{R}^2). \tag{43}$$

Clearly, the bilinear form Φ_N is coercive and continuous. On the right hand side, the dual form is well defined only if $\boldsymbol{\varphi} \in \mathbb{Y}$, since $\gamma_c v \in \mathbb{X}$. Moreover, if v is equal to one the bilinear form is zero and thus $\boldsymbol{\varphi}$ must satisfy the compatibility condition:

$$\langle [\boldsymbol{\varphi}]\;,\,1\rangle_{\Gamma_c} \;=\; 0\,. \tag{44}$$

Consequently, if $\boldsymbol{\varphi}$ belongs to $\mathbb{Y}_0$, by the Lax-Milgram theorem, the problem has a unique solution in $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$. □

Symmetric and antisymmetric Neumann problems can stated as follows:

Problem 2.9. Find $u_s, u_{as} \in W^{1,-1}(\mathbb{R}^2)$ such that

$$\begin{cases} -\Delta\, u_s \;=\; 0, & \mathbf{x} \in \Omega, \\ [\gamma_c \partial_n u_s] \;=\; \varphi, & \mathbf{x} \in \Gamma_c, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\, u_{as} \;=\; 0, & \mathbf{x} \in \Omega, \\ \gamma_c^{\pm} \partial_n u_{as} = \phi, & \mathbf{x} \in \Gamma_c, \end{cases} \tag{45}$$

for data φ in the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ and ϕ in $H^{-1/2}(\Gamma_c)$. We will refer to the maps taking Neumann data into Dirichlet traces as *Neumann-to-Dirichlet maps* (NtD).

Proposition 2.5. *The symmetric Neumann problem 2.9 has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\varphi \in \widetilde{H}_0^{-1/2}(\Gamma_c)$*. Thus, there exists a unique continuous and invertible NtD, denoted* $\mathcal{N}_s : \widetilde{H}_0^{-1/2}(\Gamma_c) \to H^{1/2}(\Gamma_c)/\mathbb{C}$*. Moreover, the energy inequality holds*

$$\langle \mathcal{N}_s\, \varphi\,,\, \varphi \rangle_{\Gamma_c} \;\geq\; C\, \|\varphi\|^2_{\widetilde{H}_0^{-1/2}(\Gamma_c)}\,. \tag{46}$$

The inverse of this application is the operator $\mathcal{D}_s$ *defined in corollary 2.6.*

Proof. We have the following variational formulation:

$$\Phi_N(u,v) \;=\; (\nabla u\,,\, \nabla v)_{\mathbb{R}^2} \;=\; \langle \varphi\,,\, \gamma_c v \rangle_{\Gamma_c} \qquad \forall\, v \in W^{1,-1}(\mathbb{R}^2) \tag{47}$$

Clearly, the bilinear form Φ_N is coercive and continuous. On the right hand side, the dual form is well defined only if $\varphi \in \widetilde{H}^{-1/2}(\Gamma_c)$ since $\gamma_c v \in H^{1/2}(\Gamma_c)$. Moreover, if v is equal to one the bilinear form is zero and thus φ must satisfy the compatibility condition:

$$\langle \varphi\,,\, 1 \rangle_{\Gamma_c} \;=\; 0\,. \tag{48}$$

Hence, if φ belongs to $\widetilde{H}_0^{-1/2}(\Gamma_c)$, by the Lax-Milgram theorem, the problem has a unique solution in $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$. □

Proposition 2.6. *The antisymmetric Neumann problem 2.9 has a unique solution in* $W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$ *if and only if* $\phi \in H^{-1/2}(\Gamma_c)$*. Hence, there exists a unique continuous and invertible* $\mathcal{N}_{as}$ *from* $H^{-1/2}(\Gamma_c) \to \widetilde{H}^{1/2}(\Gamma_c)$*. Moreover, the energy inequality holds*

$$\langle \mathcal{N}_{as}\, \phi\,,\, \phi \rangle_{\Gamma_c} \;\geq\; C\, \|\phi\|^2_{H^{-1/2}(\Gamma_c)}\,. \tag{49}$$

The inverse of this application is the operator $\mathcal{D}_{as}$ *defined in corollary 2.7.*

Proof. By Proposition 2.4, an unique application defined over $\mathbb{R}^2$ is allowed such that

$$\begin{aligned} \mathcal{T}_N \;:\; H^{-1/2}(\Gamma_c) \quad &\longrightarrow \quad W^{1,-1}(\mathbb{R}^2)/\mathbb{C}, \\ \varphi \quad &\longmapsto \quad u \;=\; \mathcal{T}_N \varphi. \end{aligned}$$

Also, since $u \in W^{1,-1}(\mathbb{R}^2)/\mathbb{C}$, by the trace theorem, there is a unique trace from either side such that $\gamma_c^+ u = -\gamma_c^- u$ and we can construct an operator $\mathcal{N}_{as} = [\gamma_c \circ \mathcal{T}_N]$ with range in $\widetilde{H}^{1/2}(\Gamma_c)$. Thus,

$$\langle \mathcal{N}_{as}\, \varphi\,,\, \varphi \rangle_{\Gamma_c} \;=\; (\nabla u\,,\, \nabla u)_{\mathbb{R}^2} \;=\; |u|^2_{1,-1,\mathbb{R}^2} \;\geq\; C_1\, \|\gamma_c u\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \tag{50}$$

by continuity of the lifting operator. This proves the invertibility of $\mathcal{N}_{as}$. Moreover, since $\mathcal{N}_{as}$ is also continuous, it holds

$$\|\varphi\|_{H^{-1/2}(\Gamma_c)} \;=\; \left\|\mathcal{N}_{as}^{-1}\, [\gamma_c u]\right\|_{H^{-1/2}(\Gamma_c)} \;\leq\; C_2\, \|[\gamma_c u]\|_{\widetilde{H}^{1/2}(\Gamma_c)}\,, \tag{51}$$

which combined with the previous inequality yields the desired result. □

3 Main results

We now present the main results of this work: explicit variational forms or regularizations for the weakly singular and hypersingular operators over an interval and its inverse; and Calderón type identities over an interval. In fact, we will show that there exist two equivalent forms for the inverse of the weakly singular operator and two equivalent representations for the hypersingular operator. Moreover, we study the mapping properties of the underlying operators. Proofs are given in the following section.

