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Real Interpolation of Spaces of Differential Forms

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Abstract

In this paper we study interpolation of Hilbert spaces of differential forms using the real method of interpolation. We show that the scale of fractional order Sobolev spaces of differential l-forms in H^s with exterior derivative in H^s can be obtained by real interpolation. Our proof heavily relies on the recent discovery of smoothed Poincaré lifting for differential forms [M. COSTABEL AND A. MCINTOSH, *On Bogovskii and regularized Poincare integral operators for de Rham complexes on Lipschitz domains*, Math. Z., (2009)]. They enable the construction of universal extension operators for Sobolev spaces of differential forms, which, in turns, pave the way for a Fourier transform based proof of equivalences of K-functionals.

Key words. Differential forms, fractional Sobolev spaces, real interpolation, K-functional, smoothed Poincaré lifting, universal extension

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1 Introduction

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ and $d \geq 2$. Let Λ^l represent the vector space of real-valued (or complex-valued), alternating, l-multilinear maps on $\mathbb{R}^d$, which is of dimension $\binom{d}{l}$. A differential form of order l on Ω is a mapping $\Omega \mapsto \Lambda^l$. Given an increasing l-permutation $I = (i_1, \dots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq d$, $1 \leq l \leq d$, we introduce the basis l-form $\boldsymbol{dx}_I = \boldsymbol{d}x_{i_1} \wedge \cdots \wedge \boldsymbol{d}x_{i_l}$, where $\boldsymbol{d}x_i$'s are the canonical coordinate forms in $\mathbb{R}^d$. This basis representation permits us to introduce the Hilbert spaces

$$\boldsymbol{H}^s(\Omega, \Lambda^l) := \left\{ \boldsymbol{\omega} = \sum\nolimits_I \boldsymbol{\omega}_I \boldsymbol{dx}_I : \ \boldsymbol{\omega}_I \in H^s(\Omega) \right\}, \quad s \in \mathbb{R}_0^+ , \tag{1.1}$$

where $H^s(\Omega) = W^{s,2}(\Omega)$ is the standard $L^2(\Omega)$-based Sobolev space (of equivalence claesses of functions $\Omega \mapsto \mathbb{R}$) of fractional order s. Throughout, Σ_I means the summation over all the increasing l-permutations I and $\mathbb{R}_0^+ := \{s \mid s \geq 0\}$. Recall that Λ^0 can be identified with $\mathbb{R}$ and $\boldsymbol{H}^s(\Omega, \Lambda^0)$ with $H^s(\Omega)$. It is known [8, Thm. B.8] that these fractional spaces form a scale of interpolation spaces, namely

$$\text{for } 0 < \theta < 1, \ \ s_0, s_1 \in \mathbb{R}, \ \ s = (1-\theta)s_0 + \theta s_1 \quad \Rightarrow \quad H^s(\Omega) = [H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta , \tag{1.2}$$

where $[X, Y]_\theta$ designates the space obtained by real interpolation between the Banach spaces X and Y, see [11], [2, Ch. 3] and Section 2. As a consequence, the spaces $\boldsymbol{H}^s(\Omega, \Lambda^l)$ also form a scale of interpolation spaces.

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Writing $\boldsymbol{d}$ for the exterior derivative, the spaces

$$\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l) := \left\{ \boldsymbol{\omega} \in \boldsymbol{H}^s(\Omega;\Lambda^l) \mid \boldsymbol{d}\boldsymbol{\omega} \in \boldsymbol{H}^s(\Omega;\Lambda^{l+1}) \right\}, \quad s \in \mathbb{R}_0^+ , \tag{1.3}$$

play a key role in the statement of second-order variational boundary value problems for differential forms, *cf.* [9]. In this article we give a positive answer to the question, whether these Hilbert spaces $\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)$, $0 \leq l \leq d$, are related by real interpolation analogous to (1.2). More precisely, in Section 4 we will prove the following main result:

Theorem 1.1. *Let Ω be a bounded Lipschitz domain. For $s_0, s_1 \in \mathbb{R}_0^+$ and $0 \leq l \leq d$,*

$$\left[\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l), \boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l) \right]_\theta = \boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l) \tag{1.4}$$

with equivalent norms, where $s = (1-\theta)s_0 + \theta s_1$ for $0 < \theta < 1$.

The policy of the proof of Theorem 1.1, which is elaborated in Section 4, is as follows: first we show the assertion for $\Omega = \mathbb{R}^d$ by means of Fourier techniques. Then the problem for general bounded domains is reduced to that case by means of a universal extension theorem for the spaces $\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)$. To that end, we rely on E. Stein's classical extension operator. How this is done employing a smoothed Poincaré mapping is outlined in Section 3.

