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A SEMI-LAGRANGIAN METHOD FOR CONVECTION OF DIFFERENTIAL FORMS

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Abstract. We propose a semi-Lagrangian discretization method for convection problems of differential forms. Our method approximates the material derivative using the nodal interpolation operator of discrete differential forms. Thereby the method is stable by construction. As application we derive a semi-Lagrangian discretization for the electromagnetic part of MHD equations.

1. Introduction

The calculus of differential forms permits us to express general time dependent convective partial differential equation as

$$-\varepsilon d * d\omega(\cdot,t) + *\partial_t\omega(\cdot,t) + *L_{\boldsymbol{\beta}}\omega(\cdot,t) = \varphi \quad \text{in } \Omega \subset \mathbb{R}^n \ . \tag{1}$$

This is an equation for an unknown l-form $\omega(\cdot,t)$, $0 \le l \le n$, on the domain Ω. The symbol $*$ stands for the so-called Hodge operator mapping an l-form to an $n-l$-form, and d denotes the exterior derivative. Together they define the principal part $d * d\omega$ of the differential operator. $L_{\boldsymbol{\beta}}$ is a convection operator for a given velocity field $\boldsymbol{\beta}$. Differential forms can be modelled by means of functions and vector fields through so-called vector proxies [5, page 132]. For $n = 3$, in the case of $*$ induced by the Euclidean metric on $\mathbb{R}^3$, the operator $d * d\omega$ becomes $-\Delta$, $\nabla\times(\nabla\times\cdot)$, and $\nabla\cdot\nabla$ in vector proxy notation, for $l = 0, 1, 2$, respectively. The convection operators for vector proxies are $\boldsymbol{\beta}\cdot\nabla$, $\nabla(\boldsymbol{\beta}\cdot\cdot) - \boldsymbol{\beta}\times(\nabla\times\cdot)$ and $\boldsymbol{\beta}(\nabla\cdot) - \nabla\times(\boldsymbol{\beta}\times\cdot)$. We refer to [10] for more details and an introduction to the calculus of differential forms.

Thinking in terms of differential forms offers considerable advantages as regards the construction of structure preserving spatial finite element discretizations of boundary value problems for $d * d\omega$ one can devise discrete counterparts of l-forms defined on triangulations of Ω, which provide suitable piecewise polynomial finite element spaces for the variational problems arising from $d * d\omega$. In particular, discrete differential forms respect the algebraic properties of the exterior derivative like $d^2 = 0$ and the DeRham exact sequence. More details are given in [2, 5, 11]. Discrete differential forms of any polynomial degree are available [1, 9]. In light of the success of discrete differential forms, it is worth exploring their use for the more general equation (1).

The convective part of the operator from (1) is formulated by means of the Lie derivative $L_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}$ is a vector field on Ω. Thus, for $l = 0$ and in terms of vector proxies, (1) becomes scalar convection diffusion. The Lie derivative operator itself can be represented as a composition of the so called contraction operators $i_{\boldsymbol{\beta}}$ and the exterior derivatives. This is the famous Cartan magic formula. The definition of the contraction is based on the notion of extrusion of manifolds and the duality pairing of forms and manifolds [3]. In [8] we started from this characterization to derive upwind discretizations.

The equivalent definition of Lie derivatives as the limit value of a difference quotient of a form and its pullback is the starting point of our semi-Lagrangian methods. We first review the definition of Lie derivatives and material derivatives for differential forms. Next, we propose a semi-Lagrangian approximation procedure for material derivatives of arbitrary discrete l-forms and discuss their stability. Finally we apply this method to solve the convective eddy current equations arising e.g. in magnetohydrodynamics. We will see, that the semi-Lagarangian discretiztion technique will yield a solver-friendly algebraic system, meaning that the linear systems that need to be solved, when evolving the inital data can be inverted fast and efficiently [15]. A related work is [14].

1.1. **Lie derivatives and material derivatives of forms.** We introduce a space-time domain $Q_T := \Omega \times [t_0, t_1]$ where $\Omega \subset \mathbb{R}^d$ and $[t_0, t_1] \subset \mathbb{R}$. Given a vector field $\boldsymbol{\beta} : Q_T \mapsto \mathbb{R}^d$ and initial values $(x, s) \in \bar{Q}_t$ we ask for solutions $X(x, s, t) : \bar{\Omega} \times [t_0, t_1]^2 \mapsto \mathbb{R}^d$ of:

$$\begin{aligned} \frac{d}{dt} X(x,s,t) &= \boldsymbol{\beta}(X(x,s,t),t), \quad t \in [t_0, t_1] \\ X(x,s,s) &= x. \end{aligned} \tag{2}$$

For fixed (x, s) the solution $X(x, s, \cdot)$ is called the characteristic curve through (x, s). A unique solution of problem (2) exists, whenever $\boldsymbol{\beta}$ is continuous in $\bar{Q}_T$ and Lipschitz continuous in $\bar{\Omega}$ for fixed $t \in [t_0, t_1]$. In this case

$$X(X(x,s,t),t,s) = x, \quad \forall x \in \Omega \tag{3}$$

hence $X(\cdot, t, s)$ is a inverse of $X(\cdot, s, t)$:

$$X(\cdot,t,s) \circ X(\cdot,s,t) = id. \tag{4}$$

For simplicity we abbreviate $X_{s,t}(\cdot) = X(\cdot, s, t)$. Before introducing the Lie derivative we recall the definition of the directional derivative for scalar functions $f : \Omega \mapsto \mathbb{R}$:

$$(\boldsymbol{\beta} \cdot \boldsymbol{\nabla} f)(x,t) := \lim_{\tau \to t} \frac{f(X(x,t,\tau)) - f(x)}{\tau - t}. \tag{5}$$

Now we write $\mathcal{D}^l$ for the space of l-forms on Ω. The scalar functions are 0–forms and the Lie derivative $L_{\boldsymbol{\beta}}$ of higher l-form $\omega \in \mathcal{D}^l$ is the generalization of the directional derivative for a scalar function. For differential forms $\omega \in \mathcal{D}^l$ of order l, $l > 0$, we replace the point evaluation of 0-forms with integration over l-dimensional oriented sub-manifolds M_l of Ω. To emphasize the duality of differential l-forms ω_l and l-dimensional oriented manifolds we introduce the notation

$$\omega(M_l) := \int_{M_l} \omega \ . \tag{6}$$

Then the Lie derivative of a l-form ω is:

$$(L_{\boldsymbol{\beta}}\omega)(M_l,t) := \lim_{\tau \to t} \frac{\omega(X_{t,\tau}(M_l)) - \omega(M_l)}{\tau - t}. \tag{7}$$

In terms of the pullback $X^*_{t,\tau}$ with

$$(X^*_{t,\tau}(\omega))(M_l) := \omega(X_{t,\tau}(M_l)) \tag{8}$$

we could also write

$$(L_{\boldsymbol{\beta}}\omega)(\cdot,t) := \lim_{\tau \to t} \frac{X^*_{t,\tau}\omega(\cdot) - \omega(\cdot)}{\tau - t}. \tag{9}$$

Following [3] the extrusion $Ext_{t,\tau}(\boldsymbol{\beta}, M_l) = \{X_{t,s}(x) \,:\, t \le s \le \tau,\, x \in M_l\}$ is the union of flux lines emerging at M_l running from t to τ (Figure 1). We define an orientation of the extrusion $Ext_{t,\tau}(\boldsymbol{\beta}, M_l)$ such that the boundary is:

$$\partial Ext_{t,\tau}(\boldsymbol{\beta}, M_l) = X_{t,\tau}(M_l) - M_l - Ext_{t,\tau}(\boldsymbol{\beta}, \partial M_l). \tag{10}$$

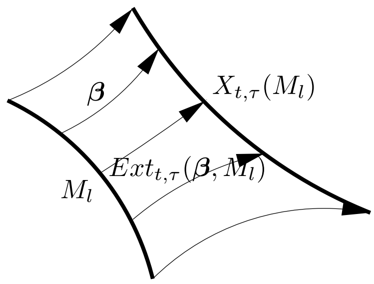

FIGURE 1. Extrusion of line segment M_l with respect to velocity field $\boldsymbol{\beta}$.