Introduce the following integral logarithmic operators for $x \in I$:

$$\mathcal{L}_1 \varphi(x) := \int_I \log \frac{1}{|x-y|} \varphi(x)\, dx \tag{52}$$

$$\mathcal{L}_2 \varphi(x) := \int_I \log \left[\frac{M(x,y)}{|x-y|} \right] \varphi(x)\, dx, \tag{53}$$

where the first one is the standard weakly singular single layer operator and where in the second:

$$M(x,y) := \frac{1}{2} \left((y-x)^2 + (w(x)+w(y))^2 \right), \tag{54}$$

with w being the weight function $w(x) := \sqrt{1 - x^2}$ for $x \in I$. Lastly, introduce the subspace $H^{1/2}_*(\Gamma_c)$ of functions g in $H^{1/2}(\Gamma_c)$ satisfying

$$\langle g\,,\, w \rangle_{\Gamma_c} = 0. \tag{55}$$

3.1 Symmetric problem and the weakly singular operator

In this case, symmetric Dirichlet and Neumann problems are given via the simple layer potential (1) with $\partial\mathcal{O}$ replaced by Γ_c. For the Neumann version, one just simply introduces the data in the potential whereas for the Dirichlet problem one needs to solve: find φ such that

$$\mathcal{L}_1 \varphi(x) = g(x), \qquad x \in I. \tag{56}$$

This integral equation admits an explicit inverse and variational formulations for the equation as well as for its inverse are given in the following proposition:

Proposition 3.1. *The symmetric variational formulation of the integral equation* (56) *in the Hilbert space* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *is*

$$\left\langle \mathcal{L}_1 \varphi\,,\, \varphi^t \right\rangle_{\Gamma_c} = \left\langle g\,,\, \varphi^t \right\rangle_{\Gamma_c}, \qquad \forall\, \varphi^t \in \widetilde{H}_0^{-1/2}(\Gamma_c) \tag{57}$$

The associated operator is $\mathcal{N}_s$ *which is a bijection between* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *and* $H^{1/2}_*(\Gamma_c)$. *Moreover, the associated bilinear form is coercive, i.e.,*

$$\langle \mathcal{L}_1 \varphi\,,\, \varphi \rangle_{\Gamma_c} \geq C \, \|\varphi\|^2_{\widetilde{H}_0^{-1/2}(\Gamma_c)} \qquad \forall\, \varphi \in \widetilde{H}_0^{-1/2}(\Gamma_c). \tag{58}$$

The inverse operator is bijective from $H^{1/2}_*(\Gamma_c)$ *onto* $\widetilde{H}_0^{-1/2}(\Gamma_c)$ *and is associated to the operator* $\mathcal{D}_s$ *which is symmetric and coercive in the space* $H^{1/2}_*(\Gamma_c)$. *It admits two variational formulations:*

$$\frac{1}{\pi^2} \left\langle \mathcal{L}_2\, g'\,,\, (g^t)' \right\rangle_{\Gamma_c} = \langle \varphi\,,\, g^t \rangle_{\Gamma_c}, \qquad \forall\, g^t \in H^{1/2}_*(\Gamma_c), \tag{59}$$

which gives a first norm on the space $H^{1/2}_*(\Gamma_c)$:

$$\frac{1}{\pi^2} \langle \mathcal{L}_2\, g'\,,\, g' \rangle_{\Gamma_c} \geq C \, \|g\|^2_{H^{1/2}_*(\Gamma_c)}, \qquad \forall\, g \in H^{1/2}_*(\Gamma_c). \tag{60}$$

The second one is

$$\frac{1}{2\pi^2}\int_I\int_I \frac{d^2}{dxdy}\log\left[\frac{M(x,y)}{|x-y|}\right](g(x)-g(y))\left(g^t(x)-g^t(y)\right)dydx \;=\; \left\langle \varphi\,,\,g^t\right\rangle_{\Gamma_c}, \tag{61}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$*, and we obtain a second norm on the space* $H_*^{1/2}(\Gamma_c)$ *which is:*

$$\int_I\int_I \frac{1-xy}{w(x)w(y)}\frac{(g(x)-g(y))^2}{(x-y)^2}dydx \;\geq\; C\,\|g\|^2_{H_*^{1/2}(\Gamma_c)}, \qquad \forall\, g\in H_*^{1/2}(\Gamma_c). \tag{62}$$

Remark 3.1. Although the Dirichlet problem 2.3 admits a unique solution for all $g^\pm = g$ in $H^{1/2}(\Gamma_c)$, the solution to a constant data, e.g. corresponding to $\gamma_m^\pm u = 1$, is such that $\varphi = 0$. Thus the integral representation (56) cannot describe this constant solution. The exact image by the operator $\mathcal{N}_s$ of the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is the subspace $H_*^{1/2}(\Gamma_c)$ which also do not contained the trace of the constant function.

3.2 Antisymmetric problem and the hypersingular operator

The solution for the antisymmetric Dirichlet problem is retrieved by direct action of the double layer potential (2). However, for the Neumann version, one must first solve the hypersingular integral equation for α (the jump of the Dirichlet trace):

$$\varphi(x) \;=\; \fint_I \frac{1}{|x-y|^2}\alpha(y)dy, \qquad \text{for} \quad x\in I. \tag{63}$$

where the dashed integral is understood as either a finite part integral for sufficiently regular α or in weak sense for functions in Sobolev spaces.

Proposition 3.2. *A symmetric variational formulation for* (63) *in the Hilbert space* $\widetilde{H}^{1/2}(\Gamma_c)$ *is given by*

$$\left\langle \mathcal{L}_1\,\alpha'\,,\,(\alpha^t)'\right\rangle_{\Gamma_c} \;=\; \left\langle \varphi\,,\,\alpha^t\right\rangle_{\Gamma_c}, \qquad \forall\,\alpha^t\in\widetilde{H}^{1/2}(\Gamma_c) \tag{64}$$

The associated operator $\mathcal{D}_{as}$ *is a bijection from* $\widetilde{H}^{1/2}(\Gamma_c)$ *to* $H^{-1/2}(\Gamma_c)$*. Moreover, this bilinear form is coercive, i.e.,*

$$\langle \mathcal{L}_1\,\alpha'\,,\,\alpha'\rangle_{\Gamma_c} \;\geq\; C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}, \qquad \forall\,\alpha\in\widetilde{H}^{1/2}(\Gamma_c). \tag{65}$$

This operator admits an alternative variational formulation:

$$\int_I\int_I \frac{(\alpha(x)-\alpha(y))\left(\alpha^t(x)-\alpha^t(y)\right)}{|x-y|^2}dxdy + 2\int_I \frac{\alpha(x)\alpha^t(x)}{1-x^2}dx \;=\; \left\langle \varphi\,,\,\alpha^t\right\rangle_{\Gamma_c}, \tag{66}$$

for all $\alpha^t\in\widetilde{H}^{1/2}(\Gamma_c)$*, and the next expression is a norm on* $\widetilde{H}^{1/2}(\Gamma_c)$

$$\int_I\int_I \frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy + 2\int_I \frac{\alpha(x)^2}{1-x^2}dx \geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}, \qquad \forall\,\alpha\in\widetilde{H}^{1/2}(\Gamma_c) \tag{67}$$

