

hp-dGFEM for Second-Order Elliptic
Problems in Polyhedra
II: Exponential Convergence

D. Schötzau*, C. Schwab and T. P. Wihler[†]

Revised: January 2012

Research Report No. 2009-29
October 2009

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*Mathematics Department, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada. This author was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[†]Mathematisches Institut, Universität Bern, 3012 Bern, Switzerland.

HP-DGFEM FOR SECOND ORDER ELLIPTIC PROBLEMS IN POLYHEDRA II: EXPONENTIAL CONVERGENCE *

D. SCHÖTZAU[†], C. SCHWAB[‡], AND T. P. WIHLE[§]

Abstract. The goal of this paper is to establish exponential convergence of hp -version interior penalty (IP) discontinuous Galerkin (dG) finite element methods for the numerical approximation of linear second-order elliptic boundary-value problems with homogeneous Dirichlet boundary conditions and piecewise analytic data in three-dimensional polyhedral domains. More precisely, we shall analyze the convergence of the hp -IP dG methods considered in [33] based on axiparallel σ -geometric anisotropic meshes and anisotropic polynomial degree distributions of μ -bounded variation.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be an open bounded polyhedron with Lipschitz boundary $\Gamma = \partial\Omega$ that consists of a finite union of plane faces. We consider the Dirichlet problem for the diffusion-reaction equation

$$\mathbf{L}u \equiv -\nabla \cdot (\mathbf{A}\nabla u) + cu = f \quad \text{in } \Omega, \quad (1.1)$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \quad (1.2)$$

where $\mathbf{A} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ is a symmetric positive definite coefficient matrix which is independent of the space coordinate \mathbf{x} and $c \geq 0$ is a given, constant reaction rate. Then, for every $f \in H^{-1}(\Omega)$, the boundary-value problem (1.1)–(1.2) admits a unique solution $u \in H_0^1(\Omega)$. If f is analytic in $\bar{\Omega}$, then u is analytic in Ω away from the singular parts of the boundary (i.e., away from edges and corners of Ω).

The hp -version of the finite element method (FEM) for the numerical solution of elliptic problems was proposed in the mid 80ies by Babuška and his coworkers. Exponential convergence rates $\exp(-b\sqrt{N})$ with respect to the number of degrees of freedom N for the hp -version of the FEM in one dimension were shown by Babuška and Gui in [13] for the model singular solution $u(x) = x^\alpha - x \in H_0^1(\Omega)$ in $\Omega = (0, 1)$, inspired by exponential convergence results in free-knot, variable order spline interpolation, e.g. [10, 28] and the references there. This result required σ -geometric meshes with a fixed subdivision ratio $\sigma \in (0, 1)$ (in particular, for $\sigma = 1/2$ geometric element sequences Ω_i are obtained by successive element bisection towards $x = 0$) while the constant b in the convergence estimate $\exp(-b\sqrt{N})$ depends on the singularity exponent α as well as on σ . Among all $\sigma \in (0, 1)$, the optimal value was shown to be $\sigma_{\text{opt}} = (\sqrt{2} - 1)^2 \approx 0.17$, see [13, Theorem 3.2], provided that the geometric mesh refinement is combined with nonuniform polynomial degrees $p_i \geq 1$ in Ω_i which are s -linear, i.e., $p_i \sim si$, with the optimal slope s being $s_{\text{opt}} = 2(\alpha - 1/2)$. In this case, the finite element error converges as $\exp(-b\sqrt{N})$ where $b = 1.76 \dots \times \sqrt{(\alpha - 1/2)}$. For the bisected geometric mesh where $\sigma = 1/2$ and for linear polynomial degree distributions with slope $s_{\text{opt}} = 0.39 \dots \times (\alpha - 1/2)$, one has $b = 1.5632 \dots \times \sqrt{(\alpha - 1/2)}$,

*This work was initiated during the workshop “Adaptive numerical methods and simulation of PDEs”, held from January 21-25, 2008, at the Wolfgang Pauli Institute in Vienna, Austria. This work was supported in part under the FP7 programme of the EU under grant No. ERC AdG 247277, by the Swiss National Science foundation under grant No. 200021_126594 and by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[†]Mathematics Department, University of British Columbia, Vancouver, BC V6T 1Z2, Canada, (schoetzau@math.ubc.ca). This author was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

[‡]Seminar for Applied Mathematics, ETH Zürich, 8092 Zürich, Switzerland (schwab@math.ethz.ch).

[§]Mathematisches Institut, Universität Bern, 3012 Bern, Switzerland (wihler@math.unibe.ch).

whereas for $\sigma = 1/2$ and uniform polynomial degree, $b = 1.1054\dots \times \sqrt{(\alpha - 1/2)}$; see [13, Table 1].

In two dimensions, exponential convergence (i.e., an upper bound of the form $C \exp(-b\sqrt[3]{N})$ on the error) for the hp -version FEM in polygons was obtained by Babuška and Guo in the mid 80ies in a series of landmark papers ([4, 16, 17] and the references therein). Key ingredients in the proof were *geometric mesh refinement* towards the singular support \mathcal{S} (the set of vertices of Ω) of the solution and nonuniform elemental polynomial degrees which increase linearly with the elements' distance to \mathcal{S} .

Starting in the 90ies, steps were undertaken to extend the analytic regularity and the a-priori error analysis of hp -FEM in [4, 16, 17] to polyhedra in \mathbb{R}^3 in [5, 15, 18, 19]; see also [24] and the references therein for recent related results. While these works were devoted to conforming hp -FEM with *isotropic elemental polynomial spaces* for second-order elliptic problems, extensions to hp -version mixed methods and conforming methods for higher-order problems in polygons were obtained in [14, 29].

Discontinuous Galerkin (dG) FEM emerged in the 70ies as stable discretizations of first-order transport-dominated problems (see [22, 23, 27]), and as nonconforming discretizations of second-order elliptic problems (cf. [1, 6, 11, 25, 35]). In the 90ies, dG methods were studied within the hp -version setting for first-order transport and for advection-reaction-diffusion problems in two- and three-dimensional domains (see, e.g., [20, 21]). There, exponential convergence rates were established for *piecewise analytic solutions* excluding, in particular, corner singularities as occurring in polygonal domains. In that context, exponential convergence was proved in [36, 37] for diffusion problems and in [34] for the Stokes equations; see also [12, 26, 38] for the analysis of dG methods under low regularity conditions.

This paper is a continuation of our work on hp -dGFEM for elliptic problems in polyhedral domains in [33], where we have shown the well-posedness and stability of hp version interior penalty (IP) discontinuous Galerkin discretizations of (1.1)–(1.2). Our analysis there covers solutions of (1.1)–(1.2) which exhibit typical corner and edge singularities in polyhedra and belong to suitably weighted Sobolev spaces. For such solutions, we have proved in [33] that hp -IP dG discretizations are well-defined and consistent for appropriate combinations of σ -geometric meshes (obtained from mapped hexahedral elements) and anisotropic elemental polynomial degrees (of μ -bounded variation). In addition, the hp -dG approximations satisfy the Galerkin orthogonality property and an abstract bound of the error measured in the dG energy norm.

In this paper, we shall prove that exponential convergence rates $\exp(-b\sqrt[5]{N})$ in terms of the number of degrees of freedom N can be obtained for hp -IP dGFEM discretizations of (1.1)–(1.2) on axiparallel σ -geometric meshes with anisotropic elemental polynomial degrees (of μ -bounded variation). Our hp -version error analysis covers, in particular, *three-dimensional generalizations of all mesh-degree combinations found to be optimal in the univariate case* in [13, Table 1], i.e., subdivision ratios $\sigma \neq 1/2$ and nonuniform polynomial degree distributions which are possibly anisotropic within each hexahedral element. Based on the one-dimensional analysis in [13], we expect that this flexibility can be used to increase the value of the constant b in the exponential convergence bound.

The outline of the article is as follows: In Section 2, we recapitulate regularity results in countably normed Sobolev spaces for the solution of (1.1)–(1.2) from [8], extending the pioneering work [3, 4] in two dimensions to the three-dimensional case. In Section 3, we define hp -dG finite element spaces on σ -*geometric axiparallel* meshes

with possibly anisotropic *polynomial degree distributions of μ -bounded variation*. Section 4 recalls the stability and quasi-optimality results of *hp*-version IP dG discretizations obtained in [33]. Furthermore, Section 5 is devoted to *hp*-interpolation estimates in the interior domains as well as in the elements abutting the singular support of the solution. Section 6 states and proves the exponential convergence of *hp*-dGFEM in \mathbb{R}^3 . Furthermore, we present some concluding remarks in Section 7.

Standard notation will be employed throughout the paper. The number of elements in a set A of finite cardinality is denoted by $|A|$. In Section 5 the function

$$\Psi_{q,r} = \frac{\Gamma(q+1-r)}{\Gamma(q+1+r)}, \quad 0 \leq r \leq q, \quad (1.3)$$

shall be used frequently, where Γ is the Gamma function satisfying $\Gamma(n+1) = n!$ for any $n \in \mathbb{N}$. Occasionally, we shall use the notations " \lesssim " or " \simeq " to mean an inequality or an equivalence containing generic positive multiplicative constants independent of any local mesh sizes and polynomial degrees. In addition, we shall use in several places notations and results from [33] which will only be briefly mentioned.

2. Regularity. Under our assumptions on the coefficients \mathbf{A} , c and the source term f in (1.1) are analytic in $\bar{\Omega}$, the weak solution $u \in H_0^1(\Omega)$ of (1.1)–(1.2) is analytic away from any corners and edges of Ω . To establish exponential convergence of *hp*-dGFEM, it is necessary to specify its precise regularity in countably normed weighted Sobolev spaces. To that end, we essentially follow [8], based on the notation already introduced in [33]. We also refer to the monograph [7]. In addition, we mention the papers [15, 18, 19] where alternative definitions of countably normed weighted Sobolev spaces in terms of local spherical coordinates have originally been defined and studied.

2.1. Subdomains and Weights. We denote by \mathcal{C} the set of corners \mathbf{c} , and by \mathcal{E} the set of open¹ edges \mathbf{e} of Ω . The singular support is given by

$$\mathcal{S} = \left(\bigcup_{\mathbf{c} \in \mathcal{C}} \mathbf{c} \right) \cup \left(\bigcup_{\mathbf{e} \in \mathcal{E}} \mathbf{e} \right) \subset \Gamma. \quad (2.1)$$

For smooth data \mathbf{A} , c and f in $\bar{\Omega}$, the set \mathcal{S} coincides with the singular support of the solution u of (1.1)–(1.2). For $\mathbf{c} \in \mathcal{C}$, $\mathbf{e} \in \mathcal{E}$ and $\mathbf{x} \in \Omega$, we define the following distance functions:

$$r_{\mathbf{c}}(\mathbf{x}) = \text{dist}(\mathbf{x}, \mathbf{c}), \quad r_{\mathbf{e}}(\mathbf{x}) = \text{dist}(\mathbf{x}, \mathbf{e}), \quad \rho_{\mathbf{c}\mathbf{e}}(\mathbf{x}) = r_{\mathbf{e}}(\mathbf{x})/r_{\mathbf{c}}(\mathbf{x}). \quad (2.2)$$

We *assume* that vertices are separated:

$$\exists \varepsilon(\Omega) > 0 : \quad \bigcap_{\mathbf{c} \in \mathcal{C}} B_{\varepsilon}(\mathbf{c}) = \emptyset, \quad (2.3)$$

where $B_{\varepsilon}(\mathbf{c})$ denotes the open ball in \mathbb{R}^3 with center \mathbf{c} and radius ε . For each corner $\mathbf{c} \in \mathcal{C}$, we define by $\mathcal{E}_{\mathbf{c}} = \{\mathbf{e} \in \mathcal{E} : \mathbf{c} \cap \bar{\mathbf{e}} \neq \emptyset\}$ the set of all edges of Ω which meet at \mathbf{c} . For any $\mathbf{e} \in \mathcal{E}$, the set of corners of \mathbf{e} is given by $\mathcal{C}_{\mathbf{e}} \equiv \partial\mathbf{e} = \{\mathbf{c} \in \mathcal{C} : \mathbf{c} \cap \bar{\mathbf{e}} \neq \emptyset\}$.

¹In this paper, all geometric objects (except points, but including, e.g., subdomains, faces, edges, elements) are assumed to be open, unless explicitly stated otherwise.

Then, for $\mathbf{c} \in \mathcal{C}$, $\mathbf{e} \in \mathcal{E}$ and $\mathbf{e}_c \in \mathcal{E}_c$, we define

$$\begin{aligned}\omega_c &= \{ \mathbf{x} \in \Omega : r_c(\mathbf{x}) < \varepsilon \wedge \rho_{c\mathbf{e}}(\mathbf{x}) > \varepsilon \quad \forall \mathbf{e} \in \mathcal{E}_c \}, \\ \omega_e &= \{ \mathbf{x} \in \Omega : r_e(\mathbf{x}) < \varepsilon \wedge r_c(\mathbf{x}) > \varepsilon \quad \forall \mathbf{c} \in \mathcal{C}_e \}, \\ \omega_{c\mathbf{e}_c} &= \{ \mathbf{x} \in \Omega : r_c(\mathbf{x}) < \varepsilon \wedge \rho_{c\mathbf{e}_c}(\mathbf{x}) < \varepsilon \}.\end{aligned}$$

When clear from the context, we write $\omega_{c\mathbf{e}}$ instead of $\omega_{c\mathbf{e}_c}$. Possibly by reducing ε in (2.3), we may partition the domain Ω into four *disjoint* parts,

$$\overline{\Omega} = \overline{\Omega_0 \dot{\cup} \Omega_{\mathcal{C}} \dot{\cup} \Omega_{\mathcal{E}} \dot{\cup} \Omega_{\mathcal{C}\mathcal{E}}}, \quad (2.4)$$

where

$$\Omega_{\mathcal{C}} = \bigcup_{\mathbf{c} \in \mathcal{C}} \omega_{\mathbf{c}}, \quad \Omega_{\mathcal{E}} = \bigcup_{\mathbf{e} \in \mathcal{E}} \omega_{\mathbf{e}}, \quad \Omega_{\mathcal{C}\mathcal{E}} = \bigcup_{\mathbf{c} \in \mathcal{C}} \bigcup_{\mathbf{e} \in \mathcal{E}_c} \omega_{c\mathbf{e}}. \quad (2.5)$$

We shall refer to the subdomains $\Omega_{\mathcal{C}}$, $\Omega_{\mathcal{E}}$ and $\Omega_{\mathcal{C}\mathcal{E}}$ as *corner*, *edge* and *corner-edge neighborhoods* of Ω , respectively, and the remaining *interior part* of the domain Ω is defined by $\Omega_0 := \Omega \setminus \Omega_{\mathcal{C}} \cup \Omega_{\mathcal{E}} \cup \Omega_{\mathcal{C}\mathcal{E}}$.

2.2. Weighted Sobolev Spaces. To each $\mathbf{c} \in \mathcal{C}$ and $\mathbf{e} \in \mathcal{E}$ we associate a corner and an edge exponent $\beta_{\mathbf{c}}, \beta_{\mathbf{e}} \in \mathbb{R}$, respectively. We collect these quantities in the multi-exponent

$$\boldsymbol{\beta} = \{ \beta_{\mathbf{c}} : \mathbf{c} \in \mathcal{C} \} \cup \{ \beta_{\mathbf{e}} : \mathbf{e} \in \mathcal{E} \} \in \mathbb{R}^{|\mathcal{C}|+|\mathcal{E}|}. \quad (2.6)$$

Inequalities of the form $\boldsymbol{\beta} < 1$ and expressions like $\boldsymbol{\beta} \pm s$, where $s \in \mathbb{R}$, are to be understood componentwise. For example, $\boldsymbol{\beta} + s = \{ \beta_{\mathbf{c}} + s : \mathbf{c} \in \mathcal{C} \} \cup \{ \beta_{\mathbf{e}} + s : \mathbf{e} \in \mathcal{E} \}$.

At the heart of exponential convergence analysis of *hp*-approximations in three dimensions is the analytic regularity of the solution u of (1.1)–(1.2) near the edges \mathcal{E} of Ω . In order to describe it, we recall from [33], for corners $\mathbf{c} \in \mathcal{C}$ and edges $\mathbf{e} \in \mathcal{E}$, the local coordinate systems in $\omega_{\mathbf{e}}$ and $\omega_{c\mathbf{e}}$ which are chosen such that \mathbf{e} corresponds to the direction $(0, 0, 1)$. Then, we denote quantities that are transversal to \mathbf{e} by $(\cdot)^\perp$, and quantities parallel to \mathbf{e} by $(\cdot)^\parallel$. In particular, if $\boldsymbol{\alpha} \in \mathbb{N}_0^3$ is a multi-index corresponding to the three local coordinate directions in a subdomain $\omega_{\mathbf{e}}$ or $\omega_{c\mathbf{e}}$, then we have $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^\perp, \alpha^\parallel)$, where $\boldsymbol{\alpha}^\perp = (\alpha_1, \alpha_2)$ and $\alpha^\parallel = \alpha_3$. Likewise notation shall be employed below in anisotropic quantities related to a face. Following [8, Definition 6.3], we introduce the anisotropically weighted semi-norm

$$\begin{aligned}|u|_{M_{\boldsymbol{\beta}}^m(\Omega)}^2 &= |u|_{H^m(\Omega_0)}^2 + \sum_{\mathbf{e} \in \mathcal{E}} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^3 \\ |\boldsymbol{\alpha}|=m}} \|r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}^\perp}|} \mathbf{D}^{\boldsymbol{\alpha}} u\|_{L^2(\omega_{\mathbf{e}})}^2 \\ &+ \sum_{\mathbf{c} \in \mathcal{C}} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^3 \\ |\boldsymbol{\alpha}|=m}} \left(\|r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|} \mathbf{D}^{\boldsymbol{\alpha}} u\|_{L^2(\omega_{\mathbf{c}})}^2 + \sum_{\mathbf{e} \in \mathcal{E}_c} \|r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|} \rho_{c\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}^\perp}|} \mathbf{D}^{\boldsymbol{\alpha}} u\|_{L^2(\omega_{c\mathbf{e}})}^2 \right),\end{aligned} \quad (2.7)$$

for $m \in \mathbb{N}_0$, and define the norm $\|\cdot\|_{M_{\boldsymbol{\beta}}^m(\Omega)}$ by $\|u\|_{M_{\boldsymbol{\beta}}^m(\Omega)}^2 = \sum_{k=0}^m |u|_{M_{\boldsymbol{\beta}}^k(\Omega)}^2$. Here, $|u|_{H^m(\Omega_0)}$ is the usual Sobolev semi-norm of order m on Ω_0 , and the operator $\mathbf{D}^{\boldsymbol{\alpha}}$ denotes the derivative in the local coordinate directions corresponding to the multi-index $\boldsymbol{\alpha}$. Finally, $M_{\boldsymbol{\beta}}^m(\Omega)$ is the weighted Sobolev space obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{M_{\boldsymbol{\beta}}^m(\Omega)}$. For subdomains $K \subset \Omega$ we shall denote by $|\cdot|_{M_{\boldsymbol{\beta}}^m(K)}$ the semi-norm (2.7) with all domains of integration replaced by their intersections with $K \subset \Omega$ and likewise also for $\|\cdot\|_{M_{\boldsymbol{\beta}}^m(K)}$.

2.3. Analytic Regularity. It is classical that the weak solution u of (1.1) is analytic in Ω and admits analytic continuations to analytic parts of the boundary. Additionally, it is also well known that analyticity may be lost near \mathcal{S} . The anisotropically weighted Sobolev spaces defined above allow us to specify precisely this possible loss. From [8, Theorem 6.8], we have the following shift theorem.

PROPOSITION 2.1. *There exist bounds $\beta_{\mathcal{E}}, \beta_{\mathcal{C}} > 0$ (depending on Ω and the coefficients in (1.1)) such that, for β satisfying*

$$0 < \beta_e < \beta_{\mathcal{E}}, \quad 0 < \beta_c < \frac{1}{2} + \beta_{\mathcal{C}}, \quad e \in \mathcal{E}, \quad c \in \mathcal{C}, \quad (2.8)$$

and for every $m \in \mathbb{N}_0$, the solution $u \in H_0^1(\Omega)$ of (1.1)–(1.2) with $\mathbf{L}u \in M_{1-\beta}^m(\Omega)$ fulfills $u \in M_{1-\beta}^m(\Omega)$. Furthermore, for every $m \in \mathbb{N}_0$ there exists a constant $C_m > 0$ (independent of f and u) such that there holds the a-priori estimate

$$\|u\|_{M_{1-\beta}^m(\Omega)} \leq C_m \|\mathbf{L}u\|_{M_{1-\beta}^m(\Omega)}. \quad (2.9)$$

Based on (2.9) and [8, Definition 6.4] (cf. also [7], as well as the recent work [24], in which all the weights are expressed in terms of r_e), for $\gamma \in \mathbb{R}^{|\mathcal{C}|+|\mathcal{E}|}$ we consider the countably normed spaces of piecewise analytic functions:

$$A_{\gamma}(\Omega) = \left\{ v \in \bigcap_{m \geq 0} M_{\gamma}^m(\Omega) : \exists C_v > 0 \text{ s.t. } |v|_{M_{\gamma}^m(\Omega)} \leq C_v^{m+1} m! \forall m \in \mathbb{N}_0 \right\}. \quad (2.10)$$

Then, the following shift theorem can be found in [8, Corollary 7.9] (see also [7]):

PROPOSITION 2.2. *If $f \in A_{1-\beta}(\Omega)$ in (1.1) for some $\beta \in \mathbb{R}^{|\mathcal{C}|+|\mathcal{E}|}$ satisfying (2.8) with $\beta_{\mathcal{E}}, \beta_{\mathcal{C}} < 1$, then we have $u \in A_{1-\beta}(\Omega)$.*

3. hp -Subspaces in Ω . In [33], we introduced a class of hp -dG spaces which involve three basic ingredients: families $\mathfrak{M}_{\sigma} = \{\mathcal{M}_{\sigma}^{(\ell)}\}_{\ell \geq 1}$ of σ -geometric meshes with ℓ layers of refinement in Ω , polynomial degree distributions which are nonuniform between elements and possibly anisotropic within each element but whose ratio across interfaces of hexahedral elements is μ -bounded. We gave a specific construction of such hp -space families in general Lipschitz polyhedra $\Omega \subset \mathbb{R}^3$ with a boundary consisting of a finite number of plane faces. Here, we restrict ourselves to axiparallel domains and meshes. In the sequel, we briefly recapitulate the construction of the corresponding hp -spaces in this special case, and refer to [33, Section 3] for details and proofs.