Plugging this into the definition of the Lie derivative (9) we get by means of Stokes' theorem

$$
\begin{aligned}
(L_{\boldsymbol{\beta}}\omega)(M_l,t) &= \lim_{\tau\to t}\frac{\omega(\partial Ext_{t,\tau}(\boldsymbol{\beta},M_l))+\omega(Ext_{t,\tau}(\boldsymbol{\beta},\partial M_l))}{\tau-t}\\
&= \lim_{\tau\to t}\frac{\mathrm{d}\omega(Ext_{t,\tau}(\boldsymbol{\beta},M_l))+\omega(Ext_{t,\tau}(\boldsymbol{\beta},\partial M_l))}{\tau-t}. \qquad (11)
\end{aligned}
$$

The contraction operator is defined as the limit of the dual of the extrusion:

$$(i_{\boldsymbol{\beta}}\omega)(M_l,t) := \lim_{\tau\to t}\frac{\omega(Ext_{t,\tau}(\boldsymbol{\beta},M_l))}{\tau-t} \qquad (12)$$

and we recover from (11) Cartan's magic formula [12, page 142, prop. 5.3] for the Lie derivative:

$$(L_{\boldsymbol{\beta}}\omega)(M_l,t) = (i_{\boldsymbol{\beta}}\mathrm{d}\omega)(M_l,t) + (\mathrm{d}i_{\boldsymbol{\beta}}\omega)(M_l,t). \qquad (13)$$

For 0-forms the second term vanishes, for top forms the first one.

Remark 1.1. *For* 1*-forms with vector proxy* $\mathbf{A}$ *in* $\mathbb{R}^3$ *this gives a general convective term*

$$L_{\boldsymbol{\beta}}\mathbf{A} \sim \boldsymbol{\beta}\times\nabla\times\mathbf{A} + \nabla(\boldsymbol{\beta}\cdot\mathbf{A})\ . \qquad (14)$$

For 2*-forms with vector proxy* $\mathbf{B}$ *in* $\mathbb{R}^3$ *this gives a general convective term*

$$L_{\boldsymbol{\beta}}\mathbf{B} \sim \boldsymbol{\beta}(\nabla\cdot\mathbf{B}) - \nabla\times\boldsymbol{\beta}\times\mathbf{B}\ . \qquad (15)$$

We refer to [4] for vector proxy representations Lie derivatives of other forms on two and tree dimensional manifolds.

What is the meaning of the limit value of (9) if we used time dependent differential forms $\omega(\cdot,t)$:

$$(D_{\boldsymbol{\beta}}\omega)(\cdot,t) := \lim_{\tau\to t}\frac{X^*_{t,\tau}\omega(\cdot,\tau)-\omega(\cdot,t)}{\tau-t} \quad ? \qquad (16)$$

This derivative is the rate of change of the action of differential forms in moving media, hence a material derivative [7, page 62]. We deduce:

$$
\begin{aligned}
(D_{\boldsymbol{\beta}}\omega)(\cdot,t) &= \lim_{\tau\to t}\frac{X^*_{t,\tau}\omega(\cdot,\tau)-X^*_{t,\tau}\omega(\cdot,t)}{\tau-t} && (17)\\
&\quad+\lim_{\tau\to t}\frac{X^*_{t,\tau}\omega(\cdot,t)-\omega(\cdot,t)}{\tau-t} && (18)\\
&= \frac{\partial}{\partial t}\omega(\cdot,t) + (L_{\boldsymbol{\beta}}\omega)(\cdot,t). && (19)
\end{aligned}
$$

Remark 1.2. *For 2-forms with vector proxy* $\mathbf{B}$ *in* $\mathbb{R}^3$ *and* $\nabla \cdot \mathbf{B} = 0$ *we have:*

$$D_{\boldsymbol{\beta}}\mathbf{B} = \partial_t \mathbf{B} + L_{\boldsymbol{\beta}}\mathbf{B} \tag{20}$$

$$= \partial_t \mathbf{B} - \nabla \times \boldsymbol{\beta} \times \mathbf{B}\,. \tag{21}$$

Remark 1.3. *Since the exterior derivative and the Lie derivative commutate we have:*

$$D_{\boldsymbol{\beta}} d = d D_{\boldsymbol{\beta}}. \tag{22}$$

As a consequence closed forms remain closed when they are advected by the material derivative. If $D_{\boldsymbol{\beta}}\omega(\cdot,t) = 0$ *and* $d\omega(\cdot,0) = 0$ *then this property is preserved:* $d\omega(\cdot,t) = 0$, $\forall t$.

There are now two different approaches to discretize material derivatives. We either use the formulation (19) in terms of partial time derivative of forms and Lie derivative or we approximate the difference quotient (16) directly. The first approach, referred to as Eulerian, reduces to a methods of lines approach with the spatial discretization of Lie derivatives from [8]. Regarding the stability properties the second semi-Lagrangian approach is much more attractive.

2. Direct and adjoint semi-Lagrange-Galerkin formulation

We focus first on the transport part of problem (1) and assume in the following that on the boundary of Ω the tangential components of the vector field $\boldsymbol{\beta}$ vanish, hence $X(\Omega,\cdot,\cdot) = \Omega$. This is a crucial but very common assumption for semi-Lagrangian methods. Only a few semi-Lagrangian methods like ELLAM [6] can handle a non-vanishing velocity field on the boundary.

If we use simple backward Euler there are two equivalent variational formulations for the transport problem: Given some $\phi(\cdot,t) \in \mathcal{D}^l$ find $\omega(\cdot,t) \in \mathcal{D}^l$, such that

$$D_{\boldsymbol{\beta}}\omega = \phi. \tag{23}$$

The *direct* variational formulation reads as: Given some $\phi(\cdot,t) \in \mathcal{D}^l$ find $\omega(\cdot,t) \in \mathcal{D}^l$, such that

$$\int_\Omega \lim_{\tau\to t} \frac{X^*_{t,\tau}\omega(\cdot,\tau) - \omega(\cdot,t)}{\tau - t} \wedge \eta = \int_\Omega \phi(\cdot,t)\wedge\eta \quad \forall \eta \in \mathcal{D}^{n-l}. \tag{24}$$

The *adjoint* variational formulation reads as: Given some $\phi(\cdot,t) \in \mathcal{D}^l$ find $\omega(\cdot,t) \in \mathcal{D}^l$, such that

$$\int_\Omega \lim_{\tau\to t} \frac{\omega(\cdot,\tau)\wedge X^*_{\tau,t}\eta - \omega(\cdot,t)\wedge\eta}{\tau - t} = \int_\Omega \phi(\cdot,t)\wedge\eta \quad \forall \eta \in \mathcal{D}^{n-l}. \tag{25}$$

Lemma 2.1. *The direct (24) and the adjoint (25) variational formulations are equivalent.*