The inverse operator is associated to the operator $\mathcal{N}_{as}^{-1} = \mathcal{D}_{as}$*, and it is a bijection of* $H^{-1/2}(\Gamma_c)$ *onto* $\widetilde{H}^{1/2}(\Gamma_c)$*, symmetric and coercive in the space* $H^{-1/2}(\Gamma_c)$*. It admits the following variational formulation:*

$$\frac{1}{\pi^2}\left\langle \mathcal{L}_2\,\varphi\,,\,\varphi^t\right\rangle_{\Gamma_c} \;=\; \left\langle \alpha\,,\,\varphi^t\right\rangle_{\Gamma_c} \qquad \forall\,\varphi\in H^{-1/2}(\Gamma_c) \tag{68}$$

and thus, the following expression is a norm on the space $H^{-1/2}(\Gamma_c)$

$$\langle \mathcal{L}_2\,\varphi\,,\,\varphi\rangle \;\geq\; C\,\|\varphi\|^2_{H^{-1/2}(\Gamma_c)}, \qquad \forall\,\varphi\in H^{-1/2}(\Gamma_c). \tag{69}$$

Proposition 3.3. *The subspace $\widetilde{H}^{1/2}(\Gamma_c)$ is exactly the functions g in the space $H_*^{1/2}(\Gamma_c)$ such that $w^{-1}g$ is in the space $L^2(\Gamma_c)$.*

The space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is exactly the image of functions which are the derivative in the usual distribution sense of functions in the space $\widetilde{H}^{1/2}(\Gamma_c)$.

3.3 Calderón-type identities

Two derivation operators have appeared in the above propositions, one whose domain lies on $\widetilde{H}^{1/2}(\Gamma_c)$ and another acting on $H_*^{1/2}(\Gamma_c)$. Since $\widetilde{H}^{1/2}(\Gamma_c)$ can be extended by zero to be a subspace of $H^{1/2}(\mathbb{R})$ which is a subspace of the distribution space $\mathbb{S}'(\mathbb{R})$ the first derivation operator, denoted by D, is defined distributionally. We will denote the second one as $-D^*$ taken in classical sense.

Proposition 3.4. *The derivation operator D is continuous and surjective from the space $\widetilde{H}^{1/2}(\Gamma_c)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_c)$, while the derivation operator $-D^*$ is continuous and surjective from the space $H_*^{1/2}(\Gamma_c)$ onto the space $H^{-1/2}(\Gamma_c)$. Moreover the operator D^* is the adjoint of the operator D with respect to the duality product in $L^2(\Gamma_c)$.*

Finally, one can prove some properties linking these derivation operators D and D^* and the logarithmic operators previously introduced just by considering the variational forms of Propositions (3.1) and (3.2).

Proposition 3.5. *The operators $\mathcal{L}_1$, $\mathcal{L}_2$, D, D^* are linked by the identities*

$$-\mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 \circ D = \mathcal{I}_{\widetilde{H}^{1/2}(\Gamma_c)}, \qquad -\mathcal{L}_1 \circ D \circ \mathcal{L}_2 \circ D^* = \mathcal{I}_{H_*^{1/2}(\Gamma_c)},$$
$$-D \circ \mathcal{L}_2 \circ D^* \circ \mathcal{L}_1 = \mathcal{I}_{\widetilde{H}_0^{-1/2}(\Gamma_c)}, \qquad -D^* \circ \mathcal{L}_2 \circ D \circ \mathcal{L}_1 = \mathcal{I}_{H^{-1/2}(\Gamma_c)}.$$

and also

$$(D^* \circ \mathcal{L}_1)^{-1} = -D \circ \mathcal{L}_2, \qquad (\mathcal{L}_2 \circ D^*)^{-1} = -\mathcal{L}_1 \circ D. \tag{71a}$$

3.4 Examples

As a by-product of these investigations, we state explicit examples of functions lying the aforementioned Sobolev spaces. These are useful to grasp the main differences. Let us introduce the following functions:

$$V_\beta(x) := \log^\beta w^2(x) \quad \text{and} \quad W_\beta(x) := w^{-2}(x) \log^\beta w^2(x), \quad \text{for } x \in \Gamma_c. \tag{72}$$

dependent on real parameters α, β.

Proposition 3.6. *The function V_β is in the space $H^{1/2}(\Gamma_c)$ if $\beta < 1/2$ and in the space $\widetilde{H}^{1/2}(\Gamma_c)$ if $\beta < -1/2$. The function W_β is in the space $H^{-1/2}(\Gamma_c)$ if $\beta < -1/2$ and in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ if $\beta < -3/2$.*

Proof. We only have to study the functions at one endpoint, let us say $(-1, 0)$. We use local polar coordinates $(r, \theta) \in \mathbb{R}^2$ with $r \in [0, \infty)$ and $\theta \in [-\pi/2, 3\pi/2)$ centered at this point and use the definition of the trace spaces by directly taking traces. Thus, when $\theta = 0$, the coordinate r along the segment is equivalent to $w^2(x)$. The function V_β is locally associated to the trace of the function $\log^\beta r$ which is in $H^1(\Omega)$ for $\beta < 1/2$ and thus V_β lies in $H^{1/2}(\Gamma_c)$. For the extension by zero, we use the function

$$Z_\beta(r, \theta) := \begin{cases} \log^\beta(r), & \text{for} \quad -\pi/2 < \theta < \pi/2, \\ \sin\theta \log^\beta(r), & \text{for} \quad \pi/2 < \theta < 3\pi/2, \end{cases} \tag{73}$$

whose trace is zero for $x < -1$ and thus belongs to $\widetilde{H}^{1/2}(\Gamma_c)$. This function is in $H^1(\Omega)$ for $\beta < -1/2$.

The results for the function W_β are a direct consequence of the properties of the operators D and D^*. □

4 Proofs for the Main Results

4.1 Analytical tools

We summarize all the results required in the proof of the above propositions. More details are provided in [11] and references therein.

4.1.1 In Hölder spaces

Denote by $\mathcal{C}^{0,\alpha}(I)$ the class of real (or complex) functions that satisfy the *Hölder condition* for every $\tau, \tau' \in I$

$$|\varphi(\tau) - \varphi(\tau')| \leq M_\alpha |\tau - \tau'|^\alpha, \qquad M_\alpha > 0,$$

for $\alpha \in]0,1[$. $\mathcal{C}_0^{0,\alpha}(I)$ is the space of $\mathcal{C}^{0,\alpha}$ functions extended by zero at the endpoints. The set $\mathcal{C}^{0,\alpha}(I)$ is a Banach space with the norm:

$$\|\varphi\|_{\mathcal{C}^{0,\alpha}(I)} = \|\varphi\|_{L^\infty(I)} + \|\varphi\|_{\alpha,I}$$

and

$$\|\varphi\|_{\alpha,I} = \sup_{\tau,\tau' \in I} \frac{|\varphi(\tau) - \varphi(\tau')|}{|\tau - \tau'|^\alpha}$$

Lemma 4.1. *Let K be compact and $0 < \alpha < \beta \leq 1$. Then the embeddings*

$$\mathcal{C}^{0,\beta}(K) \subset \mathcal{C}^{0,\alpha}(K) \subset \mathcal{C}(K)$$

are compact.