We remind that interpolation in function spaces is a powerful theoretical tool in functional analysis and numerical analysis, because estimates obtained for (simpler) special cases can instantly be extended to a whole scale of spaces. The spaces $\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)$ of differential forms discussed in this paper are isomorphic to the Sobolev spaces $\boldsymbol{H}(\mathrm{div};\Omega)$, $\boldsymbol{H}(\mathbf{curl};\Omega)$ for $d = 3$. These Sobolev spaces play a key role in the variational statement of boundary value problems in fluid mechanics and electromagnetics [4,6]. An interpolation theory for theses spaces will have significance for the mathematical and numerical analysis of these boundary value problems.

Despite the evident usefulness of Theorem 1.1 it seems not to be available in the literature. We mention the abstract framework of [1], but verifying its assumptions for the concrete setting discussed in this paper appears to be challenging.

Remark 1.2. To keep the presentation simple, we confine ourselves to the Hilbert space setting of spaces based on $L^2(\Omega)$. Extension to $L^p(\Omega)$-settings, $1 \leq p \leq \infty$ is likely possible by generalizing our approach.

2 Real method of interpolation

Let us first recall the real method of interpolation (cf. [2, Ch. 3], [8, App. B], [3, Ch. 14] for details). Assume a compatible pair of Hilbert spaces $\boldsymbol{\mathcal{X}}_0$ and $\boldsymbol{\mathcal{X}}_1$ with continuous embedding $\boldsymbol{\mathcal{X}}_1 \subset \boldsymbol{\mathcal{X}}_0$. By the real method of interpolation, we can define for $0 < s < 1$ a family of interpolation spaces $[\, \boldsymbol{\mathcal{X}}_0, \boldsymbol{\mathcal{X}}_1 \,]_s$ with the following nesting property

$$\boldsymbol{\mathcal{X}}_1 \subset [\, \boldsymbol{\mathcal{X}}_0, \boldsymbol{\mathcal{X}}_1 \,]_s \subset \boldsymbol{\mathcal{X}}_0 .$$

The $[\, \boldsymbol{\mathcal{X}}_0, \boldsymbol{\mathcal{X}}_1 \,]_s$-norm is defined through Peetre's K-functional by

$$\|\mathbf{v}\|_{[\, \boldsymbol{\mathcal{X}}_0, \boldsymbol{\mathcal{X}}_1 \,]_s} = \int_0^\infty \left(t^{-s} K(t,\mathbf{v})\right)^2 \frac{\mathrm{d}t}{t} , \tag{2.1}$$

where

$$K(t,\mathbf{v})^2 := \inf_{\substack{\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1 \\ \mathbf{v}_0\in\boldsymbol{\mathcal{X}}_0,\ \mathbf{v}_1\in\boldsymbol{\mathcal{X}}_1}} \left\{ \|\mathbf{v}_0\|^2_{\boldsymbol{\mathcal{X}}_0} + t^2 \|\mathbf{v}_1\|^2_{\boldsymbol{\mathcal{X}}_1} \right\} . \tag{2.2}$$

For well-known properties of interpolation spaces and families of linear operators defined on them, the reader is referred to [2,11].

3 Universal extension

We start from a celebrated extension theorem for Sobolev spaces due to E. M. Stein, see [10, Theorem 5, pp.181]:

Theorem 3.1. *For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geq 2$) there is an operator $\mathscr{E} : C^\infty(\overline{\Omega}) \mapsto C^\infty(\mathbb{R}^d)$ which satisfies*

1. *(extension property) $\mathscr{E}u(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$, and*
2. *(continuity) for any $m \in \mathbb{N}_0$ there exists a constant $C = C(m, \Omega)$ such that*

$$\|\mathscr{E}u\|_{H^m(\mathbb{R}^d)} \leq C \|u\|_{H^m(\Omega)} \quad \forall\, u \in C^\infty(\overline{\Omega}) .$$

Thus, $\mathscr{E}$ can be extended to a continuous extension operator $\mathscr{E} : H^m(\Omega) \mapsto H^m(\mathbb{R}^d)$ for any $m \in \mathbb{N}$ by a density argument. Furthermore, in light of of (1.2), by interpolation [8, Theorem B.2] the operator $\mathscr{E}$ can be generalized to fractional Sobolev spaces, that is, $\mathscr{E} : H^s(\Omega) \mapsto H^s(\mathbb{R}^d)$ is continuous for any $s \in \mathbb{R}_0^+$. due to the definition in (1.1), by componentwise application, we obtain an extension operator still denoted by $\mathscr{E} : \boldsymbol{H}^s(\Omega, \Lambda^l) \mapsto \boldsymbol{H}^s(\mathbb{R}^d, \Lambda^l)$ for any $s \in \mathbb{R}_0^+$ and $0 \leq l \leq d$. The operator $\mathscr{E}$ may be called "universal" for its *one-formula-fits-all* elegance.

