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A sparse composite collocation finite element method for elliptic sPDEs*

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Abstract

This work presents a stochastic collocation method for solving elliptic PDEs with random coefficients and forcing term which are assumed to depend on a finite number of random variables.

The method consists of a hierarchic wavelet discretization in space and a sequence of hierarchic collocation operators in the probability domain to approximate the solution's statistics. The selection of collocation points is based on a Smolyak construction of zeros of orthogonal polynomials w.r.t the probability density function of each random input variable. A sparse composition of levels of spatial refinements and stochastic collocation points is then proposed and analyzed, resulting in a substantial reduction of overall degrees of freedom.

Like in the Monte Carlo approach, the algorithm results in solving a number of uncoupled, purely deterministic elliptic problems, which allows the integration of existing fast solvers for elliptic PDEs.

Numerical examples on two-dimensional domains will then demonstrate the superiority of this sparse composite collocation FEM compared to the 'full composite' collocation FEM and the Monte Carlo method.

Key words. Stochastic partial differential equations, stochastic collocation methods, Smolyak approximation, multilevel approximations, sparse tensor products

AMS subject classifications. 35R60, 60H15, 65C20, 65N12, 65N15, 65N35

1 Introduction

Many engineering models of physical phenomena are subject to significant data uncertainties. We mention subsurface flow, soil mechanics, earthquake engineering, to name but a few. Neglecting model uncertainties we focus on PDEs with inherent parameter uncertainties, often modeled as random fields [1, 25], resulting in stochastic partial differential equations. Parametrizing those random fields by a finite, but possibly large number M of random variables, leads to high dimensional

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approximation problems on whose reduction of complexity a lot of recent works were focused [5, 4, 6, 19, 18, 24, 29, 28, 10].

In this work we present a *sparse composite collocation finite element algorithm* to compute the solution's random behavior. The word *sparse* has a twofold meaning here. First, we build a sequence of sparse collocation operators in the high-dimensional random parameter space as also done e.g. in [5, 19, 18, 29] resulting in a considerable reduction of the number of collocation points N_Ω to be used. The algorithms presented there, however, suffer from the fact that they are of complexity $O(N_\Omega \cdot N_D)$, N_D being the number of spatial DoFs, which is very prohibitive especially if a fine resolution of the spatial behavior is required e.g. due to short correlation lengths. Therefore, second, and more important, we additionally combine these collocation operators in a sparse fashion with a hierarchic wavelet basis in space, resulting in an algorithm of $O(N_\Omega \log N_D + N_D \log N_\Omega)$, and therefore substantially better complexity. Hence, we propose a fully discrete sparse collocation scheme where the addition of collocation points on each level of refinement is tailored to the level of resolution of the increment spaces of the hierarchic spatial wavelet discretization. We note here that the same idea of a sparse tensorization between stochastic and deterministic discretization spaces is also applicable to the stochastic Galerkin finite element method, see [4] for details. Our analysis of the sparse composite collocation method covers in particular also the case of Gaussian random variables and is therefore not only restricted to bounded variables like the above mentioned works.

By $D \subset \mathbb{R}^d$ we denote a bounded Lipschitz domain and by (Ω, Σ, P) a complete probability space with Ω denoting the outcomes, Σ the sigma-algebra of possible events and P a probability measure.

Our model problem is a stochastic elliptic boundary value problem of the form

$$\begin{cases} \mathcal{L}(u) = f & \text{in } D, \\ \mathcal{B}(u) = g & \text{on } \partial D \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is an elliptic differential operator depending on one or more random coefficients, e.g. a random diffusion coefficient $a(\omega, \mathbf{x})$, and $\mathcal{B}$ is a boundary operator. Finally, the forcing term $f(\omega, \mathbf{x}) \in L^2(\Omega; \mathcal{W}(D))$, where $\mathcal{W}(D)$ denotes a suitable Banach space, can also assumed to be random.

Our goal is to compute the statistics of u rather than the random field itself, since the latter is usually only of limited interest in practise.

The collocation approach presented here leads, like the Monte Carlo method, to a sequence of uncoupled purely deterministic problems which can be solved by any generic algorithm designed for solving problems of type (1.1) for a given realization of $a(\omega, \cdot)$. Unlike the Monte Carlo method, however, and motivated by [4, 6], those realizations are not chosen randomly but based on a hierarchic set of collocations points where in each point we solve a purely deterministic problem by a wavelet discretization adapted to the relative importance of the sample.

The outline of the paper is as follows: In Section 2 we will discuss the problem setting and impose some necessary assumptions on (1.1). An example of a problem meeting those criteria will be provided.

In Section 3 we will present the sparse composite collocation method, consisting of a thorough discussion of the stochastic semidiscretization by collocation and the wavelet discretization in space resulting in the formulation of the main result of this paper by proving algebraic approximation rates of the fully discretized problem.

Numerical examples in $D \subset \mathbb{R}^2$ and details on the implementation are then presented in Section 4. Examples concerning the hierarchical spatial and stochastic discretization as well as a comparison between sparse and full composite stochastic collocation FEM will be presented.

Appendix A contains the mathematical technicalities of our main results.

2 Problem setting

The collocation method presented in this work relies on a number of assumptions imposed on the problem (1.1), which will be introduced next. The section is concluded by giving an example of a problem satisfying all of the assumptions below which will then, in a slightly simplified form, also serve as a model problem throughout the rest of the paper.

The first assumption ensures that the problem is well posed in the sense of unique solvability.

Assumption 2.1 (Well-posedness of the problem). The random coefficients a and the forcing term f are such that the uniqueness and existence of a solution $u(\omega, \mathbf{x}) \in L^2_P(\Omega; \mathcal{W}(D))$ to (1.1) is guaranteed P-almost surely.

The next assumption restricts the model problem (1.1) to ones where the stochastic behavior of the input parameters can be described by a finite-dimensional random vector $(Y_1, \ldots, Y_M)$.

Assumption 2.2 (Finite-dimensional noise assumption). The stochastic parameters a and f depend only on a finite number M of random variables $Y_m : \Omega \to \mathbb{R}$, i.e.

$$a(\omega, \mathbf{x}) = a(Y_1, \ldots, Y_M, \mathbf{x}) \quad \text{and} \quad f(\omega, \mathbf{x}) = f(Y_1, \ldots, Y_M, \mathbf{x}) \tag{2.1}$$

This may seem very restricting at a first glance, since random fields are usually only properly described by an infinite number of random variables. However, since our goal is an approximation to the statistical moments of the solution we only need to describe the random field to a sufficiently high accuracy which is usually possible by only taking a finite number of random variables into account. This can e.g. be seen if we expand the random field a, and similarly f, into a so called Karhunen-Loève (KL) expansion:

$$a(\omega, \mathbf{x}) = \mathbb{E}_a(\mathbf{x}) + \sum_{m \geq 1} \sqrt{\lambda_m} \varphi_m(\mathbf{x}) Y_m(\omega). \tag{2.2}$$

The KL expansion is guaranteed to exist if the random field has finite second moments [12]. In that case (λ_m, φ_m) are eigenpairs of the eigenvalue problem of finding $(\lambda, \varphi) \in \mathbb{R} \times L^2(D)$ s.t.

$$\int_D V_a(\mathbf{x}, \mathbf{x}') \varphi(\mathbf{x}') \, d\mathbf{x}' = \lambda \varphi(\mathbf{x}) \tag{2.3}$$

where $V_a(\mathbf{x}, \mathbf{x}')$ denotes the 2-point covariance of the random field a. By reordering, if necessary, we can assume that the λ_m's are monotonically converging to zero if m tends to infinity, i.e. $\lambda_1 \geq \lambda_2 \geq 0$ and $\lambda_m \longrightarrow 0$. The truncated KL expansion is then given by

$$a_M(\omega, \mathbf{x}) = \mathbb{E}_a(\mathbf{x}) + \sum_{m=1}^{M} \sqrt{\lambda_m} \varphi_m(\mathbf{x}) Y_m(\omega). \tag{2.4}$$

Moreover, under certain regularity assumptions on the 2-point covariance V_a such as sufficiently high Sobolev regularity, see e.g. [4] Proposition 2.3, it can be shown that $\|\varphi_m\|_{L^\infty(D)} \leq c|\lambda_m|^{-s/2}$ with $0 < s < 1$, s depending on the smoothness of $V_a(\mathbf{x}, \mathbf{x}')$, i.e. we have a pointwise estimate on the KL eigenfunctions in terms of the associated eigenvalues. Due to this fact and the decay of the eigenvalues the random field can clearly be approximated to any prescribed accuracy by a finite number of random variables:

$$\|a - a_M\|_{L^\infty(\Omega, \mathcal{W}(D))} \; lesssim \sum_{m > M} \lambda_m^{1-s} \tag{2.5}$$

Hence, if the λ_m's are such that the last sum in (2.5) is finite, we have a-priori (pointwise) control over the truncation error of the random field. This summability of eigenvalues is in turn ensured for a lot of common covariance kernels V_a, such as Gaussian or exponential ones [15, 21, 22, 16, 23].

The case of lognormal random coefficients, which play an important role in practical applications, can be treated similarly. If e.g. a is lognormally distributed, then its logarithm can be expanded in a Karhunen-Loève series as before, hence we have

$$\ln a(\omega, \mathbf{x}) = \varphi_0(\mathbf{x}) + \sum_{m\geq 1} \sqrt{\lambda_m}\varphi_m(\mathbf{x})Y_m(\omega) \tag{2.6}$$

with $Y_m \sim \mathcal{N}(0,1)$. Following the same arguments as above, we can obtain an approximation of a to any prescribed accuracy in terms of finitely many random variables Y_m.

Assumption 2.3 (Independence).

i) The family $(Y_m)_{m\geq 1} : \Omega \to \mathbb{R}$ is *independent*,

ii) with each $Y_m(\omega)$ is associated a probability space $(\Omega_m, \Sigma_m, P_m)$, $m \in \mathbb{N}$ with the following properties:

   a) the probability measure P_m admits a probability density function $\rho_m : \mathrm{Ran}(Y_m) := \Gamma_m \longrightarrow [0,\infty)$ such that $dP_m(\omega) = \rho_m(y_m)dy_m$, $m \in \mathbb{N}$, $y_m \in \Gamma_m$ and

   b) the sigma algebras Σ_m are subsets of the Borel sets of the interval Γ_m, i.e. $\Sigma_m \subseteq \Sigma(\Gamma_m)$.

Remark 2.4. The random variables Y_m will in general not be independent, unless for Gaussian variables in a KL-expansion. One possible remedy has been proposed in [2] by introducing auxiliary probability density functions $\tilde{\rho}_m$ with $\|\rho_m/\tilde{\rho}_m\|_{L^\infty} < \infty$ such that the random variables Y_m are independent with respect to $\tilde{\rho} = \prod_{m\geq 1}\tilde{\rho}_m$.

Due to the independency in Assumption 2.3 we can consider the Y_m's as different coordinates in probability space and hence parametrize Ω by the vector $\mathbf{y} = (y_1, \ldots, y_M) \in \Gamma := \Gamma_1 \times \cdots \times \Gamma_M$ rather than $(Y_1(\omega), \ldots, Y_M(\omega))$.

We emphasize here that the parameter domains Γ_m may be bounded or unbounded. This includes in particular the case of Gaussian or exponential random variables Y_m. However, if we consider unbounded domains Γ_m, the growth at infinity has to be controlled. Following [2] we define, for a Banach space $\mathcal{W}(D)$ the function space

$$C^0_\sigma(\Gamma, \mathcal{W}(D)) := \left\{ v : \Gamma \longrightarrow \mathcal{W}(D) : \ v \ \text{ cont. in } \ \mathbf{y}, \sup_{\mathbf{y}\in\Gamma} \|\sigma(\mathbf{y})v(\mathbf{y})\|_{\mathcal{W}(D)} < \infty \right\}, \tag{2.7}$$

where $\sigma(\mathbf{y}) = \prod_{m=1}^M \sigma_m(y_m) \leq 1$ and

$$\sigma_m(y_m) = \begin{cases} 1 & \text{if } \Gamma_m \text{ bounded} \\ e^{-\alpha_m|y_m|} \text{ for some } \alpha_m > 0 & \text{if } \Gamma_m \text{ unbounded} \end{cases} \tag{2.8}$$

The associated norm is then given by $\|v\|_{C^0_\sigma} := \sup_{\mathbf{y}\in\Gamma} \|\sigma(\mathbf{y})v(\mathbf{y})\|_{\mathcal{W}(D)}$

Assumption 2.5 (Growth at infinity). i) $u \in C^0_\sigma(\Gamma, \mathcal{W}(D))$, and

ii) the joint probability density ρ satisfies

$$\rho(\mathbf{y}) \leq C_\rho e^{-\sum_{m=1}^M (\delta_m y_m)^2} \quad \forall \mathbf{y} \in \Gamma \tag{2.9}$$

for a constant $C_\rho > 0$ and δ_m strictly positive if Γ_m is unbounded and zero otherwise.

Finally we need to make some regularity assumption on the stochastic behavior of the solution. To simplify the notation we denote by y_m^* the $(M-1)$-dimensional vector $y_m^* = (y_1, \dots, y_{m-1}, y_{m+1}, \dots, y_M) \in \Gamma_m^* := \prod_{j \neq m} \Gamma_j$.

Assumption 2.6 (Stochastic regularity assumption). For each $m \in \{1, \dots, M\}$ there exists $\tau_m > 0$ such that the solution $u(y_m, y_m^*, \mathbf{x})$ as a function of $y_m \in \Gamma_m$ admits an analytic extension to the region of the complex plane

$$\Sigma(\Gamma_m, \tau_m) = \{z \in \mathbb{C} : \operatorname{dist}(z, \Gamma_m) \leq \tau_m\}, \tag{2.10}$$

To conclude this section, we will present one example where the above assumptions hold true. This example, in a simplified form, will serve as our model problem throughout the rest of the paper.

Example We consider the following stochastic diffusion-reaction problem

$$\begin{cases} -\operatorname{div}(a(\omega, \mathbf{x})\nabla u(\omega, \mathbf{x})) + c(\omega, \mathbf{x})u(\omega, \mathbf{x}) = f(\omega, \mathbf{x}) & D, \\ u(\omega, \mathbf{x})|_{\mathbf{x} \in \partial D} = 0, \end{cases} \quad P-\text{a.e. } \omega \in \Omega \tag{2.11}$$

where a and c are random fields denoting the diffusivity and reaction rate, respectively and f is a stochastic source term.

This problem can be reformulated in variational form as: Find $u \in L_P^2(\Omega, \mathcal{W}(D))$ s.t. $\forall\ v \in L_P^2(\Omega, \mathcal{W}(D))$ it holds

$$b(u, v) = l(v) \tag{2.12}$$

with

$$b(u, v) = \mathbb{E}\left[\int_D a \nabla u \nabla v + cuv \, d\mathbf{x}\right] \qquad l(v) = \mathbb{E}\left[\int_D fv \, d\mathbf{x}\right] \tag{2.13}$$

where we suppress the dependence of the coefficients and functions on $(\omega, \mathbf{x}) \in \Omega \times D$ for illustrative reasons.

The first Assumption 2.1 holds true for $\mathcal{W}(D) = H_0^1(D)$ if for the coefficients a, c, f it holds

i) $a \in L^2(\Omega \times D)$ is positive and bounded away from zero almost surely, i.e. there exist $a_* > 0$ such that

$$P\left\{\omega \in \Omega : a_* \leq \operatorname*{ess\,inf}_{\mathbf{x} \in D} a(\omega, \mathbf{x})\right\} = 1 \tag{2.14}$$

ii) $c \in L^2(\Omega \times D)$ is positive, i.e.

$$P\left\{\omega \in \Omega : 0 \leq \operatorname*{ess\,inf}_{\mathbf{x} \in D} c(\omega, \mathbf{x})\right\} = 1 \tag{2.15}$$

iii) f is square-integrable with respect to P, i.e.

$$\mathbb{E}[\|f\|_{L^2(D)}^2] < \infty \tag{2.16}$$

Indeed, by introducing the Hilbert space

$$\mathcal{H}_{a,c} := \{v \in L_P^2(\Omega, H_0^1(D)) : \mathbb{E}[\int_D a|\nabla v|^2 + cv^2 \, dx] < \infty\} \tag{2.17}$$

with the norm $\|v\|_{\mathcal{H}_{a,c}}^2 := \mathbb{E}[\int_D a|\nabla v|^2 + cv^2 \, dx]$ it follows from an application of the Lax-Milgram Lemma the existence and uniqueness of a solution $u \in \mathcal{H}_{a,c}$ to (2.12). Precisely, consider the bilinear form $b(u, v)$ given in (2.13). It follows

$$b(u, v) \leq \|u\|_{\mathcal{H}_{a,c}} \|v\|_{\mathcal{H}_{a,c}} \quad \text{and} \quad b(u, u) \geq \|u\|_{\mathcal{H}_{a,c}} \tag{2.18}$$

i.e. $b(\cdot,\cdot)$ is continuous and coercive with continuity and coercivity constant equal to one. It remains to show that $l(\cdot)$ (2.13) is continuous w.r.t. $\mathcal{H}_{a,c}$:

$$\begin{aligned} l(v) &= \mathbb{E}\left[\int_D f(\omega,\mathbf{x})v(\omega,\mathbf{x})\,d\mathbf{x}\right] \leq \mathbb{E}\left[\|f(\omega,\cdot)\|_{L^2(D)}\|v(\omega,\cdot)\|_{L^2(D)}\right] \\ &\leq \frac{C_P}{\sqrt{a_*}}\mathbb{E}\left[\|f(\omega,\cdot)\|_{L^2(D)}\|\sqrt{a(\omega,\cdot)}\nabla v(\omega,\cdot)\|_{L^2(D)}\right] \\ &\leq \frac{C_P}{\sqrt{a_*}}\|f\|_{L^2_P(\Omega,L^2(D))}\|v\|_{\mathcal{H}_{a,c}} \end{aligned}$$

where C_P denotes the Poincaré constant. Hence, by Lax-Milgram, the existence of a unique solution $u \in \mathcal{H}_{a,c}$ to (2.12) is guaranteed P-almost surely. Furthermore, since

$$\|u\|_{L^2_P(\Omega,H^1_0(D))} \leq \frac{C_P}{\sqrt{a_*}}\|u\|_{\mathcal{H}_{a,c}} \tag{2.19}$$

the space $\mathcal{H}_{a,c}$ is continuously embedded in $L^2_P(\Omega,H^1_0(D))$ and it follows the existence of a solution $u \in L^2_P(\Omega,H^1_0(D))$ to (2.12).

Remark 2.7. As pointed out in [2] it is possible to relax the condition (2.14) above to a lower bound which is itself again a random variable, i.e. there exists a $a_*(\omega)$ s.t.

$$P\{\omega\in\Omega : 0 < a_*(\omega) < \operatorname*{ess\,inf}_{\mathbf{x}\in D} a(\omega,\mathbf{x})\} = 1 \tag{2.20}$$

This case is of particular interest since it covers the case of lognormal random coefficients a (2.6).The well-posedness and hence the existence of a solution in $\mathcal{H}_{a,c}$ can then be shown in essentially the same way as above, except that we need stronger regularity assumptions on f:

$$\begin{aligned} l(v) &= \mathbb{E}\left[\int_D f(\omega,\mathbf{x})v(\omega,\mathbf{x})\,d\mathbf{x}\right] \leq \mathbb{E}\left[\|f(\omega,\cdot)\|_{L^2(D)}\|v(\omega,\cdot)\|_{L^2(D)}\right] \\ &\leq C_P\mathbb{E}\left[\frac{\|f(\omega,\cdot)\|_{L^2(D)}}{\sqrt{a_*(\omega)}}\|\sqrt{a(\omega,\cdot)}\nabla v(\omega,\cdot)\|_{L^2(D)}\right] \\ &\leq C_P\|f\|^{1/2p}_{L^{2p}_P(\Gamma,H^1_0(D))}\|1/\sqrt{a_*}\|^{1/2q}_{L^{2q}_P(\Gamma)}\|v\|_{\mathcal{H}_{a,c}} \end{aligned}$$

where we twice used Hölder in the last line and $1/p+1/q=1$, $p,q\geq 1$. Hence, if $1/\sqrt{a_*}\in L^{2q}(\Gamma)$ for some $q\geq 1$ then we must require $f\in L^{2p}(\Gamma,H^1_0(D))$ for $p=(1-1/q)^{-1}$.

Assumption 2.2 is fulfilled if we consider e.g. a truncated Karhunen-Loève expansion of a (or $\ln a$) as indicated above whereas the independency in Assumption 2.3 and the growth at infinity in Assumption 2.5 have either to be given or ensured by introducing an auxiliary density, if possible, as indicated in Remark 2.4, see also [2] for further details.

Finally, the analytic regularity of the solution $u(\mathbf{y},\mathbf{x})$, as required in Assumption 2.6 is ensured by the following

Lemma 2.8. *Assume that for every (y_m,y_m^*) there exists $\gamma_m<\infty$ satisfying*

$$\left\|\frac{\partial^k_{y_m}a(\mathbf{y})}{a(\mathbf{y})}\right\|_{L^\infty(D)} \leq \gamma_m^k k!, \quad \frac{\|\partial^k_{y_m}c(\mathbf{y})\|_{L^\infty(D)}}{\|c(y)\|_{L^\infty(D)}} \leq \gamma_m^k k!, \quad \frac{\|\partial^k_{y_m}f(\mathbf{y})\|_{L^2(D)}}{1+\|f\|_{L^2}} \leq \gamma_m^k k! \tag{2.21}$$

Assume further there exists $p,q\geq 1$ with $1/p+1/q=1$ such that there holds $c\in C^0_{\sigma^{1/p}}(\Gamma,L^\infty(D))$ and $f\in C^0_{\sigma^{1/q}}(\Gamma,L^2(D))$. Then the solution $u(y_m,y_m^*,x)$ to (2.11) as a function of y_m $u:\Gamma_m\to C^0_{\sigma^*_m}(\Gamma^*_m,H^1_0(D))$ admits an analytic extension into the region $\Sigma(\Gamma_m,\tau_m)\subset\mathbb{C}$ with $0<\tau_m<\frac{1}{2\gamma_m}$, uniformly for all $y_m^*\in\Gamma_m^*$.

We note here that this assertion is a slight generalization of Lemma 3.2 in [2] where the case $c \equiv 0$ has been studied. The proof follows closely the one given there and is omitted in this paper. Instead we refer to [3].

The conditions in (2.21) are satisfied if the coefficients can be represented by a (truncated) KL expansion, see [2] Examples 3 & 4. In fact, it is shown there, that γ_m can be chosen as $\gamma_m \sim \sqrt{\lambda_m}$ which in turn implies that the sizes of the analyticity regions are increasing if $m \to \infty$.

3 Sparse composite collocation FEM

Stochastic collocation methods for sPDEs have first been proposed in [29, 2] by using numerical quadrature for the approximate evaluation of the stochastic integrals. [29] already proposed the usage of Smolyak grids to reduce the number of collocation points. This idea was further developed and analyzed in [19] and [18]. The work [5], contrary to the other works, follows a slightly different approach by using a Hoeffding ANOVA [14] based construction of collocation points to approximate the solution's random behavior.

In this work we will follow the idea of Smolyak sparse grids in the stochastic parameter space and, in addition, also propose a sparse composition of the levels of spatial refinement and of the stages of the Smolyak cubature formula.

To simplify the presentation and discussion of the sparse composite collocation method we restrict ourselves to the model problem

$$\begin{cases} -\operatorname{div}(a(\omega,\mathbf{x})\nabla u(\omega,\mathbf{x})) = f(\mathbf{x}) \quad D, \\ u(\omega,\mathbf{x})|_{\mathbf{x}\in\partial D} = 0, \end{cases} \quad P\text{−a.e. } \omega \in \Omega \tag{3.1}$$

where the diffusion coefficient is represented by a (truncated) Karhunen-Loève expansion. Hence, we choose $\mathcal{W}(D) = H_0^1(D)$ as the Banach space of solutions. However, it is emphasized here that any elliptic problem (1.1) satisfying Assumptions 2.1 – 2.6 could, in principle, be considered instead of (3.1).

In the following subsections we will first discuss the sparse hierarchic semidiscretization in random parameter space Γ followed by the hierarchic wavelet discretization in D. Following the idea of sparse tensor products we then show by numerical analysis how to combine the increment spaces of the two hierarchic (semi-) discretizations in order to obtain a fully discrete sparse composite collocation scheme.

3.1 Smolyak collocation algorithm

In this section we will construct a hierarchic sequence of collocation operators. Due to the high dimensionality M of the random parameter space Γ, those collocation operators are based on a Smolyak construction of collocation points as also done in [18, 19, 29, 28, 10]. However, contrary to the other works, we will focus on Gaussian quadrature rules as the underlying univariate numerical integration schemes since their approximation properties allow us to avoid the introduction of a Lebesgue constant in each subspace Γ_m, resulting in an overall constant which would depend exponentially on M. Moreover, by renouncing nestedness, we gain the flexibility to increase the number of collocation points in each direction linearly w.r.t. the level of refinement instead of doubling it in each step.

3.1.1 Smolyak construction of collocation points

We choose on each level $l_1 \in \mathbb{N}_0$ a discrete, finite set of collocation points $\mathcal{Y}_{l_1} \subset \Gamma$ and a set of multiindices $\Lambda_{l_1} \subset \mathbb{N}^M$ such that the sequence is nested with increasing

l_1, i.e.

$$\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{l_1} \subset \Lambda_{l_1+1} \subset \cdots \mathbb{N}^M. \tag{3.2}$$

and such that the interpolation problem for the points $\mathcal{Y}_l$ is well-posed in $V_{l_1}^\Gamma := \text{span}\{\mathbf{y}^\alpha : \alpha \in \Lambda_{l_1}\}$. Naturally, the $V_{l_1}^\Gamma$ are also nested and we define the interpolation operators

$$\mathcal{I}_{l_1}^\Gamma : C_\sigma^0(\Gamma, H_0^1) \longrightarrow V_{l_1}^\Gamma \tag{3.3}$$

and the difference operators

$$\Delta_0^\Gamma := \mathcal{I}_0^\Gamma \qquad \Delta_{l_1}^\Gamma := \mathcal{I}_{l_1}^\Gamma - \mathcal{I}_{l_1-1}^\Gamma \tag{3.4}$$

We now solve the semidiscretized problem (3.33) for each point $\mathbf{y} \in \mathcal{Y}_l$. The random solution $u \in L_\rho^2(\Gamma) \otimes V_l^D$ is then recovered by interpolating over the collocated solutions by means of (3.3).

One way to do so is by using a full tensor Lagrange interpolation operator: Given a set of abscissae $\{y_{m,0}, \ldots, y_{m,\mu_m}\} \subset \Gamma_m$ for each $1 \le m \le M$, we denote by

$$\mathcal{X}_\mu^{\text{full}} := \{\mathbf{y_k} = (y_{1,k_1}, \ldots, y_{M,k_M}) \ : \ 0 \le k_m \le \mu_m\} \tag{3.5}$$

the tensorized grid of collocation points. The full tensor Lagrange interpolation operator is then defined as

$$\mathcal{I}^{\text{full}} = \sum_{\mathbf{y_k} \in \mathcal{X}_\mu^{\text{full}}} u(\mathbf{y_k}, \mathbf{x}) l_{\mathbf{k}}(\mathbf{y}) \tag{3.6}$$

where $u(\mathbf{y_k}, \cdot) \in V_l^D$ is the solution to (3.33) at the point $\mathbf{y_k}$ and $l_{\mathbf{k}}(\mathbf{y})$ the tensorized Lagrange interpolation polynomial at the point $\mathbf{y_k}$. The application of this interpolation formula, however, requires $\prod_{m=1}^M \mu_m$ deterministic problems to be solved. In cases where M is large, this can be very prohibitive (curse of dimension).

Hence, to keep the number of collocation points moderate, like in [19, 18, 29], we use a Smolyak-type construction but based on Gauss quadrature points. For any $1 \le m \le M$, any real number γ and a level l_m let $\{y_{m,0}, \ldots, y_{m,\mu_m}\} \subset \Gamma_m$ be the abscissae of the Gauss quadrature rule of order $\mu_m(\gamma l_m)$ with respect to the probability density ρ_m, where

$$\mu_m(\gamma l_m) = \begin{cases} \lceil \gamma l_m \rceil & \Gamma_m \text{ bounded} \\ \lceil (\gamma l_m)^2 \rceil & \Gamma_m \text{ unbounded} \end{cases} \tag{3.7}$$

The one-dimensional interpolation operators $\mathcal{I}_{l_m}^{(m)}(v)$ are then given by

$$\mathcal{I}_{l_m}^{(m)}(v) = \sum_{k_m=0}^{\mu_m(\gamma l_m)} v(y_{m,k_m}) \ell_{k_m}(y_m) \tag{3.8}$$

where ℓ_{k_m} denotes the Lagrange interpolation polynomial of degree $\mu_m(l_m)$ in the point y_{m,k_m} defined by

$$\ell_{m,k_m}(y_m) \in \mathcal{P}_{\mu_m}(\Gamma_m), \qquad \ell_{m,k_m}(y_{m,j_m}) = \delta_{k_m j_m}, \quad 0 \le k_m, j_m \le \mu_m. \tag{3.9}$$

Remark 3.1. Standard probability density functions, such as Gaussian or uniform ones, lead to well-known abscissae which are tabulated to full accuracy. For non-standard densities we refer to [11], Theorem 3.1 where an algorithm for the derivation of associated Gauss nodes and weights is described.

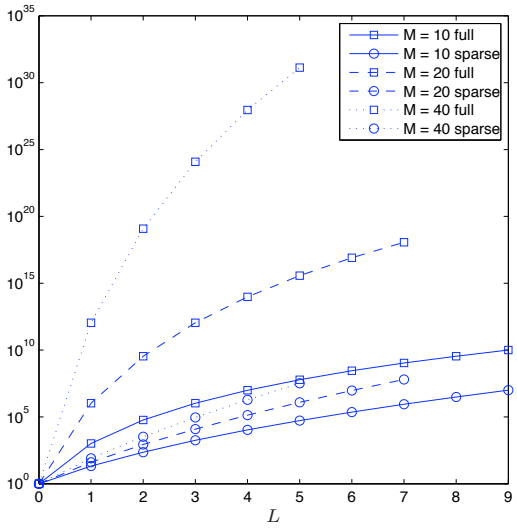

Figure 3.1: Number of collocations points in a full tensor approach (3.5) compared to the number of collocation points in the Smolyak approach (3.11) for dimensions $M = 10, 20, 40$

Next, we define the univariate differences

$$\Delta_{l_m}^{(m)} := \mathcal{I}_{l_m}^{(m)} - \mathcal{I}_{l_m-1}^{(m)}, \quad \Delta_0^{(m)} = \mathcal{I}_0^{(m)} \tag{3.10}$$

Using the multiindex notation $l = (l_1, \ldots, l_M)$ the (isotropic) M-dimensional Smolyak interpolation operator is then defined by

$$\mathcal{I}_L^{\Gamma} := \sum_{0 \le |l| \le L} (\Delta_{l_1}^{(1)} \otimes \cdots \otimes \Delta_{l_M}^{(M)}). \tag{3.11}$$

Figure 3.1 illustrates the number of collocation points of the Smolyak approach (3.11) compared to the number of collocation points in a full tensor approach (3.5).

3.1.2 Error analysis of the Smolyak collocation

For the error analysis of the Smolyak collocation algorithm described above, we assume for now that for each point $\mathbf{y_k}$ the associated elliptic problem (3.33) can be solved exactly. The approximate solution of these problems and their interaction with the Smolyak collocation method is subject of the subsequent sections.

We first provide a result on the approximation properties of the univariate interpolation operators (3.8):

Proposition 3.2. *If v solves (1.1) and satisfies the Assumptions 2.1–2.6 then the interpolation error admits the following coordinate-wise bound:*

$$\|v - \mathcal{I}_{l_m}^{(m)}(v)\|_{L^2_{\rho_m}(\Gamma_m, H^1_0(D))} \le C(r_m)\beta_m(l_m)e^{-r_m l_m}\|v\|_{C^0_{\sigma_m}(\Sigma(\Gamma_m,\tau), L^2_{\rho^*}(\Gamma_m^*, H^1_0(D)))} \tag{3.12}$$

where $\|v\|_{C^0_{\sigma_m}(\Sigma(\Gamma_m,\tau),V)} = \sigma(\mathrm{Re}z)\|v\|_V$ denotes the complex extension of the norm and

- *if* Γ_m *is bounded* $\quad \begin{cases} \beta_m \equiv 1, \\ r_m = \log\left(\frac{2\tau_m}{|\Gamma_m|} + \sqrt{1+\left(\frac{2\tau_m}{|\Gamma_m|}\right)^2}\right) \\ C(r_m) = C_1 \frac{1}{e^{r_m}-1} \end{cases}$

- *if* Γ_m *is unbounded* $\quad \begin{cases} \beta_m = l_m, \\ r_m = \tau_m \delta_m \\ C(r_m) = C_1 \frac{(1+r_m)}{r_m^2} \end{cases}$

where C_1 *is independent of* m, l_m, r_m *and with* τ_m *defined as in Lemma 2.8 and* δ_m *given by Assumption 2.5.*

We note here that a very similar result has already been proposed in [2] but the present result provides sharper estimates on the constants $C(r_m)$. In particular we see in the above proposition that in both cases the constants are decaying to zero since the size of the analyticity regions r_m is growing to infinity if $m \to \infty$, see also Lemma 2.8. The proof will be provided in Appendix A.

Following the lines of [26, 20] we will now estimate the interpolation error of the Smolyak operator (3.11). First we note that

$$\begin{aligned} \mathcal{I}_L^\Gamma &= \sum_{0\le|l^*|\le L} \Delta_{l_1}^{(1)} \otimes \cdots \otimes \Delta_{l_{M-1}}^{(M-1)} \otimes \sum_{l_M=0}^{L-|l^*|} \Delta_{l_M}^{(M)} \\ &= \sum_{0\le|l^*|\le L} \Delta_{l_1}^{(1)} \otimes \cdots \otimes \Delta_{l_{M-1}}^{(M-1)} \otimes \mathcal{I}_{L-|l^*|}^{(M)} \end{aligned} \tag{3.13}$$

where $l^* = (l_1, \ldots, l_{M-1})$. In the following denote by $\mathrm{Id}^{(m)}$ the identity operator on Γ_m and $\mathrm{Id}^M = \prod_{m=1}^M \mathrm{Id}^{(m)}$. The interpolation error can be written as

$$\mathrm{Id}^M - \mathcal{I}_L^\Gamma = (\mathrm{Id}^{M-1} - \mathcal{I}_L^{\Gamma_M^*}) \otimes \mathrm{Id}^{(M)} + \sum_{0\le|l^*|\le L} \Delta_{l_1}^{(1)} \otimes \cdots \otimes \Delta_{l_{M-1}}^{(M-1)} \otimes (\mathrm{Id}^{(M)} - \mathcal{I}_{L-|l^*|}^{(M)}) \tag{3.14}$$

where $\Gamma_M^* = \Gamma_1 \times \ldots \times \Gamma_{M-1}$. In the following, unless otherwise stated, denote by $\|\cdot\|$ the $L_\rho^2(\Gamma, H_0^1(D))$-norm or its associated operator norm. From

$$\|\Delta_{l_m}^{(m)}\| \le \|\mathrm{Id}^{(m)} - \mathcal{I}_{l_m}^{(m)}\| + \|\mathrm{Id}^{(m)} - \mathcal{I}_{l_m-1}^{(m)}\| \le 2C_m \beta_m(\gamma l_m) e^{-r_m \gamma l_m}, \tag{3.15}$$

where $C_m = C(r_m)$ as in Proposition 3.2,it follows together with (3.14)

$$\begin{aligned} \|\mathrm{Id}^M - \mathcal{I}_L^\Gamma\| &\le \|\mathrm{Id}^{M-1} - \mathcal{I}_L^{\Gamma_M^*}\| + \sum_{0\le|l^*|\le L} \|\Delta_{l_1}^{(1)}\| \cdots \|\Delta_{l_{M-1}}^{(M-1)}\| \|\mathrm{Id}^{(M)} - \mathcal{I}_{L-|l^*|}^{(M)}\| \\ &\le \|\mathrm{Id}^{M-1} - \mathcal{I}_L^{\Gamma_M^*}\| \\ &\quad + \sum_{0\le|l^*|\le L} \prod_{m=1}^{M-1} 2C_m \beta_m(\gamma l_m) e^{-r_m l_m} \cdot C_M \beta_M(\gamma l_M) e^{-r_M \gamma l_M} \\ &= \|\mathrm{Id}^{M-1} - \mathcal{I}_L^{\Gamma_M^*}\| + \sum_{0\le|l^*|\le L} \prod_{m=1}^{M} 2C_m \beta_m(\gamma l_m) e^{-r_m \gamma l_m} \end{aligned} \tag{3.16}$$

Setting $r_{\min} = \min\{r_1, \ldots, r_M\}$ we arrive at the following recursive formula

$$\|\mathrm{Id}^M - \mathcal{I}_L^\Gamma\| \le \|\mathrm{Id}^{M-1} - \mathcal{I}_L^{\Gamma_M^*}\| + (\prod_{m=1}^M 2C_m) e^{-r_{\min}\gamma L} \binom{M+L}{M} \tag{3.17}$$

where the $\beta_m(\gamma l_m)$ have been absorbed in the exponent by possibly increasing the constants C_m. We observe from Proposition 3.2 that $C_m \to 0$ if $m \to \infty$, hence the product in (3.17) can be bounded from above by a generic constant $\bar{C}$ which is independent of M. Using (3.16) repeatedly we arrive at

$$\begin{aligned} \|\mathrm{Id}^M - \mathcal{I}_L^\Gamma\| &\leq \|\mathrm{Id}^1 - \mathcal{I}^1(L)\| + \frac{1}{2}\sum_{m=2}^{M} \bar{C} e^{-r_{\min}\gamma L} \binom{m+L}{m} \\ &\leq \bar{C} e^{-r_{\min}\gamma L} \binom{M+L}{M} \end{aligned} \tag{3.18}$$

Next, we want to estimate the number of collocation points used. In order to do this we distinguish the cases where the domains Γ_m are bounded and unbounded. We first begin with the case where the random variables are bounded:

Lemma 3.3.

$$N_{\Gamma,b} = \sum_{|l|\leq L} \prod_{m=1}^{M} (1+\gamma l_m) = \binom{\gamma L + 2M}{2M} \tag{3.19}$$

Proof We first note that

$$N_{\Gamma,b} = \sum_{|l|\leq \gamma L+M} \prod_{m=1}^{M} l_m \tag{3.20}$$

Now define $\alpha_k = \sum_{|l|=k} \prod_{m=1}^{M} l_m$ and observe that

$$\sum_{k=0}^{\infty} \alpha_k x^k = (x + 2x^2 + 3x^3 + \ldots)^M \tag{3.21}$$

The generating function of the series $a_k = k$ for $k \geq 1$ is $\frac{x}{(1-x)^2}$, see [27], hence

$$\sum_{k=0}^{\infty} \alpha_k x^k = \left(\frac{x}{(1-x)^2}\right)^M. \tag{3.22}$$

This in turn means that α_k is the $(k-M)$-th coefficient of the power series of $\frac{1}{(1-x)^{2M}}$, which, by [27], Section 2.5 is equal to $\binom{k+M-1}{2M-1}$. Summing the α_k's therefore leads to

$$\sum_{|l|\leq \gamma L+M} \prod_{m=1}^{M} l_m = \sum_{k=0}^{\gamma L+M} \alpha_k = \binom{\gamma L + 2M}{2M} \tag{3.23}$$

which completes the proof. □

For the unbounded case we have that

Lemma 3.4.

$$N_{\Gamma,u} = \sum_{|l|\leq L} \prod_{m=1}^{M} (1 + (\gamma l_m)^2) \leq C \frac{2^M}{\sqrt{\pi \lceil M/2 \rceil}} \binom{\gamma L + 3M + 1}{3M+1} \tag{3.24}$$

Proof We proceed similarly to the bounded case and first note that

$$N_{\Gamma,u} \leq \sum_{|l|\leq \gamma L+M} \prod_{m=1}^{M} l_m^2$$

Defining $\alpha_k = \sum_{|l|=k} \prod_{m=1}^{M} l_m^2$ we observe

$$\sum_{k=0}^{\infty} \alpha_k x^k = (x + 4x^2 + 9x^3 + \ldots)^M = \left(\frac{x(x+1)}{(1-x)^3}\right)^M$$

where the last equation follows from the fact that the generating function of the series $a_k = k^2$ is $\frac{x(x+1)}{(1-x)^3}$ [27]. From the binomial theorem we have that

$$(x+1)^M = \sum_{k=0}^{\infty} \binom{M}{k} x^k \quad \text{and} \quad (1-x)^{-3d} = \sum_{k=0}^{\infty} \binom{k+3M-1}{3M-1} x^k.$$

Hence,

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k x^k &= x^M \left(\sum_{k=0}^{\infty} \binom{M}{k} x^k\right) \left(\sum_{k=0}^{\infty} \binom{k+3M-1}{3M-1} x^k\right) \\ &= x^M \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{l+3M-1}{3M-1} \binom{M}{k-l}\right) x^k \end{aligned}$$

and therefore $\alpha_k = \sum_{l=0}^{k-M} \binom{l+3M-1}{3M-1} \binom{M}{k-M-l}$. To estimate α_k we use the fact that $\binom{M}{k}$ attains its maximum value at $k = \lceil \frac{M}{2} \rceil$.

$$\alpha_k \leq \binom{M}{\lceil M/2 \rceil} \sum_{l=0}^{k-M} \binom{l+3M-1}{3M-1} = \binom{M}{\lceil M/2 \rceil} \binom{k+2M}{3M}$$

Finally we use Stirling's approximation and obtain

$$\sum_{|l| \leq \gamma L+M} \prod_{m=1}^{M} l_m^2 = \sum_{k=0}^{\gamma L+M} \alpha_k \leq C \frac{2^M}{\sqrt{\pi \lceil M/2 \rceil}} \binom{\gamma L + 3M + 1}{3M+1}$$

which proves (3.24). □

Inserting the estimates for $N_{\Gamma,b}$ and $N_{\Gamma,u}$ into (3.18) we obtain

Lemma 3.5. *Let u be the solution to (3.1). Then the collocation error admits the following bounds with respect to the level L and the number of collocation points:*

$$\|u - \mathcal{I}_L^{\Gamma} u\|_{L^2_{\rho}(\Gamma, H^1_0(D))} \leq C e^{-r_{\min} \gamma L} \binom{M+L}{M} \|u\|_{C^0_{\sigma}(\Sigma(\Gamma,\tau), H^1_0(D))} \tag{3.25}$$

and

$$\|u - \mathcal{I}_L^{\Gamma} u\|_{L^2_{\rho}(\Gamma, H^1_0(D))} \leq C \sqrt{N_{\Gamma,b}} \exp(-r_{\min}(N_{\Gamma,b}^{1/(2M)} - 1) \frac{2M}{1 + \ln 2M}) \|u\| \tag{3.26}$$

in the case where the Γ_m's are bounded or

$$\|u - \mathcal{I}_L^{\Gamma} u\|_{L^2_{\rho}(\Gamma, H^1_0(D))} \leq C \sqrt[3]{N_{\Gamma,u}} \exp(-r_{\min}(\frac{1}{\sqrt[3]{2}} N_{\Gamma,u}^{1/(3M+1)} - 1) \frac{3M+1}{1 + \ln(3M+1)}) \|u\| \tag{3.27}$$

in the unbounded case, where $N_{\Gamma,b}$ and $N_{\Gamma,u}$, as in Lemma 3.3 and 3.4, refer to the number of collocation points in each case and $C > 0$ is independent of M and $N_{\Gamma,b}$ or $N_{\Gamma,u}$, respectively.

Proof From (3.19) and using the inequality between geometric and arithmetic mean we have

$$\begin{aligned} N_{\Gamma,b} &\leq \binom{\gamma L + 2M}{2M} \\ &= \frac{(\gamma L + 2M)(\gamma L + 2M - 1)\cdots(\gamma L + 1)}{(2M)!} \\ &= (1 + \frac{\gamma L}{2M})(1 + \frac{\gamma L}{2M-1})\cdots(1 + \frac{\gamma L}{1}) \\ &\leq \left(\frac{2M + \gamma L \sum_{k=1}^{2M} 1/k}{2M}\right)^{2M} \\ &\leq \left(1 + \frac{\gamma L(1 + \ln 2M)}{2M}\right)^{2M} \end{aligned} \tag{3.28}$$

Hence,$L \geq (N_{\Gamma,b}^{1/2M} - 1)\frac{2M}{\gamma(1+\ln 2M)}$ from which (3.26) follows. The case of unbounded variables is treated in exactly the same way, see [3] for further details. □

Remark 3.6. As it can be seen in Lemma 3.5 the convergence rates with respect to the number of collocation points are exponential but depending on M. Since M can possibly be very large in applications, it is of interest to examine this case in some more detail. From (3.28) we have for large M

$$N_{\Gamma,b} \leq e^{\gamma L(1+\ln 2M)} \tag{3.29}$$

which, inserted into (3.25) and noting that $\binom{M+L}{M} \leq e^{M+L}$, gives

$$\|u - \mathcal{I}_L^\Gamma u\|_{L^2_\rho(\Gamma,H^1_0(D))} \leq C N_{\Gamma,b}^{-(r_{\min}-1)/(1+\ln 2M)} \|u\|_{C^0_\sigma(\Sigma(\Gamma,\tau),H^1_0(D))}$$

A similar result can be derived in the unbounded case, namely

$$\|u - \mathcal{I}_L^\Gamma u\|_{L^2_\rho(\Gamma,H^1_0(D))} \leq C N_{\Gamma,b}^{-(r_{\min}-1)/(1+\ln(3M+1))} \|u\|_{C^0_\sigma(\Sigma(\Gamma,\tau),H^1_0(D))}$$

Hence, for $M \gg 1$, the exponential convergence rates in Lemma 3.5 behave pre-asymptotically, i.e. for small L, like algebraic ones.

3.2 Wavelet discretization in space

Next we discretize our model problem in space. The (parametric) variational formulation of (3.1) is given by: Find $u \in L^2_\rho(\Gamma) \otimes H^1_0(D)$ such that $\forall\, v \in H^1_0(D)$ it holds

$$b(\mathbf{y}; u, v) := \int_D a(\mathbf{y}, \mathbf{x}) \nabla u(\mathbf{y}, \mathbf{x}) \nabla v(\mathbf{x})\, d\mathbf{x} = \int_D f(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \tag{3.30}$$

where $b(\mathbf{y}; \cdot, \cdot)$ emphasizes the dependence of the bilinear form on $\mathbf{y} \in \Gamma$. Let

$$V_0^D \subset V_1^D \subset ... \subset V_{l_2}^D \subset V_{l_2+1}^D \subset ... \subset H^1_0(D) \tag{3.31}$$

be a dense, hierarchic sequence of finite dimensional subspaces given by e.g. a piecewise linear finite element wavelet discretization on a nested sequence $\{\mathcal{T}_{l_2}\}_{l_2}$ of triangulations of D as described below and define the *detail spaces* $W_{l_2}^D$ such that

$$W_0^D := V_0^D \quad \text{and} \quad V_{l_2}^D = V_{l_2-1}^D \oplus W_{l_2}^D \text{ for } l_2 \geq 1 \tag{3.32}$$

Hence, the FE-space V_L^D admits a multilevel decomposition $V_L^D := \bigoplus_{l=0}^L W_l^D$. The semidiscrete problem on level L then reads: Find $u_L \in L^2_\rho(\Gamma) \otimes V_L^D$ such that $\forall\, v \in V_L^D$ it holds

$$b(\mathbf{y}; u_L, v) := \int_D a(\mathbf{y}, \mathbf{x}) \nabla u_L(\mathbf{y}, \mathbf{x}) \nabla v(\mathbf{x})\, d\mathbf{x} = \int_D f(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \tag{3.33}$$

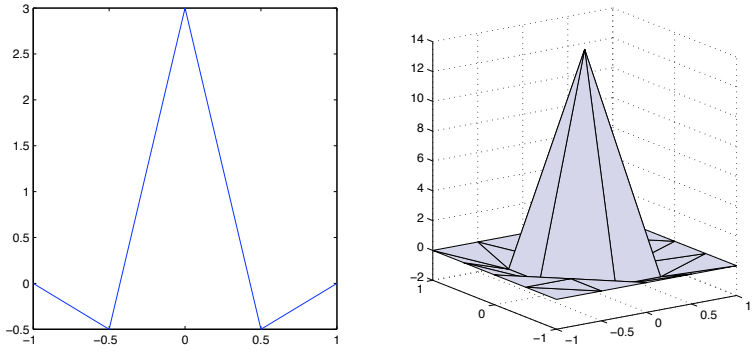

Figure 3.2: The function $\theta_{l_2}^i$ on the reference element $[-1,1]^d$ for $d=1$ (left) and $d=2$ (right)

As a hierarchical basis for the spatial discretization we choose linear finite element wavelet bases. In the following we will briefly present a construction of such bases and recapitulate basic approximation results. For a more detailed introduction into wavelets we refer to [7, 8]. The construction presented here is based on [17] which, in principle, allows us to construct wavelets on a regular simplicial triangulation of $D \subset \mathbb{R}^d$. Here, we will treat the case of $d=1$ and $d=2$ explicitly.

Let $\{\mathcal{T}_{l_2}\}_{l_2}$ be a nested sequence of regular simplicial triangulations of D and denote by $\mathcal{I}(\mathcal{T}_{l_2})$ the index set of vertices of the mesh $\mathcal{T}_{l_2}$, denoted by $\mathcal{V}(\mathcal{T}_{l_2})$, and by $\hat{\mathcal{I}}(\mathcal{T}_{l_2+1})$ the index set of vertices of the mesh $\mathcal{T}_{l_2+1}$ which do not belong to $\mathcal{T}_{l_2}$. We define

$$V_{l_2}^D := S^1(D,\mathcal{T}_{l_2}) = \left\{u \in H^1(D) : u|_T \in \mathcal{P}_1(T) \text{ for } T \in \mathcal{T}_{l_2}\right\}, \tag{3.34}$$

i.e. the space of continuous piecewise linear functions on the triangulation $\mathcal{T}_{l_2}$. Clearly, the spaces $V_{l_2}^D$ are hierarchic in the sense of (3.31). Denote by $\phi_{l_2}^k(\mathbf{x})$, $k \in \mathcal{I}(\mathcal{T}_{l_2})$ the standard hat functions on the mesh $\mathcal{T}_{l_2}$, i.e. the piecewise linear polynomials with value 1 at the vertex k and zero at the other nodes. Now we want to find a basis for the detail spaces $W_{l_2}^D$ where $l_2 = 0,1,2,\ldots$. On the level $l_2 = 0$ those are called *scaling functions* and are defined as the hat functions on the coarsest mesh $\mathcal{T}_0$, i.e. $\psi_{0k} = \phi_0^k$ and $\nabla_0 = \mathcal{I}(\mathcal{T}_0)$. On higher levels $l_2 > 0$ those basis functions are called *wavelets* and their construction is based on the meshes $\mathcal{T}_{l_2}$ and $\mathcal{T}_{l_2-1}$ as presented in the following. First, we construct a family of functions $\theta_{l_2}^i(\mathbf{x}) \in S^1(D,\mathcal{T}_{l_2})$, $i \in \mathcal{I}(\mathcal{T}_{l_2-1})$ satisfying $(\theta_{l_2}^i, \phi_{l_2-1}^k)_{L^2(D)} \simeq \delta_{ik}$. Such functions are given in $d=1$ by

$$\theta_{l_2}^i(\mathbf{v}) = \begin{cases} 3 & \mathbf{v} = \mathbf{v}_i \in \mathcal{V}(\mathcal{T}_{l_2-1}) \\ -\frac{1}{2} & \mathbf{v} \in \mathcal{V}(\mathcal{T}_{l_2}) \text{ is neighbor of } \mathbf{v}_i \\ 0 & \text{any other } \mathbf{v} \in \mathcal{V}(\mathcal{T}_{l_2}) \end{cases} \tag{3.35}$$

and in $d=2$ by

$$\theta_{l_2}^i(\mathbf{v}) = \begin{cases} 14 & \mathbf{v} = \mathbf{v}_i \in \mathcal{V}(\mathcal{T}_{l_2-1}) \\ -1 & \mathbf{v} \in \mathcal{V}(\mathcal{T}_{l_2}) \text{ is neighbor of } \mathbf{v}_i \\ 0 & \text{any other } \mathbf{v} \in \mathcal{V}(\mathcal{T}_{l_2}) \end{cases} \tag{3.36}$$

See Figure 3.2 for an example of such a $\theta_{l_2}^i$ in one and two dimensions.

The ensemble of functions $\{\theta_{l_2}^i : i \in \mathcal{I}(\mathcal{T}_{l_2-1})\} \cup \{\phi_{l_2}^i : i \in \hat{\mathcal{I}}(\mathcal{T}_{l_2})\}$ forms a L^2-Riesz basis of $V_{l_2}^D$ satisfying $(\theta_{l_2}^i, \phi_{l_2-1}^k)_{L^2(D)} = 0$ if $i \neq k$. The wavelets on level l_2

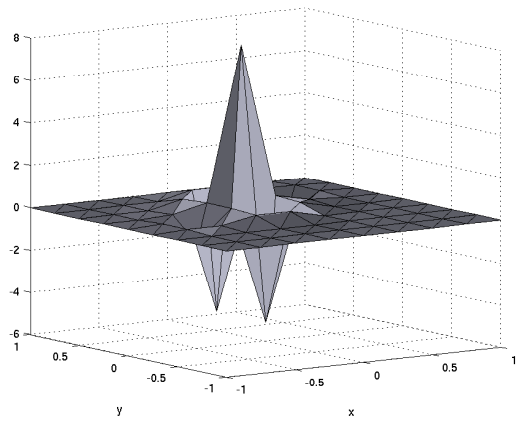

Figure 3.3: A piecewise linear finite element wavelet on a triangular mesh in 2-d

are then obtained by the formula

$$\psi_{l_2}^i(\mathbf{x}) = \phi_{l_2}^i(\mathbf{x}) - \sum_{k \in \mathcal{I}(\mathcal{T}_{l_2-1})} \frac{(\phi_{l_2}^i, \phi_{l_2-1}^k)_{L^2(D)}}{(\theta_{l_2}^k, \phi_{l_2-1}^k)_{L^2(D)}} \theta_{l_2}^k(\mathbf{x}), \quad i \in \hat{\mathcal{I}}(\mathcal{T}_{l_2}) \tag{3.37}$$

An example of such a wavelet on a triangular mesh in $D \subset \mathbb{R}^2$ can be seen in Figure 3.2

The functions $\psi_{l_2 i}$, $i \in \nabla_{l_2} := \hat{\mathcal{I}}(\mathcal{T}_{l_2})$ form a uniform L^2-Riesz basis for $W_{l_2}^D$, see [17], Proposition 3.2.10. Scaling the wavelets with a factor 2^{-l_2} i.e. defining $\tilde{\psi}_{l_2 i}(\mathbf{x}) = 2^{-l_2}\psi_{l_2 i}(\mathbf{x})$ then forms a Riesz basis for $H^1(D)$. In case of homogeneous Dirichlet boundary conditions, the above construction can be modified as follows: For $\mathbf{v}_i \in \mathcal{V}(\mathcal{T}_{l_2})$, the corresponding $\phi_{l_2-1}^i, \phi_{l_2}^i$ and $\theta_{l_2}^i$ are excluded from the ensembles and the resulting wavelets $\tilde{\psi}_{l_2 j}$ then form a uniform Riesz bases for $H_0^1(D)$.

Denote the H^1-projection $P_{l_2}^D : H^1(D) \longrightarrow V_{l_2}^D$. For functions $u \in H^{1+t}(D)$ $t \in [0,1]$ and $l_2 \geq 0$ the following approximation property is proved in [17], Lemma 2.3.1:

$$\|u - P_{l_2}^D u\|_{H^1(D)} \lesssim 2^{-l_2 t} \|u\|_{H^{1+t}(D)}, \quad t \in [0,1] \tag{3.38}$$

Noting that $N_{l_2}^D := \dim(V_{l_2}^D) \simeq 2^{l_2 d}$ we then obtain the following convergence rates with respect to the number of degrees of freedom:

$$\|u - P_{l_2}^D u\|_{H^1(D)} \lesssim (N_{l_2}^D)^{-t/d} \|u\|_{H^{1+t}(D)}, \quad t \in [0,1]. \tag{3.39}$$

3.3 Formulation of the sparse composite collocation method

Having specified the hierarchic sequence of interpolation operators $\{\mathcal{I}_{l_1}^\Gamma\}_{l_1 \geq 0}$ (3.3) and finite element spaces $\{V_{l_2}^D\}_{l_2 \geq 0}$ (3.31) we can describe the (full) tensor collocation approximation to the solution u of (3.30) as

$$u_L := \mathcal{I}_L^\Gamma(P_L^D u) = \sum_{0 \leq l_1, l_2 \leq L} \Delta_{l_1}^\Gamma((P_{l_2}^D - P_{l_2-1}^D)u) \tag{3.40}$$

Consequently, this approach uses a total number of $N_{\text{tot}}(L) = N_D(L) \cdot N_\Gamma(L)$ degrees of freedom where $N_D(L)$ denotes the number of spatial basis functions and $N_\Gamma(L)$ the number of collocation points in the random parameter space, each at level L.

To reduce this number we will consider a sparse tensor approximation instead of (3.40):

$$\hat{u}_L := \sum_{0 \le l_1 + l_2 \le L} \Delta_{l_1}^{\Gamma}((P_{l_2}^D - P_{l_2-1}^D)u) \tag{3.41}$$

Under the assumption that the dimension of the detail spaces $W_{l_2}^D$ and the number of collocation points do not grow faster than exponential with respect to the level, we can prove the following estimate on the number of total degrees of freedom

Lemma 3.7. *Assume that the number of collocation points between each level l_1 and the dimension of the detail space $W_{l_2}^D$ do not grow faster than exponential with respect to the levels l_1, l_2, i.e. there exist two geometric sequences $b_\Gamma^{l_1}$ and $b_D^{l_2}$, $l_1, l_2 = 1, 2, 3, ...$, with $b_\Gamma > 1$ and $b_D > 1$, such that*

$$(|\mathcal{Y}_{l_1}| - |\mathcal{Y}_{l_1-1}|) \lesssim b_\Gamma^{l_1} \quad \text{and} \quad \dim(W_{l_2}^D) \lesssim b_D^{l_2}$$

Then the sparse tensor approximation of the solution u to (3.30) by means of (3.41) uses

$$N_{\text{tot}} \lesssim L^\theta \max\{b_\Gamma, b_D\}^{L+1} \tag{3.42}$$

degrees of freedom where $\theta = 1$ if $b_\Gamma = b_D$ and zero otherwise.

Proof A straightforward calculation, see also [4], Lemma 3.4. □

3.4 Analysis of the sparse composite collocation method

The main focus of this section aims at proving convergence rates for the sparse tensor collocation method in both, the case of bounded and unbounded domains Γ_m.

Proposition 3.8. *Let u be the solution to (3.1) and denote by $\hat{u}_L$ its sparse tensor approximation given by (3.41). In the case where the Γ_m are bounded we obtain the convergence rates*

$$\|u - \hat{u}_L\|_{L^2_\rho(\Gamma, H_0^1)} \le C N_{tot}^{-\beta_b} \|u\|_{C^0_\sigma(\Sigma(\Gamma,\tau), H^{1+t} \cap H_0^1(D))} \tag{3.43}$$

in terms of the total number of degrees of freedom N_{tot} where as in the unbounded case we obtain

$$\|u - \hat{u}_L\|_{L^2_\rho(\Gamma, H_0^1)} \le C N_{tot}^{-\beta_u} \|u\|_{C^0_\sigma(\Sigma(\Gamma,\tau), H^{1+t} \cap H_0^1(D))} \tag{3.44}$$

with

$$\beta_b(M) = \min\{\frac{r_{\min} - 1}{1 + \ln 2M}, \frac{t \ln 2}{d}\} \quad \beta_u(M) = \min\{\frac{r_{\min} - 1}{(1 + \ln(3M+1)) + M}, \frac{t \ln 2}{d}\} \tag{3.45}$$

In other words, by the sparse composition of stochastic interpolation operators and a multilevel basis in space we retrieve the smaller of the two convergence rates provided by the spatial discretization (cf. (3.39)) and stochastic interpolation (cf. Remark 3.6).

Remark 3.9 *(Full tensor composition).* The main motivation for the sparse tensor composition (3.41) of stochastic interpolation operators and spatial Galerkin projectors lies in the reduction of degrees of freedom to $O(L^\theta \max\{N_\Gamma(L), N_D(L)\})$ as shown in Lemma 3.7 opposed to $O(N_\Omega(L) \cdot N_D(L))$ in the full composite approach

(3.40). Using the approximation results (3.38) and (3.25) we can easily derive the full composite convergence rates

$$\|u - \mathcal{I}_L^\Gamma(P_L^D u)\|_{L^2(\Gamma,H_0^1(D))} \leq (N_L)^{-\bar{\beta}} \|u\|_{C_\sigma^0(\Sigma(\Gamma,\tau),H^{1+t}\cap H_0^1(D))} \tag{3.46}$$

with $\bar{\beta} = (d/t + (1+\ln 2M)(r_{\min}-1))^{-1}$ in the case of bounded Γ and likewise in the case of unbounded stochastic domains.

Proof First, we will estimate the approximation error in terms of the refinement level L. To simplify the notation we will drop the subscript in the norms, hence $\|\cdot\| = \|\cdot\|_{L^2(\Gamma,H_0^1)}$ unless otherwise indicated. Using (3.41) we obtain

$$\begin{aligned}
\|u-\hat{u}_L\| &\leq \|\sum_{l_1=0}^{L}\sum_{l_2=L-l_1+1}^{\infty}\Delta_{l_1}(P_{l_2}^D - P_{l_2-1}^D)u\| \\
&\quad + \|\sum_{l_1=L+1}^{\infty}\sum_{l_2=0}^{\infty}\Delta_{l_1}(P_{l_2}^D - P_{l_2-1}^D)u\| \\
&= \|\sum_{l_1=0}^{L}\Delta_{l_1}(\mathrm{Id}^D - P_{L-l_1}^D)u\| + \|(\mathrm{Id}^\Gamma - \mathcal{I}_L^\Gamma)\mathrm{Id}^D u\| \\
&\leq \sum_{l_1=0}^{L}\|(\mathrm{Id}^\Gamma - \mathcal{I}_{l_1}^\Gamma)(\mathrm{Id}^D - P_{L-l_1}^D)u\| \\
&\quad + \sum_{l_1=0}^{L}\|(\mathrm{Id}^\Gamma - \mathcal{I}_{l_1-1}^\Gamma)(\mathrm{Id}^D - P_{L-l_1}^D)u\| + \|(\mathrm{Id}^\Gamma - \mathcal{I}_L^\Gamma)\mathrm{Id}^D u\| \\
&\lesssim \left(\sum_{l_1=0}^{L} C\binom{M+l_1}{M} e^{-r_{\min}\gamma l_1} e^{-\ln 2t(L-l_1)}\right. \\
&\quad \left. + C\binom{M+L}{M} e^{-r_{\min}\gamma L}\right)\|u\|_{C_\sigma^0(\Sigma(\Gamma,\tau),H^{1+t}\cap H_0^1(D))}
\end{aligned}$$

where we used the estimates (3.38) and (3.25) in the last line. Absorbing now the binomial factor into the exponent by $\binom{L+M}{M} \leq e^{L+M}$ and choosing $\gamma = \frac{t\ln 2}{r_{\min}-1}$ we arrive at

$$\|u-\hat{u}_L\|_{L^2(\Gamma,H^1(D))} \lesssim Ce^{-t(\ln 2)L}\|u\|_{C_\sigma^0(\Sigma(\Gamma,\tau),H^{1+t}\cap H_0^1(D))} \tag{3.47}$$

Next we estimate the total number of degrees of freedom. Again we treat the cases of bounded and unbounded domains Γ_m separately. In the bounded case we have from (3.29) that $N_{\Gamma,b} \leq \exp(\frac{t\ln 2}{r_{\min}-1}(1+\ln 2M)L)$, which satisfies the assumptions of Lemma 3.7 and we obtain

$$N_{\text{tot}} \leq \max\{2^d, 2^{t(1+\ln 2M)/(r_{\min}-1)}\}^L \tag{3.48}$$

Similarly, using (3.24), we get an estimate in the case of unbounded domains:

$$N_{\text{tot}} \leq \max\{2^d, 2^{t(1+\ln(3M+1))+M/(r_{\min}-1)}\}^L \tag{3.49}$$

The proof is completed by combining (3.48) and (3.49), respectively together with (3.47). □

4 Implementation and Numerical examples

In the following we present the implementation of the different components of the sparse composite collocation method and give numerical examples. The main components of the sparse composite collocation method are

- hierarchic wavelet discretization in space
- hierarchic sequence of Smolyak interpolation operators in random parameter space
- sparse composition of wavelets and interpolation operators

In the following subsections we address each of the above points in some more detail from a computational point of view. We emphasize that all computations have been performed on a *single processor* AMD64 desktop and hence without any parallelization involved. All examples are computed in a two-dimensional domain $D \subset \mathbb{R}^2$.

4.1 Wavelet discretization

As described in Subsection 3.2 we perform a Galerkin discretization with piecewise linear continuous wavelets in space by constructing the wavelets from hat basis functions on repeatedly refined, and hence nested, meshes $\mathcal{T}_{l_2}$. To build the linear system arising from the wavelet discretization of the elliptic PDE under consideration, there are two different strategies. The first and most obvious one is to rewrite the assembly routines for the stiffness and mass matrices and the load vector. However, since usually efficient and fast assembly routines based on hat basis functions are already present, we will follow a different approach incorporating these. Denote by Φ the vector of hat functions ϕ_L^i on the finest level L and by Ψ the corresponding vector of wavelets $\psi_{l_2}^i$, $0 \leq l_2 \leq L$, $i \in \nabla_{l_2}$. Furthermore let T be the transformation matrix, such that

$$\Psi = T\Phi$$

It is then well-known that the system $A\mathbf{x} = \mathbf{l}$ can equivalently be written in wavelet basis as

$$TAT^\top \hat{\mathbf{x}} = T\mathbf{l} \tag{4.1}$$

where $\hat{\mathbf{x}}$ denotes the vectors of wavelet coefficients. We note here that we never compute the triple matrix product $TAT^\top$. Instead, since we are using a conjugate gradient algorithm we compute two additional matrix-vector products in each CG-step. We also note here, that the condition number of the system (4.1) is uniformly bounded in L, see e.g. [7], meaning that the number of CG-steps does not increase when the mesh is refined. This can clearly be seen in Figure 4.2 below.

We consider the following model problem

$$-\mathrm{div}(e^{x_1}\nabla u(x_1,x_2)) = e^{x_1}\pi\cos\left(\frac{3\pi}{2}x_2\right)\left(\cos(\pi x_1) - \frac{13\pi}{4}\sin(\pi x_1)\right) \tag{4.2}$$

on $D = [-1,1]^2$. The solution to (4.2) is $u(x_1,x_2) = \sin(\pi x_1)\cos\left(\frac{3\pi}{2}x_2\right)$. The initial mesh $\mathcal{T}_0$ consists of 200 congruent triangular elements. Figure 4.1 shows the solution obtained after refining the initial mesh twice, i.e. on $\mathcal{T}_2$.

Figure 4.2 shows the convergence rate of the wavelet discretization computed up to level $l = 6$, in agreement with (3.39), as well as a comparison of the number of CG-iterations used on each level by a hat function and wavelet discretization, respectively.

4.2 Smolyak interpolation

This subsection is concerned with the Smolyak interpolation operator given in (3.11). From an implementational point of view the following reformulation (see

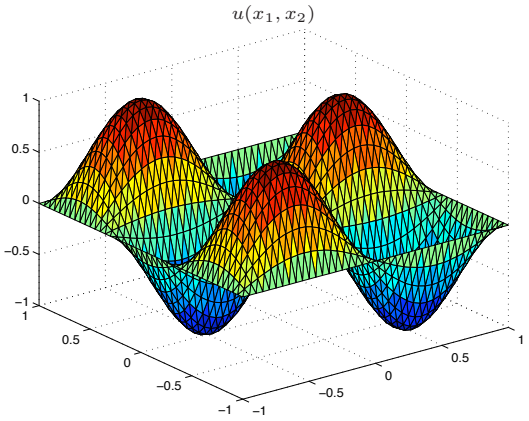

Figure 4.1: The solution $u(x_1, x_2)$ of (4.2) obtained on level $l = 2$ of the wavelet discretization

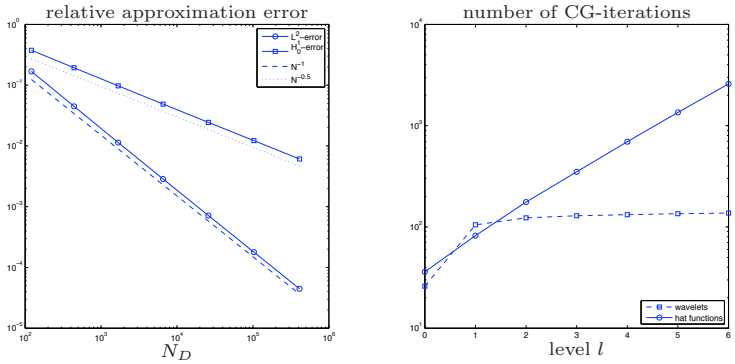

Figure 4.2: Left: Convergence rate of the wavelet discretization of (4.2) computed up to level $l = 7$. Right: Comparison between the number of CG-iterations used on each level

[26]) is more convenient:

$$\mathcal{I}_L^\Gamma = \sum_{0\le|l|\le L} (-1)^{L-|l|} \binom{M-1}{L-|l|} (\mathcal{I}_{l_1}^{(1)} \otimes \cdots \otimes \mathcal{I}_{l_M}^{(M)}) \tag{4.3}$$

Since the goal of our computations is the approximation of the mean value and possibly higher order moments of the solution u we have to compute expected values of the form

$$\mathbb{E}\left[\mathcal{I}_L^\Gamma u\right] = \textstyle\sum_{0\le|l|\le L}(-1)^{L-|l|}\binom{M-1}{L-|l|}\mathbb{E}\left[\mathcal{I}_{l_1}^{(1)} \otimes \cdots \otimes \mathcal{I}_{l_M}^{(M)} u\right]$$
$$\mathbb{E}\left[\mathcal{I}_L^\Gamma u^2\right] = \textstyle\sum_{0\le|l|\le L}(-1)^{L-|l|}\binom{M-1}{L-|l|}\mathbb{E}\left[\mathcal{I}_{l_1}^{(1)} \otimes \cdots \otimes \mathcal{I}_{l_M}^{(M)} u^2\right]$$

etc., where, according to (3.8)

$$\mathbb{E}[\mathcal{I}_{l_m}^{(m)} u] = \sum_{k_m=0}^{\mu_m(l_m)} u(y_{m,k_m}) \int_{\Gamma_m} \ell_{k_m}(y_m)\rho_m(y_m)\, dy_m = \sum_{k_m=0}^{\mu_m(l_m)} u(y_{m,k_m}) w_{k_m}$$

where w_{k_m} denotes the weights of the Gaussian quadrature formula of order $\mu_m(l_m)$.

Motivated by the zero-dimensional problem, i.e. (3.1) without dependence on the spatial variable $\mathbf{x}$ (see also [9, 5]),

$$a(\mathbf{y})u(\mathbf{y}) = 1 \tag{4.4}$$

with $a = \gamma_0 + \sum_{m=1}^M \gamma_m y_m$ and $\gamma_m \in \mathbb{R}$, we consider the example of computing the expectation $\mathbb{E}[u]$ of the solution to (4.4), i.e. the integral

$$I = \int_\Gamma \frac{\rho(\mathbf{y})d\mathbf{y}}{\gamma_0 + \sum_{m=1}^M \gamma_m y_m} \tag{4.5}$$

by a Smolyak cubature based on Legendre abscissae. Here, we choose $M = 20$ and for $m \ge 1$ we set $\gamma_m = \sqrt{\lambda_m}\|\varphi_m\|_{L^\infty(D)}$ where (λ_m, φ_m) are the largest M eigenpairs associated to the Gaussian 2-point covariance $C_a(\mathbf{x}, \mathbf{x}') = e^{-\|\mathbf{x}-\mathbf{x}'\|^2}$, see (2.3). Furthermore we set $\Gamma_m = [-1, 1]$ and ρ to be the product of the uniform probability densities on Γ_m.

The expected rate of convergence, as shown Lemma 3.5 and Remark 3.6, is strongly dependent on the smallest region of analyticity $\Sigma(\Gamma_m, r_m)$. Therefore, in our case, raising the constant γ_0 enlarges the domains of analyticity and should provide better convergence rates. This can also be seen in Figure 4.3 where we considered various choices of γ_0 and where we can see a superalgebraic convergence in all cases.

As a second example for the Smolyak procedure we have a look at the case of Gaussian random variables, hence the Smolyak algorithm is based on the zeros of Hermite polynomials while the order of the one-dimensional integration rules is increased quadratically in each step, see (3.7). Motivated again by the zero-dimensional problem (4.4) with the log-normal coefficient $a = \exp(\sum_{m=1}^M \gamma_m y_m)$ we compute the integral

$$I = \int_\Gamma \frac{\rho(\mathbf{y})d\mathbf{y}}{e^{\sum_{m=1}^M \gamma_m y_m}} \tag{4.6}$$

where $\Gamma = (-\infty, \infty)^M$ and $\rho(\mathbf{y})$ denotes the tensorized Gaussian probability density function. Since the integrand is an entire function, we expect exponential convergence which can also clearly be seen in Figure 4.4 for different choices of the dimension M as indicated.

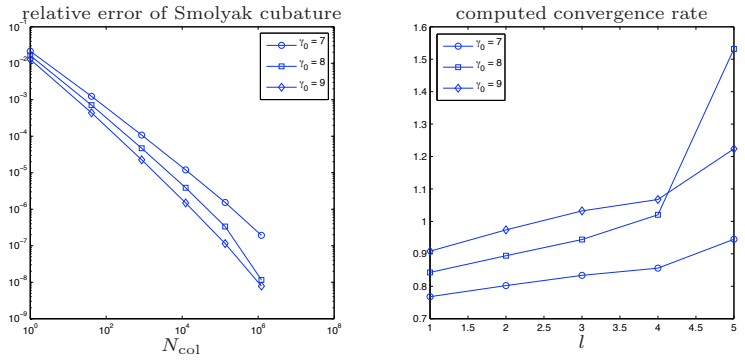

Figure 4.3: Left: Error $\epsilon = |I - \mathcal{I}_L(1/a)|$ of the Smolyak cubature computing the integral in (4.5). Right: Computed convergence rate r, if $\epsilon \leq CN^{-r}$ assumed, between two consecutive collocation levels

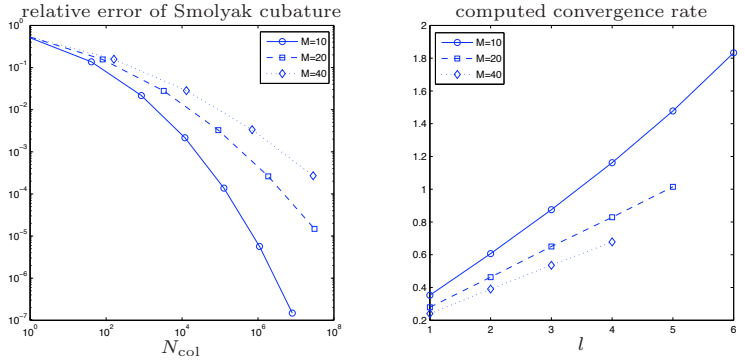

Figure 4.4: Left: Error $\epsilon = |I - \mathcal{I}_L(1/a)|$ of the Smolyak cubature computing the integral in (4.6). Right: Computed convergence rate r, if $\epsilon \leq CN^{-r}$ assumed, between two consecutive collocation levels

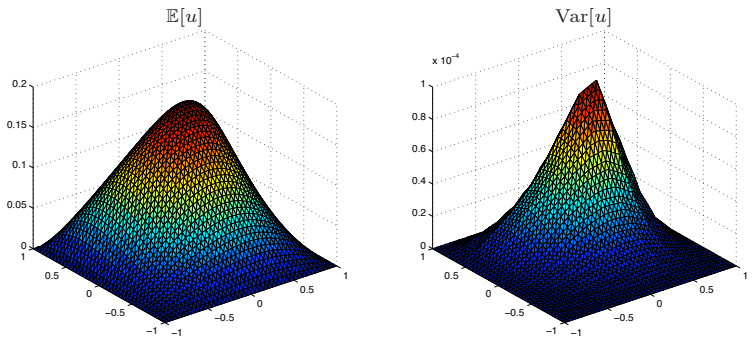

Figure 4.5: Mean and variance of the solution u to the sPDE described in Section 4.3