Definition 4.1. *Denote by $\mathcal{H}_\mu(I)$ the set of functions that can be represented as*

$$\varphi(t) = \frac{\tilde{\varphi}(t)}{w(t)} \tag{74}$$

where $\tilde{\varphi}(t) \in \mathcal{C}^{0,\mu}(I)$ and $w(t) = \sqrt{1-t^2}$, with norm

$$\|\varphi\|_{\mathcal{H}_\mu(I)} = \|w\varphi\|_{\mathcal{C}^{0,\mu}(I)}$$

Lemma 4.2 ([22, 21])**.** *Let $f \in \mathcal{H}_\mu(I)$ with $\mu < 1/2$. The solution $\varphi \in \mathcal{H}_\mu(I)$ of*

$$⨍_I \frac{\varphi(t)}{t-x}\, dt = f(x) \qquad \forall\, x \in I \tag{75}$$

is given by

$$\varphi(x) = -\left[\frac{1}{\pi^2} ⨍_I \frac{w(\tau)}{w(x)} \frac{f(\tau)\, d\tau}{\tau - x} + \frac{A_0}{w(x)}\right] \qquad \forall\, x \in I \tag{76}$$

4.1.2 Weighted L^2-spaces and Tchebychev polynomials

The Tchebychev polynomials $T_n(x)$ and $U_n(x)$ of first and second kinds, respectively, are polynomials of degree n, defined in $x \in I$ as [16]:

$$T_n(x) = \cos n\theta \qquad \text{and} \qquad U_n(x) = \frac{\sin(n+1)\,\theta}{\sin\theta} \tag{77}$$

with $x = \cos\theta$. These satisfy the recurrence relations:

$$P_n(x) \;=\; 2xP_{n-1}(x) \;-\; P_{n-2}(x)\;, \quad n \;=\; 2,3,\ldots, \tag{78}$$

together with initial conditions $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$ and $U_1(x) = 2x$. Furthermore, it holds

$$U_n(x) \;-\; U_{n-2}(x) \;=\; 2T_n(x)\;, \tag{79}$$

$$T_n'(x) \;=\; n\,U_{n-1}(x)\;, \tag{80}$$

$$(wU_{n-1})' \;=-\; n\frac{T_n}{w} \tag{81}$$

for $n \in \mathbb{N}$, with w defined as before. Moreover, the T_n are orthogonal with respect to w^{-1}:

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,w^{-1}(x)\,dx \;=\; \begin{cases} 0 & n \neq m\;, \\ \pi/2 & n = m \neq 0\;, \\ \pi & n = m = 0\;. \end{cases} \tag{82}$$

For the second kind Tchebychev polynomials U_n, it holds

$$\int_{-1}^{1} U_n(x)\,U_m(x)\,w(x)\,dx \;=\; \begin{cases} 0, & n \neq m\;, \\ \pi/2, & n = m \neq 0. \end{cases} \tag{83}$$

Based on the above, we can define the weighted function spaces and norms:

$$L^2_{1/w} \;:=\; \left\{u \text{ measurable} \;:\; \|f\|^2_{1/w} \;:=\; \int_{-1}^{1} |f(x)|^2\; w^{-1}(x)\,dx \;<\; \infty\right\},$$

$$L^2_{w} \;:=\; \left\{u \text{ measurable} \;:\; \|f\|^2_{w} \;:=\; \int_{-1}^{1} |f(x)|^2\; w(x)\,dx \;<\; \infty\right\},$$

and the associated space:

$$W \;=\; \left\{u \text{ measurable} \;:\; u \in L^2_{1/w}\;,\; u' \in L^2_w\right\} \tag{85}$$

with evident graph norm.

Theorem 4.3. *For a given $x \in I$, the logarithmic kernel admits the expansion on Tchebychev polynomials:*

$$\log\frac{1}{|x-y|} \;=\; \log 2 \;+\; \sum_{n=1}^{\infty} \frac{2}{n} T_n(x)T_n(y), \qquad \forall\, y \in I, \tag{86}$$

as a function in $L^2_{1/w}$. For all $(x, y) \in I \times I$, its derivatives has the following expressions

$$\frac{1}{x-y} \;=\; \sum_{n=1}^{\infty} 2U_{n-1}(x)T_n(y), \tag{87}$$

$$\frac{1}{x-y} \;=-\; \sum_{n=1}^{\infty} 2T_n(x)U_{n-1}(y), \tag{88}$$

$$\frac{d^2}{dxdy}\log\frac{1}{|x-y|} \;=\; \frac{1}{|x-y|^2} \;=\; \sum_{n=1}^{\infty} 2nU_{n-1}(x)U_{n-1}(y). \tag{89}$$

We also have for fixed $x \in I$, the following equality in L^2_w:

$$\log \frac{M(x,y)}{|x-y|} \;=\; \sum_{n=1}^{\infty} \frac{2w(x)w(y)}{n} U_{n-1}(x)U_{n-1}(y), \qquad \forall\, y \in I. \tag{90}$$

Derivatives of this last function are

$$\frac{d}{dx}\log\frac{M(x,y)}{|x-y|} \;=\; -\frac{w(y)}{w(x)}\frac{1}{x-y} \tag{91}$$

$$\frac{d}{dy}\log\frac{M(x,y)}{|x-y|} \;=\; \frac{w(x)}{w(y)}\frac{1}{x-y} \tag{92}$$

$$\frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|} \;=\; \frac{1-xy}{w(x)w(y)}\frac{1}{(x-y)^2} \tag{93}$$

$$\frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|} \;=\; \sum_{n=1}^{\infty} 2\,n\frac{T_n(x)T_n(y)}{w(x)w(y)}. \tag{94}$$

The following proposition links Hölder and weighted L^2-spaces:

Proposition 4.1 ([11])**.** *Any function $u \in \mathcal{H}_\mu(\Gamma_c)$, can be written as a series of weighted first kind Tchebychev polynomials. Moreover, $\mathcal{H}_\mu(\Gamma_c) \subset L^2_w$.*

4.2 Proof of Proposition 3.1

Proof. We obtain the coercivity of $\mathcal{L}_1$ via the variational formulation (58) and the coercivity of the aforementioned Neumann problem. To characterize the image of the operator, we multiply the expression (86) by w^{-1} and use the orthogonality properties of T_n, to first observe that:

$$(\mathcal{L}_1\, w)(\tau) \;=\; \pi \log 2, \qquad \forall\, \tau \;\in\; I. \tag{95}$$

We then multiply this expression by $\varphi(\tau) \in \widetilde{H}_0^{-1/2}(\Gamma_c)$, and integrate on τ over Γ_c (56). Since φ has a zero mean value, we obtain (55). The remaining statement concerning the logarithmic operator follows from Proposition 2.6.

