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Abstract
We review the design and analysis of multiresolution (wavelet) methods for the numerical solution of the Kolmogoroff equations arising, among others, in financial engineering when Lévy and Feller or Additive processes are used to model the dynamics of the risky assets.
In particular, the Dirichlet and free boundary problems connected to barrier and American style contracts are specified and solution algorithms based on wavelet representations of the Feller Processes' Dirichlet Forms are presented. Feller Processes with generators that give rise to Sobolev spaces of variable differentiation order (corresponding to a state-dependent jump intensity) are considered. A copula construction for the systematic construction of parametric multivariate Feller-Lévy processes from univariate ones is presented and the domains of the generators of the resulting multivariate Feller-Lévy processes is identified. New multiresolution norm equivalences in such Sobolev spaces allow for wavelet compression of the matrix representations of the Dirichlet forms. Implementational aspects, in particular the regularization of the process' Dirichlet form and the singularity-free, fast numerical evaluation of moments of the Dirichlet form with respect to piecewise linear, continuous biorthogonal wavelet bases are addressed. Monte Carlo path simulation techniques for such processes by FFT and symbol localization are outlined. Numerical experiments illustrate multilevel preconditioning of the moment matrices for several exotic contracts as well as for Feller-Lévy processes with variable order jump intensities. Model sensitivity of Lévy models embedded into Feller classes is studied numerically for several types of plain vanilla, barrier and exotic contracts.

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1 Introduction

We consider a certain type of multidimensional normal Markov processes, so called Feller processes. This class of Feller processes includes as special cases Lévy processes, many local volatility and, in particular, the so-called affine models in finance as special cases. Due to their nonstationarity Feller processes can exhibit qualitative behaviour that is substantially different from that of Lévy processes such as state-space dependent jump activity. The nonstationarity of Feller processes also has substantial repercussions on their computational and analytical treatment: whereas for Lévy and the closely related affine models, Fast Fourier Transformation (FFT) algorithms from [21] (in modern, hardware optimized implementations, e.g. [35]) form the basis for fast and powerful option pricing algorithms, the nonstationarity of Feller processes implies that FFT based numerical methods are, in general, not applicable in the numerical solution of their Kolmogoroff equations (with the notable exception of, for example, affine processes proposed e.g. in [31] and, for Feller processes, in connection with approaches based on "freezing" their characteristic triplet [12]). From an analytical point of view, Feller processes are rather well understood. This is due to the fact that generators of Feller processes are pseudodifferential operators with symbols that admit a Lévy-Khintchine representation (e.g. [22, 47, 48] and the references there). Contrary to Lévy processes or diffusions with local volatility, domains of the infinitesimal generators for semigroups induced by Feller processes are, generally, *variable order Sobolev spaces*. Accordingly, the use of standard discretization schemes (based on Finite Differences or Finite Elements) for numerical solution of the Kolmogoroff equations associated to such Feller processes is not straightforward; the same applies to the *numerical analysis* of these discretization schemes, i.e. the mathematical analysis of stability, consistency and convergence of these schemes. One central theme of these notes is therefore to describe recent progress in the design and the numerical analysis of discretization schemes which allow a unified numerical treatment of the Kolmogoroff equations of Feller (and more general) processes. These schemes are based on *variational, multiresolution schemes* which use spline-wavelet bases of the domains of the processes' infinitesimal generators.

Feller processes arise as natural generalizations of Lévy processes [17] and are also useful for modeling bounded processes [7]. Feller processes appear as solutions of a large class of Lévy SDEs as shown in the recent work [76, 78]. Therefore, the high dimensional problem of pricing basket options under a Lévy market model can be reduced to a low dimensional problem driven by a Feller process. Besides, due to mimicking results by [37] for continuous semimartingales and a novel result by [6] for discontinuous semimartingales, pricing of European options under general non-Markovian processes can be reduced to a Markovian setting with processes that have deterministic, but time and state-space dependent coefficients. However, we will mainly focus on the case of time-independent coefficients in the following.

The theoretical literature on Feller processes is quite extensive. We refer to the monographs by N. Jacob [47]-[49] and [50], as well as references therein for an overview of properties of the generators and the corresponding semigroups. The Martingale Problem, i.e., the problem of existence of a Feller process with a given generator, has been treated by [41, 54, 73]. Path properties have been discussed in [72, 73, 74].
On the other hand, numerical methods capable of handling general Feller processes have received little attention until now. A technique for the approximation of sample paths of Feller processes is given in [12]. A fast calibration algorithm has been proposed for a special case in [16, 23]. Fourier methods for option pricing can only be efficiently used for regular affine processes, i.e., Feller processes with coefficients that depend affinely on the state variables [32, 31]. Finite Element based pricing methods for one dimensional European pricing problems under general Feller processes are proposed by [77], while [67] considers multidimensional pricing problems of European and American type under Lévy processes using FEM.

Multidimensional Feller processes can be constructed using Lévy copulas for the construction of the jump measures. We consider European and American type contracts, as well as the calculation of sensitivities and discuss well-posedness of the corresponding pricing Partial Integro-Differential equations (PIDEs) and inequalities in the multidimensional Feller setting. In addition, we discuss wavelet based discretization schemes, that allow for efficient preconditioning. Using waveltes, the arising densely populated matrices can be compressed leading to computational schemes with essentially Black- Scholes complexity. Efficient numerical quadrature rules for the evaluation of the wavelet coefficients of the jump measures of this survey are presented.

The outline is as follows. First, we give an overview of Feller processes and state sufficient conditions on the characteristic triple for the existence of a corresponding Feller process (Section 2). Next, we define the domains of the generators, which are Sobolev spaces of variable order, we employ pseudodifferential operator (PDO) theory to characterize the spaces (Section 3). Then we describe parametric constructions of multidimensional Feller processes using Lévy copulas and state sufficient conditions on the marginals and the copula function to obtain admissible market models. We show that the class of admissible market models contains many typical examples of Lévy market models and the extensions to Feller market models (Section 4). We discuss European and American pricing problems, as well as the calculation of sensitivities with respect to model parameters and solution arguments. The Gårding inequality and the sector condition is proven for the arising bilinear forms, which yields well-posedness of the corresponding pricing problems (Section 5). In Section 6 we introduce a tensor product wavelet basis and discuss norm equivalences on the variable order Sobolev spaces. The discretization of the arising non-local operators using this basis is presented. In Section 7 we address quadrature rules for the weakly singular integrals in the Galerkin discretization and briefly survey on Monte Carlo and Fourier methods in Section 8. We conclude with uni- and bivariate numerical examples from the pricing of derivative contracts.

Throughout, we shall write $C \lesssim D$ to denote that C can be bounded by a *constant multiple* of D with a constant that is independent of parameters which C and D may depend on. Then $C \gtrsim D$ is defined as $D \lesssim C$ and $C \simeq D$ is defined as $C \lesssim D$ and $D \lesssim C$.

2 Markov processes

Semimartingales are a well-investigated class of stochastic processes that is sufficiently rich to include most of the stochastic processes commonly employed in financial modelling while still being closed under various operations such as conditional expectations, stopping etc. Semimartingales can be well understood via their (generally stochastic) semimartingale characteristic, we refer to the standard reference [51] for details. Here, we restrict ourselves to a class of processes with deterministic, but generally time- and state-space dependent characteristic triples including Lévy processes, affine processes and many local volatility models. The time-homogeneous case will be analyzed in the first part of this section, while time-inhomogeneity will be briefly discussed in the second part.

2.1 Time-homogeneous processes

We consider a Markov process X and the corresponding family of operators $(T_{s,t})$ for $0 \le s \le t < \infty$ given by

$$(T_{s,t}(f))(x) = \mathbb{E}[f(X_t)|X(s) = x],$$

for each $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, where $B_b(\mathbb{R}^d)$ denotes the space of bounded Borel measurable functions on $\mathbb{R}^d$. For *normal* Markov processes, i.e Markov processes with $T_{s,t}(B_b(\mathbb{R}^d)) \subset B_b(\mathbb{R}^d)$, we recall the following properties:

(1) $T_{s,t}$ is a linear operator on $B_b(\mathbb{R}^d)$ for each $0 \le s \le t < \infty$.
(2) $T_{s,s} = I$ for each $s \ge 0$.
(3) $T_{r,s}T_{s,t} = T_{r,t}$ whenever $0 \le r \le s \le t < \infty$.
(4) $f \ge 0$ implies $T_{s,t}f \ge 0$ for all $0 \le s \le t < \infty$, $f \in B_b(\mathbb{R}^d)$.
(5) $\|T_{s,t}\| \le 1$ for each $0 \le s \le t < \infty$, i.e. $T_{s,t}$ is a contraction.
(6) $T_{s,t}(1) = 1$ for all $t \ge 0$.

If we restrict ourselves to time-homogeneous normal Markov processes, we obtain directly from the above properties that the family of operators $T_t := T_{0,t}$ form a positivity preserving contraction semigroup. The *infinitesimal generator* $\mathscr{A}$ with domain $\mathscr{D}(\mathscr{A})$ of such a process X with semigroup $(T_t)_{t\ge 0}$ is defined by the strong limit

$$\mathscr{A}u := \lim_{t\to 0^+} \frac{1}{t}(T_t u - u) \tag{2.1}$$

for all functions $u \in \mathscr{D}(\mathscr{A}) \subset B_b(\mathbb{R}^d)$ for which the limit (2.1) exists w.r. to the sup-norm. We call $(\mathscr{A}, \mathscr{D}(\mathscr{A}))$ *generator* of X. Generators of normal Markov processes admit the *positive maximum principle*, i.e.,

$$\text{if} \quad u \in \mathscr{D}(\mathscr{A}) \quad \text{and} \quad \sup_{x\in\mathbb{R}^d} u(x) = u(x_0) > 0, \quad \text{then} \quad (\mathscr{A}u)(x_0) \le 0. \tag{2.2}$$

Furthermore, they admit a pseudodifferential representation (e.g. [22, 47, 48]):

Theorem 2.1. *Let $(\mathscr{A}, \mathscr{D}(\mathscr{A}))$ be an operator with $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(\mathscr{A})$.*
Then $\mathscr{A}|_{C_0^\infty(\mathbb{R}^d)}$ is a pseudodifferential operator,

$$(\mathscr{A}u)(x) = -a(x,D)u(x) = -(2\pi)^{-1/2}\int_{\mathbb{R}^d} a(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi \tag{2.3}$$

for $u \in C_0^\infty(\mathbb{R}^d)$. With a symbol $a(x,\xi) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ which is locally bounded in (x,ξ), $a(\cdot,\xi)$ is measurable for every ξ and $a(x,\cdot)$ is a negative definite function for every x, which admits the Lévy-Khintchine representation

$$\begin{aligned} a(x,\xi) = {} & c(x) - i\gamma(x)\xi + \xi^\top Q(x)\xi \\ & + \int_{0\ne y\in\mathbb{R}^d}\left(1 - e^{iy\xi} + \frac{iy\xi}{1+y^2}\right)\mathtt{N}(x,dy). \end{aligned} \tag{2.4}$$

Here, for $y \in \mathbb{R}^d$, $y^2 = y^\top y$ and the function

$$\mathbb{R}^d \ni x \to \int_{y \neq 0} (1 \wedge y^2) \mathtt{N}(x, dy) \tag{2.5}$$

is continuous and bounded.

The parameters $c(x), \gamma(x), Q(x), \mathtt{N}(x, dy)$ in (2.4) are called *characteristics* of the Markov process X. In the following we set $c(x) = 0$ for notational convenience and restrict ourselves to a certain kind of normal Markov processes, so called Feller processes ([2, Theorem 3.1.8] states the normality of a Feller process). These can be defined by the semigroup $(T_t)_{t \geq 0}$ generated by the corresponding process X. A semigroup $(T_t)_{t \geq 0}$ is called Feller if it satisfies

(i)T_t maps $C_0(\mathbb{R}^d)$, the continuous functions on $\mathbb{R}^d$ vanishing at infinity, into itself:

$$T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d) \quad \text{boundedly}$$

(ii)T_t is strongly continuous, i.e., $\lim_{t \to 0^+} \|u - T_t u\|_{L^\infty(\mathbb{R}^d)} = 0$ for all $u \in C_0(\mathbb{R}^d)$.

Spatially homogeneous Feller processes are Lévy-processes (e.g.[8, 69]). Their characteristics, the *Lévy characteristics*, do not depend on x explicitly.

Example 2.2 *A standard Brownian motion has the characteristics $(0, 1, 0)$. An $\mathbb{R}$-valued Lévy process has characteristics $(\gamma, Q, \mathtt{N}(\mathrm{d}y))$, for real numbers γ, $Q \geq 0$ and a jump measure $\mathtt{N}$ with $\int_{0 \neq y \in \mathbb{R}} \min(1, y^2) \mathtt{N}(\mathrm{d}y) < \infty$.*

A recent result by Schnurr [76, 78] additionally motivates the consideration of Feller processes. He proved that strong solutions of a large class of Lévy SDEs are in fact Feller processes, i.e.

Theorem 2.3. *Let Z be an n-dimensional Lévy process and $\Phi : \mathbb{R}^d \to \mathbb{R}^{d \times n}$ be bounded and globally Lipschitz. Then the solution of*

$$X_t = x + \int_0^t \Phi(X_{s-}) \, \mathrm{d}Z_s,$$

$x \in \mathbb{R}^d$ is a Feller process with $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(\mathscr{A})$.

Proof. The proof is given in [78] Theorem 2.46, Theorem 2.49 and Theorem 2.50.

Remark 2.4. The boundedness of Φ can be replaced by certain assumptions on the tail behaviour of the process Z.

This implies that pricing of a large class of basket options in a Lévy market model can be reduced to pricing under a low dimensional Feller processes.
It is also interesting to ask which symbols correspond to PDOs that are generators of Feller processes. This martingale problem is discussed in the following theorem due to [75].

Theorem 2.5. *Let $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a negative definite function, i.e., a measurable and locally bounded function that admits a Lévy-Khinchine representation for all $x \in \mathbb{R}^d$; note that this implies continuity of the symbol in ξ for all $x \in \mathbb{R}^d$. If*

(a) $\sup_{x \in \mathbb{R}^d} |a(x, \xi)| \leq \kappa(1 + |\xi|^2)$ for all $\xi \in \mathbb{R}^d$,
(b) $\xi \mapsto a(x, \xi)$ is uniformly continuous at $\xi = 0$,
(c) $x \mapsto a(x, \xi)$ is continuous for all $\xi \in \mathbb{R}^d$,

then $(-a(x, D), C_0^\infty(\mathbb{R}^d))$ extends to a Feller generator.

In the Lévy case existence of a Lévy process can be proven for any Lévy symbol. This does not hold for Feller processes. For (financial) applications it is more convenient to consider the characteristic triple instead of the symbol. We therefore make the following assumption on the characteristic triple in the remainder.

Assumption 2.6 *The characteristic triple $(\gamma(x), Q(x), \mathtt{N}(x, \mathrm{d}y))$ of a Feller process in $\mathbb{R}^d$ satisfies the following conditions:*

(I) $(\gamma(x), Q(x), \mathtt{N}(x, \mathrm{d}y))$ is a Lévy triple for all fixed $x \in \mathbb{R}^d$.

(II) The mapping $x \mapsto \mathtt{N}(x,B)$ is continuous for all $B \in \mathscr{B}(\mathbb{R}^d)$.
(III) There exists a Lévy kernel $\overline{\mathtt{N}}(y)$ s.t.

$$\mathtt{N}(x,B) \leq \overline{\mathtt{N}}(B) \quad \forall x \in \mathbb{R}^d, \quad B \in \mathscr{B}(\mathbb{R}^d).$$

(IV) The functions $x \mapsto \gamma(x)$ and $x \mapsto Q(x)$ are continuous and bounded.

We would like to conclude that there exists a Feller process whose generator is a PDO for symbols that satisfies Assumption 2.6. Therefore, it suffices to validate the prerequisites of Theorem 2.5.

Lemma 2.7. *Let $(\gamma(x), Q(x), \mathtt{N}(x,dy))$ be the characteristic triple of a process X taking values in $\mathbb{R}^d$ that satisfies Assumption 2.6. Then $(-a(x,D), C_c^\infty)$ extends to a Feller generator, where $a(x,\xi)$ is given by*

$$a(x,\xi) = -i\gamma(x)\cdot\xi + \xi^\top Q(x)\xi + \int_{\mathbb{R}^d\setminus\{0\}} \left(1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+|y|^2}\right) \mathtt{N}(x,\mathrm{d}y). \tag{2.6}$$

Proof. Condition (I) implies that the corresponding Feller symbol is negative definite. Conditions (III) and (IV) imply (a), Conditions (II) and (III) imply (b), and (c) follows from (IV) and (II).

Remark 2.8. Note that real price market models do not fit into our modeling framework due to Assumption (a) in Theoren 2.5, but the numerical methods and their numerical analysis presented in the following can in many cases be straightforwardly extended to this kind of models. As illustrative example we consider the CEV market model (e.g. [29]) throughout this survey and explain the necessary extensions. The CEV model is given by the following SDE:

$$dS_t = rS_t dt + \sigma S_t^\rho dW_t, \quad S_0 = s \geq 0, \tag{2.7}$$

where $\rho \in (0,1)$, $\sigma > 0$ and $r \geq 0$. Existence of a solution of (2.7) follows from the Skorohod existence theorem [42, Theorem IV.2.2], while pathwise uniqueness can be obtained for $\rho \geq 0.5$ from the Yamada conditions (e.g. [42, Theorem IV.3.2]). Note that the case $\rho = 0.5$ leads to equations similar to the Heston model and CIR model.

In order to apply pseudodifferential operator theory we will need stronger assumptions on the characteristic triples of the considered processes. We will state the assumptions needed at the end of Section 4. We will require in particular smoothness of the characteristic triple in the state variable x. Numerical experiments indicate that these assumptions can be weakened (Chapter 9).

2.2 Time-inhomogeneous processes

In this section we would like to drop the assumption of time-homogeneity of the processes considered and extend the framework developed above to a time-dependent setting. Using the notation from the last section, we consider a normal Markov process X with the corresponding family of operators $T_{s,t}$. The family of generators of such a process is given by

$$A_s u := \lim_{h\to 0^+} \frac{1}{h}\left(T_{s-h,s}u - u\right) \tag{2.8}$$

for all functions $u \in D(A_s) \subset B_b(\mathbb{R}^d)$. In analogy to Theorem 2.1 we obtain the following result:

Theorem 2.9. *Let $(A_s, \mathscr{D}(A_s))_{s\in\mathbb{R}^+}$ be a family of operators with $C_0^\infty(\mathbb{R}^d) \subset \mathscr{D}(A_s)$. Then $A_s|_{C_0^\infty(\mathbb{R}^d)}$ is a pseudodifferential operator for all $s \in \mathbb{R}^+$ given by*

$$(A_s u)(x) = -a(s,x,D)u(x) = -(2\pi)^{-1/2}\int_{\mathbb{R}^d} a(s,x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi$$

for $u \in C_0^\infty(\mathbb{R}^d)$. With a symbol $a(s,x,\xi): \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ which is locally bounded in (x,ξ), $a(s,\cdot,\xi)$ is measurable for every ξ, s and $a(s,x,\cdot)$ is a negative definite function for every (s,x), which admits the Lévy-Khintchine representation

$$a(s,x,\xi) = c(s,x) - i\gamma(s,x)\cdot\xi + \xi^\top Q(s,x)\xi + \int_{0\neq y\in\mathbb{R}^d}\left(1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+y^2}\right)\mathtt{N}(s,x,dy).$$

The natural question arises if we can construct a Markov process with corresponding generator for a given symbol (Martingale Problem). A general result under mild regularity assumptions on the symbol has been given by [11].

Theorem 2.10. *Let $a:\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ be a negative definite function that satisfies the following conditions for a constant $m\in\mathbb{R}$*

(1) $a(\cdot,x,\xi)$ is a continuous function for all $x,\xi\in\mathbb{R}^d$,
(2) $a(s,x,0)=0$ holds for all $(s,x)\in\mathbb{R}^+\times\mathbb{R}^d$,
(3) $\left|D_x^\beta D_\xi^\alpha a(s,x,\xi)\right| \le c_{\alpha,\beta,J}(1+|\xi|^2)^{(m-|\alpha|\wedge 2)/2}$ holds for all $s\in J\subset\mathbb{R}^+$, $x,\xi\in\mathbb{R}^d$,
(4) a is elliptic, i.e., on any compact set K it holds uniformly in s that:

$$\begin{aligned}&\text{there exists } R\in\mathbb{R}^+,\, c>0, \text{ such that } \forall x\in\mathbb{R}^d,\\ &|\xi|\ge R : \Re(a(s,x,\xi)) \ge c(1+|\xi|^2)^{m/2}.\end{aligned}$$

Then a Markov process whose family of generators are pseudodifferential operators with symbol $a(s,x,\xi)$ can be constructed.

Proof. The proof follows from [11, Theorem 4.2, Corollary 4.3].

Theorem 2.10 can be formulated in a more general setting, replacing $|\xi|^2$ in Condition (3) and (4) by any element from a certain class of negative definite functions, cf. [11, Definition 1.2].

Remark 2.11. In the following we will consider the time-homogeneous case discussed in Section 2.1, but most results can be extended to the inhomogeneous setting.

3 Function spaces

For our analysis we will need certain Sobolev-type spaces. Therefore we start with the definition of fractional order isotropic spaces. We define for a positive non-integer $s \geq 0$ and $u \in \mathscr{S}^*(\mathbb{R}^d)$

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi, \tag{3.1}$$

where $\hat{u}$ is the Fourier transform of u. Similarly, we can define anisotropic Sobolev spaces $H^s(\mathbb{R}^d)$ with norm

$$\|u\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \sum_{j=1}^{d} (1+\xi_j^2)^{s_j} \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi, \tag{3.2}$$

for any multiindex $s \geq 0$. The consideration of certain symbol classes will be useful for the definition of the variable order Sobolev spaces. We set $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$ for notational convenience.

