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Approximation properties of plane wave spaces and application to the analysis of the plane wave discontinuous Galerkin method

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Approximation properties of plane wave spaces and application to the analysis of the plane wave discontinuous Galerkin method

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Abstract

The approximation of solutions of the homogeneous Helmholtz equation by finite dimensional plane wave spaces is studied in two dimensions.

The main tool used is Vekua's theory for elliptic PDE with analytic coefficients. This leads to a general approximation result for solutions of the Helmholtz equation, using a finite dimensional space of plane wave functions, with respect to weighted Sobolev norms.

As a consequence of these estimates, two new a priori error estimates with respect to the energy and L^2 norms are proved for the plane wave discontinuous Galerkin method for the homogeneous Helmholtz equation. These estimates are sharp with respect to the order of convergence in the meshsize h. In all the bounds, the dependence of the constants on the wavenumber is made explicit. So it is possible to assess the pollution effect.

1 Introduction

The discretization of boundary value problems for the Helmholtz equation

$$-\Delta u + \omega^2 u = f \tag{1.1}$$

by standard polynomial finite element methods encounters several problems: in order to represent the oscillations of the solution, either the mesh employed has to be extremely fine or the polynomial degree has to be very high. Furthermore, the so-called pollution effect has to be dealt with: the discretization error drifts off the best approximation error, when the wavenumber ω increases [4]. Thus, satisfactory accuracy comes with exceedingly high computational costs.

The most natural idea to handle this problem is to incorporate the properties of the solutions into the discretization. This is possible by using discretization spaces built from plane wave functions, as it has been done in the partition of unity method (PUM, [3], [19]), in the discontinuous enrichment method (DEM, [12], [13]), in the variational theory of complex rays (VTCR, [24]) or in the ultraweak variational formulation (UWVF, [9]).

The ultraweak variational formulation, introduced by Cessenat and Després in [9], makes use of piecewise plane waves as basis functions and their impedance traces on the mesh skeleton as unknowns. This method can be reformulated within a more general class of methods called plane wave discontinuous Galerkin methods (PWDG), see [7], [16] and Section 4 of this paper. In the last two works, a priori h-version error estimates for these methods are also derived. In particular, for general f [16] shows that, provided that a threshold condition on the meshsize h in terms of the wavenumber ω is satisfied, linear convergence in h is obtained, in a particular mesh dependent energy norm, no matter how many plane waves are used in the local approximation spaces. The sharpness of this theoretical result is demonstrated by numerical experiments.

On the contrary, in the homogeneous case $f = 0$ it has been observed numerically ([17], [16], [7]) that the order of h-convergence improves when increasing the dimension of the local plane wave spaces. The estimate in [7] relates the convergence rate in h in the L^2–norm to the dimension of the local space but overestimates the error compared to what is experimentally observed. In this paper we prove error estimates, both in the energy and the L^2-norm, in which the rate of convergence agrees with the numerical observations. The theoretical approach is the same as the one in [16], where the analysis is based on a duality argument.

To that end, it is necessary to study how well the solutions of the two-dimensional homogeneous Helmholtz equation can be approximated by plane wave spaces in Sobolev norms. The fundamental tool is Vekua's theory for elliptic partial differential equations with analytic coefficients, developed in [26]. Following the argument of [19], we investigate how well the solutions of the Helmholtz equation can be approximated in a class of functions, called generalized harmonic polynomials (Section 2). In Section 3 we see how these generalized harmonic polynomials can be approximated by plane waves. Theorem 3.10 summarizes these approximation properties. A simple numerical experiment shows the sharpness of these best approximation estimates. The approximation results proved in these sections are of interest in their own right, independently of the application to the analysis of the PWDG method. Finally, in Section 4, we apply these results to the convergence analysis of the PWDG methods introduced in [16]. We obtain the desired error estimate, namely order m and $m + 1$ in the energy and L^2-norm, respectively, if $p = 2m + 1$ plane waves span the local approximation spaces.

The results in Section 2 are adapted from Chapter 4 of [19], but the dependence of the constants on the wavenumber is made explicit for the first time. Moreover, the approximation estimates with respect to h for an arbitrary plane wave function (Lemmata 3.1 and 3.2) and for generalized harmonic polynomials (Theorems 3.9 and 3.10), in maximum and Sobolev norms, are new.

2 The approximation by generalized harmonic polynomials

In order to study homogeneous elliptic partial differential equations with real analytic coefficients, Vekua and Bergman ([26] and [5]) independently introduced an integral operator $\mathrm{Re}\, V$ that maps holomorphic functions into solutions of the PDE under consideration. This operator is continuously invertible with respect to Sobolev norms. Since we know how to approximate a holomorphic function ϕ by harmonic polynomials, we can use the continuity of $\mathrm{Re}\, V$ to study the approximation of $\mathrm{Re}\, V(\phi)$ by the transformed polynomials, which are called generalized harmonic polynomials.

We are interested in Vekua's theory for the Helmholtz equation:

$$\Delta u + \omega^2 u = 0, \qquad \text{in } D,$$

where, throughout all this section, u is a real function. The domain $D \subset \mathbb{R}^2$ is simply connected, Lipschitz and bounded with diameter h; we identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $\mathcal{D} \subset \mathbb{C}$

be such that the closure of D is included in $\mathcal{D}$ and denote by D^* and $\mathcal{D}^*$ the complex conjugates of D and $\mathcal{D}$, respectively. The theory for the general PDEs is presented in [26] and summarized in [19]. We follow the presentation of the latter, restricted to the Helmholtz problem.

2.1 The Vekua operator and its inverse

The Vekua operator for the Helmholtz equation is defined as follows: if ϕ is an holomorphic function in D, given $z_0 \in D$ we set

$$V[\phi, z_0](z, \overline{z}) = \phi(z) - \int_{z_0}^{z} \phi(t) \frac{\partial}{\partial t} G(t, \overline{z}_0, z, \overline{z}) \, \mathrm{d}t, \tag{2.2}$$

where G is the Riemann function[1] for the Helmholtz equation, i.e.,

$$\begin{aligned} G : \mathcal{D} \times \mathcal{D}^* \times \mathcal{D} \times \mathcal{D}^* &\to \mathbb{C} \\ G(z, \zeta, t, \tau) &= J_0(\omega \sqrt{(z-t)(\zeta-\tau)}), \end{aligned} \tag{2.3}$$

J_0 being the Bessel function of the first kind of order zero.

The function $u(x, y) = \mathrm{Re}(V[\phi, z_0](x + iy, x - iy))$ is a solution of the Helmholtz equation. It is possible to prove the existence of an analytic function $U(z, \zeta)$, on $D \times D^*$, such that

$$U(z, \overline{z}) = u(\mathrm{Re}\, z, \mathrm{Im}\, z), \tag{2.4}$$

and U is a solution of $\frac{\partial^2}{\partial z \partial \zeta} U + \frac{\omega^2}{4} U = 0$.

The following theorem guarantees the invertibility of $\mathrm{Re}\, V$ and gives the expression of its inverse.

Theorem 2.1 (Vekua). *Fixed $z_0 \in D$, let u be a real solution of $\Delta u + \omega^2 u = 0$ in D, then there exists a unique function ϕ, holomorphic in D, such that*

$$\begin{aligned} \phi(z_0) &\in \mathbb{R} \\ u(x, y) &= \mathrm{Re}\, V[\phi, z_0](z, \overline{z}), \qquad z = x + iy \in D. \end{aligned}$$

Moreover, ϕ can be written as

$$\phi(z) = (\mathrm{Re}\, V)^{-1}[u, z_0](z, \overline{z}) = 2U(z, \overline{z}_0) - U(z_0, \overline{z}_0) G(z_0, \overline{z}_0, z, \overline{z}_0). \tag{2.5}$$

The proof of the theorem can be found in [26], in Section I.12.

[1] The Riemann function for the Helmholtz equation is defined as the complex function in $\mathcal{D} \times \mathcal{D}^* \times \mathcal{D} \times \mathcal{D}^*$ such that

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \zeta} G(z, \zeta, t, \tau) + \frac{\omega^2}{4} G(z, \zeta, t, \tau) &= 0, \qquad && z, t \in \mathcal{D},\ \zeta, \tau \in \mathcal{D}^*, \\ G(t, \tau, t, \tau) &= 1, && t \in \mathcal{D},\ \tau \in \mathcal{D}^*, \\ \frac{\partial}{\partial z} G(z, \tau, t, \tau) &= 0, && z, t \in \mathcal{D},\ \tau \in \mathcal{D}^*, \\ \frac{\partial}{\partial \zeta} G(t, \zeta, t, \tau) &= 0, && t \in \mathcal{D},\ \zeta, \tau \in \mathcal{D}^*. \end{aligned}$$

The remaining part of this subsection is devoted to the proof of the continuity of $\operatorname{Re} V$ and $(\operatorname{Re} V)^{-1}$ in weighted Sobolev norms defined as follows: for all $u \in H^k(D)$ we define

$$\|u\|_{k,\omega,D}^2 := \sum_{l=0}^{k} \omega^{2(k-l)} |u|_{l,D}^2 . \tag{2.6}$$

These norms are equivalent to the standard Sobolev norms, but are more appropriate for the solutions of Helmholtz equation in that all the terms are dimensionally homogeneous.

From now on, we define a constant c_* such that

$$c_* := \omega h, \tag{2.7}$$

where $h = \operatorname{diam}(D)$.

Theorem 2.2 (Continuity of $(\operatorname{Re} V)^{-1}$). *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain with diameter h, star-shaped with respect to z_0. Let $u \in H^k(D)$, $k \geq 1$, be a real solution of $\Delta u + \omega^2 u = 0$ in D and $\phi(z) = (\operatorname{Re} V)^{-1}[u, z_0](z, \overline{z})$, as in Theorem 2.1. Then there are positive constants C_0 and C_k, independent of h, ω and c_*, such that*

$$\begin{aligned} \|\phi\|_{0,D} &\leq C_0 \, e^{c_*} \left(1 + c_*^2\right)^2 \left(\|u\|_{0,D} + h \, \|\nabla u\|_{0,D} \right), \\ \|\phi\|_{k,\omega,D} &\leq C_k \, e^{c_*} \left(1 + c_*^{k+3}\right) \|u\|_{k,\omega,D} . \end{aligned} \tag{2.8}$$

Theorem 2.3 (Continuity of $\operatorname{Re} V$). *Let D be as in Theorem 2.2 with $z_0 \in D$ fixed and let $\phi \in H^k(D)$, $k \geq 0$, be holomorphic in D. Then there exist positive constants C_0, C_k, depending only on the shape of D and the position of z_0, such that*

$$\begin{aligned} \|\operatorname{Re} V[\phi, z_0]\|_{0,D} &\leq \left(1 + C_0 c_*^2 e^{c_*}\right) \|\phi\|_{0,D} , \\ \|\operatorname{Re} V[\phi, z_0]\|_{k,D} &\leq C_k e^{c_*} \left(1 + c_*^{k+2}\right) \|\phi\|_{k,\omega,D} , \qquad k \geq 1. \end{aligned} \tag{2.9}$$

Here the main novelty and the difference with the treatment given in [19] is the use of the weighted Sobolev norms and some properties of Bessel functions. This allows us to show that all the involved bounding constants depend on c_* and not on ω and h separately.

In order to prove Theorem 2.2 and Theorem 2.3 we need some intermediate results which will be presented in Sections 2.1.1 and 2.1.2. The proof of Theorems 2.2 and 2.3 will be given in Section 2.1.3.

2.1.1 The estimates for G

In this section we give estimates for the Riemann function G and its derivatives. We have

$$\begin{aligned} \frac{\partial G(z,\zeta,t,\tau)}{\partial z} &= -\frac{\partial G(z,\zeta,t,\tau)}{\partial t} = J_0'(\omega\sqrt{(z-t)(\zeta-\tau)})\frac{\omega}{2}\sqrt{\frac{\zeta-\tau}{z-t}}, \\ \frac{\partial G(z,\zeta,t,\tau)}{\partial \zeta} &= -\frac{\partial G(z,\zeta,t,\tau)}{\partial \tau} = J_0'(\omega\sqrt{(z-t)(\zeta-\tau)})\frac{\omega}{2}\sqrt{\frac{z-t}{\zeta-\tau}}. \end{aligned}$$

Since the derivatives with respect to z and t and with respect to ζ and τ are identical we can write the generic derivative of G by deriving only with respect to two variables:

$$\frac{\partial^{|\alpha|} G(z,\zeta,t,\tau)}{\partial z^{\alpha_1} \partial \zeta^{\alpha_2} \partial t^{\alpha_3} \partial \tau^{\alpha_4}} = \pm \frac{\partial^{|\alpha|} G(z,\zeta,t,\tau)}{\partial z^{\alpha_1+\alpha_3} \partial \zeta^{\alpha_2+\alpha_4}}. \tag{2.10}$$

Using either the definition of the Riemann function or the Bessel equation, we can calculate the mixed second order derivatives:

$$\frac{\partial^2 G(z,\zeta,t,\tau)}{\partial z\partial\zeta} = -\frac{\omega^2}{4}G(z,\zeta,t,\tau); \tag{2.11}$$

that is equal to other three mixed second order derivatives thanks to (2.10). We calculate pure derivatives using Bessel power expansion:

$$\begin{aligned}\frac{\partial^k G(z,\zeta,t,\tau)}{\partial z^k} &= \sum_{l\geq k} \frac{(-1)^l \omega^{2l} \frac{l!}{(l-k)!}(z-t)^{l-k}(\zeta-\tau)^l}{4^l\, l!\, l!} \\ &= \sum_{j\geq 0} \frac{(-1)^k\omega^{2k}(\zeta-\tau)^k}{2^k} \frac{(-1)^j\omega^{2j}(z-t)^j(\zeta-\tau)^j}{j!\,(j+k)!\,2^{2j+k}} \\ &= \left(-\frac{\omega^2(\zeta-\tau)}{2}\right)^k \frac{J_k(\omega\sqrt{(z-t)(\zeta-\tau)})}{\left(\omega\sqrt{(z-t)(\zeta-\tau)}\right)^k}.\end{aligned} \tag{2.12}$$

All the derivatives that involve only the variables z and t have this form; while the ones involving ζ and τ are analogous. Using again the Bessel power expansion we have

$$\left|\frac{J_k(x)}{x^k}\right| \leq \sum_{l\geq 0} \frac{1}{2^k\,k!} \frac{\left(\frac{|x|}{2}\right)^{2l}}{l!\,l!} \leq \frac{1}{2^k\,k!}e^{|x|}, \qquad k\geq 0\ \forall\, x\in\mathbb{C}. \tag{2.13}$$

Remembering that $|z-t|<h$, $|\zeta-\tau|<h$ and condition (2.7), we obtain the first bound:

$$\left|\frac{\partial^k G(z,\zeta,t,\tau)}{\partial z^k}\right| \leq \frac{1}{k!}\left(\frac{\omega c_*}{4}\right)^k e^{c_*}, \qquad k\geq 0,\ z,t\in D,\ \zeta,\tau\in\overline{D}. \tag{2.14}$$

Now we consider a differential operator D^α of order k and define

$$j = |\#(\partial_z,\partial_t) - \#(\partial_\zeta,\partial_\tau)| = |\alpha_1+\alpha_3-\alpha_2-\alpha_4|, \qquad j\in\{0,1,\dots,k\}.$$

Applying (2.10), (2.11) and finally (2.14), we have the estimate

$$\begin{aligned}|D^\alpha G(z,\zeta,t,\tau)| &= \left|\frac{\partial^j}{\partial z^j}\frac{\partial^{k-j}}{\partial z^{\frac{k-j}{2}}\partial\zeta^{\frac{k-j}{2}}}G(z,\zeta,t,\tau)\right| \\ &\leq \left|\left(\frac{-\omega^2}{4}\right)^{\frac{k-j}{2}}\frac{1}{j!}\left(\frac{\omega c_*}{4}\right)^j e^{c_*}\right| \leq \frac{1}{j!}\left(\frac{\omega}{2}\right)^k\left(\frac{c_*}{2}\right)^j e^{c_*},\end{aligned} \tag{2.15}$$

when $\alpha_1+\alpha_3\geq\alpha_2+\alpha_4$; in the other case we obtain the same bound.

