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CONVERGENCE OF THE NATURAL HP-BEM FOR THE ELECTRIC FIELD INTEGRAL EQUATION ON POLYHEDRAL SURFACES *

A. BESPALOV†, N. HEUER‡, AND R. HIPTMAIR§

Abstract. We consider the variational formulation of the electric field integral equation (EFIE) on bounded polyhedral open or closed surfaces. We employ a conforming Galerkin discretization based on div_Γ-conforming Raviart-Thomas boundary elements (BEM) of locally variable polynomial degree on shape-regular surface meshes. We establish asymptotic quasi-optimality of Galerkin solutions on sufficiently fine meshes or for sufficiently high polynomial degree.

Key words. electromagnetic scattering, electric field integral equation (EFIE), Galerkin discretization, boundary element method (BEM), *hp*-refinement, non-coercive variational problems, smoothed Poincaré mapping, projection based interpolation

AMS subject classifications. 65N38, 65N12, 78M15, 65N38

1. Introduction. Let Γ be a piecewise flat (open or closed) orientable surface equipped with a conforming triangulation $\mathcal{M} = \{K\}$, consisting of triangles. Throughout, uniform bounds on the shape-regularity of the cells will be tacitly taken for granted, see [25, Ch. 3, § 3.1]. For a fixed wave number $k > 0$, let V_k and $\mathbf{V}_k$ stand for the scalar or vectorial single layer boundary integral operator on Γ for the Helmholtz operator $-\Delta - k^2$, see [19, Sect. 4.1] or [21, Sect. 5]. The bilinear form underlying the variational formulation of the electric field integral equation ("Rumsey's principle") reads (see [3, 35], [19, Sect. 4.2] or [21, Sect. 7.2] for closed surfaces, "boundaries", and [16, Sect. 3] for open surfaces, "screens")

$$a(\mathbf{u}, \mathbf{v}) := \langle V_k \mathrm{div}_\Gamma \mathbf{u}, \mathrm{div}_\Gamma \mathbf{v} \rangle_\Gamma - k^2 \langle \mathbf{V}_k \mathbf{u}, \mathbf{v} \rangle_\Gamma \ , \tag{1.1}$$

where, as discussed in [20], $\langle \cdot, \cdot \rangle_\Gamma$ hints at a duality pairing, extending the $L^2(\Gamma)$-pairing for tangential vector fields or functions on Γ. The variational problem is posed on the Hilbert space

$$\mathbf{X} = \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \ , \tag{1.2}$$

in the case of a boundary $\Gamma = \partial\Omega$, $\Omega \subset \mathbb{R}^3$ a Lipschitz polyhedron, or on

$$\mathbf{X} = \{\mathbf{u} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) : \langle \mathbf{u}, \mathrm{grad}_\Gamma v \rangle + \langle \mathrm{div}_\Gamma \mathbf{u}, v \rangle = 0 \ \forall v \in C^\infty(\overline{\Gamma})\} \tag{1.3}$$

in the case of a screen Γ. The latter space can be understood as a space of div_Γ-conforming tangential surface vector fields with vanishing in-plane normal component on the screen edge $\partial\Gamma$. We refer to [14,17,18,20,21] for a definition and more information about $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and other trace spaces. An in-depth discussion for screens is given in [16, Sect. 2]. In this article we adopt the notations of [17]. Further, the two situations of open and closed surfaces will be treated in parallel.

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The EFIE can be recast as a linear variational problem for $a(\cdot,\cdot)$ on $\mathbf{X}$: given a source functional $f \in \mathbf{X}'$ it reads

$$\mathbf{u} \in \mathbf{X} : \quad a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X} . \tag{1.4}$$

In order to ensure uniqueness of the solution to (1.4), we make the following assumption [21, Sect. 7.1].

Assumption 1.1. *In the case of a closed surface $\Gamma = \partial\Omega$ we assume that k is different from a Dirichlet eigenvalue of the operator* $\mathbf{curl}\,\mathbf{curl}$ *on Ω.*

We opt for a natural boundary element (BE) Galerkin discretization based on conforming trial and test spaces $\mathbf{X}_N \subset \mathbf{X}$. These are obtained by using $\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$-conforming Raviart-Thomas spaces of variable local polynomial degrees $p_K \in \mathbb{N}_0$, $K \in \mathcal{M}$, on the surface triangulation $\mathcal{M}$, see Sect. 2 for a precise definition.

Refinement of the BE spaces can be achieved by raising the local polynomial degrees p_K (p-refinement) or reducing the sizes h_K of the cells of $\mathcal{M}$ (h-refinement). Thus, the proposed discretization qualifies as "*hp*-boundary element method (BEM)".

Roughly speaking, judicious *hp*-refinement can be expected to offer exponential convergence of Galerkin solutions even when the exact solution lacks global smoothness [44]. *hp*-BEM approaches have been suggested for various boundary integral equations [5,6,32,34,36,37] and are a natural idea for the EFIE as well. While convergence theory for h-refinement is well established [3,4,16,42], the extension to *hp*-refinement proved to be difficult, see [8,9] for partial results.

This article fills the gap and proves the following convergence result that translates into an a priori error estimate in the natural "energy norm" provided that information about some smoothness of the solution $\mathbf{u}$ of (1.4) is available, *cf.* [42, Sect. 8] and [16, Sect. 4] for the h-version, [9] for the p-version, and [8] for the *hp*-version with quasi-uniform meshes.

