Stochastic Galerkin discretization of the lognormal isotropic diffusion problem*

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Abstract

The stochastic Galerkin method is developed for the isotropic diffusion equation with an unbounded random diffusion coefficient. The logarithm of the diffusion coefficient is assumed to be an infinite series of Gaussian random variables.

Well-posed weak formulations of the model problem are derived on standard Bochner-Lebesgue spaces. The Galerkin solution is shown to be almost quasi-optimal in the sense that the error of the Galerkin projection can be estimated by a best approximation error in a slightly stronger norm. As a result, convergence analysis of the stochastic Galerkin method is reduced to the problem of approximating the Hermite coefficients of the exact solution by standard finite elements.

1 Introduction

Many boundary value problems arising in engineering contain parameters that are subject to significant uncertainty. If these parameters are modelled as random variables with a known joint probability distribution, the solution to the boundary value problem is a random field, which can be computed deterministically by the stochastic Galerkin method [1, 7, 5, 16, 11, 3, 9] or the stochastic collocation method [4, 5, 2, 14, 13, 16].

A commonly studied model problem is the isotropic diffusion equation

$$-\nabla \cdot (a \nabla u) = f$$

with some boundary conditions, where the diffusion coefficient a and possibly also the right hand side f are random. This equation models time-independent groundwater flow in a sediment with permeability a.

Most authors assume that the coefficient a is uniformly bounded from above and away from zero, [1, 7, 5, 14, 13, 12]. In this case, a stochastic Galerkin discretization can be performed with a continuous and coercive bilinear form, and, in particular, the Galerkin solution is quasi-optimal with respect to a problem-independent norm.

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The permeability a can be written as a countable affine combination of uncorrelated random variables, for example by means of a Karhúnen-Loève expansion. Until recently [7], this series has generally been assumed finite, [1, 4, 5, 2]. In order to ensure that a is uniformly bounded, the random variables are often assumed to be bounded, [1, 7, 5, 14]. Additionally, to obtain a product structure on the probability space, they are also assumed to be independent, [1, 5, 14].

These assumptions are often made, not because of any intrinsic merit, but out of necessity, since they greatly simplify the development and analysis of numerical methods. However, in some situations, they are questionable.

The permeability coefficient may vary drastically within a layer of sediment. Thus it seems more appropriate to expand its logarithm in an affine series of random variables. The most natural distribution for these random variables is normal, which has the advantage that the random variables are independent once they are pairwise uncorrelated, and thus independence is not a further restriction.

In this case, the bilinear forms resulting from a weak formulation of the model problem for given realizations of the parameters are not uniformly bounded from above or away from zero. Therefore, the energy space does not coincide with a standard Bochner-Lebesgue space, and standard finite element theory fails to show quasi-optimality of the Galerkin solution in any meaningful norm.

This paper lays the foundation for an analysis of the stochastic Galerkin method applied to the above model problem in the realistic setting that the logarithm of the permeability coefficient a is a *countable* affine combination of uncorrelated Gaussian random variables.

Collocation methods have been analyzed in the case of a lognormal diffusion coefficient depending on only *finitely* many random variables, [2, 4]. However, it was essential that the coefficient was bounded away from zero by a positive shift $a_* > 0$. We do not require either of these restrictions.

The model problem is presented in detail and recast as a parametric deterministic problem in Section 2. Its variational formulation is derived in Section 3. Section 4 deals with the Galerkin discretization of the model problem. In particular, it includes results on the quasi-optimality of the Galerkin solution in standard Bochner-Lebesgue spaces.

2 Parametric diffusion problem

2.1 Problem formulation

Let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $(\Omega, \mathcal{F}, P)$ a probability space. Consider the stochastic isotropic diffusion equation

$$-\nabla \cdot (a(\omega, x)\nabla u(\omega, x)) = f(\omega, x) \quad \text{for} \quad x \in D\ , \quad \omega \in \Omega\ , \tag{1}$$

where the functions $a, f : \Omega \times D \to \mathbb{R}$ are a parametric diffusion coefficient and forcing term, respectively. The differential operators $\nabla\cdot$ and ∇ are meant with respect to $x \in D$.

Denote by V a closed subspace of $H^1(D)$ on which

$$\|v\|_V := \left(\int_D |\nabla v(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \tag{2}$$

is a norm. For example, V could incorporate homogeneous essential boundary conditions. Define the parametric bilinear form

$$b(\omega; u, v) := \int_D a(\omega, x) \nabla u(x) \cdot \nabla v(x) \mathrm{d}x \qquad \text{for} \quad \omega \in \Omega, \quad u, v \in V \tag{3}$$

and reinterpret the right hand side as a map into the dual space V' defined by

$$f(\omega; v) := \int_D f(\omega, x) v(x) \mathrm{d}x \qquad \text{for} \quad \omega \in \Omega, \quad u, v \in V \; . \tag{4}$$

Then for any $\omega \in \Omega$, the weak formulation in x of (1) for homogeneous boundary conditions is

$$\text{find} \quad u(\omega) \in V \quad \text{such that} \quad b(\omega; u(\omega), v) = f(\omega; v) \qquad \forall v \in V \; . \tag{5}$$

Assumption 2.1. *There exists a sequence $(Y_m)_{m \in \mathbb{N}}$ of i.i.d. standard normal random variables on Ω and functions $a_{\min}, a_m \in L^\infty(D)$ for $m \in \mathbb{N}_0$ with $a_{\min}(x) \geq 0$, $a_0(x) \geq \underline{a}_0 > 0$ for all $x \in D$ and $(\|a_m\|_{L^\infty(D)})_m \in \ell^1(\mathbb{N})$ such that the diffusion coefficient has the form*

$$a(\omega, x) := a_{\min}(x) + a_0(x) \exp\left(\sum_{m=1}^{\infty} a_m(x) Y_m(\omega) \right) \; . \tag{6}$$

We will sometimes make the additional assumption $a_{\min} \gtrsim 1$, although we are particularly interested in the case where this assumption is not satisfied.

Define $\alpha_m := \|a_m\|_{L^\infty(D)}$ and

$$\Omega_\alpha := \left\{ \omega \in \Omega \, ; \, \sum_{m=1}^{\infty} \alpha_m \left| Y_m(\omega) \right| < \infty \right\} \; . \tag{7}$$

The following lemma ensures that the diffusion coefficient $a(\omega, x)$ is well-defined for $\omega \in \Omega_\alpha$.

Lemma 2.2. *For a.e. $x \in D$,*

$$\exp\left(\sum_{m=1}^{\infty} a_m(x) Y_m(\omega) \right) = \prod_{m=1}^{\infty} \exp(a_m(x) Y_m(\omega)) \in (0, \infty) \tag{8}$$

for all $\omega \in \Omega_\alpha$.

Proof. Let $\omega \in \Omega_\alpha$ and $x \in D$ with $|a_m(x)| \leq \alpha_m$ for all $m \in \mathbb{N}$. Then

$$\sum_{m=1}^{\infty} |a_m(x)| \, |Y_m(\omega)| \leq \sum_{m=1}^{\infty} \alpha_m \left| Y_m(\omega) \right| < \infty \, ,$$

so the sum in (8) converges absolutely. The rest of the claim follows by continuity of $\exp(\cdot)$. $\square$

Proposition 2.3. *$P(\Omega_\alpha) = 1$.*

Proof. The proof is similar to [8, Prop. 1.11]. By the monotone convergence theorem, using

$$\mathbb{E}_P(|Y_m|) = \frac{2}{\sqrt{2\pi}} \int_0^\infty y_m \exp\left(-\frac{y_m^2}{2}\right) \mathrm{d}y_m = \sqrt{\frac{2}{\pi}} \ ,$$

we have

$$\mathbb{E}_P\left(\sum_{m=1}^\infty \alpha_m |Y_m|\right) = \sum_{m=1}^\infty \alpha_m \mathbb{E}_P(|Y_m|) = \sqrt{\frac{2}{\pi}} \sum_{m=1}^\infty \alpha_m < \infty \ .$$

□

Remark 2.4. In view of Proposition 2.3 and (7) with the requirement $(\alpha_m)_m \in \ell^1(\mathbb{N})$, it is interesting to note that $P((Y_m)_m \in \ell^\infty(\mathbb{N})) = 0$. This is a consequence of $P(Y_m \leq n \,\forall m) = 0$ for all $n \in \mathbb{N}$.

Lemma 2.5. *The diffusion coefficient from Assumption 2.1 satisfies*

$$0 < \underline{a}(\omega) := \operatorname*{ess\,inf}_{x \in D} a(\omega, x) \leq \operatorname*{ess\,sup}_{x \in D} a(\omega, x) =: \overline{a}(\omega) < \infty \tag{9}$$

with

$$\overline{a}(\omega) \leq \|a_{\min}\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\sum_{m=1}^\infty \alpha_m |Y_m(\omega)|\right)$$

and

$$\underline{a}(\omega) \geq \operatorname*{ess\,inf}_{x \in D} a_{\min}(x) + \underline{a}_0 \exp\left(-\sum_{m=1}^\infty \alpha_m |Y_m(\omega)|\right)$$

for all $\omega \in \Omega_\alpha$.

Proof. This is a direct consequence of Assumption 2.1. □

Theorem 2.6. *Problem* (5) *has a unique solution $u(\omega) \in V$ for all $\omega \in \Omega_\alpha$. It satisfies*

$$\|u(\omega)\|_V \leq \frac{1}{\underline{a}(\omega)} \|f(\omega; \cdot)\|_{V'} \ . \tag{10}$$

Proof. By Lemma 2.5, the bilinear form $b(\omega; \cdot, \cdot)$ is continuous and coercive on V with coercivity constant $\underline{a}(\omega)$ for all $\omega \in \Omega_\alpha$. □

Remark 2.7. Note that the solution u does not depend on the probability measure P. However, the probability measure plays a crucial role in assessing the quality of approximations of u. Our goal is to compute u_N satisfying an a priori bound for

$$\mathbb{E}_P(\|u - u_N\|_V^p)^{\frac{1}{p}} \ . \tag{11}$$

We are particularly interested in the case $p = 2$.

2.2 Product measures on the probability space

Let $(\lambda_m)_{m\in\mathbb{N}}$ be a positive sequence with $(\log\lambda_m)_m \in \ell^1(\mathbb{N})$. The random variables $(\lambda_m^{-1}Y_m)_{m\in\mathbb{N}}$ induce a map $Y^\lambda : \Omega \to \mathbb{R}^\mathbb{N}$ by the coordinatewise definition $(Y^\lambda(\omega))_m := \lambda_m^{-1}Y_m(\omega)$. Denote by $\mathbb{R}^\mathbb{N}$ the set of sequences in $\mathbb{R}$ and by $\mathcal{B}(\mathbb{R}^\mathbb{N})$ the Borel σ-algebra on $\mathbb{R}^\mathbb{N}$.