Finally, we can bound the $W^{k,\infty}(\Omega)$ norms and seminorms:

$$|G|_{W^{k,\infty}(D\times D^*\times D\times D^*)} \leq I(k)\left(\frac{\omega}{2}\right)^k e^{c_*}\max\left\{1,\left(\frac{c_*}{2}\right)^k\right\}, \qquad k\in\{0,1,\dots\}, \tag{2.16}$$

$$\|G\|_{W^{k,\infty}(D\times D^*\times D\times D^*)} \leq C(k)\left(1+\left(\frac{\omega}{2}\right)^k\right)\left(1+\left(\frac{c_*}{2}\right)^k\right)e^{c_*}, \qquad k\in\{0,1,\dots\}, \tag{2.17}$$

where $I(k)$ is the number of multiindices in $\mathbb{N}_0^4$ of length k, and $C(k)$ is a constant that depends only on k.

2.1.2 The estimates for u and ϕ

We prove some results, following Chapter 4 of [19], in the particular case of $L = \Delta + \omega^2$ and make explicit the dependence of all the constants on ω and h.

Lemma 2.4 (Helmholtz internal estimates). *Let $u \in H^1(B(x_0, R))$ satisfy $-\Delta u - \omega^2 u = f$, with $f \in H^1(B(x_0, R))$, then there exists a positive constant C independent of ω and R such that*

$$\|u\|_{L^\infty(B_{R/2})} \le C\left(\left(R^{-1} + \omega^2 R\right)\|u\|_{0,B_R} + \|\nabla u\|_{0,B_R} + R\,\|f\|_{0,B_R}\right), \tag{2.18}$$

$$\|\nabla u\|_{L^\infty(B_{R/2})} \le C\left(\omega^2\,\|u\|_{0,B_R} + \left(R^{-1}+\omega^2 R\right)\|\nabla u\|_{0,B_R} + \|f\|_{0,B_R} + R\,\|\nabla f\|_{0,B_R}\right), \tag{2.19}$$

where B_R and $B_{R/2}$ indicate respectively the balls $B(x_0, R)$ and $B(x_0, R/2)$ with center x_0 and radius R and $R/2$.

Proof. It is enough to bound $|u(x_0)|$ and $|\nabla u(x_0)|$, because for all $x \in B_{R/2}$ we can repeat the proof using $B(x, R/2)$ instead of B_R with the same constants.

Let $\varphi : \mathbb{R}^+ \to [0, 1]$ be a smooth cut-off function such that

$$\varphi(r) = \begin{cases} 1 & |r| \le \frac{1}{4}, \\ 0 & |r| \ge \frac{3}{4}, \end{cases}$$

and $\varphi_R : \mathbb{R}^2 \to [0, 1]$, $\varphi_R(x) := \varphi\left(\frac{1}{R}|x - x_0|\right)$. Then

$$\nabla\varphi_R(x) = \varphi'\left(\tfrac{1}{R}|x - x_0|\right)\frac{x - x_0}{R|x - x_0|},$$
$$\Delta\varphi_R(x) = \frac{1}{R^2}\varphi''\left(\tfrac{1}{R}|x - x_0|\right) + \frac{1}{R|x - x_0|}\varphi'\left(\tfrac{1}{R}|x - x_0|\right).$$

We define the average of u and two auxiliary functions on B_R:

$$\overline{u} = \frac{1}{\pi R^2}\int_{B_R} u(y)\,\mathrm{d}y, \qquad g(x) = u(x)\varphi_R(x), \qquad \overline{g}(x) = (u(x) - \overline{u})\varphi_R(x);$$

their Laplacians are:

$$\begin{aligned}
\tilde{f}(x) :&= -\Delta g(x) \\
&= -\left[\frac{1}{R^2}\varphi'' + \frac{1}{R|x - x_0|}\varphi'\right]u(x) - 2\varphi'\frac{x - x_0}{R|x - x_0|}\cdot\nabla u(x) + \varphi(\omega^2 u + f), \\
\overline{f}(x) :&= \overline{f}_1(x) + \overline{f}_2(x) + \overline{f}_3(x) := -\Delta\overline{g}(x) \\
&= -\left[\frac{1}{R^2}\varphi'' + \frac{1}{R|x - x_0|}\varphi'\right](u(x) - \overline{u}) - 2\varphi'\frac{x - x_0}{R|x - x_0|}\cdot\nabla u(x) + \varphi(\omega^2 u + f),
\end{aligned}$$

where φ and his derivatives are always evaluated in $\frac{1}{R}|x - x_0|$.

It is easy to see that, for all $R > 0$, we have:

$$\int_{B(0,R)}\left(\log|x| - \log R\right)^2\mathrm{d}x = \frac{\pi}{2}R^2. \tag{2.20}$$

The fundamental solution formula for Poisson equation states that, if $-\Delta a = b$ in $\mathbb{R}^2$, then

$$a(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| b(y)\,\mathrm{d}y. \tag{2.21}$$

This equality holds for all $b \in L^2(B_R)$, thanks to Theorem 9.9 of [15]. We note that from the divergence theorem

$$\int_{B_R} \tilde{f}(y)\,\mathrm{d}y = -\int_{B_R} \Delta g(y)\,\mathrm{d}y = -\int_{\partial B_R} \nabla g(s)\cdot \boldsymbol{n}\,\mathrm{d}s = 0,$$

because $g \equiv 0$ in $\mathbb{R}^2 \setminus B_{3R/4}$. We apply (2.21) with $a = g$ and $b = \tilde{f}$; using Cauchy-Schwarz inequality and (2.20) and remembering that $\tilde{f} = 0$ in $\mathbb{R}^2 \setminus B_{3R/4}$, we obtain:

$$\begin{aligned}
|u(x_0)| = |g(x_0)| &= \left| -\frac{1}{2\pi}\int_{\mathbb{R}^2} \big(\log|x_0-y| - \log R\big)\tilde{f}(y)\,\mathrm{d}y \right| \\
&\le \frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\, R\, \left\|\tilde{f}\right\|_{0,B_{3R/4}} \\
&\le CR\left(\frac{C_{\varphi,2}}{R^2}\|u\|_{0,B_R} + \frac{C_{\varphi,1}}{R}\|\nabla u\|_{0,B_R} + C_{\varphi,0}\omega^2\|u\|_{0,B_R} + C_{\varphi,0}\|f\|_{0,B_R} \right),
\end{aligned}$$

where the constants $C_{\varphi,j}$ depend only on the $j-$th derivatives of φ and in the last step we have used the definition of $\tilde{f}$ and the fact that $\varphi'(\frac{1}{R}|x-x_0|) = \varphi''(\frac{1}{R}|x-x_0|) = 0$ in $B_{R/4}$. The estimate (2.18) easily follows.

To prove the second estimate of the lemma we note that, for all $\psi \in H_0^1(B_R)$, scaling the L^p-norm and H^1-seminorm, using the Sobolev embeddings $H_0^1(B_1) \hookrightarrow L^p(B_1)$, $2 < p < \infty$, and the Poincaré inequality, it holds

$$\begin{aligned}
\|\psi\|_{L^p(B_R)} &= \left(\int_{B_R} |\psi(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}} && = \left(\int_{B_1} |\psi(R\hat{y})|^p R^2\,\mathrm{d}\hat{y}\right)^{\frac{1}{p}} \\
&= R^{\frac{2}{p}}\left\|\hat{\psi}\right\|_{L^p(B_1)} && \le C(p)R^{\frac{2}{p}}\left\|\hat{\psi}\right\|_{H_0^1(B_1)} \\
&\le C(p)R^{\frac{2}{p}}\left\|\nabla\hat{\psi}\right\|_{L^2(B_1)} && \le C(p)R^{\frac{2}{p}}\|\nabla\psi\|_{L^2(B_R)}.
\end{aligned} \tag{2.22}$$

Now we can estimate the gradient of u in x_0 by differentiating the relation (2.21):

$$\begin{aligned}
&|\nabla u(x_0)| = |\nabla \overline{g}(x_0)| \\
&\quad = \left| -\frac{1}{2\pi}\int_{\mathbb{R}^2} \nabla_x \log|x-y|\big|_{x=x_0} \overline{f}(y)\,\mathrm{d}y \right| = \frac{1}{2\pi}\left| \int_{B_{3R/4}} \frac{x_0-y}{|x_0-y|^2}\overline{f}(y)\,\mathrm{d}y \right| \\
&\quad \le \frac{1}{2\pi}\int_{B_{3R/4}} \frac{1}{|x_0-y|}\left|\overline{f}_1(y) + \overline{f}_2(y)\right|\,\mathrm{d}y + \frac{1}{2\pi}\int_{B_{3R/4}} \frac{1}{|x_0-y|}\left|\overline{f}_3(y)\right|\,\mathrm{d}y \\
&\quad \le \frac{1}{2\pi}\frac{4}{R}\left\|\overline{f}_1 + \overline{f}_2\right\|_{L^1(B_{3R/4})} + \frac{1}{2\pi}\left\|\frac{1}{|x_0-y|}\right\|_{L^{p'}(B_{3R/4})}\left\|\overline{f}_3\right\|_{L^p(B_{3R/4})}
\end{aligned}$$

where we used Hölder inequality $L^\infty - L^1$, $L^{p'} - L^p$, $p > 2$, $\overline{f}_1 \equiv \overline{f}_2 \equiv 0$ in $B_{R/4}$,

$$\le \frac{2}{\pi}\frac{\sqrt{|B_{3R/4}|}}{R}\left\|\overline{f}_1 + \overline{f}_2\right\|_{L^2(B_{3R/4})} + C_p R^{1-\frac{2}{p}}\left\|\overline{f}_3\right\|_{L^p(B_{3R/4})}$$

because $\left\||y|^{-1}\right\|_{L^{p'}_{B(0,R)}} = \frac{R^{1-\frac{2}{p}}}{(2-p')^{\frac{1}{p'}}}$ for $p > 2$,

$$\leq C\frac{1}{R^2}\|u-\overline{u}\|_{L^2(B_{3R/4})} + C\frac{1}{R}\|\nabla u\|_{L^2(B_{3R/4})} + CR\left\|\nabla \overline{f}_3\right\|_{L^2(B_R)}$$

by definition of $\overline{f}_1$, $\overline{f}_2$, $\sqrt{|B_{3R/4}|} = \frac{3}{4}\sqrt{\pi}R$, and (2.22), $\overline{f}_3 \in H^1_0(B_R)$

$$\leq C\frac{1}{R}\|\nabla u\|_{L^2(B_{3R/4})} + CR\left\|(\nabla\overline{\varphi}_R)(\omega^2 u + f)\right\|_{L^2(B_R)} + CR\left\|\overline{\varphi}_R\nabla(\omega^2 u + f)\right\|_{L^2(B_R)}$$
$$\leq C\left(\frac{1}{R}\|\nabla u\|_{L^2(B_R)} + \omega^2\|u\|_{L^2(B_R)} + R\omega^2\|\nabla u\|_{L^2(B_R)} + \|f\|_{L^2(B_R)} + R\|\nabla f\|_{L^2(B_R)}\right),$$

where in the last but one step we have used the Poincarè–Wirtinger inequality, whose constant scales with R. The final constant depends only on the cut-off function φ but not on R and ω, so we have the second estimate in the assertion. □

Lemma 2.5. *Let D be a star-shaped domain with respect to the origin, with diameter h, such that exists $\alpha \in (0, \frac{1}{2})$ such that $B(0, 2\alpha h) \subset D$. Let $z = x + iy \in D$ and let $u \in H^1(D)$ satisfy the homogeneous equation $-\Delta u - \omega^2 u = 0$ in D. Then there exists a constant $C > 0$ such that:*

$$\begin{aligned}\int_0^1 \int_D |u(sx, sy)|^2 \,\mathrm{d}x\,\mathrm{d}y \;\; \mathrm{d}s &\leq C\frac{1}{\alpha}\left((1 + h^4\omega^4)\|u\|^2_{0,D} + h^2\|\nabla u\|^2_{0,D}\right),\\ \int_0^1 \int_D |\nabla u(sx, sy)|^2 \,\mathrm{d}x\,\mathrm{d}y \;\; \mathrm{d}s &\leq C\frac{1}{\alpha^3}\left(h^2\omega^4\|u\|^2_{0,D} + (1 + h^4\omega^4)\|\nabla u\|^2_{0,D}\right).\end{aligned} \tag{2.23}$$

Proof. Following the proof of Lemma 4.2.8 of [19] and using $|D| < h^2$ we get

$$\int_0^1 \int_D |u(sx, sy)|^2 \,\mathrm{d}x\,\mathrm{d}y \;\; \mathrm{d}s \leq \alpha h^2\|u\|^2_{L^\infty(B(0,\alpha h))} + \frac{1}{\alpha}\|u\|^2_{0,D}. \tag{2.24}$$

Using (2.18) with $f = 0$, $R = \alpha h$ and $\alpha < \frac{1}{2} < \frac{1}{\alpha}$ we have the first bound. The second bound follows from applying (2.24) to ∇u and using (2.19). □

Lemma 2.6. *Let D be as in Lemma 2.5, with $B(z_1, 2\alpha h) \subset D$, $\alpha \in (0, \frac{1}{2})$. Let $u \in H^1(D)$ be a solution of the homogeneous Helmholtz equation in D, and let $U(z, \overline{z}) = u(\operatorname{Re} z, \operatorname{Im} z)$. Let $M : D \times D \times \overline{D} \to \mathbb{C}$ be bounded. Then there exists a constant $C > 0$ independent of all the parameters such that*

$$\begin{aligned}\left\|\int_{z_1}^{z} U(t,\overline{t})M(z,t,\overline{t})\,\mathrm{d}t\right\|_{0,D} &\leq C\alpha^{-\frac{1}{2}}h\|M\|_{L^\infty}\left((1 + h^2\omega^2)\|u\|_{0,D} + h\|\nabla u\|_{0,D}\right),\\ \left\|\int_{z_1}^{z} U'(t,\overline{t})M(z,t,\overline{t})\,\mathrm{d}t\right\|_{0,D} &\leq C\alpha^{-\frac{3}{2}}h\|M\|_{L^\infty}\left(h\omega^2\|u\|_{0,D} + (1 + h^2\omega^2)\|\nabla u\|_{0,D}\right),\end{aligned} \tag{2.25}$$

where the integrals are taken on the segment between z_1 and $z \in D$ and $U'(t, \overline{t})$ denotes complex derivative with respect to t.

Proof. See the proof of Lemma 4.2.9 of [19], using (2.23). □

In the following, we will apply the previous results on a domain D that will be an element of the mesh in the finite element method. We will assume the meshes to be shape-regular, so there is no dependence of all the constants on α.