Definition 3.1 *Let $0 \leq \delta < \rho \leq 1$ and let $m(x) \in C^\infty(\mathbb{R}^d)$ be a real-valued function all of which derivatives are bounded on $\mathbb{R}^d$. The symbol $a(x,\xi)$ belongs to the class $S^{m(x)}_{\rho,\delta}$ of symbols of variable order $m(x)$ if $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $m(x) = s + \widetilde{m}(x)$ with $\widetilde{m} \in \mathscr{S}(\mathbb{R}^d)$ a tempered function, and if, for every $\alpha, \beta \in \mathbb{N}_0^d$ there is a constant $c_{\alpha,\beta}$ such that*

$$\forall x,\xi \in \mathbb{R}^d: \quad |D_x^\beta D_\xi^\alpha a(x,\xi)| \leq c_{\alpha,\beta} \langle \xi \rangle^{m(x)-\rho|\alpha|+\delta|\beta|}. \tag{3.3}$$

The variable order pseudodifferential operators $A(x,D) \in \Psi^{m(x)}_{\rho,\delta}$ correspond to symbols $a(x,\xi) \in S^{m(x)}_{\rho,\delta}$ by

$$A(x,D)u(x) := \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi, \qquad u \in C_0^\infty(\mathbb{R}^d). \tag{3.4}$$

We are now able to define an isotropic Sobolev space of variable order using the variable order Riesz potential $\Lambda^{m(x)}$ with symbol $a(x,\xi) = \langle \xi \rangle^{m(x)}$. Clearly $a(x,\xi)$ is an element of $S^{m(x)}_{1,\delta}$ for $\delta \in (0,1)$. The norm on $H^{m(x)}$ is given as

$$\|u\|^2_{H^{m(x)}} := \left\| \Lambda^{m(x)} u \right\|^2_{L^2} + \|u\|^2_{L^2}.$$

Note that for $a(x,\xi) = 1$, we obtain the usual L^2 norm, while for $a(x,\xi) = (1+|\xi|^s)$ we obtain the norm given in (3.1). The second results follows using Plancherel's theorem. Now we turn to the definition of anisotropic variable order Sobolev spaces. In analogy to Definition 3.1 we start with the definition of an appropriate symbol class.

Definition 3.2 *Let $\mathbf{m}(x) = s + \widetilde{\mathbf{m}}(x)$, $\widetilde{\mathbf{m}}(x): \mathbb{R}^d \to \mathbb{R}^d$ with each component of $\widetilde{\mathbf{m}}(x)$ being a tempered function and $s \in \mathbb{R}^d_{\geq 0}$, $0 \leq \delta < \rho \leq 1$. We define the symbol class $S^{\mathbf{m}(x)}_{\rho,\delta}$ as the set of all $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all multiindices $\alpha, \beta \in \mathbb{N}_0^d$ there exists a constant $C_{\alpha,\beta} > 0$ with*

$$\forall x,\xi \in \mathbb{R}^d: \quad \left| D_x^\beta D_\xi^\alpha a(x,\xi) \right| \leq C_{\alpha,\beta} \sum_{i=1}^{d} (1+\xi_i^2)^{(m_i(x)-\rho\alpha_i+\delta|\beta|)/2}.$$

We are now able to define an anisotropic Sobolev space of variable order using the variable order Riesz potential $\Lambda^{\mathbf{m}(x)}$ with symbol $a(x,\xi) = \langle \xi \rangle^{\mathbf{m}(x)} := \sum_{i=1}^n (1+\xi_i^2)^{\frac{1}{2}m_i(x)}$. Clearly, $a(x,\xi)$ is an element of $S^{\mathbf{m}(x)}_{1,\delta}$ for $\delta \in (0,1)$. The norm on $H^{\mathbf{m}(x)}$ is given by

$$\|u\|^2_{H^{\mathbf{m}(x)}} := \left\| \Lambda^{\mathbf{m}(x)} u \right\|^2_{L^2} + \|u\|^2_{L^2}.$$

There is an alternative representation of the above space, when $\mathbf{m}(x)$ is of the following form $\mathbf{m}(x) = (m_1(x_1), \ldots, m_d(x_d))$ which will be very useful for the proof of norm equivalences. This plays a crucial role in wavelet discretization theory. We consider the anisotropic Sobolev spaces $H_i^{m_i(x_i)}$ of variable order $m_i(x_i)$ in direction x_i, equipped with the following norms:

$$\|u\|^2_{H_i^{m(x)}} := \left\|\Lambda_i^{m_i(x_i)} u\right\|^2_{L^2} + \|u\|^2_{L^2},$$

where $\Lambda_i^{m_i(x_i)}$ is a pseudo-differential operator with symbol $(1+|\xi_i|)^{m_i(x_i)}$. It then follows by the elementary inequality

$$C_1 \left|\sum_{i=1}^d a_i\right|^2 \le \sum_{i=1}^d a_i^2 \le C_2 \left|\sum_{i=1}^d a_i\right|^2,$$

with $a_i > 0$ and C_1, C_2 only dependent on d, that

$$\|u\|^2_{H^{m(x)}} \sim \sum_{j=1}^d \|u\|^2_{H_j^{m_j(x_j)}},$$

and therefore

$$H^{m(x)} = \bigcap_{j=1}^d H_j^{m_j(x_j)}.$$

On the bounded set $D = (\mathbf{a},\mathbf{b}) = \prod_{i=1}^d (a_i, b_i) \subset \mathbb{R}^d$ we define for a variable order $\mathbf{m}(x)$, $\mathbf{a} \le x \le \mathbf{b}$ the space

$$\widetilde{H}^{\mathbf{m}(x)}(D) := \left\{ u|_D \;\middle|\; u \in H^{\mathbf{m}(x)}(\mathbb{R}^d), \quad u|_{\mathbb{R}^d \setminus \overline{D}} = 0 \right\}.$$

This space coincides with the closure of $C_0^\infty(D)$ (the space of smooth functions with support compactly contained in D) with respect to the norm

$$\|u\|_{\widetilde{H}^{\mathbf{m}(x)}(D)} := \|\widetilde{u}\|_{H^{\mathbf{m}(x)}(\mathbb{R}^d)}, \tag{3.5}$$

where $\widetilde{u}$ is the zero extension of u to all of $\mathbb{R}^d$.

Remark 3.3. In the BS case we will obtain $H^1(\mathbb{R}^d)$ as the domain of the generator and $H_0^1(D)$ in the localized case. In the Lévy case we obtain anisotropic Sobolev spaces as in (3.2) and the spaces $\widetilde{H}^{\mathbf{s}}(D)$ in the localized case for $Q = 0$. For $Q > 0$ the domains are equal to those in the BS case, cf. [67, Theorem 4.8].

4 Multivariate model setting

4.1 Copula functions

Unlike multivariate Lévy processes, not all multivariate Feller processes can be constructed in terms of univariate Feller processes using a homogeneous copula construction as in the case of Lévy processes in $\mathbb{R}^d$, cf. [52, Theorem 3.6]. However, parametric constructions of multidimensional Feller processes from the univariate margins of certain Feller processes and certain Lévy copulas are still possible, provided the univariate Feller processes and the copulas meet certain restrictions. The restrictions stem from the fact that smoothness conditions on the characteristic triple appear to be required in order to prove existence (and uniqueness) of a corresponding Feller process, cf. [75]. Therefore it would be sufficient for the parametric construction of d-dimensional Feller processes to prove that a symbol satisfies Assumption 2.6. We will only consider the construction of a d-dimensional jump measure, as the Gaussian part is standard.

Theorem 4.1. *Let F denote a d-dimensional Lévy copula for which the derivative $\partial_1 \ldots \partial_d F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ exists, is continuous and satisfies the following estimate*

$$|\partial^n F(u)| \lesssim C^{|n|}\, |n|! \min\{|u_1|, \ldots, |u_d|\} \prod_{i=1}^{d} |u_i|^{-n_i} \quad \forall u \in \mathbb{R}^d \quad n \in \mathbb{N}^d. \tag{4.1}$$

Further let $U_i(x,y)$, $i = 1, \ldots, d$ denote the tail integrals of real valued Feller processes that satisfy Assumption 2.6 and, additionally, the following conditions:

$$\left|\frac{k_i(x,y)}{U_i(x,y)}\right| \le (C \vee \frac{1}{|y|}) \quad \forall x,y \in \mathbb{R}, \tag{4.2}$$

$$\int_{\mathbb{R}\setminus B(0,1)} U_i(x,y)\, dy < \infty, \tag{4.3}$$

for $C > 0$ and $i = 1, \ldots, d$. Then there exists an $\mathbb{R}^d$-valued Feller process X whose components have tail integrals $U_1, \ldots, U_d$ and whose marginal tail integrals satisfy

$$U^I\left((x_i)_{i\in I}, (y_i)_{i\in I}\right) = F^I\left((U(x_i,y_i))_{i\in I}\right)$$

for any non-empty $I \subset \{1, \ldots, d\}$, any $(y_i)_{i\in I} \in (\mathbb{R}\setminus\{0\})^{|I|}$ and any $(x_i)_{i\in I} \in \mathbb{R}^{|I|}$. The jump measure is uniquely determined by F and U_i, $i = 1, \ldots, d$.

Proof. The proof follows [52]. As noted there, the argument is not restricted to Lévy models but can be extended to more general processes.
Since F is d-increasing and continuous, we can conclude that there exists a unique measure μ on $\overline{\mathbb{R}}^d \setminus \{\infty, \ldots, \infty\}$ such that $V_F((a,b]) = \mu((a,b])$ for any a,b with $a \le b$. For the univariate tail integrals $U(x,y)$, we define

$$U^{-1}(x,u) = \begin{cases} \inf\{y > 0 : u \ge U(x,y)\}, & u \ge 0 \\ \inf\{y < 0 : u \ge U(x,y)\} \wedge 0, & u < 0. \end{cases}$$

Let $\mathtt{N}' = f(\mu)$ be the image of μ under

$$f : (x, u_1, \ldots, u_d) \mapsto (U_1^{-1}(x_1,u_1), \ldots, U_d^{-1}(x_d,u_d))$$

and let $\mathtt{N}$ be the restriction of $\mathtt{N}'$ to $\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}$. We need to prove that $\mathtt{N}$ is a Lévy measure for all x and that the marginal tail integrals $U^I_{\mathtt{N}}$ satisfy

$$U^I_{\mathtt{N}}((x)_{i\in I}, (y_i)_{i\in I}) = F^I((U_i(x_i,y_i))_{i\in I}).$$

This will imply (I). Furthermore, we must prove continuity of the Lévy kernel in x (II) as well as boundedness in the sense of (III).

The first part follows analogously to [52]. We assume for ease of notation that $y_i > 0, i \in I$. Then

$$\begin{aligned}
U_{\mathtt{N}}^I((x_i)_{i\in I},(y_i)_{i\in I}) &= \mathtt{N}\left((x_i)_{i\in I}, \{\xi \in \mathbb{R}^d\backslash\{0\} : \xi_i \in (y_i,\infty), i\in I\}\right) \\
&= \mu\left(\{u \in \overline{R}^d : U_i^{-1}(x_i,u_i) \in (y_i,\infty), i \in I\}\right) \\
&= \mu\left(\{u \in \overline{\mathbb{R}}^d : 0 < u_i < U_i(x_i,y_i), i\in I\}\right) \\
&= \mu\left(\{u \in \overline{\mathbb{R}}^d : 0 < u_i \le U_i(x_i,y_i), i\in I\}\right) \\
&= F^I\left((U_i(x_i,y_i))_{i\in I}\right).
\end{aligned}$$

This proves in particular that the one-dimensional marginal tail integrals of $\mathtt{N}$ equal $U_1,\dots,U_d$. Since the margins of $\mathtt{N}(x,y)$ are Lévy measures on $\mathbb{R}\backslash\{0\}$ for all $x\in\mathbb{R}^d$ we obtain for every $x\in\mathbb{R}^d$:

$$\begin{aligned}
\int_{y\in\mathbb{R}^d}\left(|y|^2\wedge 1\right)\mathtt{N}(x,dy) &\le \int_{y\in\mathbb{R}^d}\sum_{i=1}^d\left(y_i^2\wedge 1\right)\mathtt{N}(x,dy) \\
&= \sum_{i=1}^d\int_{y_i\in\mathbb{R}}\left(y_i^2\wedge 1\right)\mathtt{N}_i(x_i,dy_i) < \infty.
\end{aligned}$$

Hence, for $x\in\mathbb{R}^d$, $\mathtt{N}(x,\cdot)$ is a Lévy measure on $\mathbb{R}^d$. For the second part of the proof we use Remark 2.7 in [67] which gives us:

$$k(x,y_1,\dots,y_d) = \partial_1\dots\partial_d F|_{\xi_1=U_1(x_1,y_1),\dots,\xi_d=U_d(x_d,y_d)}k_1(x_1,y_1)\dots k_d(x_d,y_d). \tag{4.4}$$

Using the properties of F and the margins we can conclude that $k(x,y_1,\dots,y_d)$ is continuous in x for all $(y_1,\dots,y_d)\in(\mathbb{R}\backslash\{0\})^d$. It remains to prove (III). Due to (4.1) we have the following estimate with $g := \partial_1\dots\partial_d F$:

$$\begin{aligned}
& k(x,y_1,\dots,y_d) && (4.5)\\
=\;& g(U_1(x_1,y_1),\dots,U_d(x_d,y_d))k_1(x_1,y_1)\dots k_d(x_d,y_d) \\
\le\;& C\min\{|U_1(x_1,y_1)|,\dots,|U_d(x_d,y_d)|\}\prod_{i=1}^d|U_i(x_i,y_i)|^{-1}\prod_{i=1}^d k_i(x_i,y_i) \\
\overset{(4.2)}{\le}\;& C\min\{|\overline{U}_1(y_1)|,\dots,|\overline{U}_d(y_d)|\}\prod_{i=1}^d\left(C\vee\frac{1}{|y_i|}\right). && (4.6)
\end{aligned}$$

Using the properties of the $\overline{\mathtt{N}}_i(dy)$ for $i=1,\dots,d$ we can conclude that (4.6) is a Lévy measure and therefore (IV) is valid for $k(x,y)$. Uniqueness of the jump measure follows from the fact that it is uniquely determined by its marginal tail integrals (cf. [52, Lemma 3.5]).

We can prove the following decay property of the jump density constructed according to the above theorem.

Lemma 4.2. *Let $k(x,y)$ be constructed according to Theorem 4.1. Besides we require the following estimate on the derivatives of $k_i(x,y)$: there exists $C>0$ s.t. $\forall x\in\mathbb{R}^d$, $y\in\mathbb{R}^d\backslash\{0\}$*

$$\left|\partial_x^n k_i(x,y)\right| \le C^{n+1}n!\,|y|^{-m_i(x)-\delta n-1}, \tag{4.7}$$

$$\left|\partial_y^n k_i(x,y)\right| \le C^{n+1}n!\,|y|^{-m_i(x)-n-1}, \tag{4.8}$$

for some $\delta\in(0,1)$ and $\max_{i=1,\dots,d}\sup_{x_i\in\mathbb{R}} m_i(x_i) = \overline{m} < 2$ as well as $\min_{i=1,\dots,d}\inf_{x_i\in\mathbb{R}} m_i(x_i) = \underline{m} > 0$. Then it holds

$$\left|\partial_x^m\partial_y^n k(x,y)\right| \le C^{|n|+1}\,|m|!\,|n|!\,\|y\|_\infty^{-\overline{m}}\prod_{i=1}^d|y_i|^{-n_i-\delta m_i-1},\quad \forall y_i\ne 0,$$

for multiindices $n,m\in\mathbb{N}_0^d$

Proof. Using the formula of Faà di Bruno [68] it can be shown

$$
\begin{aligned}
&\left|\partial_{x_i}^n\left(\partial_1 \ldots \partial_d F(U(x,y))\right)\right| \\
=&\left|\sum \frac{n!}{m_1!\ldots m_n!}(\partial_{x_i}^m \partial_1 \ldots \partial_d F)(U(x,y))\left(\frac{\partial_{x_i} U_i(x,y)}{1!}\right)^{m_1} \ldots \left(\frac{\partial_{x_i}^n U_i(x,y)}{n!}\right)^{m_n}\right| \\
&\leq \sum C_1^{n+1}\frac{n!m!}{m_1!\ldots m_n!}\|y\|_\infty^{-\alpha}\prod_j^d |y_j|^\alpha |y_i|^{\alpha m}|z_i|^{-\alpha m_1-\delta m_1}\ldots |y_i|^{-\alpha m_n-\delta n m_n} \\
&\leq C_2^{n+1} n!\,\|y\|_\infty^{-\alpha}|y_i|^{-\delta n}\prod_{j=1}^d |y_j|^\alpha ,
\end{aligned}
$$

where we sum over all multiindices $(m_1,\ldots,m_n)$, $m=\sum_i m_i$ with $n=\sum_{i=1}^n i m_i$. An analogous calculation leads to

$$
\left|\partial_{y_i}\left(\partial_1 \ldots \partial_d F(U(x,y))\right)\right| \leq C_2^{n+1} n!\,\|y\|_\infty^{-\alpha}|y_i|^{-n}\prod_{j=1}^d |y_j|^\alpha .
$$

Using the Leibniz rule we obtain

$$
\begin{aligned}
&\left|\partial_{x_i}^n k(x,y)\right| \\
&=\left|\partial_{x_i}^n\left(\partial_1 \ldots \partial_d F(U(x,y))k_1(x_1,y_1)\ldots k_d(x_d,y_d)\right)\right| \\
&=\left|\sum_{j=1}^n \frac{n!}{j!(n-j)!}\partial_{x_i}^j\left(\partial_1 \ldots \partial_d F(U(x,y))\right)\partial_{x_i}^{n-j}k_i(x_i,y_i)\prod_{m=1,m\neq d}^d k_m(x_m,y_m)\right| \\
&\leq C_3^{n+1} n!\sum_{j=1}^n \|y\|_\infty^{-\alpha}|y_i|^{-j\delta}\prod_{j=1}^d |y_j|^\alpha |y_i|^{-\alpha-1+\delta(-n+j)}\prod_{m=1,m\neq i}^d |y_m|^{-\alpha-1} \\
&\leq C_4^{n+1} n!\,\|y\|_\infty^{-\alpha}|y_i|^{-n\delta}\prod_{m=1}^d |y_m|^{-1} .
\end{aligned}
$$

It can be shown analogously for all $n\in\mathbb{N}, 0\neq y\in\mathbb{R}^d$:

$$
\left|\partial_{x_i}^n k(x,y)\right| \leq C_4^{n+1} n!\,\|y\|_\infty^{-\alpha}|y_i|^{-n}\prod_{m=1}^d |y_m|^{-1} , \tag{4.9}
$$

which completes the proof.

We will need these estimates later to prove exponential convergence of the numerical quadrature rules employed to approximate the discretized generator of the Feller process.

4.2 Sector condition

The sector condition for the symbol of the Feller process will be one of the main ingredients for proving well posedness of the initial boundary value problems for the PIDEs arising in option pricing problems. The sector condition reads:

$$
\exists \gamma > 0 \quad \text{s.t.}\ \forall x,\xi \in \mathbb{R}^d : \qquad \Re e\, a(x,\xi)+1 \geq \gamma \langle \xi \rangle^{\mathbf{m}(x)} . \tag{4.10}
$$

Verification of the sector condition is not straightforward for a general Feller process. Here, we give sufficient conditions for the sector condition to hold in terms of appropriate conditions on the marginals of the Feller process and copula function.

Definition 4.3 *The function $F:\overline{\mathbb{R}}^d\to\overline{\mathbb{R}}$ is a homogeneous Lévy copula of order 1. The functions $k_1^0,\ldots,k_d^0$ are jump measures of univariate Feller processes of order $-1-m_1,\ldots,-1-m_d$, i.e.,*

$$
k_j^0(x_j,ry_j) = r^{-1-m_j(x)}k_j^0(x_j,y_j), \quad \forall r>0 \textit{ and all } x_j\in\mathbb{R}, y_j\in\mathbb{R}\backslash\{0\}
$$

for any $j = 1,\ldots,d$ and F and $k_j^0(x_j,y_j)$, $j = 1,\ldots,d$, satisfy the assumptions of Theorem 4.1. Due to Theorem 4.1 there exists a unique Feller process with corresponding margins. We call such a d-variate Feller process $\mathbf{m}(x)$-stable, for $\mathbf{m}(x) = (m_1(x_1),\ldots,m_d(x_d))$.

For the pure jump case we will need the following additional property in order to prove a simple equivalence for the sector condition. We assume that the symmetric part of the jump measure $k^{\mathrm{sym}}(x,y) = \frac{1}{2}(k(x,y) + k(x,-y))$ admits the following estimate:

$$k^{\mathrm{sym}}(x,y) \gtrsim k^{0,\mathrm{sym}}(x,y), \quad \forall 0 < \|y\| < 1, \forall x \in \mathbb{R}^d, \tag{4.11}$$

where k^0 is the jump measure of an $\mathbf{m}(x)$-stable Feller process. We will now prove an anisotropic homogeneity property of the Feller density k^0.

Theorem 4.4. *Let the copula F and the marginal densities be as in Definition 4.3. Then the function k^0 given by (4.4) is* $\mathbf{m}$-homogeneous *in the sense that*

$$k^0\left(x, t^{-\frac{1}{m_1(x_1)}} y_1,\ldots,t^{-\frac{1}{m_d(x_d)}} y_n\right) = t^{1+\frac{1}{m_1(x_1)}+\cdots+\frac{1}{m_d(x_d)}} k^0(x,y_1,\ldots,y_n) \quad \forall t > 0.$$

Proof. The proof follows analogously to [34, Theorem 3.2].

Theorem 4.5. *Let $k^0(x,y_1,\ldots,y_d)$ be as in the previous theorem. Then the Feller symbol $(0,0,k^0(x,y_1,\ldots,y_n))$ is a real-valued anisotropic homogeneous function of type $\left(\frac{1}{m_1(x_1)},\ldots,\frac{1}{m_d(x_d)}\right)$ and order 1 for all $x \in \mathbb{R}^d$, i.e., it satisfies*

$$a(x, t^{\frac{1}{m_1(x_1)}} \xi_1,\ldots,t^{\frac{1}{m_d(x_d)}} \xi_n) = t a(x,\xi_1,\ldots,\xi_d) \quad \forall t > 0, \xi \in \mathbb{R}^d$$

Proof. The proof follows analogously to [34, Theorem 3.3] using Theorem 4.4.