Lemma 2.8. *The map $Y^\lambda : (\Omega, \mathcal{F}) \to (\mathbb{R}^\mathbb{N}, \mathcal{B}(\mathbb{R}^\mathbb{N}))$ is measurable.*

Proof. Define $Y_m^\lambda := \lambda_m^{-1}Y_m$ and let $E_m \in \mathcal{B}(\mathbb{R})$. Then $(Y_m^\lambda)^{-1}(E_m) \in \mathcal{F}$ by the measurability of Y_m^λ. Consider the cylinder set $E := \prod_{m=1}^\infty E_m$ with $E_m \in \mathcal{B}(\mathbb{R})$ and $E_m = \mathbb{R}$ for all but finitely many $m \in \mathbb{N}$. Then $(Y^\lambda)^{-1}(E) = \bigcap_{m\in\mathbb{N}}(Y_m^\lambda)^{-1}(E_m) \in \mathcal{F}$. Since the Borel σ-algebra $\mathcal{B}(\mathbb{R}^\mathbb{N})$ is generated by cylinder sets, $(Y^\lambda)^{-1}(E) \in \mathcal{F}$ for all $E \in \mathcal{B}(\mathbb{R}^\mathbb{N})$. □

Proposition 2.9. *The image of P under the map Y^λ is the countable product measure*

$$\gamma^P = \bigotimes_{m=1}^\infty \gamma_m^P \ , \tag{12}$$

where γ_m^P is the Gaussian measure on $\mathbb{R}$ with mean 0 and standard deviation λ_m^{-1}.

Proof. Consider a cylinder set $E = \prod_{m=1}^\infty E_m$ as in the proof of Lemma 2.8. By the independence of $(Y_m^\lambda)_{m\in\mathbb{N}}$,

$$\begin{aligned} P\big((Y^\lambda)^{-1}(E)\big) &= P\left(\bigcap_{m=1}^\infty (Y_m^\lambda)^{-1}(E_m)\right) \\ &= \prod_{m=1}^\infty P\big((Y_m^\lambda)^{-1}(E_m)\big) = \prod_{m=1}^\infty \gamma_m^P(E_m) = \gamma^P(E) \ , \end{aligned}$$

where all but finitely many factors in both products are equal to one. This property characterizes the product measure. □

For a sequence $(s_m)_m \in \ell^1(\mathbb{N})$, let $\widehat{\gamma}_{s,m}$ be the Gaussian measure on $\mathbb{R}$ with mean 0 and standard deviation $\sigma_m = \exp(-s_m)$, and define the product measure

$$\widehat{\gamma}_s := \bigotimes_{m=1}^\infty \widehat{\gamma}_{s,m} \tag{13}$$

on $(\mathbb{R}^\mathbb{N}, \mathcal{B}(\mathbb{R}^\mathbb{N}))$. Also, define the standard Gaussian measure $\gamma := \widehat{\gamma}_0$.

Remark 2.10. Note that $\gamma^P = \widehat{\gamma}_{s^P}$ if $s_m^P = \log\lambda_m$ for all $m \in \mathbb{N}$.

Proposition 2.11. *The probability measure $\widehat{\gamma}_s$ defined in* (13) *is equivalent to γ. The density of $\widehat{\gamma}_s$ with respect to γ is*

$$\xi_s(y) = \left(\prod_{m=1}^\infty \frac{1}{\sigma_m}\right) \exp\left(-\frac{1}{2}\sum_{m=1}^\infty (\sigma_m^{-2} - 1)y_m^2\right) \ . \tag{14}$$

Proof. Note that $\mathrm{d}\widehat{\gamma}_{s,m} = \xi_{s,m}\mathrm{d}\gamma_m$ for

$$\xi_{s,m}(y_m) = \frac{1}{\sigma_m}\exp\left(-\frac{1}{2}(\sigma_m^{-2}-1)y_m^2\right) .$$

We compute

$$\begin{aligned}\int_{\mathbb{R}} \sqrt{\xi_{s,m}(y_m)}\mathrm{d}\gamma_m(y_m) &= \frac{1}{\sqrt{2\pi\sigma_m}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}(\sigma_m^{-2}+1)y_m^2\right)\mathrm{d}y_m \\ &= \sqrt{\frac{2}{\sigma_m+\sigma_m^{-1}}} = \exp\left(\frac{1}{2}\eta_m\right)\end{aligned}$$

for some η_m with $|\eta_m| \le |s_m|$. Therefore,

$$\prod_{m=1}^{\infty}\int_{\mathbb{R}}\sqrt{\xi_{s,m}(y_m)}\mathrm{d}\gamma_m(y_m) = \exp\left(\frac{1}{2}\sum_{m=1}^{\infty}\eta_m\right) ,$$

which converges since $(s_m)_m \in \ell^1(\mathbb{N})$. Then the claim follows from Kakutani's theorem, see e.g. [8, Thm. 2.7] or [6, Thm. 2.12.7]. ∎

In particular, γ^P is equivalent to γ.

Define the set

$$\Gamma := \left\{ y \in \mathbb{R}^{\mathbb{N}} \, ; \, \sum_{m=1}^{\infty}\alpha_m |y_m| < \infty \right\} . \tag{15}$$

Lemma 2.12. $Y^\lambda(\Omega_\alpha) \subseteq \Gamma$.

Proof. Since $(\log\lambda_m) \in \ell^1(\mathbb{N})$, there exist constants $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$ with $\underline{\lambda} \le \lambda_m \le \overline{\lambda}$ for all $m \in \mathbb{N}$. Therefore,

$$\underline{\lambda}\sum_{m=1}^{\infty}\alpha_m|y_m| \le \sum_{m=1}^{\infty}\alpha_m\lambda_m|y_m| \le \overline{\lambda}\sum_{m=1}^{\infty}\alpha_m|y_m|$$

and

$$\Gamma = \left\{ y \in \mathbb{R}^{\mathbb{N}} \, ; \, \sum_{m=1}^{\infty}\alpha_m\lambda_m|y_m| < \infty \right\} .$$

Then the claim follows from (7) by inserting $y_m = (Y^\lambda(\omega))_m = \lambda_m^{-1}Y_m(\omega)$ for $\omega \in \Omega_\alpha$. ∎

We will reformulate Problem (5) with the parameter domain $\mathbb{R}^{\mathbb{N}}$ instead of Ω. For random variables $\varphi(\omega)$ that only depend on the values $(Y_m(\omega))_{m\in\mathbb{N}}$, we will write $\varphi^\lambda(y)$ or $\varphi^\lambda(Y^\lambda(\omega))$ for $\varphi(\omega)$. Note that the definition of $\varphi^\lambda(y)$ depends on $(\lambda_m)_{m\in\mathbb{N}}$. In particular,

$$a^\lambda(y,x) = a_{\min}(x) + a_0(x)\exp\left(\sum_{m=1}^{\infty}\lambda_m a_m(x)y_m\right) . \tag{16}$$

and

$$b^\lambda(y;w,v) = \int_D a^\lambda(y,x)\nabla w(x)\cdot\nabla v(x)\mathrm{d}x \qquad \text{for} \quad y\in\mathbb{R}^{\mathbb{N}}, \quad w,v\in V\ . \tag{17}$$

Then for all $v,w\in V$ and all $\omega\in\Omega_\alpha$, we have $b(\omega;w,v) = b^\lambda(Y^\lambda(\omega);w,v)$.

Assumption 2.13. *The right hand side f is of the form $f(\omega;\cdot) = f^\lambda(Y^\lambda(\omega);\cdot)$.*

Then (5) becomes the parametric deterministic problem

$$\text{find} \quad u^\lambda(y)\in V \quad \text{such that} \quad b^\lambda(y;u^\lambda(y),v) = f^\lambda(y;v) \qquad \forall v\in V\ , \quad \forall y\in\mathbb{R}^{\mathbb{N}}\ . \tag{18}$$

Theorem 2.14. *Problem* (18) *has a unique solution $u^\lambda(y)\in V$ for all $y\in\Gamma$. It is related to the solution $u(\omega)$ of* (5) *by $u(\omega) = u^\lambda(Y^\lambda(\omega))$ for all $\omega\in\Omega_\alpha$. In particular, it satisfies*

$$\left\|u^\lambda(y)\right\|_V \le \frac{1}{\underline{a}^\lambda(y)}\left\|f^\lambda(y;\cdot)\right\|_{V'} \quad \forall y\in\Gamma\ . \tag{19}$$

Proof. Let $y\in\Gamma$. By Lemma 2.5, the bilinear form $b^\lambda(y;\cdot,\cdot)$ is continuous and coercive on V with coercivity constant $\underline{a}^\lambda(y)$. The Lax-Milgram lemma implies existence and uniqueness of the solution $u^\lambda(y)$ to (18) and the bound (19).

Let $\omega\in\Omega_\alpha$. Then by definition, the solution $u(\omega)$ of (5) satisfies (18) for $y = Y^\lambda(\omega)\in\Gamma$. By uniqueness it follows that $u(\omega) = u^\lambda(Y^\lambda(\omega))$. □

Remark 2.15. By Proposition 2.3 and Proposition 2.11, Theorem 2.14 implies existence and uniqueness of the solution $u^\lambda(y)$ for γ-a.e. $y\in\mathbb{R}^{\mathbb{N}}$.

3 Weak formulation

3.1 Preliminaries

Assumption 3.1. *The right hand side $f^\lambda:\mathbb{R}^{\mathbb{N}}\to V'$ is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$-measurable.*

Product measurability of $a^\lambda(\cdot,\cdot)$ follows from Lemma 2.2 since $a^\lambda(\cdot,\cdot)$ can be written as the a.e. limit of product measurable functions.

We will consider probability measures as defined in Section 2.2 with

$$s_m = \frac{1}{2}\chi\alpha_m^\lambda = \frac{1}{2}\chi\lambda_m\alpha_m \tag{20}$$

for a parameter $\chi\ge 0$, where

$$\alpha_m^\lambda := \|\lambda_m a_m\|_{L^\infty(D)} = \lambda_m\alpha_m\ . \tag{21}$$

Denote by $\widehat{\gamma}_\chi$ the measure $\widehat{\gamma}_s$ for this choice of $(s_m)_{m\in\mathbb{N}}$, and by ξ_χ its density with respect to γ. By definition, $\gamma = \widehat{\gamma}_0$, which is compatible with the notation defined in (13).

Lemma 3.2. *If $0\le\chi\le 2\,\mathrm{e}^{-1}\min_m\alpha_m^{-1}$, then there is a unique sequence $(\lambda_m)_{m\in\mathbb{N}}$ in $[1,\mathrm{e}]$ such that $\gamma^P = \widehat{\gamma}_\chi$.*

Proof. By Remark 2.10, the values λ_m must satisfy

$$s_m^P = \log \lambda_m = \frac{1}{2}\chi\lambda_m\alpha_m \quad \forall m \in \mathbb{N} ,$$

or equivalently,

$$\frac{\log \lambda_m}{\lambda_m} = \frac{1}{2}\chi\alpha_m .$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\log x}{x} = \frac{1-\log x}{x^2} ,$$

the function $x^{-1}\log x$ takes its maximum e^{-1} at $x = \mathrm{e}$. Furthermore, $x^{-1}\log x = 0$ for $x = 1$. By continuity, $\lambda_m \in [1, \mathrm{e}]$ satisfying the claim exist if

$$\frac{1}{2}\chi\alpha_m \leq \frac{1}{\mathrm{e}}$$

and uniqueness of λ_m follows from monotonicity of $x^{-1}\log x$ on $[1, \mathrm{e}]$. □

Define

$$\begin{aligned} B_\chi^\lambda(w, v) &:= \int_{\mathbb{R}^\mathbb{N}} b^\lambda(y; w(y), v(y))\mathrm{d}\widehat{\gamma}_\chi(y) \\ &= \int_{\mathbb{R}^\mathbb{N}}\int_D a^\lambda(y, x)\nabla w(y, x)\cdot\nabla v(y, x)\mathrm{d}x\mathrm{d}\widehat{\gamma}_\chi(y) \end{aligned} \tag{22}$$

and

$$F_\chi^\lambda(v) := \int_{\mathbb{R}^\mathbb{N}} f^\lambda(y; v(y))\mathrm{d}\widehat{\gamma}_\chi(y) = \int_{\mathbb{R}^\mathbb{N}}\int_D f^\lambda(x)v(y, x)\mathrm{d}x\mathrm{d}\widehat{\gamma}_\chi(y) \tag{23}$$

for suitable w and v.