Theorem 1.2. *There is a constant $C_0 > 0$ such that for any $f \in \mathbf{X}'$ and for arbitrary mesh-degree combination satisfying $\max_K \sqrt{\frac{h_K}{p_K+1}} < C_0$ the Galerkin BE discretization of* (1.4) *admits a unique solution $\mathbf{u}_N \in \mathbf{X}_N$ and the Galerkin hp-BEM converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_N\|_{\mathbf{X}} \le C \inf_{\mathbf{v}_N \in \mathbf{X}_N} \|\mathbf{u} - \mathbf{v}_N\|_{\mathbf{X}} . \tag{1.5}$$

Both constants C_0 and C may depend only on the geometry of Γ and the shape-regularity of the surface triangulation $\mathcal{M}$.

We remark that the policy of the proof has a lot in common with recent proofs of discrete compactness for the p-version of edge elements [11,12,41]. The main tools are the same, namely, the sophisticated mathematical inventions of regularizing lifting operators [27] (see Sect. 4 below) and projection based interpolation operators [29,30] (see Sect. 5). They pave the way for verifying the assumptions of an abstract theory of Galerkin approximations for non-coercive variational problems, see [15] and Sects. 3, 6 below.

Building on these mighty foundations the present article cannot be and does not aspire to be self-contained, but will give detailed references to relevant literature. We refer to [7] for an earlier version of this paper whose analysis is based on a Hodge-decomposition of $\mathbf{X}$ which, due to regularity issues on non-smooth surfaces, requires a sophisticated projection based interpolation operator which is not needed in this paper.

In the sequel, generic constants, designated by C, C_0, C_1, etc., may depend only on the geometry of Γ and the shape-regularity of $\mathcal{M}$. They must not depend on cell sizes, local polynomial degrees, and any function.

2. Boundary element spaces. Raviart-Thomas surface elements provide an affine equivalent family of div_Γ-conforming finite elements under the Piola transformation, see [13, Sect. III.3] and [43]. We write $\boldsymbol{\mathcal{RT}}_p(K)$ for the local Raviart-Thomas space of order p on the triangle $K \in \mathcal{M}$, and $\boldsymbol{\mathcal{RT}}_{p,0}(K) \subset \mathbf{H}_0(\mathrm{div}, K)$ for the subspace of local Raviart-Thomas vector fields with vanishing normal components on ∂K. Vector fields in the latter spaces will be identified with their extensions by zero onto the whole surface Γ.

Given a polynomial degree distribution $\{p_K : p_K \in \mathbb{N}_0,\ K \in \mathcal{M}\}$, we define edge degrees according to the "maximum rule"

$$p_E := \max\{p_K :\ K \in \mathcal{M},\ E \subset \overline{K}\} \ , \quad E \in \mathcal{E} \ , \tag{2.1}$$

where $\mathcal{E}$ is the set of edges of $\mathcal{M}$. As elaborated in [39, Sect. 3.4], Raviart-Thomas spaces can be split into local "edge contributions" and "cell contributions". In detail, write ψ_1 and ψ_2 for the piecewise linear, continuous "tent/hat functions" associated with the endpoints of some edge $E \subset \mathcal{E}$. We introduce the edge space $\boldsymbol{\mathcal{RT}}_{p_E}(E)$ as

$$\boldsymbol{\mathcal{RT}}_{p_E}(E) := \mathrm{span}\{\mathbf{curl}_\Gamma(\psi_1^\alpha \psi_2^\beta),\ \alpha, \beta \in \mathbb{N},\ \alpha + \beta = p_E + 1\} \ . \tag{2.2}$$

These spaces $\boldsymbol{\mathcal{RT}}_{p_E}(E)$ obviously satisfy

$$\mathbf{u}_N \in \boldsymbol{\mathcal{RT}}_{p_E}(E) \quad \Rightarrow \quad \mathrm{supp}\, \mathbf{u}_N \subset \bigcup\{\overline{K} :\ E \subset \overline{K}\} \quad \text{and} \quad \boxed{\mathrm{div}_\Gamma \mathbf{u}_N = 0} \ . \tag{2.3}$$

Then, we *define* the boundary element space (oblivious of boundary conditions!) according to

$$\widetilde{\mathbf{X}}_N = \boldsymbol{\mathcal{RT}}_0(\mathcal{M}) + \sum_{E \in \mathcal{E}} \boldsymbol{\mathcal{RT}}_{p_E}(E) + \sum_{K \in \mathcal{M}} \boldsymbol{\mathcal{RT}}_{p_K,0}(K) \ . \tag{2.4}$$

Here, the space $\boldsymbol{\mathcal{RT}}_0(\mathcal{M})$ is the lowest order Raviart-Thomas BE space. Thanks to the maximum rule (2.1), the localized spaces $\mathbf{X}_N(K) := \widetilde{\mathbf{X}}_N|_K$, $K \in \mathcal{M}$, fulfil

$$\boldsymbol{\mathcal{RT}}_{p_K}(K) \subset \mathbf{X}_N(K) \quad \forall K \in \mathcal{M} \quad \text{and} \quad \widetilde{\mathbf{X}}_N \cdot \mathbf{n}_E|_E = \mathcal{P}_{p_E}(E) \quad \forall E \in \mathcal{E} \ , \tag{2.5}$$

with $\mathbf{n}_E$ standing for an edge normal, and $\mathcal{P}_p$ for the space of (multivariate) polynomials of degree $\leq p$, $p \in \mathbb{N}_0$. Now, we are in a position to introduce the *hp*-BEM Galerkin trial and test spaces:

- we pick $\mathbf{X}_N := \widetilde{\mathbf{X}}_N$ for closed surfaces Γ,
- we choose $\mathbf{X}_N := \widetilde{\mathbf{X}}_N \cap \mathbf{X}$ for screens, that is, in order to obtain $\mathbf{X}_N$ edge spaces for edges contained in $\partial\Gamma$ are simply discarded as well as basis functions of $\boldsymbol{\mathcal{RT}}_0(\mathcal{M})$ associated with edges on $\partial\Gamma$.