A similar operator for the spaces $\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$ will be a key technical tool in our approach to interpolation spaces. It will be based on some so-called smoothed Poincaré liftings recently introduced by M. Costabel and A. McIntosh in [5], where they used it to prove the following theorem [5, Theorem 4.6]:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, then for $l = 0, 1, \ldots, d$, there exist pseudodifferential operators R_l and K_l with the following properties:*

1. *For any $s \in \mathbb{R}$, R_l maps from $\boldsymbol{H}^s(\Omega, \Lambda^l)$ into $\boldsymbol{H}^{s+1}(\Omega, \Lambda^{l-1})$ continuously and K_l maps from $\boldsymbol{H}^s(\Omega, \Lambda^l)$ into $\boldsymbol{H}^t(\Omega, \Lambda^l)$ continuously for any $t \in \mathbb{R}$.*
2. *For any $\boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$, there holds the identity*

$$\boldsymbol{d}R_l\boldsymbol{\omega} + R_{l+1}\boldsymbol{d}\boldsymbol{\omega} + K_l\boldsymbol{\omega} = \boldsymbol{\omega} \quad \textit{in } \Omega . \tag{3.1}$$

This theorem paves the way for harnessing the classical Stein extension operator $\mathscr{E}$ from Theorem 3.1 to build universal extension operators $\mathscr{C}_l : \boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ for $s \in \mathbb{R}_0^+$, according to

$$\mathscr{C}_l := \begin{cases} \boldsymbol{d} \circ \mathscr{E} \circ R_l + \mathscr{E} \circ R_{l+1} \circ \boldsymbol{d} + \mathscr{E} \circ K_l, & l = 0, 1, \ldots, d-1; \\ \boldsymbol{d} \circ \mathscr{E} \circ R_l + \mathscr{E} \circ K_l\, . & l = d. \end{cases} \tag{3.2}$$

Now we can show a universal extension theorem for the Sobolev spaces of differential forms $\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$.

Theorem 3.3. *For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geq 2$) there is an operator $\mathscr{C}_l : \boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$, $s \in \mathbb{R}_0^+$, which satisfies*

1. *(extension property) $\mathscr{C}_l\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x})$ a.e. in Ω, and*
2. *(continuity) for any $0 \leq l \leq d$ there exists a constant $C = C(\Omega, s)$ such that*

$$\|\mathscr{C}\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)} \leq C \|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)} \quad \forall\, \boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l) .$$

Proof. Let $\boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$, namely $\boldsymbol{\omega} \in \boldsymbol{H}^s(\Omega, \Lambda^l)$ and $\boldsymbol{d}\boldsymbol{\omega} \in \boldsymbol{H}^s(\Omega, \Lambda^{l+1})$. Note that $l = d$ is a degenerate case, since then $\boldsymbol{d}\boldsymbol{\omega} = 0$, and the assertion of the theorem becomes trivial. Hence, we restrict ourselves to $0 \leq l < d$. Thanks to Theorem 3.2, there exists some $\boldsymbol{\eta} = R_{l+1}\boldsymbol{d}\boldsymbol{\omega} \in \boldsymbol{H}^{s+1}(\Omega, \Lambda^l)$ with

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{s+1}(\Omega, \Lambda^l)} \leq C \|\boldsymbol{d}\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\Omega, \Lambda^{l+1})} ,$$

and some $\boldsymbol{\rho} = R_l\boldsymbol{\omega} \in \boldsymbol{H}^{s+1}(\Omega, \Lambda^{l-1})$ such that

$$\|\boldsymbol{\rho}\|_{\boldsymbol{H}^{s+1}(\Omega, \Lambda^{l-1})} \leq C \|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\Omega, \Lambda^l)} ,$$

with both constants C independent of $\boldsymbol{\omega}$. Moreover, in view of (3.1), we have

$$\boldsymbol{d}\boldsymbol{\rho} + \boldsymbol{\eta} + K_l\boldsymbol{\omega} = \boldsymbol{\omega} \quad \text{in } \Omega .$$