Proof. Since $X_{t,s}(\Omega) = \Omega$ and $X_{\tau,t}(X_{t,\tau}(\cdot)) = id$:

$$\int_\Omega X^*_{t,\tau}\omega(\cdot,\tau)\wedge\eta = \int_{X_{\tau,t}(\Omega)} X^*_{t,\tau}\omega(\cdot,\tau)\wedge\eta \tag{26}$$

$$= \int_\Omega \omega(\cdot,\tau)\wedge X^*_{\tau,t}\eta \tag{27}$$

and the equivalence follows directly. □

Corollary 2.2. *The adjoint of a Lie derivative of a l–form is the negative Lie derivative of a $n-l$–form:*

$$\int_\Omega L_{\boldsymbol{\beta}}\omega(\cdot)\wedge\eta = -\int_\Omega \omega(\cdot)\wedge L_{\boldsymbol{\beta}}\eta.$$

Replacing the limit values in (24) and (25) with a finite difference quotient, yields semi-discrete time stepping schemes: Given $\phi(\cdot,t) \in \mathcal{D}^l$ and $\omega(\cdot,\tau) \in \mathcal{D}^l$ find $\omega(\cdot,t) \in \mathcal{D}^l$, such that

$$\int_\Omega \frac{X^*_{t,\tau}\omega(\cdot,\tau) - \omega(\cdot,t)}{\tau - t} \wedge \eta = \int_\Omega \phi(\cdot,t) \wedge \eta \quad \forall \eta \in \mathcal{D}^{n-l} \tag{28}$$

and given $\phi(\cdot,t) \in \mathcal{D}^l$ and $\omega(\cdot,\tau) \in \mathcal{D}^l$ find $\omega(\cdot,t) \in \mathcal{D}^l$, such that

$$\int_\Omega \frac{\omega(\cdot,\tau) \wedge X^*_{\tau,t}\eta - \omega(\cdot,t) \wedge \eta}{\tau - t} = \int_\Omega \phi(\cdot,t) \wedge \eta \quad \forall \eta \in \mathcal{D}^{n-l}. \tag{29}$$

Writing $\mathcal{W}^l \subset \mathcal{D}^l$ for some space of discrete l-forms on a triangulation of Ω [11] and restricting the semi discrete formulation to these spaces, we end up with the following direct and adjoint schemes: Given $\phi(\cdot,t) \in \mathcal{D}^l$ and $\omega(\cdot,\tau) \in \mathcal{D}^l$ find $\omega_h(\cdot,t) \in \mathcal{W}^l$, such that

$$\int_\Omega \frac{X^*_{t,\tau}\omega_h(\cdot,\tau) - \omega_h(\cdot,t)}{\tau - t} \wedge \eta_h = \int_\Omega \phi(\cdot,t) \wedge \eta_h \quad \forall \eta_h \in \mathcal{W}^{n-l}, \tag{30}$$

and find $\omega_h(\cdot,t) \in \mathcal{W}^l$, such that

$$\int_\Omega \frac{\omega_h(\cdot,\tau) \wedge X^*_{\tau,t}\eta_h - \omega_h(\cdot,t) \wedge \eta_h}{\tau - t} = \int_\Omega \phi(\cdot,t) \wedge \eta_h \quad \forall \eta_h \in \mathcal{W}^{n-l}. \tag{31}$$

In general $X^*_{t,\tau}\omega_h(\cdot,\tau) \notin \mathcal{W}^l$ and $X^*_{\tau,t}\eta_h \notin \mathcal{W}^{n-l}$ and the integrals can not be evaluated exactly. Since simple quadrature could cause serious problems concerning stability [13] we propose to use the nodal interpolation operators of discrete forms [11] to map the transported discrete forms $X^*_{t,\tau}\omega_h(\cdot,\tau)$ and $X^*_{\tau,t}\eta_h$ onto the space of discrete forms.

Definition 2.3 (Semi-Lagrangian time stepping)**.** *Let $\mathcal{W}^l \subset \mathcal{D}^l$ be the space of the lowest order Whitney l-form on a triangulation of Ω and Π_l the nodal interpolation operator. The direct semi-Lagrangian time stepping scheme is: Given $\omega_h(\cdot,\tau), \phi_h \in \mathcal{W}^l$, find $\omega_h(\cdot,t) \in \mathcal{W}^l$, such that*

$$\int_\Omega \omega_h(\cdot,t) \wedge \eta_h = \int_\Omega \Pi_l X^*_{t,\tau}\omega_h(\cdot,\tau) \wedge \eta_h - (\tau - t)\phi(\cdot,t) \wedge \eta_h \quad \forall \eta_h \in \mathcal{W}^{n-l}. \tag{32}$$

The adjoint semi-Lagrangian time stepping scheme is: Given $\omega_h(\cdot,\tau), \phi_h \in \mathcal{W}^l$, find $\omega_h(\cdot,t) \in \mathcal{W}^l$, such that

$$\int_\Omega \omega_h(\cdot,t) \wedge \eta_h = \int_\Omega \omega_h(\cdot,\tau) \wedge \Pi_{n-l} X^*_{\tau,t}\eta_h - (\tau - t)\phi(\cdot,t) \wedge \eta_h \quad \forall \eta_h \in \mathcal{W}^{n-l}. \tag{33}$$

If a continuous form $\omega(\cdot,t)$ is closed at some point t_0 and $D_{\boldsymbol{\beta}}\omega(\cdot,t) = 0$ then $d\omega(\cdot,t) = 0, \forall t$ (see remark (1.3)). This property is important in many physical applications and the adjoint semi-Lagrangian time-stepping full fills this in a weak sense.

Remark 2.4. *A form $\omega \in \mathcal{D}^l$ is weakly closed if $\int_\Omega \omega \wedge d\psi = 0, \forall \psi \in \mathcal{D}^{n-l-1}$. The discrete adjoint semi-Lagrangian time stepping scheme (33) with $\phi = 0$ and $\eta_h = d\psi_h$ is:*

$$\int_\Omega \omega_h(\cdot,t) \wedge d\psi_h = \int_\Omega \omega_h(\cdot,\tau) \wedge \Pi_{n-l} X^*_{\tau,t} d\psi_h \quad \forall \psi_h \in \mathcal{W}^{n-l-1}. \tag{34}$$

But since both exterior derivative and pullback and nodal interpolation and exterior derivative commutate we get:

$$\int_\Omega \omega_h(\cdot,t)\wedge d\psi_h = \int_\Omega \omega_h(\cdot,\tau)\wedge d\widetilde{\psi_h}, \tag{35}$$

with $\widetilde{\psi_h} = \Pi_{n-l-1}X^*_{\tau,t}\psi_h$. *Hence* $\omega_h(\cdot,t)$ *is weakly closed if* $\omega_h(\cdot,\tau)$ *is weakly closed.*

Remark 2.5. *The stability of the discrete semi-Lagrangian time-stepping schemes (32) and (33) hinges on the boundedness of the interpolation operator. Replacing e.g. in the homogeneous case of (32)* η_h *with* $*\omega_h(\cdot,t)$ *we see that:*

$$\|\omega_h(\cdot,t)\|^2_{L^2(\Omega)} = \int_\Omega \Pi_l X^*_{t,\tau}\omega_h(\cdot,\tau)\wedge *\omega_h(\cdot,t) \tag{36}$$

$$\leq \|\Pi_l X^*_{t,\tau}\omega_h(\cdot,\tau)\|_{L^2(\Omega)}\|\omega_h(\cdot,t)\|_{L^2(\Omega)}. \tag{37}$$

3. Nodal interpolation of transported forms

An efficient implementation of the time-stepping schemes (32) and (33) depends on the treatment of the nodal interpolation operators. We cover here algorithmic details for the interpolation operators of 0–forms and 1–forms. Given a triangulation of the domain $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. The usual nodal hat functions λ_i connected to vertices a_i for piecewise linear Lagrangian elements are a basis for discrete 0–forms. The functions $b_{\mathbf{e}} = \lambda_{e_1}\nabla\lambda_{e_2} - \lambda_{e_2}\nabla\lambda_{e_1}$ connected to edges $\mathbf{e} = (e_1, e_2)$ are a basis for discrete 1–forms.