Proposition 2.7 (Holomorphic internal estimates). *Let ϕ be a holomorphic function in D and $\phi \in L^2(D)$, then for all $z \in D$ it holds:*

$$|\phi(z)| \leq \frac{1}{\sqrt{\pi}\, d(z,\partial D)} \|\phi\|_{0,D}\,, \tag{2.26}$$

$$\left|\frac{\partial^k}{\partial z^k}\phi(z)\right| \leq \frac{(k+1)!\, e}{\sqrt{\pi}} \frac{1}{d(z,\partial D)^{k+1}} \|\phi\|_{0,D}\,, \qquad k \geq 1. \tag{2.27}$$

Proof. The first bound follows from the mean-value property of holomorphic functions and the Jensen inequality:

$$|\phi(z)|^2 = \left|\fint_{B(z,r)} \phi(w)\,\mathrm{d}w\right|^2 \leq \fint_{B(z,r)} |\phi(w)|^2\,\mathrm{d}w = \frac{1}{\pi r^2}\|\phi\|^2_{0,B(z,r)} \leq \frac{1}{\pi r^2}\|\phi\|^2_{0,D}\,,$$

with $r = d(z,\partial D)$. For the second one, we fix $r = \frac{k}{k+1} d(z,\partial D)$ so that $d(B(z,r),\partial D) = \frac{1}{1+k} d(z,\partial D)$; using the Cauchy formula and the previous inequality we have

$$\begin{aligned}\left|\frac{\partial^k}{\partial z^k}\phi(z)\right| &= \left|\frac{k!}{2\pi i}\int_{\partial B(z,r)} \frac{\phi(w)}{(w-z)^{k+1}}\,\mathrm{d}w\right| \leq \frac{k!}{2\pi r^{k+1}} 2\pi r \|\phi\|_{L^\infty(\partial B(z,r))} \\ &\leq \frac{k!}{r^k}\frac{1}{\sqrt{\pi} d(B(z,r),\partial D)}\|\phi\|_{L^2(D)} \leq \frac{(k+1)!\left(1+\frac{1}{k}\right)^k}{\sqrt{\pi}\, d(z,\partial D)^{k+1}}\|\phi\|_{L^2(D)}\,.\end{aligned}$$

□

With the hypotheses as in Lemma 2.5, the bound (2.24) holds also for a holomorphic ϕ instead of u. This fact, together with the estimate (2.26), gives an analogue of Lemma 2.6 for holomorphic functions:

$$\begin{aligned}\left\|\int_{z_1}^{z} \phi(t) M(z,t,\bar{t})\,\mathrm{d}t\right\|_{0,D} &\leq h\|M\|_{L^\infty}\left(\alpha h^2 \frac{1}{\pi(\alpha h)^2}\|\phi\|^2_{0,D} + \frac{1}{\alpha}\|\phi\|^2_{0,D}\right)^{\frac{1}{2}} \\ &\leq \left(\pi^{-1}+1\right)^{\frac{1}{2}} h\,\alpha^{-\frac{1}{2}}\|M\|_{L^\infty}\|\phi\|_{0,D}\,.\end{aligned} \tag{2.28}$$

2.1.3 Proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2. The function U satisfies

$$U(z,\zeta) = U(\bar{\zeta},\zeta)G(\bar{\zeta},\zeta,z,\zeta) + \int_{\bar{\zeta}}^{z} U(t,\bar{t})\partial_2 G(t,\bar{t},z,\zeta)\,\mathrm{d}\bar{t} + \int_{\bar{\zeta}}^{z} \partial_1 U(t,\bar{t})G(t,\bar{t},z,\zeta)\,\mathrm{d}t$$

(see [18], § 3.13), where ∂_j denotes the complex derivative with respect to the j-th variable. This, together with (2.5) and (2.3), gives

$$\phi(z) = U(z_0,\bar{z}_0) + 2\int_{z_0}^{z} U(t,\bar{t})\partial_2 G(t,\bar{t},z,\bar{z}_0)\,\mathrm{d}\bar{t} + 2\int_{\bar{z}_0}^{z} \partial_1 U(t,\bar{t})G(t,\bar{t},z,\bar{z}_0)\,\mathrm{d}t. \tag{2.29}$$

We bound the L^2-norm of ϕ using Lemma 2.6 twice with $M = G,\ \partial_2 G$:

$$\begin{aligned}
\|\phi\|_{0,D} &\le h|U(z_0,\overline{z}_0)| + C\Big[\ \|\partial_2 G\|_{L^\infty}\, h\left((1+h^2\omega^2)\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\right)\\
&\qquad + \|G\|_{L^\infty}\, h\left(h\omega^2\,\|u\|_{0,D} + (1+h^2\omega^2)\,\|\nabla u\|_{0,D}\right)\Big]\\
&\overset{(2.14)}{\le} h\,\|u\|_{L^\infty(B(z_0,\alpha h))} + C\left[\frac{\omega c_* e^{c_*}}{4} h\left((1+h^2\omega^2)\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\right)\right.\\
&\qquad \left. + e^{c_*} h\left(h\omega^2\,\|u\|_{0,D} + (1+h^2\omega^2)\,\|\nabla u\|_{0,D}\right)\right.\\
&\overset{\text{Lemma 2.4}}{\le} C\Big[(1+\omega^2h^2)\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\\
&\qquad + e^{c_*}(1+\omega h c_*)(1+c_*^2)\big(\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\,\big)\Big].
\end{aligned}$$

from which the first bound follows.

In order to obtain the second bound, we take the k-th derivative of (2.29):

$$\begin{aligned}
\frac{d^k\phi}{dz^k}(z) &= 2\sum_{n=0}^{k-1}\frac{d^{k-n-1}}{dz^{k-n-1}}\Big(U(z,\overline{z})\partial_3^n\partial_2 G(z,\overline{z},z,\overline{z}_0) + \partial_1 U(z,\overline{z})\partial_3^n G(z,\overline{z},z,\overline{z}_0)\Big)\\
&\qquad + 2\left[\int_{z_0}^{z} U(t,\overline{t})\partial_3^k\partial_2 G(t,\overline{t},z,\overline{z}_0)\,\mathrm{d}\overline{t} + \int_{\overline{z_0}}^{z}\partial_1 U(t,\overline{t})\partial_3^k G(t,\overline{t},z,\overline{z}_0)\,\mathrm{d}t\right].
\end{aligned}$$

Now we bound these derivatives using Lemma 2.6 and the estimates from Section 2.1.1:

$$\begin{aligned}
\left\|\frac{d^k}{dz^k}\phi\right\|_{0,D} &\overset{\text{Lemma 2.6}}{\le} C\Big[\sum_{j=0}^{k}|G|_{W^{k-j,\infty}(D)}\,|U|_{j,D}\\
&\quad + h\left\|\partial_3^k\partial_2 G\right\|_{L^\infty(D)}\left((1+\omega^2h^2)\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\right)\\
&\quad + h\left\|\partial_3^k G\right\|_{L^\infty(D)}\left(h\omega^2\,\|u\|_{0,D} + (1+\omega^2h^2)\,\|\nabla u\|_{0,D}\right)\Big]\\
&\overset{(2.16),\,(2.15)}{\le} C\Big[\sum_{j=0}^{k}\omega^{k-j}e^{c_*}(1+c_*^{k-j})\,|U|_{j,D}\\
&\quad + h\omega^{k+1}c_*^{k-1}e^{c_*}\left((1+\omega^2h^2)\,\|u\|_{0,D} + h\,\|\nabla u\|_{0,D}\right)\\
&\quad + h\omega^k c_*^k e^{c_*}\left(h\omega^2\,\|u\|_{0,D} + (1+\omega^2h^2)\,\|\nabla u\|_{0,D}\right)\Big]\\
&\le Ce^{c_*}\Big[(1+c_*^k)\,\|U\|_{k,\omega,D} + c_*^k\big((1+c_*^2)\omega^k\,\|u\|_{0,D} + c_*\omega^{k-1}\,\|\nabla u\|_{0,D}\,\big)\\
&\quad + c_*^{k+1}\big(c_*\omega^k\,\|u\|_{0,D} + (1+c_*^2)\omega^{k-1}\,\|\nabla u\|_{0,D}\,\big)\Big] \le Ce^{c_*}(1+c_*^{k+3})\,\|u\|_{k,\omega,D}\,,
\end{aligned}$$

because $|U|_{j,D} \le C\,|u|_{j,D}$ due to the definition of U; here, the derivatives of U are understood as $\frac{\partial^j}{\partial z^j}U(z,\overline{z})$. Finally we can bound the complete norms:

$$\begin{aligned}
\|\phi\|_{k,\omega,D}^2 &= \sum_{j=0}^{k}\omega^{2(k-j)}\,|\phi|_{j,D}^2 \le Ce^{2c_*}\left(1+c_*^{k+3}\right)^2\sum_{j=0}^{k}\omega^{2(k-j)}\,\|u\|_{j,\omega,D}^2\\
&\le Ce^{2c_*}\left(1+c_*^{k+3}\right)^2\,\|u\|_{k,\omega,D}^2\,.
\end{aligned}$$

The constant C depends only on k, z_0 and the shape of D, so we have the result. $\square$

Proof of Theorem 2.3. From the definition (2.2) of the Vekua operator we can bound the norm of $u = \operatorname{Re} V[\phi, z_0]$ using (2.28) and (2.14):

$$\|u\|_{0,D} \leq \|\phi\|_{0,D} + \left(\frac{\pi+1}{\pi\alpha}\right)^{\frac{1}{2}} h \left\|\frac{\partial}{\partial t} G\right\|_{L^\infty} \|\phi\|_{0,D} \leq \left(1 + \frac{1}{2\alpha^{\frac{1}{2}}} h\omega c_* e^{c_*}\right) \|\phi\|_{0,D}\,.$$

Let f be a holomorphic function in three variables and $z = x + iy$; a derivative of its integral can be written as

$$\frac{\partial^k}{\partial x^j \partial y^{k-j}} \int_a^z f(t, z, \overline{z})\,\mathrm{d}t = P_{k-1,j}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\right) f(z, z, \overline{z}) + \int_a^z \frac{\partial^k}{\partial x^j \partial y^{k-j}} f(t, z, \overline{z})\,\mathrm{d}t, \tag{2.30}$$

where $P_{k-1,j}$ is a formal polynomial of degree $k-1$ in two variables. We apply this formula to $f(t, z, \overline{z}) = \phi(t)\frac{\partial}{\partial t} G(t, \overline{z}_0, z, \overline{z})$:

$$\begin{aligned}
|u|_{k,D} &\leq \sum_{j=0}^{k} \left\|\frac{\partial^k}{\partial^j x \partial^{k-j} y} u\right\|_{0,D} \\
&\overset{(2.2)}{\leq} \sum_{j=0}^{k} \Bigg\| \operatorname{Re}\Bigg(\frac{\partial^k}{\partial^j x \partial^{k-j} y}\phi(z) - P_{k-1,j}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}}\right)[\phi(z)\partial_1 G(z, \overline{z_0}, z, \overline{z})] \\
&\qquad - \int_a^z \phi(t) \frac{\partial^k}{\partial^j x \partial^{k-j} y} \frac{\partial}{\partial t} G(t, \overline{z}_0, z, \overline{z})\,\mathrm{d}t\Bigg)\Bigg\|_{0,D} \\
&\overset{(2.28)}{\leq} C(k)\left[|\phi|_{k,D} + \sum_{l=0}^{k-1} |G|_{W^{k-l,\infty}(D)} |\phi|_{l,D} + h\,|G|_{W^{k+1,\infty}(D)} \|\phi\|_{0,D}\right] \\
&\overset{(2.16)}{\leq} C(k)\left[|\phi|_{k,D} + \sum_{l=0}^{k-1} e^{c_*}\omega^{k-l}(1 + c_*^{k-l})\,|\phi|_{l,D} + h\,e^{c_*}\omega^{k+1}(1 + c_*^{k+1})\,\|\phi\|_{0,D}\right] \\
&\leq C(k)\,e^{c_*}(1 + c_*^{k+2})\,\|\phi\|_{k,\omega,D}\,.
\end{aligned}$$

Summing over k, gives the result:

$$\begin{aligned}
\|u\|^2_{k,\omega,D} &= \sum_{j=0}^{k} \omega^{2(k-j)}\,|u|^2_{j,D} \leq Ce^{2c_*}\left(1 + c_*^{k+2}\right)^2 \sum_{j=0}^{k} \omega^{2(k-j)}\,\|\phi\|^2_{j,\omega,D} \\
&\leq Ce^{2c_*}\left(1 + c_*^{k+2}\right)^2 \|\phi\|^2_{k,\omega,D}\,.
\end{aligned}$$

□

From the proofs of these theorems we notice that it is easy to prove the continuity of the complete Vekua operator V with the same constants; on the contrary, the continuity of $(\operatorname{Re} V)^{-1}$ from Theorem 2.2 cannot be extended to the continuity of V^{-1} because, for a complex-valued function u, Theorem 2.1 is no longer valid (there is a version of it that involves two different holomorphic functions; see [26], Section I.10).

We highlight the main difference between these two theorems: the real Vekua operator is continuous from $L^2(D)$ in itself, while its inverse is continuous only from $H^1(D)$. The reason is that the mean-value property and the inverse estimates (2.26) hold true for holomorphic functions, while for the solutions of the homogeneous Helmholtz equation we only have (2.18), that involves the norm of the gradient of u.

2.2 The approximation by generalized harmonic polynomials

Using the Vekua transform, in order to study approximations to generic solutions of the Helmholtz equation in Sobolev norms, it is enough to study approximations to holomorphic functions in the same norms. The following theorem gives an approximation result for holomorphic functions by means of polynomials; for the proof, see [20], Theorem 2.9. We restrict ourselves to a convex domain: in the following we will apply this theorem to the cells of triangulations, which usually satisfy this requirement.

Theorem 2.8. *Let $D \subset \mathbb{R}^2$ be a bounded, Lipschitz and convex domain with diameter h and let $1 \le k \le N+1$.*

Then there exists an operator $\Pi_N^k : \{f \in H^k(D),\ \Delta f = 0\} \to \mathbb{P}^N(D)$, where $\mathbb{P}^N(D)$ denotes the space of complex polynomials in D of degree at most N, such that

$$\left|f - \Pi_N^k f\right|_{j,D} \le C h^{k-j} \left(\frac{\log(N+2)}{N+2}\right)^{k-j} |f|_{k,D} \qquad \forall\, j \in \{0,1,\dots,k\}, \tag{2.31}$$

where C depends only on k and on the shape of D.

For the complete H^j-norm, $j \in \{0,1,\dots,k\}$, we readily have:

$$\begin{aligned}\|f - f_N\|_{j,\omega,D} &\le \sum_{l=0}^{j} \omega^{j-l} |f - f_N|_{l,D} \le C \sum_{l=0}^{j} \omega^{j-l} h^{k-l} \left(\frac{\log(N+2)}{N+2}\right)^{k-l} |f|_{k,D} \\ &\le C h^{k-j} \left(1 + c_*^j\right) \left(\frac{\log(N+2)}{N+2}\right)^{k-j} |f|_{k,D}\end{aligned} \tag{2.32}$$

because $\frac{\log(N+2)}{N+2} < 1$. For every $j \in \{0,1,\dots,N+1\}$ also this bound is valid:

$$\|f - f_N\|_{j,\omega,D} \le \sum_{l=0}^{j} \omega^{j-l} |f - f_N|_{l,D} \le C \sum_{l=0}^{j} \omega^{j-l} |f|_{l,D} \le C\, |f|_{j,\omega,D}\,. \tag{2.33}$$

Definition 2.9. *Fixed $z_0 \in D$ and $1 \le k \le N+1$, for all real solution u of $\Delta u + \omega^2 u = 0$ in D, we define*

$$P_N^k(u) := \operatorname{Re} V\left[\Pi_N^k\left((\operatorname{Re} V)^{-1}[u, z_0]\right), z_0\right].$$

The functions in the form $P_N^k(u)$ are the generalized harmonic polynomials that approximate u. The functions $P_N^k(u)$ are images of the Vekua operator, so they are solutions of the homogeneous Helmholtz equation in D, when $N \le N_{max}$ they belong to a linear space of dimension $N_{max}+1$.

The main result of this section is the following theorem that provides the approximation properties of generalized harmonic polynomials.

Theorem 2.10. *Let $D \subset \mathbb{R}^2$ be a bounded, Lipschitz and convex domain with diameter h; fix $z_0 \in D$. Let $u \in H^k(D)$, $k \ge 1$, be a real solution of $\Delta u + \omega^2 u = 0$ in D. Then the generalized harmonic polynomial $P_N^k(u)$ satisfies*

$$\begin{gathered}\left\|u - P_N^k(u)\right\|_{j,\omega,D} \le C e^{2c_*} \left(1 + c_*^{k+2j+5}\right) h^{k-j} \left(\frac{\log(N+2)}{N+2}\right)^{k-j} \|u\|_{k,\omega,D} \\ \forall\, j,\, k,\, N \in \mathbb{N}_0, \quad k \ge 1, \quad 0 \le j \le k \le N+1,\end{gathered} \tag{2.34}$$

where the constant C depends only on k, z_0 and the shape of D.

Proof. It is enough to write the definition of $P_N^k(u)$ and use the Theorems 2.3, 2.8 (bound (2.32)) and 2.2:

$$\begin{aligned}\left\|u - P_N^k(u)\right\|_{j,\omega,D} &= \left\|u - \operatorname{Re} V\left[\Pi_N^k\left((\operatorname{Re} V)^{-1}[u, z_0]\right), z_0\right]\right\|_{j,\omega,D} \\ &\leq C(j) e^{c_*}\left(1 + c_*^{j+2}\right)\left\|(\operatorname{Re} V)^{-1}[u, z_0] - \Pi_N^k\left((\operatorname{Re} V)^{-1}[u, z_0]\right)\right\|_{j,\omega,D} \\ &\leq C(j,k) e^{c_*}\left(1 + c_*^{2j+2}\right) h^{k-j}\left(\frac{\log(N+2)}{N+2}\right)^{k-j}\left|(\operatorname{Re} V)^{-1}[u, z_0]\right|_{k,D} \\ &\leq C(j,k) e^{2c_*}\left(1 + c_*^{k+2j+5}\right) h^{k-j}\left(\frac{\log(N+2)}{N+2}\right)^{k-j}\|u\|_{k,\omega,D}.\end{aligned}$$

Since $j \leq k$, we can choose C depending only on k but not on j. □

In [20, Theorem 3.16] the same result is proved for a more general PDE in a starshaped and non-convex domain, using standard Sobolev norms, but the bounding constants depend on ω in a unknown way.