By applying the Stein extension componentwise, we can obtain $\widetilde{\boldsymbol{\eta}} \in \boldsymbol{H}^{s+1}(\mathbb{R}^d, \Lambda^l)$, $\widetilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{s+1}(\mathbb{R}^d, \Lambda^{l-1})$, and $\widetilde{\boldsymbol{\nu}} \in \boldsymbol{H}^{s+1}(\mathbb{R}^d, \Lambda^l)$ such that

$$\begin{aligned}
\widetilde{\boldsymbol{\eta}}_{|\Omega} &= \boldsymbol{\eta}, \qquad \|\widetilde{\boldsymbol{\eta}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^l)} \le C\,\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^l)}\,,\\
\widetilde{\boldsymbol{\rho}}_{|\Omega} &= \boldsymbol{\rho}, \quad \|\widetilde{\boldsymbol{\rho}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^{l-1})} \le C\,\|\boldsymbol{\rho}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^{l-1})}\,,\\
\widetilde{\boldsymbol{\nu}}_{|\Omega} &= K_l\boldsymbol{\omega}, \quad \|\widetilde{\boldsymbol{\nu}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^l)} \le C\,\|K_l\boldsymbol{\omega}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^l)}\,.
\end{aligned}$$

Noticing that K_l maps $\boldsymbol{H}^s(\Omega, \Lambda^l)$ continuously to $\boldsymbol{H}^{s+1}(\Omega, \Lambda^l)$ by Theorem 3.2, we see

$$\|K_l\boldsymbol{\omega}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^l)} \le C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^{s}(\Omega,\Lambda^l)}\,.$$

Define $\mathscr{C}\boldsymbol{\omega} = \boldsymbol{d}\widetilde{\boldsymbol{\rho}} + \widetilde{\boldsymbol{\eta}} + \widetilde{\boldsymbol{\nu}}$, then it is immediate to see that $(\mathscr{C}\boldsymbol{\omega})_{|\Omega} = \boldsymbol{\omega}$ and $\mathscr{C}\boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ by the following estimate:

$$\begin{aligned}
\|\mathscr{C}\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} &\le \|\boldsymbol{d}\widetilde{\boldsymbol{\rho}} + \widetilde{\boldsymbol{\eta}}\|_{\boldsymbol{H}^s(\mathbb{R}^d,\Lambda^l)} + \|\boldsymbol{d}\widetilde{\boldsymbol{\eta}}\|_{\boldsymbol{H}^s(\mathbb{R}^d,\Lambda^l)} + \|\widetilde{\boldsymbol{\nu}}\|_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}\\
&\le C\Big(\|\widetilde{\boldsymbol{\rho}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^{l-1})} + \|\widetilde{\boldsymbol{\eta}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^l)} + \|\widetilde{\boldsymbol{\nu}}\|_{\boldsymbol{H}^{s+1}(\mathbb{R}^d,\Lambda^l)}\Big)\\
&\le C\Big(\|\boldsymbol{\rho}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^{l-1})} + \|\boldsymbol{\eta}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^l)} + \|K_l\boldsymbol{\omega}\|_{\boldsymbol{H}^{s+1}(\Omega,\Lambda^l)}\Big)\\
&\le C\Big(\|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\Omega,\Lambda^l)} + \|\boldsymbol{d}\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\Omega,\Lambda^{l+1})}\Big)\\
&\le C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)}\,. \qquad (3.3)
\end{aligned}$$

This completes the proof. □

4 Interpolation in $\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$

In this section, we establish the equivalence between the interpolation spaces and fractional order Sobolev spaces $\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l)$ of differential forms.

In the first step, we establish the interpolation theorem about the equivalence between fractional Sobolev spaces $\boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and interpolation spaces for the domain $\mathbb{R}^d$. For $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, and $s = (1-\theta)s_0 + \theta s_1$, let us recall the definition of the $\big[\,\boldsymbol{H}^{s_0}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l), \boldsymbol{H}^{s_1}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)\,\big]_\theta$-norm of the interpolation space via the K-functional:

$$\|\boldsymbol{\omega}\|^2_{[\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)]_\theta} := \int_0^\infty \big(t^{-s}K(t,\mathbf{u})\big)^2\,\frac{\mathrm{d}t}{t}\,, \tag{4.1}$$

where

$$K(t,\boldsymbol{\omega})^2 := \inf_{\substack{\boldsymbol{\omega}=\boldsymbol{\omega}_0+\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_0\in\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)\\ \boldsymbol{\omega}_1\in\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}} \Big\{\,\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} + t^2\,\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}\,\Big\}. \tag{4.2}$$