3.1. 0–forms.

To determine the interpolation coefficients for a discrete transported 0-form $\omega_h(X_{t,\tau}(y),t)$ it is by linearity enough to consider the basis functions. Since $\Pi_0\omega(a_i) = \omega(a_i)$ for all vertices a_i and 0–forms ω the matrix operator $\mathbf{P}^0_{t,\tau}$ with entries

$$p^0_{ij} := \lambda_j(X_{t,\tau}(a_i)) \tag{38}$$

maps the expansions coefficients of $\omega_h(\cdot,\tau)$ to the coefficients of $\Pi_0 X^*_{t,\tau}\omega_h(\cdot,\tau)$. This means that in each time step we need to determine the points $X_{t,\tau}(a_i)$ on trajectories $X_{t,\cdot}(a_i)$ solution to the ordinary differential equation (2). We not only need to find the position but also the location within the mesh. To find the element, in which $X_{t,\tau}(a_i)$ is located we trace the path of the trajectory from one element to the next. Based on this data the matrix entries (38) can be assembled element by element (see fig. 2).

3.2. 1–forms.

The interpolation of a transported discrete 1–form $X^*_{t,\tau}\omega_h$ is again determined through the interpolation of transported basis forms and the condition $(\Pi_1 X^*_{t,\tau}\omega_h - X^*_{t,\tau}\omega_h)(\mathbf{e}) = 0, \forall \mathbf{e}\in\mathcal{E}$. This defines a matrix $\mathbf{P}^1_{t,\tau}$, mapping the expansion coefficients of $\omega_h(\cdot)$ to those of $X^*_{t,\tau}\omega_h$. The matrix entries

$$p^1_{\mathbf{e}\mathbf{e}'} := X^*_{t,\tau}b_{\mathbf{e}'}(\mathbf{e}) \tag{39}$$

are line integrals along the path from $X_{t,\tau}(a_{e_1})$ to $X_{t,\tau}(a_{e_2})$ (see fig. 3). In general the calculation of these line integrals can not be done exact. First the solution of the characteristic ODE (2) that gives the location $X_{t,\tau}(\mathbf{e})$ of the transported edge can be determined only approximately. Second we solve this ODE only for a finite number of points of the edge $\mathbf{e}$ and end up with a piecewise polynomial approximation of the transported edge. For simplicity we use here linear interpolation

$$X_{t,\tau}(\mathbf{e}) \cong [X_{t,\tau}(a_{e_1}), X_{t,\tau}(a_{e_2})] \tag{40}$$

between the transported end points $X_{t,\tau}(a_{e_1})$ and $X_{t,\tau}(a_{e_2})$. To determine the entries of the $\mathbf{e}$-th row, we trace the path from $X_{t,\tau}(a_{e_1})$ to $X_{t,\tau}(a_{e_2})$ and calculate

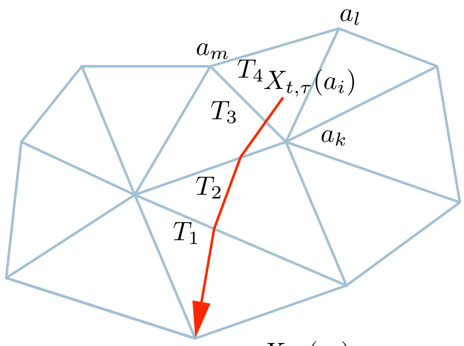

FIGURE 2. To determine the location of $X_{t,\tau}(a_i)$ we move along the trajectory $-X_{t,\cdot}(a_i)$ starting from a_i and identify the crossed elements T_1, T_2, T_3 and T_4. In this case p^0_{ik}, p^0_{il} and p^0_{im} are the only non-zero entries in the i–th row of $\mathbf{P}^0_{t,\tau}$.

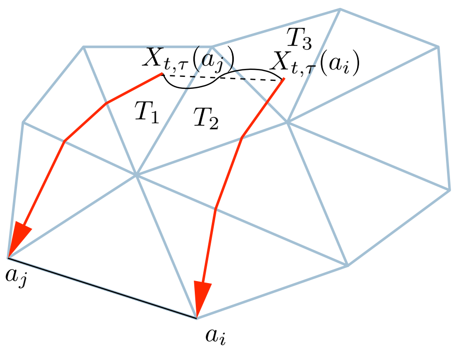

FIGURE 3. The transported edge $X^*_{t,\tau}\mathbf{e}$ (black curved line) is approximated by a straight line (black dashed line). In the case depicted here all basis function associated with edges $\mathbf{e}'$ of elements T_1, T_2 and T_3 yield a nonzero entry $p^1_{\mathbf{e},\mathbf{e}'}$

for each crossed element the line integrals for the attached basis functions. If e.g. the line crosses an element with edge $\mathbf{e}'$ from point a to point b (see fig. (4)), then the element contribution to $p^1_{\mathbf{e},\mathbf{e}'}$ is:

$$\int_{X_{t,\tau}(\mathbf{e})\cap T} b_{\mathbf{e}'} = \int_{[a,b]} b_{\mathbf{e}'} \tag{41}$$

$$= \frac{1}{2}\big(b_{\mathbf{e}'}(a) + b_{\mathbf{e}'}(b)\big)\cdot(b-a) \tag{42}$$

$$= \lambda_{e'_1}(a)\lambda_{e'_2}(b) - \lambda_{e'_2}(a)\lambda_{e'_1}(b). \tag{43}$$

In the following we will denote with $\widetilde{\mathbf{P}}^1_{t,\tau}$ an approximation to $\mathbf{P}^1_{t,\tau}$.

4. MHD MODEL

In this section we will apply both the direct and the adjoint semi-Lagrangian method to the electromagnetic part of magnetohydrodynamics models. A frequent approach in MHD neglects the displacement current in the full Maxwell's equation. This reduced model, called eddy current model, is a system of equations for the

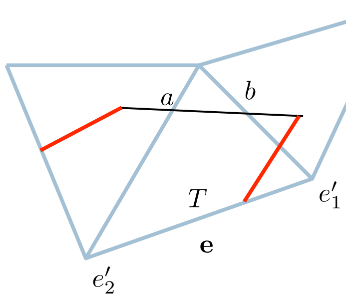

FIGURE 4. The line $[a, b]$ is the intersection of the approximation of the transported edge with element T.