3 The approximation by plane waves

In the previous section we have studied how to approximate a solution of the homogeneous Helmholtz equation by generalized harmonic polynomials. However the analytical form of these functions is not simple, so we will not use them for implementing the finite element method. Instead, we will use a finite dimensional space of plane waves that approximates generalized harmonic polynomials, that, in turn, approximate the solutions of the equation.

We start by studying the approximation of a plane wave with arbitrary direction in maximum norm (Theorem 3.3). This allows to approximate in the same norm the Herglotz functions, that are continuous linear combination of plane waves (Lemma 3.4). This class of functions includes the generalized harmonic polynomials. In order to approximate these in Sobolev norms (Theorem 3.9), we need to write them explicitly (Lemma 3.5) and to prove the continuity of the inverse of Vekua operator in L^∞-norm (Lemma 3.8). Finally we join the result of the previous section and we find the approximation of an arbitrary solution of the Helmholtz equation with the desired order of convergence in h (Theorem 3.10).

Let $p = 2m + 1 \geq 3$ be an odd integer, and let $\varphi_k \in [0, 2\pi)$, $k = 1, \dots, p$, be distinct angles. We denote the plane wave functions of wave number ω and direction $\boldsymbol{d}_\theta = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi]$ by

$$e_\theta(x, y) = e^{i\omega(x\cos\theta + y\sin\theta)} = e^{i\omega r\cos(\theta - \psi)}, \qquad (x, y) = (r\cos\psi, r\sin\psi) \in \mathbb{R}^2.$$

We introduce the discrete space

$$PW_\omega^p(\mathbb{R}^2) = \left\{v \in C^\infty(\mathbb{R}^2), \quad v(x, y) = \sum_{k=1}^p \alpha_k e_{\varphi_k}(x, y), \quad \alpha_k \in \mathbb{C}\right\}.$$

An interesting case is obtained when the angles are uniformly spaced, in this case we write

$$PW_\omega^{p,\xi}(\mathbb{R}^2) = \left\{v \in C^\infty(\mathbb{R}^2),\ v(x, y) = \sum_{k=1}^p \alpha_k e_{\varphi_k}(x, y),\ \alpha_k \in \mathbb{C},\ \varphi_k = (k-1)\frac{2\pi}{p} + \xi\right\}.$$

3.1 The approximation of generic plane waves

We first study how the space PW_ω^p approximates a plane wave with arbitrary direction. In a first step we analyze the order of approximation with respect to ω in a normalized reference element, then, scaling on the actual domain, we achieve the order of convergence with respect to h. We will make use of the following technical lemma.

Lemma 3.1. *Let $\varphi_k \in [0, 2\pi)$, $k = 1, \dots, p = 2m+1$, be different angles. Then the $p \times p$ square matrix*

$$Q = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cos\varphi_1 & \cos\varphi_2 & \cdots & \cos\varphi_p \\ \vdots & \vdots & \ddots & \vdots \\ \cos^m\varphi_1 & \cos^m\varphi_2 & \cdots & \cos^m\varphi_p \\ \sin\varphi_1 & \sin\varphi_2 & \cdots & \sin\varphi_p \\ \cos\varphi_1\sin\varphi_1 & \cos\varphi_2\sin\varphi_2 & \cdots & \cos\varphi_p\sin\varphi_p \\ \vdots & \vdots & \ddots & \vdots \\ \cos^{m-1}\varphi_1\sin\varphi_1 & \cos^{m-1}\varphi_2\sin\varphi_2 & \cdots & \cos^{m-1}\varphi_p\sin\varphi_p \end{bmatrix}$$

is invertible.

Proof. Let $\boldsymbol{v} = [b_0, \dots, b_m, c_1, \dots, c_m]^t \in \mathbb{C}^p$, be a vector. We define the function

$$\mathcal{Q}(\varphi) = \sum_{j=0}^{m} b_j \cos^j \varphi - \sum_{j=1}^{m} c_j \cos^{j-1}\varphi \sin\varphi.$$

If we can prove that $\mathcal{Q}(\varphi_k) = 0$ for each $k \in \{1, \dots, p\}$ implies $\boldsymbol{v} = \boldsymbol{0}$ then, for all $\boldsymbol{v} \in \mathbb{C}^p$, $Q^t\boldsymbol{v} = \{\mathcal{Q}(\varphi_k)\}_{k=1,\dots,p} = \boldsymbol{0}$ implies $\boldsymbol{v} = \boldsymbol{0}$, thus $\ker Q^t = \{\boldsymbol{0}\}$, i.e., Q is not singular.

For every $1 \le q \le j-1$, the following formula holds

$$\binom{j-1}{q-1} - \binom{j-1}{q} = \frac{(j-1)!(q-j+q)}{q!(j-q)!} = \frac{2q-j}{j}\binom{j}{q}.$$

We rewrite $\mathcal{Q}$ by expanding the powers of cosines as powers of binomials, rearranging the indexes in order to collect the exponentials with the same exponent $2q - j = l$, isolating and then reinserting the terms with $q = 0$ and $q = j$ in the sum over q:

$$\begin{aligned} \mathcal{Q}(\varphi) &= \sum_{j=0}^{m} b_j \sum_{q=0}^{j} \binom{j}{q} \frac{1}{2^j} e^{i\varphi(2q-j)} + \sum_{j=1}^{m} c_j \sum_{q=0}^{j-1} \binom{j-1}{q} \frac{1}{2^j i} e^{i\varphi(2q-j+1)} \big(e^{i\varphi} - e^{-i\varphi}\big) \\ &= b_0 + \sum_{j=1}^{m} \frac{b_j}{2^j} \sum_{q=0}^{j} \binom{j}{q} e^{i\varphi(2q-j)} - i \sum_{j=1}^{m} \frac{c_j}{2^j} \Bigg[\sum_{q=1}^{j} \binom{j-1}{q-1} e^{i\varphi(2q-j)} \\ &\qquad - \sum_{q=0}^{j-1} \binom{j-1}{q} e^{i\varphi(2q-j)} \Bigg] \\ &= b_0 + \sum_{j=1}^{m} \frac{1}{2^j} \Bigg[b_j e^{-i\varphi j} + b_j e^{i\varphi j} - i c_j e^{i\varphi j} + i c_j e^{-i\varphi j} \\ &\qquad + \sum_{q=1}^{j-1} \left(b_j \binom{j}{q} - i c_j \left(\binom{j-1}{q-1} - \binom{j-1}{q} \right) \right) e^{i\varphi(2q-j)} \Bigg] \end{aligned}$$

$$= b_0 + \sum_{j=1}^{m} \frac{1}{2^j} \left[\sum_{q=0}^{j} \binom{j}{q} \left(b_j - i \frac{2q-j}{j} c_j \right) e^{i\varphi(2q-j)} \right]$$

$$= \sum_{l=-m}^{m} e^{i\varphi l} \underbrace{\sum_{\substack{|l|\leq j \leq m \\ l+j \text{ even}}} \frac{1}{2^j} \binom{j}{\frac{l+j}{2}} \left(b_j - i c_j \frac{l}{j} \right)}_{=:\gamma_l}$$

where in the last step we have changed the index $l = 2q - j$; when $l = j = 0$ we understand $c_j \frac{l}{j} = 0$.

Since $\mathcal{Q}(\varphi)$ is a trigonometric polynomial, if $\mathcal{Q}(\varphi_k) = 0$ for every $k \in \{1, \dots, p\}$ then all the coefficients γ_k are zero.

Now we only have to prove that $\gamma_l = 0$, with $l = 0, \dots, p$, implies $\boldsymbol{v} = [b_0, \dots, b_m, c_1, \dots, c_m]^t = \mathbf{0}$. We notice that

$$b_m = 2^{m-1} \frac{1}{2^m} \big(b_m - i c_m + b_m + i c_m \big) = 2^{m-1} (\gamma_m + \gamma_{-m}) = 0,$$

$$c_m = i 2^{m-1} \frac{1}{2^m} \big(b_m - i c_m - (b_m + i c_m) \big) = i 2^{m-1} (\gamma_m - \gamma_{-m}) = 0.$$

We can proceed by induction with respect to decreasing l: if $b_j = c_j = 0$ for every $j \geq l > 0$ then

$$b_{l-1} = \frac{1}{2} 2^{l-1} \binom{j}{\frac{j+l-1}{2}}^{-1} (\gamma_{l-1} + \gamma_{-(l-1)}) = 0$$

$$c_{l-1} = \frac{i}{2} 2^{l-1} \binom{j}{\frac{j+l-1}{2}}^{-1} (\gamma_{l-1} - \gamma_{-(l-1)}) = 0$$

We have proved that all the coefficients, including $b_0 = \gamma_0 = 0$, vanish: $\boldsymbol{\gamma} = \mathbf{0}$ implies $\boldsymbol{v} = \mathbf{0}$, from which we have the result. □

The necessity of using an odd number p of plane waves is evident from the definition of the matrix Q. Furthermore with an even dimensional basis, in the limit $\omega \to 0$, we have an incomplete space of harmonic polynomials and we cannot apply the theory of [16]. Finally, from the simulations of [17, Sect. 3.2] we notice that increasing the dimension from $2m+1$ to $2m+2$ does not improve the order of convergence of the PWDG method.

Let D be a domain with diameter h; the reference domain is defined as $\widehat{D} = \frac{1}{h} D$, with diameter 1. A plane wave with wavenumber $\omega > 0$ scaled on $\widehat{D}$ has wavenumber $\hat{\omega} = h\omega$:

$$\hat{e}_\theta(\hat{x}, \hat{y}) := e^{i\hat{\omega}(\hat{x}\cos\theta + \hat{y}\sin\theta)} = e_\theta(h\hat{x}, h\hat{y}).$$

We have the following result.

Lemma 3.2. *Let $\widehat{D}$ be a domain with diameter 1 including the origin. Let $\varphi_k \in [0, 2\pi)$, $k = 1, \dots, p = 2m+1 \geq 3$, all different. Then, having fixed $\theta \in [0, 2\pi]$, there exist $\alpha_1, \dots, \alpha_p \in \mathbb{C}$ such that*

$$\left\| \hat{e}_\theta - \sum_{k=1}^{p} \alpha_k \hat{e}_{\varphi_k} \right\|_{L^\infty(\widehat{D})} \leq C\, \hat{\omega}^{m+1} e^{c_*}, \tag{3.35}$$

where $C > 0$ depends only on the set $\{\varphi_k\}_{k=1}^{p}$.

Proof. If we expand the exponentials in the difference between the plane wave functions $\hat{e}_\theta(x,y) = e^{i\hat{\omega}r\cos(\theta-\psi)}$ and $\hat{e}_{\varphi_k}(x,y) = e^{i\hat{\omega}r\cos(\varphi_k-\psi)}$, using polar coordinates $(x,y) = (r\cos\psi, r\sin\psi)$ we get:

$$\begin{aligned}
&\hat{e}_\theta(x,y) - \sum_{k=1}^{p} \alpha_k \hat{e}_{\varphi_k}(x,y) = \sum_{j\geq 0} \frac{1}{j!}\hat{\omega}^j (ir)^j \left[\cos^j(\theta-\psi) - \sum_{k=1}^{p}\alpha_k(\cos^j\varphi_k - \psi)\right] \\
&= \sum_{j\geq 0} \frac{1}{j!}\hat{\omega}^j (ir)^j \left[(\cos\theta\cos\psi + \sin\theta\sin\psi)^j - \sum_{k=1}^{p}\alpha_k(\cos\varphi_k\cos\psi + \sin\varphi_k\sin\psi)^j\right] \\
&= \sum_{j\geq 0} \frac{1}{j!}\hat{\omega}^j (ir)^j \left[\sum_{l=0}^{j} \frac{j!}{l!\,(j-l)!}\cos^l\psi\cos^l\theta\sin^{j-l}\psi\sin^{j-l}\theta \right.\\
&\qquad\qquad\left. - \sum_{k=1}^{p}\alpha_k \sum_{l=0}^{j}\frac{j!}{l!\,(j-l)!}\cos^l\psi\cos^l\varphi_k\sin^{j-l}\psi\sin^{j-l}\varphi_k\right] \\
&= \sum_{j\geq 0} \frac{1}{j!}\hat{\omega}^j (ir)^j \sum_{l=0}^{j}\frac{j!}{l!\,(j-l)!}\cos^l\psi\sin^{j-l}\psi\left[\cos^l\theta\sin^{j-l}\theta - \sum_{k=1}^{p}\alpha_k\cos^l\varphi_k\sin^{j-l}\varphi_k\right].
\end{aligned} \tag{3.36}$$

Lemma 3.1 guarantees that there is a vector $\boldsymbol{\alpha} \in \mathbb{C}^p$ such that

$$Q\boldsymbol{\alpha} = [1, \cos\theta, \ldots, \cos^m\theta, \sin\theta, \ldots, \cos^{m-1}\theta\sin\theta]^t,$$

that is

$$\begin{cases}
\displaystyle\sum_{k=1}^{p}\alpha_k\cos^r\varphi_k = \cos^r\theta, & r = 0,\ldots,m, \\
\displaystyle\sum_{k=1}^{p}\alpha_k\cos^{r-1}\varphi_k\sin\varphi_k = \cos^{r-1}\theta\sin\theta, & r = 1,\ldots,m.
\end{cases} \tag{3.37}$$

We choose two nonnegative integers s and t such that $s+t \leq m$; thus, when t is even,

$$\begin{aligned}
&\sum_{k=1}^{p}\alpha_k\cos^s\varphi_k\sin^t\varphi_k = \sum_{k=1}^{p}\alpha_k\cos^s\varphi_k(1-\cos^2\varphi_k)^{\frac{t}{2}} \\
&= \sum_{k=1}^{p}\alpha_k\cos^s\varphi_k\sum_{q=0}^{\frac{t}{2}}\binom{\frac{t}{2}}{q}(-1)^q\cos^{2q}\varphi_k = \sum_{q=0}^{\frac{t}{2}}\binom{\frac{t}{2}}{q}(-1)^q\sum_{k=1}^{p}\alpha_k\cos^{s+2q}\varphi_k \\
&= \sum_{q=0}^{\frac{t}{2}}\binom{\frac{t}{2}}{q}(-1)^q\cos^{s+2q}\theta = \cos^s\theta(1-\cos^2\theta)^{\frac{t}{2}} = \cos^s\theta\sin^t\theta.
\end{aligned}$$

For odd t we get the same inequality by factorising $\sin\varphi_k$ and using the second equality of (3.37) instead of the first one.

If we insert the vector $\boldsymbol{\alpha}$ in (3.36), the content of the last brackets vanishes for every $j = l + (j-l) \leq m$, so we can bound the L^∞-norm as follows:

$$\left\|\hat{e}_\theta - \sum_{k=1}^{p}\alpha_k\hat{e}_{\varphi_k}\right\|_{L^\infty(\widehat{D})} = \sup_{r,\psi}\left|\sum_{j\geq m+1}\frac{1}{j!}\hat{\omega}^j(ir)^j\left[(\cos(\theta-\psi))^j - \sum_{k=1}^{p}\alpha_k(\cos(\varphi_k-\psi))^j\right]\right|$$

$$\leq \hat{\omega}^{m+1} \sum_{j\geq 0} \frac{\hat{\omega}^j}{(j+m+1)!} \left(1 + \sum_{k=1}^{p} |\alpha_k|\right)$$
$$\leq \frac{1}{(m+1)!} \hat{\omega}^{m+1} e^{c_*} \left(1 + p \left\|Q^{-1}\right\|_1\right) \leq C(p, \{\varphi_k\})\, \hat{\omega}^{m+1}\, e^{c_*},$$

using $r < 1$, $\hat{\omega} = \omega h < c_*$ and

$$\sum_{k=1}^{p} |\alpha_k| = \|\boldsymbol{\alpha}\|_1 \leq \left\|Q^{-1}\right\|_1 \left\|[1, \cos\theta, \dots, \cos^m\theta, \sin\theta, \dots, \cos^{m-1}\theta \sin\theta]^t\right\|_1$$
$$\leq C(p, \{\varphi_k\}).$$

□

By transforming to the original domain D, we obtain the convergence in h of the best approximation in the L^∞-norm.