In order to obtain an expression for the inverse operator, we start from the expression of its kernel in term of Tchebychev polynomials given by (86)

$$\log\frac{1}{|x-y|} \;=\; \log 2 \;+\; \sum_{n=1}^{\infty}\frac{2}{n}T_n(x)T_n(y), \qquad \forall\,(x,y) \in I \times I. \tag{96}$$

The inverse operator is in fact the restriction of the operator $\mathcal{D}_s$ to the subspace of $H^{1/2}(\Gamma_c)$ whose functions satisfy $[\gamma_c g] = 0$ and condition (55). Starting from equation (56), we obtain by derivation

$$\fint_{\Gamma_c} \frac{\varphi(t)}{x\,-\,t}\,dt \;=\; g'(x), \qquad \forall\, x \;\in\; I. \tag{97}$$

Using the inverse of this operator given in Lemma 4.2, it holds

$$\varphi(x) \;=\; -\frac{1}{\pi^2}\fint_{\Gamma_c}\frac{w(\tau)}{w(x)}\frac{g'(\tau)\,d\tau}{\tau - x} \;+\; \frac{A}{w(x)}, \qquad \forall\, x \;\in\; I. \tag{98}$$

with

$$A \;=\; \frac{1}{\pi\log 2}\left[g(x) \;+\; \frac{1}{\pi^2}\int_I \log\frac{1}{|x\,-\,t|}\fint_I \frac{w(\tau)}{w(t)}\frac{g'(\tau)\,d\tau}{\tau - t}\,dt\right]. \tag{99}$$

This is also

$$\varphi(x) \;=\; \frac{1}{\pi^2 w(x)}\frac{d}{dx}\left[\int_I \log\frac{1}{|x-\tau|} w(\tau)\, g'(\tau)\, d\tau\right] + \frac{A}{w(x)}, \qquad \forall\, x \in\; I. \tag{100}$$

We now multiply (100) by a test function g^t in the space $H_*^{1/2}(\Gamma_c)$ and integrate. From (55), it holds

$$\int_I \varphi(x) g^t(x) dx \;=\; \frac{1}{\pi^2}\int_I \frac{g^t(x)}{w(x)} ⨍_I \frac{w(\tau)}{x-\tau} g'(\tau)\, d\tau\, dx, \tag{101}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$. Integration by part leads to a first expression of the variational formulation which is

$$\frac{1}{\pi^2}\int_I\int_I \log\frac{1}{|x-\tau|}\frac{d}{dx}\left[\frac{g^t(x)}{w(x)}\right]\left[w(\tau)\, g'(\tau)\right]\, d\tau\, dx \;=\; -\int_I \varphi(x) g^t(x) dx, \tag{102}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$. Unfortunately, this formulation is not symmetric. Since we know that the result is a symmetric bilinear form, we can add its adjoint to obtain a symmetric formulation. But this is not so satisfactory. To obtain a different expression, we expand both g and g^t on the Tchebychev basis:

$$g_n \;=\; \frac{2}{\pi}\int_I \frac{g(x) T_n(x)}{w(x)} dx\;, \qquad \forall\, n \;\in\; \mathbb{N}_0, \tag{103}$$

and thus $g_0 \;=\; 0$, by the definition of $H_*^{1/2}(\Gamma_c)$. Hence,

$$g(x) \;=\; \sum_{n=1}^{\infty} g_n T_n(x)\;, \qquad x \;\in\; I, \tag{104}$$

and an equivalent expansion holds for $g^t(x)$. Now the expression of the bilinear form in (101) is

$$\begin{aligned} &\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\frac{g^t(x)}{w(x)} w(\tau)\, g'(\tau)\, d\tau\, dx \\ &= \frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\left[\sum_{n=1}^{\infty} g_n^t \frac{T_n(x)}{w(x)}\right]\left[w(\tau)\, \frac{d}{d\tau}\sum_{m=1}^{\infty} g_m T_m(\tau)\right] d\tau\, dx. \end{aligned} \tag{105}$$

Using (87) and (105), it takes the form:

$$\begin{aligned} &\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\frac{g^t(x)}{w(x)} w(\tau)\, g'(\tau)\, d\tau\, dx \\ &= \frac{1}{\pi^2}\int_I\int_I \sum_{p=1}^{\infty} 2T_p(x) U_{p-1}(\tau) \sum_{n=1}^{\infty} g_n^t \frac{T_n(x)}{w(x)} w(\tau) \sum_{m=1}^{\infty} m g_m U_{m-1}(\tau)\, d\tau dx. \end{aligned} \tag{106}$$

From the identities (82) and (83), we obtain that the only non zero contributions are when $n = p$ and $m = p$ and so

$$\begin{aligned} &\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\left[\frac{g^t(x)}{w(x)}\right]\left[w(\tau)\, g'(\tau)\right]\, d\tau\, dx \\ &= \frac{2}{\pi^2}\int_I\int_I \left[\sum_{p=1}^{\infty} p\, g_p\, g_p^t \frac{T_p^2(x)}{w(x)} w(\tau)\, U_{p-1}^2(\tau)\right] d\tau\, dx \\ &= \frac{1}{2}\sum_{p=1}^{\infty} p\, g_p\, g_p^t\;. \end{aligned} \tag{107}$$

Now, we want an expression using only the derivatives of g and g^t which are

$$g'(x) \;=\; \sum_{n=1}^{\infty} n\, g_n\, U_{n-1}(x)\,, \qquad x \;\in\; I, \tag{108a}$$

$$\big(g^t(x)\big)' \;=\; \sum_{n=1}^{\infty} n\, g_n^t\, U_{n-1}(x)\,, \qquad x \;\in\; I. \tag{108b}$$

Using the orthogonality of the U_n, i.e., (83) or relation (81), we obtain

$$g_n \;=\; \frac{2}{n\pi}\int_I U_{n-1}(x)w(x)g'(x)\,dx\,, \qquad n \;\in\; \mathbb{N}, \tag{109}$$

$$g_n^t \;=\; \frac{2}{n\pi}\int_I U_{n-1}(x)w(x)\left(g^t(x)\right)'\,dx\,, \qquad n \;\in\; \mathbb{N}, \tag{110}$$

and thus, starting from (107) together with (109) and (110), we have

$$\frac{1}{2}\sum_{n=1}^{\infty} n g_n g_n^t = \frac{1}{\pi^2}\int_I\int_I\sum_{n=1}^{\infty}\frac{2w(x)w(y)}{n}U_{n-1}(x)U_{n-1}(y)g'(x)\left(g^t(y)\right)'\,dy\,dx. \tag{111}$$

Thus, due to the expression (90), the variational formulation for the inverse operator is

$$\frac{1}{\pi^2}\int_I\int_I \log\frac{M(x,y)}{|x-y|}g'(x)\left(g^t(y)\right)'\,dy\,dx \;=\; \int_I \varphi(x)g^t(x)dx, \tag{112}$$

for all $g^t \in H_*^{1/2}(\Gamma_c)$ thus giving the stated result by density arguments.

