**Eidgenössische
Technische Hochschule
Zürich**

*Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich*

Sparse tensor Galerkin discretizations for parametric and random parabolic PDEs

I: Analytic regularity and gpc-approximation[*]

V.H. Hoang[†] and Ch. Schwab

[†]Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371

Sparse Tensor Galerkin Discretizations

for

parametric and random parabolic PDEs
I: Analytic regularity and gpc-Approximation [*]

V.H. Hoang [†]and Ch. Schwab [‡]

April 26, 2010

Abstract

For initial boundary value problems of linear parabolic partial differential equations with random coefficients, we show analyticity of the solution with respect to the parameters and give an apriori error analysis for sparse tensor, space-time discretizations. The problem is reduced to a parametric family of deterministic initial boundary value problems on an infinite dimensional parameterspace by Galerkin projection onto finitely supported polynomial systems in the parameterspace. Uniform stability with respect to the support of the resulting coupled parabolic systems is established.

Analyticity of the solution with respect to the countably many parameters is established, and a regularity result of the parametric solution is proved for both, compatible as well as incompatible initial data and source terms. The present results will be used in [6] to obtain convergence rates and stability of sparse space-time tensor product Galerkin discretizations in the parameter space.

Contents

1 Introduction

The efficient numerical solution of parametric partial differential equations on high-dimensional parameter spaces has attracted considerable attention recently. We mention only the recent works [2, 8, 3] and the references there for elliptic problems and, with particular relevance to the present paper, the recent work [7] for parabolic problems. In the present paper, the first in a series of two, we investigate the regularity of a class of model parametric parabolic problems. Such problems arise, for example, in the context of diffusion in random media when the medium's permeability is a random field which is given, for example, as Karhúnen-Loève expansion. Parametrizing the random input permeability in terms of the (countably many) coefficients in the Karhúnen-Loève expansion, the solution becomes, in turn, a deterministic function which depends, as we show here, *analytically* on these input variables. Accordingly, the solution admits a so-called *generalized polynomial chaos (gpc)* expansion with respect to these input variables, with deterministic coefficients which take values in the natural Lebesgue-Bochner spaces of deterministic parabolic problems. We also prove, following the recent work [5, 4] for elliptic problems, *p*-summability of the gpc coefficient sequences of the parametric solutions, in a scale of Lebesgue-Bochner spaces in the space-time cylinder, given corresponding *p*-summability and regularity of the input's Karhúnen-Loève expansion coefficients.

We also indicate consequences of this *p*-summability of gpc coefficients for convergence rates of a class of spectral approximations in the infinite dimensional stochastic parameter space: we show that this results in large, coupled systems of deterministic parabolic equations which are well-posed independently of the selection of active stochastic modes. [7, 6]

1.1 A Class of Random Parabolic Problems

For $0 < T < \infty$, we consider in the bounded time interval $I = (0, T)$ a class of parabolic initial boundary value problems with random coefficients. Throughout, we will consider a bounded Lipschitz domain $D \subset \mathbf{R}^d$ and the associated space-time cylinder $Q_T = I \times D$. In Q_T, we consider the random parabolic initial boundary value problem

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(x, \omega) \nabla u) = g(t, x), \quad u|_{\partial D \times I} = 0, \quad u|_{t=0} = h(x). \tag{1.1}$$

At this stage, we assume the coefficient $a(x, \omega)$ to be a random field on the probability space (Ω, Σ, P) over $L^\infty(D)$. We assume in particular $a(x, \omega)$ to be independent of t (additional structural assumptions on the coefficient will be imposed shortly). The source term g and the initial data h are both assumed to be deterministic (this assumption could be relaxed without additional essential technical complications; for simplicity of exposition only we shall not pursue this here). We assume

Assumption 1.1 *There exist constants* $0 < a_{\min} \le a_{\max} < \infty$ *so that*

$$\forall x \in D, \ \forall \omega \in \Omega : \qquad 0 < a_{\min} \le a(x, \omega) \le a_{\max}.$$

In view of the sparse tensor discretizations to be investigated, we consider a *space-time variational formulation* of problem (1.1). To state it, we denote by $V = H_0^1(D)$ and $H = L^2(D)$ and identify H with its dual: $H \simeq H'$. Then $V \subset H \simeq H \subset V' = H^{-1}(D)$. For the variational formulation of (1.1) we introduce the Bochner spaces

$$\mathcal{X} = L^2(I; V) \cap H^1(I; V') \quad \text{and} \quad \mathcal{Y} = L^2(I; V) \times H. \tag{1.2}$$

In $\mathcal{X}$ and $\mathcal{Y}$ norms $\| \circ \|_{\mathcal{X}}$ and $\| \circ \|_{\mathcal{Y}}$, respectively, are for $u \in \mathcal{X}$ and $v = (v_1, v_2) \in \mathcal{Y}$ given by

$$\|u\|_{\mathcal{X}} = (\|u\|_{L^2(I;V)}^2 + \|u\|_{H^1(I;V')}^2)^{1/2} \quad \text{and} \quad \|v\|_{\mathcal{Y}} = (\|v_1\|_{L^2(I;V)}^2 + \|v_2\|_H^2)^{1/2}.$$

Given a realization $\omega \in \Omega$, a *weak solution* of problem (1.1) is a function $u(\cdot, \cdot, \omega) \in \mathcal{X}$ such that

$$\int_I \langle \frac{du}{dt}, v_1 \rangle_H dt + \int_I \int_D a(x, \omega) \nabla u(t, x, \omega) \cdot \nabla v_1(t, x) dx dt + \langle u(0, \cdot, \omega), v_2 \rangle_H$$

$$= \int_I \langle g(t, \cdot), v_1 \rangle dt + \langle h, v_2 \rangle_H, \qquad \forall v \in \mathcal{Y}. \tag{1.3}$$

The following proposition from [9] guarantees its well-posedness for all $\omega \in \Omega$, under Assumption 1.1.

1

Proposition 1.1 *Assume that $g \in L^2(I, V')$, $h \in L^2(D)$ and that Assumption 1.1 holds. Then, for every $\omega \in \Omega$, the parabolic operator $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ induced by (1.1) in the weak form (1.3) is an isomorphism: for given $(g, h) \in \mathcal{Y}'$ and every $\omega \in \Omega$, problem (1.3) has a unique solution $u(\cdot, \cdot, \omega)$ which satisfies the a-priori estimate*

$$\|u\|_{\mathcal{X}} \leq C \left(\|g\|_{L^2(I,V')} + \|h\|_{L^2(D)} \right), \tag{1.4}$$

where the constant C is bounded uniformly for all realizations.

The proof of Proposition 1.1 is based on showing that the operator $B \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ satisfies an inf-sup condition on $\mathcal{X} \times \mathcal{Y}$. Inspecting the proof in [9] one verifies that, under Assumption 1.1, the inf-sup conditions hold uniformly with respect to $\omega \in \Omega$. $\qquad \square$

In this paper, we assume that the coefficient a in (1.1) is characterized by a sequence of scalar random variables $(y_j)_{j \geq 1}$, i.e.

$$a(x, \omega) = \bar{a}(x) + \sum_{j \geq 1} y_j(\omega) \psi_j(x). \tag{1.5}$$

We assume in addition that the ψ_j are scaled in $L^\infty(D)$ such that $y_j : \Omega \to \mathbb{R}$, $j = 1, 2, \ldots$ are distributed identically and uniformly, and that the ψ_j are scaled in $L^\infty(D)$ such that the range of the y_j is $[-1, 1]$.

Then all realizations of the random vector $y = (y_1, y_2, \ldots)$ are supported in the cube $U = [-1, 1]^\mathbb{N}$. We interpret U as unit ball in $\ell^\infty(\mathbb{N})$. Via the corresponding norm $\|y\|_{\ell^\infty(\mathbb{N})}$, open subsets of U are defined in the usual way, and we denote the σ-algebra of Borel subsets of U (in the topology of $\ell^\infty(\mathbb{N})$) by $\mathcal{B}(U)$.

Assumption 1.2 *The functions $\bar{a}$ and ψ_j satisfy*

$$\sum_{j \geq 1} \|\psi_j\|_{L^\infty(D)} \leq \frac{\kappa}{1 + \kappa} \bar{a}_{\min},$$

with $\bar{a}_{\min} = \min_{x \in D} \bar{a}(x) > 0$ and $\kappa > 0$.

Assumption 1.1 is then satisfied by choosing

$$a_{\min} := \bar{a}_{\min} - \frac{\kappa}{1 + \kappa} \bar{a}_{\min} = \frac{1}{1 + \kappa} \bar{a}_{\min}. \tag{1.6}$$

1.2 Probability Spaces

Using the structural assumption (1.5) on the random coefficient, the law of the random solution u of (1.1) takes the form of a parametric deterministic function of the (in general countably many components of) $y \in U$. The variational problem can be cast in the form of a parametric family of deterministic problems for y. In the next sections, we study sparse tensor discretizations of a variational problem for u as a function of (t, x, y) in $I \times D \times U$. To do so, we need to define probability measures on U.

Let Θ be the σ-algebra defined on U which is generated from the sets of the form $\prod_{j=1}^\infty S_j$ where S_j are subintervals of $[-1, 1]$ and only a finite number of them are proper subsets of $[-1, 1]$. On Θ, we define the following measure

$$d\rho(y) := \otimes_{j \geq 1} dy_j / 2.$$

Then (U, Θ, ρ) is a probability space. As y_j are distributed uniformly, for $S = \prod_{j=1}^\infty S_j$,

$$\rho(S) = \prod_{j=1}^\infty P\{\omega : y_j(\omega) \in S_j\}.$$

We introduce Bochner spaces $\underline{\mathcal{X}} = L^2(U, \mathcal{X}, d\rho)$ and $\underline{\mathcal{Y}} = L^2(U, \mathcal{Y}, d\rho)$ and note $\underline{\mathcal{X}} \simeq L^2(U, d\rho) \otimes \mathcal{X}$, $\underline{\mathcal{Y}} \simeq L^2(U, d\rho) \otimes \mathcal{Y}$,

1.3 Parametric Deterministic Parabolic Problem

Consider the parametric family of deterministic parabolic problems: given a source term $g(t, x)$ and initial data $h(x)$, for $y \in U$, find $u(t, x, y)$ such that

$$\frac{\partial u}{\partial t}(t, x, y) - \nabla_x \cdot [a(x, y)\nabla_x u(t, x, y)] = g(t, x) \text{ in } Q_T, \quad u(t, x, y)|_{\partial D \times I} = 0, \quad u|_{t=0} = h(x), \qquad (1.7)$$

where, for every $y = (y_1, y_2, \ldots) \in U$ in $L^\infty(D)$ holds

$$a(x, y) = \bar{a}(x) + \sum_{j=1}^{\infty} y_j \psi_j(x) .$$

For the weak formulation of (1.7), we follow (1.3) and define for $y \in U$ the parametric family of bilinear forms $U \ni y \to b(y; w, (v_1, v_2)) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ by

$$b(y; w, (v_1, v_2)) = \int_I \langle \frac{dw}{dt}, v_1(t, \cdot) \rangle_H dt + \int_D \int_I a(x, y)\nabla w(t, x) \cdot \nabla v_1(t, x) dx dt + \langle w(0), v_2 \rangle_H. \qquad (1.8)$$

We also define the linear form

$$f(v) = \int_I \langle g(t), v_1(t) \rangle_H dt + \langle h, v_2 \rangle_H, \quad v = (v_1, v_2) \in \mathcal{Y}. \qquad (1.9)$$

The variational formulation for (1.7) reads: given $f \in \mathcal{Y}'$, find $u(y) : U \ni y \to \mathcal{X}$ such that

$$b(y; u, v) = f(v), \quad \forall v = (v_1, v_2) \in \mathcal{Y}. \qquad (1.10)$$

Proposition 1.2 *For each fixed $y \in U$, the operator $B(y) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ defined by $(B(y)w)(v) = b(y, w, v)$ is boundedly invertible. The norms of $B(y)$ and $B(y)^{-1}$ can be bounded uniformly by constants which only depend on $a_{\min}$, $a_{\max}$, T and the spaces $\mathcal{X}$ and $\mathcal{Y}$. In particular, the solution u of the problem (1.10) is uniformly bounded in $\mathcal{X}$ for all $y \in U$.*

The proof of this theorem can be found in Appendix A of Schwab and Stevenson [9].