3.1. Geometric hp -Meshes in Ω . We start from any coarse regular quasiuniform partition $\mathcal{M}^0 = \{Q_j\}_{j=1}^J$ of Ω into J convex axiparallel hexahedra. Each of these hexahedral elements $Q_j \in \mathcal{M}^0$ is the image under an affine mapping G_j of the reference patch $\tilde{Q} = (-1, 1)^3$, i.e.,

$$\forall Q_j \in \mathcal{M}^0 : \quad Q_j = G_j(\tilde{Q}), \quad j = 1, \dots, J. \quad (3.1)$$

In fact, since the hexahedra $\{Q_j\}_j$ are assumed axiparallel, the mappings G_j are compositions of (isotropic) dilations and translations. The hexahedral mesh \mathcal{M}^0 obtained in this fashion is shape-regular: there exists a constant $C_{\mathcal{M}^0} \geq 1$ such that

$$C_{\mathcal{M}^0}^{-1} \leq \det(DG_j) \leq C_{\mathcal{M}^0}, \quad j = 1, \dots, J. \quad (3.2)$$

Due to our assumption that the faces of Ω are plane, it is geometrically exact.

In [33], *canonical geometric mesh patches* on the reference patch \tilde{Q} have been constructed; see Figure 3.1. Geometric meshes in Ω can then be obtained by again applying the patch mappings G_j to transform these canonical geometric mesh patches on the reference patch \tilde{Q} to the patches $Q_j \in \mathcal{M}^0$. It is important to note that the geometric refinements in the canonical patches have to be suitably selected and oriented in order to achieve a proper geometric refinement towards corners and edges of Ω . Moreover, the patches Q_j with $\overline{Q_j} \cap \mathcal{S} = \emptyset$ away from the singular support \mathcal{S} are left unrefined, i.e., no refinement is considered on \tilde{Q} . In [33, Section 3.3], a specific construction of geometric meshes has been introduced in terms of four different *hp*-extensions (Ex1)–(Ex4).

Consider now the hexahedral patch $Q_j \in \mathcal{M}^0$. We denote the elements in the canonical geometric mesh patch associated with Q_j by $\tilde{\mathcal{M}}_j = \{\tilde{K}\}$, where we allow $\tilde{\mathcal{M}}_j = \{\tilde{Q}\}$ in the case of unrefined patches. The elements in $\tilde{\mathcal{M}}_j$ are then transported to the physical domain Ω via the (finitely many) affine patch maps G_j in (3.1). Moreover, for each $\tilde{K} \in \tilde{\mathcal{M}}_j$, we can write $\tilde{K} = H_{j,\tilde{K}}(\tilde{Q})$, where $H_{j,\tilde{K}} : \tilde{Q} \rightarrow \tilde{K}$ is a possibly anisotropic dilation combined with a translation of the reference cube $\hat{Q} = (-1, 1)^3$ (to be distinguished from the reference patch \tilde{Q}). Thus, the elements in the patch $Q_j \subset \Omega$ will be given by

$$\mathcal{M}_j = \left\{ K : K = (G_j \circ H_{j,\tilde{K}})(\hat{Q}), \tilde{K} \in \tilde{\mathcal{M}}_j \right\}, \quad j = 1, \dots, J;$$

A geometric mesh in Ω is now given by $\mathcal{M} = \bigcup_{j=1}^J \mathcal{M}_j$. Each hexahedral element $K \in \mathcal{M}$ is the image of the reference cube $\hat{Q} = (-1, 1)^3$ under an element mapping $\Phi_K : K = \Phi_K(\hat{Q})$, which can be written as (cf. [33, Section 3])

$$\Phi_K = G_{j(K)} \circ H_K : \hat{Q} \rightarrow K \in \mathcal{M}, \quad K \in \mathcal{M}_{j(K)}, \quad (3.3)$$

where $H_K : \hat{Q} \rightarrow \tilde{Q}$ is a possibly anisotropic dilation combined with a translation. In turn, Φ_K is also a possibly anisotropic dilation with a translation from \tilde{Q} to K . We collect all element mappings Φ_K defined in (3.3) in the *mapping vector*

$$\Phi(\mathcal{M}) := \{ \Phi_K : K \in \mathcal{M} \}. \quad (3.4)$$

With each hexahedral element $K \in \mathcal{M}$, we associate a polynomial degree vector $\mathbf{p}_K = (p_{K,1}, p_{K,2}, p_{K,3}) \in \mathbb{N}_0^3$. Its components correspond to the coordinate directions in $\hat{Q} = \Phi_K^{-1}(K)$. The polynomial degree is called *isotropic* if $p_{K,1} = p_{K,2} = p_{K,3} = p_K$. In the *hp*-error estimates, we shall be mainly concerned with the situation where $p_{K,1} = p_{K,2} =: p_K^\perp$; in this case we simply write

$$\mathbf{p}_K = (p_K^\perp, p_K^\parallel). \quad (3.5)$$

Given a mesh \mathcal{M} of hexahedral elements in Ω , we combine the elemental polynomial degrees \mathbf{p}_K into the *polynomial degree vector*

$$\mathbf{p}(\mathcal{M}) := \{ \mathbf{p}_K : K \in \mathcal{M} \}. \quad (3.6)$$

We remark that in addition to the mesh refinements, the extensions (Ex1)–(Ex4) introduced in [33] also provide appropriate polynomial degree distributions that increase linearly away from the singular set \mathcal{S} .

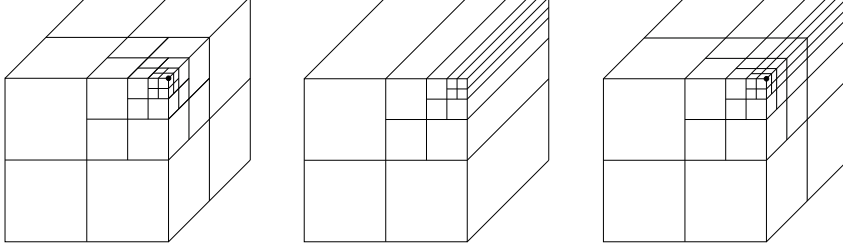


FIG. 3.1. Examples of three basic geometric mesh subdivisions in the reference patch \tilde{Q} with subdivision ratio $\sigma = \frac{1}{2}$: isotropic towards the corner \mathbf{c} (left), anisotropic towards the edge \mathbf{e} (center), and anisotropic towards the edge-corner pair \mathbf{ce} (right). The sets $\mathbf{c}, \mathbf{e}, \mathbf{ce}$ are shown in boldface.

In the sequel, we shall be working with *sequences of σ -geometrically refined meshes* denoted by $\mathcal{M}_\sigma^{(0)}, \mathcal{M}_\sigma^{(1)}, \mathcal{M}_\sigma^{(2)}, \dots$, where $\mathcal{M}_\sigma^{(0)} := \mathcal{M}^0$. Here, $\sigma \in (0, 1)$ is a fixed parameter defining the ratio of subdivision in the canonical geometric refinements in Figure 3.1. For $\ell \geq 1$, there holds: if $K \in \mathcal{M}_\sigma^{(\ell)}$, then there exists $K' \in \mathcal{M}_\sigma^{(\ell-1)}$ such that $K \subset K'$. We shall refer to the index ℓ as *refinement level* and to the sequence $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}$ of σ -geometric meshes in Ω as *σ -geometric mesh family*; see [33, Definition 3.4].

3.2. Mesh Layers. In the exponential convergence proof, we use the concept of *mesh layers*: these are partitions of $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}$ into certain subsets of elements with identical scaling properties in terms of their relative distance to the sets \mathcal{C} and \mathcal{E} . As each of the geometric reference patches shown in Figure 3.1 admits such a partition into layers, the geometric mesh families \mathfrak{M}_σ defined by the construction in [33, Section 3] also admit such a decomposition into layers. We have shown in [33]:

PROPOSITION 3.1. *Any σ -geometric mesh family \mathfrak{M}_σ obtained by iterating the basic hp-extensions (Ex1)–(Ex4) in [33] can be partitioned into a countable sequence of disjoint mesh layers $\{\mathfrak{L}_\sigma^j\}_{j=0}^{\ell-1}$, and a corresponding nested sequence of terminal layers \mathfrak{T}_σ^ℓ , such that each $\mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma$, $\ell \geq 1$, can be written as*

$$\mathcal{M}_\sigma^{(\ell)} = \mathfrak{L}_\sigma^0 \dot{\cup} \mathfrak{L}_\sigma^1 \dot{\cup} \dots \dot{\cup} \mathfrak{L}_\sigma^{\ell-1} \dot{\cup} \mathfrak{T}_\sigma^\ell. \quad (3.7)$$

Elements in the submesh

$$\mathfrak{D}_\sigma^\ell := \mathfrak{L}_\sigma^0 \dot{\cup} \mathfrak{L}_\sigma^1 \dot{\cup} \dots \dot{\cup} \mathfrak{L}_\sigma^{\ell-1} \subset \mathcal{M}_\sigma^{(\ell)} \in \mathfrak{M}_\sigma, \quad \ell \geq 1, \quad (3.8)$$

are bounded away from $\mathcal{C} \cup \mathcal{E}$, while all elements in the terminal layer \mathfrak{T}_σ^ℓ have a nontrivial intersection with $\mathcal{C} \cup \mathcal{E}$.

Evidently, $\mathcal{M}_\sigma^{(\ell)} = \mathfrak{D}_\sigma^\ell \dot{\cup} \mathfrak{T}_\sigma^\ell$ for $\ell \geq 1$. We partition \mathfrak{D}_σ^ℓ into discrete corner, edge and corner-edge neighborhoods as follows:

$$\mathfrak{D}_\sigma^\ell = \mathfrak{D}_\mathcal{C}^\ell \dot{\cup} \mathfrak{D}_\mathcal{E}^\ell \dot{\cup} \mathfrak{D}_{\mathcal{CE}}^\ell \dot{\cup} \mathfrak{D}_{\text{int}}^\ell, \quad (3.9)$$

where for $\ell \geq 1$,

$$\begin{aligned} \mathfrak{D}_{\text{int}}^\ell &:= \{K \in \mathfrak{D}_\sigma^\ell : \overline{K} \cap \Omega_0 \neq \emptyset\}, \\ \mathfrak{D}_\mathcal{C}^\ell &:= \{K \in \mathfrak{D}_\sigma^\ell : \overline{K} \cap \Omega_\mathcal{C} \neq \emptyset\} \setminus \mathfrak{D}_{\text{int}}^\ell, \\ \mathfrak{D}_\mathcal{E}^\ell &:= \{K \in \mathfrak{D}_\sigma^\ell : \overline{K} \cap \Omega_\mathcal{E} \neq \emptyset\} \setminus (\mathfrak{D}_{\text{int}}^\ell \cup \mathfrak{D}_\mathcal{C}^\ell), \\ \mathfrak{D}_{\mathcal{CE}}^\ell &:= \{K \in \mathfrak{D}_\sigma^\ell : \overline{K} \cap \Omega_{\mathcal{CE}} \neq \emptyset\} \setminus (\mathfrak{D}_{\text{int}}^\ell \cup \mathfrak{D}_\mathcal{C}^\ell \cup \mathfrak{D}_\mathcal{E}^\ell). \end{aligned} \quad (3.10)$$

Note that there exists $\ell_0 \geq 1$ (depending on ε from (2.3) and on σ) such that $\mathfrak{D}_{\text{int}}^\ell = \mathfrak{D}_{\text{int}}^{\ell_0}$ for $\ell \geq \ell_0$. Without loss of generality, we shall assume that the initial mesh is sufficiently fine so that we can choose $\ell_0 = 2$. Consequently, in what follows we shall simply write $\mathfrak{D}_{\text{int}}$ instead of $\mathfrak{D}_{\text{int}}^{\ell_0}$. In addition, we may assume without loss of generality that $\mathfrak{D}_\sigma^0 \subset \mathfrak{D}_{\text{int}}^\ell$ for $\ell \geq \ell_0 = 2$.

For an element K , we set $h_K = \text{diam}(K)$. For anisotropic elements $K \in \mathfrak{D}_\mathcal{E}^\ell \dot{\cup} \mathfrak{D}_{\mathcal{CE}}^\ell$, we recall from [33] that we denote by h_K^\parallel and h_K^\perp the elemental diameters of K parallel and transversal to the singular edge $\mathbf{e} \in \mathcal{E}$ nearest to K . For isotropic elements $K \in \mathfrak{D}_\mathcal{C}^\ell$, we have $h_K^\parallel \simeq h_K^\perp \simeq h_K$. In a sequence $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 1}$ of σ -geometric meshes, we define for any $K \in \mathcal{M}_\sigma^{(\ell)}$, $\mathbf{c} \in \mathcal{C}$ and $\mathbf{e} \in \mathcal{E}$ the quantities:

$$d_K^{\mathbf{e}} := \text{dist}(K, \mathbf{e}) = \inf_{\mathbf{x} \in K} r_{\mathbf{e}}(\mathbf{x}), \quad d_K^{\mathbf{c}} := \text{dist}(K, \mathbf{c}) = \inf_{\mathbf{x} \in K} r_{\mathbf{c}}(\mathbf{x}). \quad (3.11)$$

The geometric meshes $\mathcal{M}_\sigma^{(\ell)}$ constructed above and in [33] have the following scaling properties.

PROPOSITION 3.2. *There exists a constant $0 < \kappa_1 < 1$ (depending only on σ and on $C_{\mathcal{M}_0}$ in (3.2)) such that for all $\ell \geq 1$ we have*

$$\forall K \in \mathfrak{D}_\mathcal{C}^\ell : \quad \kappa_1 d_K^{\mathbf{c}} \leq r_{\mathbf{c}}|_K \leq \kappa_1^{-1} d_K^{\mathbf{c}}, \quad \kappa_1 h_K \leq d_K^{\mathbf{c}} \leq \kappa_1^{-1} h_K, \quad (3.12)$$

$$\forall K \in \mathfrak{D}_\mathcal{E}^\ell : \quad \kappa_1 d_K^{\mathbf{e}} \leq r_{\mathbf{e}}|_K \leq \kappa_1^{-1} d_K^{\mathbf{e}}, \quad \kappa_1 h_K^\perp \leq d_K^{\mathbf{e}} \leq \kappa_1^{-1} h_K^\perp, \quad (3.13)$$

and

$$\forall K \in \mathfrak{D}_{\mathcal{CE}}^\ell : \quad \begin{aligned} \kappa_1 d_K^{\mathbf{e}} \leq r_{\mathbf{e}}|_K \leq \kappa_1^{-1} d_K^{\mathbf{e}}, & \quad \kappa_1 d_K^{\mathbf{c}} \leq r_{\mathbf{c}}|_K \leq \kappa_1^{-1} d_K^{\mathbf{c}}, \\ \kappa_1 h_K^\perp \leq d_K^{\mathbf{e}} \leq \kappa_1^{-1} h_K^\perp, & \quad \kappa_1 h_K^\parallel \leq d_K^{\mathbf{c}} \leq \kappa_1^{-1} h_K^\parallel. \end{aligned} \quad (3.14)$$

REMARK 3.3. *It follows from Proposition 3.2 that, given $0 < \sigma < 1$ and \mathcal{M}_0 , there exists $\kappa_1(\sigma, C_{\mathcal{M}_0}) > 0$ such that for all $\ell \in \mathbb{N}$ there holds*

$$\forall K \in \mathfrak{D}_\mathcal{C}^\ell \cap \mathfrak{D}_\sigma^{\ell-1} : \quad \kappa_1^2 \sigma^\ell \leq h_K \leq \kappa_1^{-2} \sigma^\ell, \quad (3.15)$$

$$\forall K \in \mathfrak{D}_\mathcal{E}^\ell \cap \mathfrak{D}_\sigma^{\ell-1} : \quad \kappa_1^2 \sigma^\ell \leq h_K^\perp \leq \kappa_1^{-2} \sigma^\ell, \quad (3.16)$$

and, for some index $i = i(K)$, $1 \leq i \leq \ell$,

$$\forall K \in \mathfrak{D}_{\mathcal{CE}}^\ell \setminus \mathfrak{D}_{\mathcal{CE}}^{\ell-1} : \quad \kappa_1 \sigma^\ell \leq h_K^\perp \leq \kappa_1^{-1} \sigma^\ell, \quad \kappa_1 \sigma^i \leq h_K^\parallel \leq \kappa_1^{-1} \sigma^i. \quad (3.17)$$

Similarly, we partition the terminal layer \mathfrak{T}_σ^ℓ into

$$\mathfrak{T}_\sigma^\ell = \mathcal{V}_\mathcal{C}^\ell \dot{\cup} \mathcal{V}_\mathcal{E}^\ell, \quad (3.18)$$

where

$$\begin{aligned} \mathcal{V}_\mathcal{C}^\ell &= \{K \in \mathfrak{T}_\sigma^\ell : \exists \mathbf{c} \in \mathcal{C} \text{ with } \mathbf{c} \in \overline{K}\}, \\ \mathcal{V}_\mathcal{E}^\ell &= \{K \in \mathfrak{T}_\sigma^\ell : \exists \mathbf{e} \in \mathcal{E} \text{ s. t. } (\overline{K} \cap \mathcal{S})^\circ \cap \mathbf{e} \text{ is an entire edge of } K\} \setminus \mathcal{V}_\mathcal{C}^\ell. \end{aligned} \quad (3.19)$$

We notice that elements in $\mathcal{V}_\mathcal{C}^\ell$ are isotropic with $h_K \simeq h_K^\perp \simeq h_K^\parallel$, while elements in $\mathcal{V}_\mathcal{E}^\ell$ may be anisotropic. Analogous to Proposition 3.2, we have the following result.

PROPOSITION 3.4. *There exists a constant $\kappa_2(\sigma, C_{\mathcal{M}_0}) > 0$ such that for all $\ell \geq 1$ we have*

$$\forall K \in \mathcal{V}_\mathcal{C}^\ell : \quad \kappa_2 \sigma^\ell \leq h_K \leq \kappa_2^{-1} \sigma^\ell, \quad (3.20)$$

and (cf. (3.17)) for all $K \in \mathcal{V}_{\mathcal{E}}^{\ell}$ there holds

$$\kappa_2 \sigma^{\ell} \leq h_K^{\perp} \leq \kappa_2^{-1} \sigma^{\ell}, \quad \kappa_2 \sigma^{\ell-j} \leq h_K^{\parallel} \leq \kappa_2^{-1} \sigma^{\ell-j}, \quad (3.21)$$

for an exponent $j = j(K)$ with $0 \leq j \leq \ell$.

3.3. Finite Element Spaces. Let \mathcal{M} be a geometric mesh of a σ -geometric mesh family \mathfrak{M}_{σ} in Ω . Furthermore, let $\Phi(\mathcal{M})$ and $\mathbf{p}(\mathcal{M})$ be the associated element mapping and elemental polynomial degree vectors, as introduced in (3.4) and (3.6), respectively. We then introduce the discontinuous hp finite element space

$$V(\mathcal{M}, \Phi, \mathbf{p}) = \{ u \in L^2(\Omega) : u|_K \in \mathbb{Q}^{\mathbf{p}_K}(K), K \in \mathcal{M} \}. \quad (3.22)$$

Here, we define the local polynomial approximation space $\mathbb{Q}^{\mathbf{p}_K}(K)$ as follows: first, on the reference element \widehat{Q} and for a polynomial degree vector $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{N}_0^3$, we introduce the anisotropic polynomial space:

$$\mathbb{Q}^{\mathbf{p}}(\widehat{Q}) = \mathbb{P}^{p_1}(\widehat{I}) \otimes \mathbb{P}^{p_2}(\widehat{I}) \otimes \mathbb{P}^{p_3}(\widehat{I}) = \text{span} \{ \hat{\mathbf{x}}^{\alpha} : \alpha_i \leq p_i, 1 \leq i \leq 3 \}. \quad (3.23)$$

Here, for $p \in \mathbb{N}_0$, we denote by $\mathbb{P}^p(\widehat{I})$ the space of all polynomials of degree at most p on the reference interval $\widehat{I} = (-1, 1)$. Then, if K is a hexahedral element of \mathcal{M} with associated elemental mapping $\Phi_K : \widehat{Q} \rightarrow K$ and polynomial degree vector $\mathbf{p}_K = (p_{K,1}, p_{K,2}, p_{K,3})$, we define

$$\mathbb{Q}^{\mathbf{p}_K}(K) = \left\{ u \in L^2(K) : (u|_K \circ \Phi_K) \in \mathbb{Q}^{\mathbf{p}_K}(\widehat{Q}) \right\}. \quad (3.24)$$

In the case where the polynomial degree vector \mathbf{p}_K associated with K is isotropic, i.e., $p_{K,1} = p_{K,2} = p_{K,3} = p_K$, we simply write $\mathbb{Q}^{\mathbf{p}_K}(K) = \mathbb{Q}^{p_K}(K)$.