On the other hand, for any $\boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$, the $\boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$-norm of the fractional order Sobolev spaces is defined by

$$\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\mathbb{R}^d,\Lambda^l)} := \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^s |\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\,, \tag{4.3}$$

where $\widehat{\boldsymbol{\omega}}$ is the Fourier transform of $\boldsymbol{\omega}$, and $|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2 := \sum_I |\widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi})|^2$. Here the Fourier transform of a differential l-form $\boldsymbol{\omega} = \sum_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I \in \boldsymbol{L}^2(\mathbb{R}^d; \Lambda^l)$, still denoted by $\mathscr{F}$, is defined componentwise by

$$\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi}) := \mathscr{F}(\boldsymbol{\omega})(\boldsymbol{\xi}) = \sum_I \widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi})\boldsymbol{d}\boldsymbol{\xi}_I\,,$$

where

$$\widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi}) := \mathscr{F}(\boldsymbol{\omega}_I)(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \exp(-\imath\boldsymbol{\xi}\cdot\mathbf{x})\boldsymbol{\omega}_I(\mathbf{x})\,\mathrm{d}\mathbf{x}\,,$$

and $\imath$ is the imaginary unit, $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_d)^T$ is the vectorial angular frequency in $\mathbb{R}^d$ and $\boldsymbol{d}\boldsymbol{\xi}_I = \boldsymbol{d}\xi_{i_1}\wedge\cdots\wedge\boldsymbol{d}\xi_{i_l}$, with I being an increasing l-permutation. Note that (4.3) corresponds to a componentwise definition of the norm for Sobolev spaces of differential forms by means of the Fourier transform method (cf. [8, Ch. 3]).

It is easy to see that the Fourier transform converts the exterior derivative into an exterior product:

Lemma 4.1. *For any $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}, \Omega, \Lambda^l)$, we have*

$$\mathscr{F}(\boldsymbol{d}\boldsymbol{\omega}) = \imath\widehat{\boldsymbol{\xi}} \wedge \mathscr{F}(\boldsymbol{\omega}), \tag{4.4}$$

where $\widehat{\boldsymbol{\xi}}$ is the differential 1-form in the frequency domain, namely $\widehat{\boldsymbol{\xi}} = \xi_1 \boldsymbol{d}\xi_1 + \xi_2 \boldsymbol{d}\xi_2 + \cdots + \xi_d \boldsymbol{d}\xi_d$.

Thus by Lemma 4.1, we may write

$$\begin{aligned}\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} &= \|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\mathbb{R}^d,\Lambda^l)} + \|\boldsymbol{d}\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\mathbb{R}^d,\Lambda^l)} \\ &= \int_{\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^s \Big(|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2 + |\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2 \Big)\, \mathrm{d}\boldsymbol{\xi}\,. \end{aligned}\tag{4.5}$$

To show the equivalence, we need a technical lemma [8, Ex. B.4].

Lemma 4.2. *For any two constants $c_0 > 0$ and $c_1 > 0$, and a complex number $z \in \mathbb{C}$, it holds that*

$$\min_{z=z_0+z_1} \big(c_0|z_0|^2 + c_1|z_1|^2 \big) = \frac{c_0 c_1}{c_0 + c_1}|z|^2, \tag{4.6}$$

and the minimum is achieved when $c_0 z_0 = c_1 z_1 = c_0 c_1 z/(c_0 + c_1)$.

Proof. Let $z = a + b\mathrm{i}$ and $z_j = a_j + b_j\mathrm{i}$ for $j = 0, 1$. We may rewrite (4.6) as:

$$\text{minimize} \quad c_0(a_0^2 + b_0^2) + c_1(a_1^2 + b_1^2),$$

subject to

$$a = a_0 + a_1, \qquad b = b_0 + b_1.$$

This problem can be solved by the Lagrangian multiplier method by defining the Lagrangian as follows:

$$\min \big(c_0(a_0^2 + b_0^2) + c_1(a_1^2 + b_1^2) \big) + \mu(a - a_0 - a_1) + \lambda(b - b_0 - b_1)\,.$$

Necessary minimality conditions yield that

$$\begin{aligned} 2c_0 a_0 - \mu = 0, \quad 2c_1 a_1 - \mu = 0, \\ 2c_0 b_0 - \lambda = 0, \quad 2c_1 b_1 - \lambda = 0. \end{aligned}$$

We find a solution $a_j = \mu/(2c_j)$ and $b_j = \lambda/(2c_j)$ for $j = 0, 1$, $\mu = 2c_0c_1a/(c_0 + c_1)$ and $\lambda = 2c_0c_1b/(c_0 + c_1)$. Thus we see that $c_j z_j = c_0 c_1 z/(c_0 + c_1)$ for $j = 0, 1$ at the critical point. It can be easily checked that the unique minimal value $\frac{c_0c_1}{c_0+c_1}|z|^2$ is indeed attained at this critical point. ☐

Now we can establish the equivalence of the fractional Sobolev spaces of differential forms $\boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and the interpolation spaces $\big[\boldsymbol{H}^{s_0}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l), \boldsymbol{H}^{s_1}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)\big]_s$.