Lastly, one obtains the variational formulation (61) by first noticing that the finite part of the associated kernel is such that

$$⨍_I \frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|}\,dx = \;0, \qquad \forall\, y \;\in\; I. \tag{113}$$

This identity is obtained by integrating expression (94) over I and using the orthogonality of the basis T_n (82). From (113), we also have

$$g(y)g^t(y)\left\{⨍_I \frac{d^2}{dxdy}\log\frac{M(x,y)}{|x-y|}\,dx\right\} = \;0. \tag{114}$$

Now we express the kernel and the functions in the bilinear form (61) using their expansion on the Tchebychev polynomials T_n. In this long expression, all the terms related to the products $g(x)g^t(x)$ and $g(y)g^t(y)$ vanished. Thus, we recover the expression (111) with a factor two. □

Remark 4.4. The different expansions of the integral kernels in terms of Tchebychev polynomials given by (86), (87), (90), (91), (94) are absolutely essential in our proof. By doing so, one takes exactly into account all the finite parts which appear due to the non-integrable kernels.

4.3 Proof of Proposition 3.2

Proof. Using the variational formulation and the coercivity of the Neumann problem 2.9 we obtain the coercivity of the hypersingular operator. The inverse operator is the restriction of the operator $\mathcal{N}_{as}$, defined in Proposition 2.6, to the space of $\widetilde{H}^{1/2}(\Gamma_c)$ and is also coercive in this space (Proposition 2.7).

Starting from equation (63) we obtain, by integration by parts,

$$\varphi(x) \;=\; ⨍_{\Gamma_c}\frac{1}{x-y}\alpha'(y)dy, \qquad \text{for} \quad x \in \Gamma\,. \tag{115}$$

This equation can also be written as

$$\varphi(x) \;=\; -\frac{d}{dx}(\mathcal{L}_1\,\alpha')(x), \qquad \text{for} \quad x \in \Gamma\,. \tag{116}$$

Multiplying by a test function α^t and integrating by parts, we obtain the variational formulation (64):

$$\left\langle \mathcal{L}_1\,\alpha'\,,\,(\alpha^t)'\right\rangle \;=\; \left\langle \varphi\,,\,\alpha^t\right\rangle, \qquad \forall\,\alpha^t \in\, \widetilde{H}^{1/2}(\Gamma_c). \tag{117}$$

We can expand the functions α/w and α^t/w on the Tchebychev polynomials U_n. All these functions are zero at the ends of the domain Γ_c and they belong to the space $\widetilde{H}^{1/2}(\Gamma_c)$, thus the summation starts at $n=0$. We have, by density arguments,

$$\alpha(x) \;=\; \sum_{n=0}^{\infty} \alpha_n w(x) U_n(x)\,, \qquad x \,\in\, I, \tag{118a}$$

$$\text{with} \qquad \alpha_n \;=\; \frac{2}{\pi}\int_I \alpha(x) U_n(x) dx\,, \qquad n \,\in\, \mathbb{N}. \tag{118b}$$

and equivalently for a test function α^t. Thus, the quadratic form associated to the integral kernel in (115) is formally

$$⨍_I ⨍_I \frac{1}{|x-y|^2}\alpha(x)\alpha(y)dxdy = \sum_{n=0}^{\infty}(n+1)\,\alpha_n^2 \tag{119}$$

but this finite part appearing in the first hand of the equality is not clearly defined. Using the identity (81), we express the derivatives of the functions α and α^t on the functions T_n/w

$$\frac{d}{dx}(\alpha(x)) \;=\; -\sum_{n=1}^{\infty} n\,\alpha_{n-1}\frac{T_n(x)}{w(x)}\,, \qquad x \,\in\, I, \tag{120a}$$

$$\frac{d}{dy}(\alpha^t(y)) \;=\; -\sum_{n=1}^{\infty} n\,\alpha^t_{n-1}\frac{T_n(y)}{w(y)}\,, \qquad y \,\in\, I. \tag{120b}$$

Finally, using the orthogonality of the Tchebychev polynomials U_n and the expression (86), the bilinear form (66) takes the form

$$\left\langle \mathcal{L}_1\,\alpha'\,,\,(\alpha^t)'\right\rangle \;=\; \frac{\pi^2}{2}\sum_{n=0}^{\infty}(n+1)\,\alpha_n\alpha_n^t. \tag{121}$$

In order to obtain the variational formulation (66), we first remark that the finite part of the associated kernel is such that

$$⨍_I \frac{1}{x-y}dx \;=\; -\log\left(\frac{|1-y|}{|1+y|}\right), \qquad \forall\, y \,\in\, I, \tag{122}$$

and thus, by derivation in the variable y, we have

$$⨍_I \frac{1}{|x-y|^2}dx \;=\; \frac{d}{dy}⨍_I \frac{1}{x-y}dx = \frac{2}{1-y^2}, \qquad \forall\, y \,\in\, I. \tag{123}$$

From this identity we also have

$$\alpha(y)\alpha^t(y)⨍_I \frac{1}{|x-y|^2}dx \;=\; 2\frac{\alpha(y)\alpha^t(y)}{1-y^2}, \qquad \forall\, y \,\in\, I. \tag{124}$$

One can express the kernel and the function in the bilinear form (66) using their expansions on the functions wU_n, where U_n are the second type Tchebychev polynomials. In this long expression, all the terms related to the products $\alpha(x)\alpha^t(x)$ and $\alpha(y)\alpha^t(y)$ are known from the identity (124). Thus we recover the expression (119) with a factor 2.