We now introduce two families of hp -finite element spaces for the discontinuous Galerkin methods; both yield exponentially convergent approximations and are based on the σ -geometric mesh families $\mathfrak{M}_{\sigma} = \{\mathcal{M}_{\sigma}^{(\ell)}\}_{\ell \geq 1}$. The first family has uniform polynomial degree distributions, while the second (smaller) family will have linearly increasing and anisotropically distributed polynomial degrees. The *first family of hp -dG subspaces* is defined by

$$V_{\sigma}^{\ell} := V(\mathcal{M}_{\sigma}^{(\ell)}, \Phi(\mathcal{M}_{\sigma}^{(\ell)}), \mathbf{p}_1(\mathcal{M}_{\sigma}^{(\ell)})), \quad \ell \geq 1, \quad (3.25)$$

where the elemental polynomial degree vectors \mathbf{p}_K in $\mathbf{p}_1(\mathcal{M}_{\sigma}^{(\ell)})$ are isotropic and uniform, given on each element K as

$$p_K = \ell \quad \forall K \in \mathcal{M}_{\sigma}^{(\ell)}. \quad (3.26)$$

The *second family of hp -dG subspaces* is chosen as

$$V_{\sigma, \mathbf{s}}^{\ell} := V(\mathcal{M}_{\sigma}^{(\ell)}, \Phi(\mathcal{M}_{\sigma}^{(\ell)}), \mathbf{p}_2(\mathcal{M}_{\sigma}^{(\ell)})), \quad \ell \geq 1, \quad (3.27)$$

for an increment parameter $\mathbf{s} > 0$. Here the polynomial degree vectors $\mathbf{p}_2(\mathcal{M}_{\sigma}^{(\ell)})$ are linearly increasing with slope \mathbf{s} away from \mathcal{S} , i.e., specifically, the polynomial degrees within each element increase linearly with the number of mesh layers between that element and the component of the singular set \mathcal{S} nearest to it, with the factor of proportionality (“slope” in the terminology of [13]) being $\mathbf{s} > 0$; see [33, Section 3].

REMARK 3.5. *By construction, increasing the index j in the mesh layers \mathfrak{L}_σ^j corresponds to moving from inside the domain towards the singular set \mathcal{S} , with \mathfrak{L}_σ^0 being the most inner layer, and the terminal layer \mathfrak{T}_σ^ℓ being the most outer layer abutting at \mathcal{S} ; see (3.7). While this numbering takes into account the scaling properties of \mathfrak{L}_σ^j , it is in contrast to the notion of linearly increasing polynomial degrees where the polynomial degree increases from the singular set to the interior of the domain; see also [33].*

3.4. Polynomial Degree Vectors of μ -Bounded Variation. In our error analysis, we shall employ the concept of *polynomial degree vectors* $\mathbf{p}(\mathcal{M})$ of μ -bounded variation on meshes $\mathcal{M} \in \mathfrak{M}_\sigma$. To define it, for any $\mathcal{M} \in \mathfrak{M}_\sigma$, we denote the set of all interior faces in \mathcal{M} by $\mathcal{F}_\mathcal{I}(\mathcal{M})$ and the set of all boundary faces by $\mathcal{F}_\mathcal{B}(\mathcal{M})$. In addition, let $\mathcal{F}(\mathcal{M}) = \mathcal{F}_\mathcal{I}(\mathcal{M}) \cup \mathcal{F}_\mathcal{B}(\mathcal{M})$ denote the set of all (smallest) faces of \mathcal{M} . If the mesh $\mathcal{M} \in \mathfrak{M}_\sigma$ is clear from the context, we shall omit the dependence of these sets on \mathcal{M} . Furthermore, for an element $K \in \mathcal{M}$, we denote the set of its faces by $\mathcal{F}_K = \{f \in \mathcal{F} : f \subset \partial K\}$. For any element $K \in \mathcal{M}$ and any face $f \in \mathcal{F}_K$, we from now on denote by $p_{K,f}^{\parallel,(1)}, p_{K,f}^{\parallel,(2)}$ the two components of \mathbf{p}_K parallel to f , and by $p_{K,f}^\perp$ the polynomial degree of \mathbf{p}_K transversal to f . A *degree vector* $\mathbf{p}(\mathcal{M})$ is of μ -bounded variation if there is a constant $\mu \in (0, 1)$ such that

$$\mu \leq p_{K^\sharp,f}^\perp / p_{K^\flat,f}^\perp \leq \mu^{-1}, \quad (3.28)$$

uniformly for all interior faces $f = \mathcal{F}_\mathcal{I}$. A family of degree vectors is of μ -bounded variation if each vector in the family is of μ -bounded variation (uniformly for the entire family). By construction, the families based on the spaces V_σ^ℓ in (3.25) and $V_{\sigma,s}^\ell$ in (3.27) have polynomial degree vectors of μ -bounded variation.

4. Discontinuous Galerkin Discretization. In this section we present the hp -dG discretizations of (1.1)–(1.2) for which we shall prove exponential convergence. In addition, we shall briefly recall the stability and quasi-optimality results from [33, Section 4]. Throughout, $\mathcal{M} \in \mathfrak{M}_\sigma$ denotes a generic, σ -geometric mesh.

4.1. Face Operators. In order to define a dG formulation on a given mesh \mathcal{M} for the model problem (1.1)–(1.2), we shall first recall some element face operators. For this purpose, consider an interior face $f = (\partial K^\sharp \cap \partial K^\flat)^\circ \in \mathcal{F}_\mathcal{I}(\mathcal{M})$ shared by two elements $K^\sharp, K^\flat \in \mathcal{M}$. Furthermore, let v, \mathbf{w} be a scalar- respectively a vector-valued function that is sufficiently smooth inside the elements K^\sharp, K^\flat . Then we define the following jumps and averages of v and \mathbf{w} along f :

$$\begin{aligned} \llbracket v \rrbracket &= v|_{K^\sharp} \mathbf{n}_{K^\sharp} + v|_{K^\flat} \mathbf{n}_{K^\flat} & \langle\langle v \rangle\rangle &= 1/2 (v|_{K^\sharp} + v|_{K^\flat}) \\ \llbracket \mathbf{w} \rrbracket &= \mathbf{w}|_{K^\sharp} \cdot \mathbf{n}_{K^\sharp} + \mathbf{w}|_{K^\flat} \cdot \mathbf{n}_{K^\flat} & \langle\langle \mathbf{w} \rangle\rangle &= 1/2 (\mathbf{w}|_{K^\sharp} + \mathbf{w}|_{K^\flat}). \end{aligned}$$

Here, for an element $K \in \mathcal{M}$, we denote by \mathbf{n}_K the outward unit normal vector on ∂K . For a boundary face $f = (\partial K \cap \partial \Omega)^\circ \in \mathcal{F}_\mathcal{B}(\mathcal{M})$ for $K \in \mathcal{M}$, and sufficiently smooth functions v, \mathbf{w} on K , we let $\llbracket v \rrbracket = v|_K \mathbf{n}_\Omega$, $\llbracket \mathbf{w} \rrbracket = \mathbf{w}|_K \cdot \mathbf{n}_\Omega$, and $\langle\langle v \rangle\rangle = v|_K$, $\langle\langle \mathbf{w} \rangle\rangle = \mathbf{w}|_K$, where \mathbf{n}_Ω is the outward unit normal vector on $\partial \Omega$.

4.2. hp -IP dG Discretizations. The problem (1.1)–(1.2) will be discretized using an interior penalty (IP) discontinuous Galerkin method. Let $V(\mathcal{M}, \Phi, \mathbf{p})$ be an hp -dG finite element space on a σ -geometric mesh $\mathcal{M} \in \mathfrak{M}_\sigma$ with a μ -bounded degree vector \mathbf{p} . For a fixed parameter $\theta \in \mathbb{R}$, we define the hp -discontinuous Galerkin

solution u_{DG} by

$$u_{\text{DG}} \in V(\mathcal{M}, \Phi, \mathbf{p}) : \quad a_{\text{DG}}(u_{\text{DG}}, v) = \int_{\Omega} f v \, d\mathbf{x} \quad \forall v \in V(\mathcal{M}, \Phi, \mathbf{p}), \quad (4.1)$$

where $a_{\text{DG}}(u, v)$ is given by

$$\begin{aligned} a_{\text{DG}}(w, v) = & \int_{\Omega} ((\mathbf{A} \nabla_h w) \cdot \nabla_h v + c w v) \, d\mathbf{x} - \int_{\mathcal{F}(\mathcal{M})} \langle \langle \mathbf{A} \nabla_h w \rangle \rangle \cdot \llbracket v \rrbracket \, ds \\ & + \theta \int_{\mathcal{F}(\mathcal{M})} \langle \langle \mathbf{A} \nabla_h v \rangle \rangle \cdot \llbracket w \rrbracket \, ds + \gamma \int_{\mathcal{F}(\mathcal{M})} \alpha \llbracket v \rrbracket \cdot \llbracket w \rrbracket \, ds. \end{aligned}$$

Here, ∇_h is the elementwise gradient, and $\gamma > 0$ is a stabilization parameter that will be chosen sufficiently large. Furthermore, $\alpha \in L^\infty(\mathcal{F})$ is the discontinuity stabilization function

$$x\alpha(\mathbf{x}) = \begin{cases} \frac{\max(p_{K_\sharp, f}^\perp, p_{K_b, f}^\perp)^2}{\min(h_{K_\sharp, f}^\perp, h_{K_b, f}^\perp)} & \text{if } \mathbf{x} \in f = (\partial K_\sharp \cap \partial K_b)^\circ \in \mathcal{F}_{\mathcal{I}} \\ & \text{for } K_\sharp, K_b \in \mathcal{M}, \\ \frac{(p_{K, f}^\perp)^2}{h_{K, f}^\perp} & \text{if } \mathbf{x} \in f = (\partial K \cap \partial \Omega)^\circ \in \mathcal{F}_{\mathcal{B}} \text{ for } K \in \mathcal{M}. \end{cases} \quad (4.2)$$

Here, we denote by $h_{K, f}^\perp$ the height of K over face f , i.e., the diameter of K in the direction perpendicular to f . The parameter θ allows us to describe a whole range of interior penalty methods: for $\theta = -1$ we obtain the standard symmetric interior penalty (SIP) method while for $\theta = 1$ the non-symmetric (NIP) version is obtained; cf. [2] and the references therein. Let us further remark that, by our assumption that \mathbf{A} is symmetric positive definite and constant, we omit the explicit dependence of the penalty jump terms on the diffusivity.

4.3. Anisotropic Trace Inequality. In order to analyze the numerical fluxes in the dG formulation, we next recall the anisotropic trace inequality proved in [33]: let $\mathcal{M} \in \mathfrak{M}_\sigma$ for $0 < \sigma < 1$, $K \in \mathcal{M}$ axiparallel, $f \in \mathcal{F}_K$ and $s \geq 1$. Then, for any $v \in W^{1,s}(K)$, there holds

$$\|v\|_{L^s(f)}^s \leq C_s (h_{K, f}^\perp)^{-1} \left(\|v\|_{L^s(K)}^s + (h_{K, f}^\perp)^s \|\partial_{K, f, \perp} v\|_{L^s(K)}^s \right). \quad (4.3)$$

The constant $C_s > 0$ depends only on σ and on the mapping vector Φ through the patch maps G_j , but is independent of the element size and element aspect ratio. Here, $\partial_{K, f, \perp}$ signifies the partial derivative in direction transversal to $f \in \mathcal{F}_K$.

4.4. Stability and Galerkin Orthogonality. To formulate the well-posedness of the hp -dGFEM, we use the standard dG norm defined by

$$\|v\|_{\text{DG}}^2 = \int_{\Omega} (|\nabla_h v|^2 + c v^2) \, d\mathbf{x} + \gamma \int_{\mathcal{F}} \alpha \llbracket v \rrbracket^2 \, ds, \quad (4.4)$$

for any $v \in V(\mathcal{M}, \Phi, \mathbf{p}) + H^1(\Omega)$. In [33], we have shown the the following result.

THEOREM 4.1. *For any σ -geometric mesh \mathcal{M} with $0 < \sigma < 1$ and any degree vector \mathbf{p} of μ -bounded variation, the dG bilinear form $a_{\text{DG}}(\cdot, \cdot)$ is continuous and coercive on $V(\mathcal{M}, \Phi, \mathbf{p})$: there exist constants $0 < C_1 \leq C_2 < \infty$ independent of*

the refinement level ℓ , the element aspect ratios, the local mesh sizes, and the local polynomial degree vectors such that

$$|a_{\text{DG}}(v, w)| \leq C_1 \|v\|_{\text{DG}} \|w\|_{\text{DG}} \quad \forall v, w \in V(\mathcal{M}, \Phi, \mathbf{p}),$$

and, for $\gamma > 0$ sufficiently large independent of the refinement level ℓ , the element aspect ratios, the local mesh sizes, and the local polynomial degree vectors,

$$a_{\text{DG}}(v, v) \geq C_2 \|v\|_{\text{DG}}^2 \quad \forall v \in V(\mathcal{M}, \Phi, \mathbf{p}).$$

In particular, there exists a unique solution u_{DG} of (4.1).

Moreover, the following Galerkin orthogonality property holds: suppose that the solution u of (1.1)–(1.2) belongs to $M_{-1-\beta}^2(\Omega)$, where β is the weight vector from (2.6) and (2.8). Then, the dG approximation $u_{\text{DG}} \in V(\mathcal{M}, \Phi, \mathbf{p})$ satisfies

$$a_{\text{DG}}(u - u_{\text{DG}}, v) = 0 \quad \forall v \in V(\mathcal{M}, \Phi, \mathbf{p}). \quad (4.5)$$

We point out that for the non-symmetric interior penalty method corresponding to $\theta = 1$, any value of $\gamma > 0$ is sufficient for the method to be coercive.

4.5. Error Estimates. Although the error of the dG method (4.1) is not quasi-optimal with respect to the energy norm, the Galerkin orthogonality (4.5) of the error implies that the error can be bounded by a certain interpolation error of the exact solution in the dG subspace. We proceed in a standard way and split the error $e_{\text{DG}} = u - u_{\text{DG}}$ into two parts η and ξ , $e_{\text{DG}} = \eta + \xi$, with

$$\eta = u - \Pi u \in H_0^1(\Omega) + V(\mathcal{M}, \Phi, \mathbf{p}), \quad \xi = \Pi u - u_{\text{DG}} \in V(\mathcal{M}, \Phi, \mathbf{p}). \quad (4.6)$$

Here, $\Pi : M_{-1-\beta}^2(\Omega) \rightarrow V(\mathcal{M}, \Phi(\mathcal{M}), \mathbf{p})$ is a hp -(quasi)interpolant which is stable in $H^2(K)$ for each $K \in \mathfrak{D}_\sigma^\ell$ and bounded for $u \in M_{-1-\beta}^2(\Omega)$. In [33], we have proved the following error estimate.

THEOREM 4.2. *On any σ -geometric and axiparallel mesh $\mathcal{M}_\sigma^{(\ell)}$ and any degree vector of μ -bounded variation, there holds*

$$\|u - u_{\text{DG}}\|_{\text{DG}}^2 \leq C \mathbf{p}_{\max}^4 (\Upsilon_{\mathfrak{D}_\sigma^\ell}[\eta] + \Upsilon_{\mathfrak{T}_\sigma^\ell}[\eta]), \quad (4.7)$$

where

$$\begin{aligned} \Upsilon_{\mathfrak{D}_\sigma^\ell}[\eta] &= \sum_{K \in \mathfrak{D}_\sigma^\ell} \left(\max_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K)}^2 + \|\nabla \eta\|_{L^2(K)}^2 \right) \\ &\quad + \sum_{K \in \mathfrak{D}_\sigma^\ell} \sum_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^2 \|\partial_{K,f,\perp} \nabla \eta\|_{L^2(K)}^2, \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \Upsilon_{\mathfrak{T}_\sigma^\ell}[\eta] &= \sum_{K \in \mathfrak{T}_\sigma^\ell} \left(\max_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K)}^2 + \|\nabla \eta\|_{L^2(K)}^2 \right) \\ &\quad + \sum_{K \in \mathfrak{T}_\sigma^\ell} \sum_{f \in \mathcal{F}_K} |f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f)}^2. \end{aligned} \quad (4.9)$$

Here, $C = C(\sigma, \mu, \mathcal{M}_0, \theta, \gamma, \alpha_0) > 0$ is a constant independent of the refinement level ℓ , the element aspect ratios, the local mesh sizes, and the local polynomial degree vectors. Furthermore, $|f|$ is the surface measure of a face f , and $\mathbf{p}_{\max} = \max_{K \in \mathcal{M}_\sigma^{(\ell)}} \max \mathbf{p}_K$.

5. hp -Approximations. To prove exponential convergence of the hp -version dGFEM (4.1) for problem (1.1)–(1.2), it is now sufficient to construct an exponentially convergent interpolant Π for the bounds in (4.8) and (4.9) in Theorem 4.2. Since $\Pi u \in V(\mathcal{M}, \Phi, \mathbf{p})$ may be discontinuous across element interfaces, we can construct Πu separately for each element K of the discrete neighborhoods in (3.10), and of the terminal layer, i.e., on $\mathfrak{D}_{\text{int}}, \mathfrak{D}_{\mathcal{C}}^\ell, \mathfrak{D}_{\mathcal{E}}^\ell, \mathfrak{D}_{\mathcal{CE}}^\ell$, and \mathfrak{T}_σ^ℓ of $\mathcal{M}_\sigma^{(\ell)}$. We then estimate $\eta|_K = u|_K - \Pi u|_K$ for every element K under the *regularity property* $u \in A_{-1-\beta}(\Omega)$ stated in Proposition 2.2, that is, the solution u satisfies

$$|u|_{M_{-1-\beta}^m(\Omega)} \leq C_u^{m+1} m! \quad \forall m \in \mathbb{N}_0, \quad (5.1)$$

for some constant $C_u > 0$, with β satisfying (2.8) and $|\circ|_{M_{-1-\beta}^m(\Omega)}$ as in (2.7).

5.1. Anisotropic Polynomial Approximation. We begin by developing a polynomial approximation analysis based on tensor-product interpolation operators.

5.1.1. Univariate hp -Projectors and Error Bounds. Let $I = (-1, 1)$ be the unit interval. For any $k \geq 1$ we define $H^k(I)$ as the usual Sobolev space, with the norm

$$\|u\|_{H^k(I)}^2 = \sum_{j=0}^k \|u^{(j)}\|_{L^2(I)}^2. \quad (5.2)$$

For $q \geq 0$ we denote by $\pi_q : L^2(I) \rightarrow \mathbb{P}^q(I)$ the $L^2(I)$ -projection. The following C^{k-1} -conforming and univariate projector has been constructed in [9, Section 8].

LEMMA 5.1. *For any $p, k \in \mathbb{N}$ with $p \geq 2k - 1$, there is a projector $\widehat{\pi}_{p,k} : H^k(I) \rightarrow \mathbb{P}^p(I)$ that satisfies*

$$(\widehat{\pi}_{p,k})^{(k)} u = \pi_{p-k}(u^{(k)}) \quad (5.3)$$

and

$$(\widehat{\pi}_{p,k})^{(j)} u(\pm 1) := u^{(j)}(\pm 1), \quad j = 0, 1, 2, \dots, k-1. \quad (5.4)$$

Note that although conditions (5.3) and (5.4) formally overspecify the interpolating polynomial, the projector is, in fact, well defined. Moreover, we have the following stability and approximation properties, which have been proved in [9, Proposition 8.4] and [9, Theorem 8.3], respectively.

PROPOSITION 5.2.

1. *For every $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that*

$$\forall u \in H^k(I), \forall p \geq 2k - 1: \quad \|\widehat{\pi}_{p,k} u\|_{H^k(I)} \leq C_k \|u\|_{H^k(I)}. \quad (5.5)$$

2. *For integers $p, k \in \mathbb{N}$ with $p \geq 2k - 1$, $\kappa = p - k + 1$ and for $u \in H^{k+s}(I)$ with any $k \leq s \leq \kappa$ there holds the error bound*

$$\|(u - \widehat{\pi}_{p,k} u)^{(j)}\|_{L^2(I)}^2 \leq \frac{(\kappa - s)!}{(\kappa + s)!} \|u^{(k+s)}\|_{L^2(I)}^2, \quad (5.6)$$

for any $j = 0, 1, \dots, k$.