Lemma 4.3. *For $s_0, s_1 \in \mathbb{R}$ with $s_0 < s_1$, and $l \in \mathbb{N}_0$ with $0 \le l \le d$, it holds that*

$$\big[\boldsymbol{H}^{s_0}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l), \boldsymbol{H}^{s_1}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)\big]_\theta = \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)\,, \tag{4.7}$$

with equivalent norms, where $s = (1-\theta)s_0 + \theta s_1$ for $0 < \theta < 1$.

Proof. We take the cue from the proof of the interpolation theorem for standard Sobolev spaces on $\mathbb{R}^d$ [8, Thm B.7]. For any $\boldsymbol{\omega} \in \boldsymbol{H}^s(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$, let $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$ with $\boldsymbol{\omega}_j \in \boldsymbol{H}^{s_j}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ for $j = 0, 1$. We observe that

$$\begin{aligned} K(t,\boldsymbol{\omega})^2 &= \inf_{\substack{\boldsymbol{\omega}=\boldsymbol{\omega}_0+\boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_0 \in \boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l) \\ \boldsymbol{\omega}_1 \in \boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}} \|\boldsymbol{\omega}_0\|^2_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} + t^2\,\|\boldsymbol{\omega}_0\|^2_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \\ &= \inf_{\widehat{\boldsymbol{\omega}}=\widehat{\boldsymbol{\omega}}_0+\widehat{\boldsymbol{\omega}}_1} \int_{\mathbb{R}^d} \Big[(1+|\boldsymbol{\xi}|^2)^{s_0} \Big(|\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2 + |\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2 \Big) + t^2 (1+|\boldsymbol{\xi}|^2)^{s_1} \Big(|\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2 + |\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2 \Big) \Big]\, \mathrm{d}\boldsymbol{\xi} \\ &\ge \inf_{\widehat{\boldsymbol{\omega}}=\widehat{\boldsymbol{\omega}}_0+\widehat{\boldsymbol{\omega}}_1} \int_{\mathbb{R}^d} \big[(1+|\boldsymbol{\xi}|^2)^{s_0} \big(|\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2 \big)\, \mathrm{d}\boldsymbol{\xi} + t^2 (1+|\boldsymbol{\xi}|^2)^{s_1} \big(|\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2 \big) \big] \\ &\quad + \inf_{\widehat{\boldsymbol{\omega}}=\widehat{\boldsymbol{\omega}}_0+\widehat{\boldsymbol{\omega}}_1} \int_{\mathbb{R}^d} \Big[(1+|\boldsymbol{\xi}|^2)^{s_0} \Big(|\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2 \Big) + t^2 (1+|\boldsymbol{\xi}|^2)^{s_1} \Big(|\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2 \Big) \Big]\, \mathrm{d}\boldsymbol{\xi} \\ &:= \mathfrak{S} + \mathfrak{T} \end{aligned}$$

where $\widehat{\boldsymbol{\omega}}_j(\boldsymbol{\xi})$ is the Fourier transform of $\boldsymbol{\omega}_j$ for $j = 0, 1$ and $\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi}) = \widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi}) + \widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})$ by the linearity of the Fourier transform. By Lemma 4.2, we see that for each $\boldsymbol{\xi}$ the integrand in $\mathfrak{S}$ is minimized when

$$(1+|\boldsymbol{\xi}|^2)^{s_0}\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi}) = t^2(1+|\boldsymbol{\xi}|^2)^{s_1}\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi}) = \frac{t^2(1+|\boldsymbol{\xi}|^2)^{s_0+s_1}}{(1+|\boldsymbol{\xi}|^2)^{s_0}+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}}\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi}). \tag{4.8}$$

Likewise, by linearity of the operator $\boldsymbol{\xi}\wedge\cdot$, the integrand in $\mathfrak{T}$ is minimized when

$$(1+|\boldsymbol{\xi}|^2)^{s_0}\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi}) = t^2(1+|\boldsymbol{\xi}|^2)^{s_1}\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi}) = \frac{t^2(1+|\boldsymbol{\xi}|^2)^{s_0+s_1}}{(1+|\boldsymbol{\xi}|^2)^{s_0}+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}}\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi}).$$

Thanks to the special choice of splitting as given in (4.8), we have

$$\begin{aligned}
&K(t,\boldsymbol{\omega})^2 \geq \mathfrak{S}+\mathfrak{T}\\
&= \int_{\mathbb{R}^d}\frac{t^2(1+|\boldsymbol{\xi}|^2)^{s_0+s_1}}{(1+|\boldsymbol{\xi}|^2)^{s_0}+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}}|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi} + \int_{\mathbb{R}^d}\frac{t^2(1+|\boldsymbol{\xi}|^2)^{s_0+s_1}}{(1+|\boldsymbol{\xi}|^2)^{s_0}+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}}|\boldsymbol{\xi}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}\\
&\geq \inf_{\widehat{\boldsymbol{\omega}}=\widehat{\boldsymbol{\omega}}_0+\widehat{\boldsymbol{\omega}}_1}\int_{\mathbb{R}^d}\Big[(1+|\boldsymbol{\xi}|^2)^{s_0}\Big(|\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}_0(\boldsymbol{\xi})|^2\Big)+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}\Big(|\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}_1(\boldsymbol{\xi})|^2\Big)\Big]\\
&= K(t,\boldsymbol{\omega})^2\,.
\end{aligned}$$