We will study the variational problem

$$B_\chi^\lambda(u^\lambda, v) = F_\chi^\lambda(v) \tag{24}$$

for all v in a suitable space.

We will use the notation $L^p(\widehat{\gamma})$ and $L^p(\widehat{\gamma}; V)$ for the spaces of p-integrable functions on $\mathbb{R}^\mathbb{N}$ with respect to the measure $\widehat{\gamma}$ with values in $\mathbb{R}$ and V, respectively. Furthermore, we will identify $L^2(\gamma)$ with its dual.

3.2 Uniqueness of the weak solution

Define

$$\mathcal{M} := \left\{ E \in \mathcal{B}(\mathbb{R}^\mathbb{N}) \; ; \; \operatorname*{ess\,sup}_{y\in E} \max\left(\left\| f^\lambda(y; \cdot)\right\|_{V'}, \underline{a}^\lambda(y)^{-1}, \overline{a}^\lambda(y)\right) < \infty \right\} . \tag{25}$$

Then $\mathcal{M}$ contains the nested sequence of sets

$$\Gamma^n := \left\{ y \in \mathbb{R}^\mathbb{N} \; ; \; \max\left(\left\| f^\lambda(y; \cdot)\right\|_{V'}, \underline{a}^\lambda(y)^{-1}, \overline{a}^\lambda(y)\right) < n \right\} , \quad n \in \mathbb{N} , \tag{26}$$

and the union of these sets is $\mathbb{R}^\mathbb{N}$. Also, $\mathcal{M}$ contains all Borel measurable subsets of any set $E \in \mathcal{M}$. Define

$$\mathcal{D}_0 := \{ w 1_E \; ; \; E \in \mathcal{M} , \; w \in V \} . \tag{27}$$

Theorem 3.3. *The solution $u^\lambda : \mathbb{R}^{\mathbb{N}} \to V$ of* (18) *satisfies* (24) *for all $v \in \mathcal{D}_0$. Furthermore, any function $\widetilde{u} : \mathbb{R}^{\mathbb{N}} \to V$ that satisfies* (24) *for all $v \in \mathcal{D}_0$ is equal to u^λ γ-a.e. .*

Proof. For all $v = w1_E \in \mathcal{D}_0$,

$$\int_{\mathbb{R}^{\mathbb{N}}} \left| f^\lambda(y; v(y)) \right| \mathrm{d}\widehat{\gamma}_\chi(y) = \int_E \left| f^\lambda(y; w) \right| \mathrm{d}\widehat{\gamma}_\chi(y) \leq \operatorname*{ess\,sup}_{y \in E} \left\| f^\lambda(y; \cdot) \right\|_{V'} \|w\|_V < \infty \, .$$

Therefore, the function $y \mapsto f^\lambda(y; v(y))$ is integrable and by (18), it is equal to the map $y \mapsto b^\lambda(y; u^\lambda(y), v(y))$ on Γ. This implies (24).

Let $\widetilde{u}$ also satisfy (24) for all $v = w1_E \in \mathcal{D}_0$. Then

$$\int_E b^\lambda(y; \widetilde{u}(y), w) - f^\lambda(y; w) \mathrm{d}\widehat{\gamma}_\chi(y) = 0 \quad \forall E \in \mathcal{M} \, , \quad \forall w \in V \, ,$$

so the integrand vanishes for γ-a.e. $y \in \mathbb{R}^{\mathbb{N}}$, and the second part of the claim follows from uniqueness of the solution u^λ of (18). □

As a result of Theorem 3.3, deriving well-posed weak formulations of (18) is essentially equivalent to showing integrability properties of the solution u^λ. These require the following Lemma.

Lemma 3.4. *The solution $u^\lambda : \mathbb{R}^{\mathbb{N}} \to V$ of* (18) *is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$-measurable.*

Proof. Let the sets Γ^n be defined as in (26). For each $n \in \mathbb{N}$, the bilinear forms $b^\lambda(y; \cdot, \cdot)$ are uniformly continuous and coercive for $y \in \Gamma^n$, and the right hand sides $f^\lambda(y; \cdot)$ are uniformly bounded for $y \in \Gamma^n$. By the Lax-Milgram lemma, the variational problem

$$\int_{\Gamma^n} b^\lambda(y; u^n(y), v(y)) \mathrm{d}\gamma(y) = \int_{\Gamma^n} f^\lambda(y; v(y)) \mathrm{d}\gamma(y) \quad \forall v \in L^2(\gamma|_{\Gamma^n}; V)$$

has a unique solution $u^n \in L^2(\gamma|_{\Gamma^n}; V)$. This solution can be extended by zero on $\mathbb{R}^{\mathbb{N}} \backslash \Gamma^n$ to define a function on $\mathbb{R}^{\mathbb{N}}$, which we denote again by u^n. By definition, this u^n is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$-measurable.[1] As in the proof of Theorem 3.3, it follows that $u^n(y)$ is equal to the solution $u^\lambda(y)$ of (18) for γ-a.e. $y \in \Gamma^n$. Since the union of the sets Γ^n is equal to $\mathbb{R}^{\mathbb{N}}$, the function u^λ is the pointwise limit of measurable functions, so it is also measurable. □

3.3 Weak formulation on a problem-dependent space

Define the space

$$L^{b^\lambda}(\widehat{\gamma}_\chi) := \left\{ v : \mathbb{R}^{\mathbb{N}} \to V \text{ measurable} \, ; \; B^\lambda_\chi(v, v) < \infty \right\} \, . \tag{28}$$

Lemma 3.5. *The bilinear form $B^\lambda_\chi(\cdot, \cdot)$ defines an inner product on $L^{b^\lambda}(\widehat{\gamma}_\chi)$.*

[1]Since V is separable, strong and weak measurability are equivalent, see e.g. [17].

Proof. For all $y \in \Gamma$, the bilinear form $b^\lambda(y;\cdot,\cdot)$ is an inner product on V. For $w, v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$, the map $\mathbb{R}^\mathbb{N} \ni y \mapsto b_y(w(y), v(y))$ is $\widehat{\gamma}_\chi$-integrable since for γ-a.e. $y \in \mathbb{R}^\mathbb{N}$

$$\begin{aligned}
&\int_{\mathbb{R}^\mathbb{N}} \left|b^\lambda(y; w(y), v(y))\right| \mathrm{d}\widehat{\gamma}_\chi(y) \\
&\qquad \le \int_{\mathbb{R}^\mathbb{N}} \sqrt{b^\lambda(y; w(y), w(y))}\sqrt{b^\lambda(y; v(y), v(y))}\mathrm{d}\widehat{\gamma}(y)_\chi \\
&\qquad \le \sqrt{\int_{\mathbb{R}^\mathbb{N}} b^\lambda(y; w(y), w(y))\mathrm{d}\widehat{\gamma}_\chi(y)}\sqrt{\int_{\mathbb{R}^\mathbb{N}} b^\lambda(y; v(y), v(y))\mathrm{d}\widehat{\gamma}_\chi(y)} < \infty\ .
\end{aligned}$$

Therefore, $B^\lambda_\chi(\cdot,\cdot)$ is well-defined on $L^{b^\lambda}(\widehat{\gamma}_\chi)$. Symmetry, bilinearity and positive semi-definiteness of $B^\lambda_\chi(\cdot,\cdot)$ follow from the corresponding properties of $b^\lambda(y;\cdot,\cdot)$ for γ-a.e. $y \in \mathbb{R}^\mathbb{N}$. Let $0 \neq v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$. Then $\gamma\big(\{y \in \mathbb{R}^\mathbb{N}\ ;\ b^\lambda(y; v(y), v(y)) > 0\}\big) > 0$ and therefore $B^\lambda_\chi(v, v) > 0$. ☐

Define the norm induced by the inner product $B^\lambda_\chi(\cdot,\cdot)$

$$\|v\|_{\lambda,\chi} := \sqrt{B^\lambda_\chi(v, v)} \qquad \text{for} \quad v \in L^{b^\lambda}(\widehat{\gamma}_\chi) \tag{29}$$

and denote its dual norm by $\|\cdot\|'_{\lambda,\chi}$. Also define the parametric norm

$$\|v\|_{\lambda,y} := \sqrt{b^\lambda(y; v, v)} \qquad \text{for} \quad v \in V\ , \quad y \in \Gamma\ . \tag{30}$$

Note that

$$\|v\|^2_{\lambda,\chi} = \int_{\mathbb{R}^\mathbb{N}} \|v(y)\|^2_{\lambda,y}\mathrm{d}\widehat{\gamma}_\chi(y) = \|\|v(y)\|_{\lambda,y}\|^2_{L^2(\widehat{\gamma}_\chi)} \qquad \text{for} \quad v \in L^{b^\lambda}(\widehat{\gamma}_\chi)\ . \tag{31}$$

Proposition 3.6. *The space $L^{b^\lambda}(\widehat{\gamma}_\chi)$ endowed with the inner product $B^\lambda_\chi(\cdot,\cdot)$ is a Hilbert space.*

Proof. It remains to be shown that $L^{b^\lambda}(\widehat{\gamma}_\chi)$ is complete with respect to the norm $\|\cdot\|_{\lambda,\chi}$. This is a direct adaptation of an argument for the corresponding property of standard L^2 spaces, cf. e.g. [15, Thm. 3.11]. Let (v_n) be a Cauchy sequence in $L^{b^\lambda}(\widehat{\gamma}_\chi)$. Without loss of generality we can assume $\|v_{n+1} - v_n\|_{\lambda,\chi} \le 2^{-n}$. Define

$$g_k(y) := \sum_{n=1}^{k} \|v_{n+1}(y) - v_n(y)\|_{\lambda,y}\ , \qquad g(y) := \sum_{n=1}^{\infty} \|v_{n+1}(y) - v_n(y)\|_{\lambda,y}$$

for $y \in \Gamma$. Then by the triangle inequality and (31),

$$\|g_k\|_{L^2(\widehat{\gamma}_\chi)} \le \sum_{n=1}^{k} \|v_{n+1} - v_n\|_{\lambda,\chi} \le 1\ .$$

Applying Fatou's Lemma, it follows that

$$\|g\|_{L^2(\widehat{\gamma}_\chi)} \leq \liminf_{k\to\infty} \|g_k\|_{L^2(\widehat{\gamma}_\chi)} \leq 1 \; .$$

In particular, $g(y) < \infty$ for γ-a.e. $y \in \mathbb{R}^{\mathbb{N}}$ and the series

$$v(y) := v_1(y) + \sum_{n=1}^{\infty} v_{n+1}(y) - v_n(y) = \lim_{n\to\infty} v_n(y)$$

converges in V for γ-a.e. $y \in \mathbb{R}^{\mathbb{N}}$. This implies that v is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$-measurable.