Proposition 1.3 *There holds*

$$\|u(t, x, y) - u(t, x, y')\|_{\mathcal{X}} \leq C \|a(\cdot, y) - a(\cdot, y')\|_{L^\infty(D)} \quad \text{for every} \quad y, y' \in U.$$

Proof From the variational formulation (1.10), we find that the function $w = u(x, t, y) - u(x, t, y')$ satisfies the variational problem

$$\int_I \langle \frac{dw}{dt}, v_1 \rangle dt + \int_I \int_D a(x, y)\nabla w \cdot \nabla v_1(t, x) dx dt + \int_D w(x, 0)v_2(x) dx$$
$$= -\int_I \int_D (a(x, y) - a(x, y'))\nabla u(t, x, y') \cdot \nabla v_1(t, x) dx dt. \qquad (1.11)$$

From this we deduce that for every $y, y' \in U$ holds

$$\|u(t, x, y) - u(t, x, y')\|_{\mathcal{X}} \leq C \|(a(x, y) - a(x, y')) \cdot \nabla u(t, x, y)\|_{L^2(I;H)} \leq C \sup_x |a(x, y) - a(x, y')|. \ (1.12)$$

$\square$

Proposition 1.4 *The map $u(\cdot, \cdot, y) : U \to \mathcal{X}$ is measurable as a Bochner function.*

Let $h \in \mathcal{X}$. We note the $\mathcal{X}$-inner product

$$(u(y), h)_{\mathcal{X}} = \langle u(y), h \rangle_{L^2(I, V)} + \langle u(y), h \rangle_{H^1(I, V')}.$$

To show that u is measureable as a Bochner function from U to $\mathcal{X}$, it is sufficient to show that $(u(y), h)$ is measurable. Fixing a real number a, we then show that the set $Y_a = \{y : (u(y), h) > a\}$ is in the

3

σ-algebra defined on U. From Proposition 1.3 if $(u(y), h) > a$ then there is a positive number r such that if

$$\sup_x |a(x, y) - a(x, y')| < r, \tag{1.13}$$

then $(u(y'), h) > a$. We consider the set T_i of vectors $y \in U$ such that for $\bar{y} = (y_1, y_2, \ldots, y_i, z_1, z_2, \ldots)$, $(u(\bar{y}), h)_{\mathcal{X}} > a$ for all $z_j \in [-1, 1]$, $j = 1, 2, \ldots$. For each $y \in U$, from assumption (1.2),

$$|a(x, y) - a(x, \bar{y})| < r,$$

if i is large enough. Thus each vector $y \in Y_a$ belongs to a set T_i for some i. Let $R_i \subset [-1, 1]^i$ be the set of $t = (t_1, \ldots, t_i)$ such that $(t_1, \ldots, t_i, z_1, z_2, \ldots) \in T_i$ for all $z_j \in [-1, 1]$ $(j = 1, 2, \ldots)$. From (1.12) and (1.13), R_i is an open set and thus can be represented as a countable union of open cubes. Thus T_i can be represented as a countable union of cubes of the form $\prod_{j \geq 1} S_j$ where S_j is an open interval in $(-1, 1)$ and $S_j = (-1, 1)$ when j is sufficiently large. Thus T_i is measurable and so is Y_a. $\qquad\square$

With the bilinear form $B(\cdot, \cdot) : \underline{\mathcal{X}} \times \underline{\mathcal{Y}} \to \mathbb{R}$ and the linear form $F(\cdot) : \underline{\mathcal{Y}} \to \mathbb{R}$ defined by

$$B(u, v) = \int_U b(y, u, v)\rho(dy) \qquad \text{and} \qquad F(v) = \int_U f(v)\rho(dy). \tag{1.14}$$

we consider the variational problem:

$$\text{find} \quad u \in \underline{\mathcal{X}} \quad \text{such that} \quad B(u, v) = F(v) \quad \text{for all} \quad v \in \underline{\mathcal{Y}}. \tag{1.15}$$

Proposition 1.5 *The problem (1.15) has a unique solution in $L^2(U, \underline{\mathcal{X}})$.*

Proof The existence part is obvious. Moreover, from Proposition 1.2 the solution u of (1.10) belongs to $L^\infty(U, \underline{\mathcal{X}})$ so $u \in L^2(U, \underline{\mathcal{X}}, d\rho)$ is a solution of (1.15). Next we show the uniqueness of a solution $u \in L^2(U, \underline{\mathcal{X}}, d\rho)$.

Let $v(t, x, y) = (v_1(t, x)w(y), v_2(t, x)w(y))$ where $w(y) \in L^2(U, \rho(dy))$. Then

$$\int_U b(y; u, (v_1, v_2))w(y)\rho(dy) = \int_U f((v_1, v_2))w(y)\rho(dy).$$

As this holds for all $w(y) \in L^2(U, \rho(dy))$, for ρ-almost all y in U the function $u(y)$ satisfies

$$b(y; u, (v_1, v_2)) = f((v_1, v_2)) \qquad \forall v = (v_1, v_2) \in \mathcal{Y}.$$

For each $y \in U$, there is a unique function $u(t, x, y) \in \mathcal{X}$ that satisfies this equation. This function is uniformly bounded in $\mathcal{X}$ for all $y \in U$. This completes the proof. $\qquad\square$

With this theorem in hand, we recover the random solution $u(t, x, \omega)$.

Theorem 1.1 *Under Assumptions 1.1, 1.2, for given $g \in L^2(I, V')$ and $h \in H$, the variational problem: find $u \in L^2(\Omega, \mathcal{X})$ such that for every $v(t, x, \omega) = (v_1(t, x, \omega), v_2(x, \omega)) \in L^2(\Omega, \mathcal{Y})$ it holds*

$$
\begin{aligned}
& E\left\{\int_I \langle \frac{du}{dt}(t, \cdot, \cdot), v_1(t, \cdot, \cdot)\rangle_H dt\right\} + E\left\{\int_I \int_D a(x, \omega)\nabla u(t, x, \omega) \cdot \nabla v_1(t, x, \omega)dxdt\right\} \\
& + \quad E\left\{\int_D u(0, x, \omega)v_2(x, \omega)dx\right\} \\
& = \quad E\left\{\int_I \int_D g(t, x)v_1(t, x, \omega)dxdt\right\} + E\left\{\int_D h(x)v_2(x, \omega)dx\right\}
\end{aligned} \tag{1.16}
$$

admits a unique solution which satisfies the apriori estimate

$$\|u\|_{L^2(\Omega, \underline{\mathcal{X}})} \leq C(a) \left(\|g\|_{L^2(I, V')} + \|h\|_H\right) \tag{1.17}$$

2   Semidiscrete Galerkin Approximation

We discretize the parametric parabolic problem (1.7) in the variational form (1.15) by Galerkin projection onto linear combinations of N polynomials of the parameters $y \in U$ with $\mathcal{X}$-valued coefficients. We prove that this results in a coupled parabolic system of size N and establish its well-posedness regardless of the choice of particular N polynomials.

2.1 Polynomial Spaces in U

Let $(L_n)_{n\geq 0}$ be the univariate Legendre polynomials normalized according to

$$\int_{-1}^{1} |L_n(t)|^2 \frac{dt}{2} = 1. \tag{2.1}$$

Note that in this normalization, $L_0(t) = 1$. Let $\mathcal{F}$ be the countable set of sequences $\nu = (\nu_j)_{j\geq 1}$ of nonnegative integers such that only a finite number of ν_j are non-zero. For $\nu \in \mathcal{F}$, we introduce the tensorized Legendre polynomials

$$L_\nu(y) = \prod_{j\geq 1} L_{\nu_j}(y_j), \qquad \nu \in \mathcal{F}.$$

The family L_ν forms a complete orthonormal system of $L^2(U, d\rho)$. Therefore each function $u \in \underline{\mathcal{X}}$ can be represented as

$$u = \sum_{\nu\in\mathcal{F}} u_\nu L_\nu, \tag{2.2}$$

where the coefficients $u_\nu \in \mathcal{X}$ are defined by

$$u_\nu = \int_U u(\cdot,\cdot,y) L_\nu(y) \rho(dy) \in \mathcal{X},$$

the integral being understood as Bochner integral of $\mathcal{X}$-valued functions over U.

2.2 Well-posedness and quasioptimality

For every subset $\Lambda \subset \mathcal{F}$ of cardinality $N = \#\Lambda < \infty$ we define space of $\mathcal{X}$ and $\mathcal{Y}$-valued polynomial expansions

$$\underline{\mathcal{X}}_\Lambda = \{u_\Lambda(t,x,y) = \sum_{\nu\in\Lambda} u_\nu(t,x) L_\nu(y) : \quad u_\nu \in \mathcal{X}\} \subset \underline{\mathcal{X}},$$

and

$$\underline{\mathcal{Y}}_\Lambda = \{v_\Lambda(t,x,y) = \sum_{\nu\in\Lambda} v_\nu(t,x) L_\nu(y) : \quad v_\nu \in \mathcal{X}\} \subset \underline{\mathcal{Y}}.$$

In the Legendre basis $(L_\nu)_{\nu\in\mathcal{F}}$, we write

$$v_{1\Lambda}(t,x,y) = \sum_{\nu\in\Lambda} v_{1\nu}(t,x) L_\nu(y) \quad \text{and} \quad v_{2\Lambda}(x,y) = \sum_{\nu\in\Lambda} v_{2\nu}(x) L_\nu(y),$$

respectively, where $v_\nu = (v_{1\nu}, v_{2\nu}) \in \mathcal{Y}$ for all $\nu \in \mathcal{F}$. We consider the Galerkin approximation: find

$$u_\Lambda \in \underline{\mathcal{X}}_\Lambda : \qquad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \qquad \forall v_\Lambda \in \underline{\mathcal{Y}}_\Lambda. \tag{2.3}$$

Theorem 2.1 *For any subset $\Lambda \subset \mathcal{F}$, the problem (2.3) corresponds to a coupled system of $N = \#\Lambda$ many coupled, linear parabolic equations. Under Assumptions 1.1, 1.2, these systems are stable uniformly with respect to $\Lambda \subset \mathcal{F}$: for any $\Lambda \subset \mathcal{F}$, problem (2.3) admits a unique solution $u_\Lambda \in \underline{\mathcal{X}}_\Lambda$ which satisfies the apriori error bound*

$$\|u - u_\Lambda\|_{\underline{\mathcal{X}}} \leq c\Big(\sum_{\nu\notin\Lambda} \|u_\nu\|_{\mathcal{X}}^2\Big)^{1/2}.$$

Here, $u_\nu \in \mathcal{X}$ are the Legendre coefficients of the solution of the parametric problem in (2.2) and c is independent of Λ.

Proof To prove the uniform well-posedness of the coupled parabolic system resulting from the Galerkin discretization in U, we prove that the following inf-sup condition holds: there exist $\alpha, \beta > 0$ such that

for *any* $\Lambda \subset \mathcal{F}$ holds

$$\sup_{u_\Lambda \in \underline{\mathcal{X}}_\Lambda, v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \leq \alpha < \infty, \tag{2.4}$$

$$\inf_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} \sup_{0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \geq \beta > 0, \tag{2.5}$$

$$\forall 0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda : \quad \sup_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} |B(u_\Lambda, v_\Lambda))| > 0, \tag{2.6}$$

where the constants α, β are independent of $\Lambda \subset \mathcal{F}$ (the proof is provided in the Appendix).