4.3 Sparse composite collocation method

In this paragraph is subject to the implementational aspects of the sparse composition of the wavelet discretization and the Smolyak interpolation as described in Section 3.3. Again, analogous to (4.3), we rewrite (3.41) into a more suitable form for implementation as follows:

$$\hat{u}_L = \sum_{\substack{l\in\mathbb{N}^{M+1}\\ 0\le|l|\le L}} (-1)^{L-|l|}\binom{M}{L-|l|}(\mathcal{I}^{(1)}_{l_1}\otimes\cdots\otimes\mathcal{I}^{(M)}_{l_M})(P^D_{l_{M+1}}u) \tag{4.7}$$

We consider the following example problem

$$\begin{cases} -\mathrm{div}(a(\mathbf{y},\mathbf{x})\nabla u(\mathbf{y},\mathbf{x})) = f(\mathbf{x}) & D,\\ u(\omega,\mathbf{x})|_{\mathbf{x}\in\partial D} = 0, \end{cases} \quad P-\text{a.e. } \mathbf{y}\in\Gamma$$

where $D=[-1,1]^2$ and the y_m are uniformly distributed on $\Gamma_m=[-1,1]$. The diffusion coefficient a is given as a truncated Karhunen-Loève expansion

$$a(\mathbf{y},\mathbf{x}) = a_M(\mathbf{y},\mathbf{x}) = \mathbb{E}_a(\mathbf{x}) + \sum_{m=0}^{M}\sqrt{\lambda_m}\varphi_m(\mathbf{x})y_m$$

with $\mathbb{E}_a(\mathbf{x}) = 8+\sin(\pi(x+y))$ and (λ_m,φ_m) being the largest M computed eigenpairs of the eigenvalue problem (2.3) defined by the 2-point covariance $C_a(\mathbf{x},\mathbf{x}') = e^{-\|\mathbf{x}-\mathbf{x}'\|}$. Finally, we set the source term $f(\mathbf{x}) = 2e^{x+2y}$. Figure 4.5 shows the obtained solution for $M=20$ and on level $L=2$ of the refinement.

In Figure 4.6 we compare the number of total degrees of freedom and the convergence of the mean $\mathbb{E}[u]$ in the sparse composite collocation method (3.41) with the expected respective results obtained in the full composite method (3.40) for various choices of M as indicated.

Acknowledgment

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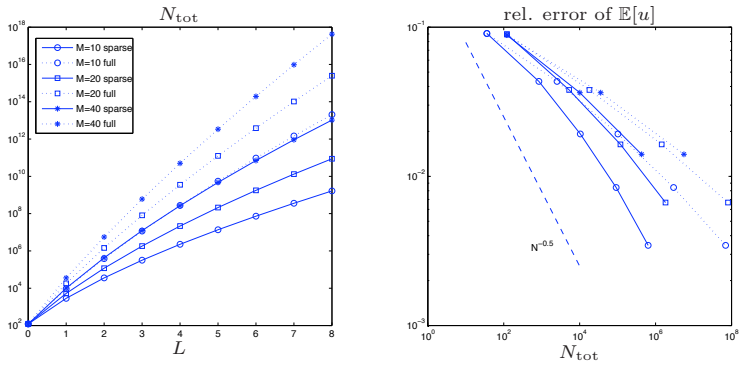

Figure 4.6: Left: Comparison between the total number of DoF in the sparse and full composite collocation method. Right: relative error of the mean $\mathbb{E}[u]$ w.r.t. N_{tot} 4.3

A Proof of Proposition 3.2

It has already been proven in [2] (cf. Lemma 4.3) that the $L^2_{\rho_m}$ interpolation error based on Gaussian abscissae can be bounded from above by a best-approximation error, i.e.

Lemma A.1. *For every function $v \in C^0_{\sigma_m}(\Gamma_m, \mathcal{W}(D))$, the interpolation error satisfies*

$$\|v - \mathcal{I}_p^{(m)}(v)\|_{L^2_{\rho_m}(\Gamma_m, \mathcal{W}(D))} \leq C \inf_{w \in \mathcal{P}_p(\Gamma_m) \otimes \mathcal{W}(D)} \|v - w\|_{C^0_{\sigma_m}(\Gamma_m, \mathcal{W}(D))} \tag{A.1}$$

where $C > 0$ is independent of p and $\mathcal{I}_p^{(m)}$ is given by (3.8) based on the zeros of orthogonal polynomials w.r.t. ρ_m.

It therefore remains to bound the best approximation error. To this end we distinguish the cases where Γ_m is bounded or unbounded. For the bounded case we refer to [2], Lemma 4.4

Lemma A.2. *Let $\Gamma_m = [-1, 1]$. For every function $v \in C^0(\Gamma_m, \mathcal{W}(D))$ which admits an analytic extension in the region $\Sigma(\Gamma_m, \tau_m) \subset \mathbb{C}$ (2.10) for some $\tau_m > 0$ it holds that*

$$\inf_{w \in \mathcal{P}_p(\Gamma_m) \otimes \mathcal{W}(D)} \|v - w\|_{C^0(\Gamma_m, \mathcal{W}(D))} \leq \frac{2}{e^{r_m} - 1} e^{-p r_m} \max_{z_m \in \Sigma(\Gamma_m, \tau_m)} \|v(z)\|_{\mathcal{W}(D)} \tag{A.2}$$

Proposition 3.2 therefore follows in the bounded case by combining Lemma A.1 and A.2 with $p = \lceil \gamma l_m \rceil$.

In the unbounded case we will, as in [2] first refer a result from [13] but then proceed in a slightly different way to obtain the result stated in Proposition 3.2. Denote by $H_n(y_m)$ the L^2_g-normalized Hermite polynomial of order n, where $g(y_m) = e^{-y_m^2}$:

$$H_n(y_m) = \sqrt{\pi^{\frac{1}{2}} 2^n n!}(-1)^n e^{y_m^2} \frac{\partial^n}{\partial y_m^n} e^{-y_m^2}$$

Furthermore we define by $h_n(y_m) = e^{-y_m^2/2} H_n(y_m)$ the Hermite functions. The following Lemma from [13] (cf. Theorem 1) then provides us with a necessary and sufficient condition for Fourier-Hermite series to converge in a complex domain containing $\mathbb{R}$.

Lemma A.3 (Hille 1940). *Let $f : \Sigma(\mathbb{R}, \tau_m) \to \mathbb{C}$ be analytic. A necessary and sufficient condition in order that the Fourier-Hermite series*

$$\sum_{k=0}^{\infty} f_k h_k(z_m), \qquad f_k = \int_{\mathbb{R}} f(y_m) h_k(y_m) dy_m$$

exists and converges to the sum $f(z_m)$ in $\Sigma(\mathbb{R}, \tau_m)$ is that for every $\beta_m \in [0, \tau_m)$ there exists a finite constant $C(\beta_m) > 0$ such that

$$|f(y_m + iw_m)| \leq C(\beta_m) e^{-|y_m|\sqrt{\beta_m^2 - w_m^2}}, \quad y_m \in \mathbb{R},\ w_m \in [-\beta_m, +\beta_m] \tag{A.3}$$

Moreover, the following bound for the Fourier coefficients holds:

$$|f_k| \leq C e^{-\tau_m \sqrt{2k+1}} \tag{A.4}$$

Recall the measure $\sigma_m(y_m) = e^{-\alpha|y_m|}$ from (2.8) and furthermore define the Gaussian measure $G_m(y_m) = e^{-(\delta_m y_m)^2/4}$ for δ_m as in Assumption 2.5. Note that Lemma A.1 holds for both of them, see [2] for details. By a change of variables $t_m = \delta_m y_m / \sqrt{2}$ we denote $\tilde{v}(t_m) = v(y_m(t))$. Expanding $\tilde{v}$ in Hermite polynomials we obtain

$$\tilde{v}(t_m) = \sum_{k=0}^{\infty} v_k H_k(t_m), \quad \text{where } v_k \in \mathcal{W}(D), \quad v_k = \int_{\mathbb{R}} \tilde{v}(t_m) H_k(t_m) e^{-t_m^2}\, dt_m$$

Next we define $f(z_m) = \tilde{v}(z_m) e^{-\frac{z_m^2}{2}}$ and observe that

$$f_k = \int_{\mathbb{R}} f(t_m) h_k(t_m)\, dt_m = \int_{\mathbb{R}} \tilde{v}(t_m) H_k(t_m) e^{-t_m^2}\, dt_m = v_k \tag{A.5}$$

f is clearly analytic in the strip $\Sigma(\mathbb{R}, \frac{\tau_m \delta_m}{\sqrt{2}})$ as it is a product of analytic functions. Furthermore we have that

$$\begin{aligned} \|f(t_m + iw_m)\|_{\mathcal{W}(D)} &= |e^{-\frac{(t_m + iw_m)^2}{2}}| \|\tilde{v}(z_m)\|_{\mathcal{W}(D)} \\ &\leq e^{-\frac{t_m^2 - w_m^2}{2}} e^{\sqrt{2}\frac{\alpha}{\delta_m}|t_m|} \|v\|_{C^0_{\sigma_m}(\Sigma(\mathbb{R},\tau_m),\mathcal{W}(D))} \end{aligned}$$

The function f thus satisfies the hypotheses of Lemma A.3 by setting

$$C(\beta_m) = \max_{\substack{t_m \in \mathbb{R} \\ w_m \in [-\beta_m, \beta_m]}} \exp\left\{ -\frac{t_m^2 - w_m^2}{2} + \sqrt{2}\frac{\alpha}{\delta_m}|t_m| + |t_m|\sqrt{\beta_m^2 - w_m^2} \right\}.$$

Hence the Hermite series of $\tilde{v}$ is converging in $\Sigma_m(\mathbb{R}, \frac{\tau_m \delta_m}{\sqrt{2}})$ and the Fourier coefficients v_k satisfy, due to (A.5) the bound

$$\|v_k\|_{\mathcal{W}(D)} \leq C e^{-\tau_m \sqrt{2k+1}} \tag{A.6}$$

Therefore we can estimate the best approximation error in Lemma A.1 by

$$E_p(v) := \inf_{w \in \mathcal{P}_p \otimes \mathcal{W}(D)} \|v - w\|_{C^0_G(\Gamma_m, \mathcal{W}(D))} \leq \max_{t_m \in \mathbb{R}} \| \sum_{k=p+1}^{\infty} v_k h_k(t_m) \|_{\mathcal{W}(D)}$$

Since $|h_k(t_m)| < 1$ for all $t_m \in \mathbb{R}$, $k \in \mathbb{N}_0$ and using (A.6) the truncated Hermite series can be bounded by

$$E_p(v) \leq \sum_{k=p+1}^{\infty} \|v_k\|_{\mathcal{W}(D)} \leq C \sum_{k=p+1}^{\infty} e^{-\frac{\tau_m \delta_m}{\sqrt{2}}\sqrt{2k+1}} \leq C \sum_{k=p+1}^{\infty} e^{-\tau_m \delta_m \sqrt{k}}$$

It remains to bound the last series. To this end we use the integral criteria for series and the formula $\int_p^\infty e^{-r\sqrt{t}}\,dt = \frac{2(1+r\sqrt{p})}{r^2}e^{-r\sqrt{p}}$ to obtain

$$E_p(v) \leq C\frac{1+\tau_m\delta_m\sqrt{p}}{(\tau_m\delta_m)^2}e^{-\tau_m\delta_m\sqrt{p}}$$

The claim now follows from (A) and Lemma A.1. □

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Research Reports

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