Let $\epsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $|\!|\!| v_n - v_m |\!|\!|_{\lambda,\chi} \leq \epsilon$ for all $n, m \geq N$. By Fatou's Lemma and (31),

$$|\!|\!| v - v_m |\!|\!|_{\lambda,\chi}^2 = \int_{\mathbb{R}^{\mathbb{N}}} \lim_{n\to\infty} |\!|\!| v_n(y) - v_m(y) |\!|\!|_{\lambda,y}^2 \mathrm{d}\widehat{\gamma}_\chi(y) \leq \liminf_{n\to\infty} |\!|\!| v_n - v_m |\!|\!|_{\lambda,\chi}^2 \leq \epsilon^2 \; .$$

It follows that $v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$ and $v_m \to v$ in $L^{b^\lambda}(\widehat{\gamma}_\chi)$. $\square$

Theorem 3.7. *For any* $f^\lambda \in \left(L^{b^\lambda}(\widehat{\gamma}_\chi)\right)'$*, the variational problem* (24) *with* $v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$ *has a unique solution* $u^\lambda \in L^{b^\lambda}(\widehat{\gamma}_\chi)$ *and*

$$|\!|\!| u^\lambda |\!|\!|_{\lambda,\chi} = |\!|\!| f^\lambda |\!|\!|_{\lambda,\chi}' \; . \tag{32}$$

Proof. This is a consequence of the Riesz isomorphism applied to the Hilbert space $L^{b^\lambda}(\widehat{\gamma}_\chi)$. $\square$

Corollary 3.8. *If* $f^\lambda \in \left(L^{b^\lambda}(\widehat{\gamma}_\chi)\right)'$*, then the solution* $u^\lambda : \mathbb{R}^{\mathbb{N}} \to V$ *of* (18) *is in* $L^{b^\lambda}(\widehat{\gamma}_\chi)$ *and satisfies* (24) *for all* $v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$.

Proof. This follows from Theorem 3.7 and Proposition 3.3 since $\mathcal{D}_0 \subseteq L^{b^\lambda}(\widehat{\gamma}_\chi)$. $\square$

Remark 3.9. Corollary 3.8 gives an alternative proof of Lemma 3.4 under the additional assumption $f^\lambda \in \left(L^{b^\lambda}(\widehat{\gamma}_\chi)\right)'$.

3.4 Weak formulation on $L^p(\gamma)$ spaces

In this section, we will consider the case $\chi = 0$ and $\lambda_m = 1$ for all $m \in \mathbb{N}$, i.e. we use the standard Gaussian measure γ to integrate over $\mathbb{R}^{\mathbb{N}}$, and this measure corresponds to the physical probability measure γ^P. All results in this section can be generalized to arbitrary $\widehat{\gamma}_s$ and (λ_m) by rescaling y and a_m.

Lemma 3.10.

$$\exp\left(\sum_{m=1}^{\infty} \alpha_m |y_m|\right) \in L^r(\gamma) \quad \forall r \in (0, \infty) \; .$$

Furthermore,

$$\left\|\exp\left(\sum_{m=1}^{\infty} \alpha_m |y_m|\right)\right\|_{L^r(\gamma)} \leq \exp\left(\frac{r}{2} \|(\alpha_m)_m\|_{\ell^2}^2 + \sqrt{\frac{2}{\pi}} \; \|(\alpha_m)_m\|_{\ell^1}\right) \; .$$

Proof. We will use the properties $(\alpha_m)_m \in \ell^1(\mathbb{N})$ and $\alpha_m \geq 0$. We will show the claim for $r = 1$; the general case follows since $(r\alpha_m)_m \in \ell^1(\mathbb{N})$ for all $r \in (0, \infty)$.

First consider the function

$$g(x) := \log(1 + \operatorname{erf}(x)) \ ,$$

where $\operatorname{erf}(\cdot)$ is the Gaussian error function

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp\bigl(-t^2\bigr) \, \mathrm{d}t \ . \tag{33}$$

For all $x \geq 0$, since $\operatorname{erf}(x) \geq 0$,

$$g'(x) = \frac{1}{1 + \operatorname{erf}(x)} \frac{2}{\sqrt{\pi}} \exp\bigl(-x^2\bigr) \leq \frac{2}{\sqrt{\pi}} \ .$$

Therefore, and since $g(0) = 0$, we have the estimate

$$g(x) \leq \frac{2}{\sqrt{\pi}} \, x \quad \forall x \geq 0 \ .$$

Using this bound, we can estimate the one-dimensional integrals

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\alpha \, |x| - \frac{x^2}{2}\right) \mathrm{d}x &= \sqrt{\frac{2}{\pi}} \exp\left(\frac{\alpha^2}{2}\right) \int_0^{\infty} \exp\left(-\frac{1}{2}(x - \alpha)\right) \mathrm{d}x \\
&= \exp\left(\frac{\alpha^2}{2}\right) \left(1 + \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}}\right)\right) \\
&= \exp\left(\frac{\alpha^2}{2} + g\left(\frac{\alpha}{\sqrt{2}}\right)\right) \\
&\leq \exp\left(\frac{\alpha^2}{2} + \sqrt{\frac{2}{\pi}} \, \alpha\right)
\end{aligned}$$

for $\alpha \geq 0$. Using dominated convergence and Prop. 2.3,

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{N}}} \exp\left(\sum_{m=1}^{\infty} \alpha_m \, |y_m|\right) \mathrm{d}\gamma(y) &= \prod_{m=1}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\alpha_m \, |y_m| - \frac{y_m^2}{2}\right) \mathrm{d}y_m \\
&\leq \prod_{m=1}^{\infty} \exp\left(\frac{\alpha_m^2}{2} + \sqrt{\frac{2}{\pi}} \, \alpha_m\right) \\
&= \exp\left(\sum_{m=1}^{\infty} \frac{\alpha_m^2}{2} + \sqrt{\frac{2}{\pi}} \, \alpha_m\right)
\end{aligned}$$

and the last term is finite since $(\alpha_m)_m \in \ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$. □

Lemma 3.11. *The functions $\overline{a}^\lambda(\cdot)$ and $\underline{a}^\lambda(\cdot)^{-1}$ are in $L^r(\gamma)$ for all $r \in (0, \infty)$. The norms satisfy*

$$\max\left(\left\|\overline{a}^\lambda\right\|_{L^r(\gamma)}, \left\|\underline{a}^\lambda(\cdot)^{-1}\right\|_{L^r(\gamma)}\right) \leq c_a \exp\left(\frac{r}{2} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right)$$

with a constant c_a independent of r.

Proof. By Lemma 3.10 and Lemma 2.5 with the bound

$$\underline{a}^\lambda(y) \geq \underline{a}_0 \exp\left(-\sum_{m=1}^{\infty} \alpha_m |y_m|\right) ,$$

the claim holds with

$$c_a = \max\left(\|a_{\min}\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)}, \left\|a_0^{-1}\right\|_{L^\infty(D)}\right) \exp\left(\sqrt{\frac{2}{\pi}} \|(\alpha_m)_m\|_{\ell^1}\right) .$$

□

Proposition 3.12. *Let p and $\widetilde{p}$ satisfy*

$$p, \widetilde{p} > 1 \quad \textit{and} \quad \frac{1}{p} + \frac{1}{\widetilde{p}} < 1 , \tag{34}$$

and let c_a be the constant from Lemma 3.11. Then for all $w \in L^p(\gamma; V)$ and $v \in L^{\widetilde{p}}(\gamma; V)$,

$$\left|B_0^\lambda(w, v)\right| \leq c_a \exp\left(\frac{p\widetilde{p}}{2(p-1)(\widetilde{p}-1) - 2} \|(\alpha_m)_m\|_{\ell^2}^2\right) \|w\|_{L^p(\gamma;V)} \|v\|_{L^{\widetilde{p}}(\gamma;V)} . \tag{35}$$

Moreover, for all $q \in (0, 2)$ and any $v \in L^q(\gamma; V)$,

$$B_0^\lambda(v, v) \geq c_a^{-1} \exp\left(\frac{-q}{2(2-q)} \|(\alpha_m)_m\|_{\ell^2}^2\right) \|v\|_{L^q(\gamma;V)}^2 . \tag{36}$$

Proof. Let $r = \frac{p\widetilde{p}}{(p-1)(\widetilde{p}-1)-1}$. Then by continuity of $b^\lambda(y; \cdot, \cdot)$ for $y \in \Gamma$,

$$\left|B_0^\lambda(w, v)\right| \leq \int_{\mathbb{R}^\mathbb{N}} \overline{a}^\lambda(y) \|w(y)\|_V \|v(y)\|_V \,\mathrm{d}\gamma(y) \leq \left\|\overline{a}^\lambda\right\|_{L^r(\gamma)} \|w\|_{L^p(\gamma;V)} \|v\|_{L^{\widetilde{p}}(\gamma;V)}$$

and (35) follows from Lemma 3.11. Now let $r = \frac{q}{2-q}$. Using coercivity of $b^\lambda(y; \cdot, \cdot)$ for $y \in \Gamma$ and the reverse Hölder inequality, we obtain

$$B_0^\lambda(v, v) \geq \int_{\mathbb{R}^\mathbb{N}} \underline{a}^\lambda(y) \|v(y)\|_V^2 \,\mathrm{d}\gamma(y) \geq \left\|\underline{a}^\lambda\right\|_{L^{-r}(\gamma)} \|v\|_{L^q(\gamma;V)}^2 .$$

Equation (36) follows from Lemma 3.11 since

$$\left\|\underline{a}^\lambda\right\|_{L^{-r}(\gamma)} = \left\|\underline{a}^\lambda(\cdot)^{-1}\right\|_{L^r(\gamma)}^{-1} .$$

□

Note that if $p = \widetilde{p}$ in Proposition 3.12, then $p > 2$ and (35) reads

$$\left|B_0^\lambda(w, v)\right| \leq c_a \exp\left(\frac{p}{2(p-2)} \|(\alpha_m)_m\|_{\ell^2}^2\right) \|w\|_{L^p(\gamma;V)} \|v\|_{L^p(\gamma;V)} . \tag{37}$$

Corollary 3.13. *For all $q < 2 < p$,*

$$L^p(\gamma;V) \subseteq L^{b^\lambda}(\gamma) \subseteq L^q(\gamma;V)$$

and

$$L^p(\gamma;V') \subseteq \left(L^{b^\lambda}(\gamma)\right)' \subseteq L^q(\gamma;V') \ .$$

Proof. The first part of the claim follows from (37) and (36). The second part is a direct consequence of the first since

$$(L^p(\gamma;V))' \cong L^{\frac{p}{p-1}}(\gamma;V') \ .$$

$\square$

Theorem 3.14. *Let $q > 0$. If $f^\lambda \in L^p(\gamma;V')$ for a $p > q$, then the solution u^λ of (18) is in $L^p(\gamma;V)$ and satisfies*

$$\left\|u^\lambda\right\|_{L^q(\gamma;V)} \leq c_a \exp\left(\frac{qp}{2(p-q)} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right) \left\|f^\lambda\right\|_{L^p(\gamma;V')} \ . \tag{38}$$

Furthermore, if q, $\widetilde{q}$ are related as in (34), then u^λ is the essentially unique solution of (24) with test functions $v \in L^{\widetilde{q}}(\gamma;V)$.