Hence we see that when (4.8) holds,

$$\begin{aligned}
K(t,\boldsymbol{\omega})^2 &= \int_{\mathbb{R}^d}\frac{t^2(1+|\boldsymbol{\xi}|^2)^{s_0+s_1}}{(1+|\boldsymbol{\xi}|^2)^{s_0}+t^2(1+|\boldsymbol{\xi}|^2)^{s_1}}\Big(|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\Big)\,\mathrm{d}\boldsymbol{\xi}\\
&= \int_{\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^{s_0}f(a(\boldsymbol{\xi})t)^2\Big(|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\Big)\,\mathrm{d}\boldsymbol{\xi},
\end{aligned}$$

where $a(\boldsymbol{\xi}) = (1+|\boldsymbol{\xi}|^2)^{(s_1-s_0)/2}$ and $f(t) = \frac{t}{\sqrt{1+t^2}}$. Therefore we derive for $0<\theta<1$,

$$\begin{aligned}
\|\boldsymbol{\omega}\|^2_{[\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)]_s} &= \int_{\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^{s_0}a(\boldsymbol{\xi})^{2\theta}\Big(\int_0^\infty\frac{t^{1-2\theta}}{1+t^2}\,\mathrm{d}t\Big)\Big(|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\Big)\,\mathrm{d}\boldsymbol{\xi}\\
&= \frac{\pi}{2\sin\pi\theta}\int_{\mathbb{R}^d}(1+|\boldsymbol{\xi}|^2)^s\Big(|\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2+|\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})|^2\Big)\,\mathrm{d}\boldsymbol{\xi} = \frac{\pi}{2\sin\pi\theta}\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^3,\Lambda^l)}\,.
\end{aligned}$$

This completes the proof. □

Then, we define the $\big[\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)\big]_\theta$-norm, for $0<\theta<1$, $s_0, s_1\in\mathbb{R}_0^+$ with $s_0<s_1$ and $s=(1-\theta)s_0+\theta s_1$, via the K-functional for a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$:

$$\|\boldsymbol{\omega}\|^2_{[\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)]_\theta} := \int_0^\infty\Big(t^{-s}\widetilde{K}(t,\mathbf{u})\Big)^2\frac{\mathrm{d}t}{t}\,, \tag{4.9}$$

where

$$\widetilde{K}(t,\boldsymbol{\omega})^2 := \inf_{\substack{\boldsymbol{\omega}=\boldsymbol{\omega}_0+\boldsymbol{\omega}_1\\ \boldsymbol{\omega}_0\in\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l)\\ \boldsymbol{\omega}_1\in\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)}}\Big\{\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l)}+t^2\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)}\Big\}. \tag{4.10}$$

Now we are in a position to prove our main result Theorem 1.1 about the equivalence of interpolation spaces in bounded Lipschitz domains.

Proof. (of Theorem 1.1) It suffices to show the norm equivalence of the two spaces under study.

(i) Let $\boldsymbol{\omega}\in\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)$, namely $\boldsymbol{\omega}\in\boldsymbol{H}^s(\Omega,\Lambda^l)$ and $\boldsymbol{d}\boldsymbol{\omega}\in\boldsymbol{H}^s(\Omega,\Lambda^{l+1})$. Thanks to Theorem 3.3, we can extend $\boldsymbol{\omega}$ to $\mathscr{C}_l\boldsymbol{\omega}\in\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)$ such that $\mathscr{C}_l\boldsymbol{\omega}_{|\Omega}=\boldsymbol{\omega}$.