Note that for $E \subset \overline{K}$, $K \in \mathcal{M}$, we may encounter $p_E > p_K$, and, consequently,

$$\mathbf{X}_N(K) \not\subset \boldsymbol{\mathcal{RT}}_{p_K}(K) \subset \mathbf{X}_N(K)\ ! \tag{2.6}$$

However, thanks to (2.4) and (2.3), we can take for granted

$$\mathrm{div}_\Gamma \mathbf{X}_N(K) = \mathrm{div}_\Gamma \boldsymbol{\mathcal{RT}}_{p_K}(K) \ . \tag{2.7}$$

From [13, §III.3, Prop. 3.2] we know that $\mathrm{div}_\Gamma \boldsymbol{\mathcal{RT}}_p(K) = \mathcal{P}_p(K)$. Thus, by (2.4), $\mathrm{div}_\Gamma : \widetilde{\mathbf{X}}_N \mapsto Q_N$, where $Q_N \subset L^2(\Gamma)$ is the space of $\mathcal{M}$-piecewise polynomials with degree p_K on every $K \in \mathcal{M}$.

By [39, Theorem 3.7], [2, Sect. 5.5], the Raviart-Thomas BE space $\widetilde{\mathbf{X}}_N$ allows for a *discrete scalar potential space* $\mathcal{S}_N \subset C^0(\Gamma)$ comprising continuous piecewise polynomial functions on $\mathcal{M}$ such that $\mathcal{S}_N|_E \subset \mathcal{P}_{p_E+1}(E)$ for all $E \in \mathcal{E}$ and the localized spaces $\mathcal{S}_N(K) = \mathcal{S}_N|_K$, $K \in \mathcal{M}$, satisfy

$$\mathbf{curl}_\Gamma \mathcal{S}_N(K) = \mathbf{X}_N(K) \cap \mathbf{H}(\mathrm{div}_\Gamma 0, K) \quad \forall K \in \mathcal{M}\ . \tag{2.8}$$

Below, we make repeated use of transformation to the reference triangle ("unit triangle") $\widehat{K} := \mathrm{convex}\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\}$, which is mapped to a generic $K \in \mathcal{M}$ by the affine mapping $\mathbf{\Phi}_K : \widehat{K} \mapsto K$, $\mathbf{\Phi}(\widehat{\boldsymbol{x}}) := \mathbf{A}_K \widehat{\boldsymbol{x}} + \boldsymbol{t}_K$, $\mathbf{A}_K \in \mathbb{R}^{3,2}$, $\boldsymbol{t}_K \in \mathbb{R}^3$. Writing $\mathbf{\Phi}_K^*$ for the associated co-variant pullback of tangential vector fields, we define spaces of functions $\widehat{K} \mapsto \mathbb{R}^2$,

$$\mathbf{X}_N(\widehat{K}) := \mathbf{\Phi}_K^* \mathbf{X}_N(K)\ , \tag{2.9}$$

which, due to non-uniform polynomial degrees, may be different for different cells K. The relevant K will be clear from the context.

Remark 2.1. *For the sake of brevity we do not include Raviart-Thomas BE spaces on (uniformly shape regular) quadrilaterals and BDM-type BE spaces on triangles into our analysis. With slight alterations the approach of this paper covers these settings. Besides, curved elements can be treated by the usual mapping techniques.*

3. Splitting technique. Owing to the infinite-dimensional kernel of div_Γ the bilinear form a from (1.1) fails to be $\mathbf{X}$-coercive, which massively compounds the difficulties of convergence analysis for Galerkin schemes, as discussed, e.g., in [21, Sect. 3] and [24]. An abstract theory for tackling a priori Galerkin error estimates for non-coercive variational problems like (1.4) was developed in [16,22] and, in particular, in [15, Sect. 3]. The latter article tells us that Theorem 1.2 will follow, once we establish

- **(A)** the existence of a stable direct splitting $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$ such that $a_{|V\times V}$ and $a_{|W\times W}$ are both $\mathbf{X}$-coercive and $a_{|V\times W}$ and $a_{|W\times V}$ are both compact,
- **(B)** the existence of a corresponding decomposition $\mathbf{X}_N = \mathbf{V}_N + \mathbf{W}_N$, $\mathbf{W}_N \subset \mathbf{W}$, that is uniformly stable with respect to cell sizes and polynomial degree p,
- **(C)** the *gap property*

$$\sup_{\mathbf{v}_N \in \mathbf{V}_N} \inf_{\mathbf{v} \in \mathbf{V}} \frac{\|\mathbf{v} - \mathbf{v}_N\|_X}{\|\mathbf{v}_N\|_X} \le C \max_K \sqrt{\frac{h_K}{p_K + 1}}. \tag{3.1}$$

We remark that the approximation property

$$\inf_{\mathbf{v}_N \in \mathbf{X}_N} \|\mathbf{u} - \mathbf{v}_N\|_X \to 0 \quad \text{as} \quad \max_K \sqrt{\frac{h_K}{p_K + 1}} \to 0, \tag{3.2}$$

dubbed CAS in [15, 23], is automatically satisfied for families of hp-Raviart-Thomas spaces on families of uniformly shape-regular meshes.

Taking the cue from the numerical analysis of electromagnetic wave equations [39, Sect. 5], one might resort to the "$L^2(\Gamma)$-orthogonal" *Hodge-decomposition* of $\mathbf{X}$ [18] in order to obtain a suitable splitting. For h-version analysis this idea was successfully applied in [16, 19, 22, 42]. Yet, on non-smooth surfaces, smoothness of functions in the $\mathbf{V}$-component may be poor, which causes substantial technical difficulties, *cf.* [7]. These are avoided when following the guideline that the analysis of boundary integral operators is often greatly facilitated by taking a detour via a volume domain, *cf.* [26]. This strategy yields decompositions with enhanced smoothness of the $\mathbf{V}$-component.