Proof. Let $r = \frac{qp}{p-q}$. By (19) and Hölder's inequality,

$$\begin{aligned}\int\limits_{\mathbb{R}^\mathbb{N}} \left\|u^\lambda(y)\right\|_V^q \,\mathrm{d}\gamma(y) &\leq \int\limits_{\mathbb{R}^\mathbb{N}} \underline{a}^\lambda(y)^{-q} \left\|f^\lambda(y;\cdot)\right\|_{V'}^q \,\mathrm{d}\gamma(y) \\ &\leq \left\|\underline{a}^\lambda(\cdot)^{-1}\right\|_{L^r(\gamma)}^q \left\|f^\lambda\right\|_{L^p(\gamma;V')}^q \ ,\end{aligned}$$

and (38) follows using Lemma 3.11.

If q, $\widetilde{q}$ satisfy (34), then (24) makes sense for $u^\lambda \in L^q(\gamma;V)$ and $v \in L^{\widetilde{q}}(\gamma;V)$ because of (35). The solution u^λ of (18) solves (24) for all $v \in L^{\widetilde{q}}(\gamma;V)$ since $b^\lambda(y;u^\lambda(y),v(y)) = f^\lambda(y;v(y))$ for all $y \in \Gamma$. Uniqueness of the solution of (24) for $v \in L^{\widetilde{q}}(\gamma;V)$ follows from Theorem 3.3 since $\mathcal{D}_0 \subseteq L^{\widetilde{q}}(\gamma;V)$. $\square$

3.5 Weak formulation with modified measures

The weak formulation of (24) presented in Section 3.4 requires Banach spaces that are not Hilbert spaces since the estimates in Proposition 3.12 do not hold for $p = \widetilde{p} = 2$ or $q = 2$. Similar results using only Hilbert spaces are preferable, in particular due to the possibility of applying Parseval's identity in connection with an orthonormal basis.

We derive such results in this section for Hilbert spaces $L^2(\widehat{\gamma};V)$ with $\widehat{\gamma}$ potentially different from, but equivalent to γ. In particular, we consider the Gaussian measures $\widehat{\gamma}_\chi$ with arbitrary $\chi > 0$. We allow general $(\lambda_m)_{m\in\mathbb{N}}$ in this section; this sequence will be specified below in such a way that our results apply to the physical probability measure γ^P. We recall the definitions

$$s_m = \frac{1}{2}\chi\alpha_m^\lambda = \frac{1}{2}\chi\lambda_m\alpha_m \quad \text{and} \quad \sigma_m = \exp(-s_m) \ .$$

Lemma 3.15. *For all $k > 0$,*

$$\xi_{k\chi}(y) \exp\left(k \sum_{m=1}^{\infty} \alpha_m^{\lambda} |y_m| \right) \leq \exp\left(k \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^{\lambda})_m \right\|_{\ell^1} \right) \quad \forall y \in \Gamma \ .$$

Proof. Let $y \in \Gamma$. Using $\sigma_m^{-2k} - 1 = \exp(2ks_m) - 1 \geq 2ks_m$, we estimate

$$\begin{aligned}
&\xi_{k\chi}(y) \exp\left(k \sum_{m=1}^{\infty} \alpha_m^{\lambda} |y_m| \right) \\
&\qquad = \exp\left(k \sum_{m=1}^{\infty} \alpha_m^{\lambda} |y_m| - \frac{1}{2} \sum_{m=1}^{\infty} (\sigma_m^{-2k} - 1) y_m^2 - k \sum_{m=1}^{\infty} \log \sigma_m \right) \\
&\qquad \leq \exp\left(k \sum_{m=1}^{\infty} \alpha_m^{\lambda} |y_m| - k \sum_{m=1}^{\infty} s_m y_m^2 + k \sum_{m=1}^{\infty} s_m \right) \\
&\qquad = \exp\left(-k \sum_{m=1}^{\infty} s_m \left(|y_m| - \frac{1}{\chi} \right)^2 + k \sum_{m=1}^{\infty} \frac{s_m}{\chi^2} + k \sum_{m=1}^{\infty} s_m \right) \\
&\qquad \leq \exp\left(k \sum_{m=1}^{\infty} \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \alpha_m^{\lambda} \right) \ .
\end{aligned}$$

□

Lemma 3.16.

$$\xi_{2\chi}(y) \xi_{\chi}(y)^{-1} \exp\left(\sum_{m=1}^{\infty} \alpha_m^{\lambda} |y_m| \right) \leq \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^{\lambda})_m \right\|_{\ell^1} \right) \quad \forall y \in \Gamma \ .$$

Proof. Let $y \in \Gamma$. By Proposition 2.11, using that the standard deviation of $\widehat{\gamma}_{2\chi,m}$ is σ_m^2,

$$\xi_{2\chi}(y) \xi_{\chi}(y)^{-1} = \left(\prod_{m=1}^{\infty} \frac{1}{\sigma_m} \right) \exp\left(-\frac{1}{2} \sum_{m=1}^{\infty} (\sigma_m^{-2} - 1) \sigma_m^{-2} y_m^2 \right) \ .$$

Then the claim follows as in the proof of Lemma 3.15 with the estimate

$$(\sigma_m^{-2} - 1) \sigma_m^{-2} = (\exp(2s_m) - 1) \exp(2s_m) \geq 2s_m \ .$$

□

Note that the right hand sides of the estimates in Lemma 3.15 and Lemma 3.16 are minimal for $\chi = 1$. They tend to ∞ as $\chi \to 0$.

Lemma 3.17. *The functions*

$$\xi_{\chi}(\cdot) \overline{a}^{\lambda}(\cdot) \ , \quad \xi_{\chi}(\cdot)^{-1} \xi_{2\chi}(\cdot) \underline{a}^{\lambda}(\cdot)^{-1} \quad \textit{and} \quad \sqrt{\xi_{2\chi}(\cdot) \underline{a}^{\lambda}(\cdot)^{-2}}$$

are in $L^{\infty}(\gamma)$. They are bounded by

$$c_a \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^{\lambda})_m \right\|_{\ell^1} \right)$$

for a constant c_a independent of χ.

Proof. By Lemmas 3.15, 3.16 and 2.5 with the bounds

$$\underline{a}^\lambda(y) \geq \underline{a}_0 \exp\left(- \sum_{m=1}^{\infty} \alpha_m^\lambda \left| y_m \right| \right)$$

and

$$\begin{aligned} \overline{a}^\lambda(y) &\leq \|a_{\min}\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\sum_{m=1}^{\infty} \alpha_m^\lambda \left| y_m \right| \right) \\ &\leq \left(\|a_{\min}\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \right) \exp\left(\sum_{m=1}^{\infty} \alpha_m^\lambda \left| y_m \right| \right) \ , \end{aligned}$$

the claim holds with

$$c_a = \max\left(\|a_{\min}\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \, , \left\| a_0^{-1} \right\|_{L^\infty(D)} \right) \ .$$

□

Proposition 3.18. *For all $w, v \in L^2(\gamma; V)$,*

$$\left| B_\chi^\lambda(w, v) \right| \leq c_a \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^\lambda)_m \right\|_{\ell^1} \right) \|w\|_{L^2(\gamma;V)} \|v\|_{L^2(\gamma;V)} \tag{39}$$

and

$$B_\chi^\lambda(v, v) \geq c_a^{-1} \exp\left(- \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^\lambda)_m \right\|_{\ell^1} \right) \|v\|_{L^2(\widehat{\gamma}_{2\chi};V)}^2 \ , \tag{40}$$

where c_a is the constant from Lemma 3.17.

Proof. By continuity of $b^\lambda(y; \cdot, \cdot)$ for $y \in \Gamma$,

$$\begin{aligned} \left| B_\chi^\lambda(w, v) \right| &\leq \int_{\mathbb{R}^\mathbb{N}} \xi_\chi(y) \overline{a}^\lambda(y) \|w(y)\|_V \|v(y)\|_V \, \mathrm{d}\gamma(y) \\ &\leq \left\| \xi_\chi \overline{a}^\lambda \right\|_{L^\infty(\gamma)} \|w\|_{L^2(\gamma;V)} \|v\|_{L^2(\gamma;V)} \end{aligned}$$

and (39) follows from Lemma 3.17. Using coercivity of $b^\lambda(y; \cdot, \cdot)$ for $y \in \Gamma$, we obtain

$$\begin{aligned} B_\chi^\lambda(v, v) &\geq \int_{\mathbb{R}^\mathbb{N}} \xi_\chi(y) \xi_{2\chi}^{-1}(y) \underline{a}^\lambda(y) \|v(y)\|_V^2 \, \xi_{2\chi}(y) \mathrm{d}\gamma(y) \\ &\geq \left\| \xi_\chi(\cdot)^{-1} \xi_{2\chi}(\cdot) \underline{a}^\lambda(\cdot)^{-1} \right\|_{L^\infty(\gamma)}^{-1} \|v\|_{L^2(\widehat{\gamma}_{2\chi};V)}^2 \end{aligned}$$

and (40) follows from Lemma 3.17. □

Corollary 3.19. *For all $\chi > 0$,*

$$L^2(\gamma; V) \subseteq L^{b^\lambda}(\widehat{\gamma}_\chi) \subseteq L^2(\widehat{\gamma}_{2\chi}; V)$$

and

$$\left(L^{b^\lambda}(\widehat{\gamma}_\chi) \right)' \subseteq L^2(\gamma; V') \ .$$

Proof. This is a direct consequence of (39) and (40). ☐

Theorem 3.20. *If $f^\lambda \in L^2(\gamma; V')$, then the solution u^λ of* (18) *is in $L^2(\widehat{\gamma}_{2\chi}; V)$ for all $\chi > 0$ and satisfies*

$$\left\|u^\lambda\right\|_{L^2(\widehat{\gamma}_{2\chi};V)} \leq c_a \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2}\right) \left\|(\alpha_m^\lambda)_m\right\|_{\ell^1}\right) \left\|f^\lambda\right\|_{L^2(\gamma;V')} . \tag{41}$$

Furthermore, if $f^\lambda \in L^p(\gamma; V')$ with $p > 2$, u^λ is the essentially unique solution of (24) *in $L^2(\gamma; V)$ with test functions $v \in L^2(\gamma; V)$.*

Proof. By (19),

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{N}}} \left\|u^\lambda(y)\right\|_V^2 \xi_{2\chi}(y)\mathrm{d}\gamma(y) &\leq \int_{\mathbb{R}^{\mathbb{N}}} \xi_{2\chi}(y)\underline{a}^\lambda(y)^{-2} \left\|f^\lambda(y;\cdot)\right\|_{V'}^2 \mathrm{d}\gamma(y) \\ &\leq \left\|\xi_{2\chi}(\cdot)\underline{a}^\lambda(\cdot)^{-2}\right\|_{L^\infty(\gamma)} \left\|f^\lambda\right\|_{L^2(\gamma;V')}^2 ,\end{aligned}$$

and (41) follows using Lemma 3.17 .