Take any splitting $\mathscr{C}_l\boldsymbol{\omega}=\boldsymbol{\eta}_0+\boldsymbol{\eta}_1$ with $\boldsymbol{\eta}_j\in\boldsymbol{H}^{s_j}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)$ for $j=0,1$. We define $\boldsymbol{\omega}_j=\boldsymbol{\eta}_j|_{\Omega_j}$, then $\boldsymbol{\omega}_j\in\boldsymbol{H}^{s_j}(\boldsymbol{d},\Omega,\Lambda^l)$ for $j=0,1$. Therefore we have

$$\begin{aligned}
\widetilde{K}(t,\boldsymbol{\omega}) &\leq \|\boldsymbol{\omega}_0\|_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l)}+t^2\|\boldsymbol{\omega}_1\|_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)}\\
&\leq \|\boldsymbol{\eta}_0\|_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}+t^2\|\boldsymbol{\eta}_1\|_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)}\,.
\end{aligned}$$

As the splitting of $\mathscr{C}_l\boldsymbol{\omega}$ was arbitrary, we have

$$\widetilde{K}(t,\boldsymbol{\omega}) \leq K(t,\mathscr{C}_l\boldsymbol{\omega}), \tag{4.11}$$

Combining (3.3), (4.11) with Lemma 4.3 implies

$$\begin{aligned}\|\boldsymbol{\omega}\|_{[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)\,]_\theta} &\leq \|\mathscr{C}_l\boldsymbol{\omega}\|_{[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)\,]_\theta} \\ &\leq C\,\|\mathscr{C}_l\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leq C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)}\,,\end{aligned}$$

Which proves $\boldsymbol{\omega} \in \big[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)\,\big]_\theta$.

(ii) For the opposite inclusion, take any $\boldsymbol{\omega} \in \big[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)\,\big]_\theta$ and any splitting $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$ with $\boldsymbol{\omega}_0 \in \boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l)$ and $\boldsymbol{\omega}_1 \in \boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)$. Now we may apply Theorem 3.3 for $\boldsymbol{H}^{s_i}(\boldsymbol{d},\Omega,\Lambda^l)$ to define $\mathscr{C}_l\boldsymbol{\omega}_i \in \boldsymbol{H}^{s_i}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)$ such that $\mathscr{C}_l\boldsymbol{\omega}_i = \boldsymbol{\omega}_i$ in Ω and

$$\|\mathscr{C}_l\boldsymbol{\omega}_i\|_{\boldsymbol{H}^{s_i}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leq C\,\|\boldsymbol{\omega}_i\|_{\boldsymbol{H}^{s_i}(\boldsymbol{d},\Omega,\Lambda^l)} \qquad \text{for } i=0,1.$$

Let $\boldsymbol{\eta} = \mathscr{C}_l\boldsymbol{\omega}_0 + \mathscr{C}_l\boldsymbol{\omega}_1$. By Theorem 3.3, we see that $\boldsymbol{\omega} = \boldsymbol{\eta}$ on Ω and

$$K(t,\boldsymbol{\eta}) \leq \|\mathscr{C}\boldsymbol{\omega}_0\|^2_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} + t^2\,\|\mathscr{C}\boldsymbol{\omega}_1\|^2_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leq C\big(\|\boldsymbol{\omega}_0\|^2_{\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l)} + t^2\,\|\boldsymbol{\omega}_1\|^2_{\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)}\big)\,.$$

Since the splitting $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$ is arbitrary, taking the infimum on the rightmost terms of the inequality above over all possible splittings we conclude

$$K(t,\boldsymbol{\eta}) \leq C\widetilde{K}(t,\boldsymbol{\omega})\,,$$

which together with Lemma 4.3 yields

$$\begin{aligned}\|\boldsymbol{\omega}\|_{\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)} \leq \|\boldsymbol{\eta}\|_{\boldsymbol{H}^s(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} &\leq C\|\boldsymbol{\eta}\|_{[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)\,]_\theta} \\ &\leq C\|\boldsymbol{\omega}\|_{[\,\boldsymbol{H}^{s_0}(\boldsymbol{d},\Omega,\Lambda^l),\boldsymbol{H}^{s_1}(\boldsymbol{d},\Omega,\Lambda^l)\,]_\theta}.\end{aligned}$$

This completes the proof. □

Remark 4.1. In three-dimensional Euclidean space $\mathbb{R}^3$, we can interpret Theorem1.1 for the vector fields modeling differential forms, see, e.g., [7, Table 2.1]. In particular for the cases $l = 1, 2$, special cases of the theorem can be stated as follows:

Lemma 4.4. *For $k, m \in \mathbb{N}$ with $k < m$, the following spaces agree*

$$\begin{aligned}\Big[\boldsymbol{H}^k(\mathbf{curl};\Omega),\boldsymbol{H}^m(\mathbf{curl};\Omega)\Big]_\theta &= \boldsymbol{H}^s(\mathbf{curl};\Omega)\,, \\ \Big[\boldsymbol{H}^k(\mathrm{div};\Omega),\boldsymbol{H}^m(\mathrm{div};\Omega)\Big]_\theta &= \boldsymbol{H}^s(\mathrm{div};\Omega)\,,\end{aligned}$$

with equivalent norms, where $s = (1-\theta)k + \theta m$ for $0 < \theta < 1$.

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