More concretely, as in [40] and [15, Sect. 4.3.1], $\mathbf{V}$ and $\mathbf{W}$ are constructed via a *regularizing projection* $\mathsf{R} : \mathbf{X} \mapsto \mathbf{X}$. To define them, we intermittently visit volume domains abutting Γ. There the construction employs H^1-regular vector potentials, see [39, Sect. 2.4] and [1, Sect. 3]:

LEMMA 3.1. *For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there are continuous mappings $\mathsf{L} : \mathbf{curl}\,\mathbf{H}(\mathbf{curl},\Omega) \mapsto (H^1(\Omega))^3$ and $\mathsf{L}_0 : \mathbf{curl}\,\mathbf{H}_0(\mathbf{curl},\Omega) \mapsto (H^1_0(\Omega))^3$ such that $\mathbf{curl}\,\mathsf{L}\,\mathbf{\Phi} = \mathbf{\Phi}$ for all $\mathbf{\Phi} \in \mathbf{curl}\,\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{curl}\,\mathsf{L}_0\,\mathbf{\Phi} = \mathbf{\Phi}$ for all $\mathbf{\Phi} \in \mathbf{curl}\,\mathbf{H}_0(\mathbf{curl},\Omega)$.*

The construction is different for boundaries and screens and yields different projection operators R_c and R_o, respectively, fortunately sharing the same pivotal properties. To begin with, fix $\mathbf{u} \in \mathbf{X}$.

(I) Case of a closed surface $\Gamma = \partial\Omega$ [40, Sect. 7]: $\mathsf{R}_c\,\mathbf{u} := \big((\mathsf{L}\,\mathbf{\Phi}) \times \boldsymbol{n}\big)\big|_\Gamma$, where

$$\mathbf{\Phi} := \operatorname{grad} w : \quad w \in H^1(\Omega) : \quad \begin{array}{rcll} -\Delta w &=& 0 & \text{in } \Omega\,, \\ \operatorname{grad} w \cdot \boldsymbol{n} &=& \operatorname{div}_\Gamma \mathbf{u} & \text{on } \Gamma\,. \end{array}$$

The fact that $\int_\Sigma \operatorname{div}_\Gamma \mathbf{u}\,\mathrm{d}S = 0$ for each connected component Σ of Γ guarantees $\mathbf{\Phi} \in \mathbf{curl}\,\mathbf{H}(\mathbf{curl},\Omega)$, and Lemma 3.1 can be applied.

(II) Case of a bounded open orientable Lipschitz surface Γ with boundary $\partial\Gamma$ and unit normal vector field $\boldsymbol{n}_\Gamma$:

ASSUMPTION 3.2. *There exist two bounded Lipschitz domains $\Omega_1, \Omega_2 \subset \mathbb{R}^3$ satisfying*
- $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \overline{\Gamma}$,
- $\Omega := \Omega_1 \cup \Gamma \cup \Omega_2$ *is a bounded Lipschitz domain with trivial topology,*
- $\Gamma \subset \partial\Omega_1$ *and* $\Gamma \subset \partial\Omega_2$.

In words, Γ is the cut chopping the sphere-like Ω into two parts Ω_1, Ω_2, see Figure 3.1.

The fact that $\int_\Gamma \operatorname{div}_\Gamma \mathbf{u}\,\mathrm{d}S = 0$ makes it possible to define for $i = 1, 2$

$$w_i \in H^1(\Omega_i) : \quad \begin{array}{rcll} -\Delta w_i &=& 0 & \text{in } \Omega_i\,, \\ \operatorname{grad} w_i \cdot \boldsymbol{n} &=& 0 & \text{on } \partial\Omega_i \setminus \Gamma\,, \\ \operatorname{grad} w_i \cdot \boldsymbol{n}_\Gamma &=& \operatorname{div}_\Gamma \mathbf{u} & \text{on } \Gamma, \end{array}$$

and then

$$\mathbf{\Phi} := \begin{cases} \operatorname{grad} w_1 & \text{in } \Omega_1 \\ \operatorname{grad} w_2 & \text{in } \Omega_2 \end{cases} \in \mathbf{H}_0(\operatorname{div} 0, \Omega)\,,$$

because the normal component of $\mathbf{\Phi}$ is continuous across Γ. Hence, we can apply Lemma 3.1 and set $\mathsf{R}_o\,\mathbf{u} := \big((\mathsf{L}_0\,\mathbf{\Phi}) \times \boldsymbol{n}_\Gamma\big)\big|_\Gamma$.

By using elliptic lifting theorems, the continuity of L and L_0, and trace theorems we conclude, $* = c, o$:

$$\exists C = C(\Gamma) > 0 : \quad \|\mathsf{R}_*\,\mathbf{u}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)} \le C\,\|\operatorname{div}_\Gamma \mathbf{u}\|_{H^{-1/2}(\Gamma)} \quad \forall \mathbf{u} \in \mathbf{X}\,, \tag{3.3}$$

where $\mathbf{H}_\perp^{1/2}(\Gamma) \subset \mathbf{X}$ is the rotated tangential trace space