We will need the following Lemma, which is a modification of [27, Lemma 2.2].

Lemma 4.6. *Let $\rho_1(x,y)$ and $\underline{\rho}_2(y) \le \rho_2(x,y) \le \overline{\rho}_2(y)$ be two anisotropic distance functions of order 1 and type $\mathbf{m}(x) = (m_1(x),\ldots,m_d(x))$ for all $x \in \mathbb{R}^d$, and let $\underline{\rho}_2(y), \overline{\rho}_2(y)$ be continuous. Furthermore, let $\Sigma := \cup_{x\in\mathbb{R}^d} \Sigma_1(x)$, where*

$$\Sigma_1(x) := \{z : \rho_1(x,z) = 1\},$$

is contained in a compact set. Then the following inequalities hold with constants $C_1, C_2 > 0$ independent of x and y:

$$C_1 \rho_1(x,y) \le \rho_2(x,y) \le C_2 \rho_1(x,y).$$

Proof. Let $y \in \mathbb{R}^d$. We set $t(x) = \frac{1}{\rho_1(x,y)}$. Then

$$\left(t(x)^{m_1(x)} y_1,\ldots,t(x)^{m_d(x)} y_d\right) \in \Sigma_1(x)$$

holds. As Σ is contained in compact set and $\overline{\rho}_2(y), \underline{\rho}_2(y)$ are continuous, we obtain

$$C_1 \le \rho_2(x,y) \le C_2 \qquad \forall x \in \mathbb{R}^d, \forall y \in \Sigma_1.$$

Hence,

$$C_1 \le \frac{1}{\rho_1(x,y)} \rho_2(x,y) = t(x)\rho_2(x,y) = \rho_2(x, t(x)^{m_1(x)} y_1,\ldots,t(x)^{m_d(x)} y_d) \le C_2.$$

Theorem 4.7. *Let X be a Feller process taking values in $\mathbb{R}^d$ with characteristic triple $(\gamma(x), Q(x), k(x,y)\mathrm{d}y)$ with density $k(x,y)$ of the jump-measure constructed parametrically as in Theorem 4.1. Further assume that either $Q > 0$ holds or that $k(x,y)$ satisfies (4.11) with an $\mathbf{m}(x)$-stable function $k^0(x,y)$. Then, there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^d$ and $\|\xi\|_\infty$ sufficiently large*

$$|\Re a(x,\xi)| \geq C \sum_{j=1}^{d} |\xi|^{m_j(x_j)}, \tag{4.12}$$

where $m_j(x_j) = 2$ *in the case* $Q > 0$.

Proof. The proof mainly follows the arguments of [85, Proposition 2.4.3]. First consider $Q = 0$. Due to Theorem 4.5 one obtains that $\Re a^0(x,\xi)$ is an anisotropic distance function of order $\left(\frac{1}{m_1(x_1)}, \dots, \frac{1}{m_d(x_d)}\right)$ for all $x \in \mathbb{R}^d$. Since all anisotropic distance functions of the same order are equivalent there exists some $C_1(x)$ such that

$$a^0(x,\xi) \geq C_1(x) \sum_{i=1}^{d} |\xi_i|^{m_i(x_i)}, \quad \forall \xi \in \mathbb{R}^d.$$

Hence,

$$\begin{aligned} |\Re a(x,\xi)| &= \int_{\mathbb{R}^n} (1-\cos(\xi \cdot y))\, k^{\mathrm{sym}}(x,y)\mathrm{d}y \\ &\geq C_2 \int_{B_1(0)} (1-\cos(\xi \cdot y)) k^{0,\mathrm{sym}}(x,y)\mathrm{d}y \\ &\geq C_2 C_1(x) \sum_{i=1}^{d} |\xi_i|^{m_i(x_i)} - C_3. \end{aligned}$$

To complete the proof, we must prove the boundedness of $C_1(x)$, i.e., we have to validate the conditions of Lemma 4.6. The compactness of Σ_1 follows from the definition of $\rho_1(x,y) = \sum_{i=1}^{d} |y_i|^{m_i(x)}$, and the estimates on $\rho_2(x,y)$ follow from the conditions imposed on k^0. Therefore, the sector condition (4.10) follows from (4.11) for a certain set of symbols. The case $Q > 0$ is trivial.

Assumption (4.11) is implied by the following conditions on the marginal jump measures and the copula function:

Assumption 4.8 *Let X be a Feller process with characteristic triple* $(\gamma, Q(x), k(x,y)\mathrm{d}y)$ *satisfying the conditions of Theorem 4.1. Let the following inequalities hold, with* F^0 *being a* 1*-homogeneous Lévy copula as in Assumption 4.3 and* $k_i^0(x,y)$ *being* $\mathbf{m}(x)$*-stable densities with tail integrals* $U_i^0(x,y)$, $i = 1,\dots,d$*:*

$$\begin{aligned} k_i(x,y) &\gtrsim k_i^0(x,y), \quad \forall 0 < |y| < 1, \forall x \in \mathbb{R}^d, \quad i = 1,\dots,d \\ \partial_1 \dots \partial_d F(U(x,y)) &\gtrsim \partial_1 \dots \partial_d F^0(U^0(x,y)) \quad \forall 0 < |y| < 1. \end{aligned}$$

4.3 A class of admissible market models

We now formulate the requirements for market models which will be admissible for our pricing schemes in terms of the marginals and the copula function. These requirements will not only ensure existence and uniqueness of a solution of the corresponding pricing problem, but also ensure that the presented FEM based algorithms are feasible.

Definition 4.9. *We call a* d*-dimensional Feller process with characteristic triple* $(\gamma(x), Q(x), \mathtt{N}(x,dy))$ *an* admissible market model *if it satisfies the following properties.*

1. *The function* $x \mapsto \gamma(x) \in \mathbb{R}^d$ *is smooth and bounded.*
2. *The function* $x \mapsto Q(x) \in \mathbb{R}^{d\times d}_{sym}$ *is smooth and bounded and* $Q(x)$ *is positive semidefinite for all* $x \in \mathbb{R}^d$.
3. *The jump measure* $\mathtt{N}(x,dy)$ *is constructed form* d *independent, univariate Feller-Lévy measures with a* 1*-homogeneous copula function* F *that fulfills the following estimate: there is a constant* $C > 0$ *such that for all* $u \in (\mathbb{R}\backslash\{0\})^d$ *and all* $n \in \mathbb{N}_0^d$ *holds*

$$|\partial^n F(u)| \leq C^{|n|+1}\, |n|! \min\{|u_1|,\dots,|u_d|\} \prod_{i=1}^{d} |u_i|^{-n_i}$$

4. *For the marginal densities* $\mathtt{N}_i(x_i,dy_i) = k_i(x,y)dy$ *the mapping* $x_i \mapsto \mathtt{N}_i(x_i,B)$ *is smooth for all* $B \in \mathscr{B}(\mathbb{R})$.
5. *There exist univariate Lévy kernels* $\overline{k}_i(y)$, $i = 1,...,d$ *with semiheavy tails, i.e., which satisfy*

$$\overline{k}_i(y) \leq C \begin{cases} e^{-\beta^-|y|}, & y < -1 \\ e^{-\beta^+ y}, & y > 1, \end{cases} \tag{4.13}$$

for some constants $C > 0$, $\beta^- > 0$ *and* $\beta^+ > 1$. *These Lévy kernels satisfy the following estimates*

$$\mathtt{N}_i(x_i,B) \leq \int_B \overline{k}_i(y)\,dy \quad \forall x_i \in \mathbb{R}, \quad B \in \mathscr{B}(\mathbb{R}),\ i = 1,\ldots,d.$$

6. *Besides, we require the following estimate on the derivatives of* $k_i(x,y)$

$$\begin{aligned} \left|\partial_x^n k_i(x,y)\right| &\leq C^{n+1} n!\, |y|^{-m_i(x)-\delta n-1}, \\ \left|\partial_y^n k_i(x,y)\right| &\leq C^{n+1} n!\, |y|^{-m_i(x)-n-1}, \end{aligned}$$

for any $\delta \in (0,1)$, *for all* $0 \neq y,x \in \mathbb{R}^d$ *and* $\overline{m} := \max_{i=1,...,d} \sup_{x_i \in \mathbb{R}} m_i(x_i) < 2$ *as well as* $\underline{m} := \min_{i=1,...,d} \inf_{x_i \in \mathbb{R}} m_i(x_i) > 0$.
7. *Finally we require* F^0 *to be a* 1*-homogeneous Lévy copula and* $k_i^0(x_i,y_i)$ *to be* $m_i(x_i)$*-stable densities with tail integrals* $U_i^0(x_i,y_i)$, $i = 1,\ldots,d$:

$$\begin{aligned} k_i(x,y) &\gtrsim k_i^0(x,y), \quad \forall 0 < |y| < 1, \forall x \in \mathbb{R}, \quad i = 1,\ldots,d \\ \partial_1 \ldots \partial_n F(U(x,y)) &\gtrsim \partial_1 \ldots \partial_n F^0(U^0(x,y)) \quad \forall 0 < |y| < 1. \end{aligned}$$

Remark 4.10. An admissible market model satisfies the requirements of Theorem 4.1 due to conditions (3), (5) and (6).

Lemma 4.11. *The symbol* $a(x,\xi)$ *of an admissible market model* X *with triple* $(0,Q(x),k(x,y)dy)$ *is contained in the symbol class* $S_{1,\delta}^{\mathbf{m}(x)}$ *for any* $\delta \in (0,1)$, *where* $\mathbf{m}(x) = \mathbf{2}$ *if* $Q(x) \geq Q_0 > 0$.

Prior to the proof of the preceding lemma we remark that the removal of the drift will be discussed in the next chapter. The proof follows analogously to [67, Proposition 3.5]. Let us illustrate the preceding, abstract developments with an example related to the so-called *tempered-stable* class of Lévy processes which were advocated in recent years in the context of financial modelling.

Example 4.12 (Feller-CGMY). We consider a d-dimensional Feller process with Clayton Lévy copula

$$F(u_1,\ldots,u_d) = 2^{2-d} \left(\sum_{i=1}^d |u_i|^{\vartheta} \right)^{-\frac{1}{\vartheta}} \left(\rho \mathbb{1}_{\{u_1,\ldots,u_d \geq 0\}} - (1-\rho) \mathbb{1}_{\{u_1,\ldots,u_d \leq 0\}} \right),$$

where $\vartheta > 0$, $\rho \in [0,1]$ together with CGMY-type densities

$$k_i(x,y) = C(x) \left(\frac{e^{-\beta_i^-(x)|y|}}{|y|^{1+m_i(x)}} \mathbb{1}_{\{y<0\}} + \frac{e^{-\beta_i^+(x)|y|}}{|y|^{1+m_i(x)}} \mathbb{1}_{\{y>0\}} \right),$$

with smooth and bounded functions $C(x) > 0$, $\beta_i^-(x) > 0$, $\beta_i^+(x) > 1$, $0 < \underline{m}_i < m_i(x) \leq \overline{m}_i < 2$, for $i = 1,\ldots,d$. We assume the Gaussian component $Q(x)$ to be positive semidefinite, smooth and bounded. The drift $\gamma(x)$ is assumed to be smooth and bounded. It is easy to see that this market model satisfies Properties (1), (2), (4)-(6) of the above definition. (3) and (7) follow analogously to the proof of [85, Proposition 2.3.7].

5 Variational PIDE formulations

A key observation at the heart of differential equation approaches to deterministic computational pricing of derivative contracts in finance is the observation (going back at least to R. Feynman and M. Kac) that conditional expectations over all sample paths of a multivariate diffusion process satisfy deterministic, parabolic partial differential equations (PDEs). The most well known representative of these PDEs in financial modelling is the classical Black-Scholes equation. This Feynman Kac correspondence holds in a much more general context, the deterministic equation being in general nonlinear, and the solution being in general understood as *viscosity solution*. Here, we will follow on linear differential equations for which the (unique) solutions will be *variational solutions* of suitable *weak formulations* of the deterministic evolution equations. As these formulations will form the basis of variational discretizations to be discussed below, we shall present their ingredients (Sobolev and Besov spaces, Dirichlet Forms, Evolution triplets and the abstract theory of parabolic evolution equations) in some detail here. Throughout, our perspective is the pricing of derivative contracts in financial models based on the Feller-Lévy processes introduced above.

5.1 European options

We consider a European option with maturity $T < \infty$ and payoff $g(S_T)$ which is assumed to be Lipschitz, where $S^i_t = S^i_0 e^{rt+X^i_t}$ and where X is a semimartingale unless specified otherwise. By the general theory of asset pricing (as, e.g., in [28]), an arbitrage free value $V(t,s)$ of this option is given by

$$V(t,s) = \mathbb{E}\left(e^{-r(T-t)} g(S_T) | S_t = s\right),$$

where the expectation is taken under the measure $\mathbb{Q}$ which is equivalent to the real world measure and under which S_T is a sigma-martingale, cf.[28]. If X is an admissible market model, we can derive a PDO and PIDE representation and prove well-posedness of the weak formulation of the problem on a bounded domain. If X is a (multidimensional) continuous semimartingale, satisfying certain non-degeneracy conditions, we can use a mimicking result due to [37] and derive a state-space and time inhomogeneous PDE. Existence and uniqueness on bounded and unbounded domains are well-known under certain smoothness and growth conditions on the coefficients. If X is a (multidimensional) discontinuous semimartingale satisfying certain non-degeneracy conditions, we can use a novel mimicking result due to [6] and derive a state-space and time dependent PDO and PIDE. Existence and uniqueness results are not yet available in this case. In the following we will concentrate on the time homogeneous Feller case.
Due to no arbitrage considerations we will require the considered processes to be martingales under a pricing measure $\mathbb{Q}$. This requirement can be expressed in terms of the characteristic triple:

Lemma 5.1. *Let X be constructed according to Theorem 4.1 with characteristic triple $(\gamma(x), Q(x), \mathtt{N}(x,dy))$. Then e^{X_j} is a $\mathbb{Q}$-martingale with respect to the canonical filtration of X if and only if*

$$\frac{Q_{jj}(x)^2}{2} + \gamma_j(x) + \int_{0\neq y\in\mathbb{R}} (e^{y_j} - 1 - y_j)\mathtt{N}_j(x,\mathrm{d}y) = 0 \quad \forall x \in \mathbb{R}. \tag{5.1}$$

Proof. For the proof of this lemma we cannot use standard arguments as e.g. in [67, Lemma 2.1], since independence of the increments is required in that proof. We consider the characteristic function of the random variable $X_t - x$ as in [46], i.e.,

$$\lambda_t(x,\xi) = E^x(e^{i(X_t-x)\cdot\xi}).$$

The martingale condition implies

$$0 = \lambda_t(x, -ie_j) - \lambda_0(x, -ie_j) \quad \forall x \in \mathbb{R}^d \text{ and } t \in \mathbb{R}_+.$$

The right-hand side can be written as

$$\begin{aligned}\lambda_t(x,-ie_j)-\lambda_0(x,-ie_j) &= t\int_0^1 \frac{d}{ds}\lambda_s(x,-ie_j)|_{s=\eta t}\,\mathrm{d}\eta\\ &= t\int_0^1 \frac{d}{ds}\left(e^{-x_j}T_s(e^{(\cdot)})(x\cdot e_j)|_{s=\eta t}\right)\mathrm{d}\eta\\ &= -t\int_0^1 e^{-x_j}T_{t\eta}(e^{(\cdot)}a(\cdot,-ie_j))(x)\,\mathrm{d}\eta,\end{aligned}$$

where we have used $(d/ds)T_s u = T_s\mathscr{A}u$ and $\mathscr{A}(e^{i(\cdot,\xi)}) = -e^{ix\xi}a(x,\xi)$. This implies $a(x,-ie_j)=0$ and the claimed result can be obtained using the analogon to [69, Proposition 11.10] for Feller processes. The reverse implication follows analogously.

We are now able to derive a PDO and PIDE for option prices. Let the stochastic process X be an admissible markovian market model and let be $g\in\mathscr{V}:=D(\mathscr{A}_X)$. Then we obtain due to semigroup theory for $u(t,x)=T_t(g)=\mathbb{E}[g(X_t)|X_0=x]$, where we have set $t_0=0$ and $r=0$ for notational convenience by differentiation in t:

$$\begin{aligned}\partial_t u-\mathscr{A}_X u &= 0 \quad \text{in } (0,T)\times\mathbb{R}^d &(5.2)\\ u_0 &= g \quad \text{in } \{0\}\times\mathbb{R}^d. &(5.3)\end{aligned}$$

Testing with a function $v\in\mathscr{V}$, we end up with the following parabolic evolution problem: Find $u\in L^2((0,T);\mathscr{V})\cap H^1((0,T);\mathscr{V}^*)$ s.t. for all $v\in\mathscr{V}$ and a.e. $t\in[0,T]$ holds

$$(\partial_t u,v)-a(u,v)=0, \qquad u_0=g, \tag{5.4}$$

where *the bilinear form $a(u,v)=(\mathscr{A}_X u,v)$ corresponds to the Dirichlet form of the stochastic process X*. Although in option pricing, only the homogeneous parabolic problem (5.4) arises, the inhomogeneous equation (5.5) is useful in many applications. We mention only the computation of the time-value of an option, or the computation of quadratic hedging strategies and the corresponding hedging error. Thus, we will in general consider the nonhomogeneuos analogon of the above equation. The general problem reads: Find $u\in L^2((0,T);\mathscr{V})\cap H^1((0,T);\mathscr{V}^*)$ s.t.

$$\begin{aligned}(\partial_t u,v)-a(u,v) &= (f,v)_{\mathscr{V}^*\times\mathscr{V}} \quad \text{in } (0,T), \forall v\in\mathscr{V} &(5.5)\\ u_0 &= g\end{aligned}$$

for some $f\in L^2((0,T);\mathscr{V}^*)$. Now we consider the localization of the unbounded problem to a bounded domain D. To be able to control the error introduced by localization we need to require the following growth condition on the payoff function: There exists some $q\geq 1$ such that

$$g(s)\lesssim\left(\sum_{i=1}^d s_i+1\right)^q, \quad \forall s\in\mathbb{R}^d_{\geq 0}. \tag{5.6}$$

For any function u with support in a bounded domain $D\subset\mathbb{R}^d$ we denote by $\widetilde{u}$ the zero extension to $\mathbb{R}^d$ and define $\mathscr{A}_D(u)=\mathscr{A}(\widetilde{u})$ with domain $\mathscr{V}_D$. The variational formulation of the pricing equation on a bounded domain $D\subset\mathbb{R}^d$ reads: Find $u\in L^2((0,T);\mathscr{V}_D)\cap H^1((0,T);(\mathscr{V}_D)^*)$ s.t. for all $v\in\mathscr{V}_D$ and a.e. $t\in[0,T]$ holds:

$$\begin{aligned}(\partial_t u,v)-a_D(u,v) &= (f,v)_{\mathscr{V}_D^*\times\mathscr{V}_D} &(5.7)\\ u_0 &= g|_D, &(5.8)\end{aligned}$$

where $a_D(u,v):=a(\widetilde{u},\widetilde{v})$ and $\widetilde{u},\widetilde{v}$ denote the zero extensions of u and v to $D^c=\mathbb{R}^d\backslash\overline{D}$. Note that the spaces $\mathscr{V}_D=:\{v\in L^2(D):\widetilde{v}\in\mathscr{V}\}$ consist of functions which vanish in a weak sense on ∂D. Under condition (5.6) pointwise convergence of the solution of the localized problem to the solution of the original problem can be shown for Lévy processes using [69, Theorem 25.18] and the semiheavy tail property. We refer to [67, Theorem 4.14] for details. A comparable result for general Feller processes does not appear to be available yet.

Remark 5.2. This formulation naturally arises for payoffs with finite support such as digital or (double) barrier options. The truncation to a bounded domain can thus be interpreted economically as the approximation of a standard derivative contract by a corresponding barrier option on the same market model. Note also that the variational framework (5.7)-(5.8) naturally allows for more general initial conditions, in particular $g \in \mathscr{H} = L^2(\mathbb{R}^d)$. Therefore, discontinuous g are admissible in the variational framework (5.7)-(5.8). This is essential for the pricing of exotic contracts such as digital options, for example.

Existence and uniqueness of weak solutions of (5.7)-(5.8) follows analogously to [77, Section 6.2], so we obtain the following theorem:

Theorem 5.3. *Let the generator $\mathscr{A}(x,D) \in \Psi^{2\mathbf{m}(x)}_{\rho,\delta}$ with $\mathbf{m}(x) = (m_1(x_1),\dots,m_d(x_d))$ be a pseudo-differential operator of variable order $2\mathbf{m}(x)$, $0 < m_i(x_i) < 1$, $i = 1,\dots,d$ with symbol $a(x,\xi) \in S^{2\mathbf{m}(x)}_{\rho,\delta}$ for some $0 < \delta < \rho \le 1$ for which there exists $\gamma > 0$ with*

$$\forall x,\xi \in \mathbb{R}^n : \qquad \Re e\, a(x,\xi) + 1 \ge \gamma \langle \xi \rangle^{2\mathbf{m}(x)}. \tag{5.9}$$

Then $\mathscr{A}(x,D) \in \Psi^{2\mathbf{m}(x)}_{\rho,\delta}$ satisfies a Gårding inequality in the variable order space $\widetilde{H}^{\mathbf{m}(x)}(D)$: There are constants $\gamma > 0$ and $C \ge 0$ such that

$$\forall u \in \widetilde{H}^{\mathbf{m}(x)}(D): \quad \Re a(u,u) \ge \gamma \|u\|^2_{\widetilde{H}^{\mathbf{m}(x)}(D)} - C\|u\|^2_{L^2(D)}, \tag{5.10}$$

and

$$\exists \lambda > 0 \quad \text{such that } \mathscr{A}(x,D) + \lambda I : \widetilde{H}^{\mathbf{m}(x)}(D) \to H^{-\mathbf{m}(x)}(D) \tag{5.11}$$

is boundedly invertible.