5.1.2. Tensor-interpolation and Error Bounds. Let now $I^d = I \times \dots \times I$ be the unit cube in \mathbb{R}^d , $d \geq 1$. Points \mathbf{x} in I^d have coordinates x_i , i.e., $\mathbf{x} = (x_1, \dots, x_d)$. On I^d we define the space

$$H_{\text{mix}}^k(I^d) = H^k(I) \otimes \dots \otimes H^k(I), \quad (5.7)$$

where \otimes denotes the tensor-product of separable Hilbert spaces. The tensor-product spaces are isomorphic to standard Bochner spaces. For example,

$$H_{\text{mix}}^k(I^d) \simeq H^k(I; H_{\text{mix}}^k(I^{d-1})) \simeq H_{\text{mix}}^k(I^{d-1}; H^k(I)). \quad (5.8)$$

We also require anisotropic spaces. For $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, we define

$$H_{\text{mix}}^{\mathbf{k}}(I^d) = H^{k_1}(I) \otimes \dots \otimes H^{k_d}(I). \quad (5.9)$$

Analogously, for $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$ we set

$$\mathbb{Q}^{\mathbf{p}}(I^d) := \mathbb{P}^{p_1}(I) \otimes \dots \otimes \mathbb{P}^{p_d}(I). \quad (5.10)$$

In I^d of dimension $d > 1$ and for $p_i \geq 2k_i - 1$, we now define the interpolation operator

$$\widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d = \bigotimes_{i=1}^d \widehat{\pi}_{p_i, k_i}^{(i)}, \quad (5.11)$$

where $\widehat{\pi}_{p_i, k_i}^{(i)}$ is the univariate operator from Lemma 5.1, acting in the variable x_i . If $p_i = p$ and $k_i = k$ for all i , we also write $\widehat{\Pi}_{p, k}^d$ in place of $\widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d$.

In what follows, we denote by $\mathbf{k}_i \in \mathbb{N}^{d-1}$ the multiindex \mathbf{k} with component k_i deleted from it.

PROPOSITION 5.3. *For $d \geq 1$ there holds:*

$$\|\widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d u\|_{H_{\text{mix}}^{\mathbf{k}}(I^d)} \leq C_{\mathbf{k}, d} \|u\|_{H_{\text{mix}}^{\mathbf{k}}(I^d)} \quad (5.12)$$

and

$$\left\| u - \widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d u \right\|_{H_{\text{mix}}^{\mathbf{k}}(I^d)} \leq C_{\mathbf{k}, d} \sum_{i=1}^d \|u - \widehat{\pi}_{p_i, k_i}^{(i)} u\|_{H^{k_i}(I; H_{\text{mix}}^{\mathbf{k}_i}(I^{d-1}))}. \quad (5.13)$$

Proof. We first prove both assertions simultaneously by induction over the dimension d . For $d = 1$, property (5.12) is just (5.5), while (5.13) is trivial. Assume next that the assertions have been proved already for $d' \geq 1$. We then verify them for $d = d' + 1$. By using the tensor-product structure, the stability property for d' , and once more (5.5), we obtain

$$\begin{aligned} \|\widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d u\|_{H_{\text{mix}}^{\mathbf{k}}(I^d)} &= \|\widehat{\Pi}_{\mathbf{p}_d, \mathbf{k}_d}^{d'} \otimes \widehat{\pi}_{p_d, k_d}^{(d)} u\|_{H_{\text{mix}}^{\mathbf{k}_d}(I^{d'}; H^{k_d}(I))} \\ &\leq C_{\mathbf{k}'_d, d'} \|\widehat{\pi}_{p_d, k_d}^{(d)} u\|_{H_{\text{mix}}^{\mathbf{k}_d}(I^{d'}; H^{k_d}(I))} \leq C_{\mathbf{k}'_d, d'} C_{k_d} \|u\|_{H_{\text{mix}}^{\mathbf{k}}(I^{d'}; H^{k_d}(I))}. \end{aligned}$$

This implies the stability property (5.12) for dimension d . Similarly, using the stability (5.12) property for d' , we find that

$$\begin{aligned}
& \|u - \widehat{\Pi}_{\mathbf{p}, \mathbf{k}}^d u\|_{H_{\text{mix}}^k(I^d)} \\
& \leq \|u - \widehat{\Pi}_{\mathbf{p}_d, \mathbf{k}_d}^{d'} u\|_{H_{\text{mix}}^k(I^d)} + \|\widehat{\Pi}_{\mathbf{p}_d, \mathbf{k}_d}^{d'}(u - \widehat{\pi}_{\mathbf{p}_d, \mathbf{k}_d}^{(d)} u)\|_{H_{\text{mix}}^k(I^d)} \\
& \leq C_{\mathbf{k}_{d'}, d'} \sum_{i=1}^{d'} \|u - \widehat{\pi}_{\mathbf{p}_i, \mathbf{k}_i}^{(i)} u\|_{H^{k_i}(I; H_{\text{mix}}^{k_i}(I^{d'}))} + \|\widehat{\Pi}_{\mathbf{p}_d, \mathbf{k}_d}^{d'}(u - \widehat{\pi}_{\mathbf{p}_d, \mathbf{k}_d}^{(d)} u)\|_{H_{\text{mix}}^{k_d}(I^{d'}; H^{k_d}(I))} \\
& \leq C_{\mathbf{k}_{d'}, d'} \sum_{i=1}^{d'} \|u - \widehat{\pi}_{\mathbf{p}_i, \mathbf{k}_i}^{(i)} u\|_{H^{k_i}(I; H_{\text{mix}}^{k_i}(I^{d'}))} + C'_{\mathbf{k}_{d'}, d'} \|u - \widehat{\pi}_{\mathbf{p}_d, \mathbf{k}_d}^{(d)} u\|_{H_{\text{mix}}^{k_d}(I^{d'}; H^{k_d}(I))} \\
& = C_{\mathbf{k}_d, d} \sum_{i=1}^d \|u - \widehat{\pi}_{\mathbf{p}_i, \mathbf{k}_i}^{(i)} u\|_{H^{k_i}(I; H_{\text{mix}}^{k_i}(I^{d'}))}.
\end{aligned}$$

This completes the proof. \square

5.1.3. Application to the reference cube. We now consider the unit cube $\widehat{K} = (-1, 1)^3 = \widehat{K}^\perp \times \widehat{K}^\parallel$, with $\widehat{K}^\perp = (-1, 1)^2$, $\widehat{K}^\parallel = (-1, 1)$. The superscripts \cdot^\perp and \cdot^\parallel denote, respectively, the coordinates $\mathbf{x}^\perp = (x_1, x_2)$ and $x^\parallel = x_3$. We wish to consider *anisotropic polynomial degrees* on \widehat{K} : for a polynomial degree vector $\mathbf{p} = (p^\perp, p^\parallel)$, we write

$$\mathbb{Q}^{\mathbf{p}}(\widehat{K}) = \mathbb{Q}^{\mathbf{p}}(\widehat{K}^\perp \times \widehat{K}^\parallel) = \mathbb{Q}^{p^\perp}(\widehat{K}^\perp) \otimes \mathbb{P}^{p^\parallel}(\widehat{K}^\parallel) = \mathbb{P}^{p^\perp}(I) \otimes \mathbb{P}^{p^\perp}(I) \otimes \mathbb{P}^{p^\parallel}(I),$$

where again $I = (-1, 1)$. We set

$$H_{\text{mix}}^{(k^\perp, k^\parallel)}(\widehat{K}) = H^{k^\perp}(\widehat{K}^\perp) \otimes H^{k^\parallel}(\widehat{K}^\parallel) = H^{k^\perp}(I) \otimes H^{k^\perp}(I) \otimes H^{k^\parallel}(I),$$

for regularity parameters $(k^\perp, k^\parallel) \in \mathbb{N}^2$.

Following the definitions and the analysis above, choosing $k = k^\parallel = k^\perp = 2$, and $p^\perp, p^\parallel \geq 3$, there is a tensor-product projector

$$\widehat{\Pi}_{(p^\perp, p^\parallel), 2}^3 : H_{\text{mix}}^2(\widehat{K}) \rightarrow \mathbb{Q}^{(p^\perp, p^\parallel)}(\widehat{K}), \quad (5.14)$$

which is well-defined and bounded in $H_{\text{mix}}^2(\widehat{K})$. By combining (5.13) and (5.6), we now obtain the following error estimate.

PROPOSITION 5.4. *For any integers $3 \leq s^\perp \leq p^\perp$, $3 \leq s^\parallel \leq p^\parallel$, there holds*

$$\begin{aligned}
\|u - \widehat{\Pi}_{(p^\perp, p^\parallel), 2}^3 u\|_{H_{\text{mix}}^2(\widehat{K})}^2 & \lesssim \Psi_{p^\parallel-1, s^\parallel-1} \sum_{\alpha_\perp^\perp \leq 2, \alpha_\perp^\parallel \leq 2} \|\mathbb{D}_\perp^{\alpha_\perp^\perp} \mathbb{D}_\parallel^{s^\parallel+1} u\|_{L^2(\widehat{K})}^2 \\
& + \Psi_{p^\perp-1, s^\perp-1} \sum_{\alpha_\perp^\perp \leq 2, \alpha_\parallel \leq 2} \|\mathbb{D}_\perp^{(\alpha_\perp^\perp, s^\perp+1)} \mathbb{D}_\parallel^{\alpha_\parallel} u\|_{L^2(\widehat{K})}^2 \\
& + \Psi_{p^\perp-1, s^\perp-1} \sum_{\alpha_\perp^\perp \leq 2, \alpha_\parallel \leq 2} \|\mathbb{D}_\perp^{(s^\perp+1, \alpha_\perp^\perp)} \mathbb{D}_\parallel^{\alpha_\parallel} u\|_{L^2(\widehat{K})}^2.
\end{aligned}$$

REMARK 5.5. *We note that in the isotropic case, we have $p = p^\perp = p^\parallel$ and $s = s^\perp = s^\parallel$, so that the error bound in Lemma 5.4 becomes*

$$\|u - \widehat{\Pi}_{p, 2}^3 u\|_{H_{\text{mix}}^2(\widehat{K})}^2 \lesssim \Psi_{p-1, s-1} \|u\|_{H^{s+5}(\widehat{K})}^2 \quad (5.15)$$

for any $3 \leq s \leq p$.

Proof. We first note that

$$\widehat{\Pi}_{(p^\perp, p^\parallel), 2}^3 = \widehat{\pi}_{p^\perp, 2}^{(1)} \otimes \widehat{\pi}_{p^\perp, 2}^{(2)} \otimes \widehat{\pi}_{p^\parallel, 2}^{(2)}.$$

To indicate the dependence on the coordinate direction, we write $\widehat{K} = I^{(1)} \times I^{(2)} \times I^{(3)}$. Hence, the result in (5.13) yields

$$\begin{aligned} \|u - \widehat{\Pi}_{(p^\perp, p^\parallel), 2}^3 u\|_{H_{\text{mix}}^2(\widehat{K})}^2 &\lesssim \|u - \widehat{\pi}_{p^\perp, 2}^{(1)} u\|_{H^2(I^{(1)}; H_{\text{mix}}^2(I^{(2)} \otimes I^{(3)}))} \\ &+ \|u - \widehat{\pi}_{p^\perp, 2}^{(2)} u\|_{H^2(I^{(2)}; H_{\text{mix}}^2(I^{(1)} \otimes I^{(3)}))} + \|u - \widehat{\pi}_{p^\parallel, 2}^{(3)} u\|_{H^2(I^{(3)}; H_{\text{mix}}^2(I^{(1)} \otimes I^{(2)}))}. \end{aligned}$$

Referring to the approximation properties in (5.6) gives

$$\begin{aligned} \|u - \widehat{\Pi}_{(p^\perp, p^\parallel), 2}^3 u\|_{H_{\text{mix}}^2(\widehat{K})}^2 &\lesssim \Psi_{p^\perp - 1, \overline{s}^\perp} \sum_{\alpha_2^\perp \leq 2, \alpha^\parallel \leq 2} \|\mathbf{D}_\perp^{(\overline{s}^\perp + 2, \alpha_2^\perp)} \mathbf{D}^{\alpha^\parallel} u\|_{L^2(\widehat{K})}^2 \\ &+ \Psi_{p^\perp - 1, \overline{s}^\perp} \sum_{\alpha_1^\perp \leq 2, \alpha^\parallel \leq 2} \|\mathbf{D}_\perp^{(\alpha_1^\perp, \overline{s}^\perp + 2)} \mathbf{D}^{\alpha^\parallel} u\|_{L^2(\widehat{K})}^2 \\ &+ \Psi_{p^\parallel - 1, \overline{s}^\parallel} \sum_{\alpha_1^\perp \leq 2, \alpha_2^\perp \leq 2} \|\mathbf{D}_\perp^{\alpha_1^\perp} \mathbf{D}_\parallel^{\overline{s}^\parallel + 2} u\|_{L^2(\widehat{K})}^2, \end{aligned}$$

for any $2 \leq \overline{s}^\perp \leq p^\perp - 1$ and $2 \leq \overline{s}^\parallel \leq p^\parallel - 1$. Thus, the substitution $s^\perp = \overline{s}^\perp + 1$, $s^\parallel = \overline{s}^\parallel + 1$ completes the proof. \square

5.1.4. Anisotropic interpolation on axiparallel hexahedra. We construct the hp -interpolant Πu on an axiparallel element $K \in \mathfrak{D}_\sigma^\ell$ as follows. Let the elemental polynomial degree vector be given by $\mathbf{p}_K = (p_K^\perp, p_K^\parallel)$. On the reference element \widehat{K} , we then define

$$\widehat{\Pi} \widehat{u} = \widehat{\Pi}_{(p_K^\perp, p_K^\parallel)}^3 \widehat{u},$$

with $\widehat{\Pi}_{(p_K^\perp, p_K^\parallel)}^3$ the projector defined in (5.14). Here, for an element $K \in \mathfrak{D}_\sigma^\ell$ we set

$$\widehat{u} = u \circ \Phi_K, \quad (5.16)$$

and define the elemental interpolation operator $(\Pi u)|_K$ by

$$(\Pi u)|_K \circ \Phi_K := \widehat{\Pi} \widehat{u}. \quad (5.17)$$

Upon possibly translating the element K , we may assume without loss of generality that $K = (0, h_K^\perp) \times (0, h_K^\perp) \times (0, h_K^\parallel)$ be an axiparallel element that is possibly anisotropic in the third coordinate direction. We denote by $\mathbf{x} = (\mathbf{x}^\perp, x^\parallel)$ and $\widehat{\mathbf{x}} = (\widehat{\mathbf{x}}^\perp, \widehat{x}^\parallel)$ the coordinates on K and \widehat{K} , respectively. Similarly, for a multi-index $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^\perp, \alpha^\parallel)$, we write $\mathbf{D}_\perp^{\boldsymbol{\alpha}^\perp}$ respectively $\mathbf{D}_\parallel^{\alpha^\parallel}$ for partial derivatives perpendicular respectively parallel to the third coordinate direction. On the reference element, we use $\widehat{\mathbf{D}}_\perp^{\boldsymbol{\alpha}^\perp}$ and $\widehat{\mathbf{D}}_\parallel^{\alpha^\parallel}$. Then,

$$d\mathbf{x}^\perp = (h_K^\perp)^2 d\widehat{\mathbf{x}}^\perp, \quad dx^\parallel = h_K^\parallel d\widehat{x}^\parallel,$$

and

$$D_{\perp}^{\alpha^{\perp}} = (h_K^{\perp})^{-|\alpha^{\perp}|} \widehat{D}_{\perp}^{\alpha^{\perp}}, \quad D_{\parallel}^{\alpha^{\parallel}} = (h_K^{\parallel})^{-\alpha^{\parallel}} \widehat{D}_{\parallel}^{\alpha^{\parallel}}.$$

Consequently, we obtain

$$\|D_{\perp}^{\alpha^{\perp}} D_{\parallel}^{\alpha^{\parallel}} u\|_{L^2(K)}^2 = (h_K^{\perp})^{2-2|\alpha^{\perp}|} (h_K^{\parallel})^{1-2\alpha^{\parallel}} \|\widehat{D}_{\perp}^{\alpha^{\perp}} \widehat{D}_{\parallel}^{\alpha^{\parallel}} \widehat{u}\|_{L^2(\widehat{K})}^2. \quad (5.18)$$

for any $\alpha^{\perp} \in \mathbb{N}_0^2$ and $\alpha^{\parallel} \in \mathbb{N}_0$. Lemma 5.4 then scales as follows:

LEMMA 5.6. *For an axiparallel element K as above, and $\eta = u - \Pi u$, there holds*

$$\|\widehat{\eta}\|_{H_{\text{mix}}^2(\widehat{K})}^2 \lesssim E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K) + E_{p^{\perp}, s^{\perp}}^{\perp}(K), \quad (5.19)$$

for any $3 \leq s^{\perp} \leq p^{\perp}$ and $3 \leq s^{\parallel} \leq p^{\parallel}$, with

$$\begin{aligned} E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K) &\lesssim \Psi_{p^{\parallel}-1, s^{\parallel}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha_2^{\perp} \leq 2} (h_K^{\perp})^{2|\alpha^{\perp}|-2} (h_K^{\parallel})^{2s^{\parallel}+1} \|D_{\perp}^{\alpha^{\perp}} D_{\parallel}^{s^{\parallel}+1} u\|_{L^2(K)}^2, \\ E_{p^{\perp}, s^{\perp}}^{\perp}(K) &\lesssim \Psi_{p^{\perp}-1, s^{\perp}-1} \sum_{s^{\perp}+1 \leq |\alpha^{\perp}| \leq s^{\perp}+3, \alpha^{\parallel} \leq 2} (h_K^{\perp})^{2|\alpha^{\perp}|-2} (h_K^{\parallel})^{2\alpha^{\parallel}-1} \|D_{\perp}^{\alpha^{\perp}} D_{\parallel}^{\alpha^{\parallel}} u\|_{L^2(K)}^2. \end{aligned}$$

Proof. From Lemma 5.4 on \widehat{K} , we see that $\|\widehat{\eta}\|_{H_{\text{mix}}^2(\widehat{K})}^2$ can be bounded by three terms. Using (5.18) the first of these terms scales as follows:

$$\begin{aligned} &\Psi_{p^{\parallel}-1, s^{\parallel}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha_2^{\perp} \leq 2} \|\widehat{D}_{\perp}^{\alpha^{\perp}} \widehat{D}_{\parallel}^{s^{\parallel}+1} \widehat{u}\|_{L^2(\widehat{K})}^2 \\ &\simeq \Psi_{p^{\parallel}-1, s^{\parallel}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha_2^{\perp} \leq 2} (h_K^{\perp})^{2|\alpha^{\perp}|-2} (h_K^{\parallel})^{2s^{\parallel}+1} \|D_{\perp}^{\alpha^{\perp}} D_{\parallel}^{s^{\parallel}+1} u\|_{L^2(K)}^2. \end{aligned}$$

Similarly, we have for the second term:

$$\begin{aligned} &\Psi_{p^{\perp}-1, s^{\perp}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha^{\parallel} \leq 2} \|\widehat{D}_{\perp}^{(\alpha_1^{\perp}, s^{\perp}+1)} \widehat{D}_{\parallel}^{\alpha^{\parallel}} \widehat{u}\|_{L^2(\widehat{K})}^2 \\ &\simeq \Psi_{p^{\perp}-1, s^{\perp}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha^{\parallel} \leq 2} (h_K^{\perp})^{2(s^{\perp}+\alpha_1^{\perp})} (h_K^{\parallel})^{2\alpha^{\parallel}-1} \|D_{\perp}^{(\alpha_1^{\perp}, s^{\perp}+1)} D_{\parallel}^{\alpha^{\parallel}} u\|_{L^2(K)}^2 \\ &\lesssim \Psi_{p^{\perp}-1, s^{\perp}-1} \sum_{s^{\perp}+1 \leq |\alpha^{\perp}| \leq s^{\perp}+3, \alpha^{\parallel} \leq 2} (h_K^{\perp})^{2|\alpha^{\perp}|-2} (h_K^{\parallel})^{2\alpha^{\parallel}-1} \|D_{\perp}^{\alpha^{\perp}} D_{\parallel}^{\alpha^{\parallel}} u\|_{L^2(K)}^2. \end{aligned}$$

The third term scales as the second term. \square

5.2. Error Bounds on $\mathfrak{D}_{\sigma}^{\ell}$. We are now ready to estimate $\Upsilon_{\mathfrak{D}_{\sigma}^{\ell}}[\eta]$ in (4.8) for $\eta = u - \Pi u$ and Π in (5.17). Due to the various element scalings, we give a separate error analysis on each of the submeshes in (3.10). To that end, we write

$$\Upsilon_{\mathfrak{D}_{\sigma}^{\ell}}[\eta] \leq \Upsilon_{\mathfrak{D}_{\text{int}}}[\eta] + \Upsilon_{\mathfrak{D}_{\varepsilon}^{\ell}}[\eta] + \Upsilon_{\mathfrak{D}_{\varepsilon}^{\ell}}[\eta] + \Upsilon_{\mathfrak{D}_{\varepsilon^{\varepsilon}}^{\ell}}[\eta], \quad (5.20)$$

where the consistency errors on the respective submeshes are defined as in [33], but with sums that are taken only over the elements in the submeshes.

We remark that we first analyze the interpolant $\Pi u \in V_{\sigma}^{\ell}$ with variable and anisotropic polynomial degree vector that defines the space $V_{\sigma, \mathbf{s}}^{\ell}$ in (3.27). The space

V_σ^ℓ in (3.25) with uniform, isotropic polynomial degree distributions will be addressed after that in Corollary 5.19 below.

In what follows, it will be further convenient to introduce the notation $T^K[\eta] := T_1^K[\eta] + T_2^K[\eta] + T_3^K[\eta]$ where

$$\begin{aligned} T_1^K[\eta] &:= \max_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K)}^2, \\ T_2^K[\eta] &:= \|\nabla \eta\|_{L^2(K)}^2, \\ T_3^K[\eta] &:= \sum_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^2 \|\partial_{K,f,\perp} \nabla \eta\|_{L^2(K)}^2. \end{aligned} \quad (5.21)$$

Hence, in light of Theorem 4.2, for any of the neighborhoods \mathfrak{D} in (5.20) there holds

$$\Upsilon_{\mathfrak{D}}[\eta] = \sum_{K \in \mathfrak{D}} T^K[\eta]. \quad (5.22)$$

In our analysis, we shall use the following result.