The projected parametric deterministic parabolic Problem (2.3) has a unique solution, and, in virtue of the independence of α, β of Λ, is well-posed and stable with stability bounds which are independent of the choice of $\Lambda \subset \mathcal{F}$. Hence, the error incurred by this projection is quasioptimal:

$$\begin{aligned}
\|u - u_\Lambda\|_{\underline{\mathcal{X}}} &\leq (1 + \beta^{-1}(\|g\|_{L^2(I, V')} + \|h\|_{L^2(D)})) \inf_{v_\Lambda \in \mathcal{X}} \|u - v_\Lambda\|_{\underline{\mathcal{X}}} \\
&\leq c\|u - \sum_{\nu \in \Lambda} u_\nu L_\nu\|_{\underline{\mathcal{X}}} = c\|\sum_{\nu \notin \Lambda} u_\nu L_\nu\|_{\underline{\mathcal{X}}}.
\end{aligned}$$

By the normalization (2.1) and Parseval's equality,

$$\|\sum_{\nu \notin \Lambda} u_\nu L_\nu\|_{\underline{\mathcal{X}}}^2 = \sum_{\nu \notin \Lambda} \|u_\nu\|_{\mathcal{X}}^2.$$

The conclusion then follows with $c = 1 + \beta^{-1}(\|g\|_{L^2(I, V')} + \|h\|_{L^2(D)})$. $\qquad\square$

3 Best N-term gpc approximations

Theorem 2.1 suggests we choose the set $\Lambda \subset \mathcal{F}$ as the set of the largest N terms $\|u_\nu\|_{\mathcal{X}}$. However, a priori, only bounds for u_ν in $\mathcal{X}$ are known. Therefore, one strategy will be to choose the set Λ according to these apriori bounds (this strategy was employed in [3] for the elliptic case). Alternatively, an optimal, adaptive Galerkin method will yield iteratively quasioptimal sequences Λ_N of active indices. We now determine such apriori bounds.

A best N-term convergence rate estimate in terms of N will result from these bounds using the following lemma.

Lemma 3.1 *Let $\alpha = (\alpha_\nu)_{\nu \in \mathcal{F}}$ be a sequence in $l^p(\mathcal{F})$. Let $q \geq p > 0$. If $\Lambda_N \subset \mathcal{F}$ is a set of indices corresponding to a set of N largest $|\alpha_\nu|$, then*

$$(\sum_{\nu \notin \mathcal{F}_N} |\alpha_\nu|^q)^{1/q} \leq \|\alpha\|_{l^p(\mathcal{F})} N^{-\sigma}, \qquad where \qquad \sigma = \frac{1}{p} - \frac{1}{q}.$$

Therefore the convergence rate of spectral approximations such as (2.3) of the parabolic problem on the infinite dimensional parameterspace U is determined by the summability of the Legendre coefficient sequence $(\|u_\nu\|_{\mathcal{X}})_{\nu \in \mathcal{F}}$. We shall now prove that summability of this sequence is determined by that of the sequence $(\psi_j(x))_{j \in \mathbb{N}}$ in the input's fluctuation expansion (1.5). Throughout, Assumptions 1.1 and 1.2 will be required to hold. In addition, we shall require

Assumption 3.1 *There exists $0 < p < 1$ such that*

$$\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(D)}^p < \infty. \tag{3.1}$$

3.1 Complex extension of the parametric problem

To estimate $\|u_\nu\|_{\mathcal{X}}$, we shall use tools from complex analysis and extend the parametric, deterministic problem (1.7) to parameter vectors taking values in the complex domain. To establish its well-posedness, we let $K < 1$ be a positive constant such that

$$K \sum_{j=1}^\infty \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min}}{8}.$$

We choose an integer J_0 such that

$$\sum_{j > J_0} \|\psi_j\|_{L^\infty(D)} < \frac{a_{\min} K}{24(1+K)}.$$

Let $E = \{1, 2, \ldots, J_0\}$ and $F = \mathbb{N} \setminus E$. We define

$$|\nu_F| = \sum_{j > J_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$ we define

$$r_m = K \text{ when } m \le J_0 \text{ and } r_m = 1 + \frac{a_{\min} \nu_m}{4 |\nu_F| \|\psi_m\|_{L^\infty(D)}} \text{ when } m > J_0, \tag{3.2}$$

where we make the convention that $\frac{|\nu_j|}{|\nu_F|} = 0$ if $|\nu_F| = 0$. We consider the open discs $\mathcal{U}_m \subset \mathbb{C}$ defined by

$$[-1, 1] \subset \mathcal{U}_m := \{z_m \in \mathbb{C} : |z_m| < 1 + r_m\} \subset \mathbb{C}. \tag{3.3}$$

We will extend the parametric deterministic problem (1.7) to parameter vector z in the polydiscs

$$\mathcal{U} = \bigotimes_{m=1}^\infty \mathcal{U}_m \subset \mathbb{C}^{\mathbb{N}}. \tag{3.4}$$

To do so, we extend the parametric, deterministic coefficient function $a(x, y)$ in (1.5) to $z \in \mathcal{U}$ by

$$a(x, z) = \bar{a}(x) + \sum_{m=1}^\infty \psi_m(x) z_m.$$

This expression is meaningful for $z \in \mathcal{U}$: we have, for almost every $x \in D$,

$$
\begin{aligned}
|a(x, z)| &\le \bar{a}(x) + \sum_{m=1}^\infty |\psi_m(x)|(1 + r_m) \\
&\le \operatorname*{ess\,sup}_{x \in D} |\bar{a}(x)| + \sum_{m=1}^{J_0} \|\psi_m\|_{L^\infty(D)}(1 + K) + \sum_{m > J_0} \left(2 + \frac{a_{\min} \nu_m}{4 |\nu_F| \|\psi_m\|_{L^\infty(D)}}\right) \|\psi_m\|_{L^\infty(D)} \\
&\le \|\bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^\infty \|\psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}.
\end{aligned}
$$

Now we consider the following *deterministic, parametric, parabolic problem*

$$\frac{\partial u(t, x, z)}{\partial t} - \nabla \cdot (a(x, z) \nabla u(t, x, z)) = g(t, x), \quad u|_{\partial D} = 0, \quad u(0, x, z) = h(x). \tag{3.5}$$

For simplicity, we denote by $\mathcal{X}$ the space $L^2(I; V) \cap H^1(I; V')$ of complex-valued functions. The solutions of (3.5) will then be complex-valued functions of x and of t. Accordingly, we understand all Hilbert spaces as spaces of complex valued functions and all innerproduct as sesquilinear forms.

7

Lemma 3.2 *The problem (3.5) has a unique solution in $\mathcal{X}$ when $z \in \mathcal{U}$. There exists a positive constant C independent of $z \in \mathcal{U}$ such that for all $z \in \mathcal{U}$*

$$\|u(\cdot, \cdot, z)\|_{\mathcal{X}} \le C(\|g\|_{L^2(I;V')} + \|h\|_{L^2(D)}). \tag{3.6}$$

Proof We note that for every $z \in \mathcal{U}$ and every $x \in D$ holds

$$
\begin{aligned}
\Re a \quad \ge \quad & \min \bar{a} - \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)}(1 + r_m) \ge (\min \bar{a}) - \frac{\kappa}{\kappa+1}(\min \bar{a}) \\
& - K \sum_{m=1}^{J_0} \|\psi_m\|_{L^\infty(D)} - \sum_{m > J_0} \|\psi_m\|_{L^\infty(D)} - \sum_{m \ge J_0} \frac{a_{\min}\nu_m}{4|\nu_F|\|\psi_m\|_{L^\infty(D)}} \|\psi_m(x)\|_{L^\infty(D)}.
\end{aligned}
$$

With the choice $a_{\min} = \min \bar{a}(x)/(1+\kappa)$, we have

$$\Re a \ge a_{\min} - \frac{a_{\min}}{8} - \frac{a_{\min}}{24} - \frac{a_{\min}}{4} > \frac{a_{\min}}{2}$$

Problem (3.5) thus has a unique solution; the proof of the apriori bound (3.6) is standard. $\qquad \square$

For each index $\nu \in \mathcal{F}$, we denote by $\text{supp}(\nu)$ its "support" and define the associated finite dimensional polydiscs

$$\mathcal{U}_\nu = \otimes_{j \in \text{supp}(\nu)} \mathcal{U}_j, \quad \text{where} \quad \text{supp}(\nu) := \{j \in \mathbb{N} : \nu_j \ne 0\}. \tag{3.7}$$

Proposition 3.1 *For $\nu \in \mathcal{F}$ and $z \in \mathcal{U}$ with fixed z_k for all indices $k \notin \text{supp}(\nu)$, the map $u : \mathcal{U}_\nu \to \mathcal{X}$ is an analytic function taking values in the function space $\mathcal{X}$.*

Proof For an index m, we fix all the coordinates z_k for $k \ne m$, and denote $z \in \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. Let $\delta \in \mathbb{C}$. We show that there exists a function $v \in \mathcal{X}$ such that

$$\lim_{\delta \to 0} \left\| \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{\mathcal{X}} = 0,$$

for all $z \in \mathcal{U}$. For $\delta > 0$ sufficiently small, $(z_m + \delta, \bar{z}_m) \in \mathcal{U}$. For such δ, the difference quotient $v^\delta = \delta^{-1}(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z_m, \bar{z}_m))$ is well-defined. The function v^δ satisfies

$$\frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, z)\nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z_m + \delta, \bar{z}_m)),$$

with the initial condition $v^\delta(0, \cdot, z_m + \delta, \bar{z}_m) = 0$. Let v be the weak solution of the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, z)\nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$\frac{\partial(v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, z)\nabla(v^\delta - v)) = \nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z))).$$

An argument similar to the proof of Proposition 1.3 shows that

$$\|u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)\|_{\mathcal{X}} \le C|\delta|.$$

Therefore

$$\|\nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)))\|_{L^2(I;V')} \le c|\delta|.$$

Standard estimations for parabolic equations show that

$$\|v^\delta - v\|_{\mathcal{X}} \le c|\delta|.$$

Thus v is the derivative of u with respect to z_m as a $\mathcal{X}$-valued function. We conclude that u is analytic as a function from $\mathcal{U}_\nu$ to $\mathcal{X}$. $\qquad \square$

3.2 Coefficient estimates

Proposition 3.2 *The following estimate holds*

$$\|u_\nu\|_{\mathcal{X}} \le C\left(\prod_{m \in \mathrm{supp}(\nu)} \frac{2(1+K)}{K} \eta_m^{-\nu_m}\right), \tag{3.8}$$

where $\eta_m := r_m + \sqrt{1+r_m^2}$ with r_m as in (3.2).

Proof The proof follows that of Lemma A.3 in Bieri, Andreev and Schwab [3]. For $\nu \in \mathcal{F}$, define $u_\nu \in \mathcal{X}$ by

$$u_\nu = \int_U u(y) L_\nu(y) \rho(dy) \tag{3.9}$$

where the integral is understood as Bochner integral of $\mathcal{X}$-valued functions. Let $S = \mathrm{supp}(\nu)$ and $\bar{S} = \mathbb{N} \setminus S$. We then denote by $\mathcal{U}_S = \otimes_{m \in S} \mathcal{U}_m$ and $\mathcal{U}_{\bar{S}} = \otimes_{m \in \bar{S}} \mathcal{U}_m$, and by $y_S = \{y_i, \ i \in S\}$ the extraction from y. Let $\mathcal{E}_m$ be the ellipse in $\mathcal{U}_m$ with foci at ± 1 and the sum of the semiaxes being η_m; and $\mathcal{E}_S = \prod_{m \in \mathrm{supp}(\nu)} \mathcal{E}_m$. We can then write (3.9) as

$$u_\nu = \frac{1}{(2\pi i)^{|\nu|_0}} \int_U L_\nu(y) \oint_{\mathcal{E}_S} \frac{u(z_S, y_{\bar{S}})}{(z_S - y_S)^{\mathbf{1}}} dz_S d\rho(y).$$

For each $m \in \mathbb{N}$, let Γ_m be a copy of $[-1,1]$ and $y_m \in \Gamma_m$. We denote by $U_S = \prod_{m \in S} \Gamma_m$ and $U_{\bar{S}} = \prod_{m \in \bar{S}} \Gamma_m$. We then have

$$u_\nu = \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_{\bar{S}}} \oint_{\mathcal{E}_S} u(z_S, y_{\bar{S}}) \int_{U_S} \frac{L_\nu(y)}{(z_S - y_S)^{\mathbf{1}}} d\rho_S(y_S) dz_S d\rho_{\bar{S}}(y_{\bar{S}}).$$

To proceed further, we recall the definitions of the Legendre functions of the second kind

$$Q_n(z) = \int_{[-1,1]} \frac{L_n(y)}{(z-y)} d\rho(y).$$

Let ν_S be the restriction of ν to S. We define

$$\mathcal{Q}_{\nu_S}(z_S) = \prod_{m \in \mathrm{supp}(\nu)} Q_{\nu_m}(z_m).$$

Making the Joukovski transformation $z_m = \frac{1}{2}(w_m + w_m^{-1})$, the Legendre polynomials of the second kind are written as

$$Q_{\nu_m}\left(\frac{1}{2}(w_m + w_m^{-1})\right) = \sum_{k=\nu_m+1}^{\infty} \frac{q_{\nu_m k}}{w_m^k}$$

with $|q_{\nu_m k}| \le \pi$. Therefore

$$|\mathcal{Q}_{\nu_S}(z_S)| \le \prod_{m \in S} \sum_{k=\nu_m+1}^{\infty} \frac{\pi}{\eta_m^k} = \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1 - \eta_m^{-1}}.$$