As noted above, we obtain $\tilde{H}^1(D) = H^1_0(D)$ as the domain of the operator if $Q(x) \ge Q_0 > 0$, i.e. if the diffusion matrix $Q(x)$ is uniformly positive definite.

Theorem 5.4. *The problem (5.7)-(5.8) for an admissible market model X with initial condition $g_D \in \mathscr{H}$ has a unique solution.*

Proof. Using the Gårding inequality (5.10) and the continuity of the operator the result follows from standard theory of parabolic evolution equations.

Remark 5.5. We obtain for finite variation pure jump models, i.e., $m(x) < 0.5$, for all $x \in D$, an advection dominated equation. Therefore we have to remove the drift for standard algorithms to be feasible. This is easy in the Lévy case as the drift coefficients in the equation are constant, cf. [67, Corollary 4.3], but more involved in the Feller case, cf. [39].

Theorem 5.6. *For $f \in C^{1,2}(J\times\mathbb{R})$, with $\partial_x f(t,x) \ne 0$, consider the change of variable $v(t,x) := u(t,f(t,x))$, where $u(t,x)$ is the solution of the following PDE*

$$\partial_t u - \Sigma(x)\partial_{xx}u + b(x)\partial_x u + c(x)u = 0 \quad in \quad J\times\mathbb{R}.$$

Let f solve the (nonlinear) PDE

$$\partial_t f - \Sigma(f(t,x))\frac{\partial_{xx} f}{\partial_x^2 f} - b(f(t,x)) = 0. \tag{5.12}$$

Then v satisfies the PDE

$$\partial_t v - \frac{\Sigma(f(t,x))}{\partial_x^2 f}\partial_{xx}v + c(f(t,x))v = 0 \quad in \quad J\times\mathbb{R}.$$

Solving the PDE (5.12) is non trivial in general.

Remark 5.7. We come back to the CEV model introduced in Remark 2.8. The generator $\mathscr{A}^{\text{CEV}}$ is given as

$$\mathscr{A}^{\text{CEV}}u(x) = \frac{1}{2}\sigma^2 x^{2\rho}\,\partial_{xx}u(x) + rx\partial_x u(x)$$

and the corresponding bilinear form a_{CEV} on a bounded domain G reads:

$$a_{\mathrm{CEV}}(u,v) = \frac{1}{2}\sigma^2 \int_G x^{2\rho}\,\partial_x u \partial_x v\,dx + \int_G (\rho\sigma^2 - rx)\partial_x u v\,dx + \int_G uv\,dx.$$

The domain of the generator is the weighted Sobolev space H_ρ, defined as the $H_\rho := \overline{C_0^\infty}(G)^{\|\cdot\|_\rho}$, where

$$\|u\|_\rho^2 = \int_G \left(x^{2\rho}\,|\partial_x u|^2 + |u|^2\right) dx.$$

A Gårding inequality and continuity of the bilinear form a_{CEV} can be shown on H_ρ.

5.2 American options

For the study of optimal stopping problems which arise e.g. from American contracts we require variational formulations of parabolic variational *inequalities*. To this end, let $\emptyset \neq \mathscr{K} \subset \mathscr{V}$ be a closed, non-empty and convex subset of $\mathscr{V}$ with *indicator function*

$$\phi(v) := I_{\mathscr{K}}(v) = \begin{cases} 0, & \text{if } v \in \mathscr{K}, \\ +\infty, & \text{else.} \end{cases} \tag{5.13}$$

This is a proper, convex lower semicontinuous (l.s.c.) function $\phi : \mathscr{V} \to \overline{\mathbb{R}}$ with domain $\mathscr{D}(\phi) = \{v \in \mathscr{V} : \phi(v) < \infty\}$. We denote by $\overline{\mathscr{K}}^{\|\circ\|_{\mathscr{H}}}$ the closure of $\mathscr{D}(\phi)$ in $\mathscr{H}$ and consider the following variational problem: Given $f \in L^2(0,T;\mathscr{V}^*)$, $u_0 \in \overline{\mathscr{K}}^{\|\circ\|_{\mathscr{H}}} \subset \mathscr{H}$,

$$\begin{aligned} &\text{find } u \in L^2(0,T;\mathscr{V}) \cap H^1(0,T;\mathscr{V}^*) \text{ such that } u \in \mathscr{D}(\phi) \text{ a.e. in } (0,T) \text{ and} \\ &\langle \partial_t u + \mathscr{A}u - f, u - v\rangle_{\mathscr{V}^*\times\mathscr{V}} + \phi(u) - \phi(v) \geq 0 \quad \forall v \in \mathscr{D}(\phi), \text{a.e. in } (0,T), \quad (5.14) \\ &\qquad u(0) = u_0 \quad \text{in} \quad \mathscr{H}. \quad (5.15) \end{aligned}$$

Existence and uniqueness results for solutions $u \in L^2(0,T;\mathscr{V})$ of (5.14)–(5.15) can be obtained from e.g. [36, Theorem 6.2.1] under rather strict conditions on the data f. To derive the well-posedness of (5.14)–(5.15) under minimal regularity conditions on f, u_0 and ϕ, the problem needs to be replaced by a *weak variational formulation*. To state it, we introduce the integral functional Φ on $L^2(0,T;\mathscr{V})$

$$\Phi(v) = \begin{cases} \displaystyle\int_0^T \phi(v(t))e^{-2\lambda t}dt, & \text{if } \phi(v) \in L^1(0,T), \\ +\infty, & \text{else,} \end{cases} \tag{5.16}$$

with $\lambda \geq 0$ as in (5.10).

Note that $\Phi(\cdot)$ is proper convex and l.s.c. with domain

$$\mathscr{D}(\Phi) = \{v \in L^2(0,T;\mathscr{V}) : \phi(v) \in L^1(0,T)\}. \tag{5.17}$$

Herewith, the weak variational formulation of (5.14)–(5.15) reads (cf. [3, 70]): Given $u_0 \in \overline{\mathscr{K}}^{\|\circ\|_{\mathscr{H}}} \subset \mathscr{H}$ and $f \in L^2(0,T;\mathscr{V}^*)$,

$$\begin{aligned} &\text{find } u \in L^\infty(0,T;\mathscr{H}) \cap \mathscr{D}(\Phi) \text{ such that } u(0) = u_0 \text{ in } \mathscr{H} \text{ and} \\ &\int_0^T \langle \partial_t v(t) + (\mathscr{A}+\lambda)u(t) - (f+\lambda v), u(t) - v(t)\rangle \cdot e^{-2\lambda t}dt + \Phi(u) - \Phi(v) \quad (5.18) \\ &\leq \frac{1}{2}\|u_0 - v(0)\|_{\mathscr{H}}^2, \\ &\text{for all } v \in \mathscr{D}(\Phi) \text{ with } \partial_t v \in L^2(0,T;\mathscr{V}^*). \end{aligned}$$

The well-posedness of (5.18) is ensured by [70, Theorem 3]:

Theorem 5.8. *Assume that the infitesimal generator $\mathscr{A}_X$ is coercive and continuous. Then problem* (5.18) *admits a unique solution*

$$u \in L^2(0,T;\mathscr{V}) \cap L^\infty(0,T;\mathscr{H}) \quad \text{such that } t \mapsto \phi(u(t,\cdot)) \in L^1(0,T).$$

Remark 5.9. As for the parabolic equality problem, also for (5.18) the initial condition is only required to hold in $\mathscr{H}$. In addition, however, in (5.18) the data u_0 must belong to the closure $\overline{\mathscr{K}}^{\|\circ\|_{\mathscr{H}}}$ of $\mathscr{K}$ in $\mathscr{H}$.

Remark 5.10. Convergence rates of backward Euler time discretizations of (5.18) for American style contracts under minimal regularity are given in [3, 59, 70].

Using the notation of the previous sections, we now consider an American option with maturity $T < \infty$ and Lipschitz continuous payoff $g(S)$. Its price $V_{\mathrm{A}}(t,S)$ is given by the optimal stopping problem

$$V_{\mathrm{A}}(t,S) = \sup_{\tau \in \mathscr{T}_{t,T}} \mathbb{E}\left(e^{-r(T-\tau)} g(S_\tau) | S_t = S\right), \tag{5.19}$$

where $\mathscr{T}_{t,T}$ denotes the set of all stopping times between t and T.

In [60, 61] it is shown how the price $V_{\mathrm{A}}(t,S)$ for $S_t^i = S_0^i e^{rt + X_t^i}$, X being a Lévy process, can be characterized as the *viscosity solution* of a corresponding Bellman equation (for details on viscosity solutions we refer to e.g. [24] and the original sources [25, 71, 82]):

Theorem 5.11. *The price $V_{\mathrm{A}}(t,S)$ of an American option defined in* (5.19) *is a viscosity solution of*

$$\min \left\{ \begin{array}{l} -\partial_t V_{\mathrm{A}}(t,S) - rV_{\mathrm{A}}(t,S) - \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j Q_{ij} \partial^2_{S_i S_j} V_{\mathrm{A}} - r \sum_{i=1}^{d} S_i \partial_{S_i} V_{\mathrm{A}}(t,S) \\ \underbrace{\quad - \int_{\mathbb{R}^d} \left(V_{\mathrm{A}}(t,Se^z) - V_{\mathrm{A}}(t,S) - \sum_{i=1}^{d} S_i (e^{z_i} - 1)\, \partial_{S_i} V_{\mathrm{A}}(t,S) \right) \mathtt{N}(\mathrm{d}z),}_{\mathscr{A}_{\mathrm{J}} V_{\mathrm{A}}} \\ V_{\mathrm{A}}(t,S) - g(S) \end{array} \right\} = 0. \tag{5.20}$$

If $V_{\mathrm{A}}(t,S)$ is uniformly continuous and there holds

$$\sup_{[0,T]\times\mathbb{R}^d_{>0}} \frac{V_{\mathrm{A}}(t,S)}{1+S} < \infty, \tag{5.21}$$

this solution is unique.

Proof. Existence of the viscosity solution follows from [61, Theorems 3.1] and its uniqueness is ensured by [61, Theorems 4.1] and [71].

Remark 5.12. Note that Theorem 5.11 holds only in the Lévy case. The solvability of the Bellman equation for more general jump measures is investigated in [1, 15].

The Bellman equation (5.20) can equivalently be restated as the following linear complementarity problem:

$$\begin{aligned} &\partial_\tau u_{\mathrm{A}}(\tau,x) + \mathscr{A}_{\mathrm{BS}}[u_{\mathrm{A}}](\tau,x) + \mathscr{A}_{\mathrm{J}}[u_{\mathrm{A}}](\tau,x) = l(\tau,x) \le 0, \\ &u_{\mathrm{A}}(\tau,x) - e^{r\tau} g_\tau(x) \ge 0, \\ &l(\tau,x)\big(u_{\mathrm{A}}(\tau,x) - e^{r\tau} g_\tau(\tau,x)\big) = 0, \end{aligned} \tag{5.22}$$

on $[0,T]\times\mathbb{R}^d$ with $\mathscr{A}_{\mathrm{BS}}$ and $\mathscr{A}_{\mathrm{J}}$ as above. The initial condition is given by $u_{\mathrm{A},0} = g(e^x)$, i.e. $u_{\mathrm{A},0} = u_0$. The function g_τ is the transformed payoff function, where we applied a transformation as in Theorem 5.6. The system (5.22) can also be considered for Feller generators and on bounded domains. An analogous localization argument to the European case can be used for Lévy market models. Furthermore, if the solution u_{A} of (5.22) satisfies $u_{\mathrm{A}} \in L^2((0,T);\mathscr{V}_D) \cap H^1((0,T);D(\mathscr{A}_D)^*)$ it can be identified with the solution of the following realization of the abstract variational inequality (5.14)–(5.15):

$$\begin{gathered} \text{Find } u_{\mathrm{A}} \in L^2((0,T);\mathscr{V}_D) \cap H^1((0,T);\mathscr{V}_D^*) \text{ such that } u_{\mathrm{A}} \in \mathscr{D}(\phi_\tau) \text{ a.e. in } (0,T) \text{ and} \\ \langle \partial_\tau u_{\mathrm{A}}, v - u_{\mathrm{A}} \rangle_{\mathscr{V}_D^*,\mathscr{V}_D} + (\mathscr{A}_D u_{\mathrm{A}}, v - u_{\mathrm{A}}) - \phi_\tau(u) + \phi_\tau(v) \ge 0\,, \\ \text{for all } v \in \mathscr{D}(\phi_\tau),\ \text{a.e. in } (0,T),\ \text{and } u_{\mathrm{A}}(0) = u_0\,, \end{gathered} \tag{5.23}$$

with $\phi_\tau := I_{\mathscr{K}_\tau}$ as in (5.13) and convex sets

$$\mathscr{K}_\tau := \{v \in \mathscr{V}_D \,:\, v \geq e^{r\tau} g_\tau\} \subset \mathscr{V}_D, \qquad \tau \in (0,T),$$

where $g_\tau : \mathbb{R}^d \to \mathbb{R}$ as above.

As illustrated above, in weak form the variational problem (5.23) reads:

$$\begin{aligned}
&\text{Find } u_{\mathrm{A}} \in L^\infty((0,T);\mathscr{V}_D) \cap H^1((0,T);\mathscr{V}_D^*) \text{ such that } u_{\mathrm{A}} \in \mathscr{D}(\Phi) \text{ a.e. in } (0,T) \text{ and}\\
&\int_0^T \langle \partial_\tau v(\tau) + (\mathscr{A}_D + \lambda) u_{\mathrm{A}}(\tau) - \lambda v(\tau), u_{\mathrm{A}}(\tau) - v(\tau)\rangle \cdot e^{-2\lambda\tau} d\tau + \Phi(u_{\mathrm{A}}) - \Phi(v)\\
&\leq \frac{1}{2}\|u_0 - v(0)\|^2_{\mathscr{H}},\\
&\text{for all } v \in \mathscr{D}(\Phi) \text{ with } \partial_\tau v \in L^2(0,T;\mathscr{V}^*).
\end{aligned} \tag{5.24}$$

Here Φ and $\mathscr{D}(\Phi)$ are depending on ϕ_τ as defined in the last section. The well-posedness of (5.24) is ensured by

Theorem 5.13. *Let X be a Feller process which is an admissible market model with state space $\mathbb{R}^d$, characteristic triplet $(\gamma, Q, \mathtt{N})$ and infinitesimal generator A. Then the weak variational inequality* (5.24) *with $u_0 \in L^2(D)$ admits a unique solution in $\mathscr{V}_D$.*

Proof. The proof follows from Theorem 5.8 using the Gårding inequality (5.10) and continuity of the corresponding Dirichlet bilinear form on $\mathscr{V}_D$ in conjunction with, e.g., [14, Remarque 3] (to account for the smooth time dependence of the convex set $\mathscr{K}_\tau$).

Remark 5.14. For $Q(x) \geq Q_0 > 0$ in D we obtain $H^1_0(D)$ as the domain of the generator. For Lévy processes an analogous result holds, cf. [67, Theorem 4.8].

A different approach to prove existence and uniqueness for the pricing problem is by discretization in time, i.e. to approximate the parabolic variational inequalities by a sequence of nonlinear elliptic equations and prove well-posedness for the solution of each equation in the sequence and apriori estimates for all elements of the sequence. Convergence of the sequence of solutions in an appropriate sense can be proven. This procedure also gives a feasible numerical scheme for the approximation of the pricing problem, for more details we refer to [43, 44] and references therein.

To approximate the solution of (5.22) and (5.23) we consider the following one-parameter family of regularized problems: for a regularization parameter $c > 0$, consider

$$\begin{aligned}
&\text{Find } u^c_{\mathrm{A}} \in L^2((0,T);\mathscr{V}_D) \cap H^1((0,T);\mathscr{V}_D^*) \text{ such that}\\
&\quad \langle \partial_\tau u^c_{\mathrm{A}}, v - u^c_{\mathrm{A}}\rangle_{\mathscr{V}_D^*,\mathscr{V}_D} + (\mathscr{A}_D u^c_{\mathrm{A}}, v - u^c_{\mathrm{A}})\\
&\quad + \left(\min\left(0, \bar{l}(\tau,x) + c(u^c_{\mathrm{A}} - e^{r\tau} g_\tau)\right), v - u^c_{\mathrm{A}}\right) \geq 0,\\
&\text{for all } v \in \mathscr{V},\ c > 0,\ \text{ a.e. in } (0,T), \text{ and } u^c_{\mathrm{A}}(0) = u_0\,.
\end{aligned} \tag{5.25}$$

Different choices for the function $\bar{l}$ are possible, if $\bar{l} \in L^2((0,T);L^2(D))$ we obtain existence of a unique solution of (5.25) for Feller models, cf. [44, Theorem 1].

Under additional assumptions on the bilinear form and the function $\bar{l}(\tau,x)$ convergence of the sequence of solutions $(u^c_{\mathrm{A}})_{c>0}$ to u_{A} with order $O(1/c)$ in the $L^\infty((0,T)\times D)$ norm can be established, cf. [44, Theorem 4.2]. After discretization in time a semi-smooth Newton method can be employed to solve the arising nonlinear systems in each time step iteratively, cf. [43].

5.3 Greeks

A key task in financial engineering is the fast and accurate calculation of sensitivities of market models with respect to model parameters. This becomes necessary for example in model calibration, but also in quantification of model uncertainty for risk analysis and in the pricing and hedging of certain derivative contracts. Classical examples are variations of option prices with respect to the spot price or with respect to time-to-maturity, the so-called "Greeks" of the model. For classical, diffusion type models and plain vanilla

type contracts, the Greeks can be obtained analytically. With the trends to more general market models of jump-diffusion type and to more complicated contracts, closed form solutions are generally not available for pricing and calibration. Thus, prices and model sensitivities have to be approximated numerically.
We will consider the sensitivity of the solution u to variation of a model parameter, like the Greek Vega ($\partial_\sigma u$) and the sensitivity of the solution u to a variation of state spaces such as the Greek Delta ($\partial_x u$).

Definition 5.15. *We call a process X a parametric Feller market model with admissible parameter set $\mathscr{S}_\eta$, if the mapping $\mathscr{S}_\eta \ni \eta \to \{\gamma, Q, \mathtt{N}\}$ is infinitely differentiable.*

For a parametric Markovian market model X in the sense of Definition 5.15 we distinguish two classes of sensitivities. In the following we assume that $X(\eta_0)$ is an admissible market model i.e. $\mathscr{A}_D(\eta_0)$ is continuous and satisfies the Gårding inequality for all $\eta_0 \in S_\eta$. Note that the domain $\mathscr{V}_D$ of the operator $\mathscr{A}_D$ might depend on η_0. For the numerical computation of sensitivities as well as for quadratic hedging it will be crucial to admit a non-trivial right hand side. Accordingly, we consider the parabolic problem

$$\partial_t u - \mathscr{A}_D(\eta_0)u = f \quad \text{in } J \times D, \quad u(0,x) = u_0 \text{ in } D, \tag{5.26}$$

with $u_0 = g$. For $f \in L^2(J;\mathscr{V}_D^*)$ and $u_0 \in \mathscr{H}$ the weak formulation of the problem (5.26) is given by:

$$\begin{aligned} &\text{Find } u \in L^2(J;\mathscr{V}_D) \cap H^1(J;\mathscr{H}) \text{ such that} \\ &(\partial_t u, v) - a\big(\eta_0; u, v\big) = \langle f, v\rangle_{\mathscr{V}_D^*,\mathscr{V}_D}, \quad \forall v \in \mathscr{V}_D, \\ &u(0,\cdot) = u_0 . \end{aligned} \tag{5.27}$$

Under the assumption that continuity and the Gårding inequality hold for every model parameter $\eta_0 \in \mathscr{S}_\eta$, the problem (5.27) admits a unique solution. We will distinguish two classes of sensitivities:

1. The sensitivity of the solution u to a variation $\mathscr{S}_\eta \ni \eta_s := \eta_0 + s\delta\eta$, $s > 0$, of a model parameter $\eta_0 \in \mathscr{S}_\eta$. Typical examples are the Greeks Vega ($\partial_\sigma u$), Rho ($\partial_r u$) and Vomma ($\partial_{\sigma\sigma} u$). Other sensitivities which are not so commonly used in the financial community are the sensitivity of the price with respect to the jump intensity or the order of the process that models the underlying.
2. The sensitivity of the solution u to a variation of arguments t, x. Typical examples are the Greeks Theta ($\partial_t u$), Delta ($\partial_x u$) and Gamma ($\partial_{xx} u$).

We note that Gamma $\partial_{xx} u$ and Vomma $\partial_{\sigma\sigma} u$ are second derivatives of u. The most straightforward approach to their numerical computation is to first obtain a numerical approximation $\tilde{u}$ of u and then to differentiate $\tilde{u}$ with respect to the respective parameters. In variational discretizations such as the ones we will introduce below, $\tilde{u}$ will be a continuous, piecewise linear function of x. Therefore, *direct computation of the Gamma $\partial_{xx} u$ by differentiation of $\tilde{u}$ is meaningless* (it will yield a finite combination of Dirac distributions) and a more sophisticated approach, based on *postprocessing the approximate variational solution $\tilde{u}$ by averaging* is required. Likewise, the stable numerical computation of sensitivities with respect to a model parameter is based on the observation that the sensitivity of interest satisfies (5.5) with a suitable f.