Theorem 3.3. *Let D be a domain of diameter h containing the origin. Let $\varphi_k \in [0, 2\pi)$, $k = 1, \dots, p = 2m+1 \geq 3$, different directions. Thus, having fixed $\theta \in [0, 2\pi]$, there exist $\alpha_1, \dots, \alpha_p \in \mathbb{C}$ such that*

$$\left\| e_\theta - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(D)} \leq C(p, \{\varphi_k\})\, h^{m+1} \omega^{m+1} e^{c_*}. \tag{3.38}$$

Proof. Choose the same α_k of Lemma 3.2 and recall that $e_\theta(x,y) = \hat{e}_\theta\left(\frac{x}{h}, \frac{y}{h}\right)$:

$$\left\| e_\theta - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(D)} = \sup_{(x,y)\in D} \left\| \hat{e}_\theta\left(\frac{x}{h}, \frac{y}{h}\right) - \sum_{k=1}^{p} \alpha_k \hat{e}_{\varphi_k}\left(\frac{x}{h}, \frac{y}{h}\right) \right\|$$
$$= \left\| \hat{e}_\theta - \sum_{k=1}^{p} \alpha_k \hat{e}_{\varphi_k} \right\|_{L^\infty(\widehat{D})} \leq C(p, \{\varphi_k\})\, \hat{\omega}^{m+1} e^{c_*} = C(p, \{\varphi_k\})\, h^{m+1} \omega^{m+1} e^{c_*}.$$

□

If $\boldsymbol{\alpha}$ is the vector of the coefficients that allows to approximate e_θ and $\boldsymbol{\beta}$ is the analogous for $e_{\theta+\pi} = e^{-i\omega(x\cos\theta + y\sin\theta)}$, then for all $\theta \in [0, 2\pi]$, with $m \geq 1$, the following bounds hold:

$$\left\| \cos\big(\omega(x\cos\theta + y\sin\theta)\big) - \sum_{k=1}^{p} \frac{\alpha_k + \beta_k}{2} e_{\varphi_k} \right\|_{L^\infty(D)}$$
$$= \frac{1}{2} \left\| e^{i\omega(x\cos\theta + y\sin\theta)} - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} + e^{-i\omega(x\cos\theta + y\sin\theta)} - \sum_{k=1}^{p} \beta_k e_{\varphi_k} \right\|_{L^\infty(D)}$$
$$\leq C(p, \{\varphi_k\})\, h^{m+1} \omega^{m+1} e^{c_*},$$

$$\left\| \sin\big(\omega(x\cos\theta + y\sin\theta)\big) - \sum_{k=1}^{p} \frac{\alpha_k - \beta_k}{2i} e_{\varphi_k} \right\|_{L^\infty(D)}$$

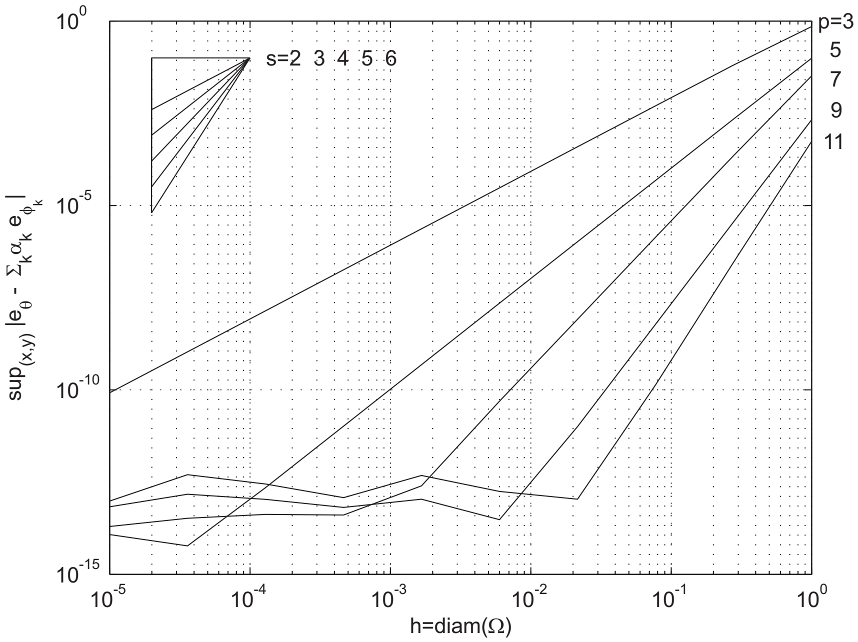
$$= \left\| \frac{1}{2i} \left(e^{i\omega(x\cos\theta + y\sin\theta)} - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} - e^{-i\omega(x\cos\theta + y\sin\theta)} + \sum_{k=1}^{p} \beta_k e_{\varphi_k} \right) \right\|_{L^\infty(D)}$$

$$\leq C(p, \{\varphi_k\})\, h^{m+1} \omega^{m+1} e^{c_*}, \quad (3.39)$$

which means that for the real and imaginary parts of a plane wave of direction θ the same best approximation estimates hold true.

Numerical experiment 1. *Figure 1 shows that the infinity norm of the approximation error for a generic plane wave in $D = (0,h)^2$ varies as h^{m+1}, so the estimates (3.38) are sharp. Evaluating the error with h approaching to zero is equivalent to evaluating it on the reference domain $\hat{D}$ with $\hat{\omega}$ going to zero, so it is not possible to compute the plane wave by using the standard basis $\{e_{\varphi_k}\}$ of $PW_\omega(\mathbb{R}^2)$ because this is not stable with respect to this limit. The basis introduced in [16] has been used instead, with a power expansion in ω truncated at the 13^{th} term.*

Figure 1: The norm $\|e_\theta - \sum_{k=1}^{p} \alpha_k e_{\varphi_k}\|_{L^\infty((0,h)^2)}$ with φ_k uniformly spaced, $\theta = 1$, $\omega = 1$, plotted against h for $p = 3, \ldots, 11$.



3.2 The approximation of the generalized harmonic polynomials

In the previous section we have derived best approximation estimates in the L^∞-norm for a plane wave function of a fixed direction in $PW^p_\omega(\mathbb{R}^2)$. By using this result, in this section we derive best approximation estimates in L^∞ and in Sobolev norms for generalized harmonic polynomials.

If a function u is a solution of the homogeneous Helmholtz equation and is also a tempered distribution, its Fourier transform satisfies

$$(|\xi|^2 - \omega^2)\hat{u}(\xi) = 0,$$

namely its support lies on the circle of radius ω centered in the origin. The Dirac delta distribution centered in $(\omega\cos\theta, \omega\sin\theta)$ is the Fourier transform of the e_θ plane wave, so, in terms of Fourier transform, Lemma 3.2 shows how it is possible to approximate a Dirac delta centered at a point on this circle by a finite number of Dirac deltas centered at fixed points on the same circle. Now we study how the same Dirac delta distributions approximate a function $g \in L^1(0, 2\pi)$.

Lemma 3.4. *Let D be a domain with diameter h and containing the origin. Let $g \in L^1(0, 2\pi)$ and*

$$u(x, y) = \int_0^{2\pi} \mathrm{Re}\left(g(\theta)e^{i\omega(x\cos\theta + y\sin\theta)}\right) \mathrm{d}\theta. \tag{3.40}$$

Let $\varphi_1 \ldots \varphi_p \in [0, 2\pi)$ be different angles, with $p = 2m + 1 \geq 3$. Then there exist $\alpha_1, \ldots, \alpha_p$ such that

$$\left\| u - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(D)} \leq C(p, \{\varphi_k\}) e^{c_*} h^{m+1} \omega^{m+1} \|g\|_{L^1(0,2\pi)} . \tag{3.41}$$

Proof. We fix $\delta > 0$ and partition the interval $[0, 2\pi)$ into measurable sets A_j such that $A_j \subset [a_j, b_j)$, $b_j - a_j < \delta$ and it is possible to approximate g with a simple function $g_\delta(\theta) = \sum_j g_j \chi_{A_j}(\theta)$ in L^1-norm:

$$\|g - g_\delta\|_{L^1(0,2\pi)} \xrightarrow{\delta\to 0} 0.$$

Plane waves are C^1 functions in θ so, thanks to the Lagrange theorem, for all $\theta \in A_j \subset [a_j, b_j)$ there exists $c \in [a_j, \theta]$ such that

$$\begin{aligned}\left| e^{i\omega(x\cos\theta + y\sin\theta)} - e^{i\omega(x\cos a_j + y\sin a_j)} \right| &= \left| (\theta - a_j) \frac{\partial}{\partial\theta} e^{i\omega(x\cos\theta + y\sin\theta)} \Big|_{\theta=c} \right| \\ &= (\theta - a_j) \left| i\omega e^{i\omega(x\cos c + y\sin c)} (-x\sin c + y\cos c) \right| \leq \delta\omega h.\end{aligned}$$

We have

$$\begin{aligned}&\left| u(x, y) - \sum_j |A_j| \,\mathrm{Re}\left(g_j e^{i\omega(x\cos a_j + y\sin a_j)}\right) \right| \\ &= \left| \int_0^{2\pi} \mathrm{Re}\left(g(\theta)e^{i\omega(x\cos\theta + y\sin\theta)} - \sum_j \chi_{A_j}(\theta) g_j e^{i\omega(x\cos a_j + y\sin a_j)}\right) \mathrm{d}\theta \right| \\ &\leq \sum_j \int_{A_j} \left| g(\theta)e^{i\omega(x\cos\theta + y\sin\theta)} - g_j e^{i\omega(x\cos a_j + y\sin a_j)} \right| \mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&\le \sum_j \int_{A_j} |g(\theta) - g_j| \left| e^{i\omega(x\cos\theta + y\sin\theta)} \right| \mathrm{d}\theta \\
&\quad + |g_j| \int_{A_j} \left| e^{i\omega(x\cos\theta + y\sin\theta)} - e^{i\omega(x\cos a_j + y\sin a_j)} \right| \mathrm{d}\theta \\
&\le \|g - g_\delta\|_{L^1(0,2\pi)} + \sum_j |g_j||A_j| \sup_{\theta\in A_j} \left| e^{i\omega(x\cos\theta + y\sin\theta)} - e^{i\omega(x\cos a_j + y\sin a_j)} \right| \\
&\le \|g - g_\delta\|_{L^1(0,2\pi)} + \|g_\delta\|_{L^1(0,2\pi)}\, \delta\omega h \xrightarrow{\delta\to 0} 0,
\end{aligned}
$$

because $\|g_\delta\|_{L^1(0,2\pi)}$ is bounded.

This convergence is uniform in (x, y). From (3.39) we approximate real and imaginary parts of $e^{i\omega(x\cos a_j + y\sin a_j)}$ and their linear combinations using plane waves belonging to $PW_\omega(\mathbb{R}^2)$, so there are $\boldsymbol{\alpha}_j \in \mathbb{C}^p$ such that:

$$
\begin{aligned}
&\left\| \sum_j \left\{ |A_j| \operatorname{Re}\left(g_j e^{i\omega(x\cos a_j + y\sin a_j)} \right) - \sum_{k=1}^p \alpha_{j,k} e_{\varphi_k} \right\} \right\|_{L^\infty(D)} \\
&= \left\| \sum_j \left\{ |A_j| \Big(\operatorname{Re}(g_j) \cos\left(\omega(x\cos a_j + y\sin a_j)\right) - \operatorname{Im}(g_j) \sin\left(\omega(x\cos a_j + y\sin a_j)\right) \Big) \right.\right. \\
&\qquad\qquad \left.\left. - \sum_{k=1}^p \alpha_{j,k} e_{\varphi_k} \right\} \right\|_{L^\infty(D)} \le \|g\|_{L^1(0,2\pi)}\, C(p, \varphi_k)\, h^{m+1} \omega^{m+1} e^{c_*}.
\end{aligned} \tag{3.42}
$$

Finally we use the triangle inequality, choosing δ small enough so that $\left| u(x,y) - \sum_j |A_j| \operatorname{Re}\left(g_j e^{i\omega(x\cos a_j + y\sin a_j)} \right) \right|$ is uniformly smaller than (3.42) and $\alpha_k := \sum_j \alpha_{j,k}$, we get

$$
\begin{aligned}
\left\| u - \sum_{k=1}^p \alpha_k e_{\varphi_k} \right\|_{L^\infty(D)} &\le \left\| u - \sum_j |A_j| \operatorname{Re}\left(g_j e^{i\omega(x\cos a_j + y\sin a_j)} \right) \right\|_{L^\infty(D)} \\
&\quad + \left\| \sum_j \left\{ |A_j| \operatorname{Re}\left(g_j e^{i\omega(x\cos a_j + y\sin a_j)} \right) - \sum_{k=1}^p \alpha_{j,k} e_{\varphi_k} \right\} \right\|_{L^\infty(D)} \\
&\le \|g\|_{L^1(0,2\pi)}\, C(p, \{\varphi_k\})\, h^{m+1} \omega^{m+1} e^{c_*}.
\end{aligned}
$$

□

We derive the following expressions for the generalized harmonic polynomials. These functions are the circular waves in the plane.

Lemma 3.5. *Let $P(z) = a_N z^N + \ldots + a_1 z + a_0$ be a complex polynomial of degree N. Denoting $z = x + iy = re^{i\psi}$, the following identities hold*

$$
\begin{aligned}
V[P,0](x,y) &= \sum_{n=0}^N a_n\, n! \left(\frac{2}{\omega} \right)^n e^{in\psi} J_n(\omega r) \\
&= \sum_{n=0}^N a_n\, (-i)^n \frac{2^n n!}{2\pi\omega^n} \int_0^{2\pi} e^{in\theta} e^{i\omega(x\cos\theta + y\sin\theta)}\, \mathrm{d}\theta.
\end{aligned} \tag{3.43}
$$

Proof. For any holomorphic ϕ, having fixed $z_0 = 0$, by the definition of the Vekua transform (2.2) we get:

$$\begin{aligned} V[\phi,0](x,y) &= \phi(z) - \int_0^z \phi(t)\frac{\partial}{\partial t}J_0\big(\omega\sqrt{(t-z)(0-\overline{z})}\big)\,\mathrm{d}t \\ &= \phi(z) - \int_0^1 \phi(sz)\frac{1}{z}\frac{\partial}{\partial s}J_0\big(\omega|z|\sqrt{1-s}\big)z\,\mathrm{d}s \\ &= \phi(z) - \int_0^1 \phi(sz)J_1\big(\omega r\sqrt{1-s}\big)\frac{\omega r}{2\sqrt{1-s}}\,\mathrm{d}s, \end{aligned} \tag{3.44}$$

because $J_0' = -J_1$.