LEMMA 5.7. *Let $K \in \mathfrak{D}_\sigma^\ell$ be an axi-parallel element, $f \in \mathcal{F}_K$, and $\mathbf{e} \in \mathcal{E}$ the nearest singular edge to K . Then we have*

$$\begin{aligned} T_1^K[\eta] &\lesssim \left((h_K^\perp)^2 (h_K^\parallel)^{-1} + h_K^\parallel \right) \|\hat{\eta}\|_{L^2(\hat{K})}^2, \\ T_2^K[\eta] &\lesssim \left((h_K^\perp)^2 (h_K^\parallel)^{-1} + h_K^\parallel \right) \|\hat{\nabla} \hat{\eta}\|_{L^2(\hat{K})}^2, \\ T_3^K[\eta] &\lesssim \left((h_K^\perp)^2 (h_K^\parallel)^{-1} + h_K^\parallel \right) |\hat{\eta}|_{H^2(\hat{K})}^2. \end{aligned}$$

Consequently, there holds

$$T^K[\eta] \lesssim \left((h_K^\perp)^2 (h_K^\parallel)^{-1} + h_K^\parallel \right) \|\hat{\eta}\|_{H_{\max}^2(\hat{K})}^2.$$

Proof. The scaling in (5.18) shows that

$$(h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K)}^2 \simeq h_{K,f}^{-2} (h_K^\perp)^2 h_K^\parallel \|\hat{\eta}\|_{L^2(\hat{K})}^2.$$

By noting that $h_{K,f}^\perp \simeq h_K^\parallel$ for $f \perp \mathbf{e}$ and $h_{K,f}^\perp \simeq h_K^\perp$ for $f \parallel \mathbf{e}$, the desired bound for $T_1^K[\eta]$ follows. A similar scaling argument shows that

$$\begin{aligned} T_2^K[\eta] &= \sum_{|\alpha^\perp|=1} \|\mathbf{D}_\perp^{\alpha^\perp} \eta\|_{L^2(K)}^2 + \|\mathbf{D}_\parallel \eta\|_{L^2(K)}^2 \\ &\simeq h_K^\parallel \sum_{|\alpha^\perp|=1} \|\hat{\mathbf{D}}_\perp^{\alpha^\perp} \hat{\eta}\|_{L^2(\hat{K})}^2 + (h_K^\perp)^2 (h_K^\parallel)^{-1} \|\hat{\mathbf{D}}_\parallel \hat{\eta}\|_{L^2(\hat{K})}^2, \end{aligned}$$

which yields the assertion for $T_2^K[\eta]$.

To bound $T_3^K[\eta]$ we distinguish the two cases $f \perp \mathbf{e}$ and $f \parallel \mathbf{e}$. In the former case, $f \perp \mathbf{e}$, we have $h_{K,f}^\perp \simeq h_K^\parallel$, and $\partial_{K,f,\perp}$ is the derivative parallel to the edge. Thus, applying the scaling in (5.18) results in

$$\begin{aligned} (h_{K,f}^\perp)^2 \|\partial_{K,f,\perp} \nabla \eta\|_{L^2(K)}^2 &\simeq (h_K^\parallel)^2 \left(\sum_{|\alpha^\perp|=1} \|\mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel \eta\|_{L^2(K)}^2 + \|\mathbf{D}_\parallel^2 \eta\|_{L^2(K)}^2 \right) \\ &\simeq (h_K^\parallel) \sum_{|\alpha^\perp|=1} \|\hat{\mathbf{D}}_\perp^{\alpha^\perp} \hat{\mathbf{D}}_\parallel \hat{\eta}\|_{L^2(\hat{K})}^2 + (h_K^\perp)^2 (h_K^\parallel)^{-1} \|\hat{\mathbf{D}}_\parallel^2 \hat{\eta}\|_{L^2(\hat{K})}^2. \end{aligned}$$

In the latter case, $f \parallel e$, we $h_{\bar{K},f}^\perp \simeq h_{\bar{K}}^\perp$, and $\partial_{K,f,\perp}$ is the derivative perpendicular to the edge. Therefore,

$$\begin{aligned} (h_{\bar{K},f}^\perp)^2 \|\partial_{K,f,\perp} \nabla \eta\|_{L^2(K)}^2 &\lesssim (h_{\bar{K}}^\perp)^2 \left(\sum_{|\alpha^\perp|=2} \|\mathbf{D}_\perp^{\alpha^\perp} \eta\|_{L^2(K)}^2 + \sum_{|\alpha^\perp|=1} \|\mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel \eta\|_{L^2(K)}^2 \right) \\ &\simeq h_K^\parallel \sum_{|\alpha^\perp|=2} \|\widehat{\mathbf{D}}_\perp^{\alpha^\perp} \widehat{\eta}\|_{L^2(\widehat{K})}^2 + (h_{\bar{K}}^\perp)^2 (h_K^\parallel)^{-1} \sum_{|\alpha^\perp|=1} \|\widehat{\mathbf{D}}_\perp^{\alpha^\perp} \widehat{\mathbf{D}}_\parallel \widehat{\eta}\|_{L^2(\widehat{K})}^2. \end{aligned}$$

The two bounds above imply the estimate for $T_3^K[\eta]$. \square

5.2.1. Submesh $\mathfrak{D}_{\text{int}}$. The submesh $\mathfrak{D}_{\text{int}}$ is independent of ℓ for $\ell \geq \ell_0 = 2$. It consists only of a finite number of shape-regular elements, whose distance to the singular support \mathcal{S} of u is at least $\epsilon(\Omega)/2$. Define $\Omega_{\text{int}} := \text{int} \left(\bigcup_{K \in \mathfrak{D}_{\text{int}}^\ell} \bar{K} \right)$. Then, on $\overline{\Omega}_{\text{int}}$ the distance functions r_e , r_c and ρ_{ce} in (2.2) appearing in the definition of the norm (2.7) of $|u|_{M_\beta^m(\Omega)}$ are bounded away from zero, and (5.1) implies that

$$\|u\|_{H^m(\Omega_{\text{int}})} \leq C^{m+1} m! \quad \forall m \in \mathbb{N}. \quad (5.23)$$

for a constant $C > 1$ (possibly different from C_u in (5.1)). Estimate (5.23) implies that there exists another constant $C > 1$ such that

$$\|\widehat{u}\|_{H^m(\widehat{K})} \leq C^{m+1} m! \quad \forall m \in \mathbb{N}. \quad (5.24)$$

We now note that the hp -extensions (Ex1)–(Ex4) which were introduced in [33] produce a uniform and isotropic polynomial degree distribution in $\mathfrak{D}_{\text{int}}$, which is denoted by p and satisfies $p \simeq \ell$.

LEMMA 5.8. *There is a constant $C > 1$ such that*

$$\Upsilon_{\mathfrak{D}_{\text{int}}}[\eta] \leq C^{2s} \Psi_{p-1,s-1} \Gamma(s+6)^2$$

for any $s \in [3, p]$.

Proof. Since there are only a finite number of shape-regular elements in $\mathfrak{D}_{\text{int}}$, Lemma 5.7, the approximation property in (5.15), and the regularity (5.24) yield

$$\begin{aligned} \Upsilon_{\mathfrak{D}_{\text{int}}}[\eta] &= \sum_{K \in \mathfrak{D}_{\text{int}}} T^K[\eta] \lesssim \sum_{K \in \mathfrak{D}_{\text{int}}} \|\widehat{\eta}\|_{H_{\text{mix}}^2(\widehat{K})}^2 \\ &\lesssim \sum_{K \in \mathfrak{D}_{\text{int}}} \Psi_{p-1,s-1} \|\widehat{u}\|_{H^{s+5}(\widehat{K})}^2 \leq C^{2s} \Psi_{p-1,s-1} ((s+5)!)^2. \end{aligned} \quad (5.25)$$

This is the desired bound for integer exponents $3 \leq s \leq p$.

Next, we interpolate the bound (5.25) to fractional regularity order s . To do so, let $s = k + \theta \in [3, p]$ with $k = \lfloor s \rfloor$ and $\theta = s - k \in (0, 1)$. We note that there exist constants $0 < C_1 < C_2 < \infty$ such that

$$0 < C_1 \leq \frac{n!(n+1)^\theta}{\Gamma(n+1+\theta)} \leq C_2 < \infty \quad \forall n \in \mathbb{N}, \quad 0 \leq \theta \leq 1. \quad (5.26)$$

Using the real method of interpolation with indices $0 < \theta < 1$ and 2 on (5.25) for the integers $k = \lfloor s \rfloor$ and $k+1 = \lfloor s \rfloor + 1$, we readily obtain the bound

$$\Upsilon_{\mathfrak{D}_{\text{int}}}[\eta] \leq C^{2(k+\theta)} (\Gamma(k+q)^2 \Psi_{p-1,k-1})^{1-\theta} (\Gamma(k+1+q)^2 \Psi_{p-1,k})^\theta,$$

where we have replaced 6 by $q \geq 0$ for future reference. Elementary manipulations using (5.26) reveal that as $p \rightarrow \infty$, we have

$$\Upsilon_{\mathcal{D}_{\text{int}}}[\eta] \lesssim C^{2(k+\theta)} \Gamma(k+\theta+q)^2 \Psi_{p-1, k+\theta-1} \simeq C^{2s} \Gamma(s+q)^2 \Psi_{p-1, s-1},$$

where the additional constants introduced depend only on C_1, C_2 in (5.26). \square

To prove exponential convergence, we let $p \rightarrow \infty$, and for each given p , minimize the bound in Lemma 5.8.

LEMMA 5.9. *For any $c > 0$ and any $q \in \mathbb{N}$, there exist constants $b > 0$ and $C > 0$ (depending only on c and q) such that*

$$\forall p \geq 3 : \quad \min_{s \in [3, p]} \{c^{2s} \Gamma(s+q)^2 \Psi_{p-1, s-1}\} \leq C^2 \exp(-2bp). \quad (5.27)$$

Proof. We begin by claiming that there are constants $b > 0$ and $C > 0$ such that

$$\forall p \geq 3 : \quad \min_{s \in [3, p]} \{c^{2s} \Gamma(s+q)^2 \Psi_{p, s}\} \leq C^2 \exp(-2bp). \quad (5.28)$$

To prove (5.28), we first consider the case $q = 1$. By Stirling's inequalities we have $\sqrt{2\pi} s^{s+1/2} e^{-s} \leq \Gamma(s+1) \leq e s^{s+1/2} e^{-s}$ for any $s \geq 1$. Hence, we estimate

$$\begin{aligned} c^{2s} \Gamma(s+1)^2 \Psi_{p, s} &\leq \frac{e^3}{\sqrt{2\pi}} c^{2s} s^{2s+1} \frac{(p-s)^{p-s}}{(p+s)^{p+s}} \left(\frac{p-s}{p+s}\right)^{\frac{1}{2}} \\ &\leq C s \left(\frac{cs}{p+s}\right)^{2s} \left(\frac{p-s}{p+s}\right)^{p-s}. \end{aligned} \quad (5.29)$$

Upon increasing the value of the constant c , we may absorb the linear factor in s in (5.29) into c^{2s} . In addition, we may assume that $p \geq [3(c+1)]$ (otherwise, we would simply increase the constant C in (5.29)). Then we choose $s = p/(c+1) \in [3, p]$. Inserting this into the previous bound gives

$$c^{2s} \Gamma(s+1)^2 \Psi_{p, s} \leq C \left(\frac{\frac{c}{c+1}}{1 + \frac{s}{p}}\right)^{\frac{2p}{c+1}} \left(\frac{1 - \frac{s}{p}}{1 + \frac{s}{p}}\right)^{p(1 - \frac{s}{p})} \leq C \left(\frac{c}{c+2}\right)^{p(1 + \frac{1}{c+1})}.$$

The bound (5.28) for $q = 1$ thus follows with $b(c) := -\frac{1}{2}(1 + \frac{1}{c+1}) \log \frac{c}{c+2} > 0$.

The case $q > 1$ is obtained analogously upon noting that by Stirling's formula, $\Gamma(s+q) \leq C(q) s^{q-1} \Gamma(s+1)$, and the polynomial factor in s may again be absorbed into c^{2s} by suitably increasing the value of c , which proves (5.28).

Now, to prove the assertion (5.27), we recall that $\Gamma(x+1) = x\Gamma(x)$ for all $x \geq 1$. Hence,

$$\Psi_{p-1, s-1} = (p-1+s)(p+s) \Psi_{p, s} \leq 4p^2 \Psi_{p, s}, \quad (5.30)$$

for any $p \geq 3$ and $s \in [3, p]$. Combining this estimate with (5.28), and absorbing the quadratic factor in p into the exponential term (by modifying the constants b and C) yield the desired result. \square

Lemma 5.8 and Lemma 5.9 immediately yield the following estimate.

PROPOSITION 5.10. *Under the regularity assumption (5.1), the hp-dG interpolation operator Πu defined in (5.17) satisfies the error bound*

$$\Upsilon_{\mathcal{D}_{\text{int}}}[\eta] \lesssim \exp(-2b\ell)$$

for some constant $b > 0$ independent of $\ell \geq 2$.

5.2.2. Submesh \mathfrak{D}_C^ℓ . Next, we consider the corner neighborhood \mathfrak{D}_C^ℓ in (3.10) and establish the analog of Proposition 5.10 for $\Upsilon_{\mathfrak{D}_C^\ell}[\eta]$. We may assume without loss of generality that $C = \{c\}$ (the general case of an arbitrary, but finite number of corners follows readily by superposition).

Elements $K \in \mathfrak{D}_C^\ell$ are shape-regular with

$$h_K \simeq h_K^\perp \simeq h_K^\parallel \quad \text{and} \quad r_c|_K \simeq d_K^c \simeq h_K, \quad (5.31)$$

according to (3.12) (see also Figure 3.1, left). We shall also use that

$$\forall 2 \leq j \leq \ell, \forall K \in \mathfrak{L}_\sigma^{\ell-j+1} \cap \mathfrak{D}_C^\ell : \quad d_K^c \simeq \sigma^{\ell-j}. \quad (5.32)$$

Let us further recall that the hp -extensions (Ex1)–(Ex4) introduced in [33] for the subspace $V_{\sigma,s}^\ell$ in (3.27) produce isotropic polynomial degrees that are uniform in each mesh layer. In the following, we simply denote these degrees by p_j , i.e.,

$$\forall 2 \leq j \leq \ell, \forall K \in \mathfrak{D}_C^\ell \cap \mathfrak{L}_\sigma^{\ell-j+1} : \quad p_K = p_j \geq 3. \quad (5.33)$$

By construction and for $\ell \rightarrow \infty$, the elemental polynomial degrees $\{p_j\}_{j \geq 2}$ form a *sequence of s -linear growth*:

$$\exists \chi \in (0, 1) : \quad \forall j \geq 2 : \quad \chi \leq p_j/j \leq \chi^{-1}. \quad (5.34)$$

We can now bound the consistency term $\Upsilon_{\mathfrak{D}_C^\ell}[\eta]$.

LEMMA 5.11. *Under the regularity assumption (5.1), we have*

$$\Upsilon_{\mathfrak{D}_C^\ell}[\eta] \lesssim \sum_{j=2}^{\ell} \sigma^{2(\ell-j) \min \beta} \Psi_{p_j-1, s_j-1} C_u^{2s_j} \Gamma(s_j + 6)^2$$

for any $s_j \in [3, p_j]$ and the constant C_u in (5.1).

Proof. Recalling that the elements in \mathfrak{D}_C^ℓ are shape-regular with $h_K \simeq h_K^\perp \simeq h_K^\parallel$, we conclude from Lemma 5.7, Lemma 5.19 and (5.31) that, for any $K \in \mathfrak{L}_\sigma^{\ell-j+1} \cap \mathfrak{D}_C^\ell$ for $2 \leq j \leq \ell$,

$$T^K[\eta] \lesssim d_K^c \|\hat{\eta}\|_{H_{\min}^2(\hat{K})}^2 \lesssim \Psi_{p_j-1, s_j-1} \sum_{s_j+1 \leq |\alpha| \leq s_j+5} (d_K^c)^{2|\alpha|-2} \|\mathbf{D}^\alpha u\|_{L^2(K)}^2. \quad (5.35)$$

Since possibly $K \cap \omega_{ce} \neq \emptyset$ we write

$$\|\mathbf{D}^\alpha u\|_{L^2(K)}^2 = \|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 + \sum_{e \in \mathcal{E}_c} \left(\|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 + \|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_{ce})}^2 \right). \quad (5.36)$$

By inserting the weight r_c , we first obtain

$$\begin{aligned} \|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 &\lesssim (d_K^c)^{2+2\beta_e-2|\alpha|} \|r_c^{-1-\beta_e+|\alpha|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 \\ &\lesssim (d_K^c)^{2+2 \min \beta-2|\alpha|} \|r_c^{-1-\beta_e+|\alpha|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2. \end{aligned} \quad (5.37)$$

Let now e be an edge in \mathcal{E}_c . By noticing that $r_e|_K \simeq d_K^e \simeq d_K^c \simeq h_K$, we then have

$$\begin{aligned} \|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 &\lesssim (d_K^c)^{2+2\beta_e-2|\alpha^+|} \|r_e^{-1-\beta_e+|\alpha^+|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2 \\ &\lesssim (d_K^c)^{2+2 \min \beta-2|\alpha^+|} \|r_e^{-1-\beta_e+|\alpha^+|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_e)}^2. \end{aligned} \quad (5.38)$$

Similarly, since $\rho_{ce} \simeq 1$ on K ,

$$\begin{aligned} \|\mathbf{D}^\alpha u\|_{L^2(K \cap \omega_{ce})}^2 &\lesssim (d_K^c)^{2+2\beta_c-2|\alpha|} \|r_c^{-1-\beta_c+|\alpha|} \rho_{ce}^{-1-\beta_e+|\alpha^\perp|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_{ce})}^2 \\ &\lesssim (d_K^c)^{2+2\min\beta_c-2|\alpha|} \|r_c^{-1-\beta_c+|\alpha|} \rho_{ce}^{-1-\beta_e+|\alpha^\perp|} \mathbf{D}^\alpha u\|_{L^2(K \cap \omega_{ce})}^2. \end{aligned} \quad (5.39)$$

Combining (5.35)–(5.39) and using (5.1) yields

$$\begin{aligned} T^K[\eta] &\lesssim \Psi_{p_j-1, s_j-1} (d_K^c)^{2\min\beta} \Psi_{p_j-1, s_j-1} \|u\|_{M_{-1-\beta}^{s_j+5}(K)}^2 \\ &\leq C_u^{2s_j} \Gamma(s_j+6)^2 (d_K^c)^{2\min\beta} \Psi_{p_j-1, s_j-1} \end{aligned} \quad (5.40)$$

for any $K \in \mathfrak{L}_\sigma^{\ell-j+1} \cap \mathfrak{D}_C^\ell$ with $2 \leq j \leq \ell$.

Next, we sum the estimate (5.40) over all mesh layers. Referring to (5.32), we arrive at

$$\begin{aligned} \Upsilon_{\mathfrak{D}_C^\ell}[\eta] &\lesssim \sum_{j=2}^{\ell} \sum_{K \in \mathfrak{L}_\sigma^{\ell-j+1} \cap \mathfrak{D}_C^\ell} T^K[\eta] \\ &\lesssim \sum_{j=2}^{\ell} \sum_{K \in \mathfrak{L}_\sigma^{\ell+j-1} \cap \mathfrak{D}_C^\ell} \Psi_{p_j-1, s_j-1} \sigma^{2(\ell-j)\min\beta} C_u^{2s_j} \Gamma(s_j+6)^2, \end{aligned} \quad (5.41)$$

for any integers $3 \leq s_j \leq p_j$. The conclusion now follows by noting that the cardinality of the set $\mathfrak{L}_\sigma^{\ell+j-1} \cap \mathfrak{D}_C^\ell$ is uniformly bounded in j and by using an interpolation argument as in Lemma 5.8. \square

Finally, to show that the bound in Lemma 5.11 is exponentially convergent, we shall make use of the following refinement of Lemma 5.9.

LEMMA 5.12. *For every $0 < \sigma < 1$, constants $\bar{\beta} > 0$, $c > 0$, $q \geq 0$ and sequences $\{p_j\}_{j=2}^\infty$, $p_j \geq 3$, of \mathbf{s} -linear growth as in (5.34), there exist constants $b > 0$ and $C > 0$ (depending only on q , σ , \mathbf{s} , χ and $\bar{\beta}$) such that*

$$\left\{ \sum_{j=2}^{\ell} \sigma^{2(\ell-j)\bar{\beta}} \min_{s_j \in [3, p_j]} c^{2s_j} \Gamma(s_j+q)^2 \Psi_{p_j-1, s_j-1} \right\} \leq C^2 \exp(-2b\ell) \quad (5.42)$$

for every $\ell \geq 2$.