An expression for the inverse of the double layer potential is retrieved by using the inverse of the operator in (115) given by Lemma (4.2). This yields

$$\alpha'(x) \;=\; -\frac{1}{\pi^2}⨍_I \frac{w(\tau)}{w(x)}\frac{\varphi(\tau)\,d\tau}{\tau - x} \;+\; \frac{A}{w(x)} \qquad \forall\, x \,\in\, I. \tag{125}$$

As the function α' has a zero coefficient on T_0, the coefficient A is zero. Thus, (125) is also

$$\alpha'(x) \;=\; -\frac{1}{\pi^2}⨍_I \frac{w(\tau)}{w(x)}\frac{\varphi(\tau)\,d\tau}{\tau - x} \qquad \forall\, x \,\in\, I. \tag{126}$$

We expand φ and φ^t on Tchebychev polynomials U_n:

$$\varphi(x) \;=\; \sum_{n=0}^{\infty}\varphi_n U_n(x)\,, \qquad x \,\in\, I, \tag{127a}$$

$$\text{with} \qquad \varphi_n \;=\; \frac{2}{\pi}\int_I \varphi(x)w(x)U_n(x)dx\,, \qquad \forall\, n \,\in\, \mathbb{N}. \tag{127b}$$

The function φ^t admits a primitive:

$$\beta^t(y) \;=\; \sum_{n=1}^{\infty}\frac{1}{n}\varphi^t_{n-1}T_n(y)\,, \qquad y \,\in\, I. \tag{128}$$

Multiplication of (126) by a test function β^t and integration by parts of the left hand side, yields

$$\int_I \alpha(x)\varphi^t(x)dx \;=\; \frac{1}{\pi^2}\int_I \frac{\beta^t(x)}{w(x)}⨍_I \frac{w(\tau)}{x-\tau}\varphi(\tau)\,d\tau\,dx, \tag{129}$$

for all $\varphi^t \in H^{-1/2}(\Gamma_c)$. Now the expression of the bilinear form in (129) is

$$\begin{aligned}&\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)}w(\tau)\,\varphi(\tau)\,d\tau\,dx\\ &= \frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\left[\sum_{n=1}^{\infty}\frac{1}{n}\varphi^t_{n-1}\frac{T_n(x)}{w(x)}\right]\left[w(\tau)\,\sum_{m=0}^{\infty}\varphi_m U_m(\tau)\right]d\tau\,dx.\end{aligned} \tag{130}$$

Using (87), (130) takes the form:

$$\begin{aligned}&\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)}w(\tau)\,\varphi(\tau)\,d\tau\,dx\\ &= \frac{1}{\pi^2}\int_I\int_I \sum_{p=1}^{\infty}2T_p(x)U_{p-1}(\tau)\sum_{n=1}^{\infty}\frac{1}{n}\varphi^t_{n-1}\frac{T_n(x)}{w(x)}w(\tau)\sum_{m=0}^{\infty}\varphi_m U_m(\tau)d\tau\,dx\end{aligned} \tag{131}$$

From the identities (82) and (83), we obtain that the only non zero contributions are when $n = p$ and $m = p-1$ and so

$$\begin{aligned}&\frac{1}{\pi^2}\int_I\int_I \frac{1}{x-\tau}\,\frac{\beta^t(x)}{w(x)}w(\tau)\,\varphi(\tau)\,d\tau\,dx\\ &= \frac{2}{\pi^2}\int_I\int_I\left[\sum_{p=1}^{\infty}\frac{1}{p}\varphi_{p-1}\,\varphi^t_{p-1}\frac{T_p^2(x)}{w(x)}w(\tau)\,U^2_{p-1}(\tau)\right]d\tau\,dx\\ &= \frac{1}{2}\sum_{p=0}^{\infty}\frac{1}{p+1}\,\varphi_p\,\varphi^t_p\,.\end{aligned} \tag{132}$$

Now, using the expression of φ and φ^t given by (127) and (127b), we obtain

$$\begin{aligned} & \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{n+1}\, \varphi_n\, \varphi_n^t\, . \\ & = \frac{1}{\pi^2}\int_I\int_I \left[\sum_{n=1}^{\infty} \frac{2w(x)w(y)}{n} U_{n-1}(x)U_{n-1}(y)\right] \varphi(x)\left(\varphi^t(y)\right)\, dy\, dx. \end{aligned} \tag{133}$$

Thus, due to the expression (90), the variational formulation for the inverse operator is

$$\frac{1}{\pi^2}\int_I\int_I \log\frac{M(x,y)}{|x-y|}\varphi(x)\varphi^t(y)\, dy\, dx \;=\; \int_I \alpha(x)\varphi^t(y)dx \tag{134}$$

for all $\varphi^t \in H^{-1/2}(\Gamma_c)$, thus giving the stated result by density arguments. □

4.4 Proof of Proposition 3.3

Proof. We consider the norms on the spaces $\widetilde{H}^{1/2}(\Gamma_c)$ and $H_*^{1/2}(\Gamma_c)$ given respectively by (67) and (62) which are respectively

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \int_I\int_I \frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dx\; dy\; + 2\int_{\Gamma_c}\frac{\alpha(x)^2}{w(x)^2}dx, \tag{135a}$$

$$\|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} \;=\; \int_I\int_I \frac{1-xy}{w(x)w(y)}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy. \tag{135b}$$

For x, y in I, it holds,

$$1 \;\leq\; \frac{1-xy}{w(x)w(y)} \;\leq\; \frac{2}{w(x)w(y)}.$$

The difference of the two squared norms (135a), (135b), is given by

$$\begin{aligned} \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} = & \int_{I\times I}\left\{1-\frac{1-xy}{w(x)w(y)}\right\}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy \\ & + 2\int_I \frac{\alpha(x)^2}{w(x)^2}dx \end{aligned} \tag{136}$$

or, equivalently,

$$\begin{aligned} \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} = & -\int_I\int_I \frac{(\alpha(x)-\alpha(y))^2}{w(x)w(y)(1-xy+w(x)w(y))}dxdy \\ & + 2\int_I \frac{\alpha(x)^2}{w(x)^2}dx. \end{aligned} \tag{137}$$

As the first term of the right hand side is negative, we have

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \leq \|\alpha\|^2_{H^{1/2}(\Gamma_c)} + 2\left\|\frac{\alpha}{w}\right\|^2_{L^2(\Gamma_c)}. \tag{138}$$

In order to obtain an inequality in the other direction, we introduce the change of variable: $x = \cos(\theta), y = \cos(\varphi)$, and write $\hat{\alpha}(\theta) = \alpha(\cos\theta)$ in the equation

$$\int_I\int_I \frac{(\alpha(x)-\alpha(y))^2}{w(x)w(y)(1-xy+w(x)w(y))}dxdy = \int_0^\pi\int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})}d\theta d\varphi. \tag{139}$$

This quantity can be decomposed as

$$\int_0^\pi \int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})} d\theta d\varphi = \int_0^\pi \int_0^\pi \frac{\hat{\alpha}^2(\theta)-\hat{\alpha}(\theta)\hat{\alpha}(\varphi)}{\sin^2(\frac{\theta+\varphi}{2})} d\theta d\varphi. \tag{140}$$

On the other hand,

$$\int_0^\pi \frac{1}{\sin^2(\frac{\theta+\varphi}{2})} d\varphi \;=\; \frac{2}{\sin(\theta)}, \tag{141}$$

and so (140) becomes

$$\int_0^\pi \int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})} d\theta d\varphi = 2\int_I \frac{\alpha(x)^2}{w(x)^2} dx - \int_0^\pi \int_0^\pi \frac{\hat{\alpha}(\theta)\hat{\alpha}(\varphi)}{\sin^2(\frac{\theta+\varphi}{2})} d\theta d\varphi, \tag{142}$$