If $f^\lambda \in L^p(\gamma; V')$, then $u^\lambda \in L^2(\gamma; V)$ by Theorem 3.14 and (24) makes sense for $v \in L^2(\gamma; V)$ because of (39). The solution u^λ of (18) solves (24) for all $v \in L^2(\gamma; V)$ since $b^\lambda(y; u^\lambda(y), v(y)) = f^\lambda(y; v(y))$ for all $y \in \mathit{\Gamma}$. Uniqueness of the solution of (24) for $v \in L^2(\gamma; V)$ follows from Theorem 3.3 since $\mathcal{D}_0 \subseteq L^2(\gamma; V)$. ☐

3.6 Coercive case

Some of the estimates in Sections 3.4 and 3.5 can be improved under the additional assumption

$$\operatorname*{ess\,inf}_{x\in D} a_{\min}(x) > 0 . \tag{42}$$

We assume for simplicity that the constant c_a from Lemma 3.11 for $\chi = 0$ and from Lemma 3.17 for $\chi > 0$ is an upper bound for $\frac{1}{a_{\min}}$.

Lemma 3.21. *If* (42) *holds, then the function $\underline{a}^\lambda(\cdot)^{-1}$ is in $L^\infty(\gamma)$ and*

$$\left\|\underline{a}^\lambda(\cdot)^{-1}\right\|_{L^\infty(\gamma)} \leq c_a .$$

Proof. This follows from (42) and Lemma 2.5 with the bound

$$\underline{a}^\lambda(y) \geq \operatorname*{ess\,inf}_{x\in D} a_{\min}(x) .$$

☐

Proposition 3.22. *Let $\chi \geq 0$. If* (42) *holds, then for all $v \in L^{b^\lambda}(\widehat{\gamma}_\chi)$,*

$$B_\chi^\lambda(v, v) \geq c_a^{-1} \left\|v\right\|_{L^2(\widehat{\gamma}_\chi;V)}^2 . \tag{43}$$

In particular,

$$L^{b^\lambda}(\widehat{\gamma}_\chi) \subseteq L^2(\widehat{\gamma}_\chi; V) \quad \textit{and} \quad L^2(\widehat{\gamma}_\chi; V') \subseteq \left(L^{b^\lambda}(\widehat{\gamma}_\chi)\right)' . \tag{44}$$

Proof. Using coercivity of $b^\lambda(y;\cdot,\cdot)$ for $y \in \Gamma$ and Lemma 3.21,

$$B_\chi^\lambda(v,v) \geq \int_{\mathbb{R}^{\mathbb{N}}} \underline{a}^\lambda(y) \|v(y)\|_V^2 \, \mathrm{d}\widehat{\gamma}_\chi(y) \geq c_a^{-1} \|v\|_{L^2(\widehat{\gamma}_\chi;V)}^2 \ .$$

$\square$

Theorem 3.23. *Let $\chi \geq 0$. If $f^\lambda \in L^p(\widehat{\gamma}_\chi;V')$ for a $p > 0$, then the solution u^λ of (18) is in $L^p(\widehat{\gamma}_\chi;V)$ and satisfies*

$$\left\|u^\lambda\right\|_{L^p(\widehat{\gamma}_\chi;V)} \leq c_a \left\|f^\lambda\right\|_{L^p(\widehat{\gamma}_\chi;V')} \ . \tag{45}$$

Proof. By (19),

$$\begin{aligned}\int_{\mathbb{R}^{\mathbb{N}}} \left\|u^\lambda(y)\right\|_V^p \mathrm{d}\widehat{\gamma}_\chi(y) &\leq \int_{\mathbb{R}^{\mathbb{N}}} \underline{a}^\lambda(y)^{-p} \left\|f^\lambda(y;\cdot)\right\|_{V'}^p \mathrm{d}\widehat{\gamma}_\chi(y) \\ &\leq \left\|\underline{a}^\lambda(\cdot)^{-1}\right\|_{L^\infty(\gamma)}^p \left\|f^\lambda\right\|_{L^p(\widehat{\gamma}_\chi;V')}^p \ ,\end{aligned}$$

and (45) follows using Lemma 3.21. $\square$

4 Galerkin discretization

4.1 Well-posed problem

Let $\chi \geq 0$ and let $\mathcal{V}_N$ be a finite-dimensional subspace of $L^{b^\lambda}(\widehat{\gamma}_\chi)$.

Theorem 4.1. *For any $f^\lambda \in \left(L^{b^\lambda}(\widehat{\gamma}_\chi)\right)'$, the variational problem (24) with test functions $v_N \in \mathcal{V}_N$ has a unique solution $u_N^\lambda \in \mathcal{V}_N$ and*

$$\|u_N^\lambda\|_{\lambda,\chi} \leq \|f^\lambda\|_{\lambda,\chi}' \ . \tag{46}$$

Proof. As a closed subspace of a Hilbert space, $\mathcal{V}_N$ endowed with the inner product $B_\chi^\lambda(\cdot,\cdot)$ is also a Hilbert space. The claim is a consequence of the Riesz isomorphism in $\mathcal{V}_N$. $\square$

Let $u^\lambda \in L^{b^\lambda}(\widehat{\gamma}_\chi)$ be as in Theorem 3.7.

Lemma 4.2. *The Galerkin solution u_N^λ of (24) with respect to the measure $\widehat{\gamma}_\chi$ is the unique element of $\mathcal{V}_N$ with*

$$\|u^\lambda - u_N^\lambda\|_{\lambda,\chi} = \inf_{v_N \in \mathcal{V}_N} \|u^\lambda - v_N\|_{\lambda,\chi} \ .$$

Proof. By definition, u_N^λ is the $B_\chi^\lambda(\cdot,\cdot)$-orthogonal projection of u^λ onto $\mathcal{V}_N$. $\square$

Note that, unlike the exact solution u^λ, the Galerkin solution u_N^λ of (24) does depend on the choice of measure $\widehat{\gamma}_\chi$.

4.2 Quasi-optimality in the general setting

Theorem 4.3. *Let $\chi = 0$ and $\lambda_m = 1$ for all $m \in \mathbb{N}$. If $f^\lambda \in L^p(\gamma; V')$ for a $p > 2$, then for all $q, \widetilde{p}$ satisfying $0 < q < 2 < \widetilde{p} < p$,*

$$\begin{aligned}
&\left\|u^\lambda - u_N^\lambda\right\|_{L^q(\gamma;V)} \\
&\qquad \le c_a \exp\left(\frac{\widetilde{p} - q}{2(2-q)(\widetilde{p}-2)} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right) \inf_{v_N \in \mathcal{V}_N} \left\|u^\lambda - v_N\right\|_{L^{\widetilde{p}}(\gamma;V)} .
\end{aligned}$$

Proof. Define the continuity constant of $B(\cdot,\cdot)$ on $L^{\widetilde{p}}(\gamma; V)$ from (37)

$$\overline{c} := c_a \exp\left(\frac{\widetilde{p}}{2(\widetilde{p}-2)} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right)$$

and the coercivity constant on $L^q(\gamma; V)$ from (36)

$$\underline{c} := c_a^{-1} \exp\left(\frac{-q}{2(2-q)} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right) .$$

Then by Lemma 4.2 and Proposition 3.12,

$$\begin{aligned}
\sqrt{\underline{c}}\left\|u^\lambda - u_N^\lambda\right\|_{L^q(\gamma;V)} &\le \left|\!\left|\!\left|u^\lambda - u_N^\lambda\right|\!\right|\!\right|_{\lambda,0} \\
&= \inf_{v_N \in \mathcal{V}_N} \left|\!\left|\!\left|u^\lambda - v_N\right|\!\right|\!\right|_{\lambda,0} \\
&\le \sqrt{\overline{c}} \inf_{v_N \in \mathcal{V}_N} \left\|u^\lambda - v_N\right\|_{L^{\widetilde{p}}(\gamma;V)} .
\end{aligned}$$

□

In particular, if $q = 2 - \epsilon$ and $\widetilde{p} = 2 + \epsilon$ for an $0 < \epsilon < \min(2, p-2)$, then

$$\left\|u^\lambda - u_N^\lambda\right\|_{L^{2-\epsilon}(\gamma;V)} \le c_a \exp\left(\frac{1}{\epsilon} \left\|(\alpha_m)_m\right\|_{\ell^2}^2\right) \inf_{v_N \in \mathcal{V}_N} \left\|u^\lambda - v_N\right\|_{L^{2+\epsilon}(\gamma;V)} .$$

Theorem 4.4. *Let $\chi > 0$ and $f^\lambda \in L^p(\gamma; V')$ for a $p > 2$. Then*

$$\begin{aligned}
&\left\|u^\lambda - u_N^\lambda\right\|_{L^2(\widehat{\gamma}_{2\chi};V)} \\
&\qquad \le c_a \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2}\right) \left\|(\alpha_m^\lambda)_m\right\|_{\ell^1}\right) \inf_{v_N \in \mathcal{V}_N} \left\|u^\lambda - v_N\right\|_{L^2(\gamma;V)} .
\end{aligned}$$

Proof. Note that the continuity constant of $B_\chi^\lambda(\cdot,\cdot)$ on $L^2(\gamma; V)$ and the inverse of the coercivity constant on $L^2(\widehat{\gamma}_{2\chi}; V)$ given in Proposition 3.18 are equal to

$$\overline{c} := c_a \exp\left(\left(\frac{1}{2\chi} + \frac{\chi}{2}\right) \left\|(\alpha_m^\lambda)_m\right\|_{\ell^1}\right) .$$

Therefore, using Lemma 4.2,

$$\begin{aligned}
\left\|u^\lambda - u_N^\lambda\right\|_{L^2(\widehat{\gamma}_{2\chi};V)} &\le \sqrt{\overline{c}} \left|\!\left|\!\left|u^\lambda - u_N^\lambda\right|\!\right|\!\right|_{\lambda,\chi} \\
&= \sqrt{\overline{c}} \inf_{v_N \in \mathcal{V}_N} \left|\!\left|\!\left|u^\lambda - v_N\right|\!\right|\!\right|_{\lambda,\chi} \\
&\le \overline{c} \inf_{v_N \in \mathcal{V}_N} \left\|u^\lambda - v_N\right\|_{L^2(\gamma;V)} .
\end{aligned}$$

□

Remark 4.5. Note that the estimate in Theorem 4.3 applies directly to γ^P since $\gamma^P = \gamma$ for $\lambda_m = 1$. For Theorem 4.4 to apply to γ^P, we need to select λ_m such that $\gamma^P = \widehat{\gamma}_{2\chi}$. By Lemma 3.2, this is possible if

$$0 \le \chi \le \frac{1}{\mathrm{e}} \min_m \alpha_m^{-1} , \tag{47}$$

and the resulting λ_m satisfy $1 \le \lambda_m \le \mathrm{e}$ for all $m \in \mathbb{N}$.

4.3 Quasi-optimality in the coercive case

If (42) holds, then Theorems 4.3 and 4.4 can be strengthened.