magnetic field $h \in \mathcal{D}^1$, the electric field $e \in \mathcal{D}^1$, the magnetic field density $b \in \mathcal{D}^2$ and the current density $j \in \mathcal{D}^2$:

$$de = -\partial_t b \qquad \text{in } \Omega, \tag{44}$$

$$dh = j \qquad \text{in } \Omega, \tag{45}$$

$$j = *_\sigma(e + i_{\boldsymbol{\beta}} b) \qquad \text{in } \Omega, \tag{46}$$

$$*_\mu h = b \qquad \text{in } \Omega. \tag{47}$$

To rewrite this system in terms of a material derivative we substitute $e = \tilde{e} + i_{\boldsymbol{\beta}} b$ and add $-i_{\boldsymbol{\beta}} db$ to Faraday's law (44). This leaves the solution unchanged, since $db = 0$. Hence we end up with the system:

$$d\tilde{e} = -D_t b \qquad \text{in } \Omega, \tag{48}$$

$$dh = j \qquad \text{in } \Omega, \tag{49}$$

$$j = *_\sigma \tilde{e} \qquad \text{in } \Omega, \tag{50}$$

$$*_\mu h = b \qquad \text{in } \Omega. \tag{51}$$

Next we state two different variational formulations, that require either the direct or the adjoint semi-Lagrangian methods. They rely on a quite general perception of material laws like (50) and (51). Given two smooth convex energy dissipation functionals

$$E_\sigma : \quad \mathcal{D}^1 \mapsto \mathbb{R} \tag{52}$$

and

$$E_\mu : \quad \mathcal{D}^1 \mapsto \mathbb{R} \tag{53}$$

the derivatives $DE_\sigma(\tilde{e}), DE_\mu(h) \in (\mathcal{D}^1)'$ are linear forms on the space of 1-forms. But since by L^2-duality we also have $(\mathcal{D}^1)' \cong (\mathcal{D}^2)$ we prefer to treat the material laws (50) and (51) as equalities on the dual space $(\mathcal{D}^1)'$:

$$< DE_\sigma(\tilde{e}), e' > = \int_\Omega j \wedge e' \qquad \forall e' \in \mathcal{D}^1, \tag{54}$$

$$< DE_\mu(h), h' > = \int_\Omega b \wedge h' \qquad \forall h' \in \mathcal{D}^1. \tag{55}$$

Likewise energy dissipation functionals:

$$E_{\sigma^{-1}} : \quad \mathcal{D}^2 \mapsto \mathbb{R} \tag{56}$$

and

$$E_{\mu^{-1}} : \quad \mathcal{D}^2 \mapsto \mathbb{R} \tag{57}$$

define the material laws:

$$< DE_{\sigma^{-1}}(j), j' > = \int_\Omega \tilde{e} \wedge j' \qquad \forall j' \in \mathcal{D}^2, \tag{58}$$

$$< DE_{\mu^{-1}}(b), b' > = \int_\Omega h \wedge b' \qquad \forall b' \in \mathcal{D}^2. \tag{59}$$

4.1. **h-based variational formulation.** For simplicity we impose homogeneous electric boundary conditions on $\partial\Omega$. Testing (48) with $h' \in \mathcal{D}^1$, integration by parts yields:

$$\int_\Omega \tilde{e} \wedge dh' + \int_\Omega D_t b \wedge h' = 0. \tag{60}$$

We eliminate $\tilde{e}$ using the material law (58) for $j = dh$ and end up with the following variational formulation: Seek $h \in \mathcal{D}^1, b \in \mathcal{D}^2$ such that:

$$\int_\Omega D_t b \wedge h' + < DE_{\sigma^{-1}}(dh), dh' > = 0 \qquad \forall h' \in \mathcal{D}^1, \tag{61}$$

$$< DE_\mu(h), h'' > - \int_\Omega b \wedge h'' = 0 \qquad \forall h'' \in \mathcal{D}^1. \tag{62}$$

In order to eliminate b as well we switch to discrete time and get according to (29) an adjoint semi-Lagrangian scheme:

(63)
$$\int_\Omega b(\cdot,\tau) \wedge X^*_{\tau,t} h' - b(\cdot,t) \wedge h' + (\tau - t) < DE_{\sigma^{-1}}(dh(\cdot,t)), dh' > = 0 \quad \forall h' \in \mathcal{D}^1.$$

Hence the variational time-discrete h-based variational formulation for general material laws reads as: Given $h(\cdot,\tau) \in \mathcal{D}^1$ find $h(\cdot,t) \in \mathcal{D}^1$ such that:

$$\begin{aligned} < DE_\mu(h(\cdot,\tau)), X^*_{\tau,t} h' > - < DE_\mu(h(\cdot,t)), h' > \\ + (\tau - t) < DE_{\sigma^{-1}}(dh(\cdot,t)), dh' >= 0 \quad \forall h' \in \mathcal{D}^1. \end{aligned} \tag{64}$$

The variational time-discrete h-based variational formulation for linear scalar material laws is: Given $h(\cdot,\tau) \in \mathcal{D}^1$ find $h(\cdot,t) \in \mathcal{D}^1$ such that:

$$\begin{aligned} \int_\Omega \mu\, h(\cdot,\tau) \wedge X^*_{\tau,t} h' - \mu\, h(\cdot,t) \wedge h' \\ + (\tau - t) \int_\Omega \sigma^{-1} dh(\cdot,t) \wedge dh' = 0 \quad \forall h' \in \mathcal{D}^1. \end{aligned} \tag{65}$$

$h(\cdot,t)$ is the solution of an elliptic PDE with a source term depending on $h(\cdot,\tau)$. An additional spatial discretization with $h(\cdot,t), h' \in \mathcal{W}^1$ yields a system of linear equations for the coefficient vector $\mathbf{h}^t$ of $h_h(\cdot,t) \in \mathcal{W}^1$:

$$(\mathbf{M}_\mu + (t - \tau)\mathbf{C}_{\sigma^{-1}})\mathbf{h}^t = (\widetilde{\mathbf{P}}^1_{\tau,t})^T \mathbf{M}_\mu \mathbf{h}^\tau. \tag{66}$$

The right-hand side is linear in $h_h(\cdot,\tau)$ and the matrix $\widetilde{\mathbf{P}}^1_{\tau,t}$ depends on solutions of the ODE (2). To calculate the evolution of some initial data $\mathbf{h}^0$ we need to solve a linear system and a number of ODEs. Just recently Xu proposed a similar approach [16], that additionally amounts the solution of ODEs for the deformation gradient $F(s,t) = \nabla X_{s,t}(x)$. The main benefit of his, but also of our discretization approach is the fact, that the resulting algebraic systems can be inverted fast and efficiently. Adopting the parlance of [15], the semi-Lagrange technique is a solver-friendly discretization.

Remark 4.1. *The same arguments as in remark (2.4) show that the solution $\mu\, h(\cdot,t)$ of the variational formulation is (65) is weakly closed if the initial data $\mu\, h(\cdot,\tau)$ is weakly closed. This will not be true for the fully discrete system (66) since the mapping $\Pi_1 X^*_{\tau,t} : \mathcal{W}^1 \mapsto \mathcal{W}^1$ requires further approximations as discussed in section (3.2).*

4.2. **a-based variational formulation.** Here we assume for simplicity homogeneous magnetic boundary conditions on $\partial\Omega$. The ansatz $\tilde{e} = -D_t a$ and $b = da$ solves Faraday's law(48). We then multiply Ampere's law with a test form $a' \in \mathcal{D}^1$ and integrate by parts:

$$\int_\Omega h \wedge da' - \int_\Omega j \wedge a' = 0 \quad \forall a' \in \mathcal{D}^1. \tag{67}$$

The material laws (54) and (59) eliminate j and h and we get the a-based variational formulation: Find $a \in \mathcal{D}^1$ such that:

$$< DE_{\mu^{-1}}(da), da' > + < DE_\sigma(D_t a), a' >= 0 \quad \forall a' \in \mathcal{D}^1. \tag{68}$$