The difference of the squared of these two norms take now the form:

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} = \int_I \int_I \frac{\alpha(x)\alpha(y)}{w(x)w(y)(1-xy+w(x)w(y))} dxdy. \tag{143}$$

Using the previous estimates, we also have the bounds

$$\int_0^\pi \int_0^\pi \frac{(\hat{\alpha}(\theta)-\hat{\alpha}(\varphi))^2}{2\sin^2(\frac{\theta+\varphi}{2})} d\theta\, d\varphi \;\le\; \int_0^\pi \int_0^\pi 2\frac{\hat{\alpha}^2(\theta)}{\sin^2(\frac{\theta+\varphi}{2})} d\theta d\varphi, \tag{144}$$

or

$$\int_I \int_I \frac{(\alpha(x)-\alpha(y))^2}{w(x)w(y)(1-xy+w(x)w(y))} dx \;\le\; 4\int_I \frac{\alpha^2(x)}{w^2(x)} dx. \tag{145}$$

Hence, we have prove that

$$\|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} \;\le\; \|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} + 2\left\|\frac{\alpha}{w}\right\|^2_{L^2(\Gamma_c)} \tag{146}$$

from which we obtain

$$\|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} \le 2\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)}. \tag{147}$$

If we consider the expression of the norms on the spaces $\widetilde{H}^{1/2}(\Gamma_c)$ and $H_*^{1/2}(\Gamma_c)$ given respectively by (65) and (60). The difference of the squared of these two norms (with one multiply by π^2) is given by

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} - \|\alpha\|^2_{H_*^{1/2}(\Gamma_c)} = \int_I \int_I \log M(x,y) g'(x) g(y)'\, dy\, dx \tag{148}$$

Using two integrations by parts, we recover the expression (143) as we have

$$\frac{d^2}{dxdy} \log M(x,y) = \frac{1}{w(x)w(y)(1-xy+w(x)w(y))} \tag{149}$$

The result concerning the dual spaces is just a direct consequence of the duality. □

4.5 Proof of Proposition 3.4

Proof. The proof consists of using Tchebychev expansions to write down functions in different spaces and then using term by term derivation to conclude. This implies the use of density and convergence results previously used. Specifically, one can

1. expand a function in $\widetilde{H}^{1/2}(\Gamma_c)$ in functions $w(x)U_n(x)$ as

$$\alpha(x) \;=\; \sum_{n=0}^{\infty} \alpha_n w(x) U_n(x)\,, \qquad x \,\in\, I. \tag{150}$$

   Then, it results from Proposition 3.2 that the following expression

$$\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty} (n+1)\,\alpha_n^2 \tag{151}$$

   defines a norm in the space $\widetilde{H}^{1/2}(\Gamma_c)$.

2. We expand a function in the space $H^{1/2}(\Gamma_c)$ on the functions $T_n(x)$ as

$$g(x) \;=\; \sum_{n=0}^{\infty} g_n T_n(x)\,, \qquad x \,\in\, I. \tag{152}$$

   and from Proposition (3.1)

$$\|g\|^2_{H^{1/2}(\Gamma_c)} \;=\; g_0^2 + \sum_{n=0}^{\infty} n\, g_n^2 \tag{153}$$

   is a norm in the space $H^{1/2}(\Gamma_c)$.

3. We expand a function in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ on the functions $\dfrac{T_n(x)}{w(x)}$ as

$$\varphi(x) \;=\; \sum_{n=0}^{\infty} \varphi_n \frac{T_n(x)}{w(x)}\,, \qquad x \,\in\, I, \tag{154}$$

   Then, from Proposition (3.1)

$$\|\varphi\|^2_{\widetilde{H}^{-1/2}(\Gamma_c)} \;=\; \varphi_0^2 \;+ \sum_{n=1}^{\infty} \frac{1}{n}\,\varphi_n^2 \tag{155}$$

   is a norm in the space $\widetilde{H}^{-1/2}(\Gamma_c)$.

4. We expand a function in the space $H^{-1/2}(\Gamma_c)$ on the functions $U_n(x)$ as

$$\varphi(x) \;=\; \sum_{n=0}^{\infty} \varphi_n U_n(x)\,, \qquad x \,\in\, I, \tag{156}$$

   Then, it results from Proposition (3.2) that the following expression

$$\|\varphi\|^2_{H^{-1/2}(\Gamma_c)} \;=\; \sum_{n=0}^{\infty} \frac{1}{n+1}\,\varphi_n^2 \tag{157}$$

   is a norm in the space $H^{-1/2}(\Gamma_c)$.

5. We choose a function in the space $\widetilde{H}^{1/2}(\Gamma_c)$ given by (150), whose norm is

$$\|\alpha\|_{\widetilde{H}^{1/2}(\Gamma_c)} \;=\; \sqrt{\sum_{n=0}^{\infty}(n+1)\,\alpha_n^2} \tag{158}$$

Its derivative is given by

$$\frac{d}{dx}\alpha(x) \;=\; -\sum_{n=1}^{\infty} n\,\alpha_{n-1}\frac{T_n(x)}{w(x)}\,, \qquad x\;\in\;I, \tag{159}$$

and its norm in the space $\widetilde{H}^{-1/2}(\Gamma_c)$ is thus

$$\left\|\frac{d}{dx}\alpha\right\|_{\widetilde{H}^{-1/2}(\Gamma_c)} \;=\; \sqrt{\sum_{n=0}^{\infty}(n+1)\,\alpha_n^2} \tag{160}$$

This proves the continuity of this operator. The surjectivity is clear on the expression of the derivative.

6. We choose now a function in the space $H_*^{1/2}(\Gamma_c)$ given by (152), which norm is

$$\|g\|_{H^{1/2}(\Gamma_c)} \;=\; \sqrt{\sum_{n=1}^{\infty} n\,g_n^2} \tag{161}$$

Its derivative is given by

$$\frac{d}{dx}g(x) \;=\; \sum_{n=1}^{\infty} n\,g_n\,U_{n-1}(x)\,, \qquad x\;\in\;I, \tag{162}$$

and its norm in the space $H^{-1/2}(\Gamma_c)$ is thus

$$\left\|\frac{d}{dx}g\right\|_{H^{-1/2}(\Gamma_c)} \;=\; \sqrt{\sum_{n=1}^{\infty} n\,g_n^2} \tag{163}$$

This proves the continuity of this operator. The surjectivity is clear on the expression of the derivative.

$\square$

5 Conclusions

We have systematically derived precise variational forms and characterizations for norms, image and ranges for the weakly and hypersingular operators arising from the Laplace equation in two dimensions with a bounded cut in two dimensional space. In particular, we observe that the derivation operator is key to understand the differences in the associated functional spaces and we provide examples for these. Moreover, we provide Calderón-type identities which will be used as preconditioners in an upcoming work.

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