We then have

$$
\begin{aligned}
\|u_\nu\|_{\mathcal{X}} &= \left\| \frac{1}{(2\pi i)^{|\nu|_0}} \int_{U_{\bar{S}}} \oint_{\mathcal{E}_S} u(z_S, y_{\bar{S}}) \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar{S}}(y_S) \right\|_{\mathcal{X}} \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \int_{U_{\bar{S}}} \oint_{\mathcal{E}_S} \|u(z_S, y_{\bar{S}})\|_{\mathcal{X}} \mathcal{Q}_{\nu_S}(z_S) dz_S d\rho_{\bar{S}}(y_S) \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \|u(z)\|_{L^\infty(\mathcal{E}_S \times U_{\bar{S}}, \mathcal{X})} \max_{\mathcal{E}_S} |\mathcal{Q}_{\nu_S}| \prod_{m \in S} \mathrm{Len}(\mathcal{E}_m) \\
&\le \frac{1}{(2\pi)^{|\nu|_0}} \|u(z)\|_{L^\infty(\mathcal{E}_S \times \mathcal{U}_{\bar{S}}, \mathcal{X})} \prod_{m \in S} \pi \frac{\eta_m^{-\nu_m-1}}{1 - \eta_m^{-1}} \mathrm{Len}(\mathcal{E}_m) \\
&\le C \prod_{m \in S} \frac{2(1+K)}{K} \eta_m^{-\nu_m},
\end{aligned}
$$

9

as $\text{Len}(\mathcal{E}_m) \leq 4\eta_m$, $\eta_m \geq 1 + K$ and $u(z)$ is uniformly bounded in $\mathcal{X}$. $\qquad\square$

To show the $l^p(\mathcal{F})$ summability of $\|u_\nu\|_\mathcal{X}$, we use the following proposition from [5].

Proposition 3.3 *For $0 < p < 1$, $\left(\dfrac{|\nu|!}{\nu!}b^\nu\right)_{\nu \in \mathcal{F}} \in l^p(\mathcal{F})$ iff (i) $\sum_{m \geq 1} b_m < 1$ and (ii) $(b_m) \in l^p(\mathbb{N})$.*

Proposition 3.4 *For $0 < p < 1$ as in Assumption 3.1, $\sum_{\nu \in \mathcal{F}} \|u_\nu\|_\mathcal{X}^p$, is finite.*

Proof We have from the previous proposition that

$$
\begin{aligned}
\|u_\nu\|_\mathcal{X} &\leq C \prod_{m \in S} \frac{2(1+K)}{K}(1 + r_m)^{-\nu_m} \\
&\leq C\Big(\prod_{m \in E, \nu_m \neq 0} \frac{2(1+K)}{K}\eta^{\nu_m} \Big)\Big(\prod_{m \in F, \nu_m \neq 0} \frac{2(1+K)}{K}\Big(\frac{4|\nu_F|\|\psi_m\|_{L^\infty(D)}}{a_{\min}\nu_m}\Big)^{\nu_m} \Big)
\end{aligned}
$$

where $\eta = 1/(1 + K)$. Let $\mathcal{F}_E = \{\nu \in \mathcal{F}: \quad \text{supp}(\nu) \subset E\}$ and $\mathcal{F}_F = \mathcal{F} \setminus E$. From this, we have

$$
\sum_{\nu \in \mathcal{F}} \|u_\nu\|_\mathcal{X}^p \leq C A_E A_F,
$$

where

$$
A_E = \sum_{\nu \in \mathcal{F}_E} \prod_{m \in E, \nu_m \neq 0} \Big(\frac{2(1+K)}{K}\Big)^p \eta^{p\nu_m},
$$

and

$$
A_F = \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F, \nu_m \neq 0} \Big(\frac{2(1+K)}{K}\Big)^p \Big(\frac{4|\nu|\|\psi_m\|_{L^\infty(D)}}{a_{\min}\nu_m}\Big)^{p\nu_m}.
$$

We now show that both A_E and A_F are finite. For A_E, we have

$$
A_E = \Big(1 + \Big(\frac{2(1+K)}{K}\Big)^p \sum_{m \geq 1} \eta^{pm} \Big)^{J_0},
$$

which is finite because $\eta < 1$. For A_F, we note that for $\nu_m \neq 0$,

$$
\frac{2(1+K)}{K} \leq \Big(\frac{2(1+K)}{K}\Big)^{\nu_m}.
$$

Therefore

$$
A_F \leq \sum_{\nu \in \mathcal{F}_F} \prod_{m \in F} \Big(\frac{|\nu|d_m}{\nu_m}\Big)^{p\nu_m},
$$

where

$$
d_m = \frac{8(1+K)\|\psi_m\|_{L^\infty(D)}}{K a_{\min}};
$$

we make the convention that $0^0 = 1$. We now proceed as in [4]: from the Stirling estimate

$$
\frac{n!e^n}{e\sqrt{n}} \leq n^n \leq \frac{n!e^n}{\sqrt{2\pi n}},
$$

we infer $|\nu|^{|\nu|} \leq |\nu|!e^{|\nu|}$ and obtain

$$
\prod_{m \in F} \nu_m^{\nu_m} \geq \frac{\nu!e^{|\nu|}}{\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\}}.
$$

Hence

$$
A_F \leq \sum_{\nu \in \mathcal{F}_F} \Big(\frac{|\nu|!}{\nu!}d^\nu\Big)^p \Big(\prod_{m \in F} \max\{1, e\sqrt{\nu_m}\}\Big)^p \leq \sum_{\nu \in \mathcal{F}_F} \Big(\frac{|\nu|!}{\nu!}\bar{d}^\nu\Big)^p,
$$

10

where $\bar{d}_m = ed_m$ and where and we have used the estimate $e\sqrt{n} \le e^n$. From this, we have

$$\sum_{m \ge 1} \bar{d}_m \le \sum_{m \in F} \frac{24(1+K)\|\psi_m\|_{L^\infty(D)}}{K a_{\min}} \le 1.$$

It is also obvious that

$$\|\bar{d}\|_{l^P(\mathbb{N})} < \infty.$$

From these estimates and from Proposition 3.3 we obtain the conclusion. □

3.3 Best N-term convergence rates

With Lemma 3.1, we have from Proposition 3.4 and Theorem 2.1 the following result:

Theorem 3.3 *If Assumptions 1.1, 1.2 and 3.1 hold for some $0 < p < 1$, there is a sequence $(\Lambda_N)_{N \in \mathbb{N}} \subset \mathcal{F}$ of index sets with cardinality not exceeding N such that the solutions u_{Λ_N} of the Galerkin semidiscretized problems (2.3) satisfy*

$$\|u - u_{\Lambda_N}\|_{\underline{\mathcal{X}}} \le C N^{-\sigma}, \qquad \sigma = \frac{1}{p} - \frac{1}{2}.$$

4 Regularity of the solutions

To get explicit error estimates for numerical schemes, we need regularity for the solution u of the problem (1.15). We will establish this in the following sections.

4.1 Compatible initial conditions

In this section, we derive a regularity estimate for the parametric solution in the case of a compatible initial condition h of the problem (1.1). In particular, we assume that

$$h \in V \cap H^2(D), \quad g \in L^2(I; H) \cap H^1(I; V'), \quad g(0, \cdot) \in H. \tag{4.1}$$

Throughout this section, Assumptions 1.1, 1.2 and (4.1) are assumed to hold. Moreoever, we impose an additional regularity assumption on the functions ψ_j.

Assumption 4.1 *We assume that $\bar{a}(x) \in W^{1,\infty}(D)$ and that for $0 < p < 1$ as in Assumption 3.1*

$$\sum_{j=1}^\infty \|\psi_j\|_{W^{1,\infty}(D)}^p < \infty.$$

With Assumption 4.1, we establish regularity for the solution u of (1.10) and the functions u_ν. We will show that under the compatibility conditions and Assumption 4.1, the solution u belongs to the space

$$\mathcal{Z} = L^2(I; H^2(D)) \cap H^1(I; V) \cap H^2(I; V') \tag{4.2}$$

equipped with the norm

$$\| \circ \|_{\mathcal{Z}} = \left(\| \circ \|_{L^2(I;H^2(D))}^2 + \| \circ \|_{H^1(I;V)}^2 + \| \circ \|_{H^2(I;V')}^2 \right)^{1/2}.$$

Proposition 4.1 *With the condition (4.1), under Assumption 4.1 it holds that $u(\cdot, \cdot, y) \in \mathcal{Z}$ for all $y \in U$ and $\|u\|_{\mathcal{Z}}$ is uniformly bounded for all $y \in U$.*

  Proof We note that

$$\nabla \cdot (a(x, y)\nabla h) = \nabla \cdot (\bar{a}(x)\nabla h) + \sum_{j=1}^\infty y_j(\psi_j \Delta h + \nabla \psi_j \cdot \nabla h) \in H.$$

The theorem follows from standard results on regularity of the solutions for parabolic equations (as, e.g., in [11] Theorem 27.4). □

The following existence and uniqueness results for u in the norm of $\mathcal{Z}$ are parametric analogues to those in the norm of $\mathcal{X}$ in Section 1.3.

Proposition 4.2 *There exists a constant $C > 0$ such that*

$$\forall y, y' \in U : \qquad \|u(\cdot, \cdot, y) - u(\cdot, \cdot, y')\|_{\mathcal{Z}} \le C\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}.$$

Proof For $y, y' \in U$, we define $w(t, x) = u(t, x, y) - u(t, x, y')$. Since, by assumption, $a(\cdot, y) \in W^{1,\infty}(D)$ for every $y \in U$, the function w is the solution of the parabolic problem

$$\frac{\partial w}{\partial t} - \nabla \cdot (a(x, y)\nabla w) = \nabla(a(x, y) - a(x, y')) \cdot \nabla u(x, t, y') + (a(x, y) - a(x, y'))\Delta u(t, x, y'),$$

$$w(t, x) = 0 \text{ when } x \in \partial D,$$

$$w(0, x) = 0.$$

In particular, the data g and h of this problem satisfy (4.1). We observe that for every $y' \in U$ we have $u(\cdot, \cdot, y') \in \mathcal{Z}$, and it holds that

$$\|\nabla(a(\cdot, y) - a(\cdot, y')) \cdot \nabla u(\cdot, \cdot; y') + (a(\cdot, y) - a(\cdot, y'))\Delta u(\cdot, \cdot, y')\|_{L^2(I;H) \cap H^1(I;V')}$$
$$\le C\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}$$

The conclusion then follows. □

Proposition 4.3 *The map $u(\cdot, \cdot, y) : U \to \mathcal{Z}$ is measurable as a Bochner function.*

Proof The proof of this proposition follows the lines of the proof of Proposition 1.4 except that here we use Proposition 4.2. □

The above results show that $u \in L^2(U, \mathcal{Z}, d\rho(y))$. Therefore

$$u_\nu = \int_U L_\nu(y) u(y) d\rho(y) \in \mathcal{Z}.$$

Next we establish a priori bounds for $\|u_\nu\|_{\mathcal{Z}}$. Let $\bar{K} < 1$ denote a positive number such that

$$\bar{K} \sum_{j=1}^{\infty} (\|\psi_j\|_{L^\infty(D)} + \|\nabla\psi_j\|_{L^\infty(D)}) < \frac{a_{\min}}{8}.$$

We again choose an integer $\bar{J}_0$ such that

$$\sum_{j > \bar{J}_0} (\|\psi_j\|_{L^\infty(D)} + \|\nabla\psi_j\|_{L^\infty(D)}) < \frac{a_{\min}\bar{K}}{24(1 + \bar{K})}.$$

Let $\bar{E} = \{1, 2, \ldots, \bar{J}_0\}$ and $\bar{F} = \mathbb{N} \setminus \bar{E}$. We define

$$|\nu_{\bar{F}}| = \sum_{j > \bar{J}_0} |\nu_j|.$$

For each $\nu \in \mathcal{F}$ we define as before $\bar{r}_m = \bar{K}$ for $m \le \bar{J}_0$ and for $m > \bar{J}_0$

$$\bar{r}_m = 1 + \frac{a_{\min}\nu_m}{4|\nu_{\bar{F}}|(\|\psi_m\|_{L^\infty(D)} + \|\nabla\psi_m\|_{L^\infty(D)})} \tag{4.3}$$

with the convention that $\frac{|\nu_j|}{|\nu_{\bar{F}}|} = 0$ if $|\nu_{\bar{F}}| = 0$. We then define the discs

$$\bar{\mathcal{U}}_m = \{z_m \in \mathbb{C} : |z_m| < 1 + \bar{r}_m\} \subset \mathbb{C}, \qquad m \in \mathbb{N}$$

and the polydisc

$$\bar{\mathcal{U}} = \bigotimes_{m=1}^{\infty} \bar{\mathcal{U}}_m \subset \mathbb{C}^{\mathbb{N}}.$$

For $z \in \bar{\mathcal{U}}$, let

$$a(x, z) = a_0(x) + \sum_{m=1}^{\infty} \psi_m(x) z_m.$$

The sum is well-defined: for $z \in \bar{\mathcal{U}}$ and a.e. $x \in D$ it holds

$$|a(x, z)| \le \max \bar{a}(x) + \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} (1 + \bar{r}_m) \le \|\bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}.$$

This is proved as the analogous inequality in Section 3.1. For $z \in \bar{\mathcal{U}}$, we again consider the problem (3.5).