Theorem 4.6. *Let $\chi = 0$ and $\lambda_m = 1$ for all $m \in \mathbb{N}$. If (42) holds and $f^\lambda \in L^p(\gamma; V')$ for a $p > 2$, then*

$$\left\| u^\lambda - u_N^\lambda \right\|_{L^2(\gamma;V)} \le c_a \exp\left(\frac{p}{4(p-2)} \left\| (\alpha_m)_m \right\|_{\ell^2}^2 \right) \inf_{v_N \in \mathcal{V}_N} \left\| u^\lambda - v_N \right\|_{L^p(\gamma;V)} .$$

Proof. The continuity constant of $B_0^\lambda(\cdot,\cdot)$ on $L^{2+\epsilon}(\gamma; V)$ from (37) is

$$\overline{c} := c_a \exp\left(\frac{p}{2(p-2)} \left\| (\alpha_m)_m \right\|_{\ell^2}^2 \right)$$

and the coercivity constant on $L^2(\gamma; V)$ from (36) is $\underline{c} := c_a^{-1}$. Then the claim follows as in the proof of Theorem 4.3. □

Theorem 4.7. *Let $\chi > 0$. If (42) holds and $f^\lambda \in L^2(\gamma; V')$, then*

$$\begin{aligned} &\left\| u^\lambda - u_N^\lambda \right\|_{L^2(\widehat{\gamma}_\chi;V)} \\ &\qquad \le c_a \exp\left(\frac{1}{2} \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \left\| (\alpha_m^\lambda)_m \right\|_{\ell^1} \right) \inf_{v_N \in \mathcal{V}_N} \left\| u^\lambda - v_N \right\|_{L^2(\gamma;V)} . \end{aligned}$$

Proof. The proof is the same as that of Theorem 4.4, with the coercivity constant c_a^{-1} on $L^2(\widehat{\gamma}_\chi; V)$. □

Remark 4.8. Note that the estimate in Theorem 4.6 applies directly to γ^P since $\gamma^P = \gamma$ for $\lambda_m = 1$. For Theorem 4.7 to apply to γ^P, we need to select λ_m such that $\gamma^P = \widehat{\gamma}_\chi$. By Lemma 3.2, this is possible if

$$0 \le \chi \le \frac{2}{\mathrm{e}} \min_m \alpha_m^{-1} , \tag{48}$$

and the resulting λ_m satisfy $1 \le \lambda_m \le \mathrm{e}$ for all $m \in \mathbb{N}$.

4.4 Tensor-product Hermite basis

Let h_n denote the Hermite polynomial of degree $n \in \mathbb{N}_0$ on $\mathbb{R}$, scaled such that $(h_n)_{n\in\mathbb{N}_0}$ forms an orthonormal basis of $L^2(\mathbb{R})$ with respect to the standard Gaussian measure. It can be expressed as

$$h_n(t) = \frac{(-1)^n}{\sqrt{n!}} \exp\left(\frac{t^2}{2} \right) \frac{\mathrm{d}^n}{\mathrm{d}t^n} \exp\left(\frac{-t^2}{2} \right) , \tag{49}$$

see for example [6, Sec. 1.3] or [8, Sec 9.2.1]. For $\mu \in \mathbb{N}_0^{\mathbb{N}}$, define the support

$$\operatorname{supp}\mu := \{m \in \mathbb{N} \,;\, \mu_m \neq 0\} \ . \tag{50}$$

Define the index set of finitely supported sequences in $\mathbb{N}_0$,

$$\Lambda := \left\{\mu \in \mathbb{N}_0^{\mathbb{N}} \,;\, \#\operatorname{supp}\mu < \infty\right\} \ . \tag{51}$$

For $\mu \in \Lambda$, we define the tensor product Hermite polynomial of degree μ by

$$H_\mu := \bigotimes_{m=1}^{\infty} h_{\mu_m} \ , \tag{52}$$

i.e. for any $y \in \mathbb{R}^{\mathbb{N}}$, since $h_0 = 1$,

$$H_\mu(y) = \prod_{m=1}^{\infty} h_{\mu_m}(y_m) = \prod_{m \in \operatorname{supp}\mu} h_{\mu_m}(y_m) \ . \tag{53}$$

Lemma 4.9. *The set $\boldsymbol{H} := (H_\mu)_{\mu\in\Lambda}$ is an orthonormal basis of $L^2(\gamma)$.*

We refer to [8, Theorem 9.7] for a proof of Lemma 4.9. By Parseval's identity, the map

$$T_{\boldsymbol{H}} : \ell^2(\Lambda) \to L^2(\gamma) \ , \quad \boldsymbol{v} \mapsto \sum_{\mu\in\Lambda} v_\mu H_\mu \tag{54}$$

is an isometric isomorphism of Hilbert spaces. As a direct consequence, the map

$$T_{\boldsymbol{H}}^V := T_{\boldsymbol{H}} \otimes \mathrm{id}_V : \ell^2(\Lambda; V) \to L^2(\gamma; V) \ , \quad \boldsymbol{v} \mapsto \sum_{\mu\in\Lambda} H_\mu \otimes v_\mu \tag{55}$$

is also an isometric isomorphism of Hilbert spaces.[2] Its inverse expands any $v \in L^2(\gamma; V)$ into a square summable sequence of Hermite coefficients $v_\mu \in V$ for $\mu \in \Lambda$.

4.5 Error bounds

In this section, we will apply the quasi-optimality results from Sections 4.2 and 4.3 to finite element spaces constructed by approximating each Hermite coefficient in a given deterministic finite element space. We will formulate the results on the original probability space $(\Omega, \mathcal{F}, P)$. Note that if $u_N^\lambda : \mathbb{R}^{\mathbb{N}} \to V$ is an approximation of u^λ, then $u_N := u_N^\lambda \circ Y^\lambda$ is the corresponding approximation of u.

Assumption 4.10.

$$\mathcal{V}_N = \left\{v \in L^2(\gamma; V) \,;\, v_\mu \in V_{N,\mu} \ \ \forall \mu \in \Lambda\right\} \ , \tag{56}$$

where $V_{N,\mu} \subseteq V$ is a finite dimensional subspace for all $\mu \in \Lambda$, and $V_{N,\mu} = \{0\}$ for all but finitely many $\mu \in \Lambda$.

[2] We identify $L^2(\gamma; V)$ with $L^2(\gamma) \otimes V$ and $\ell^2(\Lambda; V)$ with $\ell^2(\Lambda) \otimes V$.

Theorem 4.11. *Let $f^\lambda \in L^p(\gamma; V')$ for a $p > 2$. If $\chi = 0$ and $\lambda_m = 1$ for all $m \in \mathbb{N}$, then for all $q, \widetilde{p}$ satisfying $0 < q < 2 < \widetilde{p} < p$,*

$$\mathbb{E}_P(\|u - u_N\|_V^q)^{\frac{1}{q}} \leq c_a \exp\left(\frac{\widetilde{p} - q}{2(2-q)(\widetilde{p}-2)} \|(\alpha_m)_m\|_{\ell^2}^2 \right) \sum_{\mu \in \Lambda} \|H_\mu\|_{L^{\widetilde{p}}(\gamma)} \inf_{w_N \in V_{N,\mu}} \left\| u_\mu^\lambda - w_N \right\|_V . \quad (57)$$

If $0 < \chi \leq \mathrm{e}^{-1} \min_m \alpha_m^{-1}$ and $(\lambda_m)_{m \in \mathbb{N}}$ is chosen as in Remark 4.5, then

$$\mathbb{E}_P\left(\|u - u_N\|_V^2 \right)^{\frac{1}{2}} \leq c_a \exp\left(\mathrm{e} \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \|(\alpha_m)_m\|_{\ell^1} \right) \left(\sum_{\mu \in \Lambda} \inf_{w_N \in V_{N,\mu}} \left\| u_\mu^\lambda - w_N \right\|_V^2 \right)^{\frac{1}{2}} . \quad (58)$$

Proof. Equation (57) follows from Theorem 4.3 by applying the triangle inequality to the right hand side. Equation (58) follows from Theorem 4.4 and Lemma 4.9 using Parseval's identity. □

Note that (58) can be used to bound the convergence of the covariance function

$$C_u(x, x') := \mathbb{E}_P((u(x) - \mathbb{E}_P(u(x)))\,(u(x') - \mathbb{E}_P(u(x')))) \quad (59)$$

in $V \otimes V$. This follows immediately by using the tensor-product structure of the spaces involved. Because of the restriction $q < 2$, (57) does not apply to the second moment. However, it is still sufficient for bounding the convergence of the mean field $E_u(x) := \mathbb{E}_P(u(x))$ in V with $q = 1$. The following theorem uses an additional assumption and applies to the second moment in both cases.

Theorem 4.12. *Let (42) hold. If $\chi = 0$ and $\lambda_m = 1$ for all $m \in \mathbb{N}$, and if $f^\lambda \in L^p(\gamma; V')$ for a $p > 2$, then*

$$\mathbb{E}_P\left(\|u - u_N\|_V^2 \right)^{\frac{1}{2}} \leq c_a \exp\left(\frac{p}{4(p-2)} \|(\alpha_m)_m\|_{\ell^2}^2 \right) \sum_{\mu \in \Lambda} \|H_\mu\|_{L^p(\gamma)} \inf_{w_N \in V_{N,\mu}} \left\| u_\mu^\lambda - w_N \right\|_V . \quad (60)$$

If $0 < \chi \leq 2\,\mathrm{e}^{-1} \min_m \alpha_m^{-1}$ and $(\lambda_m)_{m \in \mathbb{N}}$ is chosen as in Remark 4.8, and if $f^\lambda \in L^2(\gamma; V')$, then

$$\mathbb{E}_P\left(\|u - u_N\|_V^2 \right)^{\frac{1}{2}} \leq c_a \exp\left(\frac{\mathrm{e}}{2} \left(\frac{1}{2\chi} + \frac{\chi}{2} \right) \|(\alpha_m)_m\|_{\ell^1} \right) \left(\sum_{\mu \in \Lambda} \inf_{w_N \in V_{N,\mu}} \left\| u_\mu^\lambda - w_N \right\|_V^2 \right)^{\frac{1}{2}} . \quad (61)$$

Proof. Equation (60) follows from Theorem 4.6 by applying the triangle inequality to the right hand side. Equation (61) follows from Theorem 4.7 and Lemma 4.9 using Parseval's identity. □

4.6 Implications for convergence rates

Under assumptions on the regularity and decay of the Hermite coefficients u_μ^λ, the error bounds in Section 4.5 have direct implications on the convergence rates of the Galerkin solution. These follow as in [7, Sec. 8]. We first cite a result due to Stechkin, [7, 10].

Lemma 4.13. *Let $0 < p \leq q$ and let $(c_\mu)_{\mu\in\Lambda} \in \ell^p(\Lambda)$. Furthermore, for $N \in \mathbb{N}$, define Λ_N as the set of the first N indices in a decreasing rearrangement of $(|c_\mu|)_{\mu\in\Lambda} \in \ell^p(\Lambda)$. Then*

$$\left(\sum_{\mu\in\Lambda\setminus\Lambda_N} |c_\mu|^q\right)^{\frac{1}{q}} \leq \|(c_\mu)\|_{\ell^p(\Lambda)} N^{-r} \quad \textit{for} \quad r = \frac{1}{p} - \frac{1}{q} \geq 0 \; .$$

We define Λ_N as in Lemma 4.13 for $c_\mu := \left\|u_\mu^\lambda\right\|_V$, i.e. Λ_N contains N indices $\mu \in \Lambda$ such that $\left\|u_\mu^\lambda\right\|_V \geq \left\|u_\nu^\lambda\right\|_V$ for any $\nu \in \Lambda\setminus\Lambda_N$.