The time discrete case for scalar linear material laws is: Given $a(\cdot,\tau) \in \mathcal{D}^1$ find $a(\cdot,t) \in \mathcal{D}^1$ such that:

$$(\tau - t)\int_\Omega \mu^{-1}\, da(\cdot,t) \wedge da' + \int_\Omega \sigma(X^*_{t,\tau}a(\cdot,\tau) - a(\cdot,t)) \wedge a' = 0 \quad \forall a' \in \mathcal{D}^1. \tag{69}$$

This is according to (28) a direct semi-Lagrangian method. The spatial discretization with $a(\cdot,t), a' \in \mathcal{W}^1$ yields the linear system:

$$((t-\tau)\mathbf{C}_{\mu^{-1}} + \mathbf{M}_\sigma)\mathbf{a}^t = \mathbf{M}_\sigma \widetilde{\mathbf{P}}^1_{t,\tau}\mathbf{a}^\tau. \tag{70}$$

5. Numerical experiments

To illustrate the favourable stability properties of the semi-Lagrangian method, we augment the h-based variational formulation (65) with a non-zero right-hand side source term $\mathbf{f}$. In the vector proxy notation this reads as:

$$\begin{aligned} \int_\Omega \mu\, \mathbf{h}(\cdot,\tau)\, X^*_{\tau,t}\mathbf{h}' - \mu\, \mathbf{h}(\cdot,t)\, \mathbf{h}' \\ + (\tau - t)\int_\Omega \sigma^{-1}\nabla\times\mathbf{h}(\cdot,t)\,\nabla\times\mathbf{h}' = (\tau - t)\int_\Omega \mathbf{f}\,\mathbf{h}' \quad \forall \mathbf{h}' \in \mathcal{D}^1. \end{aligned} \tag{71}$$

The domain Ω will be the square $[-1,1]^2$ and the velocity

$$\boldsymbol{\beta} = (1-x^2)(1-y^2)\begin{pmatrix} 0.66 \\ 1 \end{pmatrix} \tag{72}$$

vanishes on the boundary of Ω. If we then take

$$\mathbf{h} = \cos(2\pi t)\begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ (1-x^2)(1-y^2) \end{pmatrix} \tag{73}$$

(see fig. 5) and constant material laws $\mu \equiv 1$ and $\sigma \equiv 1$ the right-hand side $\mathbf{f}$ calculates as:

$$\mathbf{f} := \partial_t\mathbf{h} + \boldsymbol{\beta}(\nabla\cdot\mathbf{h}) - \nabla\times\boldsymbol{\beta}\times\mathbf{h} + \nabla\times\nabla\times\mathbf{h}. \tag{74}$$

For simplicity we approximate the solutions of the characteristic ODE (2) with implicit Euler. In figures (6) and (7) we monitor the error for semi-Lagrangian time stepping (66) for different CFL-numbers and mesh sizes. We find stability even for CFL-numbers larger than one. These experiments suggest further that the convergence is of first order (Table (1)).

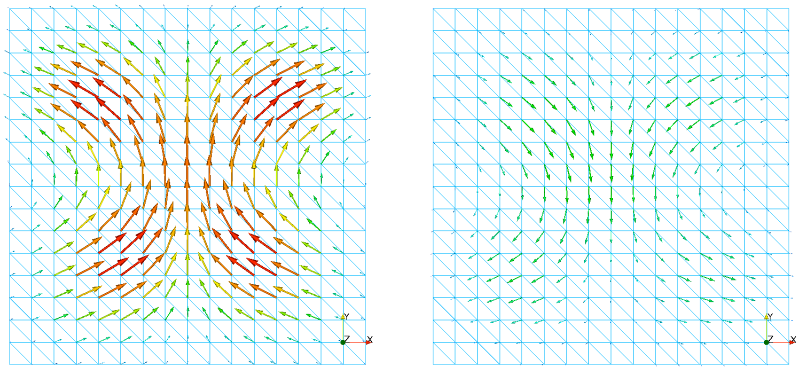

FIGURE 5. Solution $\mathbf{h}$ (see (73)) at $t = 0$ (left) and $t = 1.66$ computed with CFL-number 1.

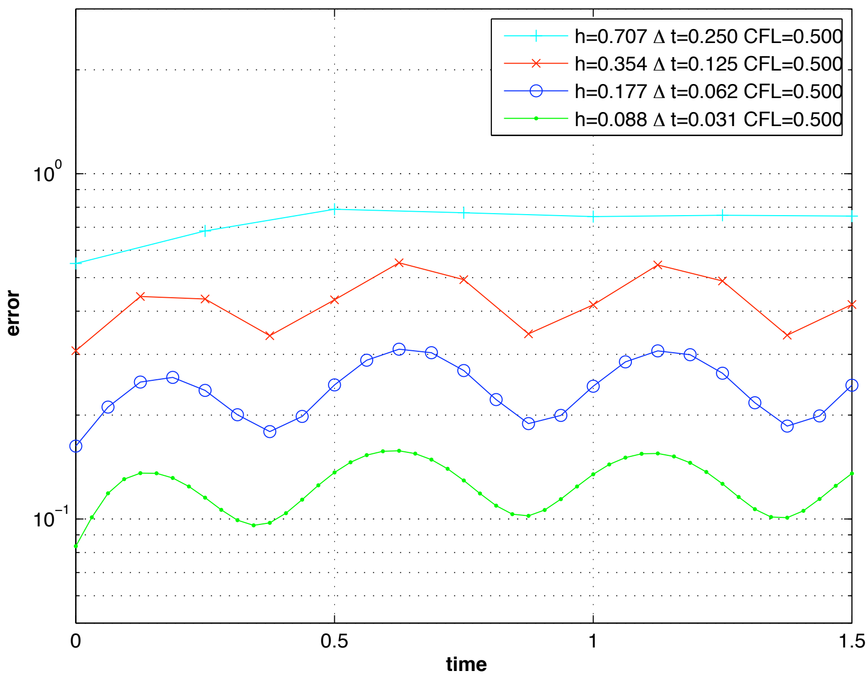

FIGURE 6. Evolution of L^2-error for problem (71) and small CFL-number.

We even observe stability for the pure transport problem, e.g. in the direct formulation:

$$\int_\Omega X_{t,\tau}\mathbf{h}(\cdot,\tau)\,\mathbf{h}' - \mu\,\mathbf{h}(\cdot,t)\,\mathbf{h}' = 0 \quad \forall \mathbf{h}' \in \mathcal{D}^1, \tag{75}$$

where now $X_{t,\tau}$ is the characteristic curve for the velocity field:

$$\boldsymbol{\beta} = \chi_{[-0.9,0.9]^2}\begin{pmatrix} 1 \\ 0.66 \end{pmatrix}. \tag{76}$$