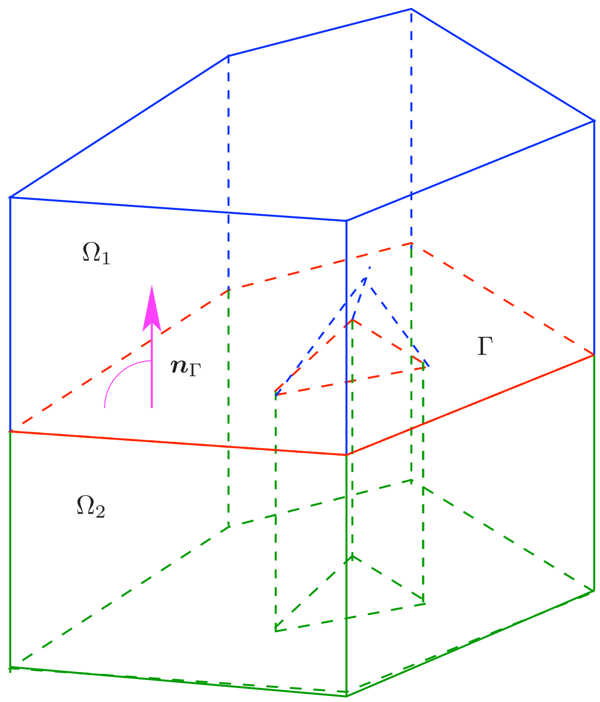

FIG. 3.1. *Screen Γ with attached domains Ω_1 and Ω_2. Note the nontrivial topology of Γ and how it can be dealt with in the construction of Ω.*

- of $(H^1(\Omega))^3$ on $\Gamma := \partial\Omega$ for closed surfaces [17, 20],
- of $(H^1_0(\Omega))^3$ on the screen Γ, see [17, Sect. 3.2].

Moreover, by construction, on Γ

$$\mathrm{div}_\Gamma \, \mathsf{R}_* \, \mathbf{u} = \mathrm{div}_\Gamma \mathbf{u} \quad \forall \mathbf{u} \in \mathbf{X} \quad \Rightarrow \quad \mathsf{R}_*^2 = \mathsf{R}_* \ . \tag{3.4}$$

Now we are in a position to define

$$\mathbf{V} := \mathsf{R}_*(\mathbf{X}) \subset \mathbf{H}_\perp^{1/2}(\Gamma), \quad \mathbf{W} := (Id - \mathsf{R}_*)(\mathbf{X}) \overset{\text{by (3.4)}}{=} \mathbf{X} \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma) \ . \tag{3.5}$$

In light of (3.3), the continuous embedding $\mathbf{H}_\perp^{1/2}(\Gamma) \hookrightarrow \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ ensures stability of the splitting.

Note that the embedding $\mathbf{V} \hookrightarrow \mathbf{L}^2_t(\Gamma)$ is compact by (3.3) and Rellich's theorem. Thus, thanks to the $(H^{1/2}(\Gamma))'$-coercivity (resp., $(\mathbf{H}_\perp^{1/2}(\Gamma))'$-coercivity) of the single layer boundary integral operator V_k (resp., $\mathbf{V}_k$), see [19, Prop. 2], [21, Lemma 8], [21, Lemma 7], and [16, Proof of Thm. 3.4], we infer the $\mathbf{X}$-coercivity of $a_{|V\times V}$ and $a_{|W\times W}$. Again, appealing to the compact embedding $\mathbf{V} \hookrightarrow (\mathbf{H}_\perp^{1/2}(\Gamma))'$, the compactness of $a_{|V\times W}$ and $a_{|W\times V}$ is immediate [21, Lemma 9]. This yields **(A)**.

REMARK 3.3. *Assumption 3.2 is easily verified for piecewise smooth Lipschitz screens through extension in normal direction followed by patching holes by means of thick cutting surfaces in order to mend topological defects. Yet, to keep the paper focused, we will not elaborate on this, but prefer to retain Assumption 3.2.*

REMARK 3.4. *We recall from [17] that $\mathbf{X}$ is the natural tangential trace space of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ for a closed surface Γ, and of $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$ for a screen Γ.*

4. Smoothed Poincaré lifting. For a domain $D \subset \mathbb{R}^2$ that is star-shaped with respect to $\boldsymbol{a} \in D$, the Poincaré lifting

$$(\mathsf{P}_{\boldsymbol{a}}\, u)(\boldsymbol{x}) := \int_0^1 \tau u(\boldsymbol{a} + \tau(\boldsymbol{x} - \boldsymbol{a}))(\boldsymbol{x} - \boldsymbol{a})\, \mathrm{d}\tau \tag{4.1}$$

provides a right inverse of the 2D divergence-operator $\operatorname{div} \mathbf{u} := (\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2})$ for continuous functions: $\operatorname{div} \mathsf{P}_{\boldsymbol{a}}\, u = u$ for all $u \in C^0(\overline{D})$, see [29, Sect. 3]. In [27] M. Costabel and A. McIntosh demonstrated how to mend the somewhat insufficient continuity properties of $\mathsf{P}_{\boldsymbol{a}}$ by local averaging:

ASSUMPTION 4.1. *D is star-shaped with respect to a ball $B \subset D$.*

Then define the *smoothed Poincaré lifting* [27, Sect. 3] as

$$(\mathsf{P}\, u)(\boldsymbol{x}) := \int_B \psi(\boldsymbol{a})(\mathsf{P}_{\boldsymbol{a}}\, u)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{a}\ , \tag{4.2}$$

where $\psi \in C^\infty(\mathbb{R}^2)$, $\operatorname{supp}(\psi) \subset B$, $\int_B \psi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 1$. We get the following powerful mapping properties from [27, Cor. 3.3].