Proposition 4.4 *The problem (3.5) admits a unique solution, which is uniformly bounded in $\mathcal{Z}$ for all $z \in \bar{\mathcal{U}}$.*

Proof For $z \in \bar{\mathcal{U}}$, by an argument analogous to what we did in Section 3.1, we estimate

$$\Re a \ge \min a(x) - \sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} (1 + \bar{r}_m) \ge \frac{a_{\min}}{2}.$$

The problem (3.5) thus has a unique solution under Assumptions 1.1, 1.2, 4.1 and (4.1). Furthermore for every $z \in \overline{\mathcal{U}}$ and a.e. $x \in D$,

$$
\begin{aligned}
|\nabla a(x, z)| &\le \|\nabla \bar{a}\|_{L^\infty(D)} + \sum_{m=1}^{\infty} (1 + \bar{r}_m) \|\nabla \psi_m\|_{L^\infty(D)} \\
&\le \|\nabla \bar{a}\|_{L^\infty(D)} + \sum_{m=1}^{\bar{J}_0} \|\nabla \psi_m\|_{L^\infty(D)} (1 + \bar{K}) + \sum_{j > \bar{J}_0} \left(2 + \frac{a_{\min} \nu_j}{4 |\nu_F| \|\psi_j\|_{L^\infty(D)}} \right) \|\nabla \psi_j\|_{L^\infty(D)} \\
&\le \|\nabla \bar{a}\|_{L^\infty(D)} + 2 \sum_{m=1}^{\infty} \|\nabla \psi_m\|_{L^\infty(D)} + \frac{a_{\min}}{4}.
\end{aligned}
$$

Therefore

$$g(0, \cdot) - \nabla \cdot (a(\cdot, z) \nabla h(\cdot)) \in H,$$

for all $z \in \bar{\mathcal{U}}$ and its H-norm is uniformly bounded. For $z \in \bar{\mathcal{U}}$ the solution of (3.5) is thus uniformly bounded in $\mathcal{Z}$. $\qquad \square$

For each index $\nu \in \mathcal{F}$ we define the polydiscs

$$\bar{\mathcal{U}}_\nu = \bigotimes_{j \in \text{supp}(\nu)} \bar{\mathcal{U}}_j. \tag{4.4}$$

Proposition 4.5 *For $\nu \in \mathcal{F}$, fixing z_k where $k \notin \text{supp}(\nu)$, the map $u : \bar{\mathcal{U}}_\nu \to \mathcal{Z}$ is analytic as a $\mathcal{Z}$-valued function.*

Proof For a fixed index $m \in \text{supp}(\nu)$, we fix all the coordinates z_k when $k \neq m$, and denote $z \in \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. Let $\delta \in \mathbb{C}$. We show that there exists a function $v \in \mathcal{Z}$ such that

$$\lim_{\delta \to 0} \left\| \frac{u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{\mathcal{Z}} = 0,$$

for all $z \in \mathcal{U}$. Let

$$v^\delta = \frac{u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, (z_m, \bar{z}_m))}{\delta}.$$

The function v^δ satisfies

$$\frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, (z_m + \delta, \bar{z}_m))),$$

13

with the initial condition $v^\delta(0, \cdot, (z_m + \delta, \bar{z}_m)) = 0$. Let v denote the solution of the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, z)\nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$\frac{\partial(v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, z)\nabla(v^\delta - v)) = \nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z))).$$

An argument similar to the proof of Proposition 4.2 shows that

$$\|u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)\|_{\mathcal{Z}} \leq C|\delta|.$$

Therefore

$$\|\nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z)))\|_{L^2(I;H) \cap H^1(I;V')} \leq C|\delta|.$$

Further,

$$\nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, (z_m + \delta, \bar{z}_m)) - u(\cdot, \cdot, z))) = 0,$$

at $t = 0$. Thus

$$\|v^\delta - v\|_{\mathcal{Z}} \leq C|\delta|.$$

This shows that v is the derivative of u as a $\mathcal{Z}$-valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{\mathcal{U}}_\nu$ to $\mathcal{Z}$. $\qquad\square$

From the preceding arguments, we deduce the following result. Its proof is similar to those of Propositions 3.2 and 3.4.

Theorem 4.1 *Under Assumptions 1.1, 1.2, 4.1, and for data g and h satisfying the compatibility conditions (4.1), the coefficients $u_\nu \in \mathcal{Z}$ satisfy the estimates*

$$\|u_\nu\|_{\mathcal{Z}} \leq C \prod_{m \in \mathrm{supp}(\nu)} \frac{2(1 + \bar{K})}{\bar{K}} \bar{\eta}_m^{-\nu_m}, \quad \nu \in \mathcal{F} \tag{4.5}$$

where $\bar{\eta}_m = r_m + \sqrt{1 + r_m^2}$. Moreover, for the same constant $0 < p < 1$ as in Assumption 4.1 $\sum_{\nu \in \mathcal{F}} \|u_\nu\|_{\mathcal{Z}}^p$, is finite.

4.2 Incompatible initial conditions

We now consider the case where the initial condition does not satisfy the compatible condition (4.1). We define the following spaces by interpolation: for $0 < \theta < 1$,

$$H^\theta = (H, V)_{\theta,2}, \quad H^{1+\theta} = (V, V \cap H^2(D))_{\theta,2}, \quad H^{-1+\theta} = (V', H)_{\theta,2}.$$

We assume throughout this section that Assumption 4.1 holds. For each $y \in U$, we consider the parametric eigenvalue problem: find $\lambda(y) \in \mathbb{R}$ and $0 \neq \phi(y) \in V$ such that for all $\psi \in V$ holds

$$\int_D a(x, y)\nabla\phi(y) \cdot \nabla\psi dx = \lambda(y) \int_D \phi(y)\psi dx, \ \forall \psi \in V \tag{4.6}$$

By the spectral theorem, for every $y \in U$ exists a countable family $(\lambda_i(y), \phi_i(y))_{i=1}^\infty$ of eigenpairs such that $\phi_i(y) \in V$ are an orthonormal basis of H. From the Poincaré inequality, $\lambda_i(y) \geq \bar{\lambda}$ for a positive constant $\bar{\lambda}$ which only depends on the constants $a_{\max}$ and $a_{\min}$ in Assumption 1.1 and is independent of y.

For each $t \in I$ and every $y \ni U$, we define the parametric evolution operator $T(y; t)$ in terms of the eigenfunctions $\phi_i(y)$ by

$$T(y; t)u = \sum_{i=1}^\infty e^{-\lambda_i(y)t}(u, \phi_i(y))_H \phi_i(y). \tag{4.7}$$

The solution of the problem (1.10) can be represented as

$$u(t, \cdot, y) = T(y; t)h + \int_0^t T(y; t - s)g(s)ds, \ 0 \leq t \leq T. \tag{4.8}$$

In this section, we assume that $g : I \to H$ is analytic on $[0, T]$.

14

Proposition 4.6 *For $0 \leq \theta \leq 1$ and $0 \leq s \leq 1$, there exists $c > 0$ such that for every $y \in U$ and every $0 < t \leq T$ holds*

$$\|T^{(l)}(y;t)\|_{H^\theta, H^{\pm 1+s}} \leq ct^{-l \mp 1/2 + \theta/2 - s/2}.$$

Proof For every $y \in U$, we define on V the norm

$$\|\phi\|^2_{y;E} = \int_D a(x,y) |\nabla \phi(x,y)|^2 dx, \qquad y \in U.$$

As $a(x,y)$ satisfies Assumption 1.1, this is an equivalent norm in V. Specifically, there are positive constants c_1 and c_2 independent of y such that

$$c_1 \| \circ \|_V \leq \| \circ \|_E \leq c_2 \| \circ \|_V. \tag{4.9}$$

For $v \in H$ and $i \in \mathbb{N}$, define $v_i(y) = (v, \phi_i(y))_H$. For all $y \in U$, we have

$$\|v\|^2_H = \sum_{i=1}^{\infty} (v_i(y))^2.$$

From (4.7):

$$\|T^{(l)}(y;t)v\|^2_{y;E} = \sum_{i=1}^{\infty} \lambda_i(y)^{2l+1} e^{-2\lambda_i t} (v_i(y))^2.$$

The maximum value of $e^{-2\lambda t} \lambda^{2l+1}$ is $C(l)t^{-(2l+1)}$. Thus

$$\|T^{(l)}(y;t)v\|^2_{y;E} \leq t^{-(2l+1)} \sum_{i=1}^{\infty} (v_i(y))^2 = C(l)t^{-(2l+1)} \|v\|^2_H.$$

Therefore

$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H,V)} \leq ct^{-l-1/2}. \tag{4.10}$$

For $v \in V$,

$$\|v\|^2_{y;E} = \sum_{i=1}^{\infty} (v_i(y))^2 \lambda_i(y).$$

The maximum value of $e^{-2\lambda t} \lambda^{2l}$ is $C(l)t^{-2l}$. From this and (4.9) we get

$$\|T^{(l)}(y;t)\|_{\mathcal{L}(V,V)} \leq C(l)t^{-l}. \tag{4.11}$$

Similarly, we obtain for every $v \in H$ the bound

$$\|T^{(l)}(y;t)v\|^2_H = \sum_{i=1}^{\infty} \lambda_i^{2l} e^{-2\lambda_i t} v_i^2.$$

Therefore, we also have

$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H,H)} \leq C(l)t^{-l}. \tag{4.12}$$

Further

$$\|T^{(l)}(y;t)v\|^2_H = \sum_{i=1}^{\infty} \lambda_i^{2l-1} e^{-2\lambda_i t} v_i^2 \lambda_i.$$

A similar argument shows that for every $l \geq 1$

$$\|T^{(l)}\|_{\mathcal{L}(V,H)} \leq C(l)t^{-l+1/2}. \tag{4.13}$$

We observe that for every $y \in U$, the function $w = T(y;t)v$ is the solution of the Cauchy problem

$$\frac{\partial w}{\partial t} - \nabla \cdot (a(x,y)\nabla w) = 0, \qquad w(0, \cdot) = v.$$

15

Therefore,
$$w^{(l+1)}(t, \cdot) - \nabla \cdot (a(x, y)\nabla w^{(l)}(t, \cdot)) = 0,$$

i.e.
$$-\Delta w^{(l)}(t, \cdot) = -\frac{1}{a}w^{(l+1)}(t, \cdot) + \frac{1}{a}\nabla a \cdot \nabla w^{(l)}(t, \cdot).$$

As the domain D is convex,
$$\|w^{(l)}(t, .)\|_{H^2(D)} \leq c(\|w^{(l+1)}(t, .)\|_H + \|w^{(l)}(t, .)\|_V)$$

which is bounded by $C(l)t^{-(l+1)}\|v\|_H$ and $C(l)t^{-(l+1/2)}\|v\|_V$. Therefore
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H,H^2(D))} \leq C(l)t^{-(l+1)}, \tag{4.14}$$

and
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(V,H^2(D))} \leq C(l)t^{-(l+1/2)}. \tag{4.15}$$