Proof. By Lemma 5.9,

$$\min_{s_j \in [3, p_j]} c^{2s_j} \Gamma(s_j+q)^2 \Psi_{p_j-1, s_j-1} \leq C \exp(-2bp_j),$$

with constants $b > 0$ and $C > 0$ only depending on the values of q and c in that Lemma. Hence, since $p_j \simeq s_j$, the sum (5.42) can be bounded by

$$\sum_{j=2}^{\ell} \sigma^{2(\ell-j)\bar{\beta}} \min_{s_j \in [3, p_j]} c^{2s_j} \Gamma(s_j+q)^2 \Psi_{p_j-1, s_j-1} \leq C \sum_{j=2}^{\ell} e^{-2\bar{\beta}|\log \sigma|(\ell-j)-2bs_j}. \quad (5.43)$$

We split the sum into two partial sums as follows: first, a sum over $2 \leq j \leq \theta\ell$ (corresponding to mesh layers with ‘small’ elements) and second, a sum over $\theta\ell \leq j \leq$

ℓ with a parameter $0 < \theta < 1$ which is independent of ℓ . Then we estimate the sum in (5.43),

$$\begin{aligned} \sum_{j=2}^{\ell} e^{-2\bar{\beta}|\log \sigma|(\ell-j)-2bsj} &\leq \sum_{j=2}^{\lfloor \theta\ell \rfloor} e^{-2\bar{\beta}|\log \sigma|(\ell-j)-2bsj} + \sum_{j=\lceil \theta\ell \rceil}^{\ell} e^{-2\bar{\beta}|\log \sigma|(\ell-j)-2bsj} \\ &\lesssim \lceil \theta\ell \rceil e^{-2\bar{\beta}|\log \sigma|(1-\theta)\ell} + \lceil (1-\theta)\ell \rceil e^{-2bs\theta\ell}, \end{aligned}$$

from where the assertion (5.42) follows. \square

Lemma 5.11 and Lemma 5.12 give the following result.

PROPOSITION 5.13. *Under the regularity assumption (5.1), the hp-dG interpolation operator Π_u defined in (5.17) satisfies the error bound*

$$\Upsilon_{\mathcal{D}_{\mathcal{E}}^{\ell}}[\eta] \lesssim \exp(-2b\ell)$$

for some constant $b > 0$ independent of $\ell \geq 2$.

5.2.3. Submesh $\mathcal{D}_{\mathcal{E}}^{\ell}$. In this section, we consider the edge neighborhood $\Omega_{\mathcal{E}}$ in (2.5) and shall bound $\Upsilon_{\mathcal{D}_{\mathcal{E}}^{\ell}}[\eta]$. Again, we may assume without loss of generality that $\mathcal{E} = \{e\}$, i.e., that we are dealing with a single edge e (the general case follows by superposition).

We recall that for elements $K \in \mathcal{D}_{\mathcal{E}}^{\ell}$, the diameters h_K^{\parallel} parallel to e are of order one, while the diameters h_K^{\perp} perpendicular to edge e satisfy

$$r_e|_K \simeq d_K^e \simeq h_K^{\perp}, \quad (5.44)$$

according to (3.13) (see also Figure 3.1, middle). We also observe that

$$\forall 2 \leq j \leq \ell, \forall K \in \mathfrak{L}_{\sigma}^{\ell-j+1} \cap \mathcal{D}_{\mathcal{E}}^{\ell}: \quad d_K^e \simeq \sigma^{\ell-j}. \quad (5.45)$$

The hp-extensions (Ex1)–(Ex4) of [33] for the space $V_{\sigma,s}^{\ell}$ in (3.27) yield anisotropic elemental polynomial degrees p_K^{\perp} and p_K^{\parallel} that are identical over all elements K in $\mathfrak{L}_{\sigma}^{\ell-j+1} \cap \mathcal{D}_{\mathcal{E}}^{\ell}$; we thus simply denote them by p_j^{\perp} and p_j^{\parallel} , respectively. The variation of the polynomial degrees p_j^{\perp} across mesh layers is s -linear as in (5.34), while the polynomial degrees $p_j^{\parallel} \geq 3$ parallel to the edge e are constant and proportional to ℓ , i.e., $p_j^{\parallel} = p^{\parallel} \geq 3$, with $p^{\parallel} \simeq \ell$. Next, we bound $T^K[\eta]$ for $K \in \mathfrak{L}_{\sigma}^{\ell-j+1} \cap \mathcal{D}_{\mathcal{E}}^{\ell}$.

LEMMA 5.14. *Let $K \in \mathfrak{L}_{\sigma}^{\ell-j+1} \cap \mathcal{D}_{\mathcal{E}}^{\ell}$ for $2 \leq j \leq \ell$. Then, under the regularity assumption (5.1), there holds*

$$T^K[\eta] \lesssim \sigma^{2(\ell-j)\beta_e} \left(\Psi_{p_j^{\perp}-1, s_j^{\perp}-1} C^{2s_j^{\perp}} \Gamma(s_j^{\perp} + 6)^2 + \Psi_{p^{\parallel}-1, s^{\parallel}-1} C^{2s^{\parallel}p} \Gamma(s^{\parallel} + 6)^2 \right)$$

for any $s_j^{\perp} \in [3, p_j^{\perp}]$ and $s^{\parallel} \in [3, p^{\parallel}]$.

Proof. Using that h_K^{\parallel} is of order one, Lemma 5.7 and Lemma 5.6 yield

$$T^K[\eta] \lesssim \|\widehat{\eta}\|_{H_{\text{mix}}^2(\widehat{K})}^2 \lesssim E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K) + E_{p_j^{\perp}, s_j^{\perp}}^{\perp}(K),$$

with $E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K)$ and $E_{p_j^{\perp}, s_j^{\perp}}^{\perp}(K)$ defined in Lemma 5.6. Taking into account (5.44) and that h_K^{\parallel} is of order one, we can bound $E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K)$ as follows:

$$E_{p^{\parallel}, s^{\parallel}}^{\parallel}(K) \simeq \Psi_{p^{\parallel}-1, s^{\parallel}-1} \sum_{\alpha_1^{\perp} \leq 2, \alpha_2^{\perp} \leq 2} (d_K^e)^{2|\alpha^{\perp}|-2} \|\mathbb{D}_{\perp}^{\alpha^{\perp}} \mathbb{D}_{\parallel}^{s^{\parallel}+1} u\|_{L^2(K)}^2.$$

Since $K \cap \Omega_0 = \emptyset$ and $K \cap \omega_c = \emptyset$ for all $c \in \mathcal{C}$, we may write

$$\|D^\alpha u\|_{L^2(K)}^2 = \|D^\alpha u\|_{L^2(K \cap \omega_e)}^2 + \sum_{c \in \mathcal{C}_e} \|D^\alpha u\|_{L^2(K \cap \omega_{ce})}^2$$

We then proceed by inserting the weight function r_e and employing (5.44):

$$\|D^\alpha u\|_{L^2(K \cap \omega_e)}^2 \lesssim (d_K^e)^{2+2\beta_e-2|\alpha^+|} \|r_e^{-1-\beta_e+|\alpha^+|} D^\alpha u\|_{L^2(K \cap \omega_e)}^2.$$

If $K \cap \omega_{ce} \neq \emptyset$ for a corner $c \in \mathcal{C}_e$, then $r_c|_K \simeq h_K^\parallel$ is bounded away from zero, and $\rho_{ce} \simeq r_e$. Hence, we readily obtain

$$\|D^\alpha u\|_{L^2(K \cap \omega_{ce})}^2 \lesssim (d_K^e)^{2+2\beta_e-2|\alpha^+|} \|r_c^{-1-\beta_c+|\alpha^+|} \rho_{ce}^{-1-\beta_e+|\alpha^+|} D^\alpha u\|_{L^2(K \cap \omega_{ce})}^2.$$

Combining these estimates with (5.45) and (5.1), we find that

$$\begin{aligned} E_{p^\parallel, s^\parallel}^\parallel(K) &\lesssim \Psi_{p^\parallel-1, s^\parallel-1} (d_K^e)^{2\beta_e} \|u\|_{M_{-1-\beta_e}^{s^\parallel+5}(K)}^2 \\ &\lesssim \Psi_{p^\parallel-1, s^\parallel-1} \sigma^{2(\ell-j)\beta_e} C^{2s^\parallel} \Gamma(s^\parallel + 6)^2. \end{aligned} \quad (5.46)$$

Similarly, we bound $E_{p_j^\perp, s_j^\perp}^\perp(K)$:

$$E_{p_j^\perp, s_j^\perp}^\perp(K) \lesssim \Psi_{p_j^\perp-1, s_j^\perp-1} \sum_{s_j^\perp+1 \leq |\alpha^+| \leq s_j^\perp+3, \alpha^\parallel \leq 2} (d_K^e)^{2|\alpha^+|-2} \|D_\perp^{\alpha^+} D_\parallel^{\alpha^\parallel} u\|_{L^2(K)}^2.$$

Proceeding as before, we see that

$$\begin{aligned} E_{p_j^\perp, s_j^\perp}^\perp(K) &\lesssim \Psi_{p_j^\perp-1, s_j^\perp-1} (d_K^e)^{2\beta_e} \|u\|_{M_{-1-\beta_e}^{s_j^\perp+5}(K)}^2 \\ &\leq \Psi_{p_j^\perp-1, s_j^\perp-1} \sigma^{2(\ell-j)\beta_e} C^{2s_j^\perp} \Gamma(s_j^\perp + 6)^2. \end{aligned} \quad (5.47)$$

The bounds (5.2.3), (5.46) and (5.47) imply the desired estimate for integer regularity exponents. An interpolation argument as in (5.26) proves the assertion. \square

PROPOSITION 5.15. *Under the regularity assumption (5.1), the hp-dG interpolation operator Πu defined in (5.17) satisfies the error bound*

$$\Upsilon_{\mathcal{D}_\varepsilon^\ell}[\eta] \lesssim \exp(-2b\ell)$$

for some constant $b > 0$ independent of $\ell \geq 2$.

Proof. Summing the bound in Lemma 5.14 over all mesh layers and noting that the cardinality of the sets $\mathcal{L}_\sigma^{\ell-j+1} \cap \mathcal{D}_\varepsilon^\ell$ are uniformly bounded in j result in

$$\Upsilon_{\mathcal{D}_\varepsilon^\ell}[\eta] \lesssim S^\parallel + S^\perp, \quad (5.48)$$

where

$$\begin{aligned} S^\perp &= \sum_{j=2}^{\ell} \Psi_{p_j^\perp-1, s_j^\perp-1} \sigma^{2(\ell-j)\beta_e} C_u^{2s_j^\perp} \Gamma(s_j^\perp + 6)^2, \\ S^\parallel &= \sum_{j=2}^{\ell} \Psi_{p^\parallel-1, s^\parallel-1} \sigma^{2(\ell-j)\beta_e} C_u^{2s^\parallel} \Gamma(s^\parallel + 6)^2. \end{aligned}$$

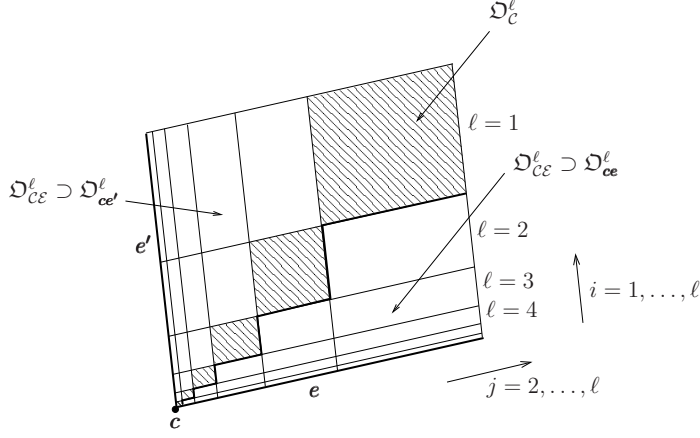


FIG. 5.1. Elements in $\mathfrak{D}_{C\mathcal{E}}^\ell$.

Due to the \mathbf{s} -linearity of the degrees p_j^\perp and the fact that $\beta_e > 0$, Lemma 5.12 allows us to find parameters $s_j^\perp \in [3, p_j^\perp]$ such that $S^\perp \lesssim e^{-2b\ell}$.

In the sum S^\parallel , the polynomial degree p^\parallel parallel to the edge e is constant and proportional to ℓ . Applying Lemma 5.9, we can find $s^\parallel \in [3, p^\parallel]$ such that

$$S^\parallel \lesssim e^{-2b\ell} \sum_{j=2}^{\ell} \sigma^{2(\ell-j)\beta_e} \lesssim e^{-2\bar{b}\ell},$$

which completes the proof. \square

5.2.4. Submesh $\mathfrak{D}_{C\mathcal{E}}^\ell$. Finally, we consider the corner-edge neighborhood $\Omega_{C\mathcal{E}}$ defined in (3.10) and prove the exponential convergence of $\Upsilon_{\mathfrak{D}_{C\mathcal{E}}^\ell}[\eta]$. Again, it is sufficient to consider a single corner \mathbf{c} with a single edge $e = e_{\mathbf{c}}$ originating from it, i.e., $\mathcal{C} = \{\mathbf{c}\}$ and $\mathcal{E} = \{e\}$. In view of (3.10), an element $K \in \mathfrak{D}_{C\mathcal{E}}^\ell$ has empty intersection with Ω_0 , Ω_C , and $\Omega_{\mathcal{E}}$. Hence, if the edges and vertices are sufficiently separated by the initial mesh, we may assume $K \subseteq \omega_{ce}$.

It will be convenient in the error analysis to group elements $K \in \mathfrak{D}_{C\mathcal{E}}^\ell$ into sets $\mathfrak{L}_{C\mathcal{E}}^{ij}$ of elements whose aspect ratios are equivalent uniformly with respect to ℓ . To this end, we observe that there exists $\kappa(\sigma) > 0$ such that for all $\ell \geq 2$ and elements $K \in \mathfrak{D}_{C\mathcal{E}}^\ell$ there are indices $2 \leq i \leq j \leq \ell$ such that

$$\kappa^{-1}\sigma^j \leq h_K^\perp \leq \kappa\sigma^j, \quad \kappa^{-1}\sigma^i \leq h_K^\parallel \leq \kappa\sigma^i. \quad (5.49)$$

We say that $K \in \mathfrak{D}_{C\mathcal{E}}^\ell$ belongs to $\mathfrak{L}_{C\mathcal{E}}^{ij}$ if it satisfies (5.49) with indices (i, j) (refer to (3.17) for notation). Then we have (with a possibly nondisjoint union)

$$\mathfrak{D}_{C\mathcal{E}}^\ell = \bigcup_{j=2}^{\ell} \bigcup_{i=2}^j \mathfrak{L}_{C\mathcal{E}}^{ij}. \quad (5.50)$$

We refer to Figure 5.1 for the notation and illustration and observe that the cardinality of all $\mathfrak{L}_{C\mathcal{E}}^{ij}$ is uniformly bounded, independently of i, j .

From the hp -extensions (Ex1)–(Ex4) in [33] we obtain polynomial degree distributions for the dG subspace $V_{\sigma, \mathbf{s}}^\ell$ that satisfy

$$\forall K \in \mathfrak{L}_{\mathcal{CE}}^{ij} : \quad \mathbf{p}_K = (p_i^\perp, p_j^\parallel) \simeq (\mathbf{s}i, \mathbf{s}j), \quad 2 \leq i \leq j \leq \ell. \quad (5.51)$$

Moreover,

$$\forall K \in \mathfrak{L}_{\mathcal{CE}}^{ij} : \quad r_e|_K \simeq d_K^e \simeq h_K^\perp \simeq \sigma^{\ell-i}, \quad r_c|_K \simeq d_K^c \simeq h_K^\parallel \simeq \sigma^{\ell-j}. \quad (5.52)$$

LEMMA 5.16. *Let $K \in \mathfrak{L}_{\mathcal{CE}}^{ij}$ with $\mathbf{p}_K = (p_i^\perp, p_j^\parallel)$ and with $p_i^\perp, p_j^\parallel \geq 3$. Then, under the regularity assumption (5.1), for any $s_i^\perp \in [3, p_i^\perp]$, $s_j^\parallel \in [3, p_j^\parallel]$, there holds*

$$T^K[\eta] \leq \sigma^{2(\ell-j)(\beta_c - \beta_e)} \sigma^{2(\ell-i)\beta_e} (1 + \sigma^{2(j-i)}) N[u]_{ij}, \quad (5.53)$$

where

$$N[u]_{ij} = \Psi_{p_i^\perp-1, s_i^\perp-1} C^{2s_i^\perp} \Gamma(s_i^\perp + 6)^2 + C^{2s_j^\parallel} \Psi_{p_j^\parallel-1, s_j^\parallel-1} \Gamma(s_j^\parallel + 6)^2. \quad (5.54)$$

Proof. By combining Lemma 5.7, (5.52), and Lemma 5.6, we obtain

$$\begin{aligned} T^K[\eta] &\lesssim ((d_K^e)^2 (d_K^c)^{-1} + d_K^c) \|\widehat{\eta}\|_{H_{\text{mix}}^2(\widehat{K})}^2 \\ &\lesssim d_K^c (1 + (d_K^e)^2 (d_K^c)^{-2}) \left(E_{p_j^\parallel, s_j^\parallel}^\parallel(K) + E_{p_j^\perp, s_j^\perp}^\perp(K) \right), \end{aligned} \quad (5.55)$$

with $E_{p_j^\parallel, s_j^\parallel}^\parallel(K)$ and $E_{p_j^\perp, s_j^\perp}^\perp(K)$ defined in Lemma 5.6. Using (5.52), we estimate $E_{p_j^\parallel, s_j^\parallel}^\parallel(K)$ as follows:

$$E_{p_j^\parallel, s_j^\parallel}^\parallel(K) \simeq \Psi_{p_j^\parallel-1, s_j^\parallel-1} \sum_{\alpha_1^\perp \leq 2, \alpha_2^\perp \leq 2} (d_K^e)^{2|\alpha^\perp|-2} (d_K^c)^{2s_j^\parallel+1} \|\mathbf{D}^{\alpha^\perp} \mathbf{D}_\parallel^{s_j^\parallel+1} u\|_{L^2(K)}^2.$$

Then, inserting the weight ρ_{ce} ,

$$\begin{aligned} \|\mathbf{D}^{\alpha^\perp} \mathbf{D}_\parallel^{s_j^\parallel+1} u\|_{L^2(K)}^2 &\lesssim (d_K^c)^{2+2\beta_c-2(|\alpha^\perp|+s_j^\parallel+1)-2-2\beta_e+2|\alpha^\perp|} \\ &\times (d_K^e)^{2+2\beta_e-2|\alpha^\perp|} \|r_c^{-1-\beta_c+(|\alpha^\perp|+s_j^\parallel+1)} \rho_{ce}^{-1-\beta_e+|\alpha^\perp|} \mathbf{D}^{\alpha^\perp} \mathbf{D}_\parallel^{s_j^\parallel+1} u\|_{L^2(K)}^2. \end{aligned}$$

Hence,

$$E_{p_j^\parallel, s_j^\parallel}^\parallel(K) \lesssim \Psi_{p_j^\parallel-1, s_j^\parallel-1} (d_K^c)^{2(\beta_c - \beta_e) - 1} (d_K^e)^{2\beta_e} \|u\|_{M_{-1-\beta}^{s_j^\parallel+5}(K)}^2. \quad (5.56)$$

Similarly,

$$E_{p_j^\perp, s_j^\perp}^\perp(K) \simeq \Psi_{p_i^\perp-1, s_i^\perp-1} \sum_{s_i^\perp+1 \leq |\alpha^\perp| \leq s_i^\perp+3, \alpha^\parallel \leq 2} (d_K^e)^{2|\alpha^\perp|-2} (d_K^c)^{2\alpha^\parallel-1} \|\mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel^{\alpha^\parallel} u\|_{L^2(K)}^2,$$

where

$$\begin{aligned} \|\mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel^{\alpha^\parallel} u\|_{L^2(K)}^2 &\lesssim (d_K^c)^{2\alpha^\parallel-1+2+2\beta_c-2(|\alpha^\perp|+\alpha^\parallel)-2-2\beta_e+2|\alpha^\perp|} \\ &\times (d_K^e)^{2+2\beta_e-2|\alpha^\perp|} \|r_c^{-1-\beta_c+(|\alpha^\perp|+\alpha^\parallel)} \rho_{ce}^{-1-\beta_e+|\alpha^\perp|} \mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel^{\alpha^\parallel} u\|_{L^2(K)}^2. \end{aligned}$$

Thus,

$$E_{p_j^\perp, s_j^\perp}^\perp(K) \lesssim \Psi_{p_i^\parallel - 1, s_i^\parallel - 1}(d_K^c)^{2(\beta_c - \beta_e) - 1} (d_K^e)^{2\beta_e} \|u\|_{M_{-1-\beta}^{s_i^\perp + 5}(K)}^2. \quad (5.57)$$

Referring to (5.55), (5.56), (5.57), and to the regularity property (5.1) shows that

$$T^K[\eta] \lesssim (d_K^c)^{2(\beta_c - \beta_e)} (d_K^e)^{2\beta_e} (1 + (d_K^e)^2 (d_K^c)^{-2}) N[u]_{ij}, \quad (5.58)$$

which is the assertion for integer regularity exponents. An interpolation argument as in Lemma 5.8 and using the relations in (5.52) once more finish the proof. \square

We are now ready to prove exponential convergence of $\Upsilon_{\mathfrak{D}_{\mathcal{C}\mathcal{E}}^\ell}[\eta]$.

PROPOSITION 5.17. *Under the regularity assumption (5.1), the hp-dG interpolation operator Πu defined in (5.17) satisfies the error bound*

$$\Upsilon_{\mathfrak{D}_{\mathcal{C}\mathcal{E}}^\ell}[\eta] \lesssim \exp(-2b\ell)$$

for some constant $b > 0$ independent of $\ell \geq 2$.