THEOREM 4.2. *Under Assumption 4.1, the smoothed Poincaré lifting* P *according to* (4.2) *provides a continuous operator* $\mathsf{P} : H^s(D) \mapsto (H^{s+1}(D))^2$ *for any $s \in \mathbb{R}$ and satisfies* $\operatorname{div} \mathsf{P}\, \varphi = \varphi$ *for all $\varphi \in L^2(D)$.*

A crucial property of the smoothed Poincaré mapping is the preservation of the local boundary element spaces: the smoothed Poincaré mapping P on the (star-shaped) reference triangle $\widehat{K}$ fulfills, *cf.* [41, Sect. 3], [38], [27, Sect. 4.2],

$$\mathsf{P}(\operatorname{div} \boldsymbol{\mathcal{RT}}_p(\widehat{K})) \subset \boldsymbol{\mathcal{RT}}_p(\widehat{K}) \quad \overset{\text{by (2.7)}}{\Rightarrow} \quad \boxed{\mathsf{P}(\operatorname{div} \mathbf{X}_N(\widehat{K})) \subset \mathbf{X}_N(\widehat{K})}\ . \tag{4.3}$$

5. Projection based interpolation. Following [40] and [15, Sect. 4.3.1] again, local projection operators will be used to build a suitable splitting of $\mathbf{X}_N$. However, p-refinement entails a more subtle approach that resorts to so-called commuting projection based interpolation operators, see [29–31], [39, Sect. 3.6], and [28] for a comprehensive exposition. Commuting projectors link different finite element spaces on $\mathcal{M}$, the spaces S_N and $\widetilde{\mathbf{X}}_N$ in the current setting. Employing the relatively simple construction of [29] will be sufficient for our purposes and the following results from that article and from [10, 30] will be used:

1. There are projection operators (with domains $\mathcal{D}(\cdot)$)

$$\boldsymbol{\Pi}_X : \mathcal{D}(\boldsymbol{\Pi}_X) \subset H_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \mapsto \widetilde{\mathbf{X}}_N\ , \tag{5.1}$$

$$\Pi_S : \mathcal{D}(\Pi_S) \subset H^1(\Gamma) \mapsto S_N\ , \tag{5.2}$$

$$\Pi_Q : L^2(\Gamma) \mapsto Q_N\ , \tag{5.3}$$

satisfying the *commuting diagram properties* [29, Prop. 3]

$$\mathbf{curl}_\Gamma \circ \Pi_S = \boldsymbol{\Pi}_X \circ \mathbf{curl}_\Gamma \quad \text{on} \quad \mathcal{D}(\Pi_S)\ , \tag{5.4}$$

$$\operatorname{div}_\Gamma \circ \boldsymbol{\Pi}_X = \Pi_Q \circ \operatorname{div}_\Gamma \quad \text{on} \quad \mathcal{D}(\boldsymbol{\Pi}_X)\ . \tag{5.5}$$

For an open surface Γ the interpolation operator complies with boundary conditions:

$$\boldsymbol{\Pi}_X(\mathbf{X} \cap \mathcal{D}(\boldsymbol{\Pi}_X)) = \mathbf{X}_N\ . \tag{5.6}$$

2. As typical for the finite element interpolation operators, $\mathbf{\Pi}_X$ and Π_S are strictly local in the sense that both these projectors can be obtained by patching together purely local cell based projectors $\mathbf{\Pi}_{K,X}$ and $\Pi_{K,S}$, $K \in \mathcal{M}$. This is because for any edge E of $\mathcal{M}$ with in-plane normal $\boldsymbol{n}_E$ the traces $\mathbf{\Pi}_X \mathbf{u} \cdot \boldsymbol{n}_E|_E$ and $\Pi_S \varphi|_E$ depend only on $(\mathbf{u} \cdot \boldsymbol{n}_E)|_{\overline{E}}$ and $\varphi|_{\overline{E}}$, respectively. Furthermore, pullback commutes with local interpolation:
$$\mathbf{\Pi}_{\widehat{K},X} \circ \mathbf{\Phi}_K^* = \mathbf{\Phi}_K^* \circ \mathbf{\Pi}_{K,X} \quad \text{on} \quad \mathcal{D}(\mathbf{\Pi}_{K,X}) \,. \tag{5.7}$$
3. The projectors $\Pi_{K,S}$ enjoy the approximation property (this follows from [10, Thm. 4.1] with a scaling argument)
$$|\varphi - \Pi_{K,S}\varphi|_{H^1(K)} \le C \sqrt{\frac{h_K}{p_K+1}}\, |\varphi|_{H^{3/2}(K)} \qquad \forall \varphi \in H^{3/2}(K) \,. \tag{5.8}$$

These facts can be used to establish a special projection error estimate for $\mathbf{\Pi}_{K,X}$, *cf.* [41, Sect. 5], [39, Lemma 4.6], [1, Sect. 4].

Lemma 5.1. *With $C > 0$ depending only on the shape-regularity of the triangle $K \in \mathcal{M}$ there holds*
$$\|\mathbf{u} - \mathbf{\Pi}_{K,X}\mathbf{u}\|_{L^2(K)} \le C \sqrt{\frac{h_K}{p_K+1}}\, \|\mathbf{u}\|_{\mathbf{H}^{1/2}(K)} \ ,$$
for all $\mathbf{u} \in \mathbf{H}^{1/2}(K)$ with $\boxed{\operatorname{div}_\Gamma \mathbf{u} \in \operatorname{div}_\Gamma \mathbf{X}_N(K)}$.