5.3.1 Sensitivity with respect to the model parameter

Let $\mathscr{C}$ be a Banach space over a domain $D \subset \mathbb{R}^d$. $\mathscr{C}$ is the space of parameters or coefficients in the operator $\mathscr{A}$ and $\mathscr{S}_\eta \subseteq \mathscr{C}$ is the set of admissible coefficients. We denote by $u(\eta_0)$ the unique solution to (5.27) and introduce the derivative of $u(\eta_0)$ with respect to $\eta_0 \in \mathscr{S}_\eta$ as the mapping $D_{\eta_0} u(\eta_0) : \mathscr{C} \to \mathscr{V}_D$

$$\widetilde{u}(\delta\eta) := D_{\eta_0} u(\eta_0)(\delta\eta) := \lim_{s \to 0^+} \frac{1}{s}\big(u(\eta_0 + s\delta\eta) - u(\eta_0)\big), \quad \delta\eta \in \mathscr{C}.$$

We also introduce the derivative of $\mathscr{A}_D(\eta_0)$ with respect to $\eta_0 \in \mathscr{S}_\eta$

$$\widetilde{\mathscr{A}_D}(\delta\eta)\varphi := D_{\eta_0}\mathscr{A}(\eta_0)(\delta\eta)\varphi := \lim_{s \to 0^+} \frac{1}{s}\big(\mathscr{A}_D(\eta_0 + s\delta\eta)\varphi - \mathscr{A}_D(\eta_0)\varphi\big),$$

where $\varphi \in \mathscr{V}_D$, $\delta\eta \in \mathscr{C}$. We assume that $\widetilde{\mathscr{A}_D}(\delta\eta) \in \mathscr{L}(\widetilde{\mathscr{V}_D}, \widetilde{\mathscr{V}_D}^*)$ with $\widetilde{\mathscr{V}_D}$ a real and separable Hilbert space satisfying

$$\widetilde{\mathscr{V}_D} \subseteq \mathscr{V}_D \subset \mathscr{H} \cong \mathscr{H}^* \subset \mathscr{V}_D^* \subseteq \widetilde{\mathscr{V}_D^*}.$$

We further assume that there exists a real and separable Hilbert space $\overline{\mathscr{V}_D} \subseteq \widetilde{\mathscr{V}_D}$ such that $\widetilde{\mathscr{A}_D} v \in \mathscr{V}_D^*$, $\forall v \in \overline{\mathscr{V}_D}$. We have the following relation between $D_{\eta_0} u(\eta_0)(\delta\eta)$ and u.

Lemma 5.16. *Let* $\widetilde{\mathscr{A}_D}(\delta\eta) \in \mathscr{L}(\widetilde{\mathscr{V}_D}, \widetilde{\mathscr{V}_D}^*)$, $\forall \delta\eta \in \mathscr{C}$ *and* $u(\eta_0) : J \to \overline{\mathscr{V}_D}$, $\eta_0 \in \mathscr{S}_\eta$ *be the unique solution to*

$$\partial_t u(\eta_0) - \mathscr{A}_D(\eta_0) u(\eta_0) = 0 \quad \text{in } J \times D, \quad u(\eta_0)(0,\cdot) = g(x) \quad \text{in } D. \tag{5.28}$$

Then, $\widetilde{u}(\delta\eta)$ *solves*

$$\partial_t \widetilde{u}(\delta\eta) - \mathscr{A}_D(\eta_0)\widetilde{u}(\delta\eta) = \widetilde{\mathscr{A}_D}(\delta\eta) u(\eta_0) \quad \text{in } J \times D, \quad \widetilde{u}(\delta\eta)(0,\cdot) = 0 \quad \text{in } D. \tag{5.29}$$

Proof. Since $u(\eta_0)(0) = g$ does not depend on η_0 its derivative with respect to η is 0. Now let $\eta_s := \eta_0 + s\delta\eta$, $s > 0$, $\delta\eta \in \mathscr{C}$. Subtract from the equation $\partial_t u(\eta_s)(t) - \mathscr{A}_D(\eta_s) u(\eta_s)(t) = 0$ equation (5.28) and divide by s to obtain

$$\begin{aligned}\partial_t \, \frac{1}{s}\big(u(\eta_s)(t) - u(\eta_0)(t)\big) - \frac{1}{s}\big(\mathscr{A}_D(\eta_s) - \mathscr{A}_D(\eta_0)\big) u(\eta_s)(t) \\ - \frac{1}{s}\mathscr{A}_D(\eta_0)\big(u(\eta_s)(t) - u(\eta_0)(t)\big) = 0.\end{aligned}$$

Taking $\lim_{s\to 0^+}$ gives equation (5.29).

We associate to the operator $\widetilde{\mathscr{A}_D}(\delta\eta)$ the Dirichlet form $\widetilde{a}(\delta\eta;\cdot,\cdot) : \widetilde{\mathscr{V}_D} \times \widetilde{\mathscr{V}_D} \to \mathbb{R}$ which is given by

$$\widetilde{a}(\delta\eta; u, v) = \langle \widetilde{\mathscr{A}_D}(\delta\eta) u, v \rangle_{\widetilde{\mathscr{V}_D}^*, \widetilde{\mathscr{V}_D}}.$$

The variational formulation to (5.29) reads:

$$\begin{aligned}&\text{Find } \widetilde{u}(\delta\eta) \in L^2(J;\mathscr{V}_D) \cap H^1(J;\mathscr{H}) \text{ such that} \\ &(\partial_t \widetilde{u}(\delta\eta), v)_{\mathscr{H}} - a\big(\eta_0; \widetilde{u}(\delta\eta), v\big) = +\widetilde{a}\big(\delta\eta; u(\eta_0), v\big), \quad \forall v \in \mathscr{V}_D, \\ &\widetilde{u}(\delta\eta)(0) = 0.\end{aligned} \tag{5.30}$$

Note that (5.30) has a unique solution $\widetilde{u}(\delta\eta) \in \mathscr{V}_D$ due to the assumptions on $a(\eta_0;\cdot,\cdot)$, $\widetilde{\mathscr{A}_D}$ and $u(\eta_0) \in \overline{\mathscr{V}_D}$. The numerical solution of (5.30) will be discussed in the next chapters.

5.3.2 Sensitivity with respect to solution arguments

We discuss the computation of $\mathscr{D}^{\mathbf{n}} u = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d} u$ for arbitrary multi-index $\mathbf{n} \in \mathbb{N}_0^d$, where $\mathbf{n} = (n_1, \ldots, n_d)$. For $\mu \in \mathbb{Z}^d$ and $h > 0$ we define the translation operator $T_h^\mu \varphi(x) = \varphi(x + \mu h)$ and the forward difference quotient $\partial_{h,j}\varphi(x) = h^{-1}(T_h^{e_j}\varphi(x) - \varphi(x))$, where e_j, $j = 1, \ldots, d$, denotes the j-th standard basis vector in $\mathbb{R}^d$. For $\mathbf{n} \in \mathbb{N}_0^d$ we denote by $\partial_h^{\mathbf{n}}\varphi = \partial_{h,1}^{n_1} \cdots \partial_{h,d}^{n_d}\varphi$ and by $\mathscr{D}_h^{\mathbf{n}}$ the difference operator of order $n \geq 0$

$$\mathscr{D}_h^{\mathbf{n}}\varphi := \sum_{\gamma, |\mathbf{n}| = n} C_{\gamma,\mathbf{n}} T_h^\gamma \partial_h^{\mathbf{n}}\varphi.$$

Definition 5.17. *The difference operator* $\mathscr{D}_h^{\mathbf{n}}$ *of order* $|\mathbf{n}| = n$ *and mesh width* h *is called an approximation to the derivative* $\mathscr{D}^{\mathbf{n}}$ *of order* $s \in \mathbb{N}_0$ *if for any* $G_0 \subset G$ *there holds*

$$\|\mathscr{D}^{\mathbf{n}}\varphi - \mathscr{D}_h^{\mathbf{n}}\varphi\|_{\widetilde{H}^r(G_0)} \leq Ch^s \|\varphi\|_{\widetilde{H}^{s+r+n}(G)}, \forall \varphi \in \widetilde{H}^{s+r+n}(G). \tag{5.31}$$

Using finite elements for the discretization with basis $b_1, \ldots, b_N$ of V_N, the action of $\mathscr{D}_h^{\mathbf{n}}$ to $v_N \in V_N$ can be realized as matrix-vector multiplication $\underline{v}_N \mapsto \mathbf{D}_h^{\mathbf{n}} \underline{v}_N$, where

$$\mathbf{D}_h^{\mathbf{n}} = \big(\mathscr{D}_h^{\mathbf{n}} b_1, \cdots, \mathscr{D}_h^{\mathbf{n}} b_N\big) \in \mathbb{R}^{N \times N},$$

and $\underline{v}_N$ is the coefficient vector of v_N with respect to the basis of V_N.

Example 5.18. Let V_N be, the space of piecewise linear continuous functions on $[0,1]$ vanishing at the end points $0, 1$. For $\alpha, \beta, \gamma \in \mathbb{R}$ and $\mu \in \mathbb{N}_0$ we denote by $\mathrm{diag}_\mu(\alpha,\beta,\gamma)$ the matrices

$$\mathrm{diag}_\mu(\alpha,\beta,\gamma) = \begin{pmatrix} \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots & & \\ & & \cdots & 0 & \alpha & \beta & \gamma & 0 & \cdots \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where the entries β are on the μ-th lower diagonal. Then, the matrices $\mathbf{Q}_h$ of the forward difference quotient ∂_h and $\mathbf{T}_\mu$ of the translation operator T_h^μ respectively are given by

$$\mathbf{Q}_h = h^{-1}\mathrm{diag}_0(0,-1,1), \quad \mathbf{T}_\mu = \mathrm{diag}_\mu(0,1,0).$$

Hence, for example, we have for the centered finite difference quotient

$$\mathscr{D}_h^2\varphi(x) = h^{-2}(\varphi(x+h) - 2\varphi(x) + \varphi(x-h)),$$

of order 2 in one dimension $\mathbf{D}_h^2 = \mathbf{T}_{-1}\mathbf{Q}_h^2 = h^{-2}\mathrm{diag}_0(1,-2,1)$. In the multidimensional case the matrix $\mathbf{D}_h^{\mathbf{n}}$ is given by

$$\mathbf{D}_h^{\mathbf{n}} = \sum_{\gamma,|\mathbf{n}|=n} C_{\gamma,\mathbf{n}}\mathbf{T}_{\gamma_1} \otimes \cdots \otimes \mathbf{T}_{\gamma_d}\mathbf{Q}_h^{n_1} \otimes \cdots \otimes \mathbf{Q}_h^{n_d}.$$

6 Wavelets

For the numerical solution, we discretize the parabolic equation (5.7)-(5.8) in $(0,T) \times D$ in the spatial variable with spline wavelet bases for $V = \widetilde{H}^{\mathbf{m}(x)}(D)$ an in the time parameter by the θ-scheme or the more sophisticated discontinuous Galerkin timestepping which allows to expoit the time-analycity of the processes' semigroups. To present the spatial discretizations, we briefly recapitulate basic definitions and results on wavelets from, e.g., [20] and the references there. For specific spline wavelet constructions on a bounded interval I, we refer to e.g. [30], [63] and [83]. Since for all infinitesimal generators arising in connection with Markov processes the Sobolev order $2m(x)$ of the generator satisfies $0 \leq m(x) \leq 1$, the full machinery of multiresolution analyses in Sobolev spaces of arbitrary order is not required; we confine ourselves therefore to *continuous, piecewise polynomial multiresolution systems in* $\mathbb{R}^1$. For wavelet discretizations of Kolmogoroff equations for multivariate models, we shall employ tensor products of these univariate, piecewise polynomials multiresolution systems.

Our use of compactly supported, piecewise polynomial multiresolution systems (rather than the more commonly employed B-spline Finite Element spaces) for the Galerkin discretization of Kolmogoroff equations is motivated by the following key properties of these spline wavelet systems: a) the approximation properties of the multiresolution sytems equal those of the B-spline systems, b) the spline wavelet systems form *Riesz bases* of the domains of the infinitesimal generators of the Markov processes, thereby allowing for simple and efficient *preconditioning of the matrices arising in wavelet representations of the processes' Dirichlet forms*, c) the spline wavelet systems can be designed to have a large number of *vanishing moments*, thereby allowing for a *compression* of the wavelet matrix for the jump measure.

6.1 Spline wavelets on an interval

Our Galerkin discretizations of Kolmogoroff equations for Feller processes are based on biorthogonal wavelet bases on a bounded interval $I \subset \mathbb{R}$.

We recapitulate the basic definitions from, e.g., [20, 83] to which we also refer for further references and additional details, such as the construction of higher order wavelets.

Our wavelet systems are two-parameter systems $\{\psi_{l,k}\}_{l=-1,\ldots,\infty,k\in\nabla_l}$ of compactly supported functions $\psi_{l,k}$. Here the first index, l, denotes "level" of refinement resp. resolution: wavelet functions $\psi_{l,k}$ with large values of the level index are well-localized in the sense that $\mathrm{diam}(\mathrm{supp}\psi_{l,k}) = O(2^{-l})$. The second index, $k \in \nabla_l$, measures the *localization* of wavelet $\psi_{l,k}$ within the interval I at scale l and ranges in the index set ∇_l. In order to achieve maximal flexibility in the construction of wavelet systems (which can be used to satisfy other requirements, such as minimizing their support size or to minimize the size of constants in norm equivalences), we will consider wavelet systems which are *biorthogonal* in $L^2(I)$, consisting of a primal wavelet system $\{\psi_{l,k}\}_{l=-1,\ldots,\infty,k\in\nabla_l}$ which is a Riesz basis of $L^2(I)$ (and which will enter explicitly in the Galerkin discretizations of the Markov processes) and a corresponding dual wavelet system $\{\widetilde{\psi}_{l,k}\}_{l=-1,\ldots,\infty,k\in\nabla_l}$ (which will never be used explicitly in our algorithms). Notice that construction of *fully* $L^2(I)$ *orthonormal wavelet systems* is feasible, but results in function systems which are either nonpolynomial or have larger supports or fewer vanishing moments.

The primal wavelet bases $\psi_{l,k}$ span finite dimensional spaces

$$\mathscr{W}^l := \mathrm{span}\,\{\psi_{l,k} : k \in \nabla_l\}\,, \quad \mathscr{V}^L := \bigoplus_{l=-1}^{L-1} \mathscr{W}^l \quad l = -1,0,1\ldots,$$

and the dual spaces are defined analogously in terms of the dual wavelets $\widetilde{\psi}_{l,k}$ by

$$\widetilde{\mathscr{W}^l} := \mathrm{span}\,\{\widetilde{\psi}_{l,k} : k \in \nabla_l\}\,, \quad \widetilde{\mathscr{V}}^L := \bigoplus_{l=-1}^{L-1} \widetilde{\mathscr{W}}^l \quad l = -1,0,1\ldots,$$

In the sequel we require the following properties of the wavelet functions to be used on our Galerkin discretization schemes, we assume wlog $I = (0,1)$.

1. Biorthogonality: the basis functions $\psi_{l,k}$, $\widetilde{\psi}_{l,k}$ satisfy

$$\langle \psi_{l,k}, \widetilde{\psi}_{l',k'} \rangle = \delta_{l,l'} \delta_{k,k'}. \tag{6.1}$$

2. Local support: the diameter of the support is proportional to the meshsize 2^{-l},

$$\text{diam supp } \psi_{l,k} \sim 2^{-l} \ , \ \text{diam supp } \widetilde{\psi}_{l,k} \sim 2^{-l}. \tag{6.2}$$

3. Conformity: the basis functions should be sufficiently regular, i.e.

$$\mathscr{W}^l \subset \widetilde{H}^1(I) \ , \ \widetilde{\mathscr{W}^l} \subset H^\delta(I) \ \text{ for some } \delta > 0 \ , \ \ l \geq -1. \tag{6.3}$$

Furthermore $\bigoplus_{l=-1}^{\infty} \mathscr{W}^l$, $\bigoplus_{l=-1}^{\infty} \widetilde{\mathscr{W}^l}$ are supposed to be dense in $L_2(I)$

4. Vanishing moments: The primal basis functions $\psi_{l,k}$ are assumed to satisfy vanishing moment conditions up to order $p^*+1 \geq p$

$$\langle \psi_{l,k}, x^\alpha \rangle = 0 \ , \ \alpha = 0, \ldots, d = p^*+1, l \geq 0, \tag{6.4}$$

and for all dual wavelets, except the ones at each end point, one has

$$\langle \widetilde{\psi}_{l,k}, x^\alpha \rangle = 0 \ , \ \alpha = 0, \ldots, d = p+1, l \geq 0. \tag{6.5}$$

At the end points the dual wavelets satisfy only

$$\langle \widetilde{\psi}_{l,k}, x^\alpha \rangle = 0 \ , \ \alpha = 1, \ldots, d = p+1, l \geq 0. \tag{6.6}$$

We remark that the third condition implies that the wavelets satisfy the zero Dirichlet condition, namely $\psi_{l,k}(0) = \psi_{l,k}(1) = 0$; the representation of this boundary condition by the subspace is important in the pricing of barrier contracts. To satisfy the homogeneous Dirichlet condition by the wavelet basis, we sacrifice the vanishing moment property of those wavelets whose supports include the endpoints of I, i.e. $x = 0$ or $x = 1$. For example, $\psi_{l,0}$, $l = 0, \ldots$, at the end point $x = 0$ (assuming that the localization index $k \in \nabla_l$ enumerates the wavelets in the direction of increasing values of x).

A systematic and general construction for arbitrary order biorthogonal spline wavelets is presented in [26]. Sufficiently far apart from the end points of $(0,1)$, biorthogonal wavelet (e.g. [20] and the references there) bases are used in this approach. In the recent paper [38] a wavelet bases was constructed with slightly smaller support at the end points. Using biorthogonal wavelets in the case $p = 1$, piecewise linear spline wavelets vanishing outside $I = (0,1)$ are obtained by simple scaling. The interior wavelets have two vanishing moments and are obtained from the mother wavelet $\psi(x)$ which takes the values $(0, -\frac{1}{6}, -\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{6}, 0, 0, 0)$ at the points $(0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1)$ by scaling and translations: $\psi_{l,k}(x) := 2^{l/2}\psi(2^{l-3}x - k + 2)$ for $2 \leq k \leq 2^l - 3$ and $l \geq 3$. At the left boundary $k = 1$, we use the piecewise linear function ψ_{left} defined by the nodal values $(0, \frac{5}{8}, \frac{-3}{4}, \frac{-1}{4}, \frac{1}{4}, \frac{1}{8}, 0, 0, 0)$ and $\psi_{right}(x) = \psi_{left}(1-x)$. For additional details we refer to [38].

The following particular system of biorthogonal spline wavelet basis functions are Riesz bases for all constant or variable order Sobolev spaces of order $s \in [0,1]$ (and only these spaces arise as domains of the infinitesimal generators of Feller-Lévy processes) and have proved efficient for our present applications [57]. They are a biorthogonal system of piecewise linear, continuos polynomial spline wavelets which were optimized for having small support. Their dual wavelets do not permit compact support, but they are nevertheless exponentially decaying, i.e.,

$$\left| \widetilde{\Psi}_e(x) \right| \leq C \exp(-\kappa \, |x|), \quad \kappa > 0, x \in \mathbb{R}. \tag{6.7}$$

Note that the dual wavelets never enter the Galerkin discretization schemes explicitly. The biorthogonal wavelets in the case $p = 1$ are continuous, piecewise linear spline wavelets vanishing outside $I = (0,1)$ (for general intervals $I = (a,b)$, they are obtained by simple scalings). The interior wavelets have two vanishing moments and are obtained from one piecewise linear, continuous mother wavelet function $\psi(x)$ taking values $(0, -\frac{1}{2}, 1, -\frac{1}{2}, 0)$ at $(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$ by scaling and translation: $\psi_{l,k}(x) := 2^{l/2}\psi(2^{l-1}x - (2k-1)2^{-2})$ for $1 \leq k \leq 2^l - 2$ and $l \geq 2$.

The boundary wavelets are likewise constructed from the continuous, piecewise linear functions ψ_*, with values $(0, 1, -\frac{1}{2}, 0)$ at $(0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4})$, and ψ^*, taking values $(0, -\frac{1}{2}, 1, 0)$ at $(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1)$: $\psi_0^l = \psi_*(2^{l-1}x)$ and

$\psi_{l,2^l-1} = 2^{l/2}\psi^*(2^{l-1}x - 2^{l-1}+1)$. The following results are known for wavelets satisfying the above requirements (e.g., [20]).

Any function $v \in \widetilde{H}^s(I)$, $0 \le s \le p+1$, and, due to the embeddings $\widetilde{H}^{\underline{m}}(I) \subset \widetilde{H}^{m(x)}(I) \subset \widetilde{H}^{\overline{m}}(I)$, in particular any function $v \in \widetilde{H}^{m(x)}(I)$ can be represented in the wavelet series

$$v = \sum_{l=0}^{\infty}\sum_{k=1}^{M^l} v_{l,k}\psi_{l,k} = \sum_{\lambda\in\mathscr{I}} v_\lambda \psi_\lambda, \qquad v_\lambda = \int_I v\widetilde{\psi_\lambda}\,dx. \tag{6.8}$$

Here, we used the symbol $\lambda = (l,k)$ to denote a generic index in the index set

$$\mathscr{I} := \{\lambda = (l,k) : l = 0,1,2\dots\ ,\ k = 1,\dots M_l\}.$$

Approximations v_h of functions $v \in \widetilde{H}^{m(x)}(I)$ can be obtained by truncating the wavelet expansion (6.8). In this way, a "quasi-interpolating" approximation operator $Q_h : \widetilde{H}^{m(x)}(I) \to V_h$, can be defined by truncating the wavelet expansion, i.e. by

$$Q_h v = \sum_{l=0}^{L-1}\sum_{k=1}^{M^l} v_{l,k}\psi_{l,k}. \tag{6.9}$$

For all $v_h = \sum_{l=0}^{L-1}\sum_{k=1}^{M^l} v_{l,k}\psi_{l,k} \in V_h = \mathscr{V}^L$, $h \sim 2^{-L}$, there holds the norm equivalence

$$\|v_h\|^2_{\widetilde{H}^s(I)} \sim \sum_{l=0}^{L-1}\sum_{k=1}^{M^l} |v_{l,k}|^2 2^{2ls}, \tag{6.10}$$

for all $0 \le s < \frac{3}{2}$. This result is sharp in the sense that the norm equivalence fails in the upper limit $s = 3/2$; spline-wavelet systems consisting of higher order, piecewise polynomials with higher regularity across interval boundaries are known, but are not required in the present context, as the arguments in Dirichlet forms of Feller processes must belong locally to $H^1(\mathbb{R}^d)$, at best.