Proof. Summing up the estimate in Lemma 5.16 over all the mesh layers (using the fact that the cardinalities of the sets $\mathfrak{L}_{\mathcal{C}\mathcal{E}}^{ij}$ are uniformly bounded) results in

$$\begin{aligned} \Upsilon_{\mathfrak{D}_{\mathcal{C}\mathcal{E}}^\ell}[\eta] &\lesssim \sum_{j=2}^{\ell} \sum_{i=2}^j \sigma^{2(\ell-j)(\beta_c - \beta_e)} \sigma^{2(\ell-i)\beta_e} (1 + \sigma^{2(j-i)}) N[u]_{ij} \\ &\lesssim \sum_{j=2}^{\ell} \sum_{i=2}^j \sigma^{2(\ell-j)(\beta_c - \beta_e)} \sigma^{2(\ell-i)\beta_e} N[u]_{ij} \lesssim S^\perp + S^\parallel, \end{aligned}$$

where

$$\begin{aligned} S^\perp &= \sum_{j=2}^{\ell} \sigma^{2(\ell-j)(\beta_c - \beta_e)} \sum_{i=2}^j \sigma^{2(\ell-i)\beta_e} \Psi_{p_i^\perp - 1, s_i^\perp - 1} C^{2s_i^\perp} \Gamma(s_i^\perp + 6)^2, \\ S^\parallel &= \sum_{j=2}^{\ell} \sigma^{2(\ell-j)(\beta_c - \beta_e)} \sum_{i=2}^j \sigma^{2(\ell-i)\beta_e} \Psi_{p_j^\parallel - 1, s_j^\parallel - 1} C^{2s_j^\parallel} \Gamma(s_j^\parallel + 6)^2. \end{aligned}$$

Let us first bound S^\perp . To do so, we write

$$\begin{aligned} S^\perp &= \sum_{j=2}^{\ell} \sigma^{2(\ell-j)(\beta_c - \beta_e + \beta_e)} \left(\sum_{i=2}^j \sigma^{2(\ell-i)\beta_e} \sigma^{2(j-i)\beta_e} \Psi_{p_i^\perp - 1, s_i^\perp - 1} C^{2s_i^\perp} \Gamma(s_i^\perp + 6)^2 \right) \\ &= \sum_{j=2}^{\ell} \sigma^{2(\ell-j)\beta_c} \left(\sum_{i=2}^j \sigma^{2(j-i)\beta_e} \Psi_{p_i^\perp - 1, s_i^\perp - 1} C^{2s_i^\perp} \Gamma(s_i^\perp + 6)^2 \right). \end{aligned}$$

Then, we use the fact that $\beta_e > 0$, Lemma 5.12 with ℓ replaced by j , and the s-linearity of p_i^\perp to obtain parameters $s_i^\perp \in [3, p_i^\perp]$ and a constant $b_1 > 0$ such that

$$\sum_{i=2}^j \sigma^{2(j-i)\beta_e} \Psi_{p_i^\perp - 1, s_i^\perp - 1} C^{2s_i^\perp} \Gamma(s_i^\perp + 6)^2 \lesssim e^{-2b_1 j}, \quad j \geq 2.$$

Therefore, we conclude that there is second constant $b_2 > 0$ such that

$$S^\perp \lesssim \sum_{j=2}^{\ell} \sigma^{2(\ell-j)(\beta_c - \beta_e)} e^{-2b_1 j} \lesssim \sum_{j=2}^{\ell} \sigma^{2(\ell-j)(\beta_c - \beta_e + \beta_e)} e^{-2b_1 j} = \sum_{j=2}^{\ell} e^{-2b_2(\ell-j) - 2b_1 j}.$$

With $b = \min\{b_1, b_2\}$, we thus obtain

$$S^\perp \lesssim \sum_{j=2}^{\ell} e^{-2b(\ell+j-j)} \lesssim \ell e^{-2b\ell} \lesssim e^{-2\bar{b}\ell}. \quad (5.59)$$

To prove exponential convergence of S^\parallel , we first note that

$$\sum_{i=2}^j \sigma^{2(\ell-i)\beta_e} = \sigma^{2(\ell-j)\beta_e} \frac{1 - \sigma^{2\beta_e(j-1)}}{1 - \sigma^{2\beta_e}} \leq C(\sigma, \beta_e) \sigma^{2(\ell-j)\beta_e}, \quad j \geq 2. \quad (5.60)$$

To see (5.60), we sum the geometric series as follows:

$$\sum_{i=2}^j \sigma^{2(\ell-i)\beta_e} = \sigma^{2(\ell-j)\beta_e} \frac{1 - \sigma^{2\beta_e(j-1)}}{1 - \sigma^{2\beta_e}} \leq \frac{1}{1 - \sigma^{2\beta_e}} \sigma^{2(\ell-j)\beta_e}.$$

Hence, by (5.60) and Lemma 5.12,

$$S^\parallel \lesssim \sum_{j=2}^{\ell} \sigma^{2(\ell-j)\beta_e} \Psi_{p_j^\parallel - 1, s_j^\parallel - 1} C^{2s_j^\parallel} \Gamma(s_j^\parallel + 6)^2 \lesssim e^{-2b\ell}$$

This completes the proof. \square

5.3. Approximation in \mathfrak{D}_σ^ℓ . By combining the bound in (5.20) with the results in Propositions 5.10, 5.13, 5.15 and 5.17, we now immediately obtain the following approximation property in \mathfrak{D}_σ^ℓ .

THEOREM 5.18. *Consider a family $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell=1}^\infty$ of axi-parallel σ -geometric meshes with an anisotropic \mathbf{s} -linear polynomial degree vector $\mathbf{p}_2(\mathcal{M}_\sigma^{(\ell)})$ as in (3.27) with degrees greater or equal to 3. Then for $u \in A_{-1-\beta}(\Omega)$ and $\ell \geq 2$, there is a projection $\Pi_\ell : A_{-1-\beta}(\Omega) \rightarrow V(\mathfrak{D}_\sigma^\ell, \Phi(\mathfrak{D}_\sigma^\ell), \mathbf{p}_2(\mathfrak{D}_\sigma^\ell))$ that satisfies the error bound*

$$\Upsilon_{\mathfrak{D}_\sigma^\ell} [u - \Pi_\ell u] \leq C \exp(-2b\ell), \quad \ell \geq 2. \quad (5.61)$$

Here, the constants $b > 0$ and $C > 0$ are independent of ℓ (but depend on the weight vector β , the μ -variation of the degree vectors, the geometric grading factor σ , the slope \mathbf{s} , the regularity constant C_u in (5.1), and on the initial mesh \mathcal{M}_0).

Moreover, in general there holds

$$N_\ell(\mathbf{s}) := \dim(V_{\sigma, \mathbf{s}}^\ell) \simeq b \ell^5 + O(\ell^4), \quad \ell \rightarrow \infty, \quad (5.62)$$

and the approximation bound (5.61) can be written as

$$\Upsilon_{\mathfrak{D}_\sigma^\ell} [u - \Pi_\ell u] \leq C \exp(-2b\sqrt[5]{N}), \quad N = \dim(V_{\sigma, \mathbf{s}}^\ell) \rightarrow \infty. \quad (5.63)$$

By repeating verbatim the proofs of Propositions 5.13, 5.15 and 5.17 with uniform polynomial degrees, and by replacing in these proofs each reference to Lemma 5.12 by a reference to Lemma 5.9, we obtain the following corollary.

COROLLARY 5.19. *Under the assumptions of Theorem 5.18, but for uniform polynomial degrees $p_K = p \simeq \ell$ (with $p \geq 3$) in all elements $K \in \mathfrak{D}_\sigma^\ell$, $\ell \geq 2$, the error bounds (5.61), (5.63) remain valid, however, in (5.62) and (5.63), the constant b is replaced by a smaller value \bar{b} .*

5.4. Error Bounds on \mathfrak{T}_σ^ℓ . We now address the error in elements in the terminal layers $\mathfrak{T}_\sigma^\ell \subset \mathcal{M}_\sigma^{(\ell)}$. Again, we proceed separately for elements near edges and vertices and recall, to this end, the partition (3.19): $\mathfrak{T}_\sigma^\ell = \mathcal{V}_c^\ell \cup \mathcal{V}_e^\ell$. Based on this, we write

$$\Upsilon_{\mathfrak{T}_\sigma^\ell}[\eta] \leq \Upsilon_{\mathcal{V}_c^\ell}[\eta] + \Upsilon_{\mathcal{V}_e^\ell}[\eta], \quad (5.64)$$

where $\Upsilon_{\mathfrak{T}_\sigma^\ell}[\eta]$ is from (4.9) in Theorem 4.2 and the consistency errors $\Upsilon_{\mathcal{V}_c^\ell}[\eta]$, $\Upsilon_{\mathcal{V}_e^\ell}[\eta]$ are taken on the respective submeshes of \mathfrak{T}_σ^ℓ .

The construction of the hp -interpolant Πu from (4.6) in \mathfrak{T}_σ^ℓ will exploit the homogeneous essential boundary conditions: for functions in $M_{-1-\beta}^1(\Omega)$, the corresponding L^2 - and H^1 -terms in the norm (2.7) on $M_{-1-\beta}^1(\Omega)$ carry weights with negative exponents. Consequently, it will be sufficient to approximate the solution of (1.1)–(1.2) on the exponentially small elements in \mathfrak{T}_σ^ℓ by the zero function: we set $\Pi u|_K \equiv 0$ for all $K \in \mathfrak{T}_\sigma^\ell$ and, hence, $\eta = u$ in (4.6), and (4.9).

In the sequel, the interpolation errors on the subsets \mathcal{V}_c^ℓ , \mathcal{V}_e^ℓ will be analyzed separately. We first prove the following auxiliary result:

LEMMA 5.20. *Let $K \in \mathfrak{T}_\sigma^\ell$, $u \in M_{-1-\beta}^2(K)$, and $0 \leq j \lesssim \ell$ be chosen such that $h_K^\perp \simeq \sigma^\ell$ and $h_K^\parallel \simeq \sigma^{\ell-j}$ (cf. Proposition 3.4). Then*

$$\|\nabla u\|_{L^1(K)} \leq C \sigma^{\ell(\frac{3}{2} + \min \beta) - \frac{j}{2}} \|u\|_{M_{-1-\beta}^1(K)},$$

and

$$\sigma^{-\ell} \|\mathbf{D} \nabla u\|_{L^1(K)} + \|\mathbf{D}_\perp \nabla u\|_{L^1(K)} \leq C \sigma^{\ell(\frac{1}{2} + \min \beta) - \frac{j}{2}} \|u\|_{M_{-1-\beta}^2(K)},$$

where the constant $C > 0$ does not depend on $u, \sigma, \ell, \mathbf{p}$ and β .

Proof. We may assume that there is at most one corner $\mathbf{c} \in \mathcal{C}$ such that $K \cap \omega_{\mathbf{c}} \neq \emptyset$ or $K \cap (\omega_{\mathbf{c}e} \cup \omega_e) \neq \emptyset$ for some edge $e \in \mathcal{E}_{\mathbf{c}}$. Then we write

$$\|\nabla u\|_{L^1(K)} = \|\nabla u\|_{L^1(K \cap \omega_{\mathbf{c}})} + \sum_{e \in \mathcal{E}_{\mathbf{c}}} \left(\|\nabla u\|_{L^1(K \cap \omega_e)} + \|\nabla u\|_{L^1(K \cap \omega_{\mathbf{c}e})} \right).$$

First, note that if $K \cap \omega_{\mathbf{c}} \neq \emptyset$, then K must be isotropic with $h_K^\perp \simeq h_K^\parallel \simeq h_K \simeq \sigma^\ell$ (i.e., $j = 0$). Thus, Hölder's inequality and the fact that $r_{\mathbf{c}} \lesssim \sigma^\ell$, $|K| \simeq \sigma^{3\ell}$ yield

$$\|\nabla u\|_{L^1(K \cap \omega_{\mathbf{c}})} \leq \|r_{\mathbf{c}}^{\beta_{\mathbf{c}}}\|_{L^2(K \cap \omega_{\mathbf{c}})} \|r_{\mathbf{c}}^{-1-\beta_{\mathbf{c}}+1} \nabla u\|_{L^2(K \cap \omega_{\mathbf{c}})} \lesssim \sigma^{\ell(\frac{3}{2} + \beta_{\mathbf{c}})} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{\mathbf{c}})}.$$

Then, if $K \cap \omega_e \neq \emptyset$ for $e \in \mathcal{E}_{\mathbf{c}}$, we have similarly $r_e \lesssim \sigma^\ell$ and $|K| \simeq \sigma^{2\ell} \sigma^{\ell-j}$ so that

$$\begin{aligned} \|\nabla u\|_{L^1(K \cap \omega_e)} &\leq \sum_{|\alpha|=1} \left\| r_e^{1+\beta_e-|\alpha^\perp|} \right\|_{L^2(K \cap \omega_e)} \left\| r_e^{-1-\beta_e+|\alpha^\perp|} \mathbf{D}_\perp^{\alpha^\perp} \mathbf{D}_\parallel^{\alpha^\parallel} u \right\|_{L^2(K \cap \omega_e)} \\ &\lesssim \|r_e^{\beta_e}\|_{L^2(K \cap \omega_e)} \|u\|_{M_{-1-\beta}^1(K \cap \omega_e)} \lesssim \sigma^{\ell(\frac{3}{2} + \beta_e) - \frac{j}{2}} \|u\|_{M_{-1-\beta}^1(K \cap \omega_e)}. \end{aligned}$$

Furthermore, if $K \cap \omega_{ce} \neq \emptyset$ for $e \in \mathcal{E}_c$, we have

$$\begin{aligned} \|\nabla u\|_{L^1(K \cap \omega_{ce})} &\leq \sum_{|\alpha|=1} \left\| r_c^{\beta_c} \rho_{ce}^{1+\beta_e-|\alpha^+|} \right\|_{L^2(K \cap \omega_{ce})} \\ &\quad \times \left\| r_c^{-1-\beta_c+1} \rho_{ce}^{-1-\beta_e+|\alpha^+|} D_{\perp}^{\alpha^+} D_{\parallel}^{\alpha^+} u \right\|_{L^2(K \cap \omega_{ce})}. \end{aligned}$$

With $r_c \simeq \sigma^{\ell-j}$ on $K \cap \omega_{ce}$ and $r_e \lesssim \sigma^{\ell}$, we find that

$$\begin{aligned} \left\| r_c^{\beta_c} \rho_{ce}^{1+\beta_e-|\alpha^+|} \right\|_{L^2(K \cap \omega_{ce})} &\lesssim \left\| r_c^{\beta_c} \rho_{ce}^{\beta_e} \right\|_{L^2(K \cap \omega_{ce})} = \left\| r_c^{\beta_c-\beta_e} r_e^{\beta_e} \right\|_{L^2(K \cap \omega_{ce})} \\ &\lesssim \sigma^{(\ell-j)(\beta_c-\beta_e)} \left\| r_e^{\beta_e} \right\|_{L^2(K \cap \omega_{ce})} \\ &\lesssim \sigma^{(\ell-j)(\beta_c-\beta_e)+\ell\beta_e} \sigma^{\ell} \sigma^{\frac{1}{2}(\ell-j)} \\ &= \sigma^{\frac{3}{2}\ell-\frac{j}{2}} \sigma^{(\ell-j)\beta_c+j\beta_e} \lesssim \sigma^{\frac{3}{2}\ell-\frac{j}{2}} \sigma^{\ell \min \beta}. \end{aligned}$$

Hence,

$$\|\nabla u\|_{L^1(K \cap \omega_{ce})} \lesssim \sigma^{\ell(\frac{3}{2}+\min \beta)-\frac{j}{2}} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{ce})},$$

and thus the first bound follows. The proof of the second inequality is similar. \square

5.4.1. Interpolation on \mathcal{V}_c^{ℓ} . We first consider elements $K \in \mathcal{V}_c^{\ell}$ which abut at exactly one corner $c \in \mathcal{C}$. Such elements are isotropic with $h_K \simeq h_K^{\perp} \simeq h_K^{\parallel} \simeq \sigma^{\ell}$. For convenience, let us suppose that the mesh is fine enough, so that $\mathcal{V}_c^{\ell} \subset \Omega_C \cup \Omega_{C\mathcal{E}}$.

PROPOSITION 5.21. *Let $K \in \mathcal{V}_c^{\ell}$ and $u \in M_{-1-\beta}^2(K)$. Then there holds*

$$\Upsilon_{\mathcal{V}_c^{\ell}}[\eta] \leq C \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(\Omega)}^2 \lesssim \exp(-2b\ell) \|u\|_{M_{-1-\beta}^2(K)}^2. \quad (5.65)$$

Proof. Let $K \in \mathcal{V}_c^{\ell}$ abut at corner $c \in \mathcal{C}$ with $K \cap \omega_c \neq \emptyset$. We shall use that $\sup_K r_c \simeq h_K \simeq \sigma^{\ell}$. Then,

$$h_K^{-2} \|\eta\|_{L^2(K \cap \omega_c)}^2 \lesssim h_K^{-2} \sup_K r_c^{2+2\beta_c} \|r_c^{-1-\beta_c} u\|_{L^2(K \cap \omega_c)}^2 \lesssim \sigma^{2\ell\beta_c} \|u\|_{M_{-1-\beta}^1(K \cap \omega_c)}^2,$$

and

$$\|\nabla \eta\|_{L^2(K \cap \omega_c)}^2 \lesssim \sup_K r_c^{2\beta_c} \|r_c^{-1-\beta_c+1} \nabla u\|_{L^2(K \cap \omega_c)}^2 \lesssim \sigma^{2\ell\beta_c} \|u\|_{M_{-1-\beta}^1(K \cap \omega_c)}^2.$$

Furthermore, if $K \cap \omega_{ce} \neq \emptyset$ for $e \in \mathcal{E}_c$, we use that ρ_{ce} is bounded on $K \cap \omega_{ce}$ to conclude that

$$\begin{aligned} h_K^{-2} \|\eta\|_{L^2(K \cap \omega_{ce})}^2 &\lesssim h_K^{-2} \sup_{K \cap \omega_{ce}} r_c^{2+2\beta_c} \rho_{ce}^{2+2\beta_e} \|r_c^{-1-\beta_c} \rho_{ce}^{-1-\beta_e} u\|_{L^2(K \cap \omega_{ce})}^2 \\ &\lesssim \sigma^{2\ell\beta_c} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{ce})}^2, \end{aligned}$$

and

$$\begin{aligned} \|\nabla \eta\|_{L^2(K \cap \omega_{ce})}^2 &\lesssim \sup_K r_c^{2\beta_c} \sum_{|\alpha|=1} \sup_{K \cap \omega_{ce}} \rho_{ce}^{2+2\beta_e-2|\alpha^+|} \left\| r_c^{-1-\beta_c+|\alpha^+|} \rho_{ce}^{-1-\beta_e+|\alpha^+|} D^{\alpha} u \right\|_{L^2(K \cap \omega_{ce})}^2 \\ &\lesssim \sigma^{2\ell\beta_c} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{ce})}^2. \end{aligned}$$

Moreover, for $f \in \mathcal{F}_K$, we have that

$$|f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f)}^2 \lesssim h_K^{-1} \|\nabla u\|_{L^1(f)}^2,$$

and employing the trace inequality (4.3) with $s = 1$, we obtain

$$|f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f)}^2 \lesssim h_K^{-3} \|\nabla u\|_{L^1(K)}^2 + h_K^{-1} \|\nabla^2 u\|_{L^1(K)}^2.$$

Next, using Lemma 5.20 with $j = 0$ results in

$$|f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f)}^2 \lesssim \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(K)}^2.$$

Summing up the above bounds completes the proof. \square

5.4.2. Interpolation on $\mathcal{V}_\mathcal{E}^\ell$. Elements K along Dirichlet edges may be anisotropic. They are parallel to some edge $\mathbf{e} \in \mathcal{E}$, with maximal length $h_K^\parallel \simeq \sigma^{\ell-j}$, for some $0 \leq j \lesssim \ell$, in the direction parallel to \mathbf{e} ; their diameter in the direction orthogonal to \mathbf{e} is $h_K^\perp \simeq \sigma^\ell$; see Proposition 3.4.

PROPOSITION 5.22. *Let $K \in \mathcal{V}_\mathcal{E}^\ell$ and $u \in M_{-1-\beta}^2(K)$. Then there holds*

$$\Upsilon_{\mathcal{V}_\mathcal{E}^\ell}[\eta] \leq C \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(\Omega)}^2 \lesssim \exp(-2b\ell) \|u\|_{M_{-1-\beta}^2(K)}^2,$$

where $\sigma \in (0, 1)$ is the geometric refinement parameter and where ℓ denotes the refinement level. The constant $C > 0$ is independent of u , σ , ℓ , the polynomial degree vector \mathbf{p} , and $\min \beta > 0$ (cf. (2.8)), and the constant $b > 0$ depends on σ, β .

Proof. We distinguish three cases.

Case 1. If $K \cap \omega_{\mathbf{c}} \neq \emptyset$, for some $\mathbf{c} \in \mathcal{C}$, then K is isotropic, and we may proceed as in the previous section. This leads to an estimate very similar to (5.65) (with the left-hand side restricted to $\mathcal{V}_\mathcal{E}^\ell \cap \Omega_{\mathbf{c}}$).