Proof. Write P for the smoothed Poincaré lifting (see Sect. 4) on $\widehat{K}$. Fix $K \in \mathcal{M}$ and pick $\mathbf{u} \in \mathbf{H}^{1/2}(\widehat{K})$ with $\operatorname{div}\mathbf{u} \in \operatorname{div}\mathbf{X}_N(\widehat{K})$. This vector field is split according to
$$\mathbf{u} = \mathsf{P}\operatorname{div}\mathbf{u} + \underbrace{(\mathbf{u} - \mathsf{P}\operatorname{div}\mathbf{u})}_{\text{div-free}} = \mathsf{P}\operatorname{div}\mathbf{u} + \mathbf{curl}_{2D}\,\varphi \,, \tag{5.9}$$
where $\mathbf{curl}_{2D}$ denotes a rotated gradient and the existence of the scalar potential $\varphi \in \{\psi \in H^1(\widehat{K}) : \int_{\widehat{K}} \psi \,\mathrm{d}\mathbf{x} = 0\}$ is a consequence of $\operatorname{div}(\mathbf{u} - \mathsf{P}\operatorname{div}\mathbf{u}) = 0$, which follows from Theorem 4.2. Theorem 4.2 also supplies the continuity of $\mathsf{P} : H^{-1/2}(\widehat{K}) \mapsto \mathbf{H}^{1/2}(\widehat{K})$, which paves the way for estimating
$$\begin{aligned} |\varphi|_{H^{3/2}(\widehat{K})} &\le C\, |\mathbf{curl}_{2D}\,\varphi|_{\mathbf{H}^{1/2}(\widehat{K})} \le C \left(\|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K})} + \|\mathsf{P}\operatorname{div}\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K})} \right) \\ &\le \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K})} + C\, \|\operatorname{div}\mathbf{u}\|_{H^{-1/2}(\widehat{K})} \le C\, \|\mathbf{u}\|_{\mathbf{H}^{1/2}(\widehat{K})} \ , \end{aligned} \tag{5.10}$$
where the first step is justified by interpolation between $H^1(\widehat{K})$ and $H^2(\widehat{K})$. Then, by the projector property of $\mathbf{\Pi}_{\widehat{K},X}$, imbedding (4.3), and the discrete nature of $\operatorname{div}\mathbf{u}$, there holds
$$\begin{aligned} \mathbf{u} - \mathbf{\Pi}_{\widehat{K},X}\mathbf{u} &= \underbrace{(Id - \mathbf{\Pi}_{\widehat{K},X})\,\mathsf{P}\operatorname{div}\mathbf{u}}_{=0} + (Id - \mathbf{\Pi}_{\widehat{K},X})\,\mathbf{curl}_{2D}\,\varphi \\ &\overset{\text{by (5.4)}}{=} \mathbf{curl}_{2D}(Id - \Pi_{\widehat{K},S})\varphi \ , \end{aligned}$$

where we owe the last identity to the commuting diagram property (5.4) on $\widehat{K}$. This makes it possible to apply (5.8)

$$\begin{aligned}\left\|\mathbf{u} - \boldsymbol{\Pi}_{\widehat{K},X}\mathbf{u}\right\|_{L^2(\widehat{K})} &= \left|\varphi - \Pi_{\widehat{K},S}\varphi\right|_{H^1(\widehat{K})} \\ &\leq C\,(p_K+1)^{-1/2}\,|\varphi|_{H^{3/2}(\widehat{K})} \overset{(5.10)}{\leq} C\,(p_K+1)^{-1/2}\,|\mathbf{u}|_{\mathbf{H}^{1/2}(\widehat{K})}\ .\end{aligned}$$

Here, switching to the semi-norm in $\mathbf{H}^{1/2}(\widehat{K})$ can be justified by a fractional Bramble-Hilbert lemma [33, Prop. 6.1]. Eventually, (5.7) and a scaling argument take the estimate to the cell K. $\square$

We have implicitly proved that the (p-dependent !) local projectors $\boldsymbol{\Pi}_{K,X} : \{\mathbf{u} \in \mathbf{H}^{1/2}(K) : \mathrm{div}_\Gamma\mathbf{u} \in \mathrm{div}_\Gamma\mathbf{X}_N(K)\} \mapsto \mathbf{X}_N(K)$ are continuous with norm *independent of* p.

Remark 5.2. *In fact, the projector $\boldsymbol{\Pi}_X$ is closely linked to the splitting* (2.4). *From [29] and [28, Sect. 4.] we extract the particular form*

$$\boldsymbol{\Pi}_X = \boldsymbol{\Pi}_0 + \sum_{E\in\mathcal{E}} \boldsymbol{\Pi}_E(Id - \boldsymbol{\Pi}_0) + \sum_K \boldsymbol{\Pi}_K(Id - \boldsymbol{\Pi}_E)(Id - \boldsymbol{\Pi}_0)\ ,$$

where $\boldsymbol{\Pi}_0$, $\boldsymbol{\Pi}_E$, $\boldsymbol{\Pi}_K$ are suitable projection operators into $\boldsymbol{\mathcal{RT}}_0(\mathcal{M})$, $\boldsymbol{\mathcal{RT}}_{p_E}(E)$, and $\boldsymbol{\mathcal{RT}}_{p_K,0}(K)$, respectively.

6. Discrete splitting. Since $\mathrm{div}_\Gamma\,\mathsf{R}_*\,\mathbf{X}_N = \mathrm{div}_\Gamma\mathbf{X}_N$, (5.1) confirms that the following definitions are valid for $* = c, o$:

$$\mathbf{V}_N := \boldsymbol{\Pi}_X\,\mathsf{R}_*(\mathbf{X}_N) \quad , \quad \mathbf{W}_N := (Id - \boldsymbol{\Pi}_X \circ \mathsf{R}_*)\mathbf{X}_N\ . \tag{6.1}$$

By the commuting diagram property (5.5) and (3.4), we find

$$\begin{aligned}&\mathrm{div}_\Gamma\boldsymbol{\Pi}_X\,\mathsf{R}_*\,\mathbf{u}_N = \Pi_Q\mathrm{div}_\Gamma\,\mathsf{R}_*\,\mathbf{u}_N = \Pi_Q\underbrace{\mathrm{div}_\Gamma\mathbf{u}_N}_{\in Q_N} = \mathrm{div}_\Gamma\mathbf{u}_N \quad \forall\mathbf{u}_N \in \mathbf{X}_N\ .\\ &\Rightarrow \quad \mathsf{R}_*\,\boldsymbol{\Pi}_X\,\mathsf{R}_* = \mathsf{R}_* \quad \text{on } \mathbf{X}_N\ . \end{aligned}\tag{6.2}$$