From interpolation of Hilbert spaces using the real method (see, e.g., [10]), we deduce from (4.10), (4.11), (4.14) and (4.15) that for all $y \in U$:
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H^\theta, H^{1+s})} \leq ct^{-l-1/2+\theta/2-s/2}. \tag{4.16}$$

Now we consider $\|T^{(l)}(y;t)v\|_{V'}$. We have (with $\langle \cdot, \cdot \rangle$ denoting the $V \times V'$-duality pairing obtained by extending the H-inner product by continuity)

$$
\begin{aligned}
\|T^{(l)}(y;t)v\|_{V'} &= \sup_{\psi \in V} \langle T^{(l)}(y;t)v, \psi \rangle / \|\psi\|_V \\
&= \sup_{\psi \in V} a\Big(\sum_{i=1}^{\infty}(-1)^l e^{-\lambda_i(y)t}(\lambda_i(y))^{l-1}v_i(y)\phi_i(y), \psi\Big) / \|\psi\|_V \\
&\leq c\Big(\sum_{i=1}^{\infty}(\lambda_i(y))^{2l-1}(v_i(y))^2 e^{-2\lambda_i(y)t}\Big)^{1/2}.
\end{aligned}
$$

Therefore for $l \geq 1$ for all $y \in U$
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H,V')} \leq C(l)t^{-l+1/2}. \tag{4.17}$$

Similarly, we have that for $l \geq 1$
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(V,V')} \leq C(l)t^{-l+1}. \tag{4.18}$$

From interpolation, we get from (4.12), (4.13), (4.17) and (4.18) that for $l \geq 1$
$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H^\theta, H^{-1+s})} \leq C(l)t^{-l+1/2+\theta/2-s/2}. \tag{4.19}$$

$\square$

Proposition 4.7 *Assume that $h \in H^\theta$, for some $0 < \theta < 1$. Then for every $s < \theta$*
$$u(\cdot, \cdot, y) \in L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$$

and there exists $C_1 > 0$ independent of y such that
$$\forall y \in U: \qquad \|u(\cdot, \cdot, y)\|_{L^2(I;H^{1+s})\cap H^1(I;H^{-1+s})} \leq C_1.$$

Proof From Proposition 4.6, we have that
$$\|T(y;t)h\|_{H^{1+s}} \leq ct^{-1/2+\theta/2-s/2}.$$

Furthermore as $\|g(s)\|_H < c$ for all s
$$\Big\|\int_0^t T(y;t-s)g(s)ds\Big\|_{H^{1+s}} \leq c\int_0^t (t-s)^{-1/2-s/2}ds \leq ct^{1/2-s/2}.$$

16

Therefore for $0 < t \le 1$,
$$\|u(t, \cdot, \cdot)\|_{H^{1+s}} \le ct^{-1/2+\theta/2-s/2}. \tag{4.20}$$

From (4.8),
$$u'(t, \cdot, y) = T'(y; t)h + g(t) + \int_0^t T'(y; t-s)g(s)ds.$$

From Proposition 4.6:
$$\|T'(y; t)h\|_{H^{-1+s}} \le ct^{-1/2+\theta/2-s/2}.$$

We also have
$$\| \int_0^t T'(y; t-s)g(s)ds\|_{H^{-1+s}} \le c \int_0^t (t-s)^{-1/2-s/2}ds \le ct^{1/2-s/2}.$$

Therefore for $0 < t \le 1$:
$$\|u'(t, \cdot, \cdot)\|_{H^{-1+s}} \le ct^{-1/2+\theta/2-s/2}. \tag{4.21}$$

This completes the proof. $\qquad\square$

Proposition 4.8 *For $s < \theta$, there exists a constant $C > 0$ such that $\forall\, y, y' \in U$:*
$$\|u(\cdot, \cdot, y) - u(\cdot, \cdot, y')\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} \le C\|a(., y) - a(., y')\|_{W^{1,\infty}(D)}.$$

Proof Define $w(t, x, y, y') = u(t, x, y) - u(t, x, y')$ and
$$G(t, x) = \nabla(a(x, y) - a(x, y')) \cdot \nabla u(x, t, y') + (a(x, y) - a(x, y'))\Delta u(t, x, y'). \tag{4.22}$$

The function $G \in L^2(I; V')$. Therefore w is the weak solution of the problem
$$\frac{\partial w}{\partial t} - \nabla \cdot (a(x, y)\nabla w) = G(t, x), \qquad w(t, x) = 0 \text{ when } x \in \partial D, \quad w(0, x) = 0. \tag{4.23}$$

From (4.20), we deduce that for all t and for every $y, y' \in U$ holds
$$\|G(t, .)\|_{H^{-1+s}} \le c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}t^{-1/2+\theta/2-s/2}.$$

Note that $G \notin L^2(I; H)$. However, for every $0 < t_0 < t$ and for every $y, y' \in U$ we have
$$w(t, \cdot, y, y') = T(y; t - t_0)w(t_0) + \int_{t_0}^t T(y; t-r)G(r)dr.$$

Therefore
$$
\begin{aligned}
\|w(t, \cdot, y, y')\|_{H^{1+s}} &\le \|T(y; t-t_0)w(t_0)\|_{H^{1+s}} \\
&\quad + c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} \int_0^t (t-r)^{-1/2-s/2}r^{-1+\theta/2}dr.
\end{aligned}
$$

Passing to the limit $t_0 \to 0$, as $\|w(t_0)\|_H \to 0$, from Proposition 4.6 we obtain
$$\|w(t, \cdot, y, y')\|_{H^{1+s}} \le c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}t^{-1/2+\theta/2-s/2}.$$

Similarly,
$$
\begin{aligned}
\|w'(t, \cdot, y, y')\|_{H^{-1+s}} &\le \|T'(y; t-t_0)w(t_0)\|_{H^{-1+s}} + \|G(t)\|_{H^{-1+s}} \\
&\quad + c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)} \int_0^t (t-r)^{-1/2-s/2}r^{-1+\theta/2}dr
\end{aligned}
$$

Letting $t_0 \to 0$, we get
$$\|w'(t, \cdot, y, y')\|_{H^{-1+s}} \le c\|a(\cdot, y) - a(\cdot, y')\|_{W^{1,\infty}(D)}t^{-1/2+\theta/2-s/2}.$$

This completes the proof. $\qquad\square$

17

Proposition 4.9 *The map $u(\cdot, \cdot, y) : U \to L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$ is strongly measurable as a Bochner function.*

Proof The proof is by exactly the same argument as the proof of Proposition 1.4. We use Proposition 4.8 in place of Proposition 1.3. □

Thus we conclude that $u(\cdot, \cdot, y) \in L^2(U, L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s}), d\rho)$. Therefore the function

$$u_\nu(\cdot, \cdot) = \int_U L_\nu(y) u(\cdot, \cdot, y) d\rho(y) \in L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s}).$$

To describe the analyticity of the parametric solutions, we employ the complex domain $\bar{\mathcal{U}} \in \mathbb{C}^{\mathbb{N}}$ defined in (4.4).

For $z \in \bar{\mathcal{U}}$, we consider again problem (3.5). For each index $\nu \in \mathcal{F}$, we also recall the domain $\bar{\mathcal{U}}_\nu$ defined in (3.7).

Proposition 4.10 *Given $\nu \in \mathcal{F}$, for every $z \in \mathbb{C}^{\mathbb{N}}$ with fixed coordinates z_i where $i \notin \text{supp}(\nu)$, the map $u : \bar{\mathcal{U}}_\nu \to L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$ is analytic.*

Proof For a fixed index m, we fix all the coordinates z_k when $k \neq m$, and denote $z \in \mathbb{C}^{\mathbb{N}}$ as $z = (z_m, \bar{z}_m)$. Let $\delta \in \mathbb{C}$. We show that there exists a function $v \in \mathcal{Z}$ such that

$$\lim_{\delta \to 0} \left\| \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z)}{\delta} - v(\cdot, \cdot, z) \right\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} = 0,$$

for all $z \in \mathcal{U}$. Let

$$v^\delta = \frac{u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z_m, \bar{z}_m)}{\delta}.$$

The function v^δ satisfies

$$\frac{\partial v^\delta}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v^\delta) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z_m + \delta, \bar{z}_m)),$$

with the initial condition $v^\delta(0, \cdot, z_m + \delta, \bar{z}_m) = 0$. Let v satisfy the equation

$$\frac{\partial v}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla v) = \nabla \cdot (\psi_m \nabla u(\cdot, \cdot, z)),$$

with $v(0, \cdot, z) = 0$. Then

$$\frac{\partial (v^\delta - v)}{\partial t} - \nabla \cdot (a(\cdot, z) \nabla(v^\delta - v)) = \nabla \cdot (\psi_m \nabla(u(\cdot, \cdot, z_m + \delta, \bar{z}_m) - u(\cdot, \cdot, z))).$$

An argument similar to the proof of Proposition 4.8 shows that

$$\|u(t, \cdot, (z_m + \delta, \bar{z}_m)) - u(t, \cdot, z)\|_{H^{1+s}} \leq c|\delta| t^{-1/2 + \theta/2 - s/2}.$$

Therefore

$$\|\nabla \cdot (\psi_m \nabla(u(t, \cdot, z_m + \delta, \bar{z}_m) - u(t, \cdot, z)))\|_{H^{-1+s}} \leq c|\delta| t^{-1/2 + \theta/2 - s/2}.$$

Thus

$$\|v^\delta - v\|_{L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})} \leq c|\delta|.$$

This shows that v is the derivative of u with respect to z_m as a $L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$-valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{\mathcal{U}}_\nu$ to $L^2(I; H^{1+s}) \cap H^1(I; H^{-1+s})$. □

Proposition 4.11 *For any interval (a, b) such that $0 < a < b < \min(1, T)$ and for every $y \in U$ hold the bounds*

$$\|u(\cdot, \cdot, y)\|^2_{H^l((a,b);V)} \leq C(l) \int_a^b t^{-2l-1+\theta} dt, \tag{4.24}$$

and

$$\|u(\cdot, \cdot, y)\|^2_{H^l((a,b);V')} \leq C(l) \int_a^b t^{-2l+1+\theta} dt, \tag{4.25}$$

where $C(l)$ does not depend on a, b and on y.

18

Proof From (4.10) and (4.11),

$$\|T^{(l)}(y;t)\|_{\mathcal{L}(H^\theta,V)} \le C(l)t^{-l-1/2+\theta/2}. \tag{4.26}$$

From (4.8) we obtain for every $y \in U$

$$u^{(l)}(t,\cdot,y) = T^{(l)}(y;t)h + \sum_{i=0}^{l-1} T^{(i)}(y;t)g^{(l-1-i)}(0) + \int_0^t T(y;r)g^{(l)}(t-r)dr. \tag{4.27}$$

Then

$$\|T^{(l)}(y;t)h\|_V \le C(l)t^{-l-1/2+\theta/2}.$$

Further

$$\|\sum_{i=0}^{l-1} T^{(i)}(y;t)g^{(l-1-i)}(0)\|_V \le c\sum_{i=0}^{l-1} t^{-i-1/2} \le clt^{-l+1/2},$$

and

$$\|\int_0^t T(y;r)g^{(l)}(t-r)dr\|_V \le c\int_0^t r^{-1/2}dr \le ct^{1/2}.$$

From these bounds we deduce

$$\|u(\cdot,\cdot,y)\|_{H^l((a,b);V)}^2 \le C(l)\int_a^b t^{-2l-1+\theta}dt.$$

The proof for $\|u(\cdot,\cdot,y)\|_{H^l((a,b);V')}^2$ is similar.