Validity of (6.10) in the variable order spaces $\widetilde{H}^{m(x)}(I)$ was shown in [77, Theorem 3]. There, it was in particular shown that for $u \in \widetilde{H}^{m(x)}$ it holds

$$\|u\|^2_{\widetilde{H}^{m(x)}(I)} \sim \sum_{l=0}^{\infty}\sum_{k=1}^{M^l} \left|u_{l,k}\right|^2 2^{2\underline{m}_\lambda l}, \tag{6.11}$$

where we recall the notation $\lambda = (l,k) \in \mathscr{I}$ and $\underline{m}_\lambda$ which is defined as

$$\underline{m}_\lambda := \inf\{m(x) : x \in \Omega_\lambda\} \quad \text{and} \quad \overline{m}_\lambda := \sup\{m(x) : x \in \Omega_\lambda\} \tag{6.12}$$

for the extended support Ω_λ of a wavelet basis function ψ_λ defined by

$$\Omega_\lambda := \Omega_{l,k} = \bigcup_{\lambda'\in\mathscr{I}:l'\ge l} \{\operatorname{supp}\psi_{\lambda'} : \operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'} \neq \emptyset\}. \tag{6.13}$$

For $0 \le s < \frac{3}{2} \le t \le p+1$, we have the approximation property (e.g. [20])

$$\|v - Q_h v\|_{\widetilde{H}^s(I)} \le Ch^{t-s}\|v\|_{H^t(I)}. \tag{6.14}$$

Remark 6.1. The pricing equations can also be considered in real price variables and not in log-price variables as described above, this leads to pseudodifferential operators, whose domains are weighted Sobolev spaces with possibly degenerated weights. An example for such kind of equations is given by the CEV model, cf. Remark 2.8 and 5.7. Norm equivalences and efficient preconditioning for this kind of equations has been considered by [10]. The corresponding norm equivalence for a weighted space $L^2_w(0,1)$ with norm $\|u\|^2_{L^2_w} = \int_0^1 w(x)^2 u(x)^2\,dx$ reads:

$$\|v\|^2_{L^2_w(0,1)} \sim \sum_{l=0}^{\infty}\sum_{k=1}^{M^l} |v_{l,k}|^2 w^2(2^{-l}k), \tag{6.15}$$

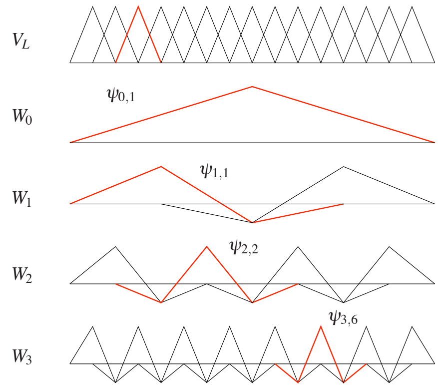


Fig. 1 Single-scale space V_L and its decomposition into multiscale wavelet spaces W_ℓ for $L = 3$ and $p = 1$.

for $v \in L^2_w$ and w being a possibly singular weight function fulfilling weak smoothness assumptions, cf. [10, Assumption 3.1]. Note that we obtain a variable weight in the exponent in (6.11) for the variable order Sobolev space, while we obtain a variable weight in the case of the weighted space (6.15). It is also possible to combine the variable order Sobolev spaces and the weighted Sobolev spaces, leading to weighted variable order spaces with analogous norm equivalences to (6.11) and (6.15).

6.2 Tensor product spaces

On $D = [0,1]^d$, $d > 1$, we define the subspace V_L of $\widetilde{H}^{\mathbf{m}(x)}(D)$ as the full tensor product of d univariate approximation spaces, i.e. $V_L := \bigotimes_{1\le i\le d} \mathscr{V}^{l_i}$, which can be written as

$$V_L = \left\{\psi_{\mathbf{l},\mathbf{k}} : 0 \le l_i \le L-1, k_i \in \nabla_{l_i}, i = 1,\dots,d\right\},$$

with basis functions $\psi_{\mathbf{l},\mathbf{k}} = \psi_{l_1,k_1}\cdots\psi_{l_d,k_d}$, $0 \le l_i \le L-1$, $k_i \in \nabla_{l_i}$, $i = 1,\dots,d$. We can write V_L in terms of increment spaces

$$V_L = \bigoplus_{0\le l_i\le L-1} \mathscr{W}^{l_1} \otimes \dots \otimes \mathscr{W}^{l_d}.$$

Therefore, we have for any function $u \in L^2(D)$ the series representation

$$u = \sum_{l_i=0}^{\infty} \sum_{k_i\in\nabla_{l_i}} u_{\mathbf{l},\mathbf{k}} \psi_{\mathbf{l},\mathbf{k}}.$$

Using the one dimensional norm equivalences and the intersection structure we obtain

$$\|u\|^2_{H^{\mathbf{m}(x)}} \sim \sum_{\lambda} \left(2^{2\underline{m}^1_{\lambda_1} l_1} + \dots + 2^{2\underline{m}^d_{\lambda_d} l_d}\right) |u_\lambda|^2. \tag{6.16}$$

Corollary 6.2. *Let $u \in H^s(D) \cap \widetilde{H}^1(D)$ for some $1 \le s \le p+1$. Then for the quasi-interpolant $u_h = Q_h u = \sum_{l_i=0}^{L-1}\sum_{k=1}^{M^{l_i}} u_{\mathbf{l},\mathbf{k}}\psi_{\mathbf{l},\mathbf{k}}$ there holds for $0 < \overline{m} < 1 \le s \le p+1$ the Jackson estimate*

$$\begin{aligned}\|u-u_h\|^2_{\widetilde{H}^{m(x)}(D)} &\lesssim \int_I \left(2^{2L(m_1(x_1)-s)} + \dots + 2^{2L(m_d(x_d)-s)}\right)(|D^s u(x)|^2 + |u(x)|^2)dx \\ &\lesssim 2^{2L(\overline{m}-s)} \|u\|^2_{H^s(D)},\end{aligned}$$

where $\overline{m} = \max_{i=1,\dots,d} \overline{m}_i$.

Proof. For multi-indices $\lambda = (l,k), \mu = (L,k') \in \mathscr{I}$, we introduce the notation $\lambda \succeq \mu$ if $l_i \ge L_i$ and $\operatorname{supp}\psi_{\lambda_i} \cap \operatorname{supp}\psi_{\mu_i} \ne \emptyset$ for all $i = 1,\dots,d$. For $s \ge \frac{3}{2}$ we choose $s' < s$ with $1 \le s' < \frac{3}{2}$, otherwise we set $s' = s$. We observe that $\underline{m}_{\lambda_i} - s' \le \overline{m}_{\mu_i} - s' < 0$ holds for all $\lambda_i \succeq \mu_i$. Therefore we conclude from the norm equivalence (6.16)

$$\begin{aligned}
\|u-u_h\|^2_{\widetilde{H}^{m(x)}(D)} &\sim \sum_{l_i \ge L} \sum_{k_i=1}^{M_{l_i}} \left(2^{2l_1 \underline{m}^1_{\lambda_1}} + \ldots + 2^{2l_d \underline{m}^d_{\lambda_d}} \right) |u_\lambda|^2 \\
&= \sum_{l_i \ge L} \sum_{k_i=1}^{M_{l_i}} \left(2^{2l_1(\underline{m}^1_{\lambda_1} - s')} 2^{2l_1 s'} + \ldots + 2^{2l_d(\underline{m}^d_{\lambda_d} - s')} 2^{2l_d s'} \right) |u_\lambda|^2 \\
&\lesssim \sum_{\mu \in \nabla_L} \left(2^{2L(\overline{m}^1_{\mu_1} - s')} + \ldots + 2^{2L(\overline{m}^d_{\mu_d} - s')} \right) \\
&\quad \times \sum_{\lambda \succeq \mu} \left(2^{2s' l_1} + \ldots + 2^{2s' l_d} \right) |u_\lambda|^2 ,
\end{aligned}$$

where $\nabla_L = \{\mu = (L,k') : k'_i = 1,\ldots,M_L, i = 1,\ldots,d\}$.

Let $\mu = (L,k')$, $L = |\mu|$ and $\square_\mu := \Pi_{i=1}^d [2^{-L} k'_i, 2^{-L}(k'_i+1)]$. Then, by the norm equivalence (6.16) and the approximation property (6.14), we have

$$\sum_{\mu \in \nabla_L} \sum_{\lambda \succeq \mu} \left(2^{2s' l_1} + \ldots + 2^{2s' l_d} \right) |u_\lambda|^2 \lesssim \sum_{\mu \in \nabla_L} 2^{2L(s'-s)} \|u\|^2_{H^s(\square_\mu)} .$$

Recalling that $2^{L\underline{m}^i_\mu} \sim 2^{Lm^i(x_i)} \sim 2^{L\overline{m}^i_\mu}$ holds for all $x \in \square_\mu$, we obtain the final result

$$\begin{aligned}
\|u-u_h\|^2_{\widetilde{H}^{m(x)}(D)} &\lesssim \int_I \left(2^{2L(m_1(x_1)-s)} + \ldots + 2^{2L(m_d(x_d)-s)} \right) (|D^s u(x)|^2 + |u(x)|^2) dx \\
&\lesssim 2^{2L(\overline{m}-s)} \|u\|^2_{H^s(D)} .
\end{aligned}$$

6.3 Space discretization

For computational reasons it is convenient to consider the PIDE formulation of the Feller generator $\mathscr{A}_X$ with symbol $a(x,\xi)$. For $u \in S(\mathbb{R}^d)$ we can write as in (2.6):

$$\begin{aligned}
\mathscr{A}_X u(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x,\xi) \hat{u}(\xi)\, \mathrm{d}\xi \\
&= -\gamma(x) \cdot \nabla u(x) + \sum_{k,l}^d Q_{kl}(x) \partial_{kl} u \\
&\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \int_{\mathbb{R}^d} \left(1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+|y|^2} \right) \mathtt{N}(x,\mathrm{d}y) \hat{u}(\xi)\, \mathrm{d}\xi \\
&= -\gamma(x) \cdot \nabla u(x) + \sum_{k,l}^d Q_{kl} \partial_{kl} u \\
&\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \left(1 - e^{iy\cdot\xi} + \frac{iy\cdot\xi}{1+|y|^2} \right) \hat{u}(\xi)\, \mathrm{d}\xi\, \mathtt{N}(x,\mathrm{d}y) \\
&= -\gamma(x) \cdot \nabla u(x) + \sum_{k,l}^d Q_{kl} \partial_{kl} u \\
&\quad + \underbrace{\int_{\mathbb{R}^d} \left(u(x) - u(x+y) + \frac{y \cdot \nabla_x u(x)}{1+|y|^2} \right) \mathtt{N}(x,\mathrm{d}y)}_{\mathscr{A}^C_J u} .
\end{aligned}$$

If we assume X to be an admissible market model, we can drop the damping factor $\frac{1}{1+|y|^2}$ and replace the measure $\mathtt{N}$ by the corresponding jump kernel k in the above calculation, due to (4.13). We convert the canonical jump operator $\mathscr{A}_J^C$ into the integrated jump operator $\mathscr{A}_J$, due to an analogous argument to [85, Lemma 2.2.7],

$$\begin{aligned}\mathscr{A}_J u(x) = &\sum_{i=1}^{d} \int_{\mathbb{R}} (u(x+y_i e_i) - u(x) - y_i \partial_i u(x))\, k_i(x_i, y_i) \mathrm{d}y_i \\ &+ \sum_{j=2}^{d} \sum_{\substack{|I|=j \\ I_1<\ldots<I_j}} \int_{\mathbb{R}^j} \frac{\partial^j u}{\partial y^I}(x+y^I) F^I((U_k(x_k,y_k))_{k\in I}) \mathrm{d}y^I,\end{aligned}$$

where F is the Lévy copula of X and U_k are the corresponding tail integrals. The corresponding bilinear form reads

$$\begin{aligned}\mathscr{E}_J(\cdot,\cdot) = &\underbrace{\sum_{i=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (u(x+y_i e_i) - u(x) - y_i \partial_i u(x))\, v(x)\, k_i(x_i, y_i)\, \mathrm{d}x \mathrm{d}y_i}_{(I)} \\ &+ \sum_{j=2}^{d} \sum_{\substack{|I|=j \\ I_1<\ldots<I_j}} \int_{\mathbb{R}^j} \int_{\mathbb{R}^d} \frac{\partial^j u}{\partial y^I}(x+y^I) v(x) F^I\left((U_k(x_k,y_k))_{k\in I}\right) \mathrm{d}x \mathrm{d}y^I.\end{aligned}$$

Remark 6.3. The marginal jump kernels $k_i(x,y)$ can have a singularity at $y_i = 0$ of order $1+\alpha_i < 3$, so a numerical evaluation of the discretized bilinear form for nonsmooth arguments (such as the continuous, piecewise linears) is not straightforward, since the condition (2.5) on the jump measure $\mathtt{N}(x,dy)$ implies that possibly nonintegrable singularities could arise in the densities $k_i(x_i,y_i)$ in the representation of $A_J u(x)$, unless u is locally in $C^{1,1}$ in that representation. This is not the case, however, for the piecewise linear, continuous wavelet functions. Therefore, in our implementations, we use antiderivatives of k^i to remove nonintegrable singularities that are easier to handle. Specifically, we use the following equality for (I) for smooth u and v:

$$\begin{aligned}&\int_{\mathbb{R}} \int_{\mathbb{R}^d} (u(x+y_i e_i) - u(x) - y_i \partial_i u(x))\, v(x)\, k_i(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_i^2 u(x+y_i e_i) v(x)\, k_i^{(-2)}(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_i u(x+y_i e_i) \partial_i v(x)\, k_i^{(-2)}(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i \\ &\quad -\int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_i u(x+y_i e_i) v(x)\, \partial_i k_i^{(-2)}(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i.\end{aligned} \tag{6.17}$$

Therefore, the jump part of the bilinear form can be written as

$$\begin{aligned}\mathscr{E}_J(u,v) = &-\sum_{i=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_i u(x+y_i e_i) \partial_i v(x)\, k_i^{(-2)}(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \partial_i u(x+y_i e_i) v(x)\, \partial_i k_i^{(-2)}(x_i,y_i)\, \mathrm{d}x \mathrm{d}y_i \\ &+ \sum_{j=2}^{d} \sum_{\substack{|I|=j \\ I_1<\ldots<I_j}} \int_{\mathbb{R}^j} \int_{\mathbb{R}^d} \frac{\partial^j u}{\partial y^I}(x+y^I) v(x) F^I\left((U_k(x_k,y_k))_{k\in I}\right) \mathrm{d}x \mathrm{d}y^I.\end{aligned}$$

This representation is well-defined also for compactly supported functions u, v which have only local integrable, weak first derivatives, such as the spline wavelet functions introduced above.

Remark 6.4. The major differences to the Lévy case reside in the second term of (6.17) and the x-dependence of the integration kernels. In order for the described procedure to be feasible, the second antiderivatives of the jump kernel need to be available. This is the case for many processes such as CGMY-type or Variance Gamma-type Feller processes, for instance.

Using the wavelet basis described above, we have to compute the stiffness matrix of the diffusion part and of the jump part of the bilinear form. The computation of the stiffness matrix for the diffusion part is standard (e.g. [85]). In what follows, we therefore focus on the jump part

$$\begin{aligned}\mathbf{A}^J_{(\mathbf{l}',\mathbf{k}'),(\mathbf{l},\mathbf{k})} = &-\sum_{i=1}^{d}\int_{\mathbb{R}}\int_{D_R}\partial_i\psi_{\mathbf{l},\mathbf{k}}(x+y_ie_i)\partial_i\psi_{\mathbf{l}',\mathbf{k}'}(x)\,k_i^{(-2)}(x_i,y_i)\,\mathrm{d}x\mathrm{d}y_i\\ &+\int_{\mathbb{R}}\int_{D_R}\partial_i\psi_{\mathbf{l},\mathbf{k}}(x+y_ie_i)\psi_{\mathbf{l}',\mathbf{k}'}(x)\,\partial_ik_i^{(-2)}(x_i,y_i)\,\mathrm{d}x\mathrm{d}y_i\\ &+\sum_{j=2}^{d}\sum_{\substack{|I|=j\\ I_1<\ldots<I_j}}\int_{\mathbb{R}^j}\int_{D_R}\frac{\partial^j\psi_{\mathbf{l},\mathbf{k}}}{\partial y^I}(x+y^I)\psi_{\mathbf{l}',\mathbf{k}'}(x)F^I((U_k(x_k,y_k))_{k\in I})\,\mathrm{d}x\mathrm{d}y^I.\end{aligned}$$

We define the one-dimensional matrices $\mathbf{M}^i$:

$$\mathbf{M}^i_{(l',k'),(l,k)} := \int_{-R}^{R}\psi_{l,k}\psi_{l',k'}\mathrm{d}x.$$

Additionally, we define

$$\begin{aligned}\mathbf{A}^i_{(l',k'),(l,k)} := &-\int_{\mathbb{R}}\int_{-R}^{R}\psi'_{l,k}(x+z)\psi'_{l',k'}(x)k_i^{(-2)}(x,z)\,\mathrm{d}x\,\mathrm{d}z\\ &-\int_{\mathbb{R}}\int_{-R}^{R}\psi'_{l,k}(x+z)\psi_{l',k'}(x)\,\partial_xk_i^{(-2)}(x,z)\,\mathrm{d}x\,\mathrm{d}z\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^I_{(\mathbf{l}'_I,\mathbf{k}'_I),(\mathbf{l}_I,\mathbf{k}_I)} := &\int_{\mathbb{R}^{|I|}}\int_{[-R,R]^{|I|}}\partial^I\psi_{\mathbf{l_I},\mathbf{k_I}}(x+z)\psi_{\mathbf{l}'_\mathbf{I},\mathbf{k}'_\mathbf{I}}(x)\\ &\times F^I((U_k(x_k,y_k))_{k\in I})\,\mathrm{d}x\mathrm{d}z.\end{aligned}$$

We can now write the jump stiffness matrix as

$$\mathbf{A}^J_{(\mathbf{l}',\mathbf{k}'),(\mathbf{l},\mathbf{k})} = \sum_{i=1}^{d}\sum_{\substack{|I|=j\\ I_1<\ldots<I_j}}\mathbf{A}^I_{(\mathbf{l_I}',\mathbf{k}'_\mathbf{I}),(\mathbf{l_I},\mathbf{k_I})}\prod_{i\in I^c}M^i_{(l'_i,k'_i),(l_i,k_i)}.$$

The matrix $\mathbf{A}^J$ is densely populated due to the non-local character of the operator A^J.

6.4 Wavelet compression

Compression schemes of the matrix $\mathbf{A}^J$ aim at reducing of the complexity of the computation to essentially the complexity of discretizations for generators of diffusion processes (as the appear, e.g., in the Black-Scholes model). This can be achieved by defining an appropriate approximation matrix $\widetilde{\mathbf{A}}^J$ corresponding to a bilinear form $\widetilde{a}(u,v)$. The analysis of compression schemes for high-dimensional anisotropic Lévy type operators was done by [64, 65, 66]. Processes with state spaces in $\mathbb{R}^1$ whose generators are Sobolev spaces of variable order were treated by [77]. The analysis in [77] can be extended to processes with state spaces in $\mathbb{R}^1$ whose generators are Sobolev spaces of variable order. The extension of norm equivalences for generators of processes in $\mathbb{R}^d$ with domains that are *Sobolev spaces of anisotropic, variable orders* is in progress.

7 Computational scheme

7.1 Time discretization

In order to obtain a fully discrete approximation (in space and time) to the parabolic problem (5.7), we have to discretize the semi-discrete formulation in time. This can be done for example via discontinuous Galerkin time stepping as in [79] or by the θ-scheme. We will present the preconditioning for the θ-scheme in more detail. Multilevel preconditioning in the implementation of dG-time stepping is analogous to [77, Section 6.3.2].

At each time step, we need to solve a linear system

$$(\mathbf{M}+\theta\Delta t\mathbf{A})\underline{u}_L^{m+1} = (\mathbf{M}-(1-\theta)\Delta t\mathbf{A})\underline{u}_L^m,$$

at each time step $m=0,\dots,M-1$, with $\underline{u}_L^0 = \underline{u}_{L,0}$, where $\underline{u}_L^m$ denotes the coefficient vector of $u_L(t_m,\cdot)$, $\mathbf{M}$ the mass matrix and $\mathbf{A}$ the stiffness matrix in the corresponding basis. For the iterative solution of these systems we use multilevel preconditioning obtained through the wavelet norm equivalences. We obtain for $u\in V_{L+1}$ with coefficient vector $\underline{u}$

$$|\underline{u}|^2 \lesssim (\underline{u},\mathbf{M}\underline{u}) \lesssim |\underline{u}|^2,$$

due to (6.16). We denote by $\mathbf{D}_A$ the diagonal matrix with entries $2^{2\underline{m}^1_{\lambda_1} l_1}+\ldots+2^{2\underline{m}^d_{\lambda_d} l_d}$. Then we obtain, from (6.16) and the well-posedness:

$$(\underline{u},\mathbf{D}_A\underline{u}) \lesssim (\underline{u},\mathbf{A}\underline{u}) \lesssim (\underline{u},\mathbf{D}_A\underline{u}).$$

Thus, we have

$$(\underline{u},\mathbf{D}\underline{u}) \lesssim (\underline{u},\mathbf{B}\underline{u}) \lesssim (\underline{u},\mathbf{D}\underline{u}),$$

with $\mathbf{D}=\mathbf{I}+\theta\Delta t\mathbf{D}_A$ and $\mathbf{B}=\mathbf{M}+\theta\Delta t\mathbf{A}$. Finally we obtain for $\widehat{\underline{u}}=\mathbf{D}^{1/2}\underline{u}$:

$$|\widehat{\underline{u}}|^2 \lesssim \left(\widehat{\underline{u}},\mathbf{D}^{-1/2}\mathbf{B}\mathbf{D}^{-1/2}\widehat{\underline{u}}\right) \lesssim |\widehat{\underline{u}}|^2.$$

Therefore, we can iteratively solve the linear system $\widehat{\mathbf{B}}\widehat{\underline{u}}=\widehat{b}$ with GMRES in a number of steps that is independent of the level index L, where $\widehat{\mathbf{B}}=\mathbf{D}^{-1/2}\mathbf{B}\mathbf{D}^{-1/2}$ and $\widehat{\underline{b}}=\mathbf{D}^{-1/2}\widehat{\underline{b}}$. The discretization of (5.22) can be performed in a similar fashion with $\theta=1$, i.e., an implicit Euler scheme, and leads to the following system to be solved at every timestep:

$$(\mathbf{M}+\theta\Delta t\mathbf{A})\underline{u}_L^{m+1} \geq (\mathbf{M}-(1-\theta)\Delta t\mathbf{A})\underline{u}_L^m, \tag{7.1}$$

$$(\underline{u}_L^{m+1}-\underline{\widetilde{g}}_L)^\top\left(\mathbf{M}(\underline{u}_L^{m+1}-\underline{u}_L^m)+(\Delta t\mathbf{A})(\theta\underline{u}_L^{m+1}-(1-\theta)\underline{u}_L^m)\right)=0 \tag{7.2}$$

with $\underline{u}_L^0=\underline{u}_{L,0}$ and $\underline{u}_L^{m+1}\in\underline{\mathrm{K}}^{m+1}$, $\underline{\mathrm{K}}^{m+1}:=\{\underline{u}\in\mathbb{R}^{N_L}|\underline{u}\geq\underline{\widetilde{g}}^{m+1}\}$, where $\underline{\widetilde{g}}^m$ denotes the coefficient vector of $\widetilde{g}_{t_m}$ in the corresponding basis. The system (7.1)-(7.2) can be solved using the projected successive over-relaxation (PSOR) method, for example. The use of PSOR for a hierarchical discretization is not optimal as the sign of the coefficient vector entries is crucial. The discretization of (5.25) can be carried out analogously. Using this method we do not have to rely on the PSOR method and can exploit the hierarchical structure of the basis.