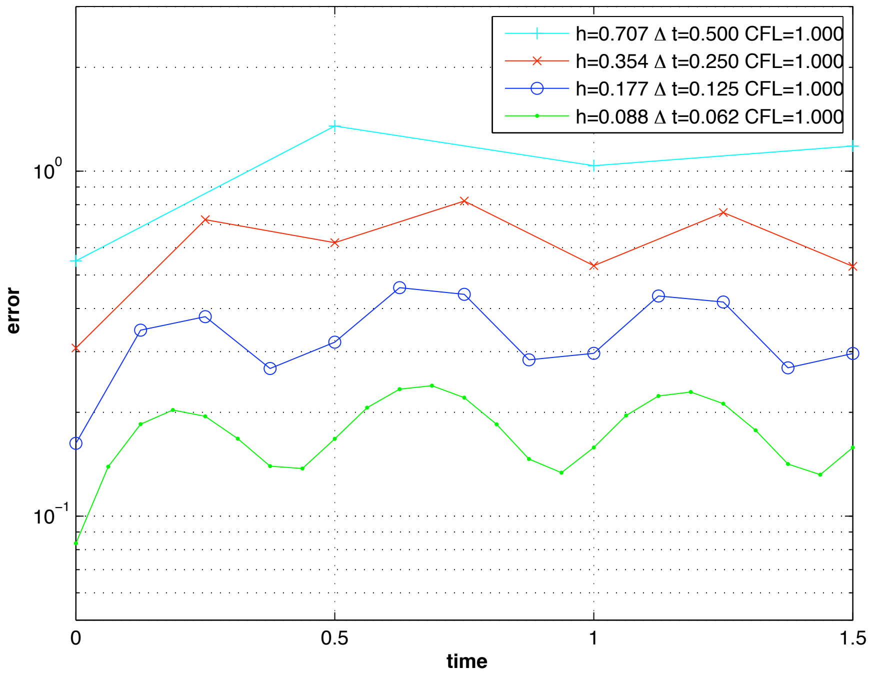

FIGURE 7. Evolution of L^2-error for problem (71) and large CFL-number.

h	Δt	L^2-error
0.707	0.250	0.7543
0.354	0.125	0.4184
0.177	0.062	0.2441
0.088	0.031	0.1355

h	Δt	L^2-error
0.707	0.500	1.1830
0.354	0.250	0.5302
0.177	0.125	0.2962
0.088	0.062	0.1581

TABLE 1. Error measured in L^2-norm at $T = 1.5$ for the experiments in figure (6) (left) and figure (7).

The inital data (fig. (8)) has compact support $[-0.8, 0.2]^2$, hence the solution is not effected by the discontinuity of $\boldsymbol{\beta}$. Figure (9) and (10) show the computed solution of the transport problem on a structured mesh for different CFL-numbers. The pulse like initial shape of the field strength is smoothed and the amplitude decreases. This phenomena is less strong for higher CFL-numbers. Figure (11) shows further that the discrete L^2-norm decreases monotone.

The semi-Lagrangian time stepping scheme is stable even in the case of a vanishing *curl-curl*-operator. This stability is a feature of the semi-Lagrangian framework and not due to the definition of Lie derivatives. If we use the spatial discretization of Lie derivatives [8] based on the finite difference quotient (9) we only need to integrate a system of ODEs. The implicit Euler scheme then reads as

$$(\mathbf{M}_\mu + (t-\tau)\mathbf{C}_{\sigma^{-1}} - (t-\tau)\mathbf{L}^T\mathbf{M}_\mu)\mathbf{h}^t = \mathbf{M}_\mu\mathbf{h}^\tau, \tag{77}$$

where $\mathbf{L}$ is the discretization of the Lie derivative of 1-forms. A second scheme, that resembles more the semi-Lagrangian scheme, is a splitting method that treats

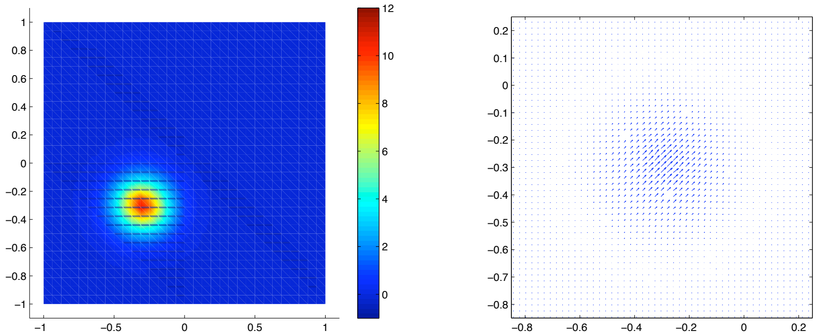

FIGURE 8. Field strength (left) and orientation of initial data for transport problem (75).

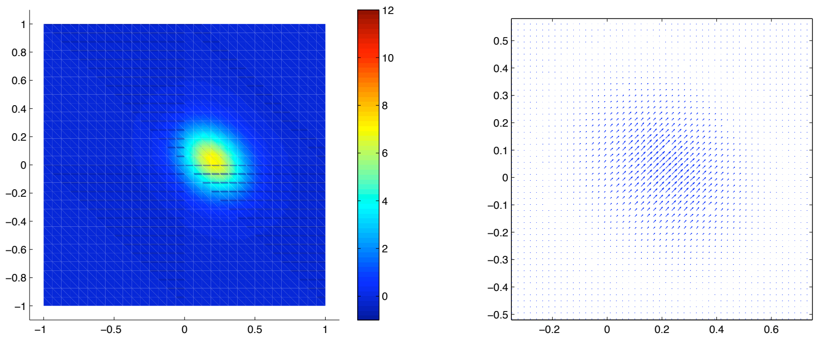

FIGURE 9. Transport of initial data, depicted in figure (8) at $T = 0.5$ for CFL-number 0.5 on a mesh with mesh size $h = 0.0884$.

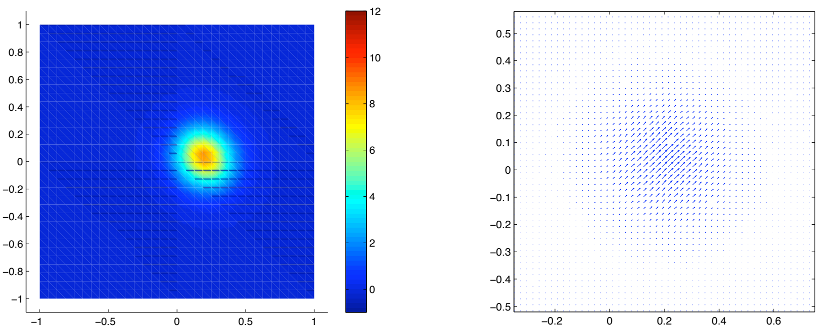

FIGURE 10. Transport of initial data, depicted in figure (8) at $T = 0.5$ for CFL-number 1 on a mesh with mesh size $h = 0.0884$.

the convective term explicitly:

$$(\mathbf{M}_\mu + (t-\tau)\mathbf{C}_{\sigma^{-1}})\mathbf{h}^t = (\mathbf{M}_\mu - (t-\tau)\mathbf{L}^T\mathbf{M}_\mu)\mathbf{h}^\tau. \tag{78}$$

Figure (12) shows that the scheme based on operator splitting is unstable for vanishing *curl-curl* operator. The semi-Lagrangian scheme is as stable as the Eulerian one using implicit Euler.