Proposition 4.12 *For all y, $y' \in U$,*

$$\|u(\cdot,\cdot,y) - u(\cdot,\cdot,y')\|_{H^l((a,b);V)} \le C\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)}.$$

Proof For $t_0 < t$, the solution w of (4.23) is written as

$$w(t,\cdot) = T(y;t-t_0)w(t_0) + \int_0^{t-t_0} T(y;r)G(t-r)dr. \tag{4.28}$$

Therefore,

$$w^{(l)}(t,\cdot,y) = T^{(l)}(y;t-t_0)w(t_0) + \sum_{i=0}^{l-1} T^{(i)}(y;t-t_0)G^{(l-1-i)}(t_0) + \int_0^{t-t_0} T(y;r)G^{(l)}(t-r)dr. \tag{4.29}$$

From (4.27) and (4.16) we get

$$\|u^{(l)}(t,\cdot,y)\|_V \le ct^{-l-1/2+\theta/2} + c\sum_{i=0}^{l-1} t^{-i-1/2} + c\int_0^t r^{-1/2}dr \le C(l)t^{-l-1/2+\theta/2}.$$

From (4.27) and (4.19)

$$\|u^{(l)}(t,\cdot,y)\|_H \le ct^{-l+\theta/2} + c\sum_{i=0}^{l-1} t^{-i} + c\int_0^t dr \le C(l)t^{-l+\theta/2}.$$

From

$$-\Delta u^{(l)}(t,\cdot,y) = \frac{1}{a}g^{(l)}(t) + \frac{1}{a}\nabla a \cdot \nabla u^{(l)}(t,\cdot,y) - \frac{1}{a}u^{(l+1)}(t,\cdot,y)$$

and the convexity of the domain D, we deduce that

$$\|u^{(l)}(t,\cdot,y)\|_{H^2(D)} \le C(l)t^{-l-1+\theta/2}.$$

Thus for all i, $\|G^{(i)}(t)\|_H \le C(i)t^{-i-1+\theta/2}\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)}$. With $t_0 = 0$ we get from (4.28) the bound

$$\|w(t,\cdot,y)\|_H \le c\int_0^t (t-r)^{-1+\theta/2}dr\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)} \le ct^{\theta/2}.$$

From (4.29) we obtain for every $y \in U$ the estimate

$$\begin{aligned}
\|w^{(l)}(t,\cdot,y)\|_V &\le C(l)\Big((t-t_0)^{-l-1/2}t_0^{\theta/2} + c\sum_{i=0}^{l-1}(t-t_0)^{-i-1/2}t_0^{-l+i+\theta/2} \\
&\quad + \int_0^{t-t_0} r^{-1/2}(t-r)^{-l-1+\theta/2}dr\Big)\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)}.
\end{aligned}$$

Let $t_0 = t/2$, we deduce that

$$\|w^{(l)}(t,\cdot,y)\|_V \le C(l)t^{-l-1/2+\theta/2}\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)}.$$

This completes the proof. $\qquad\square$

Proposition 4.13 *The function $u : U \to H^l((a,b);V)$ is measurable as a Bochner function.*

Moreover, for every $\nu \in \mathcal{F}$, for $z \in \mathbb{C}^{\mathbb{N}}$ with fixed coordinates z_i where $i \notin \mathrm{supp}(\nu)$, the map $u : \bar{\mathcal{U}}_\nu \to H^l((a,b);V)$ is analytic.

Proof The first assertion is proved similarly to Proposition 1.4: here, however, we use Proposition 4.12 in place of Proposition 1.3.

To show analyticity of the mapping $u : U \to H^l((a,b);V)$, we fix all z_k where $k \ne m$ and show that there exists a function $v \in H^l((a,b);V)$ such that

$$\lim_{\delta \to 0}\left\| \frac{u(\cdot,\cdot,z_m+\delta,\bar{z}_m) - u(\cdot,\cdot,z)}{\delta} - v(\cdot,\cdot,z)\right\|_{H^l((a,b);V)} = 0,$$

for all $z \in \mathcal{U}$. Let

$$v^\delta = \frac{u(\cdot,\cdot,z_m+\delta,\bar{z}_m) - u(\cdot,\cdot,z_m,\bar{z}_m)}{\delta}.$$

The function v^δ satisfies

$$\frac{\partial v^\delta}{\partial t} - \nabla\cdot(a(\cdot,z)\nabla v^\delta) = \nabla\cdot(\psi_m\nabla u(\cdot,\cdot,z_m+\delta,\bar{z}_m)),$$

with the initial condition $v^\delta(0,\cdot,z_m+\delta,\bar{z}_m) = 0$. Let v satisfy the equation

$$\frac{\partial v}{\partial t} - \nabla\cdot(a(\cdot,\cdot,z)\nabla v) = \nabla\cdot(\psi_m\nabla u(\cdot,\cdot,z)),$$

with $v(0,\cdot,z) = 0$. Then

$$\frac{\partial(v^\delta - v)}{\partial t} - \nabla\cdot(a(\cdot,\cdot,z)\nabla(v^\delta - v)) = \nabla\cdot(\psi_m\nabla(u(\cdot,\cdot,z_m+\delta,\bar{z}_m) - u(\cdot,\cdot,z))).$$

An argument similar to the proof of Proposition 4.12, using (4.16) for $s = 1$ shows that

$$\|u^{(i)}(t,\cdot,(z_m+\delta,\bar{z}_m)) - u^{(i)}(t,\cdot,z)\|_{H^2(D)} \le c|\delta|t^{-i-1+\theta/2}.$$

Therefore

$$\|\nabla\cdot(\psi_m\nabla(u^{(i)}(t,\cdot,z_m+\delta,\bar{z}_m) - u^{(i)}(t,\cdot,z)))\|_H \le c|\delta|t^{-i-1+\theta/2}.$$

Thus using a formula similar to (4.29), we get

$$\|(v^\delta - v)^{(l)}(t)\|_V \le c|\delta|t^{-l-1/2+\theta/2}.$$

This shows that v is the derivative of u as a $H^l((a,b);V)$-valued function. From Hartogs' theorem, we conclude that u is analytic as a function from $\bar{\mathcal{U}}_\nu$ to $H^l((a,b);V)$. $\qquad\square$

Proposition 4.14 *There exists a constant $c > 0$ such that for all constants $k \in (0,1)$ and for all $y \in U$ holds*

$$\|u(\cdot,\cdot,y) - u(k,\cdot,y)\|_{L^2((0,k);V) \cap H^1((0,k);V')} \leq ck^{\theta/2}.$$

Proof The result follows from estimates (4.20) and (4.21). □

Proposition 4.15 *Fixing $k \in (0,1)$, $u(k,\cdot,\cdot)$ as a map from U to $L^2((0,k);V) \cap H^1((0,k);V')$ is measurable.*

Proof From Proposition 4.12 we obtain

$$\|u(k,\cdot,y) - u(k,\cdot,y')\|_V \leq c\|a(\cdot,y) - a(\cdot,y')\|_{W^{1,\infty}(D)}.$$

The remaining proof of this proposition is similar to that of Proposition 1.4. □

An argument analogous to the proof of Proposition 4.13 shows that

Proposition 4.16 *Fixing the coordinates z_i where $i \notin \mathrm{supp}(\nu)$, the map $u(\cdot,\cdot,\cdot) - u(k,\cdot,\cdot) : \bar{\mathcal{U}}_\nu \to L^2((0,k);V) \cap H^1((0,k);V')$ is analytic.*

As $u_\nu \in H^l((a,b),V)$, $u_\nu(k,\cdot)$ is uniquely determined and given by

$$u_\nu(k,\cdot) = \int_D u(k,\cdot,y) L_\nu(y) \rho(dy),$$

with the integral to be understood as a Bochner integral over U of $\mathcal{X}$-valued functions. We then deduce the following results.

Proposition 4.17 *The coefficient sequences*

$$\|u_\nu\|_{L^2(I;H^{1+s}) \cap H^1(I;H^{-1+s})}, \ \|u_\nu\|_{H^l((a,b);V)} \ and \ \|u_\nu(\cdot,\cdot,y) - u_\nu(k,\cdot,y)\|_{L^2((0,k);V) \cap H^1((0,k);V')}$$

are in $l^p(\mathcal{F})$.

Proof A similar argument to the proof of Proposition 3.2 shows that for every $\nu \in \mathcal{F}$ hold

$$\|u_\nu\|_{L^2(I;H^{1+s}) \cap H^1(I;H^{-1+s})} \leq \mathcal{B}(\nu)\|u\|_{L^\infty(U;L^2(I;H^{1+s}) \cap H^1(I;H^{-1+s}))},$$

$$\|u_\nu\|^2_{H^l((a,b);V)} \leq \mathcal{B}(\nu)^2 C(l) \int_a^b t^{-2l-1+\theta} dt,$$

and

$$\|u_\nu(\cdot,\cdot) - u_\nu(k,\cdot)\|_{L^2((0,k);V) \cap H^1((0,k);V')} \leq C\mathcal{B}(\nu)k^{\theta/2},$$

where we denote

$$\mathcal{B}(\nu) := \prod_{m \in \mathrm{supp}(\nu)} \frac{2(1+\bar{K})}{\bar{K}} \bar{\eta}_m^{-\nu_m} \tag{4.30}$$

with $\bar{\eta}_m$ as in (4.5). Proceeding similarly to the proof of Proposition 3.4, we obtain that for $0 < p < 1$ as in Assumption 4.1, $\sum_{\nu \in \mathcal{F}} (\mathcal{B}(\nu))^p$ is finite.

Appendix In this appendix, we prove the inf-sup conditions (2.4), (2.5) and (2.6) for the semidiscretized parametric problems. Specifically, we show that under Assumptions 1.1, 1.2, there exist constants $0 < \beta \leq \alpha < \infty$ such that for *any* subset $\Lambda \subset \mathcal{F}$ holds

$$\text{(A)} \quad \sup_{u_\Lambda \in \underline{\mathcal{X}}_\Lambda, v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \leq \alpha < \infty,$$

$$\text{(B)} \quad \beta = \inf_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} \sup_{0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda} \frac{|B(u_\Lambda, v_\Lambda)|}{\|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}} \geq \beta > 0,$$

$$\text{(C)} \quad \forall 0 \neq v_\Lambda \in \underline{\mathcal{Y}}_\Lambda : \sup_{0 \neq u_\Lambda \in \underline{\mathcal{X}}_\Lambda} |B(u_\Lambda, v_\Lambda))| > 0.$$

21

We first note that

$$\|u_\Lambda\|_{\underline{\mathcal{X}}}^2 = \sum_{\nu \in \Lambda}(\|u_\nu\|_{L^2(I;V)}^2 + \|u_\nu\|_{H^1(I;V')}^2), \quad \text{and} \quad \|v_\Lambda\|_{\underline{\mathcal{Y}}}^2 = \sum_{\nu \in \Lambda}(\|v_{1\nu}\|_{L^2(I;V)}^2 + \|v_{2\nu}\|_H^2).$$

First we show the continuity condition (A). We have

$$
\begin{aligned}
|B(u_\Lambda, v_\Lambda)| &\leq \int_U \left\{ \int_I \left(\left\| \frac{du_\Lambda}{dt}(t,\cdot,y) \right\|_{V'} \|v_{1\Lambda}(t,\cdot,y)\|_V + C\|\nabla u_\Lambda(t,\cdot,y)\|_H \|\nabla v_{1\Lambda}(t,\cdot,y)\|_H \right) dt \right. \\
&\qquad \left. + \|u_\Lambda(0,\cdot,y)\|_H \|v_{2\Lambda}(\cdot,y)\|_H \right\} d\rho(y) \\
&\leq \int_U \left\{ \int_I \left(\left\| \frac{du_\Lambda(t,\cdot,y)}{dt} \right\|_{V'} + C\|u_\Lambda(t,\cdot,y)\|_V \right) \|v_{1\Lambda}\|_V dt + M\|u_\Lambda(\cdot,\cdot,y)\|_{\mathcal{X}} \cdot \|v_{2\Lambda}(\cdot,y)\|_H \right\} d\rho(y) \\
&\leq \left\{ \int_U \int_I \left(\left\| \frac{du_\Lambda(t,\cdot,y)}{dt} \right\|_{V'} + C\|u_\Lambda(t,\cdot,y)\|_V \right)^2 dt d\rho(y) \right\}^{1/2} \\
&\quad \cdot \left\{ \int_U \int_I \|v_{1\Lambda}(t,\cdot,y)\|_V^2 dt d\rho(y) \right\}^{1/2} + M\|u_\Lambda\|_{L^2(U;\mathcal{X})} \cdot \|v_{2\Lambda}\|_{L^2(U,H)} \\
&\leq C\|u_\Lambda\|_{\underline{\mathcal{X}}} \cdot \|v\|_{\underline{\mathcal{Y}}},
\end{aligned}
$$

where we have used that the "initial value trace operator" from $\mathcal{X}$ is bounded, i.e. the constant

$$M = \sup_{0 \neq w \in \mathcal{X}} \frac{\|w(0)\|_H}{\|w\|_{\mathcal{X}}}$$

is finite. To show the inf-sup condition (B), we note that

$$\int_U \int_I \langle \frac{du_\Lambda}{dt}, v_{1\Lambda} \rangle_H dt = \sum_{\nu \in \Lambda} \int_I \langle \frac{du_\nu(t,\cdot)}{dt}, v_{\nu 1} \rangle_H dt;$$

and

$$
\begin{aligned}
&\int_U \int_I \int_D a(x,y) \nabla u_\Lambda(t,x,y) \cdot \nabla v_{1\Lambda}(t,x,y) dt dx d\rho(y) \\
&= \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D \left(\int_U a(x,y) L_\nu(y) L_\mu(y) d\rho(y) \right) \nabla u_\nu(t,x) \cdot \nabla v_{1\nu}(t,x) dx dt \\
&= \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D A^{\nu\mu}(x) \nabla u_\nu(t,x) \cdot \nabla v_{1\nu}(t,x) dx dt
\end{aligned}
\tag{4.31}
$$

where

$$A^{\nu\mu}(x) = \int_U a(x,y) L_\nu(y) L_\mu(y) d\rho(y),$$

and

$$\int_U \langle u_\Lambda(0), v_{2\Lambda} \rangle_H d\rho(y) = \sum_\nu \langle u_\nu(0,\cdot), v_{2\nu} \rangle_H.$$