In the case $\phi(z) = z^n$, expanding the Bessel function J_1 and using $\int_0^1 s^n(1-s)^j\,\mathrm{d}s = \beta(1+n,1+j) = \frac{n!\,j!}{(n+j+1)!}$ we obtain the first identity for a monomial:

$$\begin{aligned} V[z^n,0](x,y) &= z^n - \int_0^1 s^n z^n \sum_{j=0}^{\infty}\frac{(-1)^j\big(\omega r\sqrt{1-s}\big)^{2j+1}}{j!\,(j+1)!\,2^{2j+1}}\frac{\omega r}{2\sqrt{1-s}}\,\mathrm{d}s \\ &= z^n\left(1-\sum_{j=0}^{\infty}\frac{(-1)^j\big(\omega r\big)^{2j+2}}{j!\,(j+1)!\,2^{2j+2}}\int_0^1 s^n(1-s)^j\,\mathrm{d}s\right) \\ &= z^n\left(1-\frac{2^n\,n!}{r^n\omega^n}\sum_{j=0}^{\infty}\frac{(-1)^j\big(\omega r\big)^{2j+2+n}}{(n+j+1)!\,(j+1)!\,2^{2j+2+n}}\right) \\ &= e^{in\psi}\left(r^n+\frac{2^n\,n!}{\omega^n}\sum_{l=1}^{\infty}\frac{(-1)^l\big(\omega r\big)^{2l+n}}{(l+n)!\,l!\,2^{2l+n}}\right) \\ &= e^{in\psi}\frac{2^n\,n!}{\omega^n}\sum_{l=0}^{\infty}\frac{(-1)^l\big(\omega r\big)^{2l+n}}{(l+n)!\,l!\,2^{2l+n}} = e^{in\psi}n!\left(\frac{2}{\omega}\right)^n J_n(\omega r). \end{aligned}$$

In order to prove the second identity we use

$$\int_0^{2\pi} e^{il\theta}\,\mathrm{d}\theta = \begin{cases} 0 & l\in\mathbb{Z},\ l\neq 0, \\ 2\pi & l=0, \end{cases}$$

and obtain the result:

$$\begin{aligned} (-i)^n\frac{2^n n!}{2\pi\omega^n}\int_0^{2\pi} e^{in\theta}e^{i\omega r\cos(\theta-\psi)}\,\mathrm{d}\theta &= (-i)^n\frac{2^n n!}{2\pi\omega^n}\int_0^{2\pi} e^{in\theta}\sum_{j=0}^{\infty}\frac{1}{j!}\big(i\omega r\cos(\theta-\psi)\big)^j\,\mathrm{d}\theta \\ &= (-i)^n\frac{2^n n!}{2\pi\omega^n}\sum_{j=0}^{\infty}\frac{1}{j!}(i\omega r)^j\int_0^{2\pi} e^{in\theta}\left(\frac{e^{i(\theta-\psi)}+e^{-i(\theta-\psi)}}{2}\right)^j\mathrm{d}\theta \\ &= (-i)^n\frac{2^n n!}{2\pi\omega^n}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{i\omega r}{2}\right)^j\sum_{k=0}^{j}\frac{j!}{k!\,(j-k)!}\int_0^{2\pi} e^{i(n+2k-j)\theta}\,\mathrm{d}\theta\; e^{i\psi(j-2k)} \\ &= (-i)^n\frac{2^n n!}{2\pi\omega^n}\sum_{\substack{j\geq 0\\ 0\leq k\leq j\\ n+2k-j=0}}\left(\frac{i\omega r}{2}\right)^j\frac{1}{k!\,(j-k)!}2\pi e^{i\psi(j-2k)} \end{aligned}$$

$$= (-i)^n \frac{2^n n!}{\omega^n} \sum_{k=0}^{\infty} (-1)^k i^n \left(\frac{\omega r}{2}\right)^{2k+n} \frac{1}{k!\,(k+n)!} e^{i\psi n} = n! \left(\frac{2}{\omega}\right)^n e^{i\psi n} J_n(\omega r)$$

Finally we get the assertion for a generic polynomial by linearity. □

Thanks to this lemma, we get that

$$\operatorname{Re} V[P,0](x,y) = \sum_{n=0}^{N} \frac{2^n n!}{2\pi\omega^n} \int_0^{2\pi} \operatorname{Re}\left(a_n (-i)^n e^{in\theta} e^{i\omega(x\cos\theta + y\sin\theta)}\right) \mathrm{d}\theta$$

can be written in the form (3.40), with

$$g(\theta) = \sum_{n=0}^{N} \frac{2^n n!}{2\pi\omega^n} a_n (-i)^n e^{in\theta}, \tag{3.45}$$

with norm

$$\|g\|_{L^1(0,2\pi)} \leq \sum_{n=0}^{N} \frac{2^n n!}{\omega^n} |a_n|. \tag{3.46}$$

So Lemma 3.4 applies to the generalized polynomials, which means that the space $PW_\omega(\mathbb{R}^2)$ approximates these functions and generic plane waves with the same order in h. The result is stated the following theorem.

Theorem 3.6. *Let D be a domain of diameter h containing the origin. Let $P(z) = a_N z^N + \ldots + a_1 z + a_0$ be a complex polynomials of degree N and let $\varphi_1, \ldots, \varphi_p \in [0, 2\pi)$ be different angles, with $p = 2m+1 \geq 3$. Then there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ such that*

$$\left\| \operatorname{Re} V[P,0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(D)} \leq C(p, \{\varphi_k\}) h^{m+1} \sum_{n=0}^{N} 2^n n! \omega^{m+1-n} |a_n|.$$

Numerical experiment 2. *The numerical studies show that with the optimal choice of the α_ks the error behaves like*

$$\left\| \operatorname{Re} V[z^N,0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty((0,h)^2)} \sim C\, h^{\max(p-N,N)},$$

as in Figure 2. When $N = m, m+1$ we get exactly the order h^{m+1} we proved for the generic function u, while with other values of the degree N the exponent of h is better.

We end this section by proving best approximation estimates in Sobolev norms. In order to do that, we need the following two lemmata, about the link between holomorphic and real harmonic functions and the continuity of $(\operatorname{Re} V)^{-1}$ in maximum norm.

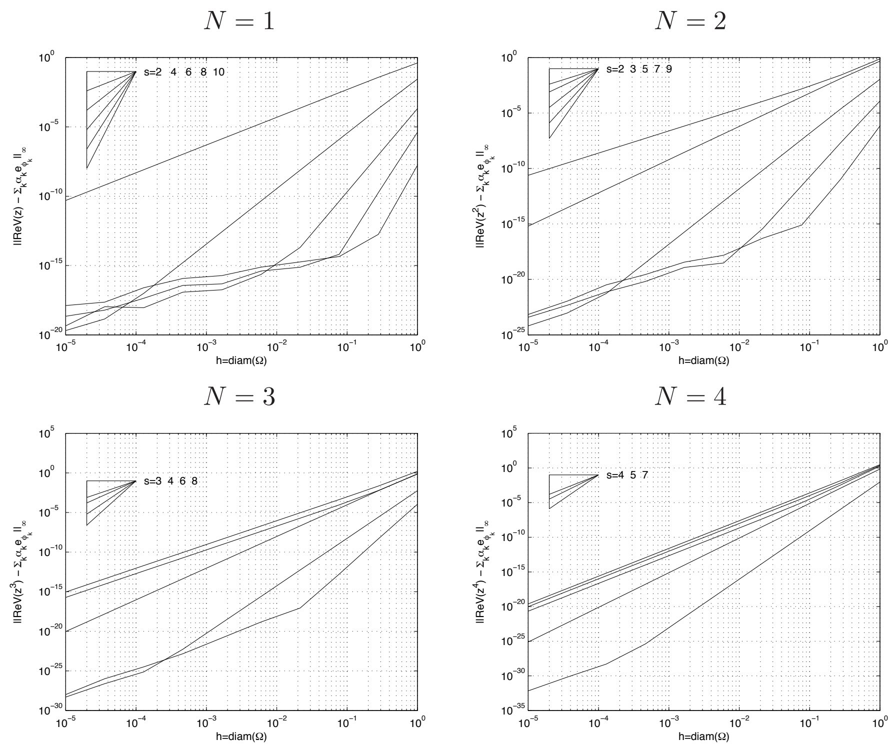
Lemma 3.7. *Let u_0 be a real harmonic function in the ball $B(0,3h)$ and let*

$$v_0(x,y) := \int_0^y D_1 u_0(x,t)\,\mathrm{d}t - \int_0^x D_2 u_0(t,0)\,\mathrm{d}t, \tag{3.47}$$

where D_1 and D_2 denote the real derivatives with respect to x and y, respectively. Then $\phi = u_0 + iv_0$ is the unique holomorphic function in $B(0,3h)$ such that $\operatorname{Re}\phi = u_0$ and $\phi(0,0) \in \mathbb{R}$. Furthermore there is a constant C independent of h such that

$$\|\phi\|_{L^\infty(B(0,2h))} \leq C \|u_0\|_{L^\infty(B(0,3h))}. \tag{3.48}$$

Figure 2: The norm $\left\| \operatorname{Re} V[z^N, 0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty((0,h)^2)}$ with $N = 1, \ldots, 4$, $p = 3, \ldots, 11$, φ_k equally spaced, $\omega = 1$ and varying h. The order of convergence is $h^{\max(p-N,N)}$, better than h^{m+1} when $N \neq m, m+1$.



Proof. From the definition of v_0 it follows that $\phi(0,0) = u_0(0,0) \in \mathbb{R}$ and ϕ is holomorphic since the Cauchy-Riemann conditions hold. By contradiction, suppose there exists another holomorphic function $\tilde{\phi} \neq \phi$ satisfying the same condition, then $\tilde{\phi} = u_0 + i\tilde{v}_0$, so $D_1\tilde{v}_0(x,y) = -D_2u_0(x,y) = D_1v_0(x,y)$ and $D_2\tilde{v}_0(x,y) = D_1u_0(x,y) = D_2v_0(x,y)$, with $\tilde{v}_0(0,0) = v_0(0,0) = 0$. Thus v_0 and $\tilde{v}_0$ are functions with the same gradient everywhere and the same value in the origin, so they have to coincide and the uniqueness of ϕ follows.

In order to prove the bound (3.48), we use the internal estimate:

$$|D_j v(x,y)| \leq C\frac{1}{R} \|v\|_{L^\infty(B((x,y),R))}, \qquad j = 1,2,$$

that holds for all harmonic functions v in $B((x,y),R)$. The proof can be find in Theorem 2.4 of [2]. Using this estimate and (3.47) we get the results:

$$\begin{aligned}
\|\phi\|_{L^\infty(B(0,2h))} &\leq \|u_0\|_{L^\infty(B(0,2h))} + \|v_0\|_{L^\infty(B(0,2h))} \\
&\leq \|u_0\|_{L^\infty(B(0,2h))} + 2\,h\,\|D_1u_0\|_{L^\infty(B(0,2h))} + 2\,h\,\|D_2u_0\|_{L^\infty(B(0,2h))} \\
&\leq \|u_0\|_{L^\infty(B(0,2h))} + h\,C\,\frac{1}{h}\,\|u_0\|_{L^\infty(B(0,3h))} \leq C\,\|u_0\|_{L^\infty(B(0,3h))}.
\end{aligned}$$

□

Lemma 3.8. *Let u be a real function satisfying $\Delta u+\omega^2 u=0$ in $B(0,3h)$. Then there exists a constant $C>0$ independent of h and ω such that*

$$\left\|(\operatorname{Re} V)^{-1}[u,0]\right\|_{L^\infty(B(0,2h))} \leq C(1+c_*^2 e^{\frac{3}{2}c_*})\left\|u\right\|_{L^\infty(B(0,3h))}. \tag{3.49}$$

Proof. Set $z=x+iy$. It is possible to prove ([26], §13.2) that there exists a unique complex harmonic function u_0 in $B(0,3h)$ such that the following identities hold:

$$\begin{aligned}
u(x,y) &= u_0(x,y) - \int_0^1 u_0(sx,sy)\frac{\partial}{\partial s}J_0\big(\omega r\sqrt{1-s}\big)\,\mathrm{d}s,\\
u_0(x,y) &= u(x,y) + \frac{1}{2}\omega r\int_0^1 \frac{u(xs,ys)}{\sqrt{s(1-s)}} I_1(\omega r\sqrt{s(1-s)}))\,\mathrm{d}s,
\end{aligned}$$

where $r=|x+iy|$ and $I_1(z)=-iJ_1(iz)$ is the modified Bessel function of order one. Since u is real, also u_0 is real. Thanks to (2.13) we have

$$\left|\frac{I_1(x)}{x}\right| = \left|\frac{J_1(ix)}{x}\right| \leq \frac{1}{2}e^{|x|} \qquad \forall\, x\in\mathbb{C},$$

and using $r\sqrt{s(1-s)}<3h\cdot\frac{1}{2}$ the following bound holds

$$\begin{aligned}
\|u_0\|_{L^\infty(B(0,3h))} &\leq \|u\|_{L^\infty(B(0,3h))} + \left\|\frac{\omega^2 r^2}{2}\int_0^1 u(xs,ys)\frac{I_1(\omega r\sqrt{s(1-s)})}{\omega r\sqrt{s(1-s)}}\,\mathrm{d}s\right\|_{L^\infty(B(0,3h))}\\
&\leq \left(1+\frac{1}{4}\omega^2(3h)^2 e^{\omega\frac{3}{2}h}\right)\|u\|_{L^\infty(B(0,3h))} \leq \left(1+\frac{9}{4}c_*^2 e^{\frac{3}{2}c_*}\right)\|u\|_{L^\infty(B(0,3h))}.
\end{aligned} \tag{3.50}$$

Lemma 3.7 guarantees that there exists only one ϕ holomorphic, with $\operatorname{Re}\phi=u_0$ and $\phi(0,0)\in\mathbb{R}$. Thanks to (3.44) we have

$$V[\phi,0](x,y) = \phi(z) - \int_0^1 \phi(sz)\frac{\partial}{\partial s}J_0\big(\omega|z|\sqrt{1-s}\big)\,\mathrm{d}s,$$

and taking the real part

$$\operatorname{Re} V[\phi,0](x,y) = u_0(z) - \int_0^1 u_0(sz)\frac{\partial}{\partial s}J_0\big(\omega|z|\sqrt{1-s}\big)\,\mathrm{d}s = u(x,y).$$

thus, $\phi(z)=(\operatorname{Re} V)^{-1}[u,0](x,y)$ and we can conclude using (3.48) and (3.50)

$$\begin{aligned}
\left\|(\operatorname{Re} V)^{-1}[u,0]\right\|_{L^\infty(B(0,2h))} &= \|\phi\|_{L^\infty(B(0,2h))} \leq C\,\|u_0\|_{L^\infty(B(0,3h))}\\
&\leq C(1+c_*^2 e^{\frac{3}{2}c_*})\,\|u\|_{L^\infty(B(0,3h))}.
\end{aligned}$$

□

Now we are able to approximate the generalized polynomials in Sobolev norms. The following theorem proves the existence of a linear combination of plane waves in the space $PW_\omega(\mathbb{R}^2)$ that approximates any given generalized harmonic polynomial with order h^{m+1-j} simultaneously in the different Sobolev norms.

Theorem 3.9. *Let D be a domain of diameter h containing the origin and such that $d(0, \partial D) \geq \tau h$ for a constant τ independent of h. Let $P(z)$ be a complex polynomial of degree N an let $\varphi_1, \ldots, \varphi_p \in [0, 2\pi)$, $p = 2m+1 \geq 3$, be different angles. Then there exist $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ such that*

$$\left\| \operatorname{Re} V[P, 0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{j,\omega,D} \leq C e^{\frac{7}{2}c_*} \left(1 + c_*^{2j+4}\right) h^{m+1-j}\, \omega^{m+1-N}\, \|P\|_{N,\omega,D}\,, \quad (3.51)$$

for every $j \geq 0$, where the constant $C > 0$ is independent of h, ω, c_* and the polynomial P.