Remark 7.1. An analogous discretization scheme can be applied for time-inhomo- geneous problems, cf. Section 2.2. Using an implicit Euler scheme convergence of the solution of the discretized problem to the weak solution (in an appropriate sense) can be shown under weak smoothness assumptions on the time-dependence of the coefficients.

7.2 Numerical quadratures

As seen in the previous section we have to compute matrix entries of the form:

$$\mathbf{A}_{(\mathbf{l}',\mathbf{k}'),(\mathbf{l},\mathbf{k})} = \int_{\mathbb{R}^d}\int_{D_R} \partial_1 \ldots \partial_d \psi_{\mathbf{l},\mathbf{k}}(x+y)\psi_{\mathbf{l}',\mathbf{k}'}(x)\kappa(x,y)\mathrm{d}x\mathrm{d}y \tag{7.3}$$

We consider the following class of function. The kernels we consider fall into this class due to Theorem 4.1 and Lemma 4.2.

Assumption 7.2 *Let* $f \in L^1([0,1]^d \times [0,1]^d)$. *There exist* $0 < \alpha < d$, $\alpha \notin \mathbb{N}$, $C > 0$, $\delta \in (0,1)$, *such that for* $k,m \in \mathbb{N}_0$, $i = 1,\ldots,d$

$$\left|\partial^k_{\xi_i}\partial^m_{x_j} f(x,\xi)\right| \lesssim k!m!C^{k+m}\,\|\xi\|_\infty^{-\alpha}\,\xi_i^{-k}\xi_j^{-\delta}, \quad \forall \xi, x \in (0,1)^d. \tag{7.4}$$

We are now able to prove the exponential convergence in the number of quadrature points of a quadrature rule for the matrix entries $\mathbf{A}_{(\mathbf{l}',\mathbf{k}'),(\mathbf{l},\mathbf{k})}$. Therefore we denote the Gauss-Legendre integration rule on $[0,1]$ by $Q_g^{[0,1]} f = \sum_{j=1}^g \omega_{g,j} f(\xi_{g,j})$ and obtain the following error estimate for $f \in C^{2g}([0,1])$ using Stirling's formula:

$$\left|E_g^{[0,1]} f\right| := \left|I^{[0,1]} f - Q_g^{[0,1]} f\right| \lesssim \frac{2^{-4g}}{(2g)!} \max_{\xi\in[0,1]} \left|f^{(2g)}(\xi)\right|.$$

In the multidimensional case we obtain a similar error estimate for $f \in C^{2g}([0,1]^d)$ using [85, Lemma 5.1.1.]

$$\left|E_g^{[0,1]^d} f\right| \lesssim \frac{2^{-4g}}{(2g)!} \sum_{i=1}^d \max_{\xi\in[0,1]^d} \left|\partial_i^{(2g)} f(\xi)\right|. \tag{7.5}$$

We are now able to define a composite quadrature rule as in [81]. Let a geometric partition on $[0,1]$ be given by $0 < \sigma^n < \sigma^{n-1} < \ldots < \sigma < 1$, for $n \in \mathbb{N}$, $\sigma \in (0,1)$. We denote the subdomains by $\Lambda_j := [\sigma^{n+1-j}, \sigma^{n-j}]$, with $j = 1,\ldots,n$ and $\Lambda_0 = [0,\sigma^n]$. Given a linear degree vector $q \in \mathbb{N}^d$ and $q_j = \lceil \mu j \rceil$ with slope $\mu > 0$, we use on each subdomain Λ_j, $j = 1,\ldots,n$ a Gauss quadrature with degree q_j and no quadrature points in Λ_0. The composite Gauss quadrature rule is defined by

$$Q_\sigma^{n,q} f = \sum_{j=1}^n Q_{q_j}^{\Lambda_j} f$$

and exponential convergence can be proven.

Theorem 7.3. *Let f satisfy (7.4). consider*

$$\sigma \in (0,1) \quad \textit{such that } w = \frac{C(1-\sigma)}{4\sigma} < 1, \tag{7.6}$$

and linear degree vectors $(q^1,\ldots,q^d)$, $q^j = (q^j_1,\ldots,q^j_n)$,

$$q^i_j = \lceil \mu^i_j \rceil, \quad \textit{with slopes } \mu^i > \frac{(1-\frac{\alpha}{d})\ln\sigma}{2\ln w}. \tag{7.7}$$

Then we obtain for any fixed $x \in [0,1]^d$

$$\left|I^{[0,1]^d} f - Q_\sigma^{n,(q^1,\ldots,q^d)} f\right| \lesssim e^{-\gamma \sqrt[2d]{N}}. \tag{7.8}$$

Proof. The proof can be found in [85, Theorem 5.2.3].

We will use composite Gauss quadrature rules in the y-variable and standard Gauss quadratures in the x-variable.

Theorem 7.4. *We consider the following quadrature rule for an f with Property 7.2:*

$$Q = Q_\sigma^{n,(q^1,\ldots,q^d)} \otimes Q_g$$

and prove the following estimate for the error defined as

$$E[f] = \left| I^{[0,1]^d} f - Qf \right| \lesssim e^{\sqrt[2d]{N}},$$

for $g = \sqrt[8d]{N}$.

Proof.

$$E[f] = I_1^{[0,1]^d} \otimes \left(I_2^{[0,1]^d} - Q_g \right) f + \left(I_1^{[0,1]^d} - Q_\sigma^{n,(q^1,\ldots,q^d)} \right) \otimes Q_g f \tag{7.9}$$

$$\lesssim \int_{[0,1]^d} \frac{2^{-4g}}{(2g)!} \max_{x\in[0,1]^d} \left| \frac{\partial^{(2g)} f}{\partial x}(x,\xi) \right| d\xi + e^{-\gamma n} \sum_{k=1}^{d} \sum_{j=1}^{g} \omega_{g,j,k} \tag{7.10}$$

$$\lesssim e^{-4g^2} + e^{-\gamma n}. \tag{7.11}$$

The number of quadrature points N can be bounded by $N \lesssim n^{2d}$. Therefore we obtain exponential convergence in the number of quadrature points choosing $g = \sqrt[8d]{N}$.

Remark 7.5. We can obtain exponentially converging integration schemes under weaker assumptions on the integration kernel using [19].

8 Alternative pricing approaches

In the following we briefly survey on two different approaches to pricing under Feller processes.

8.1 Monte Carlo simulation

In [12] a very general framework for the simulation of paths for Feller processes is presented. For an analysis of path properties of Feller processes we refer to [72, 74, 73]. Under minimal assumptions on coefficients, they provide a method for path simulation. Their algorithm is based on an approximation of a Feller process by a Markov chain with Lévy increments. In each time step the corresponding transition operator is a PDO with constant symbol. This method presents to our knowledge the only approach to the simulation of a general Feller process. Unfortunately it is not feasible for pricing as approximation rates cannot be obtained as convergence is observed in the Skorohod topology without a rate. The algorithm consists of the following steps:

(1) Freeze the Feller symbol at $X_{t_i} = x_i$.
(2) Simulate a Lévy increment with characteristic function $a(x_i, \xi)$.
(3) Obtain an approximation at $X_{t_{i+1}}$ and go back to (1).

We illustrate the algorithm with two examples. First we consider a NIG type model that was proposed by [5].

Example 8.1. The symbol of the NIG-type Feller process is given by

$$a(x,\xi) = -i\gamma(x)\xi + \delta(x)\left[(\alpha(x)^2 - (\beta(x)+i\xi)^2)^{1/2} - (\alpha(x)^2 - \beta(x)^2)^{1/2}\right],$$

where we obtain the characteristic function of an NIG process omitting the dependence on x. The parameters of the NIG process have a nice interpretation. The parameter β describes the asymmetry, while $\alpha - \beta$ and $\alpha + \beta$ describe the rate of exponential decay at the right and left tail, δ is a scaling parameter and γ describes the drift. For details on the NIG distribution we refer to [4]. A variable function $\beta(x)$ is chosen in order to model mean-reverting behaviour, i.e.,

$$\beta(x) = -2\frac{\chi}{\pi}\arctan(\varepsilon x).$$

The other parameters are chosen to be constant. Due to properties of the NIG distribution we have to set χ s.t. $|\chi| < \alpha$ holds. NIG random numbers can be generated by sampling a standard Brownian motion and an IG process, we refer to [80] for details. Figure 2 shows 10 sample paths of this Feller NIG-type process.

Now we consider a CGMY-type model as described in Example 4.12. The simulation of CGMY distributed random numbers is not straightforward. We therefore use Fourier techniques, i.e., we apply the discrete Fourier transform to the characteristic function of the corresponding random variable and obtain an approximation of the probability distribution function. Figure 3 shows sample paths generated by this method.

8.2 Fourier pricing

A large amount of literature has been devoted to regular affine processes, we refer to the fundamental monograph [31] for a theoretical introduction. These are multidimensional jump diffusions with coefficients that depend affinely on the state variables. Due to [31, Theorem 2.7] regular affine processes are Feller processes with $C_0^\infty(\mathbb{R}^d)$ contained in the domain of the generator. The characteristic function $y \to \mathbb{E}[\exp(i\langle y, X_t\rangle)]$ is for each y available as the solution of a (multidimensional) Riccati equation. Therefore, Fourier pricing methods can be employed to price a variety of contracts. We refer to [18, 9, 33] for details and briefly describe the approach discussed in [33] for the one dimensional case, the multidimensional extension is straightforward with a more involved notation.

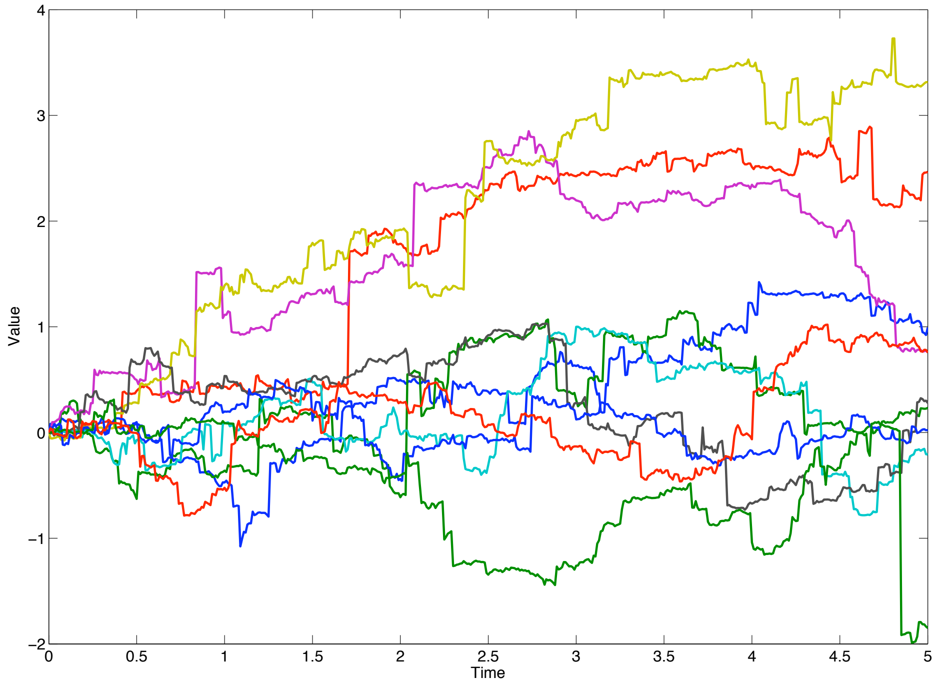


Fig. 2 Sample paths in of a Feller NIG model as in Example 8.1 with $\alpha = 2$, $\chi = 1.5$, $\delta = 1$ and $\gamma = 0$.

Consider an option with payoff $g(X_T) = \widetilde{g}(X_T + s)$, where X is the underlying regular affine process and $s = \log S_0$. For sake of notational simplicity suppose $t = 0$. Since the payoff $g(k)$ may tend to a positive constant or infinity as $k \to \pm\infty$, the Fourier transformation of $g(k)$ does not exist, in general. Therefore, instead of $g(k)$ one has to consider the damped payoff $h(k) = e^{\alpha k} g(k)$ with an appropriately chosen damping constant $\alpha \in \mathbb{R}$, such that $h \in L^1_{bc}(\mathbb{R})$, $\widehat{h} \in L^1(\mathbb{R})$ and $E[\exp(\alpha X_T)] < \infty$ holds, where $L^1_{bc}(\mathbb{R})$ denotes the space of all bounded continuous functions in $L^1(\mathbb{R})$. The option price $u(t,x)$ is then given by, cf. [33, Theorem 2.2]

$$\begin{aligned} u(0,s) &= \frac{e^{\alpha s}}{2\pi} \int_{\mathbb{R}} e^{-izs} \varphi_{X_T}(-z - i\alpha) \widehat{g}(z + i\alpha)\, \mathrm{d}z \\ &= e^{\alpha s} \mathscr{F}^{-1}[\mathscr{F}[g(y)](z) \varphi_{X_T}(-z)](s), \end{aligned} \tag{8.1}$$

where φ_{X_T} is the extended characteristic function of the random variable X_T, $\mathscr{F}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ denotes the generalized inverse Fourier transformation shifted by $i\alpha$.

Remark 8.2. The assumptions on the payoff function g can be relaxed such as to include discontinuous payoffs like barrier options, cf. [33, Remark 2.3]. Theoretically, the pricing problem could be considered for much more general market models, i.e., in a general semimartingale setting, leading to analogous pricing formulas to (8.1). But the characteristic function will not be easily available in this case, making the pricing approach non feasible. Pricing formulas for options on the supremum or infimum of the underlying process, such as lookback or one-touch options, can be derived in an analogous way, considering them as plain vanilla options on the supremum or infimum process. The (extended) characteristic function for the supremum and infimum process can be derived in the Lévy case, cf [33, Section 5].

In most cases the Fourier transformation of $g(y)$ has to be evaluated numerically and one hence has to calculate both $\mathscr{F}$ and $\mathscr{F}^{-1}$ in (8.1) numerically. In dimension $d = 1$ however the Fourier transformation of most payoffs can be obtained analytically, cf. [13, 33, 56] and only one Fourier transformation, i.e., $\mathscr{F}^{-1}$ in (8.1), has to be evaluated numerically.

We now briefly describe the discretization of the arising multidimensional Fourier integrals. The multi-

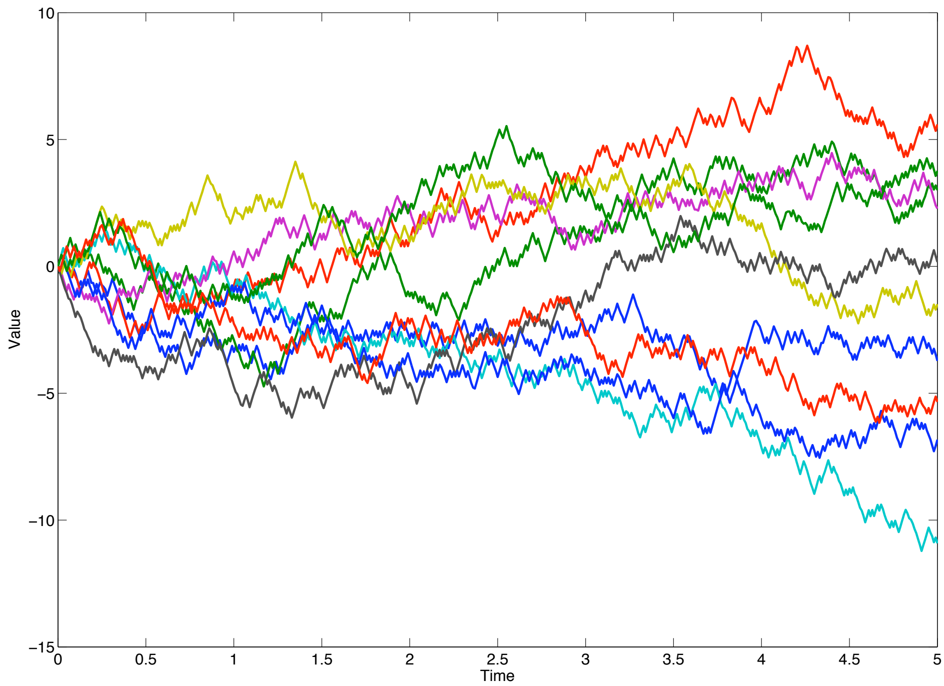


Fig. 3 Sample paths in of a Feller CGMY model as in Example 4.12 with $m(x) = \exp(-x^2) + 0.2$, $\beta^- = 9$, $\beta^+ = 7$ and $C = 1$.

dimensional discrete Fourier transform of a given series of data points $f_{\underline{j}}$ is given by the collection

$$\widehat{f}_{\underline{k}} = \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} e^{2\pi i \langle \underline{k}, \underline{j} \rangle / N} f_{\underline{j}}, \quad k_n = 0, \ldots, N-1, n = 1, \ldots, d. \tag{8.2}$$

To compute $\widehat{f}_{\underline{k}}, k_n = 0, \ldots, N-1, n = 1, \ldots, d$ one a-priori needs $O(N^{2d})$ operations. Utilizing the so-called *fast Fourier transform* [21, 62, 84] this computational cost can be reduced to $O(N^d \log N)$, we refer to [35] for details on the implementation. For instance, suppose we want to approximate the inverse Fourier transform of a function $f(z)$ with a discrete Fourier transform (to solve (8.1) one may choose $f(z) = \mathscr{F}[g(y)](z + i\alpha)\varphi_{X_T}(-z - i\alpha)$). Then, the integral can be truncated and discretized using the trapezoidal rule:

$$\begin{aligned} \mathscr{F}^{-1}[f(z)](x) = \int_{\mathbb{R}^d} e^{-i\langle x,z \rangle} f(z) \mathrm{d}z &\approx \int_{[-R,R]^d} e^{-i\langle x,z \rangle} f(z) \mathrm{d}z \\ &\approx \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} \omega_{\underline{j}} f(z_{\underline{j}}) e^{-i\langle x, z_{\underline{j}} \rangle}, \end{aligned}$$

with discretization step $\Delta z = \frac{2R}{N-1}$, $z_{j_n}^n = -R + j_n \Delta z$ in Fourier space and suitable weights $w_{\underline{j}}$, see e.g. [45]. Herewith, in order to obtain an approximate value of $u(t,x)$ in (8.1) for any $x \in \mathbb{R}^d$, we also have to discretize the spot price or x-domain $\mathbb{R}^d$. For this, we define an additional grid by setting $x_{j_n}^n = -R_2 + k_n \Delta x$ with step size $\Delta x = \frac{2R_2}{N-1}$ and given $R_2 > 0$. With the relation

$$\Delta z \cdot \Delta x = \frac{2\pi}{N} \tag{8.3}$$

we then find

$$\mathscr{F}^{-1}[f(z)](x_{\underline{k}}) \approx e^{iR\langle x_{\underline{k}},\mathbf{1}\rangle} \sum_{j_1=0}^{N-1} \cdots \sum_{j_d=0}^{N-1} e^{-i2\pi\langle \underline{k},\underline{j}\rangle/N} \omega_{\underline{j}} f(z_{\underline{j}})\, e^{iR_2\Delta z\langle \underline{1},\underline{j}\rangle} = e^{iR\langle x_{\underline{k}},\underline{1}\rangle} \widehat{f}_{\underline{k}}.$$

This expression can now be evaluated very efficiently using the fast Fourier transform as mentioned above. Also note that by (8.3) the discretization of the Fourier space and the spot price space are related and cannot be chosen independently. No time stepping is required and for $d = 1$ dimension only $O(N \log N)$ work is needed to obtain the price at N spot prices. For convergence rates and error analysis see [55].

The main advantage of Fourier based pricing in comparison with FEM based algorithms is the computation speed in certain cases. For plain vanilla contracts in market models where the characteristic function is available in closed form Fourier methods are significantly more efficient. Note that Fourier techniques can also be employed for a wide range of stochastic volatility models, cf. [58, 33]. Although many of these models fall into the class of Feller processes, we cannot directly employ the FEM methods developed above, as the symbols of these processes do not fulfill property (a) of Theorem 2.5. FEM techniques for stochastic volatility models have been considered, e.g., by [39].
Shortcomings of the Fourier approach are that the solution of the Riccati equations can be computationally expensive, if the characteristic function cannot be computed analytically, besides the pricing of exotic and early exercise contracts is not trivial in many cases. Finally, Fourier techniques are to our knowledge not feasible for general Feller processes. For details on regular affine processes we refer to [31, 32, 53].