Then

$$
\begin{aligned}
B(u_\Lambda, v_\Lambda) &= \sum_{\nu \in \Lambda} \int_I \langle \frac{du_\nu(t,\cdot)}{dt}, v_{\nu 1} \rangle_H dt \\
&\quad + \sum_{\nu \in \Lambda, \mu \in \Lambda} \int_I \int_D A^{\nu\mu}(x) \nabla u_\nu(t,x) \cdot \nabla v_{1\nu}(t,x) dx dt + \sum_{\nu \in \Lambda} \langle u_\nu(0,\cdot), v_{2\nu} \rangle_H.
\end{aligned}
$$

To show the ellipticity of $A^{\nu\mu}(x)$, we observe that for any array of vectors $\xi^\nu \in \mathbf{R}^d$ and for any $\mu, \nu \in \Lambda$ holds

$$A^{\nu\mu}(x) \xi_i^\nu \xi_i^\mu = \int_U a(x,y)(L_\nu(y)\xi_i^\nu)(L_\mu(y)\xi_i^\mu) d\rho(y) \geq a_{\min} \sum_{i=1}^d \sum_{\nu \in \Lambda} (\xi_i^\nu)^2$$

22

(repeated indices indicate summation).

Now we follow the approach of Babuska and Janik [1]: we consider the operator

$$\mathbf{A}: \ [V]^N \to [V']^N$$

defined by (with convention of summation over repeated indices)

$$(\mathbf{A}w)^\nu = -\nabla(A^{\nu\mu}(x)\nabla w_\mu(x)),$$

for each vector $\mathbf{w} = \{(w_\nu(x)): \ \nu \in \Lambda\} \in [V]^N$. Let λ_i^2 $(i = 1, 2, \ldots)$ be the eigenvalues of this operator. The corresponding eigenvectors are $\mathbf{w}_1, \mathbf{w}_2, \ldots$. For each $\phi \in (V)^N$:

$$\int_D A^{\nu,\mu}\nabla\mathbf{w}_{i\mu}(x) \cdot \nabla\phi_\nu(x)dx = \lambda_i \int_D \mathbf{w}_i^\nu(x)\phi_\nu(x)dx,$$

where summation is taken over μ and ν. Denote the vector $\mathbf{u}_\Lambda = \{(u_\nu(t,x))\}$. Then

$$\mathbf{u}_\Lambda(t,x) = \sum_{i=1}^\infty a_i(t)\mathbf{w}_i(x).$$

We choose $\mathbf{w}_i$ as an orthonormal base of $[H]^N$. We denote by

$$\mathbf{v}_1(x) = \{(v_{1\nu}(x)), \ \nu \in \Lambda\} = \sum_{i=1}^\infty b_i(t)\mathbf{w}_i.$$

Then

$$(a_{\max})^{-1}\int_0^T \sum_{i=1}^\infty \lambda_i(a_i(t))^2 dt \le \|\mathbf{u}\|_{L^2(I,[V]^N)}^2 \le (a_{\min})^{-1}\int_0^T \sum_{i=1}^\infty \lambda_i(a_i(t))^2 dt,$$

and

$$a_{\min}\int_0^T \sum_{i=1}^\infty \frac{(\dot{a}_i(t))^2}{\lambda_i}dt \le \left\|\frac{du}{dt}\right\|_{V'}^2 \le a_{\max}\int_0^T \sum_{i=1}^\infty \frac{(\dot{a}_i(t))^2}{\lambda_i}dt.$$

Thus

$$\int_0^T \left(\sum_{i=1}^\infty \frac{(\dot{a}_i(t))^2}{\lambda_i} + \lambda_i(a_i(t))^2\right)dt,$$

is equivalent to $\|\mathbf{u}\|_{(\mathcal{X})^N}^2 = \|u_\Lambda\|_{\underline{\mathcal{X}}_\Lambda}^2$. Let

$$\mathbf{v}_2(x) = \{(v_{2\nu}(x)), \ \nu \in \Lambda\}.$$

We write

$$\mathbf{v}_2(x) = \sum_{i=1}^\infty c_i\mathbf{w}_i(x).$$

Then

$$\int_0^T \sum_{i=1}^\infty \lambda_i(b_i(t))^2 dt + \sum_{i=1}^\infty c_i^2$$

is equivalent to $\|v_\Lambda\|_{\underline{\mathcal{Y}}}^2$ and we have

$$B(u_\Lambda, v_\Lambda) = \int_I \sum_{i=1}^\infty \lambda_i^{1/2}b_i(t)\left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2}a_i(t)\right]dt + \sum_{i=1}^\infty a_i(0)c_i.$$

We choose $b_i(t) \in L^2(0,T)$ such that

$$\lambda_i^{1/2}b_i(t) = \frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2}a_i(t), \quad \text{and} \quad c_i = a_i(0).$$

The bilinear form then becomes

$$\int_I \sum_{i=1}^{\infty} \left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t)\right]^2 dt + \sum_{i=1}^{\infty} a_i(0) c_i = \int_I \sum_{i=1}^{\infty} \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}}\right)^2 + (\lambda_i^{1/2} a_i(t))^2\right]^2 dt + \sum_{i=1}^{\infty} (a_i(T))^2$$

$$\geq \int_I \sum_{i=1}^{\infty} \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}}\right)^2 + (\lambda_i^{1/2} a_i(t))^2\right]^2 dt \geq C_1 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2.$$

With the above choice of $b_i(t)$, we also have

$$
\begin{aligned}
\|v_{1\Lambda}\|_{L^2(U, L^2(I, V))}^2 &\leq (a_{\min})^{-1} \sum_{i=1}^{\infty} \int_I \lambda_i (b_i(t)^2 dt \\
&= (a_{\min})^{-1} 2 \sum_{i=1}^{\infty} \int_I \left[\left(\frac{\dot{a}_i(t)}{\lambda_i^{1/2}}\right)^2 + (\lambda_i^{1/2} a_i(t))^2\right]^2 dt \\
&\leq C_2 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2.
\end{aligned}
\tag{4.32}
$$

We also have

$$\|v_{2\Lambda}\|_{L^2(U;H)}^2 = \|u_\Lambda(0, ., .)\|_{L^2(U;H)}^2 \leq M^2 \|u_\Lambda\|_{\underline{\mathcal{X}}}^2.$$

Thus

$$\|v_\Lambda\|_{\underline{\mathcal{Y}}}^2 \leq (C_2 + M^2) \|u_\Lambda\|_{\underline{\mathcal{X}}}^2.$$

Therefore, in this case

$$B(u_\Lambda, v_\Lambda) \geq C_1 (C_2 + M^2)^{-1/2} \|u_\Lambda\|_{\underline{\mathcal{X}}} \|v_\Lambda\|_{\underline{\mathcal{Y}}}.$$

We now show condition (C). We again use

$$B(u_\Lambda, v_\Lambda) = \int_I \left(\sum_{i=1}^{\infty} \left[\frac{\dot{a}_i(t)}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t)\right] \lambda_i^{1/2} b_i(t)\right) dt + \sum_{i=1}^{\infty} a_i(0) c_i.$$

For each v_Λ, we choose $a_i(t)$ such that

$$\frac{\dot{a}_i}{\lambda_i^{1/2}} + \lambda_i^{1/2} a_i(t) = \lambda b_i(t), \quad a_i(0) = c_i.$$

It then follows that $B(u_\Lambda, v_\Lambda) > 0$.

References

[1] Ivo Babuška and Tadeusz Janik. The h-p version of the finite element method for parabolic equations. I. The p-version in time. *Numer. Methods Partial Differential Equations*, 5(4):363–399, 1989.

[2] Ivo Babuška, Raúl Tempone, and Georgios E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.*, 42(2):800–825 (electronic), 2004.

[3] Marcel Bieri, Roman Andreev, and Christoph Schwab. Sparse tensor discretization of elliptic spdes. *SIAM J. Sci. Computing*, 31(6):4281–4304, 2009.

[4] Albert Cohen, Ronald DeVore, and Christoph Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs. Report 2010-03, Seminar for Applied Mathematics, ETH Zürich, 2010.

[5] Albert Cohen, Ronald DeVore, and Christoph Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. Report 2009-01, Seminar for Applied Mathematics, ETH Zürich, 2010. *To appear in J. FoCM*, 2010.

[6] Viet-Ha Hoang and Christoph Schwab. Sparse Tensor Galerkin discretizations for parametric and random parabolic PDEs II: Implementation of sparse tensor space-time discretizations. *Tech. Report*, page (in preparation), 2010.

[7] Fabio Nobile and Raul Tempone. Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients. *Int. J. Numer. Meth. Engg.*, 80(5):979–1006, 2009.

[8] Fabio Nobile, Raul Tempone, and Clayton G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2411–2442, 2008.

[9] Christoph Schwab and Rob Stevenson. Space-time adaptive wavelet methods for parabolic evolution problems. *Math. Comp.*, 78(267):1293–1318, 2009.

[10] H. Triebel. *Theory of Function Spaces, 2nd Ed.* Joh. Ambrosius Barth, Leipzig, Germany, 1995.

[11] J. Wloka. *Partial differential equations.* Cambridge University Press, Cambridge, 1987. Translated from the German by C. B. Thomas and M. J. Thomas.

Research Reports

No.	Authors/Title

10-11 *V.H. Hoang and C. Schwab*
Sparse tensor Galerkin discretizations for parametric and random parabolic PDEs. I: Analytic regularity and gpc-approximation

10-10 *V. Gradinaru, G.A. Hagedorn, A. Joye*
Exponentially accurate semiclassical tunneling wave functions in one dimension

10-09 *B. Pentenrieder and C. Schwab*
hp-FEM for second moments of elliptic PDEs with stochastic data.
Part 2: Exponential convergence

10-08 *B. Pentenrieder and C. Schwab*
hp-FEM for second moments of elliptic PDEs with stochastic data.
Part 1: Analytic regularity

10-07 *C. Jerez-Hanckes and J.-C. Nédélec*
Asymptotics for Helmholtz and Maxwell solutions in 3-D open waveguides

10-06 *C. Schwab and O. Reichmann*
Numerical analysis of additive, Lévy and Feller processes with applications to option pricing

10-05 *C. Schwab and R. Stevenson*
Fast evaluation of nonlinear functionals of tensor product wavelet expansions

10-04 *B.N. Khoromskij and C. Schwab*
Tensor-structured Galerkin approximation of parametric and stochastic elliptic PDEs

10-03 *A. Cohen, R. DeVore and C. Schwab*
Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs

10-02 *V. Gradinaru, G.A. Hagedorn, A. Joye*
Tunneling dynamics and spawning with adaptive semi-classical wave-packets

10-01 *N. Hilber, S. Kehtari, C. Schwab and C. Winter*
Wavelet finite element method for option pricing in highdimensional diffusion market models

09-41 *E. Kokiopoulou, D. Kressner, N. Paragios, P. Frossard*
Optimal image alignment with random projections of manifolds: algorithm and geometric analysis