The formulation of the eddy current model in terms of material derivatives hinges on $\nabla\cdot\mathbf{b} = 0$ (see section (4)). We know that the solution of the continuous problem fulfills this constraint, when ever it is true for the initial data. Here we study this

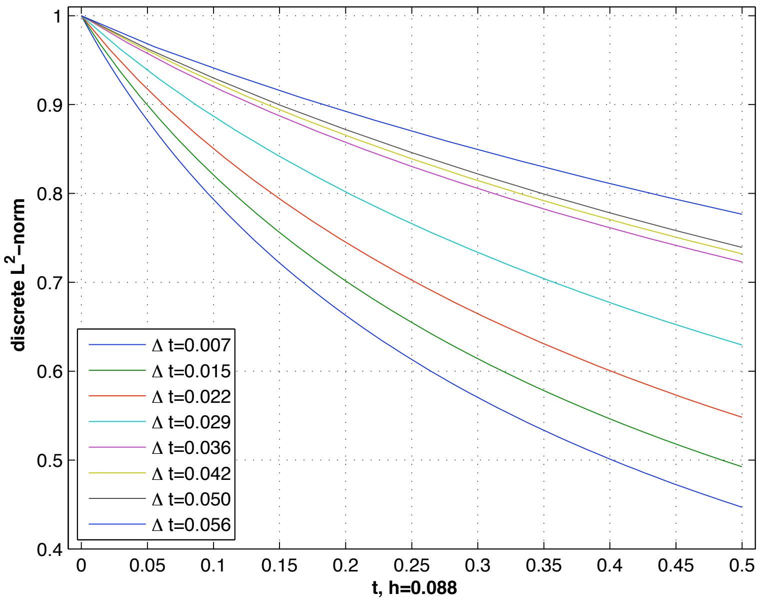

FIGURE 11. Monotone decrease of discrete L^2-norm for pure transport (75).

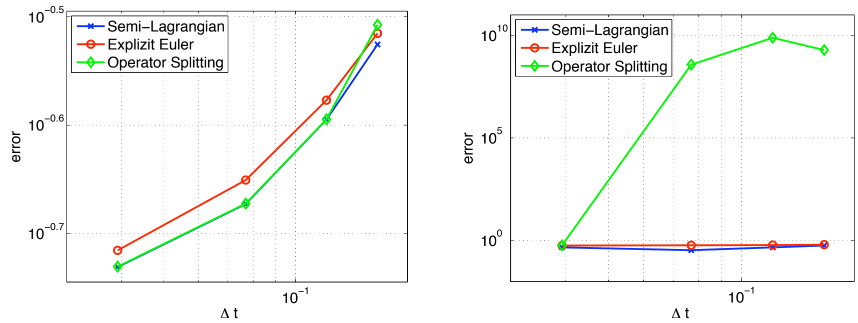

FIGURE 12. L^2-error for different time step sizes for non homogeneous problem (71) with exact solution (73) on a mesh with mesh size $h = 0.177$. With moderate *curl-curl*-term ($\sigma = 1$, $\mu = 1$) all three methods yield similar results (left). For small *curl-curl* term ($\sigma = 10^4$, $\mu = 1$) the solution of the splitting scheme blows up.

conservation property for the eddy current model in h-formulation (see section (4.1)), with $\sigma = \mu = 1$ and $\boldsymbol{\beta} = (1 - x^2)(1 - y^2)(0.66, 1)^T$. In figure (13) we monitor this conservation property on a regular and a distorted mesh. All three schemes, the semi-Lagrangian scheme (66), the implicit Euler scheme (77) and the splitting scheme (78) preserve the weak divergence. The approximation described in (3.2) does not destroy this conservation property.

REFERENCES

[1] Mark Ainsworth and Joe Coyle. Hierarchic finite element bases on unstructured tetrahedral meshes. *International Journal for Numerical Methods in Engineering*, 58:2103–2130, 2003.

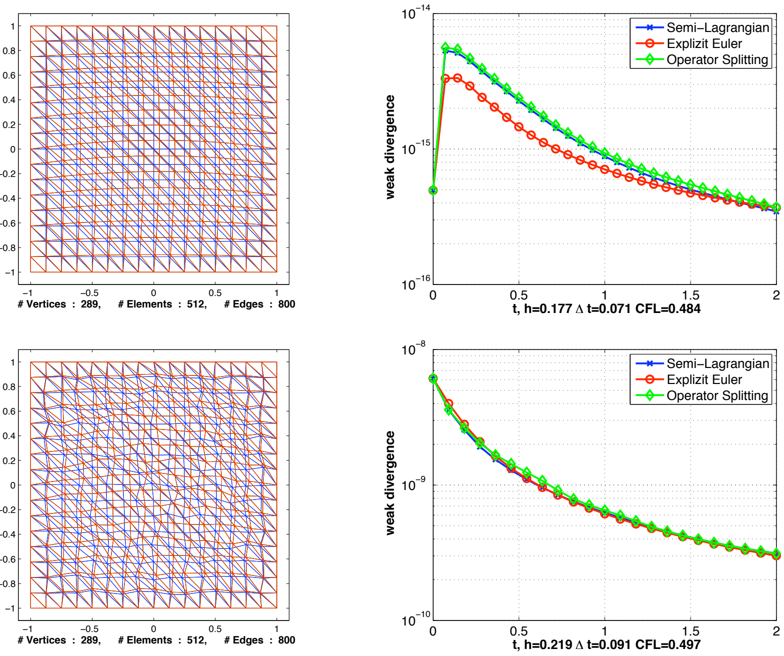

FIGURE 13. The semi-Lagrangian time stepping (66) aswell as implicit Euler scheme (77) and the splitting scheme (78) do preserve the weak divergence. Blue and red lines indicate the fixed FEM mesh and the transported mesh.

[2] D.N. Arnold, R.S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, 15:1–155, 2006.

[3] Alain Bossavit. Extrusion, contraction: their discretization via Whitney forms. *The International Journal for Computation and Mathematics in the Electrical and Electronic Engineering*, 22(2):470–480, 2003.

[4] Alain Bossavit. Applied differential geometry. http://www.em.tut.fi/ bossavit/BackupICM /Compendium.pdf, 2005.

[5] Alain Bossavit. Discretization of electromagnetic problems: The "generalized finite differences". In W.H.A. Schilders and W.J.W. ter Maten, editors, *Numerical Methods in Electromagnetics*, volume XIII of *Handbook of numerical analysis*, pages 443–522. Elsevier, Amsterdam, 2005.

[6] M.A. Celia, T.F. Russell, I. Herrera, and R.Ewing. An Eulerian-Lagrangian localized adjoint method for the advection–diffusion equation. *Adv. Wat. Resour.*, 13:187–206, 1990.

[7] Morton E. Gurtin. *An introduction to continuum mechnaics*. Academic Press.

[8] Holger Heumann and Ralf Hiptmair. Extrusion contraction upwind schemes for convection-diffusion problems. Technical Report 2008-30, Seminar for Applied Mathematics, ETH Zürich, 2008.

[9] Ralf Hiptmair. Canonical construction of finite elements. *Math. Comp.*, 68:1325–1346, 1999.

[10] Ralf Hiptmair. Discrete Hodge operators. *Numerische Mathematik*, 90:265–289, 2001.

[11] Ralf Hiptmair. Finite elements in computational electromagnetism. *Acta Numerica*, 11:237–339, 2002.

[12] Serge Lang. *Fundamentals of Differential Geometry*. Springer, 1999.

[13] K.W. Morton, A. Priestley, and E. Süli. Convergence of the Lagrange-Galerkin method with non-exact integration. *Mathematical Modelling and Numerical Analysis*, 22(4):625–653, 1988.

[14] R.N. Rieben, D.A. White, B.K. Wallin, and J.M. Solberg. An arbitrary Lagrangian–Eulerian discretization of MHD on 3D unstructured grids. *Journal of Computational Physics*, 226:534–570, 2007.

[15] Jinchao Xu. Optimal algorithms for discretized partial differential equations. In *ICIAM*, 2007.

[16] Jinchao Xu. private communication, 2008.

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