9 Numerical examples

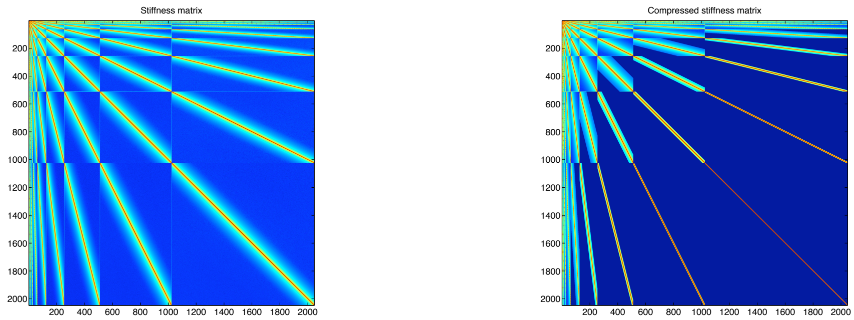
We conclude our presentation with several computational examples to illustrate the performance of our solution algorithms. All applications are taken from the pricing of derivative contracts on Lévy and Feller-Lévy market models. Our emphasis of the examples given below is to demonstrate that a) the methods handle Lévy as well as Feller-Lévy and Sato processes in a unified approach, with comparable efficiency (i.e. numerical accuracy versus computational work) and b) to present examples where the capability of our methods is applied to quantify model risk. Specifically, to assess the impact of pricing under Lévy models versus more general, Feller-Lévy models.

9.1 Univariate case

In this section we describe the implementation numerical solution methods for the Kolmogoroff equations for Feller processes taking values in $\mathbb{R}^1$ using the techniques described above. We assume the risk-neutral dynamics of the underlying asset to be given by

$$S_t = S_0 e^{rt+X_t},$$

where X is a Feller process with characteristic triple $(\gamma(x), \sigma(x), k(x,y)dy)$ under a risk neutral measure $\mathbb{Q}$ such that e^X is a martingale with respect to the canonical filtration of X. In the following we set $r = 0$ for notational convenience. We will only consider Feller processes X that are admissible market models. In



(a) Stiffness matrix (b) Compressed stiffness matrix

Fig. 4 Stiffness matrices for the pure jump case with CGMY-type Lévy kernel $\left(Y(x) = 1.25e^{-x^2} + 0.5\right)$.

(a) Stiffness matrix (b) Mass matrix

Fig. 5 Stiffness and Mass matrices for the Black-Scholes model with $\sigma = 0.3$ and $r = 0$.

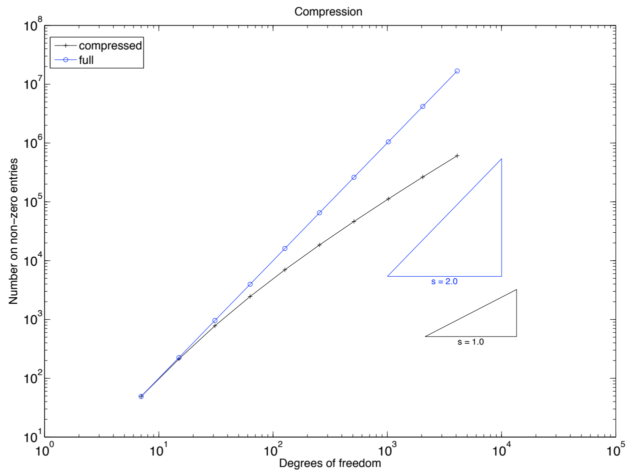


Fig. 6 Number of non-zero entries of the compressed/uncompressed stiffness matrix versus number of degrees of freedom corresponding to the Lévy kernel in Example 9.1 and $k = 1.25$.

the following we will consider a special family of Feller processes to confirm the theoretical results of the previous chapters.

Example 9.1. We consider a CGMY-type Feller process with jump kernel

$$k(x,y) = C \begin{cases} e^{-\beta^+ y} y^{-1-m(x)}, & y > 0 \\ e^{-\beta^- |y|} |y|^{-1-m(x)}, & y < 0, \end{cases} \qquad m(x) = ke^{-x^2} + 0.5.$$

This process has no Gaussian component and the drift $\gamma(x)$ is chosen according to (5.1).

We will also consider the following family of processes that do not satisfy the conditions of the theory developed above, since the variable order is assumed to be Lipschitz continuous only.

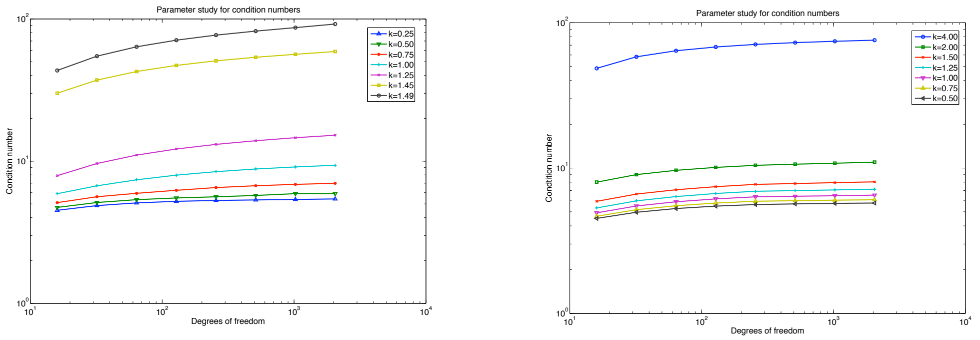
Example 9.2. We consider again a CGMY-type Feller process with jump kernel

$$k(x,y) = C \begin{cases} e^{-\beta^+ y} y^{-1-m(x)}, & y > 0 \\ e^{-\beta^- |y|} |y|^{-1-m(x)}, & y < 0, \end{cases}$$

$$m(x) = 0.5 + k \begin{cases} 0.4x, & 0.25 > x > 0 \\ 0.8x - 0.1, & 0.5 > x \geq 0.25 \\ -0.4x + 0.5, & 0.75 > x \geq 0.5 \\ -0.8x + 0.8, & 1 > x \geq 0.75 \\ 0.5, & \text{else} \end{cases}.$$

This process has no Gaussian component and the drift $\gamma(x)$ is chosen according to (5.1).

In Figure 4 the stiffness matrix for the process in Example 9.1 is depicted. Note that the uncompressed stiffness matrix is densely populated, but structurally very similar to the matrix in the Black-Scholes model. In a next step we study the number of non-zero entries of the uncompressed and compressed stiffness matrix. Due to Section 6.4 we expect essentially linear growth of the number of non-zero elements for the compressed matrix (Figure 6).
The condition numbers of the preconditioned stiffness matrices have to be uniformly bounded in the number of levels due to Section 7.1. A parameter study for various choices of k in Example 9.1 and Example 9.2 is shown in Figure 7. The condition numbers are uniformly bounded and of order 10^1 in most cases, although the norm equivalences (6.16) only apply to Example 9.1. For variable orders with $1.95 \leq \overline{m}$ we obtain condition numbers of order 10^2. Note that the condition numbers are not only influenced by the order of the singularity of the jump kernel at $z = 0$, but also by the rates of exponential decay β^+ and β^-. Fast decaying tails, i.e., large β^+ and β^- may lead to larger constants.



(a) Example 9.1 (b) Example 9.2

Fig. 7 Condition numbers for different levels and choices of k.

Figure 8 shows the price of a European put option for several Lévy processes and one Feller process. In the Feller case we choose $m(x) = 0.8e^{-x^2} + 0.1$ in Example 9.1 and for the Lévy models we set $m \in \{0.1, 0.5, 0.7, 0.8, 0.9\}$. In all cases we set $C = 1$, $\beta^+ = \beta^- = 10$ and use truncation parameters $a = -3$, $b = 3$ in log-moneyness coordinates. The prices in the Feller model are significantly different from the prices in the different Lévy models. This can be explained by the ability of the Feller model to account for different tail behaviour for different states of the process, which is not possible using Lévy processes. Figure 9 shows the prices of American put option for a Feller process and a several Lévy models. We use a

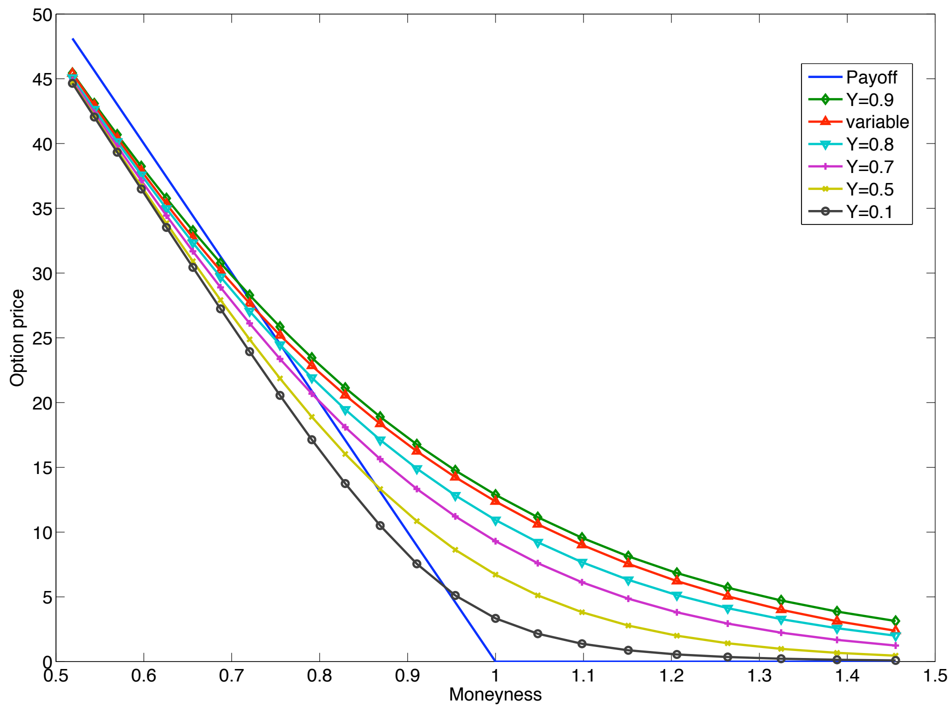


Fig. 8 Option prices for several models for a European put option with $T = 1$ and $K = 100$.

Lagrangian multiplier approach as described in Section 5.2 and refer to [43, 44] for more details, analogous results were obtained using the PSOR algorithm. The parameters were chosen as above.
We now consider model sensitivities. For the computation of sensitivities w.r. to model parameters we consider a special case of Example 9.1, i.e. $k = 0$ and therefore $Y = 0.5$ and calculate the sensitivity of

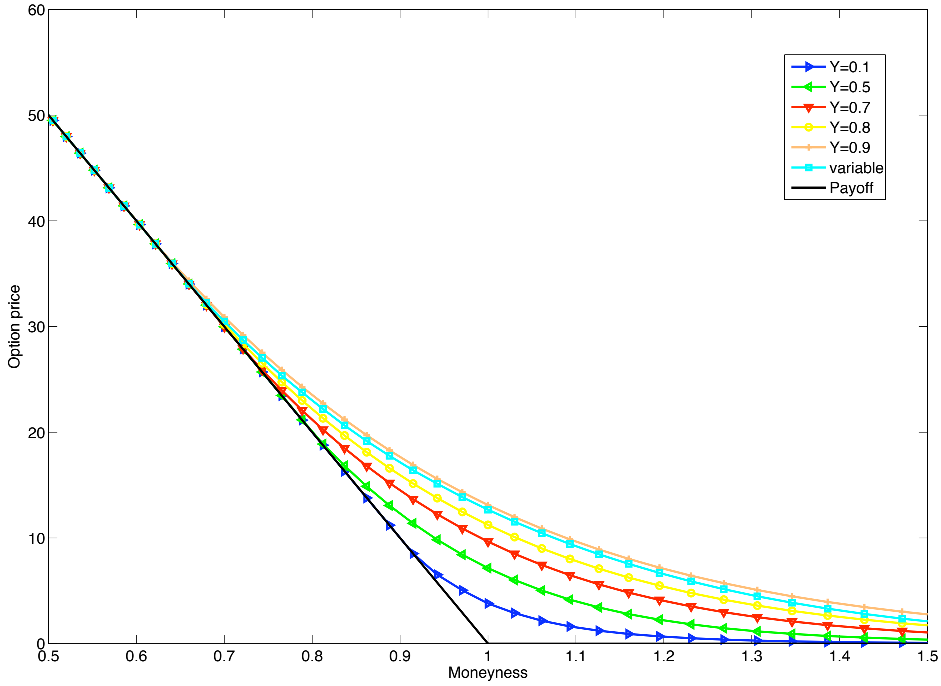


Fig. 9 Option prices for several models for an American put option with $T = 1$ and $K = 100$.

the price w.r. to the jump intensity parameter m, where we let $0 < m < 2$, the rest of the parameters being chosen as above. Then we have $\mathscr{S}_\eta = (0,2)$ with $\eta = m$ and

$$\widetilde{\mathscr{A}}(\delta m)\varphi = -\delta m \int_{\mathbb{R}} \{\varphi(x+z) - \varphi(x) - z\partial_x\varphi(x)\}\widetilde{k}(z)\mathrm{d}z \in \mathscr{L}(\widetilde{V}, \widetilde{V}^*)$$

where the kernel $\widetilde{k}$ is given by

$$\widetilde{k}(z) := -\ln|z|k(z).$$

It is easy to check that it holds $\int_{|z|\le 1} z^2\widetilde{k}(z)\mathrm{d}z < \infty$, $\int_{|z|>1}\widetilde{k}(z)\mathrm{d}z < \infty$ due to $m < 2$. In this setting, $\widetilde{V} = V = \widetilde{H}^1(D)$, if $\sigma > 0$, and $\widetilde{V} = \widetilde{H}^{m/2+\varepsilon}(D) \subset \widetilde{H}^{m/2}(D) = V, \forall \varepsilon > 0$, if $\sigma = 0$. We refer to [40] for more details. Figure 10 shows the sensitivity in this model w.r. to the parameter $\eta = m$. As expected from Figure 8, we observe a positive sensitivity which is significantly larger at the money, than deep out or in the money.

9.2 Multidimensional case

We consider a special case of the model presented in Example 4.12 with constant model parameters and no diffusion, i.e., a multidimensional pure jump Lévy model. We are interested in option prices as well as the sensitivity with respect to the copula parameter ϑ. We have $\mathscr{S}_\eta = (0,\infty)$ with $\eta = \vartheta$ and

$$\widetilde{\mathscr{A}}(\delta\vartheta) = \delta\vartheta \int_{\mathbb{R}^2} \frac{\partial^2 u}{\partial y_1 \partial y_2}(x+y)F_\vartheta(U_1(y_1), U_2(y_2))\,dy, \tag{9.1}$$

where F_ϑ is given by:

$$F_\vartheta(u) = \frac{1}{\vartheta^2}F(u)\left(\ln\left(\sum_{i=1}^d |u_i|^{-\vartheta}\right) + \frac{\vartheta\sum_{i=1}^d |u_i|^{-\vartheta}\ln|u_i|}{\sum_{i=1}^d |u_i|^{-\vartheta}}\right).$$

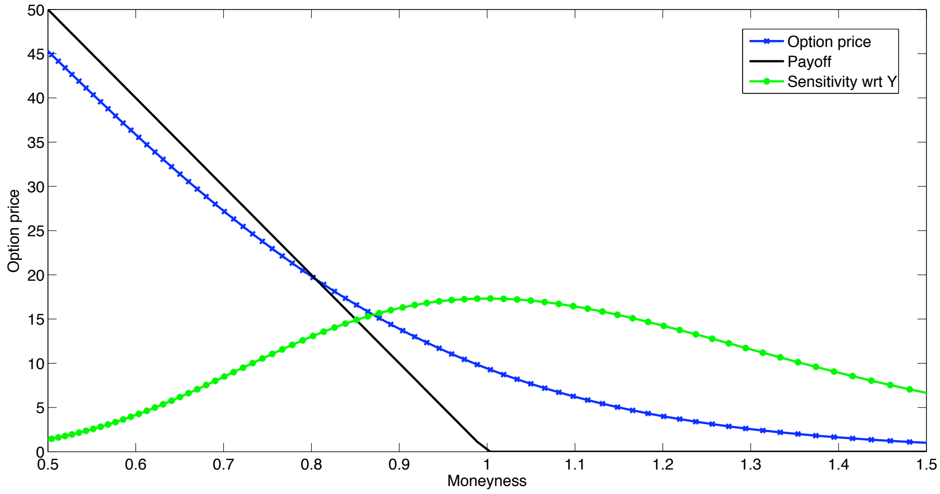


Fig. 10 Computed sensitivity of a European put w.r.t. the jump intensity parameter m in the CGMY model.

The following parameters were chosen $C = 1$, $\beta^- = [10, 9]$, $\beta^+ = [15, 16]$, $m = [0.5, 0.7]$, $\vartheta = 0.5$, $\rho = 0.5$. The option price is depicted in Figure 11 and the sensitivity is depicted in Figure 12.

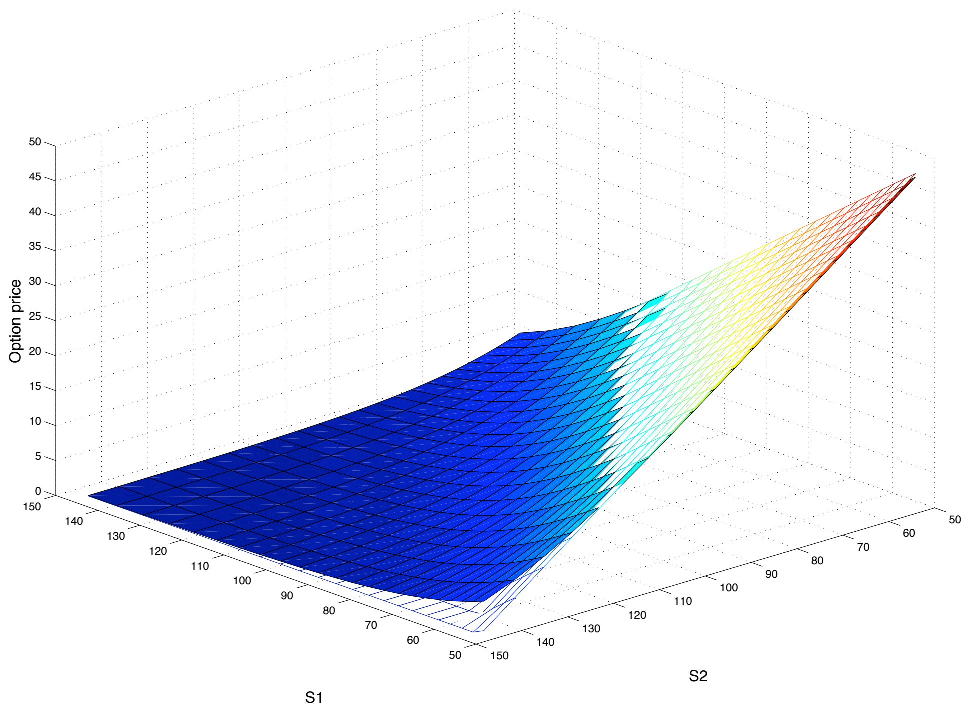


Fig. 11 European put basket price with payoff $g(S_1, S_2) = (K - 0.5S_1 - 0.5S_2)$ in a multidimensional CGMY model with Clayton copula with $K = 100$ and $T = 1$.

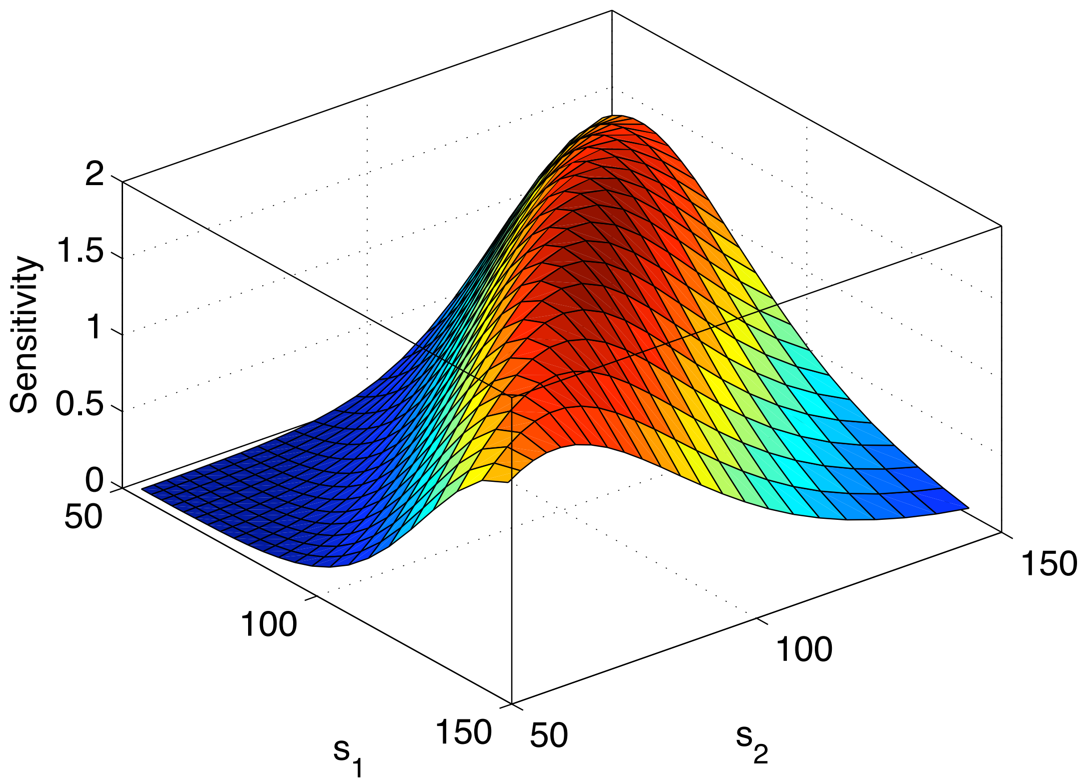


Fig. 12 Sensitivity with respect to the copula parameter ϑ of a European put basket price with payoff $g(S_1, S_2) = (K - 0.5S_1 - 0.5S_2)$ in a multidimensional CGMY model with Clayton copula with $K = 100$ and $T = 1$..

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