Proof. Let be $P(z) = a_N z^N + \ldots + a_1 z + a_0$; using (2.26), we have

$$|a_n| = \left| \frac{1}{n!} \frac{\partial^n P}{\partial z^n}(0) \right| \leq \frac{1}{n!} \frac{1}{\sqrt{\pi} d(0, \partial D)} \left\| \frac{\partial^n}{\partial z^n} P \right\|_{0,D} \leq \frac{1}{\sqrt{\pi} n!\, \tau h} |P|_{n,D}\,,$$

where we have exploited the regularity of the domain. This and (3.46) imply that, with g defined as in (3.45), the following bound holds

$$\|g\|_{L^1(0,2\pi)} \leq \sum_{n=0}^{N} \frac{2^n n!}{\omega^n} |a_n| \leq \sum_{n=0}^{N} \frac{2^n}{\sqrt{\pi} \tau h} \frac{|P|_{n,D}}{\omega^n} \leq C(\tau, N) \frac{\|P\|_{N,\omega,D}}{h\, \omega^N}. \quad (3.52)$$

In order to get the result, we transform the term to be bounded with Vekua operator and use its continuity (2.9): for every $j \geq 0$ we have

$$\begin{aligned}
&\left\| \operatorname{Re} V[P, 0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{j,\omega,D} \\
&\leq C(j) e^{c_*} (1 + c_*^{j+2}) \left\| P - \sum_{k=1}^{p} \alpha_k (\operatorname{Re} V)^{-1} [e_{\varphi_k}, 0] \right\|_{j,\omega,D} \\
&\leq C(j) e^{c_*} (1 + c_*^{j+2}) \sum_{l=0}^{j} \omega^{j-l} h \left| P - \sum_{k=1}^{p} \alpha_k (\operatorname{Re} V)^{-1} [e_{\varphi_k}, 0] \right|_{W^{l,\infty}(D)} \\
&\leq C(j) e^{c_*} (1 + c_*^{j+2}) \sum_{l=0}^{j} \omega^{j-l} h\, h^{-l-1} \left\| P - \sum_{k=1}^{p} \alpha_k (\operatorname{Re} V)^{-1} [e_{\varphi_k}, 0] \right\|_{0,B(0,2h)}
\end{aligned}$$

where we used the internal estimates for the derivatives of holomorphic functions (2.27), with $d(D, \partial B(0, 2h)) \geq h$, recall that we don't have analogue estimates for generalized harmonic polynomials,

$$\begin{aligned}
&\leq C(j) e^{c_*} (1 + c_*^{j+2})(1 + c_*^{j}) h^{-j} h \left\| P - \sum_{k=1}^{p} \alpha_k (\operatorname{Re} V)^{-1} [e_{\varphi_k}, 0] \right\|_{L^\infty(B(0,2h))} \\
&\leq C(j) e^{c_*} (1 + c_*^{2j+2}) h^{1-j} (1 + e^{\frac{3}{2}c_*} c_*^2) \left\| \operatorname{Re} V[P, 0] - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(B(0,3h))}
\end{aligned}$$

where we have to use the L^∞-continuity of $(\operatorname{Re} V)^{-1}$ (3.49), because Theorem 2.2 gives an unsatisfactory estimate for the L^2-norm; in this way we have bounded all the norms we want to estimate using the L^∞-norm of the same expression on a larger domain,

$$\leq C(j) e^{\frac{5}{2}c_*}(1+c_*^{2j+4}) h^{1-j} \left\| \int_0^{2\pi} \operatorname{Re}\left(g(\theta) e^{i\omega(x\cos\theta + y\sin\theta)}\right) \mathrm{d}\theta - \sum_{k=1}^{p} \alpha_k e_{\varphi_k} \right\|_{L^\infty(B(0,3h))}$$

with g in the form (3.45); finally, using (3.41) and (3.52), we get

$$\begin{aligned}
&\leq C(j,p,\{\varphi_k\},N,\tau) e^{\frac{5}{2}c_*}(1+c_*^{2j+4}) h^{1-j} h^{m+1} \omega^{m+1} e^{c_*} \frac{\|P\|_{N,\omega,D}}{h\,\omega^N} \\
&= C(j,p,\{\varphi_k\},N,\tau) e^{\frac{7}{2}c_*}(1+c_*^{2j+4}) h^{m+1-j} \omega^{m+1-N} \|P\|_{N,\omega,D}\,.
\end{aligned}$$

□

3.3 The approximation of the solutions of the homogeneous Helmholtz equation

Now we have all the tools necessary to study the approximation of a generic solution of the homogeneous Helmholtz equation by the plane waves of the space $PW_\omega^p(\mathbb{R}^2)$. The key instruments are Theorems 2.10 and 3.9.

Theorem 3.10. *Let D be a bounded, Lipschitz and convex domain with diameter h, containing $B(0,\tau h)$ with τ independent from h. Let $u \in H^{k+1}(D)$, $k \geq 0$, be a complex-valued function satisfying $\Delta u + \omega^2 u = 0$ in D. Let $\varphi_1, \dots, \varphi_p \in [0, 2\pi)$ different angles, with $p = 2m+1 \geq 3$. Then there exists $\alpha_1, \dots, \alpha_p \in \mathbb{C}$ such that*

$$\left\| u - \sum_{l=1}^{p} \alpha_l e_{\varphi_l} \right\|_{j,\omega,D} \leq C e^{\frac{9}{2}c_*}\left(1 + c_*^{k+2j+7+|m-k|}\right) h^{\min(k,m)+1-j}\, \omega^{-(k-m)^+}\, \|u\|_{k+1,\omega,D} \tag{3.53}$$

for every integer $0 \leq j \leq k+1$ and where C is a positive constant depending only on the shape of D, on p, on the choice of the angles $\{\varphi_l\}$ and on the indices j and k.

Proof. We fix the degree N of the generalized harmonic polynomials to be used in this proof equal to k and define two holomorphic function

$$\phi_1 := (\operatorname{Re} V)^{-1}[\operatorname{Re} u, 0], \qquad \phi_2 := (\operatorname{Re} V)^{-1}[\operatorname{Im} u, 0],$$

their approximating polynomials

$$P_1 := \Pi_k^k(\phi_1), \qquad P_2 := \Pi_k^k(\phi_2),$$

and the corresponding generalized harmonic polynomials

$$Q_1 := \operatorname{Re} V[P_1, 0] = P_k^k(\operatorname{Re} u), \qquad Q_2 := \operatorname{Re} V[P_2, 0] = P_k^k(\operatorname{Im} u).$$

Theorem 2.10 leads to

$$\begin{aligned}
\|u - Q_1 - iQ_2\|_{j,\omega,D} &\leq \left\|\operatorname{Re} u - P_k^k(\operatorname{Re} u)\right\|_{j,\omega,D} + \left\|\operatorname{Im} u - P_k^k(\operatorname{Im} u)\right\|_{j,\omega,D} \\
&\leq C(k) e^{2c_*}\left(1 + c_*^{k+2j+6}\right) h^{k+1-j} \left(\frac{\log(k+2)}{k+2}\right)^{k+1-j} \|u\|_{k+1,\omega,D}\,.
\end{aligned}$$

The hypotheses about the indexes are verified because $k+1 \geq j$.

According to Theorem 3.9, there exist $\alpha_1^1, \ldots, \alpha_p^1$ such that

$$\begin{aligned}\left\| Q_1 - \sum_{l=1}^{p} \alpha_l^1 e_{\varphi_l} \right\|_{j,\omega,D} &= \left\| \operatorname{Re} V[P_1, 0] - \sum_{l=1}^{p} \alpha_l^1 e_{\varphi_l} \right\|_{j,\omega,D} \\ &\leq C e^{\frac{7}{2}c_*} \left(1 + c_*^{2j+4}\right) h^{m+1-j}\, \omega^{m+1-k} \left\| P_1 \right\|_{k,\omega,D},\end{aligned}$$

and similarly, for iQ_2, there exist $\alpha_1^2, \ldots, \alpha_p^2$ that gives a similar bound. In order to replace the norm of u on the right hand side, we apply Theorem 2.8 with $j = k$, and we use (2.33) and the continuity of $(\operatorname{Re} V)^{-1}$ (2.8):

$$\begin{aligned}\left\| P_1 \right\|_{k,\omega,D} &= \left\| \Pi_k^k \phi_1 \right\|_{k,\omega,D} \leq \left\| \phi_1 \right\|_{k,\omega,D} + \left\| \phi_1 - \Pi_k^k \phi_1 \right\|_{k,\omega,D} \leq C \left\| \phi_1 \right\|_{k,\omega,D} \\ &\leq C e^{c_*} (1 + c_*^{k+3}) \left\| \operatorname{Re} u \right\|_{k,\omega,D} \leq C e^{c_*} (1 + c_*^{k+3})\, \omega^{-1} \left\| \operatorname{Re} u \right\|_{k+1,\omega,D};\end{aligned}$$

the analogous holds for P_2.

Finally we combine the three last bounds written using triangle inequality and we get:

$$\begin{aligned}&\left\| u - \sum_{l=1}^{p} (\alpha_l^1 + i\alpha_l^2) e_{\varphi_l} \right\|_{j,\omega,D} \\ &\quad\leq \left\| u - Q_1 - iQ_2 \right\|_{j,\omega,D} + \left\| Q_1 - \sum_{l=1}^{p} \alpha_l^1 e_{\varphi_l} \right\|_{j,\omega,D} + \left\| Q_2 - \sum_{l=1}^{p} \alpha_l^2 e_{\varphi_l} \right\|_{j,\omega,D} \\ &\quad\leq C e^{2c_*} \left(1 + c_*^{k+2j+6}\right) h^{k+1-j} \left(\frac{\log(k+2)}{k+2} \right)^{k+1-j} \left\| u \right\|_{k+1,\omega,D} \\ &\qquad\quad + C e^{\frac{7}{2}c_*} \left(1 + c_*^{2j+4}\right) h^{m+1-j}\, \omega^{m+1-k} \cdot e^{c_*} \left(1 + c_*^{k+3}\right) \omega^{-1} \left\| u \right\|_{k+1,\omega,D} \\ &\quad\leq C e^{\frac{9}{2}c_*} \left(1 + c_*^{k+2j+7}\right) \left(h^{k+1-j} + h^{m+1-j}\, \omega^{m-k}\right) \left\| u \right\|_{k+1,\omega,D} \\ &\quad\leq C e^{\frac{9}{2}c_*} \left(1 + c_*^{k+2j+7}\right) h^{\min(k,m)+1-j} \left\{ \begin{array}{ll} 1 + (h\omega)^{m-k} & m \geq k \\ h^{k-m} + \dfrac{1}{\omega^{k-m}} & m < k \end{array} \right\} \left\| u \right\|_{k+1,\omega,D} \\ &\quad\leq C e^{\frac{9}{2}c_*} \left(1 + c_*^{k+2j+7+|m-k|}\right) h^{\min(k,m)+1-j} \omega^{-(k-m)^+} \left\| u \right\|_{k+1,\omega,D}\end{aligned}$$

with C independent of ω, h and c_*. $\square$

This theorem shows that the approximation properties depends on the measure of the domain, on the dimension of the local space of plane waves and on the regularity of the solution. The two latter parameters limit the order of convergence with respect to h.

Remark 3.11. *If the domain D is small enough such that $c_* \leq 1$, the* (3.53) *becomes*

$$\left\| u - \sum_{l=1}^{p} \alpha_l e_{\varphi_l} \right\|_{j,\omega,D} \leq C\, h^{k+1-j} \left\| u \right\|_{k+1,\omega,D}, \qquad 0 \leq j \leq k+1 \leq m+1.$$

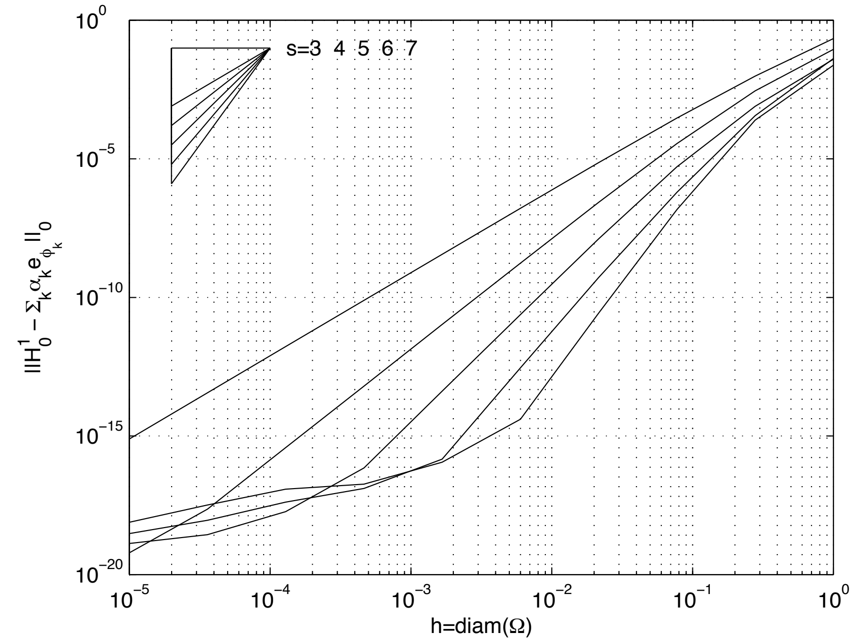
This situation will be relevant in the study of the PWDG method where D will be any element of the mesh.

Numerical experiment 3. *A simple numerical study demonstrates the sharpness of the bound in Theorem 3.10. The function to approximate is the cylindrical wave* $u(x,y) = H_0^{(1)}(\omega|(x,y)-(-1/4,0)|)$, *where* $H_0^{(1)}$ *is the Hankel function of the first kind of order zero and the domain is the square* $D=(0,h)^2$. *Since it is a* C^∞ *function we take* $k=m$. *We notice from Figure 3 that*

$$\frac{\left\|H_0^{(1)} - \sum_{k=1}^p \alpha_k e_{\varphi_k}\right\|_{0,D}}{\left\|H_0^{(1)}\right\|_{m+1,\omega,D}} \sim \frac{h^{m+2}}{h} = h^{m+1},$$

as predicted by Theorem 3.10.

Figure 3: The norm $\left\|H_0^{(1)} - \sum_{k=1}^p \alpha_k e_{\varphi_k}\right\|_{0,(0,h)^2}$ with $p = 3,\ldots,11$, φ_k equally spaced, $\omega = 1$ and h varying. The order of the error is h^{m+2}.



4 The PWDG method for the homogeneous Helmholtz equation

The theory of the approximation of homogeneous Helmholtz solutions using plane waves functions can be used in the analysis of the plane wave discontinuous Galerkin method (PWDG) derived in [16] as a a generalization of the ultraweak variational formulation (UWVF) by Cessenat and Després (see [9]). The analysis of the homogeneous case given below follows the same line as that in [16].

Theorem 3.10 allows us to prove a sharp best approximation estimate. This leads to new a priori error estimates for the method, both in a particular energy norm and in the L^2-norm. These estimates improve the one proved in [7] and are sharp with respect to the meshsize h, as confirmed by the numerical results presented in [17].

Also in this section all the bounding constant are independent of the wavenumber, this allows to make explicit the pollution effect.

4.1 The formulation of the method

We consider the usual homogeneous Helmholtz equation with impedance boundary condition in a bounded, polygonal, convex domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u - \omega^2 u = 0 & \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n} + i\omega u = g & \text{on } \partial\Omega, \end{cases} \tag{4.54}$$

where $g \in L^2(\partial\Omega)$ and $\omega > 0$ is the wavenumber.

We introduce a triangular mesh $\mathcal{T}_h$ where every element K satisfies $h_K = \operatorname{diam} K \leq h$. Introducing a space V_h of (discontinuous) finite elements on this mesh, integrating by parts and following the steps of [16] we can write the method as follows: we want to find a function $u_h \in V_h$ such that

$$\int\limits_K (\nabla u_h \cdot \nabla \overline{v}_h - \omega^2 u_h \overline{v}_h)\,\mathrm{d}V - \int\limits_{\partial K} (u_h - \hat{u}_h)\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int\limits_{\partial K} i\omega\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = 0, \tag{4.55}$$

for every $v_h \in V_h$ and for every $K \in \mathcal{T}_h$. In this equation $\hat{u}_h$ and $\hat{\sigma}_h$ are the so-called numerical fluxes, approximations of the traces of u and of its gradient on the skeleton of the mesh. If we integrate by parts once more and we choose Trefftz-type test functions (i.e. satisfying the homogeneous Helmholtz equation on each element), we obtain the ultraweak form of the discontinuous Galerkin method:

$$\int_{\partial K} \hat{u}_h\,\overline{\nabla v_h \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}\,\overline{v}_h\,\mathrm{d}S = 0.$$

In order to define $\hat{u}_h$ and $\hat{\boldsymbol{\sigma}_h}$, we need to introduce some notation. Let u_h and $\boldsymbol{\sigma}_h$ be a function and a vector field that are regular in each element of $\mathcal{T}_h$, $u_h^\pm$ and $\boldsymbol{\sigma}_h^\pm$ their trace on $\partial K^+ \cap \partial K^-$, and let $\boldsymbol{n}^+, \boldsymbol{n}^- = -\boldsymbol{n}^+$ be the outgoing normal unit vectors to $\partial K^\pm$; we define:

$$\begin{aligned} &\text{the averages:} \quad \{\!\{u_h\}\!\} := \tfrac{1}{2}(u_h^+ + u_h^-)\,, && \{\!\{\boldsymbol{\sigma}_h\}\!\} := \tfrac{1}{2}(\boldsymbol{\sigma}_h^+ + \boldsymbol{\sigma}_h^-)\,, \\ &\text{the jumps:} \quad [\![u_h]\!]_N := u_h^+\boldsymbol{n}^+ + u_h^-\boldsymbol{n}^-\,, && [\![\boldsymbol{\sigma}_h]\!]_N := \boldsymbol{\sigma}_h^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\sigma}_h^- \cdot \boldsymbol{n}^-\,. \end{aligned}$$

We define also a function that represents the local meshsize:

$$\forall\, \boldsymbol{x} \in \operatorname{int}(\partial K^- \cap \partial K^+) \qquad \mathtt{h}(\boldsymbol{x}) = \min(h_{K^-}, h_{K^+}).$$

We choose the numerical fluxes depending on three parameters $\mathtt{a}, \mathtt{b}, \mathtt{d}$ that are real, bounded and independent of the meshsize and ω:

$$\mathtt{a} \geq \mathtt{a}_{\min} > 0 \text{ in } \mathcal{F}_h^{\mathcal{I}}, \qquad \mathtt{b} \geq \mathtt{b}_{\min} > 0 \text{ in } \mathcal{F}_h^{\mathcal{I}}, \qquad 0 < \mathtt{d}_{\min} \leq \mathtt{d} < \tfrac{1}{2\omega \mathtt{h}} \text{ in } \mathcal{F}_h^{\mathcal{B}},$$

$$\begin{cases} \hat{\boldsymbol{\sigma}}_h = \dfrac{1}{i\omega} \{\!\{\nabla_h u_h\}\!\} - \dfrac{\mathtt{a}}{\omega \mathtt{h}} [\![u_h]\!]_N, \\ \hat{u}_h = \{\!\{u_h\}\!\} - \dfrac{\mathtt{bh}}{i} [\![\nabla_h u_h]\!]_N, \end{cases} \qquad \text{in } \partial K^- \cap \partial K^+ \subset \mathcal{F}_h^{\mathcal{I}},$$

$$\begin{cases} \hat{\boldsymbol{\sigma}}_h = \dfrac{1}{i\omega} \nabla_h u_h - (1 - \mathtt{d}\omega\mathtt{h}) \left(\dfrac{1}{i\omega} \nabla_h u_h + u_h \boldsymbol{n} - \dfrac{1}{i\omega}\, g\, \boldsymbol{n} \right), \\ \hat{u}_h = u_h - \mathtt{d}\omega\mathtt{h} \left(\dfrac{1}{i\omega} \nabla_h u_h \cdot \boldsymbol{n} + u_h - \dfrac{1}{i\omega}\, g \right), \end{cases} \qquad \text{in } \partial K \subset \mathcal{F}_h^{\mathcal{B}}. \tag{4.56}$$

If we implement this method using plane wave basis functions, we obtain the PWDG method.