Case 2. If $K \cap \omega_{\mathbf{e}} \neq \emptyset$, for some $\mathbf{e} \in \mathcal{E}_{\mathbf{c}}$, then the weighted Sobolev norm from (2.7) close to an edge $\mathbf{e} \in \mathcal{E}$ behaves locally like

$$\|u\|_{M_{-1-\beta}^2(K \cap \omega_{\mathbf{e}})}^2 \simeq \sum_{|\alpha| \leq 2} \left\| r_{\mathbf{e}}^{-1-\beta_{\mathbf{e}}+|\alpha^\perp|} \mathbf{D}^\alpha u \right\|_{L^2(K \cap \omega_{\mathbf{e}})}^2.$$

Noting that $\sup_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-1} \lesssim (h_K^\perp)^{-1}$, there holds:

$$\begin{aligned} & \max_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K \cap \omega_{\mathbf{e}})}^2 + \|\nabla \eta\|_{L^2(K \cap \omega_{\mathbf{e}})}^2 + |f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f \cap \omega_{\mathbf{e}})}^2 \\ & \lesssim (h_K^\perp)^{-2} \|u\|_{L^2(K \cap \omega_{\mathbf{e}})}^2 + \|\nabla u\|_{L^2(K \cap \omega_{\mathbf{e}})}^2 + |f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f \cap \omega_{\mathbf{e}})}^2, \end{aligned}$$

for any $f \in \mathcal{F}_K$. Here, using that $\sup_{K \cap \omega_{\mathbf{e}}} r_{\mathbf{e}} \simeq h_K^\perp \simeq \sigma^\ell$, we have

$$(h_K^\perp)^{-1} \|u\|_{L^2(K \cap \omega_{\mathbf{e}})} \lesssim (h_K^\perp)^{\beta_{\mathbf{e}}} \|r_{\mathbf{e}}^{-1-\beta_{\mathbf{e}}} u\|_{L^2(K \cap \omega_{\mathbf{e}})} \lesssim \sigma^{\ell \beta_{\mathbf{e}}} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{\mathbf{e}})},$$

and

$$\begin{aligned} \|\nabla u\|_{L^2(K \cap \omega_{\mathbf{e}})} & \lesssim \sum_{|\alpha|=1} (h_K^\perp)^{1+\beta_{\mathbf{e}}-|\alpha^\perp|} \left\| r_{\mathbf{e}}^{-1-\beta_{\mathbf{e}}+|\alpha^\perp|} \mathbf{D}^\alpha u \right\|_{L^2(K \cap \omega_{\mathbf{e}})} \\ & \lesssim \sigma^{\ell \beta_{\mathbf{e}}} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{\mathbf{e}})}. \end{aligned}$$

Next, by means of (4.3) with $s = 1$, we see that

$$|f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f\cap\omega_e)}^2 \lesssim |f|^{-1} (h_{K,f}^\perp)^{-1} \left(\|\nabla u\|_{L^1(K)}^2 + (h_{K,f}^\perp)^2 \|\mathbf{D}_\perp \nabla u\|_{L^1(K)}^2 \right).$$

Therefore, if f is parallel to \mathbf{e} it holds that $h_{K,f}^\perp \simeq \sigma^\ell$ and so $|f|^{-1} (h_{K,f}^\perp)^{-1} \simeq (\sigma^\ell \sigma^{\ell-j})^{-1} \sigma^{-\ell} = \sigma^{j-3\ell}$. In this case,

$$|f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f\cap\omega_e)}^2 \lesssim \sigma^{j-3\ell} \left(\|\nabla u\|_{L^1(K)}^2 + \sigma^{2\ell} \|\mathbf{D}_\perp \nabla u\|_{L^1(K)}^2 \right).$$

Invoking Lemma 5.20, leads to

$$|f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f\cap\omega_e)}^2 \lesssim \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(K)}^2.$$

If f is orthogonal to \mathbf{e} it holds that $h_{K,f}^\perp \simeq \sigma^{\ell-j}$ and therefore $|f|^{-1} (h_{K,f}^\perp)^{-1} \simeq \sigma^{-2\ell} \sigma^{-(\ell-j)} = \sigma^{j-3\ell}$. In this case,

$$|f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f\cap\omega_e)}^2 \lesssim \sigma^{j-3\ell} \left(\|\nabla u\|_{L^1(K)}^2 + \sigma^{2(\ell-j)} \|\mathbf{D}_\parallel \nabla u\|_{L^1(K)}^2 \right).$$

Thence, with the aid of Lemma 5.20, we obtain

$$|f|^{-1} h_{K,f}^\perp \|\nabla u\|_{L^1(f\cap\omega_e)}^2 \lesssim \sigma^{2\ell \min \beta} \sigma^{2(\ell-j)} \|u\|_{M_{-1-\beta}^2(K)}^2 \lesssim \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(K)}^2.$$

Case 3. Finally, if $K \cap \omega_{ce} \neq \emptyset$, then the norm from (2.7) behaves locally as follows:

$$\|u\|_{M_{-1-\beta}^2(K \cap \omega_{ce})}^2 \simeq \sum_{|\alpha| \leq 2} \left\| r_c^{-1-\beta_c+|\alpha|} \rho_{ce}^{-1-\beta_e+|\alpha^\perp} \mathbf{D}^\alpha u \right\|_{L^2(K \cap \omega_{ce})}^2.$$

Here, for the volume terms, we may proceed similarly as in Case 2 by carefully taking into account the weight ρ_{ce} . More precisely, using that $\sup_{K \cap \omega_{ce}} r_c \simeq \sigma^{\ell-j}$ and $\sup_{K \cap \omega_{ce}} r_e \simeq h_{K,f}^\perp \simeq \sigma^\ell$, we have

$$\begin{aligned} (h_{K,f}^\perp)^{-1} \|u\|_{L^2(K \cap \omega_{ce})} &\lesssim (h_{K,f}^\perp)^{-1} \sup_{K \cap \omega_{ce}} r_c^{1+\beta_c} \rho_{ce}^{1+\beta_e} \|r_c^{-1-\beta_c} \rho_{ce}^{-1-\beta_e} u\|_{L^2(K \cap \omega_{ce})} \\ &\lesssim \sigma^{(\ell-j)(\beta_c-\beta_e)} \sigma^{\ell\beta_e} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{ce})} \\ &\lesssim \sigma^{\ell \min \beta} \|u\|_{M_{-1-\beta}^1(K \cap \omega_{ce})}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla u\|_{L^2(K \cap \omega_{ce})} &\lesssim \sup_{K \cap \omega_{ce}} r_c^{\beta_c} \rho_{ce}^{\beta_e} \sum_{|\alpha|=1} \left\| r_c^{-1-\beta_c+|\alpha|} \rho_{ce}^{-1-\beta_e+|\alpha^\perp} \mathbf{D}^\alpha u \right\|_{L^2(K \cap \omega_{ce})} \\ &\lesssim \sigma^{(\ell-j)(\beta_c-\beta_e)+\ell\beta_e} \|u\|_{M_{-1-\beta}^2(K \cap \omega_{ce})} \\ &\lesssim \sigma^{\ell \min \beta} \|u\|_{M_{-1-\beta}^2(K \cap \omega_{ce})}. \end{aligned}$$

Furthermore, the face expressions are again estimated by employing Lemma 5.20. This leads to

$$\begin{aligned} \max_{f \in \mathcal{F}_K} (h_{K,f}^\perp)^{-2} \|\eta\|_{L^2(K \cap \omega_{ce})}^2 + \|\nabla \eta\|_{L^2(K \cap \omega_{ce})}^2 + |f|^{-1} h_{K,f}^\perp \|\nabla \eta\|_{L^1(f \cap \omega_{ce})}^2 \\ \lesssim \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(K)}^2. \end{aligned}$$

This completes the proof. \square

5.4.3. Exponential Convergence in Terminal Layers. Summing up the above estimates for all terminal layer elements, we have proved the following result:

PROPOSITION 5.23. *Let $u \in M_{-1-\beta}^2(\Omega)$. Then there holds*

$$\Upsilon_{\mathcal{V}_c^\ell}[\eta] + \Upsilon_{\mathcal{V}_\varepsilon^\ell}[\eta] \leq C \sigma^{2\ell \min \beta} \|u\|_{M_{-1-\beta}^2(\Omega)}^2 \simeq \exp(-2b\ell) \|u\|_{M_{-1-\beta}^2(\Omega)}^2,$$

where σ is the geometric refinement parameter and ℓ the refinement level. The constant $C > 0$ is independent of u , σ , ℓ , the polynomial degree vector \mathbf{p} , and $\min \beta > 0$ (cf. (2.8)), and the constant $b > 0$ depends on σ, β .

6. Exponential Convergence. The hp -interpolation error estimates from the previous section together with Theorem 4.2 allow us to deduce exponential convergence rates of the hp -dG discretizations in (4.1), provided the solution u of (1.1)–(1.2) admits analytic regularity.

THEOREM 6.1. *Assume that the right hand side f of the reaction-diffusion equation (1.1)–(1.2) in the axiparallel polyhedron $\Omega \subset \mathbb{R}^3$ belongs to the analytic space $A_{1-\beta}(\Omega)$ with a weight vector β satisfying (2.8).*

Let $\mathfrak{M}_\sigma = \{\mathcal{M}_\sigma^{(\ell)}\}_{\ell \geq 0}$ be family of axiparallel σ -geometric meshes generated by the hp -extensions (Ex1)–(Ex4) in [33, Section 3], and consider the hp -dG discretizations in (4.1) based on the sequences of approximating subspaces

$$V_\sigma^\ell = V(\mathcal{M}_\sigma^{(\ell)}, \Phi(\mathcal{M}_\sigma^{(\ell)}), \mathbf{p}_1(\mathcal{M}_\sigma^{(\ell)})) \quad \text{respectively} \quad V_{\sigma, \mathbf{s}}^\ell = V(\mathcal{M}_\sigma^{(\ell)}, \Phi(\mathcal{M}_\sigma^{(\ell)}), \mathbf{p}_2(\mathcal{M}_\sigma^{(\ell)}))$$

defined in (3.25) respectively (3.27), with the mapping vectors $\Phi(\mathcal{M}_\sigma^{(\ell)})$ as in (3.3), and with the vector $\mathbf{p}_1(\mathcal{M}_\sigma^{(\ell)})$ in (3.26) of constant, isotropic and uniform polynomial degrees proportional to ℓ for the space V_σ^ℓ , respectively the linear, anisotropic degree distribution $\mathbf{p}_2(\mathcal{M}_\sigma^{(\ell)})$ generated by sequence of (σ, \mathbf{s}) -extensions (Ex1)–(Ex4) from [33, Section 3] for $V_{\sigma, \mathbf{s}}^\ell$.

Then for each $\ell \geq 0$, the hp -dG approximation u_{DG} is well-defined, and as $\ell \rightarrow \infty$, the approximate solutions u_{DG} satisfy the error estimate

$$\|u - u_{DG}\|_{DG} \leq C \exp(-b\sqrt[5]{N}) \quad (6.1)$$

where $N = \dim(V(\mathcal{M}_\sigma^{(\ell)}, \Phi(\mathcal{M}_\sigma^{(\ell)}), \mathbf{p}(\mathcal{M}_\sigma^{(\ell)})))$ denotes the number of degrees of freedom of the discretization for any of the two spaces V_σ^ℓ or $V_{\sigma, \mathbf{s}}^\ell$.

The constants $b > 0$ and $C > 0$ are independent of N , but depend on σ , μ , $C_{\mathcal{M}^0}$, θ , γ , α_0 , $\min \beta > 0$, and on which of the polynomial degree vectors $\mathbf{p}_1(\mathcal{M}_\sigma^{(\ell)})$ or $\mathbf{p}_2(\mathcal{M}_\sigma^{(\ell)})$ are used.

Proof. This follows readily from Theorem 4.2, Theorem 5.18, and Proposition 5.23. Note that the constants $b > 0$ and $C > 0$ have to be suitably modified to absorb the algebraic factor \mathbf{p}_{\max}^4 in (4.7). \square

7. Concluding Remarks. In the present paper, we have proved exponential convergence of a class of IP hp -dGFEM for linear scalar diffusion problems with homogeneous Dirichlet boundary conditions in axiparallel polyhedral domains $\Omega \subset \mathbb{R}^3$ under the provision of analytic regularity of the solution u in a family of countably normed Sobolev spaces.

While this setting appears fairly particular, we emphasize that most components of our error analysis directly apply also to more general situations. Let us mention some of them: a) analogous regularity results are also available for boundary-value

problems of linear second-order strongly elliptic systems in polyhedral domains, such as three-dimensional elasticity in [7, 24]. Our convergence analysis generalizes, with obvious modifications, to the Dirichlet problem for the second-order elliptic systems considered in [7, 24]; b) mixed and Neumann boundary conditions: here the regularity of the solutions in weighted spaces is available in [8, 7, 24]. For such boundary conditions, analogous results hold, but the error bounds and the design of the hp -dGFEM in the terminal layers \mathfrak{T}_σ^ℓ will require modifications which will be addressed elsewhere.

So far, we considered polyhedra with plane, axiparallel faces and built element mapping vectors out of dilation-translation mappings of hexahedra. Most of our analysis, however, remains valid if these mappings are combined with a finite number of patch-mappings as in the stability analysis of [33]. These generalizations will also be presented elsewhere.

For *mixed hp-dGFEM in \mathbb{R}^3* on σ -geometric meshes of (affinely mapped) hexahedral elements for the Stokes problem, divergence stability was shown in [30, 31, 32]. Based on analytic regularity estimates for these mixed problems, the consistency analysis of the present paper will allow to establish exponential convergence rates of hp -dGFEM also for these problems; cf. [39].

REFERENCES

- [1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.
- [2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2001.
- [3] I. Babuška and B. Q. Guo. The h - p version of the finite element method for domains with curved boundaries. *SIAM J. Numer. Anal.*, 25(4):837–861, 1988.
- [4] I. Babuška and B. Q. Guo. Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order. *SIAM J. Math. Anal.*, 19(1):172–203, 1988.
- [5] I. Babuška and B. Q. Guo. Approximation properties of the h - p version of the finite element method. *Comput. Methods Appl. Mech. Engrg.*, 133(3-4):319–346, 1996.
- [6] G. Baker. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.*, 31:44–59, 1977.
- [7] M. Costabel, M. Dauge, and S. Nicaise. *Corner Singularities and Analytic Regularity for Linear Elliptic Systems*. In preparation.
- [8] M. Costabel, M. Dauge, and S. Nicaise. Analytic regularity for linear elliptic systems in polygons and polyhedra. *Math. Models Methods Appl. Sci.*, 22(8), 2012.
- [9] M. Costabel, M. Dauge, and C. Schwab. Exponential convergence of hp -FEM for Maxwell’s equations with weighted regularization in polygonal domains. *Math. Models Methods Appl. Sci.*, 15(4):575–622, 2005.
- [10] W. Dahmen and K. Scherer. Best approximation by piecewise polynomials with variable knots and degrees. *J. Approx. Theory*, 26(1):1–13, 1979.
- [11] J. Douglas and T. Dupont. Interior penalty procedures for elliptic and parabolic Galerkin methods. In *Computing Methods in Applied Sciences (Second Internat. Sympos., Versailles, 1975)*, volume 58 of *Lecture Notes in Phys.*, pages 207–216. Springer, Berlin, 1976.
- [12] T. Gudi. Some nonstandard error analysis of discontinuous Galerkin methods for elliptic problems. *Calcolo*, 47(4):239–261, 2010.
- [13] W. Gui and I. Babuška. The h , p and h - p versions of the finite element method in 1 dimension. II. The error analysis of the h - and h - p versions. *Numer. Math.*, 49(6):613–657, 1986.
- [14] B. Q. Guo. The h - p version of the finite element method for elliptic equations of order $2m$. *Numer. Math.*, 53(1–2):199–224, 1988.
- [15] B. Q. Guo. The h - p version of the finite element method for solving boundary value problems in polyhedral domains. In *Boundary Value Problems and Integral Equations in Nonsmooth Domains*, volume 167 of *Lecture Notes in Pure and Appl. Math.*, pages 101–120. Dekker, New York, 1995.

- [16] B. Q. Guo and I. Babuška. The hp -version of the finite element method. Part I: The basic approximation results. *Comp. Mech.*, 1:21–41, 1986.
- [17] B. Q. Guo and I. Babuška. The hp -version of the finite element method. Part II: General results and applications. *Comp. Mech.*, 1:203–220, 1986.
- [18] B. Q. Guo and I. Babuška. Regularity of the solutions for elliptic problems on nonsmooth domains in \mathbb{R}^3 . I. Countably normed spaces on polyhedral domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 127(1):77–126, 1997.
- [19] B. Q. Guo and I. Babuška. Regularity of the solutions for elliptic problems on nonsmooth domains in \mathbb{R}^3 . II. Regularity in neighbourhoods of edges. *Proc. Roy. Soc. Edinburgh Sect. A*, 127(3):517–545, 1997.
- [20] P. Houston, C. Schwab, and E. Süli. Stabilized hp -finite element methods for first-order hyperbolic problems. *SIAM J. Numer. Anal.*, 37:1618–1643, 2000.
- [21] P. Houston, C. Schwab, and E. Süli. Discontinuous hp -finite element methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.*, 39:2133–2163, 2002.
- [22] C. Johnson and J. Pitkäranta. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Math. Comp.*, 46:1–26, 1986.
- [23] P. LeSaint and P.A. Raviart. On a finite element method for solving the neutron transport equation. In C. de Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 89–145. Academic Press, New York, 1974.
- [24] A. L. Mazzucato and V. Nistor. Well-posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. *Arch. Ration. Mech. Anal.*, 2008.
- [25] J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abh. Math. Sem. Univ. Hamburg*, 36:9–15, 1971.
- [26] D. A. Di Pietro and A. Ern. Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations. *Math. Comp.*, 79(271):1303–1330, 2010.
- [27] W.H. Reed and T.R. Hill. Triangular mesh methods for the neutron transport equation. Technical Report Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.
- [28] K. Scherer. On optimal global error bounds obtained by scaled local error estimates. *Numer. Math.*, 36:257–277, 1981.
- [29] D. Schötzau and C. Schwab. Exponential convergence in a Galerkin least squares hp -FEM for Stokes flow. *IMA J. Numer. Anal.*, 21:53–80, 2001.
- [30] D. Schötzau, C. Schwab, and A. Toselli. Mixed hp -DGFEM for incompressible flows. *SIAM J. Numer. Anal.*, 40:2171–2194, 2003.
- [31] D. Schötzau, C. Schwab, and A. Toselli. Stabilized hp -DGFEM for incompressible flow. *Math. Models Methods Appl. Sci.*, 13(10):1413–1436, 2003.
- [32] D. Schötzau, C. Schwab, and A. Toselli. Mixed hp -DGFEM for incompressible flows. II. Geometric edge meshes. *IMA J. Numer. Anal.*, 24(2):273–308, 2004.
- [33] D. Schötzau, C. Schwab, and T. P. Wihler. hp -dGFEM for elliptic problems in polyhedra I: Stability and quasioptimality on geometric meshes. *Submitted*.
- [34] D. Schötzau and T. P. Wihler. Exponential convergence of mixed hp -DGFEM for Stokes flow in polygons. *Numer. Math.*, 96:339–361, 2003.
- [35] M. F. Wheeler. An elliptic collocation finite element method with interior penalties. *SIAM J. Numer. Anal.*, 15:152–161, 1978.
- [36] T. P. Wihler. *Discontinuous Galerkin FEM for Elliptic Problems in Polygonal Domains*. PhD thesis, Swiss Federal Institute of Technology Zurich, 2002. Diss. ETH No. 14973.
- [37] T. P. Wihler, P. Frauenfelder, and C. Schwab. Exponential convergence of the hp -DGFEM for diffusion problems. *Comput. Math. Appl.*, 46:183–205, 2003.
- [38] T. P. Wihler and B. Rivière. Discontinuous Galerkin methods for second-order elliptic PDE with low-regularity solutions. *J. Sci. Comput.*, 46(2):151–165, 2011.
- [39] T. P. Wihler and M. Wirz. Mixed hp -discontinuous Galerkin FEM for linear elasticity and Stokes flow in three dimensions. *Math. Models Methods Appl. Sci.*, 2012. in press.

Research Reports

No.	Authors/Title
09-29	<i>D. Schötzau, C. Schwab, T. Wihler</i> <i>hp</i> -dGFEM for second-order elliptic problems in polyhedra. II: Exponential convergence
09-28	<i>D. Schötzau, C. Schwab, T. Wihler</i> <i>hp</i> -dGFEM for second-order elliptic problems in polyhedra. I: Stability and quasioptimality on geometric meshes
09-27	<i>A. Moiola, R. Hiptmair, I. Perugia</i> Approximation by plane waves
09-26	<i>M. Karow, E. Kokiopoulou, D. Kressner</i> On the computation of structured singular values and pseudospectra
09-25	<i>M. Durán, M. Guarini, C.F. Jerez-Hanckes</i> Hybrid FEM/BEM modeling of finite-sized photonic crystals for semiconductor laser beams
09-24	<i>A. Bespalov, N. Heuer, R. Hiptmair</i> Convergence of the natural <i>hp</i> -BEM for the electric field integral equation on polyhedral surfaces
09-23	<i>R. Hiptmair, J. Li, J. Zou</i> Real interpolation of spaces of differential forms
09-22	<i>R. Hiptmair, J. Li, J. Zou</i> Universal extension for Sobolev spaces of differential forms and applications
09-21	<i>T. Betcke, D. Kressner</i> Perturbation, computation and refinement of invariant pairs for matrix polynomials
09-20	<i>R. Hiptmair, A. Moiola, I. Perugia</i> Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the <i>p</i> -version
09-19	<i>C. Winter</i> Wavelet Galerkin schemes for multidimensional anisotropic integrodifferential operators
09-18	<i>C.J. Gittelsohn</i> Stochastic Galerkin discretization of the lognormal isotropic diffusion problem
09-17	<i>A. Bendali, A. Tizaoui, S. Tordeux, J. P. Vila</i> Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole