Eidgenössische
Technische Hochschule
Zürich

Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

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C. Winter

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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C. Winter*

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Abstract

We consider a wavelet Galerkin scheme for solving partial integrodifferential equations arising from option pricing in multidimensional Lévy models. Sparse tensor product spaces are applied for the discretization to reduce the complexity in the number of degrees of freedom and wavelet compression methods are used to decrease the number of non-zero matrix entries. We focus on algorithmic details of the scheme, in particular on the numerical integration of the matrix coefficients.

Keywords: Composite Gauss quadrature, multivariate Lévy models, wavelets

1 Introduction

Finite element methods have successfully been applied to integral operators of the type, $\mathcal{A}u(x) = \int_D \kappa(x,y)u(y)\,\mathrm{d}y$, where the kernel functions $\kappa(x,y)$ are piecewise smooth apart form the diagonal $\{(x,y) \in D \times D : x = y\}$. Several schemes have been developed to solve these problems in dimension $d \leq 3$, see [6, 11, 13] and the references therein. For tensor product domains D of the type $D = [-R,R]^d$, $R > 0$, we extend these methods to the anisotropic case with singularities in each direction $\{(x,y) \in D \times D : x_i = y_i,\ i = 1\ldots,d\}$ for $d \geq 1$. In particular, we consider integrodifferential operators $\mathcal{A}$ arising in finance given by

$$\mathcal{A}u(x) = -\frac{1}{2}\sum_{i,j=1}^{d} \mathcal{Q}_{ij}\partial_{x_i x_j}u(x) - \int_{\mathbb{R}^d} \big(u(x+z) - u(x) - z \cdot \nabla u(x) 1_{\{|z|\leq 1\}}\big)k(z)\,\mathrm{d}z,$$

*Seminar for Applied Mathematics, ETH Zurich, 8092 Zurich, Switzerland

where $\mathcal{Q}$ is the covariance matrix and k a multidimensional Lévy density satisfying $\int_{\mathbb{R}^d} 1 \wedge |z|^2 \, k(z) \, \mathrm{d}z < \infty$. The Lévy density k can for example be (isotropic) α-stable like $k(z) = |z|^{-d-\alpha}$, with $0 < \alpha < 2$ or (anisotropic) $\boldsymbol{\alpha}$-stable like

$$k(z) = \prod_{i=1}^{d} |z_i|^{\alpha_i \vartheta - 1} \big(\sum_{i=1}^{d} |z_i|^{\alpha_i \vartheta} \big)^{-\frac{1}{\vartheta} - d},$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $0 < \alpha_i < 2$, $i = 1, \ldots, d$ and $\vartheta > 0$. More examples are given in [16].

Following [5, 9] we consider a wavelet Galerkin scheme where sparse tensor product spaces are applied for the discretization to reduce the complexity in the number of degrees of freedom from $\mathcal{O}(h^{-d})$ to $\mathcal{O}(h^{-1} |\log h|^{d-1})$. Here, h denotes the mesh width of the finite element discretization. The resulting matrices are dense since the jump operator is non-local. Therefore, wavelet compression methods are used to reduce the number of non-zero matrix entries. We focus on algorithmic details of the scheme, in particular on the numerical integration of the matrix coefficients. Since the multidimensional Lévy densities have singularities at the origin and on the axes, variable order composite Gauss quadrature formulas are employed. We show that the quadrature rule leads to exponential convergence for Lévy densities which are piecewise analytic. Using an hierarchical data structure, an adaptive numerical scheme is developed which computes each matrix entry with a given accuracy. The accuracy is chosen by an a-priori numerical analysis of the scheme such that the solution of the perturbed problem still converges at the optimal rate and the computational complexity is log linear $\mathcal{O}(h^{-1} |\log h|^{4d-2})$.

2 Preliminaries

Let D be a non-empty bounded open subset of $\mathbb{R}^d$. If a function $u : D \to \mathbb{R}$ is sufficiently smooth, we denote the partial derivatives of u by $\partial^{\mathbf{n}} u = \partial_1^{n_1} \cdots \partial_d^{n_d} u$, where $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ is a multiindex. Throughout, we write $x \lesssim y$ to express that the scalar x is bounded by a constant multiple of y, i.e., there exists a $c > 0$ such that $x \leq c\, y$. Correspondingly $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. For a non-empty set $\mathcal{I} \subset \{1, \ldots, d\}$ we define its complement by $\mathcal{I}^c = \{1, \ldots, d\} \backslash \mathcal{I}$. We set $x^{\mathcal{I}} = (x_i)_{i \in \mathcal{I}}$ and use the notation

$$x + y^{\mathcal{I}} = z \in \mathbb{R}^d \quad \text{with } z_i = \begin{cases} x_i & \text{if } i \notin \mathcal{I} \\ x_i + y_i & \text{else,} \end{cases}$$

for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{|\mathcal{I}|}$. Furthermore, $\partial^{\mathcal{I}} = \partial_{i_1} \cdots \partial_{i_k}$ for $\mathcal{I} = \{i_1, \ldots, i_k\}$, $k \in \{1, \ldots, d\}$. We need the tail integrals of the Lévy density k which is the function

$U:\mathbb{R}^d\backslash\{0\}\to\mathbb{R}$ given by

$$U(x_1,\dots,x_d)=\prod_{i=1}^{d}\operatorname{sgn}(x_j)\int_{I(x_1)}\cdots\int_{I(x_d)}k(z)\,\mathrm{d}z_d\dots\mathrm{d}z_1,$$
$$\text{with}\quad I(x)=\begin{cases}(x,\infty) & \text{if } x\geq 0,\\ (-\infty,x] & \text{if } x<0.\end{cases}$$

The $\mathcal{I}$-marginal tail integrals $U^{\mathcal{I}}$ are the tail integral of marginal Lévy densities $k^{\mathcal{I}}$. Furthermore for $0<\alpha_i<2$, $i=1,\dots,d$ we call a Lévy density $\boldsymbol{\alpha}$-stable if

$$k^{\mathrm{sym}}(r^{-\frac{1}{\alpha_1}}z_1,\dots,r^{-\frac{1}{\alpha_d}}z_d)=r^{1+\frac{1}{\alpha_1}+\dots+\frac{1}{\alpha_d}}k^{\mathrm{sym}}(z_1,\dots,z_d),\quad\forall r>0,$$

where $k^{\mathrm{sym}}(z)=(k(z)+k(-z))/2$ is the symmetric part of k. We make the following assumptions on the Lévy density k where we denote with k_i, $i=1,\dots,d$ the marginal Lévy densities.

Assumption 2.1. *Let k be a Lévy density with marginal Lévy densities k_i.*

i) There are constants $\beta_i^->0$, $\beta_i^+>0$, $i=1,\dots,d$ such that

$$k_i(z)\lesssim\begin{cases}e^{-\beta_i^-|z|}, & z<-1,\\ e^{-\beta_i^+z}, & z>1.\end{cases}\tag{2.1}$$

ii) There exist an $\boldsymbol{\alpha}$-stable Lévy density k^0 such that

$$k(z)\lesssim k^0(z),\quad 0<|z|<1.\tag{2.2}$$

iii) If $\mathcal{Q}$ is not positive definite, we assume additionally that

$$k^{\mathrm{sym}}(z)\gtrsim k^{0,\mathrm{sym}}(z),\quad 0<|z|<1,\tag{2.3}$$

iv) We require that the density k is real analytic outside $z_i=0$, $i=1,\dots,d$,

$$|\partial^{\mathbf{n}}k(z)|\lesssim C^{|\mathbf{n}|}\,|\mathbf{n}|!\,\|z\|_\infty^{-\alpha}\prod_{i=1}^{d}|z_i|^{-n_i-1},\quad\forall z_i\neq 0.\tag{2.4}$$

for $C>0$, $\alpha=\|\boldsymbol{\alpha}\|_\infty$ and multiindex $\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{N}_0^d$.

For $u,v\in C_0^\infty(D)$ we associate with $\mathcal{A}$ the bilinear form

$$\begin{aligned}\mathcal{E}(u,v)=&\frac{1}{2}\sum_{i,j=1}^{d}\mathcal{Q}_{ij}\int_D\partial_iu(x)\partial_jv(x)\,\mathrm{d}x\\&-\int_D\int_{\mathbb{R}^d}\left(u(x+z)-u(x)-z\cdot\nabla u(x)\right)v(x)k(z)\,\mathrm{d}z\,\mathrm{d}x,\end{aligned}\tag{2.5}$$

where we extend u by zero outside of the domain D. We omitted the cut-of function $1_{\{|z|\leq 1\}}$ since we assume semi-heavy tails (2.1). The variational formulation is given by,

$$\begin{aligned} &\text{Find } u \in L^2((0,T);\mathcal{V}) \cap H^1((0,T);\mathcal{V}^*) \text{ such that} \\ &\langle \partial_t u, v\rangle_{\mathcal{V}^*,\mathcal{V}} + \mathcal{E}(u,v) = 0\,,\ t \in (0,T),\ \forall v \in \mathcal{V}, \\ &u(0) = u_0, \end{aligned} \tag{2.6}$$

with initial condition $u_0 \in L^2(D)$. The well-posedness of (2.6) is ensured by the next theorem. A proof is given in [9, Theorem 4.8].

Theorem 2.2. *Assume either $\mathcal{Q} > 0$ or $\mathcal{Q} = 0$ and the Lévy density k satisfies* (2.2), (2.3)*, with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. Then, the variational equation* (2.6) *admits a unique solution in $\mathcal{V}$. For $\mathcal{Q} > 0$ there holds $\mathcal{V} = \widetilde{H}^1(D)$ and for $\mathcal{Q} = 0$ one obtains the anisotropic Sobolev space $\mathcal{V} = \widetilde{H}^{\boldsymbol{\alpha}/2}(D)$ defined by $\widetilde{H}^{\mathbf{s}}(D) := \{u|_D : u \in H^{\mathbf{s}}(\mathbb{R}^d), u|_{\mathbb{R}^d\setminus D} = 0\}$.*

Using partial integration we can rewrite the bilinear form $\mathcal{E}(\cdot,\cdot)$ as

$$\begin{aligned} \mathcal{E}(u,v) = &\frac{1}{2}\sum_{i,j=1}^{d} \mathcal{Q}_{ij}\int_D \partial_i u(x)\partial_j v(x)\,\mathrm{d}x \\ &+ \sum_{i=1}^{d}\int_{\mathbb{R}}\int_D \partial_i u(x+z_i)\partial_i v(x) k_i^{-2}(z_i)\,\mathrm{d}x\,\mathrm{d}z_i \\ &- \sum_{i=2}^{d}\sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1<\cdots<\mathcal{I}_i}}\int_{\mathbb{R}^i}\int_D \partial^{\mathcal{I}} u(x+z^{\mathcal{I}})v(x)U^{\mathcal{I}}(z^{\mathcal{I}})\,\mathrm{d}x\,\mathrm{d}z^{\mathcal{I}}, \end{aligned}$$

where $k^{-2}(x) = \mathrm{sgn}(x)\int_{I(x)} U(z)\,\mathrm{d}z$ are the second antiderivative of k vanishing at $\pm\infty$. We refer to [9, Lemma 2.5] for more details.

3 Wavelet basis

We start by explaining wavelets in one dimension, following the construction described in [2]. The d-variate bases are obtained by tensor product construction.

3.1 Spline wavelets on the interval

The one-dimensional interval $D = [0,1]$ is partitioned into an equidistant mesh $\mathcal{T}_\ell$ with mesh width $h_\ell = 2^{-\ell}$, $\ell \in \mathbb{N}$. We define $\mathcal{V}_\ell$ as the space of piecewise polynomials of degree $p-1 \in \mathbb{N}$ on the mesh $\mathcal{T}_\ell$ which vanish at the endpoints and denote with $N_\ell = \dim\, \mathcal{V}_\ell = O(2^\ell)$. The spaces $\mathcal{V}_\ell$ are nested, $\mathcal{V}_\ell \subset \mathcal{V}_{\ell+1}$, and generated by single scale bases $\Phi_\ell := \{\phi_{\ell,k} : k \in \Delta_\ell\}$ with suitable index set Δ_ℓ. We assume that the basis functions $\phi_{\ell,k} \in \Phi_\ell$, $\ell \in \mathbb{N}$, have compact support of size $|\text{supp}\, \phi_{\ell,k}| \lesssim 2^{-\ell}$ and are normalized in L^2, $\|\phi_{\ell,k}\|_{L^2([0,1])} = 1$. The approximation order of Φ_ℓ is given by p. In addition, we associated with $\Phi_{\ell,k}$ a dual basis, $\widetilde{\Phi}_\ell := \{\widetilde{\phi}_{\ell,k} : k \in \Delta_\ell\}$, i.e., one has $\langle \phi_{\ell,k}, \widetilde{\phi}_{\ell,k'}\rangle = \delta_{k,k'}$, $k, k' \in \Delta_\ell$. The approximation order of $\widetilde{\Phi}_\ell$ is denoted by $\widetilde{p}$, and we assume $p \le \widetilde{p}$.

Given the single-scale basis Φ_ℓ, we can construct a biorthogonal complement or wavelet basis $\Psi_\ell = \{\psi_{\ell,k} : k \in \nabla_\ell\}$, $\widetilde{\Psi}_\ell = \{\widetilde{\psi}_{\ell,k} : k \in \nabla_\ell\}$ with $\nabla_\ell = \Delta_{\ell+1} \backslash \Delta_\ell$ such that

$$\begin{aligned} &\mathcal{V}_{\ell+1} = \mathcal{V}_\ell \oplus \mathcal{W}_\ell, \quad \widetilde{\mathcal{V}}_{\ell+1} = \widetilde{\mathcal{V}}_\ell \oplus \widetilde{\mathcal{W}}_\ell, \quad \ell \in \mathbb{N}, \\ &\mathcal{V}_\ell = \mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{\ell-1}, \quad \ell \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where the increment spaces $\mathcal{W}_\ell$, $\widetilde{\mathcal{W}}_\ell$ are the span of Ψ_ℓ, $\widetilde{\Psi}_\ell$ for $\ell > 0$, and $\mathcal{W}_0 := \mathcal{V}_1$, $\widetilde{\mathcal{W}}_0 := \widetilde{\mathcal{V}}_1$. We suppose the wavelets $\psi_{\ell,k}$ have compact support $|\text{supp}\, \psi_{\ell,k}| \lesssim 2^{-\ell}$ and are normalized in $L^2([0,1])$.

Any function $u \in \mathcal{V}_{L+1}$ has the representation

$$u = \sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} u_{\ell,k} \psi_{\ell,k} = \sum_{\ell=0}^{L} \sum_{k \in \nabla_\ell} \langle u, \widetilde{\psi}_{\ell,k}\rangle \psi_{\ell,k} \,.$$

For $u \in \widetilde{H}^s([0,1])$, $0 \le s \le p$ one obtains an infinite series

$$u = \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} u_{\ell,k} \psi_{\ell,k}, \tag{3.2}$$

which converges in $\widetilde{H}^s([0,1])$. There holds the norm equivalence

$$\|u\|^2_{\widetilde{H}^s([0,1])} \lesssim \sum_{\ell=0}^{\infty} \sum_{k \in \nabla_\ell} 2^{2s\ell} \, |u_{\ell,k}|^2 \lesssim \|u\|^2_{\widetilde{H}^s([0,1])} \,, \quad 0 \le s < p - 1/2 \,. \tag{3.3}$$

Example 3.1. *We give an example of wavelet basis for $\widetilde{H}^s([0,1])$, $0 \le s < 3/2$ using piecewise linear continuous functions, $p = 2$, on $[0,1]$ vanishing at the endpoints. The mesh $\mathcal{T}_\ell$ is defined by the nodes $x_{\ell,k} := k2^{-\ell-1}$ with $k = 0, \dots, 2^{\ell+1}$. Let $N_\ell = 2^{\ell+1} - 1$ and $c_\ell := \sqrt{3} \cdot 2^{\ell/2-1}, \ell \in \mathbb{N}_0$. We define the wavelets $\psi_{\ell,k}$ for level $\ell \in \mathbb{N}_0$, $k = 1, \dots, 2^\ell$. For $\ell = 0$ we have $N_0 = 1$ and $\psi_{0,1}$ is the function with value $2c_0$ at $x_{1,0} = 1/2$. For $\ell \ge 1$ the wavelet $\psi_{\ell,1}$ has the values $\psi_{\ell,1}(x_{\ell,1}) = 2c_\ell$, $\psi_{\ell,1}(x_{\ell,2}) = -c_\ell$ and zero at all other nodes. The wavelet $\psi_{\ell,2^\ell}$ has the values $\psi_{\ell,2^\ell}(x_{\ell,N_\ell}) = 2c_\ell$, $\psi_{\ell,2^\ell}(x_{\ell,N_\ell-1}) = -c_\ell$ and zero at all other nodes. The wavelet $\psi_{\ell,k}$ with $1 < k < 2^\ell$ has the values $\psi_{\ell,k}(x_{\ell,2k-2}) = -c_\ell$, $\psi_{\ell,k}(x_{\ell,2k-1}) = 2c_\ell$, $\psi_{\ell,k}(x_{\ell,2k}) = -c_\ell$ and zero at all other nodes.*

3.2 Sparse tensor product space

We define the subspace V_L on $D = [0,1]^d$, $d > 1$ as the full tensor product of the one-dimensional spaces $V_{L+1} := \bigotimes_{1\le i\le d} \mathcal{V}_{L+1}$ which can be written as

$$V_{L+1} = \operatorname{span}\{\psi_{\boldsymbol{\ell},\mathbf{k}} \; : \; 0 \le \ell_i \le L,\, k_i \in \nabla_{\ell_i}, i = 1,\ldots,d\}\,,$$

with basis functions $\psi_{\boldsymbol{\ell},\mathbf{k}} = \psi_{\ell_1,k_1}\cdots\psi_{\ell_d,k_d}$, $0 \le \ell_i \le L$, $k_i \in \nabla_{\ell_i}$, $i = 1,\ldots,d$. Using (3.1) we can write V_{L+1} again in terms of increment spaces

$$V_{L+1} = \bigoplus_{0\le\ell_i\le L} \mathcal{W}^{\ell_1}\otimes\cdots\otimes\mathcal{W}^{\ell_d}.$$

Therefore, together with (3.2) for any function $u \in L^2([0,1]^d)$ we have the series representation

$$u = \sum_{\ell_i=0}^{\infty}\sum_{k_i\in\nabla_{\ell_i}} u_{\boldsymbol{\ell},\mathbf{k}}\psi_{\boldsymbol{\ell},\mathbf{k}}.$$

Using the norm equivalences (3.3) we obtain

$$\|u\|^2_{\widetilde{H}^{\mathbf{s}}([0,1]^d)} \lesssim \sum_{\ell_i=0}^{\infty}\sum_{k_i\in\nabla_{\ell_i}} (2^{2s_1\ell_1}+\ldots+2^{2s_d\ell_d})\,|u_{\boldsymbol{\ell},\mathbf{k}}|^2 \lesssim \|u\|^2_{\widetilde{H}^{\mathbf{s}}([0,1]^d)}\,, \tag{3.4}$$

for $0 \le s_i \le p - 1/2$, $i = 1,\ldots,d$.

Remark 3.2. *To obtain a multilevel preconditioner we only need these norm equivalences for $\widetilde{H}^{\boldsymbol{\alpha}/2}([0,1]^d)$, i.e., $0 \le s_i = \alpha_i/2 \le 1$, $i = 1,\ldots,d$. Therefore, $p = 2$ is sufficient.*

The space V_L has $\mathcal{O}(2^{Ld})$ degrees of freedom which grow exponentially with increasing dimension d. To avoid this "curse of dimension" we introduce the sparse tensor product space

$$\begin{aligned}\widehat{V}_{L+1} &= \operatorname{span}\{\psi_{\boldsymbol{\ell},\mathbf{k}} \; : \; 0 \le \ell_1+\cdots+\ell_d \le L,\, k_i \in \nabla_{\ell_i}, i = 1,\ldots,d\} \\ &= \bigoplus_{0\le\ell_1+\cdots+\ell_d\le L} \mathcal{W}_{\ell_1}\otimes\cdots\otimes\mathcal{W}_{\ell_d}\,.\end{aligned}$$

As $L \to \infty$ we have $N = \dim(V_{L+1}) = \mathcal{O}(2^{dL})$ and $\widehat{N} = \dim(\widehat{V}_{L+1}) = \mathcal{O}(L^{d-1}\,2^L)$, i.e., the spaces $\widehat{V}_L$ have considerably smaller dimension than V_L. On the other hand, they do have similar approximation properties as V_L, provided the function to be approximated is sufficiently smooth. As shown in [14] there holds for $u \in \mathcal{H}^s(D)$ with $0 \le r < p - 1/2$, $r \le s \le p$,

$$\inf_{u_L\in\widehat{V}_L} \|u - u_L\|_{\widetilde{H}^r} \lesssim \begin{cases} h^{s-r}\,|\log h|^{\frac{d-1}{2}} & \text{if } r = 0, s = p \\ h^{s-r} & \text{else.}\end{cases} \tag{3.5}$$

Here, $\mathcal{H}^s(D)$ denotes the mixed Sobolev space $\mathcal{H}^s(D) = \widetilde{H}^s(D)\otimes\cdots\otimes\widetilde{H}^s(D)$.

3.3 Wavelet discretization

Let $D = [-R, R]^d$. Since we are mainly interested in jump part we explain the discretization only for $\mathcal{Q} = 0$. Using the basis $\psi_{\boldsymbol{\ell},\mathbf{k}} = \psi_{\ell_1,k_1} \cdots \psi_{\ell_d,k_d}$, $0 \leq \ell_1 + \cdots + \ell_d \leq L$, $k_i \in \nabla_{\ell_i}$ of $\widehat{V}_{L+1}$ we need to compute the stiffness matrix for the jump part

$$\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \mathcal{E}(\psi_{\boldsymbol{\ell},\mathbf{k}}, \psi_{\boldsymbol{\ell}',\mathbf{k}'}) = \sum_{i=1}^{d} \int_{\mathbb{R}} \int_{D} \partial_i \psi_{\boldsymbol{\ell},\mathbf{k}}(x + z_i) \partial_i \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) k_i^{-2}(z_i) \,\mathrm{d}x \,\mathrm{d}z_i$$
$$- \sum_{i=2}^{d} \sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1 < \cdots < \mathcal{I}_i}} \int_{\mathbb{R}^i} \int_{D} \partial^{\mathcal{I}} \psi_{\boldsymbol{\ell},\mathbf{k}}(x + z^{\mathcal{I}}) \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) U^{\mathcal{I}}(z^{\mathcal{I}}) \,\mathrm{d}x \,\mathrm{d}z^{\mathcal{I}} \,.$$

We define

$$\begin{aligned}
\mathbf{M}^i_{(\ell',k'),(\ell,k)} &:= \int_{-R}^{R} \psi_{\ell,k} \psi_{\ell',k'} \,\mathrm{d}x, \\
\mathbf{A}^i_{(\ell',k'),(\ell,k)} &:= - \int_{\mathbb{R}} \int_{-R}^{R} \psi'_{\ell,k}(x + z) \psi'_{\ell',k'}(x) k_i^{-2}(z) \,\mathrm{d}x \,\mathrm{d}z, \\
\mathbf{A}^{\mathcal{I}}_{(\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}),(\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}})} &:= \int_{\mathbb{R}^{|\mathcal{I}|}} \int_{[-R,R]^{|\mathcal{I}|}} \partial^{\mathcal{I}} \psi_{\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}}}(x + z) \psi_{\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}}(x) U^{\mathcal{I}}(z) \,\mathrm{d}x \,\mathrm{d}z \,.
\end{aligned} \tag{3.6}$$

where $i = 1, \ldots, d$, $\boldsymbol{\ell}_{\mathcal{I}} = (\ell_i)_{i \in \mathcal{I}}$, $0 \leq \ell_i \leq L$, $\mathbf{k}_{\mathcal{I}} = (k_i)_{i \in \mathcal{I}}$, $k_i \in \nabla_{\ell_i}$, $\mathcal{I} \subset \{1, \ldots, d\}$, $|\mathcal{I}| > 1$, and write the jump stiffness matrix as

$$\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = - \sum_{i=1}^{d} \sum_{\substack{|\mathcal{I}|=i \\ \mathcal{I}_1 < \cdots < \mathcal{I}_i}} \mathbf{A}^{\mathcal{I}}_{(\boldsymbol{\ell}'_{\mathcal{I}},\mathbf{k}'_{\mathcal{I}}),(\boldsymbol{\ell}_{\mathcal{I}},\mathbf{k}_{\mathcal{I}})} \prod_{j \in \mathcal{I}^c} \mathbf{M}^j_{(\ell'_j,k'_j),(\ell_j,k_j)} \,.$$

Applying the θ-scheme in time, we can write the problem (2.6) in fully discrete form

$$\begin{aligned}
&\text{Find } \underline{u}_L^{m+1} \in \mathbb{R}^{\widehat{N}} \text{ such that for } m = 0, \ldots, M-1, \\
&\Delta t^{-1} \mathbf{M}(\underline{u}_L^{m+1} - \underline{u}_L^m) + \theta \mathbf{A} \underline{u}_L^{m+1} + (1 - \theta) \mathbf{A} \underline{u}_L^m = 0, \\
&\underline{u}_L^0(0) = \underline{u}_{L,0} \,.
\end{aligned} \tag{3.7}$$

with $u_L^m = \sum_{0 \leq |\boldsymbol{\ell}| \leq L} \sum_{k_i \in \nabla_{\ell_i}} u^m_{\boldsymbol{\ell},\mathbf{k}} \psi_{\boldsymbol{\ell},\mathbf{k}}$ and degree of freedoms $\widehat{N} = \dim(\widehat{V}_{L+1}) = \mathcal{O}(2^L L^{d-1})$. The stiffness matrix is, in general, densely populated. Using wavelet compression we can reduce the number of non-zero entries in $\mathbf{A}$ to $\mathcal{O}(2^L L^{2(d-1)})$.

3.4 Wavelet compression of the Lévy measure

Wavelet compression for isotropic domains has been studied extensively by various authors, e.g., [3, 2, 6, 15]. It is shown that compression yields asymptotically optimal complexity (on not necessarily tensor product domains) in the sense that the number of non-zero entries in the resulting matrices grows linearly with the number of degrees of freedom. These results are extended to anisotropic spaces on sparse tensor product spaces in [8].

To define the compression scheme we need to introduce some notation. Consider tensor product wavelets $\psi_{\boldsymbol{\ell},\mathbf{k}} = \psi_{\ell_1,k_1} \otimes \ldots \otimes \psi_{\ell_d,k_d}$, $\psi_{\boldsymbol{\ell}',\mathbf{k}'} = \psi_{\ell_1',k_1'} \otimes \ldots \otimes \psi_{\ell_d',k_d'}$. The distance of support in each coordinate direction is denoted by

$$\delta_{x_i} := \operatorname{dist}\{\operatorname{supp}\psi_{\ell_i,k_i}, \operatorname{supp}\psi_{\ell_i',k_i'}\},$$

for $i = 1, \ldots, d$ and the distance of singular support

$$\delta_{x_i}^{\text{sing}} := \begin{cases} \operatorname{dist}\{\operatorname{singsupp}\psi_{\ell_i,k_i}, \operatorname{supp}\psi_{\ell_i',k_i'}\}, & \text{if } \ell_i \le \ell_i' \\ \operatorname{dist}\{\operatorname{supp}\psi_{\ell_i,k_i}, \operatorname{singsupp}\psi_{\ell_i',k_i'}\}, & \text{else} \end{cases}$$

Let $0 < \alpha < p - \frac{1}{2}$, define

$$\begin{aligned} \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} := & \begin{cases} L(p-\alpha/2) - p\,|\boldsymbol{\ell}| & \text{if } p(L-|\boldsymbol{\ell}|) \ge \alpha/2(L-|\boldsymbol{\ell}|_\infty) \\ -\alpha/2\,|\boldsymbol{\ell}|_\infty & \text{else,} \end{cases} \\ & + \begin{cases} L(p-\alpha/2) - p\,|\boldsymbol{\ell}'| & \text{if } p(L-|\boldsymbol{\ell}'|) \ge \alpha/2(L-|\boldsymbol{\ell}'|_\infty) \\ -\alpha/2\,|\boldsymbol{\ell}'|_\infty & \text{else,} \end{cases} \end{aligned}$$

and $m_i := \ell_i + \ell_i' - 2\min\{\ell_i, \ell_i'\}$. Furthermore, we denote the index sets $\mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^c, \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \subset \{1, \ldots, d\}$ by

$$\mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^c = \left\{ i \in \{1,\ldots,d\} \;:\; \delta_{x_i} > 2^{-\min\{\ell_i,\ell_i'\}} \right\}, \quad \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \{1,\ldots,d\} \backslash \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^c,$$

and set

$$\begin{aligned} \beta_{\boldsymbol{\ell},\boldsymbol{\ell}}^i &= \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} - \widetilde{p}(\ell_i + \ell_i') + \alpha \sum_{j \ne i} \min\{\ell_j, \ell_j'\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}} m_j - \widetilde{p} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^c \backslash \{i\}} m_j, \\ \widetilde{\beta}_{\boldsymbol{\ell},\boldsymbol{\ell}}^i &= \widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'} - \widetilde{p}\max\{\ell_i, \ell_i'\} + \alpha \sum_{j \ne i} \min\{\ell_j, \ell_j'\} + \frac{1}{2} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \backslash \{i\}} m_j - \widetilde{p} \sum_{j \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^c} m_j. \end{aligned}$$

The cut-off parameter are now defined by

$$\begin{aligned} \mathcal{B}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^i &= a \max\left\{ 2^{-\min\{\ell_i,\ell_i'\}}, 2^{\beta_{\boldsymbol{\ell},\boldsymbol{\ell}}^i/(2\widetilde{p}+\alpha)} \right\}, \quad a > 0, \\ \widetilde{\mathcal{B}}_{\boldsymbol{\ell},\boldsymbol{\ell}'}^i &= a' \max\left\{ 2^{-\max\{\ell_i,\ell_i'\}}, 2^{\widetilde{\beta}_{\boldsymbol{\ell},\boldsymbol{\ell}}^i/(\widetilde{p}+\alpha)} \right\}, \quad a' > 0. \end{aligned}$$

The compression scheme is based on the fact that the matrix entries $\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \mathcal{E}(\psi_{\boldsymbol{\ell},\mathbf{k}}, \psi_{\boldsymbol{\ell}',\mathbf{k}'})$ can be estimated a-priori and therefore neglected if these are smaller than some cut-off parameter. There are two reasons for an entry to be omitted. Either the distance of the supports $\operatorname{supp}\psi_{\ell_i,k_i}$ and $\operatorname{supp}\psi_{\ell_i',k_i'}$ or the distance of the singular supports is large enough for some $i \in \{1, \ldots, d\}$.

Theorem 3.3. *Assume $\mathcal{Q} > 0$ and that the Lévy density k satisfies* (2.4) *with* $0 < \alpha < p - 1/2$. *Define the compression scheme by*

$$\widetilde{\mathbf{A}}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \begin{cases} 0, & \text{if } \exists i \in \mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'} \ : \ \delta_{x_i} > \mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} \\ 0, & \text{if } \exists i \in \mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'} \ : \ \delta^{\text{sing}}_{x_i} > \widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} \\ \mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})}, & \text{else.} \end{cases}$$

If $\widetilde{p} > 2dp - (d+1)\alpha$ and $\alpha \leq 2/d$, the number of non-zero entries for the compressed matrix $\widetilde{\mathbf{A}}$ is $\mathcal{O}(2^L L^{2(d-1)})$.

Proof. See [8, Theorem 4.6.3]. □

Remark 3.4. *Here, we only stated the isotropic case $\alpha_1 = \ldots = \alpha_d = \alpha$, i.e., in each direction the same compression is used. Although we still get asymptotically optimal complexity, the number of matrix entries can further be reduced using anisotropic compression. The corresponding compression scheme is defined in [8].*

We now consider the fully discrete problem (3.7) where we replace the matrix $\mathbf{A}$ with the compressed matrix $\widetilde{\mathbf{A}}$.

$$\begin{aligned} &\text{Find } \underline{\widetilde{u}}^{m+1}_L \in \mathbb{R}^{\widehat{N}} \text{ such that for } m = 0, \ldots, M-1, \\ &\Delta t^{-1}\mathbf{M}(\underline{\widetilde{u}}^{m+1}_L - \underline{\widetilde{u}}^m_L) + \theta\widetilde{\mathbf{A}}\underline{\widetilde{u}}^{m+1}_L + (1-\theta)\widetilde{\mathbf{A}}\underline{\widetilde{u}}^m_L = 0, \\ &\underline{\widetilde{u}}^0_L(0) = \underline{u}_{L,0}\,. \end{aligned} \tag{3.8}$$

There exists a unique solution $\widetilde{u}^m_L$ of the perturbed scheme (3.8) and the solution converges at the optimal rate.

Theorem 3.5. *Assume the Lévy density k satisfies Assumption 2.1. Consider $\widetilde{\mathbf{A}}$ as given in Theorem 3.3 and let all assumptions of Theorem 3.3 hold. Then, there exists a unique solution $\widetilde{u}^m_L$ of the perturbed θ-scheme* (3.8). *Furthermore, if $u \in C^1([0,T],\mathcal{H}^p(D)) \cap C^3([0,T],\mathcal{V}^*)$ and the approximation $\underline{u}_{L,0} \in \widehat{V}_{L+1}$ of the initial data u_0 is quasi-optimal in $L_2(D)$, then for $\theta = 1/2$*

$$\left\|u^M - \widetilde{u}^M_L\right\|^2_{L^2(D)} + \Delta t \sum_{m=0}^{M-1} \left\|u^{m+1/2} - \widetilde{u}^{m+1/2}_L\right\|^2_{\mathcal{V}} \leq C(u)\left(\Delta t^4 + 2^{-2L(p-\alpha/2)}\right),$$

where u is the solution of (2.6) *and the constant $C(u) > 0$ depends on higher space and time derivatives of u.*

Proof. See [8, Theorem 2.2.3, Theorem 3.3.8]. □

These convergence rates are shown in the next example. We only look at independent margins because here the matrix entries $\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})}$ can be computed exactly.

Example 3.6. *Let $d = 2$ and consider*

$$k_i(z_i) = c_i \frac{e^{-\beta_i^- |z_i|}}{|z_i|^{1+\alpha_i}} 1_{\{z_i<0\}} + c_i \frac{e^{-\beta_i^+ z_i}}{z_i^{1+\alpha_i}} 1_{\{z_i>0\}}, \quad i = 1, 2, \quad U(z_1, z_2) = 0.$$

We solve the elliptic problem $\mathcal{A}[u] = f$ on $D = [0, 1]^2$, where f is chosen such that the exact solution is

$$u(x) = \begin{cases} (x_1^2 - 2x_1^3 + x_1^4)(x_2^2 - 2x_2^3 + x_2^4) & \text{if } x \in D \\ 0 & \text{else.} \end{cases}$$

We set the model parameter $c_1 = c_2 = 1$, $\beta_1^- = 10$, $\beta_1^+ = 15$, $\beta_2^- = 9$, $\beta_2^+ = 16$, $\alpha_1 = 0.5$, $\alpha_2 = 0.7$ and the compression parameter $a = 1$, $a' = 1$, $p = 2$, $\widetilde{p} = 2$. For $L = 8$ the absolute value of the entries in the stiffness matrix $\mathbf{A}$ and the compressed matrix $\widetilde{\mathbf{A}}$ are shown in Figure 1. Here, large entries are colored red. For the stiffness matrix blue entries are small but non-zero whereas for the compressed matrix blue entries are zero either due to the first or second compression. One clearly sees that the compression scheme neglects small entries.

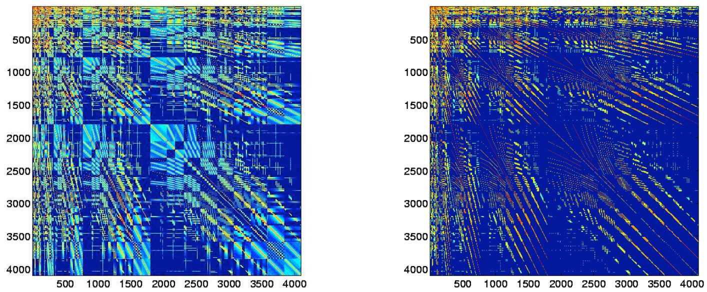

Figure 1: Stiffness matrix $\mathbf{A}$ (left) and compressed matrix $\widetilde{\mathbf{A}}$ (right) for level $L = 8$

To compare the convergence rates in Figure 2 we also solve the problem on full grid. In the left picture it can be seen that sparse grid has (up to log terms) the same rate as full grid and that the compression scheme preserves the convergence rate. We additionally plot the convergence rate in terms of degrees of freedom. For full grid we have $N = \mathcal{O}(2^{2L})$ and for sparse grid $\widehat{N} = \mathcal{O}(L\, 2^L)$. The convergence rate in full grid shows the "curse of dimension", whereas for the sparse grid we still obtain the optimal rate (up to log terms).

Since in general the matrix entries $\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})}$ cannot be computed exactly, we need to approximate these with a numerical quadrature rule. To still retain the optimal order of convergence, we require a certain accuracy.

Theorem 3.7. *Consider $\widetilde{\mathbf{A}}$ as given in Theorem 3.3 and let $\widehat{\mathbf{A}}$ be a perturbed matrix such that*

$$\left| (\widetilde{\mathbf{A}} - \widehat{\mathbf{A}})_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} \right| \lesssim \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}, \quad \text{with} \quad \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'} \lesssim 2^{-(|\boldsymbol{\ell}|+|\boldsymbol{\ell}'|)/2}\, 2^{-\widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'}}. \tag{3.9}$$

Then, Theorem 3.5 still holds with $\widehat{\mathbf{A}}$ instead of $\widetilde{\mathbf{A}}$.

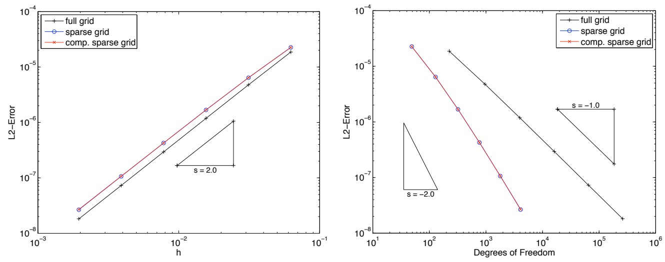

Figure 2: Convergence rate of the wavelet discretization in terms of the mesh width h (left) and in terms of degrees of freedom (right)

Proof. We need to show that the error satisfies $\|\mathbf{S}\|_2 \lesssim 2^{-\widetilde{L}_{\boldsymbol{\ell},\boldsymbol{\ell}'}}$, as shown in [8, Theorem 2.5.2]. Let $\mathbf{S} = \left|\widetilde{\mathbf{A}}_{\boldsymbol{\ell}',\boldsymbol{\ell}} - \widehat{\mathbf{A}}_{\boldsymbol{\ell}',\boldsymbol{\ell}}\right|$ where $\mathbf{A}_{\boldsymbol{\ell}',\boldsymbol{\ell}}$ is the block matrix with entries $(\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})})_{k_i' \in \nabla_{\ell_i'}, k_i \in \nabla_{\ell_i}}$. Estimating for each row (or column), the sum over all entries yields

$$\sum_{k_1 \in \nabla_{\ell_1}} \cdots \sum_{k_d \in \nabla_{\ell_d}} \mathbf{S}_{\mathbf{k}',\mathbf{k}} \lesssim 2^{|\boldsymbol{\ell}|} \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'} \quad \text{and} \quad \sum_{k_1' \in \nabla_{\ell_1'}} \cdots \sum_{k_d' \in \nabla_{\ell_d'}} \mathbf{S}_{\mathbf{k}',\mathbf{k}} \lesssim 2^{|\boldsymbol{\ell}'|} \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'} \,,$$

We can rewrite this as

$$\begin{aligned} \sum_{k_1 \in \nabla_{\ell_1}} \cdots \sum_{k_d \in \nabla_{\ell_d}} w_{\mathbf{k}} \mathbf{S}_{\mathbf{k}',\mathbf{k}} &\lesssim w_{\mathbf{k}'} 2^{(|\boldsymbol{\ell}|+|\boldsymbol{\ell}'|)/2} \, \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'} \,, \\ \sum_{k_1' \in \nabla_{\ell_1'}} \cdots \sum_{k_d' \in \nabla_{\ell_d'}} w_{\mathbf{k}'} \mathbf{S}_{\mathbf{k}',\mathbf{k}} &\lesssim w_{\mathbf{k}} 2^{(|\boldsymbol{\ell}|+|\boldsymbol{\ell}'|)/2} \, \epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'} \,, \end{aligned}$$

with weights $w_{\mathbf{k}} = 2^{(|\boldsymbol{\ell}'|-|\boldsymbol{\ell}|)/4}$ and $w_{\mathbf{k}'} = 2^{(|\boldsymbol{\ell}|-|\boldsymbol{\ell}'|)/4}$. Using the Schur lemma [7, Lemme 4] we obtain the required result. □

3.5 Multilevel preconditioning

We have to solve a linear system

$$\left(\mathbf{M} + \theta \Delta t \widetilde{\mathbf{A}}\right) \underline{\widetilde{u}}_L^{m+1} = \left(\mathbf{M} - \Delta t(1-\theta)\Delta t \widetilde{\mathbf{A}}\right) \underline{\widetilde{u}}_L^{m} \,,$$

at each time step $m = 0, \ldots, M-1$. For an iterative solution of these systems, $\mathbf{B}\underline{u} = \underline{b}$, we use multilevel preconditioning. The preconditioner is obtained by using the wavelet norm equivalences. With (3.4) for $s = 0$ we have for every $u \in \widehat{V}_{L+1}$ with coefficient vector $\underline{u} \in \mathbb{R}^{\widehat{N}}$ that

$$\langle \underline{u}, \underline{u} \rangle \lesssim \langle \underline{u}, \mathbf{M}\underline{u} \rangle \lesssim \langle \underline{u}, \underline{u} \rangle \,,$$

Denote by $\mathbf{D}_A$ the diagonal matrix with entries $2^{\alpha_1 \ell_1} + \ldots + 2^{\alpha_d \ell_d}$ for an index corresponding to level $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$. Then, Theorem 3.5 and (3.4) for $s_i = \alpha_i/2$, $i = 1, \ldots, d$ imply that

$$\langle \underline{u}, \mathbf{D}_A \underline{u} \rangle \lesssim \langle \underline{u}, \mathbf{A} \underline{u} \rangle \lesssim \langle \underline{u}, \mathbf{D}_A \underline{u} \rangle \,,$$

Thus, we have $\langle \underline{u}, \mathbf{D} \underline{u} \rangle \lesssim \langle \underline{u}, \mathbf{B} \underline{u} \rangle \lesssim \langle \underline{u}, \mathbf{D} \underline{u} \rangle$, with the diagonal matrix $\mathbf{D} = \mathbf{I} + \theta \Delta t \mathbf{D}_A$. Written in terms of $\underline{\widehat{u}} = \mathbf{D}^{1/2} \underline{u}$ we finally obtain

$$|\underline{\widehat{u}}|^2 \lesssim \langle \underline{\widehat{u}}, \mathbf{D}^{-1/2} \mathbf{B} \mathbf{D}^{-1/2} \underline{\widehat{u}} \rangle \lesssim |\underline{\widehat{u}}|^2 \;.$$

The linear system $\widehat{\mathbf{B}}\, \underline{\widehat{u}} = \underline{\widehat{b}}$ with preconditioned matrix $\widehat{\mathbf{B}} = \mathbf{D}^{-1/2} \mathbf{B} \mathbf{D}^{-1/2}$ and right hand side $\underline{\widehat{b}} = \mathbf{D}^{-1/2} \underline{b}$ can be solved with GMRES [10] in a number of steps which is independent of level index L.

Lemma 3.8. *For the linear system $\widehat{\mathbf{B}}\, \underline{\widehat{u}} = \underline{\widehat{b}}$ let $\underline{\widehat{u}}_j$ denote the iterate obtained by the* GMRES *method with initial guess $\underline{\widehat{u}}_0$. There is a constant $0 < r < 1$ independent of L and Δt such that*

$$\left| \underline{\widehat{b}} - \widehat{\mathbf{B}} \underline{\widehat{u}}_j \right| \lesssim r^j \left| \underline{\widehat{b}} - \widehat{\mathbf{B}} \underline{\widehat{u}}_0 \right| \;.$$

Proof. See [5]. □

4 Composite Gauss quadrature rules

As seen in the last section we have to evaluate integrals $\int_{[-1,1]^{|\mathcal{I}|}} u(z^{\mathcal{I}}) U^{\mathcal{I}}(z) \,\mathrm{d}z$ for $\mathcal{I} \subset \{1, \ldots, d\}$. The tail integrals $U^{\mathcal{I}}(z)$ have a singularity at the origin and possibly on each axis. Therefore, we can not use standard quadrature rules for integration since these depend on the smoothness of the function. Instead, we use a composite Gauss quadrature rule as proposed in [12]. Elementary Gauss quadrature formulas of varying orders on subdomains are combined. The size of these subdomains decreases geometrically towards the singular support of the integrand. Multidimensional quadrature rules are obtained by using tensor products of one-dimensional quadrature formulas. We start recalling error estimates for the basic Gauss-Legendre quadrature rules.

4.1 Gauss-Legendre quadrature

For a given function $f \in C([0,1])$ we set $I^{[0,1]} f := \int_0^1 f(s) \,\mathrm{d}s$ and denote the g-point Gauss-Legendre integration rule on $[0,1]$ by $Q_g^{[0,1]} f := \sum_{j=1}^{g} \omega_{g,j} f(\xi_{g,j})$. If $f \in C^{2g}([0,1])$ we obtain the following error estimate (see, e.g., [4])

$$\left| E_g^{[0,1]} f \right| := \left| I^{[0,1]} f - Q_g^{[0,1]} f \right| = \frac{(g!)^4}{(2g+1)[(2g)!]^3} \left| f^{(2g)}(\xi) \right|, \quad \xi \in [0,1] \,.$$

We use the Stirling formula $g! \sim \sqrt{2\pi g}\, g^g e^{-g}$ to obtain the estimate

$$\left|E_g^{[0,1]} f\right| \lesssim \frac{2^{-4g}}{(2g)!} \max_{\xi\in[0,1]} \left|f^{(2g)}(\xi)\right| . \tag{4.1}$$

On $[0,1]^d$ we approximate the integral, $I^{[0,1]^d} f := \bigotimes_{1\le i\le d} I^{[0,1]} f = \int_{[0,1]^d} f(s)\,\mathrm{d}s$, for $f \in C([0,1]^d)$ by a tensor product Gauss-Legendre quadrature rule

$$Q_g^{[0,1]^d} f := \bigotimes_{1\le i\le d} Q_g^{[0,1]} f = \sum_{j_1,\dots,j_d=1}^{g} \prod_{i=1}^{d} \omega_{g,j_i} f(\xi_{g,j_1},\dots,\xi_{g,j_d}) ,$$

and obtain the following error bound.

Lemma 4.1. *If $f \in C^{2g}([0,1]^d)$, the quadrature error $E_g^{[0,1]^d} f := I^{[0,1]^d} f - Q_g^{[0,1]^d} f$ is bounded by*

$$\left|E_g^{[0,1]^d} f\right| \lesssim \frac{2^{-4g}}{(2g)!} \sum_{i=1}^{d} \max_{\xi\in[0,1]^d} \left|\partial_i^{2g} f(\xi)\right| . \tag{4.2}$$

Proof. We prove this lemma by induction over the dimension d. With (4.1) it is true for $d=1$. For $d>1$ we have

$$\begin{aligned}
\left|E_g^{[0,1]^d} f\right| &= \left(\bigotimes_{1\le i\le d} I^{[0,1]} - \bigotimes_{1\le i\le d} Q_g^{[0,1]}\right) f \\
&= \left(\bigotimes_{1\le i\le d} I^{[0,1]} - \bigotimes_{1\le i\le d-1} I^{[0,1]} \otimes Q_g^{[0,1]} + \bigotimes_{1\le i\le d-1} I^{[0,1]} \otimes Q_g^{[0,1]} - \bigotimes_{1\le i\le d} Q_g^{[0,1]}\right) f \\
&= \bigotimes_{1\le i\le d-1} I^{[0,1]} \otimes \left(I^{[0,1]} - Q_g^{[0,1]}\right) f + \left(\bigotimes_{1\le i\le d-1} I^{[0,1]} - \bigotimes_{1\le i\le d-1} Q_g^{[0,1]}\right) \otimes Q_g^{[0,1]} f \\
&\lesssim \frac{2^{-4g}}{(2g)!} \max_{\xi\in[0,1]^d} \left|\partial_d^{2g} f(\xi)\right| + \frac{2^{-4g}}{(2g)!} \sum_{i=1}^{d-1} \max_{\xi\in[0,1]^d} \left|\partial_i^{2g} f(\xi)\right| .
\end{aligned}$$

□

For our analysis we consider a class of functions which have singularities on the origin and on the axes.

Assumption 4.2. *Let $f \in L^1([0,1]^d)$. There exist $0 < \alpha < d$, $\alpha \notin \mathbb{N}$, $C_f > 0$, such that for $k \in \mathbb{N}_0$, $i = 1,\dots,d$*

$$\left|\partial_i^k f(\xi)\right| \lesssim k!\, C_f^k \, \|\xi\|_\infty^{-\alpha} \, \xi_i^{-k}, \quad \forall \xi \in (0,1)^d . \tag{4.3}$$

Equation (4.3) is satisfied by all tail integrals corresponding to a Lévy density which satisfy Assumption 2.1. We introduce the notation

$$I^{[0,1]} f_{\xi_i} := \int_0^1 f(\xi_1,\dots,s_i,\dots,\xi_d)\,\mathrm{d}s_i,$$

where we just integrate over the i-th dimension, $i \in \{1, \dots, d\}$. Similarly $Q_g^{[0,1]} f_{\xi_i}$ and $E_g^{[0,1]} f_{\xi_i} := I^{[0,1]} f_{\xi_i} - Q_g^{[0,1]} f_{\xi_i}$. We can now state the basic error estimates on rectangular domains.

Proposition 4.3. *Let $i \in \{1, \dots, d\}$, interval $[a, b]$ with $a, b \in \mathbb{R}$, $0 \le a \le b \le 1$ and $h = b - a$. Assume f satisfies* (4.3) *and set $\mathcal{I} = \{1, \dots, d\} \backslash i$. Then,*

$$\left| E_g^{[a,b]} f_{\xi_i} \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha + \frac{\alpha}{d}} \, h \left(\frac{C_f h}{4a} \right)^{2g} a^{-\frac{\alpha}{d}}, \quad \text{for } a > 0, \quad \xi^{\mathcal{I}} \in (0,1)^{d-1}\,, \tag{4.4}$$

$$\left| E_0^{[a,b]} f \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha + \frac{\alpha}{d}} \, h^{1 - \frac{\alpha}{d}}, \quad \text{for } a = 0, \quad \xi^{\mathcal{I}} \in (0,1)^{d-1}\,. \tag{4.5}$$

Proof. Consider the transformation $\varphi : [0, 1] \to [a, b]$, $\varphi(\xi) = a + h\xi$. Then, with $I^{[a,b]} f_\xi = I^{[0,1]}(f_\xi \circ \varphi) h$ and $\partial_i^k f(\xi_1, \dots, \varphi(\xi_i), \dots, \xi_d) = \partial_i^k f \, h^k$ we get (4.4) by

$$\begin{aligned} \left| E_g^{[a,b]} f_{\xi_i} \right| &= h \left| E_g^{[0,1]} (f_{\xi_i} \circ \varphi) \right| \lesssim h \, \frac{2^{-4g}}{(2g)!} \max_{\xi_i \in [a,b]} \left| \partial_i^{2g} f(\xi) \right| h^{2g} \\ &\lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha + \frac{\alpha}{d}} \, h \left(\frac{C_f h}{4a} \right)^{2g} a^{-\frac{\alpha}{d}}\,. \end{aligned}$$

With $|f| \lesssim \|\xi\|_\infty^{-\alpha}$ one obtains (4.5) since

$$\begin{aligned} \left| E_0^{[0,h]} f_{\xi_i} \right| &= \left| \int_{[0,h]} f(\xi_1, \dots, s_i, \dots, \xi_d) \, \mathrm{d}s_i \right| \\ &\lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha + \frac{\alpha}{d}} \left| \int_{[0,h]} s_i^{-\frac{\alpha}{d}} \, \mathrm{d}s_i \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha + \frac{\alpha}{d}} \, h^{1 - \frac{\alpha}{d}}\,. \end{aligned}$$

□

4.2 Composite Gauss quadrature

On $[0, 1]$ a geometric partition is given by $0 < \sigma^n < \sigma^{n-1} < \ldots < \sigma < 1$ for $n \in \mathbb{N}$, $\sigma \in (0, 1)$. We denote the subdomains by $\Lambda_j := [\sigma^{n+1-j}, \sigma^{n-j}]$, with $j = 1, \dots, n$ and $\Lambda_0 := [0, \sigma^n]$. Given a linear degree vector $\mathbf{q} \in \mathbb{N}^n$, $q_j = \lceil \mu j \rceil$ with slope $\mu > 0$, we use on each subdomain Λ_j, $j = 1, \dots, n$, a Gauss quadrature with degree q_j and no quadrature points in Λ_0. The composite Gauss quadrature rule in the i-th direction is defined by

$$Q_\sigma^{n,\mathbf{q}} f_{\xi_i} = \sum_{j=1}^{n} Q_{q_j}^{\Lambda_j} f_{\xi_i}, \quad i \in \{1, \dots, d\}, \tag{4.6}$$

and converges exponentially.

Theorem 4.4. *Let $i \in \{1, \dots, d\}$ and f satisfy* (4.3). *Consider*

$$\sigma \in (0,1), \quad \textit{such that} \quad w = \frac{C_f(1-\sigma)}{4\sigma} < 1, \tag{4.7}$$

and a linear degree vector $\mathbf{q} = (q_1, \dots, q_n)$*,*

$$q_j = \lceil \mu j \rceil, \quad \textit{with slope} \quad \mu > \frac{(1-\frac{\alpha}{d})\ln\sigma}{2\ln w}. \tag{4.8}$$

Then,

$$\left| I^{[0,1]} f_{\xi_i} - Q_\sigma^{n,\mathbf{q}} f_{\xi_i} \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \, \sigma^{n(1-\frac{\alpha}{d})}. \tag{4.9}$$

Proof. On each Λ_j, $j = 1, \dots, n$ we have the following estimate using (4.4) with $a = \sigma^{n+1-j}$ and $h = \sigma^{n-j}(1-\sigma)$

$$\left| E_g^{\Lambda_j} f_{\xi_i} \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \, h \left(\frac{C_f(1-\sigma)}{4\sigma} \right)^{2g} \sigma^{-(n+1-j)\frac{\alpha}{d}} \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \, w^{2g} \sigma^{(n-j)(1-\frac{\alpha}{d})}.$$

Summing over all subdomains $j = 1, \dots, n$, yields

$$\begin{aligned} \sum_{j=1}^n \left| E_{q_j}^{\Lambda_j} f_{\xi_i} \right| &\lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \sum_{j=1}^n w^{2q_j} \sigma^{(n-j)(1-\frac{\alpha}{d})} \\ &\lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \, \sigma^{n(1-\frac{\alpha}{d})} \sum_{j=1}^\infty \left(w^{2\mu} \sigma^{\frac{\alpha}{d}-1} \right)^j. \end{aligned}$$

The last sum converges since $\mu > \frac{(1-\frac{\alpha}{d})\ln\sigma}{2\ln w}$. We neglected the subdomains Λ_0 in the composite Gauss quadrature. Using (4.5) we have

$$\left| E_0^{[0,\sigma^n]} f_{\xi_i} \right| \lesssim \|\xi^{\mathcal{I}}\|_\infty^{-\alpha+\frac{\alpha}{d}} \, \sigma^{n(1-\frac{\alpha}{d})}.$$

□

Remark 4.5. *Condition* (4.7) *is suboptimal. Using [12, Theorem 4.1] or [1, Proposition 2.8] we can obtain exponential convergence for any $\sigma \in (0,1)$.*

We define the composite Gauss quadrature on $[0,1]^d$ by the tensor product of one-dimensional composite Gauss quadrature rules $Q_\sigma^{n,(\mathbf{q}_1,\dots,\mathbf{q}_d)} f = \bigotimes_{1 \le i \le d} Q_\sigma^{n,\mathbf{q}_i} f_{\xi_i}$. The composite Gauss quadrature rule converges exponentially with respect to the number N of Gauss points.

Theorem 4.6. *Let f satisfy* (4.3). *Consider a grading factor $\sigma \in (0,1)$ satisfying* (4.7) *and linear degree vectors* $(\mathbf{q}_1, \dots, \mathbf{q}_d)$ *satisfying* (4.8). *Then, there exist a $\gamma > 0$ such that the quadrature error decays exponentially*

$$\left| I^{[0,1]^d} f - Q_\sigma^{n,(\mathbf{q}_1,\dots,\mathbf{q}_d)} f \right| \lesssim e^{-\gamma \sqrt[2d]{N}}.$$

Proof. We prove this theorem in two steps.

1. As in proof of Lemma 4.1 we prove

$$\left|I^{[0,1]^d} f - Q_\sigma^{n,(\mathbf{q}_1,\ldots,\mathbf{q}_d)} f\right| \lesssim e^{-\gamma n},$$

by induction over the dimension d. With (4.9) it is true for $d = 1$. For $d > 1$ we have with (4.9)

$$\begin{aligned}
\left|I^{[0,1]^d} f - Q_\sigma^{n,(\mathbf{q}_1,\ldots,\mathbf{q}_d)} f\right| &= \left(\bigotimes_{1\le i\le d} I^{[0,1]} - \bigotimes_{1\le i\le d} Q_\sigma^{n,\mathbf{q}_i}\right) f \\
&= \bigotimes_{1\le i\le d-1} I^{[0,1]} \otimes \left(I^{[0,1]} - Q_\sigma^{n,\mathbf{q}_d}\right) f \\
&\quad + \left(\bigotimes_{1\le i\le d-1} I^{[0,1]} - \bigotimes_{1\le i\le d-1} Q_\sigma^{n,(\mathbf{q}_1,\ldots,\mathbf{q}_{d-1})}\right) \otimes Q_\sigma^{n,\mathbf{q}_d} f \\
&\lesssim \int_{[0,1]^{d-1}} \left\|\xi^{\mathcal{I}}\right\|_\infty^{-\alpha+\frac{\alpha}{d}} \,\mathrm{d}\xi^{\mathcal{I}} \sigma^{n(1-\frac{\alpha}{d})} + e^{-\gamma n} \sum_{j=1}^{n} \sum_{m=1}^{q_{d,j}} \omega_{j,m} \xi_{j,m}^{-\frac{\alpha}{d}} \\
&\lesssim e^{-\widetilde{\gamma} n}.
\end{aligned}$$

2. Let $\mu_1 = \max\{\mu_1, \ldots, \mu_d\}$. We estimate the number of quadrature points by

$$N \le \left(\sum_{j=1}^{n} q_{j,1}\right)^d \lesssim \left(\sum_{j=1}^{n} j\right)^d \lesssim n^{2d}.$$

□

We give a numerical example which shows the exponential convergence of the composite Gauss quadrature formula.

Example 4.7. *Consider the function* $f(x) = \left(\sum_{i=1}^d x_i^{\beta_i \vartheta}\right)^{-\frac{1}{\vartheta}}$ *on the domain* $[0,1]^d$ *for* $\alpha = \beta_1 = \ldots = \beta_d = 0.5$. *We apply a composite Gauss quadrature formula with grading factor* $\sigma = 0.2$ *and linear degree vectors with slope* $\mu_1 = \ldots = \mu_d = 0.5$. *For* $\vartheta = 0.5$ *the relative quadrature error* $\left|I^{[0,1]^d} f - Q_\sigma^{n,(\mathbf{q}_1,\ldots,\mathbf{q}_d)} f\right| / \left|I^{[0,1]^d} f\right|$ *versus* $\sqrt[2d]{N}$ *is plotted in logarithmic scale in Figure 3. Additionally, we also plot the relative error for* $d = 2$ *and various* σ. *As already seen in the proof of Theorem 4.6 the convergence rate depends on* $1 - \alpha/d$ *which increases in* d.

5 Computational scheme

As seen in (3.6) we need to compute matrix entries of the type

$$\mathbf{B}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \int_{\mathbb{R}^d} \int_D \partial_1 \cdots \partial_d \psi_{\boldsymbol{\ell},\mathbf{k}}(x+z) \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) \kappa(z) \,\mathrm{d}x \,\mathrm{d}z\,, \tag{5.1}$$

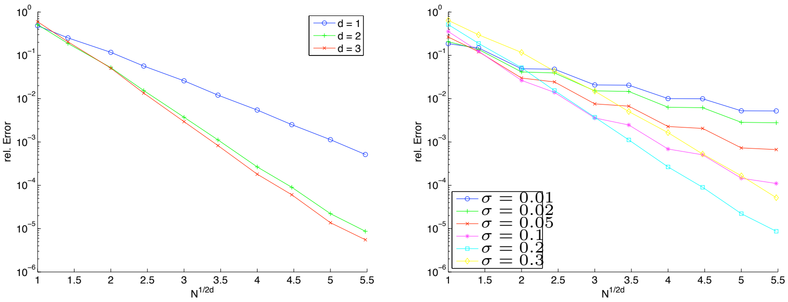

Figure 3: Exponential convergence of the composite Gauss quadrature for $\vartheta = 0.5$, $\sigma = 0.2$ and $d = 2, 3$ (left) and $d = 2$ and various σ (right)

where the kernel κ satisfies (4.3), i.e.,

$$\left|\partial_i^k \kappa(z)\right| \lesssim k!\, C_f^k\, \|z\|_\infty^{-\alpha}\, z_i^{-k}, \quad \forall z \in \mathbb{R}^d, \quad k \in \mathbb{N}_0, \quad i = 1, \dots, d\,,$$

for $0 < \alpha < d$, $\alpha \notin \mathbb{N}$ and $C_f > 0$. Introducing a new variable $y = x + z$ we can write the integral (5.1) as

$$\mathbf{B}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \int_{\Sigma_{\boldsymbol{\ell},\mathbf{k}}} \int_{\Sigma_{\boldsymbol{\ell}',\mathbf{k}'}} \partial_1 \cdots \partial_d \psi_{\boldsymbol{\ell},\mathbf{k}}(y) \psi_{\boldsymbol{\ell}',\mathbf{k}'}(x) \kappa(y - x)\, \mathrm{d}y\, \mathrm{d}x\,, \tag{5.2}$$

where $\Sigma_{\boldsymbol{\ell},\mathbf{k}} = \operatorname{supp} \psi_{\boldsymbol{\ell},\mathbf{k}}$. Similar equations have been studied for the boundary element methods, although only in the isotropic setting. Several schemes have been developed to solve these problems in dimension $d \leq 3$, see [6, 11, 13] and the references therein. We adapt these methods to the anisotropic case for $d \geq 1$. Throughout this section we consider wavelets as described in Example 3.1 which are piecewise linear.

5.1 Hierarchical data structure

For an efficient implementation of the compression scheme it is necessary to have an hierarchical data structure. Therefore, we introduce an hierarchical element tree up to a given level $L \in \mathbb{N}$.

We start with $D_{(0,\dots,0),(1,\dots,1)} = D$ as the first generation. On the ℓ-th generation we consider the elements $D_{\boldsymbol{\ell},\mathbf{k}}$ where the multiindices are given by $\boldsymbol{\ell} = (\ell_1, \dots, \ell_d)$, $\ell_i = 0, \dots, \ell$, $i = 1, \dots, d$ with $\|\boldsymbol{\ell}\|_\infty = \ell$, and $\mathbf{k} = (k_1, \dots, k_d)$, $k_i = 1, \dots, 2^{\ell_i}$, $i = 1, \dots, d$. Each element $D_{\boldsymbol{\ell},\mathbf{k}}$ has sons $D_{\boldsymbol{\ell}+\widetilde{\boldsymbol{\ell}},\mathbf{k}+\widetilde{\mathbf{k}}}$ where $\left|\boldsymbol{\ell} + \widetilde{\boldsymbol{\ell}}\right| \leq L$,

$$\widetilde{\ell}_i = \begin{cases} 0 & \text{if } \ell_i \neq \ell \\ \in \{0, 1\} & \text{if } \ell_i = \ell \end{cases}, \quad i = 1, \dots, d, \quad \text{with } \|\widetilde{\boldsymbol{\ell}}\|_\infty = 1\,,$$

and $\widetilde{k}_i = 2^{\widetilde{\ell}_i}(k_i - 1) + 1, \dots, 2^{\widetilde{\ell}_i} k_i$, $i = 1, \dots, d$. Since there exists a bijective mapping which indicates each element $D_{\boldsymbol{\ell},\mathbf{k}}$ uniquely by an integer λ, we write shortly $D_\lambda = D_{\boldsymbol{\ell},\mathbf{k}}$ and set $|\lambda| = |\ell|$, $\|\lambda\|_\infty = \|\ell\|_\infty$. Similar to the standard single-scale finite element method, we do not compute the matrix entry (5.2) directly over $\operatorname{supp}\psi_{\boldsymbol{\ell},\mathbf{k}}$, since $\psi_{\boldsymbol{\ell},\mathbf{k}}$ is not smooth. Instead, we decompose $\Sigma_{\boldsymbol{\ell},\mathbf{k}}$ into a set of elements $\bigcup D_\lambda$ such that $\psi_{\boldsymbol{\ell},\mathbf{k}}|_{D_\lambda}$ is smooth. More precisely, consider the set

$$\mathcal{L}_{\boldsymbol{\ell},\mathbf{k}} = \left\{ D_{\boldsymbol{\ell}+1,\widetilde{\mathbf{k}}} \ :\ \widetilde{k}_i = \max\{2(k_i-1),1\},\ldots,\min\{2k_i+1,2^{l+1}\},\ i=1,\ldots,d \right\}.$$

Then,

$$\Sigma_{\boldsymbol{\ell},\mathbf{k}} = \operatorname{supp}\psi_{\boldsymbol{\ell},\mathbf{k}} = \bigcup_{D_\lambda\in\mathcal{L}_{\boldsymbol{\ell},\mathbf{k}}} D_\lambda, \quad \Sigma^{\text{sing}}_{\boldsymbol{\ell},\mathbf{k}} = \operatorname{singsupp}\psi_{\boldsymbol{\ell},\mathbf{k}} = \bigcup_{D_\lambda\in\mathcal{L}_{\boldsymbol{\ell},\mathbf{k}}} \partial D_\lambda\,,$$

and

$$\psi_{\boldsymbol{\ell},\mathbf{k}}(x)|_{D_\lambda} = \sum_{n_1,\ldots,n_d=1}^{2} 2^{|\boldsymbol{\ell}|/2}\omega_{\boldsymbol{\ell},\mathbf{k},\mathbf{n},\lambda}\,\widehat{\phi}_{\mathbf{n}}(\varphi_\lambda^{-1}(x)), \quad D_\lambda\in\mathcal{L}_{\boldsymbol{\ell},\mathbf{k}}\,, \tag{5.3}$$

with weights $\omega_{\boldsymbol{\ell},\mathbf{k},\mathbf{n},\lambda} = \prod_{i=1}^d \omega_{\ell_i,k_i,n_i,\lambda}$, shape functions $\widehat{\phi}_{\mathbf{n}}(z) = \prod_{i=1}^d \widehat{\phi}_{n_i}(z_i)$ and diffeomorphism $\varphi_\lambda : D_\lambda \to [0,1]^d$. The one-dimensional weights follow immediately from Example 3.1 and the one-dimensional shape functions are $\widehat{\phi}_1(z) = 1-z$, $\widehat{\phi}_2(z) = z$. To set up the compression scheme we need to check the distance criteria $\delta_{x_i} > \mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}$, $i\in\mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ and $\delta^{\text{sing}}_{x_i} > \widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}$, $i\in\mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$. Checking these criteria for each matrix coefficient would require $\mathcal{O}(\widehat{N}^2)$ operations. For an efficient computation we exploit the tree structure described above. We denote by $\sigma_{\ell_i,k_i} = \operatorname{supp}\psi_{\ell_i,k_i}$, $\sigma^{\text{sing}}_{\ell_i,k_i} = \operatorname{singsupp}\psi_{\ell_i,k_i}$, $i=1,\ldots,d$ and say that the wavelet $\psi_{\widetilde{\boldsymbol{\ell}},\text{son}}$ is the son of $\psi_{\boldsymbol{\ell},\text{father}}$ if $\sigma_{\widetilde{\ell}_i,\text{son}} \subseteq \sigma_{\ell_i,\text{father}}$, $i=1,\ldots,d$ and there exists $i\in\{1,\ldots,d\}$ such that $\widetilde{\ell}_i = \ell_i+1$. Then, the following lemmas hold.

Lemma 5.1. *Let* $\operatorname{dist}\{\sigma_{\ell_i,father},\sigma_{\ell'_i,father}\} > \mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ *for* $i\in\mathcal{I}^c_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ *and* $\sigma_{\ell_i+1,son}\subseteq\sigma_{\ell_i,father}$*,* $\sigma_{\ell'_i+1,son}\subseteq\sigma_{\ell'_i,father}$

Then $\operatorname{dist}\{\sigma_{\ell_i+1,son},\sigma_{\ell'_i,father}\} > \mathcal{B}^i_{\widetilde{\boldsymbol{\ell}},\boldsymbol{\ell}'}$ *and* $\operatorname{dist}\{\sigma_{\ell_i+1,son},\sigma_{\ell'_i+1,son}\} > \mathcal{B}^i_{\widetilde{\boldsymbol{\ell}},\widetilde{\boldsymbol{\ell}}'}$ *where* $\widetilde{\boldsymbol{\ell}} = (\ell_1,\ldots,\ell_i+1,\ldots,\ell_d)$ *and* $\widetilde{\boldsymbol{\ell}}' = (\ell'_1,\ldots,\ell'_i+1,\ldots,\ell'_d)$*.*

Proof. The result follows from $\mathcal{B}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} \geq \mathcal{B}^i_{\widetilde{\boldsymbol{\ell}},\boldsymbol{\ell}'} \geq \mathcal{B}^i_{\widetilde{\boldsymbol{\ell}},\widetilde{\boldsymbol{\ell}}'}$. □

Lemma 5.2. *Let* $\operatorname{dist}\{\sigma_{\ell_i,father},\sigma^{\text{sing}}_{\ell'_i,k'_i}\}\widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ *for* $i\in\mathcal{I}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$*,* $\ell_i > \ell'_i$ *and* $\sigma_{\ell_i+1,son}\subseteq\sigma_{\ell_i,father}$*.*

Then $\operatorname{dist}\{\sigma_{\ell_i+1,son},\sigma^{\text{sing}}_{\ell'_i,k'_i}\}\widetilde{\mathcal{B}}^i_{\widetilde{\boldsymbol{\ell}},\boldsymbol{\ell}'}$ *where* $\widetilde{\boldsymbol{\ell}} = (\ell_1,\ldots,\ell_i+1,\ldots,\ell_d)$ *and* $\widetilde{\boldsymbol{\ell}}' = (\ell'_1,\ldots,\ell'_i+1,\ldots,\ell'_d)$*.*

Proof. The result follows from $\widetilde{\mathcal{B}}^i_{\boldsymbol{\ell},\boldsymbol{\ell}'} \geq \widetilde{\mathcal{B}}^i_{\widetilde{\boldsymbol{\ell}},\boldsymbol{\ell}'}$. □

Remark 5.3. *Similar results for different wavelets in* $d=2$ *are given in [6].*

Using Lemma 5.1 and 5.2 we have only to check the distance criteria for coefficients which have a non-zero father. The number of operations for setting up the compression scheme is then obviously of log linear complexity $\mathcal{O}(2^L L^{2(d-1)})$.

5.2 Matrix computation

Replacing the wavelets in (5.2) by the element representation (5.3) leads to

$$\mathbf{B}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})} = \sum_{D_\lambda \in \mathcal{L}_{\boldsymbol{\ell},\mathbf{k}}} \sum_{D_{\lambda'} \in \mathcal{L}_{\boldsymbol{\ell}',\mathbf{k}'}} \sum_{n_1,\ldots,n_d=1}^{2} \sum_{n_1',\ldots,n_d'=1}^{2} \omega_{\boldsymbol{\ell},\mathbf{k},\mathbf{n},\lambda} \omega_{\boldsymbol{\ell}',\mathbf{k}',\mathbf{n}',\lambda'} \mathbf{Q}_{(\lambda,\mathbf{n}),(\lambda',\mathbf{n}')} \,,$$

with

$$\mathbf{Q}_{(\lambda,\mathbf{n}),(\lambda',\mathbf{n}')} = 2^{|\lambda|/2+|\lambda'|/2} \int_{D_\lambda} \int_{D_{\lambda'}} \partial_1 \cdots \partial_d \widehat{\phi}_{\mathbf{n}}(\varphi_\lambda^{-1}(y)) \widehat{\phi}_{\mathbf{n}'}(\varphi_{\lambda'}^{-1}(x)) \kappa(y-x) \,\mathrm{d}y \,\mathrm{d}x \,,$$

or in terms of the reference interval

$$\mathbf{Q}_{(\lambda,\mathbf{n}),(\lambda',\mathbf{n}')} = (-1)^{|\mathbf{n}|} \, 2^{|\lambda|/2+|\lambda'|/2} \prod_{i=1}^{d} h_i' \int_{[0,1]^d} \int_{[0,1]^d} \widehat{\phi}_{\mathbf{n}'}(\widehat{x}) \widehat{\kappa}_{\lambda,\lambda'}(\widehat{x},\widehat{y}) \,\mathrm{d}\widehat{y} \,\mathrm{d}\widehat{x} \,, \quad (5.4)$$

where $h_i' = R\, 2^{-\ell_i'}$, $i = 1, \ldots, d$, and $\widehat{\kappa}_{\lambda,\lambda'}(\widehat{x},\widehat{y}) = \kappa(\varphi_\lambda(\widehat{y}) - \varphi_{\lambda'}(\widehat{x}))$. Therefore, computing the matrix entries reduces to computing the element-element interactions $\mathbf{Q}_{(\lambda,\mathbf{n}),(\lambda',\mathbf{n}')}$.

We can again use the hierarchical data structure to obtain an entry of a father element from the son elements. For example, for a father element $D_{\text{father}} = D_{(\boldsymbol{\ell},\mathbf{k})}$ with the two sons $D_{\text{son}_1} = D_{(\ell_1,\ldots,\ell_i+1,\ldots,\ell_d),(k_1,\ldots,2k_i-1,\ldots,k_d)}$ and $D_{\text{son}_2} = D_{(\ell_1,\ldots,\ell_i+1,\ldots,\ell_d),(k_1,\ldots,2k_i,\ldots,k_d)}$, we get

$$\mathbf{Q}_{(\text{father},\mathbf{n}),(\lambda',\mathbf{n}')} = \big(\mathbf{Q}_{(\text{son}_1,\mathbf{n}),(\lambda',\mathbf{n}')} + \mathbf{Q}_{(\text{son}_2,\mathbf{n}),(\lambda',\mathbf{n}')}\big)\, 2^{-3/2} \,. \quad (5.5)$$

Similarly,

$$\begin{aligned} \mathbf{Q}_{(\lambda,\mathbf{n}),(\text{father},n_1',\ldots,1,\ldots,n_d')} = \big(&\mathbf{Q}_{(\lambda,\mathbf{n}),(\text{son}_1,n_1',\ldots,1,\ldots,n_d')} + \mathbf{Q}_{(\lambda,\mathbf{n}),(\text{son}_1,n_1',\ldots,2,\ldots,n_d')}/2 \\ &+ \mathbf{Q}_{(\lambda,\mathbf{n}),(\text{son}_2,n_1',\ldots,1,\ldots,n_d')}/2\big) 2^{-1/2} \,. \end{aligned} \quad (5.6)$$

5.3 Numerical integration

Consider $\boldsymbol{\ell}, \mathbf{k}, \boldsymbol{\ell}', \mathbf{k}' \in \mathbb{N}^d$ with the corresponding λ, λ', fix $\mathbf{n}, \mathbf{n}'$ and introduce the notation $\delta_i = \operatorname{dist}\{D_\lambda^i, D_{\lambda'}^i\}$ where $D_\lambda = D_\lambda^1 \times \cdots \times D_\lambda^d$. Let $\epsilon > 0$, $i \in \{1, \ldots, d\}$ and set $\mathcal{I} = \{1, \ldots, d\} \backslash i$ and $z = y - x$. We distinguish several cases: The integrand $\widehat{\kappa}_{\lambda,\lambda'}(x,y)$ is non-singular in $y_i - x_i$, i.e., $\delta_i > 0$, the elements are identical, $D_\lambda = D_{\lambda'}$, or the elements share a common vertex.

1. Let $\delta_i > C_f \max\{h_i, h_i'\}/4$. Consider

$$\begin{aligned} g &\gtrsim \frac{\ln\epsilon + \frac{\alpha}{d}\ln\delta_i + (|\lambda'|/2 - |\lambda|/2)\ln 2}{2\ln w}, \\ g' &\gtrsim \frac{\ln\epsilon + (\frac{\alpha}{d} - 1)\ln\delta_i + (|\lambda'|/2 - |\lambda|/2 - \ell_i')\ln 2}{2\ln w'}, \end{aligned} \tag{5.7}$$

number of Gauss points where $w = \frac{h_i C_f}{4\delta_i}$ and $w' = \frac{h_i' C_f}{4\delta_i}$. Furthermore, let the standard Gauss quadrature points and weights on $[0,1]$ be given by $\underline{\xi}_g, \underline{\omega}_g \in \mathbb{R}^g$. Then, we define quadrature points $\underline{\xi}^i \in \mathbb{R}^{gg'}$ and weights $\underline{\omega}^i \in \mathbb{R}^{gg'}$ by

$$\underline{\xi}^i = \underline{\xi}_g^\lambda \otimes \underline{1}_{g'} - \underline{1}_g \otimes \underline{\xi}_{g'}^{\lambda'}, \quad \underline{\omega}^i = 2^{\ell_i/2 + \ell_i'/2} h_i^{-1} \widehat{\phi}_{n_i'}(\underline{1}_g \otimes \underline{\xi}_{g'}) .* \underline{\omega}_g^\lambda \otimes \underline{\omega}_{g'}^{\lambda'}, \tag{5.8}$$

where $\underline{1}_g = (1,\dots,1)^\top \in \mathbb{R}^g$, $\xi_{g,j}^\lambda = (k_i - 1)h_i + h_i\xi_{g,j}$ and $\omega_{g,j}^\lambda = h_i\omega_{g,j}$. Here, we used the Kronecker tensor product $\underline{x} = \underline{y} \otimes \underline{z} \in \mathbb{R}^{ml}$, for $\underline{y} \in \mathbb{R}^m$, $\underline{z} \in \mathbb{R}^l$ and the vector multiplication $\underline{x} = \underline{y} .* \underline{z} \in \mathbb{R}^n$ where $x_j = y_j z_j$, $j = 1,\dots,n$ for $\underline{z}, \underline{y} \in \mathbb{R}^n$.

2. Let $\delta_i = 0$, $\ell_i = \ell_i'$ and $k_i = k_i'$. Consider

$$n \gtrsim \frac{\ln\epsilon + (|\lambda'|/2 - |\lambda|/2 - \ell_i)\ln 2}{(1 - \frac{\alpha}{d})\ln(h_i\,\sigma)}. \tag{5.9}$$

refinements for the composite Gauss quadrature and σ, $\mathbf{q}$ satisfying (4.7), (4.8). Furthermore, let the composite Gauss quadrature points and weights on $[0,1]$ be given by $\underline{\xi}_n, \underline{\omega}_n \in \mathbb{R}^N$. Then, we define quadrature points $\underline{\xi}^i \in \mathbb{R}^{2N}$ and weights $\underline{\omega}^i \in \mathbb{R}^{2N}$ by

$$\begin{aligned} &(\xi_j^i)_{1\le j\le N} = h_i\underline{\xi}_n, \quad (\xi_j^i)_{N+1\le j\le 2N} = -h_i\underline{\xi}_n, \\ &(\omega_j^i)_{1\le j\le N} = h_i\int_0^1 \widehat{\phi}_{n_i'}(\underline{\xi}_n + x(1-\underline{\xi}_n))\,\mathrm{d}x .* (1-\underline{\xi}_n) .* \underline{\omega}_n, \\ &(\omega_j^i)_{N+1\le j\le 2N} = h_i\int_0^1 \widehat{\phi}_{n_i'}(x(1-\underline{\xi}_n))\,\mathrm{d}x .* (1-\underline{\xi}_n) .* \underline{\omega}_n. \end{aligned} \tag{5.10}$$

3. Let $\delta_i = 0$, $\ell_i = \ell_i'$ and $k_i = k_i' - 1$. Consider

$$\begin{aligned} g &\gtrsim \frac{\ln\epsilon + \frac{\alpha}{d}\ln h_i + (|\lambda'|/2 - |\lambda|/2)\ln 2}{2\ln w}, \\ n &\gtrsim \frac{\ln\epsilon + (|\lambda'|/2 - |\lambda|/2 - \ell_i)\ln 2}{(2 - \frac{\alpha}{d})\ln(h_i\,\sigma)}, \end{aligned} \tag{5.11}$$

number of Gauss points or refinements, respectively. We define quadrature points $\underline{\xi}^i \in \mathbb{R}^{g+N}$ and weights $\underline{\omega}^i \in \mathbb{R}^{g+N}$ by

$$\begin{aligned} &(\xi_j^i)_{1\le j\le g} = h_i + h_i\underline{\xi}_g, \quad (\xi_j^i)_{g+1\le j\le g+N} = h_i\underline{\xi}_n, \\ &(\omega_j^i)_{1\le j\le g} = h_i\int_0^1 \widehat{\phi}_{n_i'}(\underline{\xi}_g + x(1-\underline{\xi}_g))\,\mathrm{d}x .* (1-\underline{\xi}_g) .* \underline{\omega}_g, \\ &(\omega_j^i)_{g+1\le j\le g+N} = h_i\int_0^1 \widehat{\phi}_{n_i'}(x\underline{\xi}_n)\,\mathrm{d}x .* \underline{\xi}_n .* \underline{\omega}_n. \end{aligned} \tag{5.12}$$

Using a tensor product quadrature formula we have the following error estimate.

Theorem 5.4. *Assume that the kernel κ satisfies* (4.3). *Consider $\epsilon > 0$ and assume either $\delta_i > C_f \max\{h_i, h_i'\}/4$ or $\delta_i = 0$, $\ell_i = \ell_i'$, $k_i = k_i'$ or $\delta_i = 0$, $\ell_i = \ell_i'$, $k_i = k_i' - 1$ for $i = 1, \ldots, d$. Define the d-dimensional quadrature points and weights by*

$$\underline{\xi}_i = \widehat{\bigotimes_{1\le j\le i-1}} \underline{1}^j \,\widehat{\otimes}\, \underline{\xi}^i \,\widehat{\otimes} \widehat{\bigotimes_{i+1\le j\le d}} \underline{1}^j, \quad \underline{\omega} = \widehat{\bigotimes_{1\le j\le d}} \underline{\omega}^j,$$

where the one-dimensional quadrature points and weights $\underline{\xi}^i$, $\underline{\omega}^i$, $i = 1, \ldots, d$ are given by (5.8), (5.10) *or* (5.12). *Then, we obtain*

$$\left| 2^{|\lambda|/2+|\lambda'|/2} \prod_{i=1}^{d} h_i' \int_{[0,1]^d} \int_{[0,1]^d} \widehat{\phi}_{\mathbf{n}'}(\widehat{x}) \widehat{\kappa}_{\lambda,\lambda'}(\widehat{x}, \widehat{y}) \,\mathrm{d}\widehat{y} \,\mathrm{d}\widehat{x} - \langle \underline{\omega}, \kappa(\underline{\xi}_1, \ldots, \underline{\xi}_d) \rangle \right| \lesssim \epsilon.$$

Proof. We again distinguish three cases.

1. Let $\delta_i > C_f \max\{h_i, h_i'\}/4$ and define $f(\widehat{x}, \widehat{y}) = 2^{|\lambda|/2-|\lambda'|/2} \widehat{\phi}_{\mathbf{n}'}(\widehat{x}) \widehat{\kappa}_{\lambda,\lambda'}(\widehat{x}, \widehat{y})$. Using the standard product rule

$$\partial_{x_i}^n f(\widehat{x}, \widehat{y}) = \partial_{x_i}^n \widehat{\kappa}_{\lambda,\lambda'}(\widehat{x}, \widehat{y}) \widehat{\phi}_{\mathbf{n}'}(\widehat{x}) + \partial_{x_i}^{n-1} \widehat{\kappa}_{\lambda,\lambda'}(\widehat{x}, \widehat{y})\, \partial_i \widehat{\phi}_{\mathbf{n}'}(\widehat{x}) \,,$$

there holds for $h_i = R\, 2^{-\ell_i}$,

$$\left|\partial_{y_i}^n f(\widehat{x}, \widehat{y})\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} n!\, (h_i C_f)^n \, \delta_i^{-\frac{\alpha}{d}-n} \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha-\frac{\alpha}{d}}, \quad n \in \mathbb{N}_0 \,,$$

and for $\delta_i \gtrsim h_i'$ and $h_i' = R\, 2^{-\ell_i'}$,

$$\left|\partial_{x_i}^n f(\widehat{x}, \widehat{y})\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} n!\, (h_i' C_f)^{n-1} \, \delta_i^{-\frac{\alpha}{d}-n+1} \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha-\frac{\alpha}{d}}, \quad n \in \mathbb{N}_0 \,.$$

Therefore, we obtain similar to (4.4)

$$\left|E_{g,g'}^{[0,1]^2} f_{\widehat{x}_i,\widehat{y}_i}\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha+\frac{\alpha}{d}} \left(w_i^{2g} \delta_i^{-\frac{\alpha}{d}} + (w_i')^{2g'} \delta_i^{-\frac{\alpha}{d}+1} 2^{\ell_i'}\right),$$

with $w_i = h_i C_f/(4\delta_i)$, $w_i' = h_i' C_f/(4\delta_i)$. Choosing the number of Gauss points according to (5.7) we have

$$\left|E_{g,g'}^{[0,1]^2} f_{\widehat{x}_i,\widehat{y}_i}\right| \lesssim \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha+\frac{\alpha}{d}} \epsilon.$$

2. Let $\delta_i = 0$, $\ell_i = \ell_i'$ and $k_i = k_i'$. The integrand $\widehat{\kappa}_{\lambda,\lambda'}(\widehat{x}, \widehat{y})$ is singular on the diagonal $x_i = y_i$. We first transform this singularity to the axis. Let $\kappa_i(s-t) = \kappa(z_1, \ldots, h_i(s-t), \ldots, z_d)$ and consider the integral

$$I = \int_{[0,1]} \int_{[0,1]} \phi(s)\psi(t)\kappa_i(s-t) \,\mathrm{d}s \,\mathrm{d}t \,.$$

Introducing the variable $z = s - t$ and splitting the integral yields

$$
\begin{aligned}
I &= -\int_{[0,1]} \int_{s}^{s-1} \phi(s)\psi(s-z)\kappa_i(z)\,\mathrm{d}z\,\mathrm{d}s \\
&= \int_{[0,1]} \int_{s-1}^{0} \phi(s)\psi(s-z)\kappa_i(z)\,\mathrm{d}z\,\mathrm{d}s + \int_{[0,1]} \int_{0}^{s} \phi(s)\psi(s-z)\kappa_i(z)\,\mathrm{d}z\,\mathrm{d}s\,.
\end{aligned}
$$

With $x = s - z$, $y = -z$ we have

$$
\int_{[0,1]} \int_{s-1}^{0} \phi(s)\psi(s-z)\kappa_i(z)\,\mathrm{d}z\,\mathrm{d}s = \int_{[0,1]} \int_{0}^{x} \phi(x-y)\psi(x)\kappa_i(-y)\,\mathrm{d}y\,\mathrm{d}x\,,
$$

and therefore,

$$
I = \int_{[0,1]} \int_{0}^{x} \phi(x)\psi(x-y)\kappa_i(y)\,\mathrm{d}y\,\mathrm{d}x + \int_{[0,1]} \int_{0}^{x} \phi(x-y)\psi(x)\kappa_i(-y)\,\mathrm{d}y\,\mathrm{d}x\,.
$$

Finally setting $x = \xi + \eta(1-\xi)$, $y = \xi$, we obtain

$$
\begin{aligned}
I = &\int_{[0,1]} \int_{[0,1]} \phi(\xi + \eta(1-\xi))\psi(\eta(1-\xi))\kappa_i(\xi)(1-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&+ \int_{[0,1]} \int_{[0,1]} \phi(\eta(1-\xi))\psi(\xi + \eta(1-\xi))\kappa_i(-\xi)(1-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \qquad (5.13)
\end{aligned}
$$

The function

$$
\begin{aligned}
f(\widehat{x},\widehat{y}) = 2^{|\lambda|/2-|\lambda'|/2}(1-\widehat{y}_i)\big(&\widehat{\phi}_{n_i'}(\widehat{y}_i + \widehat{x}_i(1-\widehat{y}_i))\kappa_i(\widehat{y}_i) \\
&+ \widehat{\phi}_{n_i'}(\widehat{x}_i(1-\widehat{y}_i))\kappa_i(-\widehat{y}_i)\big)\,,
\end{aligned}
$$

has a singularity at $\widehat{y}_i = 0$ and satisfies (4.3) with respect to $\widehat{y}_i$, i.e.,

$$
\left|\partial^k_{\widehat{y}_i} f(\widehat{x},\widehat{y})\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} k!\,(h_i C_f)^k\,(h_i\widehat{y}_i)^{-\frac{\alpha}{d}-k}\,\left\|z^{\mathcal{I}}\right\|_{\infty}^{-\alpha+\frac{\alpha}{d}}\,, \quad k \in \mathbb{N}_0\,.
$$

The integrand f is polynomial in the $\widehat{x}_i$ and can be integrated exactly. Thus, similar to Theorem 4.4 we obtain

$$
\left|I^{[0,1]^2} f_{\widehat{y}_i} - Q^{n,\mathbf{q}}_{h_i\sigma} f_{\widehat{y}_i}\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2}\, 2^{\ell_i}\left\|z^{\mathcal{I}}\right\|_{\infty}^{-\alpha+\frac{\alpha}{d}}\,(h_i\,\sigma)^{n(1-\frac{\alpha}{d})}\,,
$$

where σ, $\mathbf{q}$ satisfy (4.7), (4.8). Choosing the number of refinements according to (5.9) we have

$$
\left|I^{[0,1]^2} f_{\widehat{y}_i} - Q^{n,\mathbf{q}}_{h_i\sigma} f_{\widehat{y}_i}\right| \lesssim \left\|z^{\mathcal{I}}\right\|_{\infty}^{-\alpha+\frac{\alpha}{d}}\,\epsilon.
$$

3. Let $\delta_i = 0$, $\ell_i = \ell_i'$ and $k_i = k_i' - 1$. Similar to the case of identical elements we have $\kappa_i(s+t) = \kappa(z_1,\ldots,h_i(s+t),\ldots,z_d)$ and transform the integral

$$
I = \int_{[0,1]} \int_{[0,1]} \phi(s)\psi(t)\kappa_i(s+t)\,\mathrm{d}s\,\mathrm{d}t\,.
$$

into

$$I = \int_{[0,1]} \int_{[0,1]} \phi(\xi + \eta(1-\xi))\psi(1+\eta(\xi-1))\kappa_i(\xi+1)(1-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+ \int_{[0,1]} \int_{[0,1]} \phi(\eta\xi)\psi(\xi(1-\eta))\kappa_i(\xi)\xi\,\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{5.14}$$

The function

$$f(\widehat{x},\widehat{y}) = 2^{|\lambda|/2-|\lambda'|/2}\widehat{\phi}_i(\widehat{y}_i + \widehat{x}_i(1-\widehat{y}_i))\kappa_i(\widehat{y}_i+1)(1-\widehat{y}_i)\,,$$

can be integrated exactly in the $\widehat{x}_i$ direction and has no singularity in $\widehat{y}_i$, i.e.,

$$\left|\partial^k_{\widehat{y}_i} f(\widehat{x},\widehat{y})\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} k!\,(h_i C_f)^k\, h_i^{-\frac{\alpha}{d}-k} \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha-\frac{\alpha}{d}}\,, \quad k \in \mathbb{N}_0\,.$$

The function

$$f(\widehat{x},\widehat{y}) = 2^{|\lambda|/2-|\lambda'|/2}\widehat{\phi}(\widehat{x}_i\widehat{y}_i)\kappa_i(\widehat{y}_i)\widehat{y}_i\,,$$

can again be integrated exactly in the $\widehat{x}_i$ direction and has singularity in $\widehat{y}_i$, i.e.,

$$\left|\partial^k_{\widehat{y}_i} f(\widehat{x},\widehat{y})\right| \lesssim 2^{|\lambda|/2-|\lambda'|/2} k!\,(h_i C_f)^k\, (h_i\widehat{y}_i)^{-\frac{\alpha}{d}-k+1} \left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha+\frac{\alpha}{d}}\,, \quad k \in \mathbb{N}_0\,.$$

Choosing the number of Gauss points and refinements according to (5.11) we again obtain an error in the i-th direction of order $\left\|z^{\mathcal{I}}\right\|_\infty^{-\alpha+\frac{\alpha}{d}} \epsilon$.

Finally, tensorization arguments as in Theorem 4.6 yield the required result. □

5.4 Adaptive strategy

As proposed in [6] we define an adaptive strategy to compute the element-element interactions $\mathbf{Q}_{(\lambda,\mathbf{n}),(\lambda',\mathbf{n}')}$ with the precision $\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ given by (3.9).

We loop over the dimension $i = 1, \ldots, d$. For each i we do:

1. *Starting point.* If $\delta_i > C_f \max\{h_i, h_i'\}/4$ we define quadrature points in the i-th direction according to (5.7). Else if $\delta_i = 0$, $\ell_i = \ell_i'$ and $k_i = k_i'$ or $k_i = k_i' - 1$, $k_i' = k_i - 1$ define quadrature points according to (5.9) or (5.11). Otherwise go to item 2 if $\ell_i > \ell_i'$, item 3 if $\ell_i' > \ell_i$ and item 4 if $\ell_i = \ell_i'$.

2. *Case* $\ell_i > \ell_i'$. Replace the larger element $D_{\lambda'}$ by its two sons and compute the associated element-element interaction with precision $2^{-3/2}\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ according to item 1. The desired element-element interaction is calculated via formula (5.6).

3. *Case* $\ell_i' > \ell_i$. Replace the larger element D_λ by its two sons and compute the associated element-element interaction with precision $2^{-1/2}\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ according to item 1. The desired element-element interaction is calculated via formula (5.5).

4. *Case* $\ell_i = \ell_i'$. Replace both elements D_λ by their two sons and compute the associated element-element interaction with precision $\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ according to item 1. The desired element-element interaction is calculated via formulas (5.5) and (5.6).

Note that using this strategy we only have to compute an element-element interaction where Theorem 5.4 holds. The next lemma shows that the algorithm stops after, at the most, $\mathcal{O}(\|\ell_i - \ell_i'\|_\infty)$ steps.

Lemma 5.5. *Let $i \in \{1, \dots, d\}$. The following statements concerning the computation of the element-element interaction by the above algorithm are valid:*

1. *The given element-element interaction is subdivided into at most $\mathcal{O}(|\ell_i - \ell_i'|)$ interactions $\mathbf{Q}_{(\widehat{\lambda},\mathbf{n}),(\widehat{\lambda}',\mathbf{n}')}$ where $\widehat{\ell_i} \geq \ell_i$, $\widehat{\ell_i'} \geq \ell_i'$.*

2. *If $\ell_i \leq \ell_i'$, there holds $\ell_i \leq \widehat{\ell_i} \leq \widehat{\ell_i'} \sim \ell_i'$. The analogous result holds if $\ell_i' \leq \ell_i$.*

3. *On a fixed level $\widehat{\ell_i}$ and $\widehat{\ell_i'}$ the number of directly computed as well as subdivided element-element interactions is $\mathcal{O}(1)$.*

Proof. See [6, Lemma 9.7]. □

Now with formulas (5.5), (5.6), Lemma 5.5 and Theorem 5.4 it follows that the proposed quadrature algorithm computes the desired element-element interactions with a precision that stays proportional to $\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$.

Corollary 5.6. *Assume the Lévy density $k(z)$ satisfies (2.4), i.e., is real analytic outside of $z_i = 0$, $i = 1, \dots, d$. Let $\epsilon_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ be given by (3.9). Then, the number of quadrature points to compute an entry $\mathbf{A}_{(\boldsymbol{\ell}',\mathbf{k}'),(\boldsymbol{\ell},\mathbf{k})}$ is at most $\mathcal{O}(L^{2d})$ and the overall operations to compute the stiffness matrix $\mathbf{A}$ at most of log linear complexity $\mathcal{O}(2^L L^{4d-2})$.*

Proof. We have for the one-dimensional Gauss points in (5.8), $g, g' \lesssim L$, for the refinements in (5.10), $n \lesssim L$ and for the quadrature points and refinements in (5.12) again, $g, n \lesssim L$. Therefore, we need at most $\mathcal{O}(L^2)$ quadrature points in each direction $i = 1, \dots, d$. □

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