Let $(V_j)_{j\in\mathbb{N}_0}$ be a sequence of finite element spaces in V with $M_j := \dim V_j$ satisfying

$$M_0 = 0 \; , \quad 1 \leq M_1 \leq \eta \quad \text{and} \quad M_j \leq M_{j+1} \leq \eta M_j \quad \forall j \geq 1 \; , \tag{62}$$

for a constant $\eta \geq 1$. Furthermore, we assume that for an $s > 0$ there is a constant $C > 0$ such that

$$\inf_{v_j\in V_j} \|v - v_j\|_V \leq C M_j^{-s} \|v\|_W \quad \forall j \geq 1 \tag{63}$$

and

$$\inf_{v_0\in V_0} \|v - v_0\|_V = \|v\|_V \leq C \|v\|_W \; , \tag{64}$$

for all $v \in W$, where W is a dense subspace of V with norm $\|\cdot\|_W$ stronger than $\|\cdot\|_V$. For example, W could be equal to $V \cap H^2(D)$. Note that, for all $j \in \mathbb{N}_0$ and all $M_j \leq \widetilde{M} \leq M_{j+1}$,

$$\inf_{v_j\in V_j} \|v - v_j\|_V \leq C\eta^s \widetilde{M}^{-s} \|v\|_W \; . \tag{65}$$

For $j \geq 1$, this follows from $\widetilde{M} \leq M_{j+1} \leq \eta M_j$, and for $j = 0$, we use $\widetilde{M} \leq \eta$.

Assumption 4.14. *$V_{N,\mu} = V_{j(\mu)}$ for all $\mu \in \Lambda$, and $j(\mu) = 0$ for all $\mu \in \Lambda\setminus\Lambda_N$.*

The other values of $j(\mu)$ will be specified below in such a way that the bounds in Section 4.5 lead to optimal convergence rates.

We will first consider the bounds (58) and (61). We assume that $(u_\mu^\lambda)_\mu \in \ell^p(\Lambda; V)$ for some $p < 2$ and define

$$r := \frac{1}{p} - \frac{1}{2} > 0 \; . \tag{66}$$

By Lemma 4.13 and Assumption 4.14, (58) and (61) each imply

$$\mathbb{E}_P\left(\|u - u_N\|_V^2\right) \lesssim N^{-2r} \left\|(u_\mu^\lambda)_\mu\right\|_{\ell^p(\Lambda;V)}^2 + \sum_{\mu\in\Lambda_N} \inf_{v_j\in V_{j(\mu)}} \left\|u_\mu^\lambda - v_j\right\|_V^2 \; . \tag{67}$$

We would like to determine minimal values of $j(\mu)$ such that the second term in the right hand side of (67) is dominated by the first. Following [7], we initially allow the dimensions of the finite element spaces to be continuous variables and consider the minimization problem

$$\text{minimize} \quad \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu \quad \text{under the constraint} \quad \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu^{-2s} \left\|u_\mu^\lambda\right\|_W^2 \leq N^{-2r} , \tag{68}$$

where $\widetilde{M}_\mu \in (0,\infty)$ for all $\mu \in \Lambda_N$. For a Lagrange multiplier A, the solution is given by

$$\widetilde{M}_\mu = A^{\frac{1}{1+2s}} \left\|u_\mu^\lambda\right\|_W^{\frac{2}{1+2s}} \quad \forall \mu \in \Lambda_N , \tag{69}$$

and A is determined by

$$N^{-2r} = \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu^{-2s} \left\|u_\mu^\lambda\right\|_W^2 = A^{-1} \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu = A^{-\frac{2s}{1+2s}} \sum_{\mu\in\Lambda_N} \left\|u_\mu^\lambda\right\|_W^{\frac{2}{1+2s}} . \tag{70}$$

We define

$$j(\mu) := \max\left\{ j \in \mathbb{N}_0 \,;\, M_j \leq \widetilde{M}_\mu \right\} \quad \text{for} \quad \mu \in \Lambda_N . \tag{71}$$

Then the total number N_{dof} of degrees of freedom is

$$N_{\mathrm{dof}} := \dim \mathcal{V}_N = \sum_{\mu\in\Lambda_N} M_{j(\mu)} \leq \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu \tag{72}$$

and by (65),

$$\sum_{\mu\in\Lambda_N} \inf_{v_j\in V_{j(\mu)}} \left\|u_\mu^\lambda - v_j\right\|_V^2 \leq C\eta^{2s} \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu^{-2s} \left\|u_\mu^\lambda\right\|_W^2 = C\eta^{2s} N^{-2r} . \tag{73}$$

Inserting this into (67), we get a convergence rate of r with respect to the number N of Hermite 'coefficients' in the discrete solution. This N can be compared to the number of deterministic problems to be solved in other methods.

As in [7], we can relate N to the dimension N_{dof} of $\mathcal{V}_N$. If $(u_\mu^\lambda)_\mu \in \ell^q(\Lambda;W)$ for $q = \frac{2}{1+2s}$, then $N_{\mathrm{dof}}^s \lesssim N^r$ and by (67),

$$\mathbb{E}_P\Big(\|u - u_N\|_V^2\Big)^{\frac{1}{2}} \lesssim N_{\mathrm{dof}}^{-s} , \tag{74}$$

which is the same convergence rate as for a single deterministic problem. If $(u_\mu^\lambda)_\mu \in \ell^q(\Lambda;W)$ for some $q > \frac{2}{1+2s}$, then

$$N_{\mathrm{dof}} \lesssim N^{\frac{2r+\delta(1+2s)}{2s}} \quad \text{for} \quad \delta := 1 - \frac{2}{q+2sq} > 0 \tag{75}$$

and (67) implies

$$\mathbb{E}_P\Big(\|u - u_N\|_V^2\Big)^{\frac{1}{2}} \lesssim N_{\mathrm{dof}}^{-\frac{2sr}{2r+\delta(1+2s)}} . \tag{76}$$

For example, if it is known that $(u_\mu^\lambda)_\mu \in \ell^2(\Lambda;W)$, then the convergence rate with respect to N_{dof} is $\frac{sr}{r+s}$.

A similar analysis can be performed for (57) and (60). If $(u_\mu^\lambda)_\mu \in \ell^p(\Lambda; V)$ for a $p < 1$, then Lemma 4.13 implies

$$\mathbb{E}_P\Big(\|u - u_N\|_V^{\widetilde{q}}\Big)^{\frac{1}{\widetilde{q}}} \lesssim N^{-r} \left\|(u_\mu^\lambda)_\mu\right\|_{\ell^p(\Lambda;V)} + \sum_{\mu\in\Lambda_N} \|H_\mu\|_{L^{\widetilde{p}}(\gamma)} \inf_{v_j \in V_{j(\mu)}} \left\|u_\mu^\lambda - v_j\right\|_V \tag{77}$$

with $r := \frac{1}{p} - 1 > 0$. We define $j(\mu)$ as in (71) for $\widetilde{M}_\mu \in (0,\infty)$ minimizing

$$\sum_{\mu\in\Lambda_N} \widetilde{M}_\mu \quad \text{under the constraint} \quad \sum_{\mu\in\Lambda_N} \widetilde{M}_\mu^{-s} \left\|u_\mu^\lambda\right\|_W \|H_\mu\|_{L^{\widetilde{p}}(\gamma)} \leq N^{-r} \,. \tag{78}$$

As in [7] and by similar arguments as above, it follows that if the sequence $(\|H_\mu\|_{L^{\widetilde{p}}(\gamma)} \left\|u_\mu^\lambda\right\|_W)_\mu$ is in $\ell^q(\Lambda)$ with $q = \frac{1}{1+s}$, then

$$\mathbb{E}_P\Big(\|u - u_N\|_V^{\widetilde{q}}\Big)^{\frac{1}{\widetilde{q}}} \lesssim N_{\text{dof}}^{-s} \,, \tag{79}$$

and if $q > \frac{1}{1+s}$, then

$$\mathbb{E}_P\Big(\|u - u_N\|_V^{\widetilde{q}}\Big)^{\frac{1}{\widetilde{q}}} \lesssim N_{\text{dof}}^{-\frac{sr}{r+\delta(1+s)}} \quad \text{for} \quad \delta := 1 - \frac{1}{q+sq} > 0 \,. \tag{80}$$

Remark 4.15. Note that in both of the cases discussed above, the values of $j(\mu)$ depend on the norms $\left\|u_\mu^\lambda\right\|_W$ of Hermite coefficients of the exact solution. In practice, a priori bounds can be used instead. Also, a priori bounds for $\left\|u_\mu^\lambda\right\|_V$ can be used to determine Λ_N. The derivation of such bounds will be the topic of future work.

It is assumed that $(u_\mu^\lambda)_\mu \in \ell^p(\Lambda; V)$ for some p, and the value of r depends on p. Without the knowledge of p, it is only possible to compute relative values of $\widetilde{M}_\mu$ and it is not guaranteed that the two error components on (67) or (77) are balanced. The assumptions of the form $(u_\mu^\lambda)_\mu \in \ell^q(\Lambda; W)$ are only used to determine the convergences rate and are thus less essential.

5 Conclusion

Since the diffusion coefficient $a(\cdot,\cdot)$ is not bounded from above or away from zero, the Lax-Milgram lemma does not directly apply to the variational problem (24) on standard Bochner-Lebesgue spaces. Nevertheless, the pointwise solution u is the essentially unique solution of (24) for certain choices of test and trial spaces, which need not be contained in the energy space $L^{b^\lambda}(\widehat{\gamma}_\chi)$. This is a consequence of pointwise application of the Lax-Milgram lemma for each $\omega \in \Omega_\alpha$.

If the variational formulation is derived in the most straightforward manner, using the physical probability measure to integrate over the parameter domain, then it is well-posed on $L^p(\gamma; V)$ spaces with p potentially different from 2. One can remain in a Hilbert-space setting by integrating with respect to a modified Gaussian measure. All of the spaces involved are standard Bochner-Lebesgue spaces; only the choice of the measure depends on problem parameters.

For general finite element spaces, the Galerkin solution exists and is almost quasi-optimal with respect to standard Bochner-Lebesgue norms, that is, the error of the Galerkin solution is bounded by a best approximation error in a slightly stronger norm. In particular, if tensorized Hermite polynomials are used to discretize the parameter-dependence of the solution, then the error of the Galerkin solution in a mean-square sense with respect to the physical probability measure is bounded by the individual best approximation errors of the Hermite coefficients of the exact solution.

The convergence analysis of the stochastic Galerkin method is thus reduced to the question of approximability of the Hermite coefficients in $H^1(D)$. Results concerning regularity and decay properties of the Hermite coefficients will immediately imply convergence rates of the Galerkin solution and allow an a priori selection of optimal finite element spaces.

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