Hence, $\boldsymbol{\Pi}_X\,\mathsf{R}_* : \mathbf{X}_N \mapsto \mathbf{X}_N$ is a projection, which confirms that $\mathbf{X}_N = \mathbf{V}_N + \mathbf{W}_N$. Stability, in the sense of

$$\|\boldsymbol{\Pi}_X\,\mathsf{R}_*\,\mathbf{u}_N\|_X \leq C\,\|\mathbf{u}_N\|_X\ , \tag{6.3}$$

with $C > 0$ depending on Γ and the shape-regularity of $\mathcal{M}$ only, is another consequence of Lemma 5.1 together with $\mathrm{div}_\Gamma\boldsymbol{\Pi}_X\,\mathsf{R}_*\,\mathbf{u}_N = \mathrm{div}_\Gamma\mathbf{u}_N$. This latter property also implies $\mathbf{W}_N \subset \mathbf{W} = \mathbf{X} \cap \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. This verifies assumption **(B)** from Sect. 3.

It remains to establish **(C)**, the gap property (3.1), which will be an immediate consequence of the following lemma.

Lemma 6.1. *There is a constant $C > 0$ depending only on the geometry of Γ and the shape-regularity of $\mathcal{M}$, such that for $* = c, o$*

$$\|(Id - \boldsymbol{\Pi}_X)\,\mathsf{R}_*\,\mathbf{u}_N\|_X \leq C \max_K \sqrt{\frac{h_K}{p_K+1}}\,\|\mathbf{u}_N\|_X \quad \forall\mathbf{u}_N \in \mathbf{X}_N\ .$$

Proof. By construction, we know that $\operatorname{div}_\Gamma \mathsf{R}_* \mathbf{u}_N = \operatorname{div}_\Gamma \mathbf{u}_N$, which permits us to apply the estimate of Lemma 5.1 to $\mathsf{R}_* \mathbf{u}_N|_K$, $K \in \mathcal{M}$:

$$\|(Id - \mathbf{\Pi}_{K,X}) \mathsf{R}_* \mathbf{u}_N\|^2_{L^2(K)} \le C \frac{h_K}{p_K+1} |\mathsf{R}_* \mathbf{u}_N|^2_{\mathbf{H}^{1/2}(K)} .$$

Patching together the local projectors and using sub-additivity of the $|\cdot|_{\mathbf{H}^{1/2}_\perp}$-semi-norm, we arrive at (we remind that $\mathsf{R}_* \mathbf{u}_N \in \mathbf{H}^{1/2}_\perp(\Gamma)$)

$$\begin{aligned} \|(Id - \mathbf{\Pi}_X) \mathsf{R}_* \mathbf{u}_N\|_{L^2(\Gamma)} &\le C \max_K \sqrt{\frac{h_K}{p_K+1}} |\mathsf{R}_* \mathbf{u}_N|_{\mathbf{H}^{1/2}_\perp(\Gamma)} \\ &\overset{(3.3)}{\le} C \max_K \sqrt{\frac{h_K}{p_K+1}} \|\operatorname{div}_\Gamma \mathbf{u}_N\|_{H^{-1/2}(\Gamma)} . \end{aligned}$$

Since $\operatorname{div}_\Gamma((Id - \mathbf{\Pi}_X) \mathsf{R}_* \mathbf{u}_N) = 0$, this is sufficient for the assertion of the lemma.

We point out that when we deal with an open surface Γ, we recall that $\mathsf{R}_o \mathbf{u}_N \in \mathbf{X}$ is guaranteed by the construction of Sect. 3. For a continuous tangential vector field $\mathbf{u} \in \mathbf{X}$ that is smooth on the faces of Γ, the constraint in (1.3) implies vanishing in-plane normal components on $\partial\Gamma$. By locality of $\mathbf{\Pi}_X$, this will carry over to $\mathbf{\Pi}_X \mathbf{u}$, which means $\mathbf{\Pi}_X \mathbf{u} \in \mathbf{X}_N$, *cf.* (5.6). Further, we know from Sect. 5, that $\mathsf{R}_o \mathbf{u}_N$ is in the domain of $\mathbf{\Pi}_X$. A simple density argument then confirms $\mathbf{\Pi}_X \mathsf{R}_o \mathbf{u}_N \in \mathbf{X}_N$, without adjusting the interpolation operator $\mathbf{\Pi}_X$, and the above proof carries over unaltered. ∎

The gap property (3.1) now immediately follows from the estimate of Lemma 6.1:

$$\begin{aligned} \sup_{\mathbf{v}_N \in \mathbf{V}_N} \inf_{\mathbf{v} \in \mathbf{V}} \frac{\|\mathbf{v} - \mathbf{v}_N\|_X}{\|\mathbf{v}_N\|_X} &\le \sup_{\mathbf{v}_N \in \mathbf{V}_N} \frac{\|\mathsf{R}_* \mathbf{v}_N - \mathbf{v}_N\|_X}{\|\mathbf{v}_N\|_X} \\ &\overset{(6.2)}{=} \sup_{\mathbf{v}_N \in \mathbf{V}_N} \frac{\|\mathsf{R}_* \mathbf{v}_N - \mathbf{\Pi}_X \mathsf{R}_* \mathbf{v}_N\|_X}{\|\mathbf{v}_N\|_X} \\ &\overset{\text{Lemma 6.1}}{\le} C \max_K \sqrt{\frac{h_K}{p_K+1}}. \end{aligned}$$

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