Choosing the fluxes in a slightly different way, we can recover the UWVF (see [16]). This choice however does not satisfies the hypothesis needed for the stability analysis.

We suppose $h < 1$ and that the mesh satisfies a regularity assumption: there exists $\alpha_0 > 0$ such that, for every angle α of every triangle $K \in \mathcal{T}_h$, $\alpha > \alpha_0$ holds.

Define the discontinuous finite element space of piecewise linear combination of plane wave functions:

$$V_h = \{v \in L^2(\Omega) : \; v|_K \in PW_\omega^{p,0}(\mathbb{R}^2) \; \forall \, K \in \mathcal{T}_h\},$$

and the bilinear form on $(H^2(\Omega) + V_h) \times (H^2(\Omega) + V_h)$:

$$\begin{aligned} a_h(u,v) :=& (\nabla_h u, \nabla_h v) - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![u]\!]_N \cdot \{\!\{\overline{\nabla_h v}\}\!\} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S \\ & - \int_{\mathcal{F}_h^{\mathcal{B}}} \mathtt{d}\omega\mathtt{h}\, u\, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{B}}} \mathtt{d}\omega\mathtt{h}\, \nabla_h u \cdot \boldsymbol{n}\, \overline{v} \, \mathrm{d}S \\ & + i \int_{\mathcal{F}_h^{\mathcal{I}}} \mathtt{bh} [\![\nabla_h u]\!]_N [\![\overline{\nabla_h v}]\!]_N \, \mathrm{d}S + i \int_{\mathcal{F}_h^{\mathcal{B}}} \mathtt{dh}\, \nabla_h u \cdot \boldsymbol{n}\, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S \\ & + i \int_{\mathcal{F}_h^{\mathcal{I}}} \frac{\mathtt{a}}{\mathtt{h}} [\![u]\!]_N \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1 - \mathtt{d}\omega\mathtt{h})\, u\, \overline{v} \, \mathrm{d}S \,. \end{aligned}$$

We also define a seminorm and two norms related to the Sobolev H^1-norm:

$$\begin{aligned} |v|_{DG}^2 :=& \|\nabla_h v\|_{0,\Omega}^2 + \left\| (\mathtt{bh})^{1/2} [\![\nabla_h v]\!]_N \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \left\| \mathtt{a}^{1/2} \mathtt{h}^{-1/2} [\![v]\!]_N \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 \\ & + \left\| (\mathtt{dh})^{1/2} \nabla_h v \cdot \boldsymbol{n} \right\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2 + \omega \left\| (1 - \mathtt{d}\omega\mathtt{h})^{1/2} v \right\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2, \\ \|v\|_{DG}^2 :=& |v|_{DG}^2 + \omega^2 \|v\|_{0,\Omega}^2, \\ \|v\|_{DG^+}^2 :=& \|v\|_{DG}^2 + \left\| (\mathtt{bh})^{-1/2} \{\!\{v\}\!\} \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 \\ & + \left\| \mathtt{a}^{-1/2} \mathtt{h}^{1/2} \{\!\{\nabla_h v\}\!\} \right\|_{0,\mathcal{F}_h^{\mathcal{I}}}^2 + \left\| (\mathtt{dh})^{-1/2} v \right\|_{0,\mathcal{F}_h^{\mathcal{B}}}^2 . \end{aligned}$$

Finally we can reformulate the method in a new way, equivalent to the (4.55): find $u_h \in V_h$ such that, for all $v_h \in V_h$,

$$a_h(u_h, v_h) - \omega^2 (u_h, v_h) = \int_{\mathcal{F}_h^{\mathcal{B}}} i \mathtt{dh}\, g\, \overline{\nabla_h v_h \cdot \boldsymbol{n}} \, \mathrm{d}S + \int_{\mathcal{F}_h^{\mathcal{B}}} (1 - \mathtt{d}\omega\mathtt{h})\, g\, \overline{v}_h \, \mathrm{d}S \,, \tag{4.57}$$

and the method is consistent, i.e.,

$$a_h(u-u_h,v_h)-\omega^2(u-u_h,v_h)=0 \qquad \forall v_h\in V_h, \tag{4.58}$$

where u is the analytical solution of (4.54). The coercivity of the imaginary part of a_h guarantees the existence and the uniqueness of the discrete solution u_h to (4.57).

Now we list the properties proved in [16] we need in order to obtain the error estimates.

Theorem 4.1 ([16], Theorems 8 and 9). *Let $p=2m+1\geq 3$, then there exists two constants C_{tinv} and C_{inv} independent of K and ω, such that, for all $v\in PW_\omega^{p,0}(\mathbb{R}^2)$ and for all $K\in\mathcal{T}_h$*

$$\begin{aligned}\|v\|_{0,\partial K} &\leq C_{tinv}h_K^{-1/2}\|v\|_{0,K}\,,\\ \|\nabla v\|_{0,K} &\leq C_{inv}(\omega h_K+1)h_K^{-1}\|v\|_{0,K}\,.\end{aligned}$$

The bounds of Theorem 4.1 were used to derive the following abstract error estimate.

Proposition 4.2 ([16], Proposition 19). *Let u be the analytical solution of (4.54), u_h the discrete solution of (4.57) and $\mathtt{a}\geq\mathtt{a}_{min}>C_{tinv}^2$. Then there exists $C_{abs}>0$, independent of ω and the mesh, such that*

$$\|u-u_h\|_{DG}\leq C_{abs}\left(\inf_{v_h\in V_h}\|u-v_h\|_{DG^+}+\sup_{0\neq w_h\in V_h}\frac{\omega\ |(u-u_h,w_h)|}{\|w_h\|_{0,\Omega}}\right).$$

Finally, an estimate of the second term on the right-hand side of the bound in Proposition 4.2 was derived by a duality technique. We report here the result in the homogeneous case.

Proposition 4.3. *Let Ω be a bounded convex domain. Let u and u_h be as in Proposition 4.2, then*

$$\sup_{0\neq w_h\in V_h}\frac{\omega\,|(u-u_h,w_h)|}{\|w_h\|_{0,\Omega}}\leq C_{dual}\,\omega h(1+\omega h)^{\frac{3}{2}}(1+\mathrm{diam}(\Omega)\omega)\,\|u-u_h\|_{DG}\,,$$

holds, with $C_{dual}>0$ independent of ω and the mesh.

4.2 The order of convergence of the method

We can use the results of Section 3.3 to get a best approximation estimate of homogeneous Helmholtz solutions in the space V_h with respect to the DG^+ norm. We know how the functions belonging to $PW_\omega^{p,0}$ approximate a solution of the homogeneous Helmholtz equation, now we need the analogous approximation properties for functions in V_h.

Proposition 4.4. *Let $u\in H^{k+1}(\Omega)$, $1\leq k\leq m$, be the solution of the homogeneous Helmholtz equation in a bounded and convex domain Ω, if $\omega h\leq c_*$ then*

$$\inf_{v_h\in V_h}\|u-v_h\|_{DG^+}\leq C_{BA}e^{\frac{9}{2}c_*}\left(1+c_*^{m+10}\right)\,h^k\,\|u\|_{k+1,\omega,\Omega}\,, \tag{4.59}$$

with $C_{BA}>0$ independent of ω and $\mathcal{T}_h$.

Proof. For each $K \in \mathcal{T}_h$ we have the inequality

$$\|u\|_{0,\partial K}^2 \leq C \|u\|_{0,K} \left(h_K^{-1} \|u\|_{0,K} + |u|_{1,K} \right) \quad \forall u \in H^1(K) , \tag{4.60}$$

obtained by scaling the norms in the thesis of Theorem 1.6.6 in [6] on a reference domain with unitary diameter.

We define $z_h \in V_h$ such that for every triangle K, $z_h|_K$ is the linear combination of plane wave functions given by Theorem 3.10. The hypothesis $B(0, \tau h) \subset K$ is guaranteed for each K by the regularity assumption on the mesh. We bound the approximation error of the traces using the same theorem:

$$\begin{aligned}
\|u - z_h\|_{0,\partial K}^2 &\overset{(4.60)}{\leq} C \|u - z_h\|_{0,K} \left(h_K^{-1} \|u - z_h\|_{0,K} + \|u - z_h\|_{1,\omega,K} \right) \\
&\leq C e^{9c_*} (1 + c_*^{2m+16}) h_K^{2k+1} \|u\|_{k+1,\omega,K}^2 , \\
\|\nabla_h (u - z_h)\|_{0,\partial K}^2 &\leq C |u - z_h|_{1,K} \left(h_K^{-1} |u - z_h|_{1,K} + |u - z_h|_{2,K} \right) \\
&\leq C \|u - z_h\|_{1,\omega,K} \left(h_K^{-1} \|u - z_h\|_{1,\omega,K} + \|u - z_h\|_{2,\omega,K} \right) \\
&\leq C e^{9c_*} (1 + c_*^{2m+20}) h_K^{2k-1} \|u\|_{k+1,\omega,K}^2 .
\end{aligned}$$

Now we bound the error in the DG^+ norm (note that the terms with the traces of $u - z_h$ contain $\mathtt{h}^{-\frac{1}{2}}$ and the ones with the trace of the gradient contain $\mathtt{h}^{\frac{1}{2}}$):

$$\begin{aligned}
\inf_{v_h \in V_h} \|u - v_h\|_{DG^+}^2 \leq \|u - z_h\|_{DG^+}^2 \quad &\leq \omega^2 \|u - z_h\|_{0,\Omega}^2 + \|\nabla_h (u - z_h)\|_{0,\Omega}^2 \\
+ \sum_{K \in \mathcal{T}_h} \left(\sup \mathtt{a} + h_K \omega \sup (1 - \mathtt{d} \omega \mathtt{h}) + \sup \mathtt{b}^{-1} + \sup \mathtt{d}^{-1} \right) \frac{1}{h_K} \|u - z_h\|_{0,\partial K}^2 \\
+ \left(\sup \mathtt{b} + \sup \mathtt{d} + \sup \mathtt{a}^{-1} \right) h_K \|\nabla_h (u - z_h)\|_{0,\partial K}^2 \\
\leq C e^{9c_*} (1 + c_*^{2m+20}) h^{2k} \|u\|_{k+1,\omega,K}^2 ,
\end{aligned}$$

by taking the square roots we have the result. $\square$

First, we prove an a priori error estimate with respect to the DG norm, that corresponds to an energy norm. The order of convergence with respect to the meshsize is k, the regularity of the exact solution u. When u is sufficiently regular, namely $u \in H^{m+1}(\Omega)$, the order of convergence is $O(h^m)$, with $m = (p-1)/2$, exactly as predicted by the numerical simulation in [17]. In order to have this convergence is necessary that the mesh $\mathcal{T}_h$ is fine enough.

Theorem 4.5. *Let Ω be a bounded convex domain, $u \in H^{k+1}(\Omega)$, $1 \leq k \leq m$, be the analytical solution of the Helmholtz homogeneous equation, the condition* (4.56) *on the flux parameters be satisfied and $\mathtt{a}_{\min} > C_{tinv}^2$. Then, if this threshold condition is satisfied:*

$$\omega h \big(1 + \mathrm{diam}(\Omega)\, \omega \big) (1 + \omega h)^{\frac{3}{2}} < \frac{1}{C_{abs} C_{dual}}, \tag{4.61}$$

the estimate

$$\|u - u_h\|_{DG} \leq C e^{\frac{9}{2} c_*} \left(1 + c_*^{m+10} \right) \, h^k \, \|u\|_{k+1,\omega,\Omega} \tag{4.62}$$

holds with $C > 0$ independent of ω and $\mathcal{T}_h$.

Proof. Combining the results of the Propositions 4.2, 4.3 and 4.4 we have

$$\|u-u_h\|_{DG} \le C_{abs}\Big(C_{BA}e^{\frac{9}{2}c_*}\left(1+c_*^{m+10}\right)\ h^k\ \|u\|_{k+1,\omega,\Omega} \\ + C_{dual}\,\omega h(1+\omega h)^{\frac{3}{2}}\big(1+\mathrm{diam}(\Omega)\,\omega\big)\,\|u-u_h\|_{DG}\Big).$$

Taking to the left-hand side the last term, if (4.61) is verified, the coefficient of $\|u-u_h\|_{DG}$ is positive and we can conclude. ☐

Remark 4.6. *In order to obtain the best approximation estimate, it is enough to require that ωh is small, because the* (4.59) *depends exponentially on c_*. For the convergence of the method we need the more restrictive condition* (4.61) *which, with $\omega > 1$, requires a small $\omega^2 h$. This is exactly the* pollution effect *as described in [4]. Also in the PWDG method this phenomenon imposes a strong constrain on the mesh, and consequentially on the computational cost of the method. When the mesh satisfies this constraint the order of convergence is optimal.*

We conclude with an a priori error estimate with respect to the $L^2(\Omega)$-norm.

Theorem 4.7. *With the same hypotheses of Theorem 4.5, the bound*

$$\|u-u_h\|_{0,\Omega} \le Ce^{\frac{9}{2}c_*}\left(1+c_*^{m+\frac{23}{2}}\right)\ \big(1+\mathrm{diam}(\Omega)\,\omega\big)\ h^{k+1}\ \|u\|_{k+1,\omega,\Omega}\,, \tag{4.63}$$

is verified, with $C > 0$ independent of the wavenumber ω and the mesh $\mathcal{T}_h$.

The proof can be carried out along the same lines as Theorem 28 of [16].

With a fixed ω, the order of convergence in the L^2-norm is optimal; the constant in the estimate increases linearly with ω as a consequence of the pollution effect.

Remark 4.8. *In the regular case, when $u \in H^{m+1}(\Omega)$, and when $c_* < 1$ the error estimates for the method become*

$$\|u-u_h\|_{DG} \le C\,h^m\ \|u\|_{m+1,\omega,\Omega}\,,$$
$$\|u-u_h\|_{0,\Omega} \le C\,\big(1+\mathrm{diam}(\Omega)\,\omega\big)\,h^{m+1}\ \|u\|_{m+1,\omega,\Omega}$$

Both this estimates are sharp with respect to h, as it can be verified by the numerical simulation in [17].

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